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# Global Uniform Ultimate Boundedness of Semi-Passive Systems Interconnected over Directed Graphs

Anes Lazri   Mohamed Maghenem   Elena Panteley   Antonio Loría

**Abstract**—We analyze the solutions of networked heterogeneous nonlinear systems under diffusive consensus control. We assume that the individual systems are state strictly semi-passive and the closed-loop interconnected systems form a network with an underlying connected directed graph that contains a directed spanning tree. For these systems, we establish global uniform ultimate boundedness of the solutions. We provide an illustrative example involving a network of Stuart-Landau oscillators.

## I. INTRODUCTION

The analysis of interconnected nonlinear systems, given its complexity, has been widely addressed in various fields of research, including biology [1], sociology [2], and power engineering [3]. In that context, synchronization corresponds to the case in which all the systems asymptotically follow the same trajectory –see [4]. This notion, in addition to consensus, continue to attract growing interest in control community –see [5] [6].

The behavior adopted by interconnected agents depends on two main factors: the nature of the systems and the type of interconnection. For the latter, while linear coupling is the most commonly assumed, nonlinear coupling, such as that found in Kuramoto’s oscillator model [7], can also be found. Various forms of nonlinear coupling can be observed in contexts like neuronal cell modeling [8]. Furthermore, as far as system properties are concerned, the theory of passivity and dissipativity is among the most widely considered aspects for studying the synchronization of interconnected systems in [9], [10].

The main interest of this paper is to present results on uniform ultimate boundedness of semi-passive systems interconnected over directed networks. Roughly speaking, these are systems that define a passive map in regions of the state space that do not include a compact centered at the origin [11], [12], [9]. In [13], it can be seen that these properties are used to establish the asymptotic stability of the origin, as well as almost global orbital asymptotic stability for a directed network of heterogeneous systems. Furthermore, these results are essential in analyzing the uniform global asymptotic stability and uniform global practical asymptotic stability of the synchronization set, as shown in [4] for heterogeneous systems on a strongly connected network. Moreover, in [14], the ultimate boundedness of solutions is used to prove the global asymptotic stability of the

synchronization set for semi-passive systems interconnected over a strongly connected graph with communication delays. These results motivate and shed light on the importance of analyzing uniform ultimate boundedness for interconnected systems on a directed network without being strongly connected or balanced.

One of the first works to deal with the boundedness of solutions of a network of semi-passive units is [9], where the network is undirected. The ultimate boundedness of strictly semi-passive networked systems is, in turn, studied in [12]. In the latter, the authors prove the result for an undirected network. The result is then used in [15] to analyze the global asymptotic stability for such a network. With the same assumption on the topology, the authors of [1] show that conductance-based neuronal model (Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo, and Hindmarsh-Rose) that satisfy the semi-passivity property, are guaranteed to possess ultimately bounded solutions.

Many works analyze the ultimate boundedness of semi-passive systems over strongly connected graphs. In [16], the author proves that despite delays in the interconnections, the solutions of a network of semi-passive systems are globally ultimately stable when the network is strongly connected. Moreover, in [17], the authors address the problem of controlled synchronization of interconnected robotic systems with dynamic uncertainty and prove that in the general case of heterogeneous agents communicating on balanced graphs, the proposed control law that renders the closed-loop system semi-passive guarantees ultimate boundedness of the synchronization and the tracking errors. This again underlines the importance of this property in the field of control, whether in systems analysis or even in systems control.

However, it is observed that even in the case of a connected graph, solutions are uniformly ultimately bounded, which is the main objective of this paper. This paper extends these results to the case of a generic directed graph containing a spanning tree, without assuming the graph to be strongly connected or balanced.

In the next Section, we introduce some essential properties of interconnected systems. In Section II, we present the main problem before giving the principal statements in Section III. Our theoretical findings are illustrated with numerical simulations in Section IV, and we provide some closing remarks in Section V.

M. Maghenem is with University of Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, France. E-mail: mohamed.maghenem@cnrs.fr; E. Panteley and A. Loría are with L2S, CNRS, 91192 Gif-sur-Yvette, France. E-mail: elena.panteley@cnrs.fr and antonio.loria@cnrs.fr. A. Lazri is with L2S, CNRS, Univ Paris-Saclay, France (e-mail: anes.lazri@centralesupelec.fr)

## II. PROBLEM STATEMENT AND MAIN RESULT

Consider a set of heterogeneous nonlinear systems of the form<sup>1</sup>

$$\dot{x}_i = f_i(x_i) + u_i \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad (1)$$

where each  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  is continuous for all  $i \in \{1, 2, \dots, n\}$ ,  $f_i \neq f_j$  for all applying  $i$  and  $j$ , and  $u_i$  are control inputs. Let these systems enjoy the passivity property described below.

*Assumption 1 (State strict semi-passivity):* For each  $i \in \{1, 2, \dots, n\}$ , the input-output map  $u_i \mapsto x_i$  defined by the dynamics (1) is state strict semipassive. More precisely, there exist a continuously differentiable storage function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^+$ , a class  $\mathcal{K}_\infty$  function  $\underline{\alpha}_i$ , a constant  $\rho_i > 0$ , a continuous function  $H_i : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous function  $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ , such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad (2)$$

$$\dot{V}_i(x_i) \leq 2u_i x_i - H_i(x_i), \quad (3)$$

and  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . •

*Remark 1:* In [11] the property described in Assumption 1 is called strict *quasipassivity*. In [12] a similar concept is named strict semi-passivity, but radial unboundedness of the storage function is not imposed. See also [9]. •

We address the non-obvious question of whether the solutions of a set of heterogeneous nonlinear, state strictly semi-passive systems (1), remain bounded under a wide range of possible interconnection configurations. For instance, it is well-known that two feedback interconnected systems remain passive and, therefore, the solutions are bounded. Passivity, however, is lost under a cascaded interconnection and boundedness does not come for free, in general (*e.g.*, a cascade of integrators). Note that two feedback-interconnected systems form a simple undirected network, while a cascade may be assimilated to a directed path. In this paper, we address the question of boundedness of solutions for large-scale systems interconnected through directed graphs, under the effect of control inputs that are set to

$$u_i := -\gamma \sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\}, \quad (4)$$

where  $a_{ij} \geq 0$  are interconnection weights that are different in either direction, *i.e.*,  $a_{ij} \neq a_{ji}$  and  $\gamma > 0$  is a coupling gain. That is, in closed loop, the systems form a network with an underlying topology described by a graph  $\mathcal{G}$  that satisfies the following.

*Assumption 2:* The digraph  $\mathcal{G}$  is connected and contains a directed spanning tree. •

*Remark 2:* Particular interesting cases of graphs satisfying Assumption 2 include cascaded systems and leader-follower formation-control systems. We stress, however, that we do

<sup>1</sup>For simplicity, but without loss of generality, we assume that  $x \in \mathbb{R}$ ; all statements hold after pertinent changes in the notation, if  $x \in \mathbb{R}^p$ , with  $p > 1$ .

not assume that the network necessarily has a leader node with no incoming edges. •

In other words, for the closed-loop system in compact form

$$\dot{x} = F(x) - \gamma Lx, \quad (5)$$

where the elements of  $L$  are defined as

$$[L]_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^N a_{i\ell}, & i = j, \quad i, j \leq N, \end{cases}$$

$x := [x_1 \ \dots \ x_n]^\top$ , and  $F(x) := [f_1(x_1) \ f_2(x_2) \ \dots \ f_n(x_n)]^\top$ , we are interested in assessing the following.

**GUUB** The solutions  $t \rightarrow x(t)$  to (5) are globally ultimately bounded, uniformly in  $\gamma$ , if given  $\gamma_o > 0$ , there exists  $r = r(\gamma_o) > 0$  such that, for all  $r_o > 0$ , there exists  $T = T(r_o, \gamma_o) \geq 0$  such that, for all  $\gamma \geq \gamma_o$ ,

$$|x(t_o)| \leq r_o \Rightarrow |x(t)| \leq r \quad \forall t \geq T. \quad (6)$$

Our main statement is the following

*Theorem 1 (main result):* The solutions of the networked system (1)-(4) under Assumptions 1 and 2 are globally uniformly ultimately bounded, *i.e.*, the GUUB property holds. □

## III. RATIONALE OF THE MAIN RESULT

The statement of Theorem 1, for generic connected directed graphs, resides on a fact of interest in its own right. Roughly speaking, that an arbitrary connected graph contains a strongly-connected component—subgraph—that acts on the rest of the network's nodes as a leader. In that light the overall system (1) takes a cascade form. To illustrate this fact let us consider a network of  $n$  single integrators  $\dot{x}_i = u_i$ , that is, without the drift  $f_i(x_i)$ . Let these systems be interconnected according to the classical consensus protocol

$$u_i := -\sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\},$$

which corresponds to (4) with unitary gain  $\gamma = 1$ , as it is more commonly considered in related literature. In closed loop, the networked system is governed by the well-studied linear differential equation

$$\dot{x} = -Lx, \quad (7)$$

where  $L \in \mathbb{R}^n$  is the Laplacian matrix of a connected digraph  $\mathcal{G}$  that contains a directed spanning tree.

Now, if the digraph  $\mathcal{G}$  is connected and contains a spanning tree (Assumption 2), then it admits a decomposition into a leading strongly connected subgraph  $\mathcal{G}_\ell \neq \emptyset$  and a subgraph  $\mathcal{G}_f := \mathcal{G} \setminus \mathcal{G}_\ell$  of followers. Namely, the agents that do not belong to the leading component, and which we call the follower agents. In this case, up to a permutation, the Laplacian  $L$  admits the lower-block decomposition

$$L = \begin{bmatrix} L_\ell & 0 \\ -A_{\ell f} & M_f \end{bmatrix}, \quad (8)$$

where  $L_\ell := D_\ell - A_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$  is the Laplacian matrix of the strongly connected component  $\mathcal{G}_\ell$ , the lower-left block  $A_{\ell f} \in \mathbb{R}^{n_\ell \times n_f}$ ,  $n_f := n - n_\ell$ , is a non-negative matrix, and the lower-right block  $M_f \in \mathbb{R}^{n_f \times n_f}$  is a non-singular matrix [18]. The block  $M_f$  can be seen as the sum of the Laplacian matrix  $L_f$  corresponding to  $\mathcal{G}_f$  and a diagonal matrix  $D_{\ell f}$  gathering the weights of the interconnections between nodes in  $\mathcal{G}_\ell$  and the nodes in  $\mathcal{G}_f$ . That is,  $M_f = L_f + D_{\ell f}$ , where  $L_f = D_f - A_f$ .

It follows that, up to a permutation of the states  $x_i$  in  $x$ , the latter may be decomposed into  $x^\top := [x_\ell^\top \ x_f^\top]$ , where  $x_\ell \in \mathbb{R}^{n_\ell}$  gathers the states of the leading component and is governed by

$$\Sigma_\ell : \dot{x}_\ell = -L_\ell x_\ell, \quad (9)$$

and  $x_f \in \mathbb{R}^{n_f}$  contains the states of the nodes in the non-leading component, whose dynamics is modelled by

$$\Sigma_f : \dot{x}_f = -M_f x_f + A_{\ell f} x_\ell. \quad (10)$$

We see that the leading states follow their own dynamics and, because  $L_\ell$  is the graph of a strongly connected di-graph, achieve consensus. In particular, the solutions  $t \mapsto x_\ell(t)$  are bounded. On the other hand, the system  $\Sigma_f$  is perturbed by the (bounded) trajectories of the leading component. The two systems being linear autonomous, the boundedness of the solutions  $t \mapsto x_f(t)$  resides on a simple cascades argument and, certainly, on the properties of the matrix  $M_f$ . We explore this in finer detail next.

#### A. Lyapunov analysis over a digraph

It is known that consensus is achieved for  $\Sigma_\ell$  since  $L_\ell$  has exactly one single null eigen-value and others have positive real parts. More interestingly, we can provide a constructive proof, via Lyapunov's direct method, to establish uniform exponential stability of the consensus manifold

$$\mathcal{A}_\ell := \{x_\ell \in \mathbb{R}^{n_\ell} : x_{\ell 1} = x_{\ell 2} = \dots = x_{\ell n_\ell}\}. \quad (11)$$

Let  $v_o := [v_1 \ v_2 \ \dots \ v_{n_\ell}]^\top$  be a left eigenvector associated to  $\lambda_1(L_\ell) = 0$  and let  $V_o := \text{blkdiag}\{v_o\}$ . The vector  $v_o$  has the fundamental property that all its components are positive. More precisely, we recall the following statement that follows from [19, Theorem 4.31]. A proof is provided in the Appendix.

*Lemma 1:* Let  $L \in \mathbb{R}^{n \times n}$  be the Laplacian matrix of a directed and strongly connected graph. Let  $v_o := [v_1 \ v_2 \ \dots \ v_n]^\top \in \mathbb{R}^n$  be the left eigenvector of  $L$  associated with the null eigenvalue of  $L$ ; namely,  $v_o^\top L = 0$ . Then,  $v$  is positive,  $\text{Ker}(V_o L + L^\top V_o) = \text{Span}\{\mathbf{1}_n\}$ , and  $V_o L + L^\top V_o$  is positive semi-definite, where  $V_o := \text{blkdiag}\{v_o\}$ .  $\square$

Based on Lemma 1,  $Q_o := L_\ell^\top V_o + V_o L_\ell$  is symmetric and positive semi-definite and its kernel is spanned by  $\mathbf{1}_{n_\ell}$ . As a result, the derivative of the Lyapunov function candidate

$$W(x_\ell) := x_\ell^\top V_o x_\ell,$$

<sup>2</sup>For a vector  $v = [v_1 \ v_2 \ \dots \ v_{n_\ell}]^\top$  we define  $\text{blkdiag}\{v\}$  as the diagonal matrix whose diagonal elements correspond to those of  $v$  in the same order.

along the solutions to  $\Sigma_\ell$  in (9), satisfies

$$\dot{W}(x_\ell) = -x_\ell^\top [L_\ell^\top V_o + V_o L_\ell] x_\ell \quad (12)$$

$$\leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2, \quad (13)$$

where  $|x_\ell|_{\mathcal{A}_\ell}$  denotes the distance of  $x_\ell$  to the set  $\mathcal{A}_\ell$ , that is  $|x_\ell|_{\mathcal{A}_\ell} := \min\{|x_\ell - y| : y \in \mathcal{A}_\ell\}$ .

Now, let

$$Z(x_\ell) := [x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell]^\top V_o [x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell],$$

which is positive definite in the space of the consensus errors  $[x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell]$ . Indeed,  $v_o^\top x_\ell$  may be regarded as a weighted average of the states  $x_\ell$ —cf. [20], so consensus is achieved if and only if  $x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell = 0$ . Then, we use  $v_o^\top L = 0$ ,  $v_o^\top \mathbf{1}_{n_\ell} = 1$ , that  $\mathbf{1}_{n_\ell}$  is in the kernel of  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$ , and that  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$  is the Laplacian matrix of an all-to-all graph, so  $\mathbf{1}_{n_s}$  spans the kernel of  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$ . It follows that there exist  $\bar{z}, \underline{z} > 0$  such that

$$\underline{z} |x_\ell|_{\mathcal{A}_\ell}^2 \leq Z(x_\ell) \leq \bar{z} |x_\ell|_{\mathcal{A}_\ell}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (14)$$

On the other hand, the total derivative of  $Z$  along the solutions of  $\dot{x}_\ell = -L_\ell x_\ell$  satisfies

$$\dot{Z}(x_\ell) = -x_\ell^\top Q_o x_\ell \quad (15)$$

$$\leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2. \quad (16)$$

To obtain the previous expression we used the identities previously described. Uniform exponential stability of  $\mathcal{A}_\ell$  follows from (15), (14), and standard Lyapunov theory on stability of sets.

#### B. Exponential stability for $\Sigma_f$ on $\{x_\ell = 0\}$

We turn our attention now, to the followers' equation (10). For the sake of clarity, we restrict our analysis to this equation on the manifold  $\{x_\ell = 0\}$ . The purpose is to construct a Lyapunov function for the nominal part of the system (10), that serves to assess the robustness of the latter in the presence of bounded disturbances  $t \mapsto x_\ell(t)$ .

We start by remarking that the matrix  $M_f$  is a non-singular  $M$ -matrix. We recall that a matrix  $M := [m_{ij}]$ ,  $(i, j) \in \{1, 2, \dots, n\}^2$ , is an  $M$ -matrix if  $m_{ij} \leq 0$  for all  $i \neq j$  and its eigenvalues have non-negative real parts. Equivalently,  $M := \lambda I_n - B$ , where  $B$  is a non-negative matrix and  $\lambda \geq \rho(B)$ , where  $\rho(B) := \max\{|\lambda_i(B)| : i \in \{1, 2, \dots, n\}\}$  is the spectral radius of  $B$ —see [21], [18] for more details. This property is important to us in view of the following result, which is inspired from [19, Section 4.3.5]—a short original proof is provided in the Appendix, for completeness.

*Lemma 2:* Let  $M \in \mathbb{R}^{n \times n}$  be a non-singular  $M$ -matrix. Then, the matrices  $S := RM + M^\top R$  and  $R := \text{blkdiag}\{M^{-\top} \mathbf{1}_n\} (\text{blkdiag}\{M^{-1} \mathbf{1}_n\})^{-1}$  are positive definite. Moreover, if  $M \mathbf{1}_n \geq 0$  then  $\bar{S} := \bar{R}M + M^\top \bar{R}$  and  $\bar{R} := \text{blkdiag}\{M^{-\top} \mathbf{1}_n\}$  are positive definite.  $\square$

Based on Lemma 2, since  $M_f$  is a non-singular  $M$ -matrix and  $M_f \mathbf{1}_n \geq 0$ , then we can use the Lyapunov function candidate  $Y(x_f) := x_f^\top R_f x_f$ , where  $R_f :=$

$\text{blkdiag}\{M_f^{-\top}\mathbf{1}_{n_f}\}$ , which is positive definite. Furthermore, along the solutions to  $\Sigma_f$ , we have

$$\dot{Y}(x_f) = -x_f^\top [M_f^\top R_f + R_f M_f] x_f.$$

Now, since  $(M_f^\top R_f + R_f M_f)$  is positive definite, exponential stability of the origin for  $\Sigma_f$  follows. The interest of the function  $Y$  above is that it serves to study the robustness of  $\Sigma_f$  relative to bounded inputs  $x_\ell$ . We show this below, where we provide the main guidelines of the proof of our main statement.

### C. Sketch of Proof of Theorem 1

Consider now the system (5). Under Assumption 2, the Laplacian matrix  $L$  admits a permutation such that (8) holds. Therefore, akin to the case of Equation (7), the state  $x$  may be decomposed into  $x := [x_\ell^\top x_f^\top]^\top$  and the system (5) takes the cascaded form

$$\dot{x}_\ell = f_\ell(x_\ell) - \gamma L_\ell x_\ell, \quad (17a)$$

$$\dot{x}_f = f_f(x_f) + \gamma A_{\ell f} x_\ell - \gamma M_f x_f, \quad (17b)$$

where

$$f_\ell(x_\ell) := [f_1(x_{\ell_1}) \quad \cdots \quad f_{n_\ell}(x_{\ell_{n_\ell}})]^\top$$

$$f_f(x_f) := [f_{n_\ell+1}(x_{f_1}) \quad \cdots \quad f_{n_\ell+n_f}(x_{f_{n_f}})]^\top.$$

Equation (17a) corresponds to the dynamics of a *leading* component, a networked system with an underlying strongly connected graph  $\mathcal{G}_\ell$ , and a *follower* component, with dynamics (17b). The proof of the statement is constructed using a cascades argument and proving, firstly, global uniform ultimate boundedness for the solutions of (17a) and, consequently, the same property for (17b).

To that end, let  $r_o > 0$  be arbitrarily fixed and let  $|x(0)| \leq r_o$ . Then,  $|x_\ell(0)| \leq r_o$  and  $|x_f(0)| \leq r_o$ .

1) *Uniform ultimate boundedness for the leading component:* after Assumption 1, for each  $i \in \{1, 2, \dots, n_\ell\}$ , there exists a storage function  $V_i$  such that its total derivative along the trajectories of (1) satisfies

$$\dot{V}_i(x_{\ell_i}) \leq 2u_i^\top x_{\ell_i} - H_i(x_{\ell_i}) \quad (18)$$

$$H_i(x_{\ell_i}) \geq \psi_i(|x_{\ell_i}|) \quad \forall |x_{\ell_i}| \geq \rho_i. \quad (19)$$

Next, let  $W(x_\ell) := \sum_{i=1}^{n_\ell} v_i V_i(x_{\ell_i})$ , where  $v_i$  corresponds to the  $i$ th element of  $v_o$ , which is the left eigenvector associated to the zero eigenvalue of  $L_\ell$ . Since the graph  $\mathcal{G}_\ell$  is strongly connected, then  $v_i > 0$  for all  $i \in \{1, 2, \dots, n_\ell\}$ , so  $W$  is positive definite and radially unbounded. Now, from (18), we obtain

$$\dot{W}(x_\ell) \leq 2 \sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell_i} - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell_i}) \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (20)$$

The first term on the right-hand side of (20) satisfies

$$\sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell_i} = u^\top V_o x_\ell, \quad (21)$$

where  $V_o := \text{blkdiag}\{v_o\}$  and, since  $u = -\gamma L_\ell x_\ell$ , it follows that

$$\begin{aligned} \dot{W}(x_\ell) &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell_i}) - \gamma x_\ell^\top [L_\ell^\top V_o + V_o L_\ell] x_\ell \\ &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell_i}) - \gamma x_\ell^\top Q_o x_\ell, \end{aligned} \quad (22)$$

with  $Q_o := V_o L_\ell + L_\ell^\top V_o$ , which is positive semi-definite—see Lemma 1. Furthermore, we note that

$$\begin{aligned} -\gamma x_\ell^\top Q_o x_\ell &= - [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell]^\top Q_o [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell] \\ &\leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2, \end{aligned}$$

where  $|x_\ell|_{\mathcal{A}}$  denotes the distance of  $x_\ell$  to the set  $\mathcal{A}$  and  $\lambda_2(Q_o)$  is the second smallest eigenvalue of  $Q_o$ .

Now, on one hand, we have that  $v_i > 0$  for all  $i \in \{1, 2, \dots, n_\ell\}$  and, on the other,  $-H_i(x_{\ell_i}) > 0$  only if  $|x_{\ell_i}| \leq \rho_i$ . Therefore, introducing the constant

$$H_\ell := - \sum_{i=1}^{n_\ell} \max_{|x_i| \leq \rho_i} \{v_i H_i(x_{\ell_i})\} > 0,$$

after (22), we get

$$\dot{W}(x_\ell) \leq H_\ell - \gamma \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (23)$$

In turn, given  $\gamma_o > 0$  and  $\epsilon > 0$ , for all  $\gamma \geq \gamma_o$ , we have

$$\dot{W}(x_\ell) \leq H_\ell - \gamma_o \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \leq -\epsilon \quad \forall x_\ell \notin \mathcal{C}, \quad (24)$$

where

$$\mathcal{C} := \left\{ x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell|_{\mathcal{A}} \leq \sqrt{n_\ell} R_\epsilon := \sqrt{\frac{\epsilon + H_\ell}{\gamma_o \lambda_2(Q_o)}} \right\}.$$

From (24) it follows that the states  $x_\ell$  converge to a residual set which is a neighbourhood of the consensus manifold. That is,  $|x_\ell|_{\mathcal{A}}$  converges to the compact  $\mathcal{C}$ . Now, since  $|x_\ell|_{\mathcal{A}} = |x_\ell - \mathbf{1}_{n_\ell} (\mathbf{1}_{n_\ell}^\top x_\ell) / n_\ell|$  we also conclude that  $x_\ell$ , which satisfies

$$|x_\ell| \leq |x_\ell|_{\mathcal{A}} + |\mathbf{1}_{n_\ell}^\top x_\ell| / \sqrt{n_\ell}. \quad (25)$$

converges to a compact set and is ultimately bounded.

2) *Uniform ultimate boundedness for the followers:* Following up the previous computations and arguments, we establish global uniform ultimate boundedness for the non-leading component, determined by (17b).

Since  $M_f$  is a non-singular  $M$ -matrix, after Lemma 2, we conclude that  $S := P M_f + M_f^\top P$  and  $P := \text{blkdiag}\{M_f^{-\top} I_{n_f}\}$  are symmetric and positive definite. We also note that  $P$  is diagonal. Then, let  $p_i$ ,  $i \in \{1, 2, \dots, n_f\}$ , be the  $i$ th diagonal element of  $P$ . In addition, let

$$Z(x_f) := \sum_{i=1}^{n_f} p_i V_i(x_{f_i}).$$

Its total derivative along the trajectories of (17b) satisfies

$$\begin{aligned} \dot{Z}(x_f) &\leq - \sum_{i=1}^{n_f} p_i H_i(x_{f_i}) - \gamma x_f^\top [P M_f + M_f^\top P] x_f \\ &\quad + 2\gamma x_f^\top [P A_{\ell f}] x_\ell. \end{aligned} \quad (26)$$

Now, since the leader trajectories are ultimately bounded, it suffices to consider the latter inequality for all  $|x_\ell| \leq r_\ell$ , i.e.,

$$\dot{Z}(x_f) \leq H_f - \gamma \lambda_1(S) |x_f|^2 + 2\gamma \bar{p} r_\ell |x_f|, \quad (27)$$

where  $\bar{p} := |PA_{\ell f}|$  and

$$H_f := \sum_{i=1}^{n_f} \max_{|x_{fi}| \leq \rho_i} \{p_i H_i(x_{fi})\}.$$

The latter follows under Assumption 1. After (27) it follows that  $\dot{Z}(x_f) \leq -\epsilon$  for all  $\gamma \geq \gamma_o$  and for all  $x_f$  and  $x_\ell$  such that

$$|x_f| > \beta_1 := 1 + \frac{2\bar{p}r_\ell}{\lambda_1(S)} + \sqrt{\frac{\epsilon + H_f}{\gamma_o \lambda_1(S)}} \quad \text{and} \quad |x_\ell| \leq r_\ell.$$

Ultimate boundedness follows.

#### IV. EXAMPLE

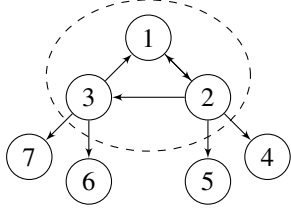


Fig. 1. Interconnection graph of Stuart-Landau oscillators with a strongly connected component

Consider a network of  $N = 7$  interconnected Stuart-Landau oscillators,

$$\dot{x}_i = \alpha_i x_i - \omega_i y_i - x_i(x_i^2 + y_i^2) + u_{1i} \quad (28a)$$

$$\dot{y}_i = \omega_i x_i + \alpha_i y_i - y_i(x_i^2 + y_i^2) + u_{2i}, \quad (28b)$$

where  $\alpha_i, \omega_i, x_i$ , and  $y_i \in \mathbb{R}$  for all  $i \in \{1, 2, \dots, N\}$ , and  $x_i, y_i$  are Cartesian coordinates on the plane. Relative to such models,  $x_i$  and  $y_i$  represent the real and the imaginary parts of each oscillator's state. Constants  $\alpha_i$  and  $\omega_i$  are randomly chosen in the interval  $[0.5, 3]$ .

Let  $x_i := [x_i \ y_i]^\top$  and  $u := -\gamma[L \otimes I_2]x$ , where  $L$  is the Laplacian matrix corresponding to the interconnection graph shown in Figure 1, and for the purpose of simulation, we set  $\gamma = 5$ .

The graph is not strongly connected, but it satisfies Assumption 2. Moreover, the first  $n_\ell = 3$  nodes of this Graph form a strongly connected subnetwork. Furthermore, Assumption 1 is also satisfied with  $V_i(x_i) = x_i^2 + y_i^2$ . The total derivative of  $V_i$  along the solutions of (28) yields

$$\dot{V}_i(x_i) = -2(x_i^2 + y_i^2)(x_i^2 + y_i^2 - \alpha_i) + 2x_i^\top u,$$

where  $H_i(x_i) := 2(x_i^2 + y_i^2)(x_i^2 + y_i^2 - \alpha_i) \geq \psi(|x|) := \alpha_i |x|^2$  for all  $|x| \geq \sqrt{2\alpha_i}$ .

Figure 2 shows that the trajectories of the system are globally uniformly ultimately bounded.

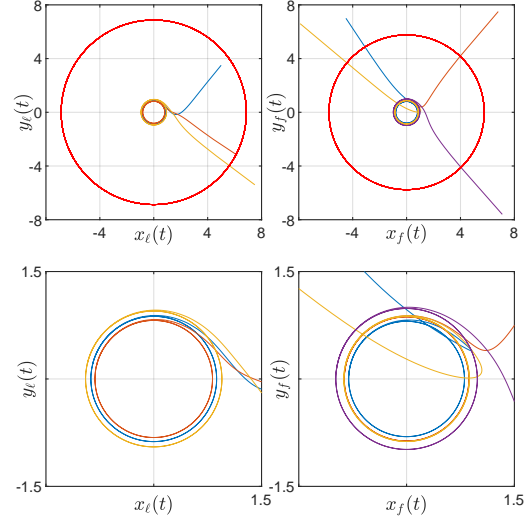


Fig. 2. Trajectories of the solutions to (28) on the  $(x,y)$ -axis. The oscillators do not achieve dynamic consensus (see the upper plots), but the trajectories are uniformly ultimately bounded (see the zoomed lower plots). The thick curves represent estimates of the ultimate bounds.

#### V. FURTHER DISCUSSION

##### A. Generalization to Weakly Connected Di-Graphs

Our results can be easily extended to graphs that are weakly connected. Indeed, a weakly connected di-graph  $\mathcal{G}$  can be decomposed into  $S \in \{1, 2, \dots, n\}$  strongly connected subgraphs  $\{\mathcal{G}^s(\mathcal{V}^s, \mathcal{E}^s)\}_{s=1}^S$ . Among the strongly connected subgraphs in  $\mathcal{G}$ , we identify the *leading* strongly connected subgraphs  $\{\mathcal{G}^s\}_{s \in \{1, \dots, S^l\}}$ ,  $S^l \leq S$ , with no in-neighbors from any other strongly connected subgraph. The Laplacian matrix  $L$  in this case can be expressed, up to some permutation, in the more general lower-block triangular form

$$L = \begin{bmatrix} \text{blkdiag} \{L_{\ell s}\} & 0 \\ s \in \{1, \dots, S^l\} & \\ & -A_{lf} & M_f \end{bmatrix},$$

where each  $L_{\ell s} := D_{\ell s} - A_{\ell s} \in \mathbb{R}^{n_s \times n_s}$  corresponds to the Laplacian matrix of the strongly connected graph  $\mathcal{G}_s$ , the lower-left block  $A_{lf} \in \mathbb{R}^{n_f \times n - n_f}$ ,  $n_f := n - \sum_{s \in S^l} n_s$ , is a non-negative matrix, and the lower-right block  $M_f \in \mathbb{R}^{n_f \times n_f}$  is a non-singular  $M$ -matrix.

The extension consists in determining an ultimate bound for each leading subgroup, as in part 1) in the proof of Theorem 1. Then, the same arguments as in part 2) allow to deduce the ultimate bound for the followers. The only difference, now, is that the dynamics of  $x_f$  may admit multiple entries  $x_{\ell s}$ ,  $s \in \{1, 2, \dots, S^l\}$ , instead of only one.

##### B. Discrete-Time Systems

For a network of discrete-time system of the form

$$x_i^+ = x_i + f_i(x_i) + u_i \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}. \quad (29)$$

The protocol  $u = -\delta Lx$  in this case generates the closed-loop dynamics

$$x^+ = F(x) + (I_n - \delta L)x. \quad (30)$$

In such a discrete-time setting, it is important to choose  $\delta$  within the interval  $(0, \bar{\delta}]$ , where  $\bar{\delta}$  is such that the matrices  $L_\ell^\top V_o + V_o L_\ell - 2\bar{\delta}(L_\ell^\top V_o L_\ell)$  and  $M_f^\top P + P M_f - 2\bar{\delta}(M_f^\top P M_f)$  are positive definite, where  $V_o$  and  $P$  are introduced in the sketch of proof of Theorem 1. Indeed, for large values of  $\delta$ , the solutions to (30) can diverge, even when  $F \equiv 0$ ; see [22]. When respecting the aforementioned bounds on  $\delta$ , our results apply to (30) under the following discrete-time version of input-to-state semi-passivity property.

*Definition 1:* For each  $i \in \{1, 2, \dots, n\}$ , the input-output map  $u_i \mapsto x_i$  defined by the dynamics (30) is state strict semi-passive if there exists a continuously differentiable storage function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^+$ , a constant  $\rho_i > 0$ , a continuous function  $H_i : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous function  $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ , such that

$$V(x^+) - V(x) \leq 2u_i x_i + u_i^2 - H_i(x_i), \quad (31)$$

and  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . •

Definition 1 is derived from the definition of discrete-time passivity [23], and discrete-time quasi-passivity [24] for systems with structure (29).

### C. Global Asymptotic Practical Stability

Global uniform ultimate boundedness played a key role in [25] in order to deduce more refined results for networks of the form (5). In particular, it is shown in the aforementioned reference that the consensus set, for (5), is globally practically asymptotically stable provided that the following additional assumption is satisfied.

*Assumption 3:* The functions  $f_i$  are continuously differentiable and satisfy  $f_i(0) = 0$  for all  $i \in \{1, 2, \dots, n\}$ . Furthermore, the origin  $\{x_m := \frac{1}{n} \mathbf{1}^\top x / n = 0\}$  of the averaged dynamics  $\dot{x}_m = v_o^\top F(\frac{1}{n} \mathbf{1} x_m)$  is globally asymptotically stable. •

Further assumptions also allowed to establish almost global asymptotic stability of a limit cycle for (5), leading to frequency synchronization of the network. The proofs take advantage of the uniform ultimate boundedness property established here.

## REFERENCES

- [1] E. Steur, I. Tyukin, and H. Nijmeijer, "Semi-passivity and synchronization of diffusively coupled neuronal oscillators," *Physica D: Nonlinear Phenomena*, vol. 238, no. 21, pp. 2119–2128, 2009.
- [2] C. Altafini, "Dynamics of opinion forming in structurally balanced social networks," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 5876–5881.
- [3] N. I. Sieber, "Time-scale modeling of dynamic networks with applications to power systems," *Zamm-zeitschrift Fur Angewandte Mathematik Und Mechanik*, vol. 64, pp. 340–340, 1984.
- [4] E. Panteley and A. Loria, "Synchronization and dynamic consensus of heterogeneous networked systems," *IEEE Trans. on Automatic Control*, vol. 62, no. 8, pp. 3758–3773, 2017.
- [5] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1511–1523, 2013.
- [6] J. Wang and X. Hu, "Distributed consensus in multi-vehicle cooperative control: Theory and applications," *IEEE Control Systems Magazine*, vol. 30, pp. 85–86, 06 2010.
- [7] J. E. Mota de Oliveira and M. G. Quiles, "Community detection in complex networks using coupled kuramoto oscillators," in *2014 14th International Conference on Computational Science and Its Applications*, 2014, pp. 85–90.
- [8] N. Corson, G. Rawan, S. Balev, and C. Bertelle, "Modelling the dynamics of complex interaction systems: from morphogenesis to control," *International Journal of Bifurcation and Chaos*, vol. 22, 04 2012.
- [9] A. Pogromsky, "Passivity-based design of synchronizing systems," *International Journal of Bifurcation and Chaos*, vol. 8, 02 1998.
- [10] N. Chopra and M. Spong, "Passivity-based control of multi-agent systems," *Passivity-Based Control of Multi-agent Systems*, vol. 34, 01 2006.
- [11] I. G. Polushin, D. Hill, and A. L. Fradkov, "Strict quasipassivity and ultimate boundedness for nonlinear control systems," *IFAC Proceedings Volumes*, vol. 31, no. 17, pp. 505–510, 1998, 4th IFAC Symposium on Nonlinear Control Systems Design, Enschede, Netherlands, 1-3 July 1998.
- [12] A. Y. Pogromsky, T. Glad, and H. Nijmeijer, "On diffusion driven oscillations in coupled dynamical systems," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 9, no. 4, pp. 629–644, 1999.
- [13] M. Maghenem, E. Panteley, and A. Loria, "Singular-Perturbations-Based Analysis of Dynamic Consensus in Directed Networks of Heterogeneous Nonlinear Systems," Aug. 2022, submitted for publication to *IEEE Transactions on Automatic Control*. [Online]. Available: <https://hal.science/hal-03752340>
- [14] E. Steur and H. Nijmeijer, "Synchronization in networks of diffusively time-delay coupled (semi-)passive systems," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, no. 6, pp. 1358–1371, 2011.
- [15] A. Y. Pogromski and H. Nijmeijer, "Cooperative oscillatory behavior of mutually coupled dynamical systems," *IEEE Trans. on Circuits and Systems I: Fundamental Theory and Applications*, vol. 48, no. 2, pp. 152–162, 2001.
- [16] N. Chopra, "On ultimate boundedness of delay synchronization algorithms for semi-passive systems," in *2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 2010, pp. 1663–1668.
- [17] Y.-C. Liu and N. Chopra, "Robust controlled synchronization of interconnected robotic systems," in *Proceedings of the 2010 American Control Conference*, 2010, pp. 1434–1439.
- [18] M. U. Javed, J. I. Poveda, and X. Chen, "Excitation conditions for uniform exponential stability of the cooperative gradient algorithm over weakly connected digraphs," *IEEE Control Systems Letters*, vol. 6, pp. 67–72, 2021.
- [19] Z. Qu, *Cooperative control of dynamical systems: applications to autonomous vehicles*. London, UK: Springer Verlag, 2009.
- [20] E. Panteley, A. Loria, and S. Sukumar, "Strict lyapunov functions for consensus under directed connected graphs," in *Proc. European Control Conference (ECC)*, St. Petersburg, Russia, 2020, pp. 935–940.
- [21] X. Chen, B. Xudong, M.-A. Belabbas, and T. Basar, "Controllability of formations over directed time-varying graphs," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 3, pp. 407–416, 2017.
- [22] F. Cacace, M. Mattioni, S. Monaco, and D. Normand-Cyrot, "A new distributed protocol for consensus of discrete-time systems," *European Journal of Control*, p. 100833, 2023.
- [23] Z. Liu, M. Zhang, A. Saberi, and A. A. Stoorvogel, "Passivity based state synchronization of homogeneous discrete-time multi-agent systems via static protocol in the presence of input delay," *European Journal of Control*, vol. 41, pp. 16–24, 2018.
- [24] E. M. Navarro-López, "Local feedback passivation of nonlinear discrete-time systems through the speed-gradient algorithm," *Automatica*, vol. 43, no. 7, pp. 1302–1306, 2007.
- [25] M. Maghenem, E. Panteley, and A. Loria, "Singular-perturbations-based analysis of synchronization in heterogeneous networks: a case-study," in *Proc. 55th IEEE Conf. Decision and Control*, Las Vegas, NV, USA, 2016, pp. 2581–2586.
- [26] W. Ren and R. W. Beard, *Distributed consensus in multivehicle cooperative control*. Springer verlag, 2005.
- [27] S. U. Pillai, T. Suel, and C. Seunghun, "The perron-frobenius theorem: some of its applications," *IEEE Signal Processing Magazine*, vol. 22, no. 2, pp. 62–75, 2005.

APPENDIX

*Proof of Lemma 1:* After [26, Theorem C.3.], the matrix  $L$  is irreducible if and only if the corresponding graph is strongly connected. Furthermore, since the off-diagonal elements of  $L$  are nonpositive and  $L\mathbf{1}_n = 0$ , we conclude that  $L$  is a singular  $M$ -matrix. Hence, using [19, Theorem 4.31], we conclude that the only left eigenvector of  $L$  corresponding to the null eigenvalue has strictly positive entries and the matrix  $V_oL + L^\top V_o$  is positive semi-definite. Finally, to show that  $\mathbf{1}_n$  spans the kernel of  $V_oL + L^\top V_o$ , we start noticing that  $(V_oL + L^\top V_o)\mathbf{1}_n = 0$ . Furthermore, each off-diagonal element of  $V_oL + L^\top V_o$  is given by  $a_{ij}v_i + a_{ji}v_j$ ,  $(i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ . As a result,  $V_oL + L^\top V_o$  is the Laplacian matrix associated with a bi-directional graph that is as connected as the graph of  $L$  disregarding the direction of the interconnections. Since the graph of  $L$  is already a strongly connected graph, we conclude that  $\text{rank}(V_oL + L^\top V_o) = n - 1$ . ■

*Proof of Lemma 2:* We start noting that if  $M$  is a non-singular  $M$ -matrix, so is  $M^{-1}$ . As a result,  $M^{-1}$  and  $M^{-\top}$  are both non-negative matrices (see Lemma 3). Moreover, since  $M^{-1}$  and  $M^{-\top}$  are non-singular, it follows that  $M^{-1}\mathbf{1}_n$  and  $M^{-\top}\mathbf{1}_n$  are positive vectors. Hence, the matrices  $R$  and  $\bar{R}$  are positive definite. To complete the proof, we show that  $S$  is a non-singular  $M$ -matrix, which is enough to conclude that  $S$  is positive definite since it is symmetric. Indeed,

$$\begin{aligned} S(M^{-1}\mathbf{1}_n) &= R\mathbf{1}_n + M^\top R M^{-1}\mathbf{1}_n \\ &= R\mathbf{1}_n + M^\top R \text{blkdiag}\{M^{-1}\mathbf{1}_n\}\mathbf{1}_n \\ &= R\mathbf{1}_n + M^\top \text{blkdiag}\{M^{-\top}\mathbf{1}_n\}\mathbf{1}_n \\ &= R\mathbf{1}_n + M^\top M^{-\top}\mathbf{1}_n \\ &= R\mathbf{1}_n + \mathbf{1}_n > 0. \end{aligned}$$

That is, we just showed that the multiplication of  $S$  by a positive vector is a positive vector. Since the off-diagonal elements of  $S$  are non-positive, the latter is enough to conclude that  $S$  is non-singular  $M$ -matrix according to Lemma 3.

Finally, we show that, when  $M\mathbf{1}_n \geq 0$ , then  $\bar{S}$  is a non-singular  $M$ -matrix, which is enough to conclude that  $\bar{S}$  is positive definite since it is symmetric. To do so, it is enough to compute the product  $\bar{S}\mathbf{1}_n$  and use the same arguments as before. ■

*Lemma 3:* Let the off-diagonal elements of  $M \in \mathbb{R}^{n \times n}$  be non-positive. Then, the following properties are equivalent:

- a)  $M$  is a non-singular  $M$ -matrix.
- b)  $M$  is non-singular and  $M^{-1}$  is a non-negative matrix.
- c) There exists  $x > 0$  such that  $Mx > 0$ . □

*Proof:* Property a) is equivalent to the existence of  $\lambda > 0$  and  $B \geq 0$  such that  $M = \lambda I_n - B$  and  $\lambda > \rho(B)$ . In other words, by letting  $T := B/\lambda$ , we conclude that  $\rho(T) < 1$ . The latter inequality implies the existence of the limit  $\lim_{k \rightarrow \infty} \sum_{i=0}^k T^i$ , which is equal to  $(I_n - T)^{-1} = \lambda M^{-1}$ . Since  $T \geq 0$ , we conclude that so is the limit  $\lim_{k \rightarrow \infty} \sum_{i=0}^k T^i$ ; thus, a) implies b) To prove that b) implies c), it is enough to take  $x := M^{-1}\mathbf{1}_n$ .

Now, to prove that c) implies a), we let  $X := \text{blkdiag}\{x\}$  with  $x := M^{-1}\mathbf{1}_n$ . As a result,  $Mx = MX\mathbf{1}_n > 0$ , and since  $X$  is positive definite, we conclude that  $(X^{-1}MX)\mathbf{1}_n > 0$ , which implies that  $X^{-1}MX$  is strictly diagonally dominant; hence, by Lemma 4, we conclude that  $X^{-1}MX$  is a non-singular  $M$ -matrix; thus, so is  $M$ . ■

*Lemma 4:* If  $M \in \mathbb{R}^{n \times n}$  is symmetric and diagonally dominant, then it is positive semi-definite. If  $M$  is symmetric and strictly diagonally dominant, then it is positive definite. □

*Proof:* It is enough to observe that, based on Girshgorin Desk Theorem [27], for every eigenvalue  $\lambda$  of a general symmetric matrix  $M \in \mathbb{R}^{n \times n}$ , there exists  $i \in \{1, 2, \dots, n\}$  such that  $m_{ii} - \sum_{j \neq i} |m_{ij}| \leq \lambda \leq m_{ii} + \sum_{j \neq i} |m_{ij}|$ . ■