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A NOTE ON CONTRACTIVE SEMI-GROUPS ON A 1:1 JUNCTION FOR SCALAR CONSERVATION LAWS AND HAMILTON-JACOBI EQUATIONS

P. CARDALIAGUET

ABSTRACT. We show that any continuous semi-group on L^1 which is (i) L^1 -contractive, (ii) satisfies the conservation law $\partial_t \rho + \partial_x(H(x, \rho)) = 0$ in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ (for a space discontinuous flux $H(x, p) = H^l(p)\mathbf{1}_{x < 0} + H^r(p)\mathbf{1}_{x > 0}$), and (iii) satisfies natural continuity and scaling properties, is necessarily given by a germ condition at the junction: $\rho(t, 0) \in \mathcal{G}$ a.e., where \mathcal{G} is a maximal, L^1 -dissipative and complete germ. In a symmetric way, we prove that any continuous semi-group on L^∞ which is (i) L^∞ -contractive, (ii) satisfies with the Hamilton-Jacobi equation $\partial_t u + H(x, \partial_x u) = 0$ in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ (for a space discontinuous Hamiltonian H as above), and (iii) satisfies natural continuity and scaling properties, is necessarily given by a flux limited solution of the Hamilton-Jacobi equation.

1. INTRODUCTION

The aim of this note is to identify the conservation law (CL) semi-groups and the Hamilton-Jacobi (HJ) semi-groups which are L^1 - (for the CL) or L^∞ - (for HJ) contracting, in the context of a 1:1 junction. Namely we consider a discontinuous flux (or Hamiltonian) $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$H(x, p) = H^l(p)\mathbf{1}_{\{x < 0\}} + H^r(p)\mathbf{1}_{\{x > 0\}}$$

and look at the CL

$$\partial_t \rho + \partial_x(H(x, \rho)) = 0 \quad \text{in } \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$$

or the HJ equation

$$\partial_t u + H(x, \partial_x u) = 0 \quad \text{in } \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}).$$

The problem is to find conditions that have to be put at the junction $x = 0$ to ensure that the equation is well-posed and L^1 -contractive (for the CL) or L^∞ -contractive (for the HJ). Many papers have built such semi-groups for the CL: see, for instance, [1, 2, 3, 4, 5, 10, 14, 15]. The main contribution of [2] is to identify a general class of such semi-groups, expressed through the notion of \mathcal{G} -entropy solutions: the additional condition takes the form $(\rho(t, 0^-), \rho(t, 0^+)) \in \mathcal{G}$ for a.e. $t \geq 0$, where the set $\mathcal{G} \subset \mathbb{R}^2$ is a maximal, L^1 -dissipative and complete germ (see [2]). A somewhat symmetric construction is done in [11] for the HJ equation, introducing the notion of flux limited solutions: the additional condition is now takes the form:

$$\partial_t u + \min\{\bar{A}, (H^l)^+(\partial_x^- u), (H^r)^-(\partial_x^+ u)\} = 0 \quad \text{in } (0, \infty) \times \{0\},$$

for a suitable flux limiter \bar{A} (see below for the notations). The aim of this short note is to show that these semi-groups—described through germs (for CL) or flux limiters (for HJ)—are actually the only contractive ones.

Our main results (Theorem 2.1 for the HJ equation and Theorem 3.1 for the conservation law) state that, if a continuous semi-group is L^1 - or L^∞ - contractive, coincides locally with the CL or HJ semi-group outside $x = 0$ and satisfies natural continuity and scaling properties, then it is necessarily given by a germ (for the CL) or a flux limiter (for the HJ). The choice to present at the same time the case of CL and of HJ is related to the fact that there is a strong relationship between both semi-groups: see [6] (for convex Hamiltonians) and [9] (for coercive ones). We make a use of this property in the proofs.

These results are somewhat expected: indeed, the introduction of [2] suggests that the notion of \mathcal{G} -entropy solutions was introduced to describe all the L^1 -dissipative semi-groups for conservation law on an 1:1 junction. On the other hand, [9, 11] show that, for the HJ equation, general conditions of the form $\partial_t u + F(\partial_x u) = 0$ at the junction can be expressed in terms of flux limiter conditions. The main interest of this note is to actually *prove* the fact that natural contraction properties are enough to identify the semi-groups. To fix the ideas we work in the framework of traffic flow, i.e., assume that the

fluxes/Hamiltonians $H^{l/r}$ are strictly concave. The results can probably be extended to more general fluxes or Hamiltonians, but to the expense of more technical proofs.

Let us finally say a word about more complex junctions, i.e., junctions with more than one incoming branch or more than one outgoing one. The situation is then much more delicate: approaches by HJ and by CL are no longer equivalent (see for instance a counterexample in [6]), and the class of conditions that have to be put at the junction is much richer and—yet—far from understood.

2. IDENTIFICATION OF THE HJ EQUATIONS

We fix two smooth, Lipschitz and uniformly concave Hamiltonians $H^l : [0, R^l] \rightarrow \mathbb{R}$ and $H^r : [0, R^r] \rightarrow \mathbb{R}$, with $H^l(R^l) = H^r(R^r) = H^l(0) = H^r(0) = 0$ (for some $R^l, R^r > 0$) and set

$$H(x, p) = H^l(p)\mathbf{1}_{\{x < 0\}} + H^r(p)\mathbf{1}_{\{x > 0\}}.$$

We define Lip as the set of Lipschitz maps $u : \mathbb{R} \rightarrow \mathbb{R}$ with $u' \in [0, R^l]$ a.e. in $(-\infty, 0)$ and $u' \in [0, R^r]$ a.e. in $(0, \infty)$. The set Lip is endowed with the topology of local uniform convergence. We denote by \bar{S}^{HJ} the semi-group associated with the Hamilton-Jacobi equation on the line: given $u_0 \in Lip$, $u(t, x) := \bar{S}^{HJ}(t, u_0)(x)$ is the *minimal* solution to

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 & \text{in } (0, \infty) \times \mathbb{R} \\ u(0, x) = u_0(x) & \text{in } \mathbb{R} \end{cases}$$

Then $u(t, \cdot) \in Lip$ for any $t \geq 0$ and \bar{S}^{HJ} is a continuous semi-group. Note that, if $H^l = H^r$, then \bar{S}^{HJ} the standard semi-group associated with the Hamilton-Jacobi equation on the line. Here we need to specify that it is the minimal solution as the Hamiltonian is discontinuous.

Our aim is to identify any semi-group S^{HJ} satisfying the following conditions:

- (H1) $S^{HJ} : [0, \infty) \times Lip \rightarrow Lip$ is a continuous homogeneous semi-group,
- (H2) S^{HJ} is L^∞ -contractive: if $u_0^1, u_0^2 \in Lip \cap L^\infty(\mathbb{R})$, $\|S^{HJ}(t, u_0^1) - S^{HJ}(t, u_0^2)\|_\infty \leq \|u_0^1 - u_0^2\|_\infty$,
- (H3) S^{HJ} permutes with constants: $S^{HJ}(t, u_0 + c) = S^{HJ}(t, u_0) + c$ for any $c \in \mathbb{R}$ and $u_0 \in Lip$,
- (H4) S^{HJ} has a finite propagation property: there exists $C_0 > 0$ such that, if $u_0^1 = u_0^2$ on $[a, b]$, then $S^{HJ}(t, u_0^1) = S^{HJ}(t, u_0^2)$ on $[a + C_0 t, b - C_0 t]$ for $t \in [0, C_0^{-1}(b - a)]$,
- (H5) S^{HJ} coincides locally in $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with \bar{S}^{HJ} : for any $u_0 \in Lip$ and $x \neq 0$,
 $S^{HJ}(t, u_0)(y) = \bar{S}^{HJ}(t, u_0)(y)$ for any $t \in (0, C_0^{-1}|x|)$ and $y \in \mathbb{R}$ with $|y - x| < |x| - C_0 t$,
- (H6) S^{HJ} is scale invariant: for any $\epsilon, t > 0$ and $u_0 \in Lip$, $\epsilon S^{HJ}(t/\epsilon, \epsilon u_0(\cdot/\epsilon))(x/\epsilon) = S^{HJ}(t, u_0)(x)$.

Theorem 2.1. *Under the conditions above, there exists $\bar{A} \in [0, A_{\max}]$ such that $S^{HJ} = S^{HJ, \bar{A}}$, where $A_{\max} = \min\{\max H^l, \max H^r\}$ and $S^{HJ, \bar{A}}$ is the HJ semi-group with flux-limiter \bar{A} , i.e., for a given initial condition $u_0 \in Lip$, the viscosity solution to*

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 & \text{in } (0, \infty) \times (\mathbb{R} \setminus \{0\}) \\ \partial_t u + \min\{\bar{A}, (H^l)^+(\partial_x^- u), (H^r)^-(\partial_x^+ u)\} = 0 & \text{in } (0, \infty) \times \{0\} \\ u(0, x) = u_0(x) & \text{in } \mathbb{R} \end{cases} \quad (1)$$

Above $(H^{l/r})^+$ (respectively $(H^{l/r})^-$) denotes the smallest nondecreasing (resp. nonincreasing) map above $H^{l/r}$. The notion of flux-limited viscosity solution of (1) is defined as usual, except that the test functions $\phi = \phi(t, x)$, which are continuous in $\mathbb{R}_+ \times \mathbb{R}$ and C^1 in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, have only a left- and right- space derivative at $x = 0$, denoted respectively by $\partial_x^- \phi(t, 0)$ and $\partial_x^+ \phi(t, 0)$. This notion of viscosity flux limited solution for (1) was introduced in [11], which also established that, for any $A \in [0, A_{\max}]$, there exists a unique viscosity solution to (1). Note that $\bar{S}^{HJ} = S^{HJ, A_{\max}}$. Recall finally that, by [8], under (H3), (H2) is equivalent to a comparison principle: $u_0^1 \leq u_0^2 \Rightarrow S^{HJ}(t, u_0^1) \leq S^{HJ}(t, u_0^2)$.

Proof. To simplify notation, we set throughout the proof $S = S^{HJ}$, $\bar{S} = \bar{S}^{HJ}$ and $S^A = S^{HJ, A}$. Let us start with preliminary remarks. For $u_0 \in Lip$, the map $(t, x) \rightarrow S(t, u_0)(x)$ is continuous and a solution of the HJ equation in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$: by [11, Theorem 2.7] it is therefore a supersolution to the HJ equation on the whole space $(0, \infty) \times \mathbb{R}$. We infer by comparison that $S(t, u_0) \geq \bar{S}(t, u_0)$.

Given $A \in [0, A_{\max}]$, let $p_A^{r/l,-}$ and $p_A^{r/l,+}$ be respectively the smallest and the largest solutions to $H^{r/l}(p) = A$. We define the maps

$$\check{\phi}_A(x) = p_A^{l,-} x \mathbf{1}_{x \leq 0} + p_A^{r,+} x \mathbf{1}_{x \geq 0}, \quad \hat{\phi}_A(x) = p_A^{l,+} x \mathbf{1}_{x \leq 0} + p_A^{r,-} x \mathbf{1}_{x \geq 0}.$$

We note that

$$S^A(t, \hat{\phi}_A) = \hat{\phi}_A - tA.$$

Indeed, the map $w(t, x) := \hat{\phi}_A(x) - tA$ is a classical solution to the HJ equation in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ and satisfies the initial condition. As for the condition at the vertex, it holds in a classical sense since

$$\partial_t w(t, 0) + \min\{A, H^{l,+}(\partial_x^- w(t, 0)), H^{r,-}(\partial_x^+ w(t, 0))\} = -A + \min\{A, H^{l,+}(p_A^{l,+}), H^{r,-}(p_A^{r,-})\} = 0.$$

The idea of the proof is to first identify $S(t, \cdot)$ for the particular functions $\check{\phi}_A$ and $\hat{\phi}_A$, and then to use the $\check{\phi}_A$ and $\hat{\phi}_A$ as test functions. We do this through several steps: we first compute $S^A(t, \hat{\phi}_0)$ (Step 1), then define \bar{A} and show, thanks to the first step, that $S(t, \hat{\phi}_0) = S^{\bar{A}}(t, \hat{\phi}_0)$ (Step 2): as $u \geq \hat{\phi}_0 + u(0)$ for any $u \in Lip$, this will give the supersolution property. To prove the subsolution property, we show that $S(t, \check{\phi}_A) = S^{\bar{A}}(t, \check{\phi}_A)$ (Step 3). We conclude in Step 4 by the proof of the equality $S = S^{\bar{A}}$.

Step 1. We first claim that, for any $A \in [0, A_{\max}]$,

$$S^A(t, \hat{\phi}_0)(x) = \begin{cases} \hat{\phi}_A(x) - tA & \text{if } x \in [t(H^l)'(p_A^{l,+}), t(H^r)'(p_A^{r,-})], \\ \bar{S}(t, \hat{\phi}_0)(x) & \text{otherwise} \end{cases} \quad (2)$$

Moreover $u(t, x) := S^A(t, \hat{\phi}_0)(x)$ is the unique viscosity solution to the boundary value problem:

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 & \text{in } (0, \infty) \times (\mathbb{R} \setminus \{0\}), \\ u(t, 0) = -tA & \text{in } (0, \infty) \times \{0\}, \\ u(0, x) = \hat{\phi}_0(x) & \text{in } \mathbb{R}. \end{cases} \quad (3)$$

Proof of the claims: Let us first compute $\bar{S}(t, \hat{\phi}_0)(x)$. We claim that

$$\bar{S}(t, \hat{\phi}_0)(x) = \begin{cases} 0 & \text{if } x \geq t(H^r)'(0), \\ -t(H_{A_{\max}}^l)^*(x/t) \mathbf{1}_{x \leq 0} - t(H_{A_{\max}}^r)^*(x/t) \mathbf{1}_{x \geq 0} & \text{if } x \in [t(H^l)'(R^l), t(H^r)'(0)], \\ R^l x & \text{if } x \leq t(H^l)'(R^l), \end{cases}$$

where $(H_{A_{\max}}^{l/p})^*(p) = \sup_{y \in [0, R]} -py + \min\{H^{l/p}(y), A_{\max}\}$. Indeed, let us denote by $w(t, x)$ the function in the right-hand side. Then w is continuous at $(t, 0)$ because

$$w(t, 0^\pm) = -t(H_{A_{\max}}^{l/p})^*(0) = -t \sup_{y \in [0, R]} \min\{H^{l/p}(y), A_{\max}\} = -tA_{\max}.$$

Moreover, w is continuous at (t, x) for $x := t(H^r)'(0) > 0$ since

$$w(t, x^-) = -t(H_{A_{\max}}^r)^*(x/t) = -t \sup_{y \in [-R, 0]} -(H^r)'(0)y + \min\{H^r(y), A_{\max}\} = tH^r(0) = 0,$$

because the maximum in the second formula is reached at $y = 0$ by the optimality condition. In the same way, for $x = t(H^l)'(R) < 0$,

$$w(t, x^+) = -t(H_{A_{\max}}^l)^*((H^l)'(R)) = t(H^l)'(R^l)R^l = R^l x.$$

In addition, w is actually C^1 outside $x = 0$, because w is C^1 outside $x \in \{t(H^l)'(R^l), t(H^r)'(0), 0\}$ and

$$\partial_x w(t, (t(H^r)'(0))^-) = -((H_{A_{\max}}^r)^*)'((H^r)'(0)) = 0,$$

while $\partial_x w(t, (t(H^l)'(R^l))^+) = -((H_{A_{\max}}^l)^*)'((H^l)'(R^l)) = R^l$. In addition, for $x = t(H^r)'(0)$,

$$\partial_t w(t, x^-) = -(H_{A_{\max}}^r)^*(x/t) + (x/t)((H_{A_{\max}}^r)^*)'(x/t) = -(H_{A_{\max}}^r)^*((H^r)'(0)) + (H^r)'(0)((H_{A_{\max}}^r)^*)'((H^r)'(0)) = 0,$$

and, for $x = t(H^l)'(R^l)$,

$$\partial_t w(t, x^+) = -(H_{A_{\max}}^l)^*((H^l)'(R^l)) + (H^l)'(R^l)((H_{A_{\max}}^l)^*)'((H^l)'(R^l)) = (H^l)'(R^l)R^l - (H^l)'(R^l)R^l = 0.$$

One also easily checks that w satisfies the HJ equation in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ in a classical sense and the initial condition. Finally, at $x = 0$,

$$\partial_t w(t, 0) + \min\{A_{\max}, H^{l,+}(\partial_x^- w(t, 0)), H^{r,-}(\partial_x^+ w(t, 0))\} = -A_{\max} + A_{\max} = 0,$$

since $\partial_x^- w(t, 0) = -((H_{A_{\max}}^l)^*)'(0^-) = p_{A_{\max}}^{l,-}$ and $\partial_x^+ w(t, 0) = p_{A_{\max}}^{r,+}$. So $w(t, x) = \bar{S}(t, \hat{\phi}_0)(x)$.

Next we denote by u the function in the right-hand side of (2). We note that, if $A = A_{\max}$, the result is obvious because $\bar{S} = S^{A_{\max}}$ and we have seen that $w(t, 0^\pm) = -tA_{\max}$. Let us now assume that $A \in [0, A_{\max})$. Let us check that u is continuous at the points $(t, t(H^l)'(p_A^{l,+}))$ and $(t, t(H^r)'(p_A^{r,-}))$. Indeed, for $x = t(H^l)'(p_A^{l,+}) \in [t(H^l)'(R^l), t(H^r)'(0)]$,

$$\hat{\phi}_A(x) - tA = p_A^{l,+} t(H^l)'(p_A^{l,+}) - t \min\{H^l(p_A^{l,+}), A_{\max}\} = -t(H_{A_{\max}}^l)^*((H^l)'(p_A^{l,+})) = w(t, x).$$

A symmetric argument shows that u is continuous at $(t, t(H^r)'(p_A^{r,-}))$. We now claim that u is C^1 near the points $(t, t(H^l)'(p_A^{l,+}))$ and $(t, t(H^r)'(p_A^{r,-}))$. As the maps $(t, x) \rightarrow \hat{\phi}_A(x) - tA$ and w are C^1 near these points, we just need to check that the derivatives coincide at these points. Indeed, for $x = t(H^l)'(p_A^{l,+})$,

$$\partial_x w(t, x) = ((H_{A_{\max}}^l)^*)'((H^l)'(p_A^{l,+})) = p_A^{l,+} = \partial_x \hat{\phi}_A(t(H^l)'(p_A^{l,+})),$$

while

$$\begin{aligned} \partial_t w(t, x) &= -(H_{A_{\max}}^l)^*((H^l)'(p_A^{l,+})) + (H^l)'(p_A^{l,+})((H_{A_{\max}}^l)^*)'((H^l)'(p_A^{l,+})) \\ &= (H^l)'(p_A^{l,+})p_A^{l,+} - H^l(p_A^{l,+}) - (H^l)'(p_A^{l,+})p_A^{l,+} = -A. \end{aligned}$$

A symmetric argument shows that u is C^1 near $(t, t(H^r)'(p_A^{r,-}))$. This also implies that u is a solution to the HJ equation in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ and satisfies the initial condition. As $u(t, 0) = -tA$, u solves therefore the boundary value problem (3). Finally, u coincides near $x = 0$ with $\hat{\phi}_A(x) - tA$, which is a solution to the HJ equation with flux limiter A . Thus u is also a solution to the HJ equation with flux limiter A : this proves the claim.

Step 2. Let us set $\bar{A} = -S(1, \hat{\phi}_0)(0)$. We now claim that $\bar{A} \in [0, A_{\max}]$ and that

$$S(t, \hat{\phi}_0) = S^{\bar{A}}(t, \hat{\phi}_0). \quad (4)$$

while

$$S(t, \hat{\phi}_{\bar{A}}) = \hat{\phi}_{\bar{A}} - t\bar{A} = S^{\bar{A}}(t, \hat{\phi}_{\bar{A}}). \quad (5)$$

Proof of the claim: Let $u(t, x) := S(t, \hat{\phi}_0)(x)$. By the HJ equation, $\partial_t u \in [-A_{\max}, 0]$ a.e.. Hence $\bar{A} \in [0, A_{\max}]$. We now prove (4). By assumption (H6), the map $w(t, x) := S(t, \hat{\phi}_{\bar{A}})(x)$ is scale invariant: for any $\epsilon > 0$, $\epsilon w(t/\epsilon, x/\epsilon) = w(t, x)$. Thus $w(t, x) = tw(1, x/t)$. As a consequence, $w(t, 0) = -t\bar{A}$ for any $t \geq 0$ and thus w is the unique viscosity solution to the boundary value problem (3) with $A := \bar{A}$. By Step 1, we obtain $w(t, x) = S^{\bar{A}}(t, \hat{\phi}_0)(x)$, which is (4).

To prove (5), we use again the self similarity to obtain that $S(t, \hat{\phi}_{\bar{A}})(x) = tS(1, \hat{\phi}_{\bar{A}})(x/t)$. Now, by (2) and for any $t > 0$, there is some $\delta > 0$ such that $u(s, x) = S^{\bar{A}}(s, \hat{\phi}_0)(x)$ coincides with $\hat{\phi}_{\bar{A}}(x) - s\bar{A}$ in $[t, t + \delta] \times [-\delta, \delta]$. Thus, by finite speed of propagation (H4) and semi-group properties,

$$S(h, \hat{\phi}_{\bar{A}}(\cdot) - t\bar{A})(x) = S(h, u(t, \cdot))(x) = u(t + h, x) = \hat{\phi}_{\bar{A}}(x) - (t + h)\bar{A}$$

for $h > 0$ and $|x|$ small enough. This implies (using (H3)) that $S(h, \hat{\phi}_{\bar{A}}(\cdot))(0) = -h\bar{A}$, and thus that $S(1, \hat{\phi}_{\bar{A}})(0) = -\bar{A}$. We infer that $(t, x) \rightarrow S(t, \hat{\phi}_{\bar{A}})(x)$ is a solution to the boundary value problem (3) with $A := \bar{A}$ and initial condition $\hat{\phi}_{\bar{A}}$. But $(t, x) \rightarrow \hat{\phi}_{\bar{A}}(x) - t\bar{A}$ is the unique solution to this boundary value problem, which proves (5).

Step 3. We next claim that, for any $A \in [0, \bar{A}]$,

$$S(t, \check{\phi}_A)(0) = -At.$$

Proof of the claim: By scale invariance, we know that $S(t, \check{\phi}_A)(x) = tS(1, \check{\phi}_A)(x/t)$. Recall also that $S(t, \check{\phi}_A) \geq \bar{S}(t, \check{\phi}_A) = \check{\phi}_A - tA$. Thus $S(1, \check{\phi}_A)(0) \geq -A$. Our aim is to prove the equality. Let us assume for a while that $\beta := -S(1, \check{\phi}_A)(0) < A$. Then $(t, x) \rightarrow S(t, \check{\phi}_A)(x)$ is the unique viscosity solution to the boundary value problem

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 & \text{in } (0, \infty) \times (\mathbb{R} \setminus \{0\}) \\ u(t, 0) = -t\beta & \text{in } (0, \infty) \\ u(0, x) = \check{\phi}_A(x) & \text{in } \mathbb{R} \end{cases} \quad (6)$$

We note that

$$S(t, \check{\phi}_A)(x) = \max\{\check{\phi}_A(x) - At, \hat{\phi}_\beta(x) - \beta t\}. \quad (7)$$

Indeed, the function on the right-hand side satisfies the boundary conditions at $x = 0$ (since $\beta < A$) and at $t = 0$ (since $\check{\phi}_A \geq \hat{\phi}_\beta$), and is a viscosity solution in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ as the maximum of two viscosity solutions, the Hamiltonian being concave. Thus (7) holds.

Given $t > 0$, there exists $\delta > 0$ such that $S(s, \check{\phi}_A)(x) = \hat{\phi}_\beta(x) - \beta s$ on $[t, t + \delta] \times [-\delta, \delta]$. Hence, by localization,

$$S(h, \hat{\phi}_\beta)(x) = S(t + h, \check{\phi}_A)(x) + \beta(t + h) = \hat{\phi}_\beta(x) - \beta h$$

for $h > 0$ small and $|x|$ small. Thus $S(h, \hat{\phi}_\beta)(0) = -\beta h$ for $h > 0$ small. Using the scaling invariance, we infer that $S(1, \hat{\phi}_\beta)(0) = -\beta$. On the other hand, as $\hat{\phi}_\beta \leq \hat{\phi}_{\bar{A}}$ (because $\beta < A \leq \bar{A}$), we have by comparison and (5):

$$-\beta = S(1, \hat{\phi}_\beta)(0) \leq S(1, \hat{\phi}_{\bar{A}})(0) = -\bar{A},$$

a contradiction with $\beta < \bar{A}$. So we have proved the equality $S(1, \check{\phi}_A) = -A$.

Step 4. We finally prove that $S = S^{\bar{A}}$. Fix $u_0 \in Lip$ and set $u(t, x) = S(t, u_0)(x)$. Let us check that u solves (1). Recall that u solves the HJ equation in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ and the initial condition u_0 at $t = 0$. It remains to check the viscosity solution property at $x = 0$.

We first prove that u is a supersolution to (1). As $u(t, \cdot) \in Lip$, we have, for any $t \geq 0$,

$$u(t, x) \geq u(t, 0) + \hat{\phi}_0(x).$$

Hence, by comparison (H2), commutation with constants (H3) and semi-group property (H1),

$$u(t + h, x) \geq u(t, 0) + S(h, \hat{\phi}_0)(x) \quad \forall h > 0, \forall x \in \mathbb{R}.$$

So, by (4) and (2) (applied to $x = 0 \in [t(H^l)'(p_A^{l,+}), t(H^r)'(p_A^{r,-})]$),

$$u(t + h, 0) \geq u(t, 0) - h\bar{A},$$

which shows that $\partial_t u(t, 0) + \bar{A} \geq 0$ in the viscosity sense. By [11, Theorem 2.7] (written in [11] for convex Hamiltonians while here H is concave), u is therefore a viscosity supersolution to (1).

We now prove that u is a subsolution to (1). Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 test function on the junction (meaning that φ is continuous in \mathbb{R} , C^1 in $(-\infty, 0]$ and in $[0, \infty)$) and ψ be a C^1 test function such that $(t, x) \rightarrow u - \varphi(x) - \psi(t)$ has a local maximum at some $(\bar{t}, 0)$ with $\bar{t} > 0$. We have to check that

$$\partial_t \psi(\bar{t}) + \min\{\bar{A}, H^{l,+}(\partial_x^- \varphi(0)), H^{r,-}(\partial_x^+ \varphi(0))\} \leq 0. \quad (8)$$

Without loss of generality, we suppose that $\varphi(0) = 0$ and $\psi(\bar{t}) = u(\bar{t}, 0)$. Following [11, Theorem 2.7], we can also assume that

$$H^l(\partial_x^- \varphi(0)) = H^r(\partial_x^+ \varphi(0)) = \bar{A}. \quad (9)$$

Note that (8) is obvious if $\bar{A} = 0$ because u is nonincreasing in time and thus $\psi'(\bar{t}) \leq 0$. We now suppose that $\bar{A} > 0$. Fix $A \in [0, \bar{A}]$. Then, because of (9), we have locally around $x = 0$, $\varphi(x) \leq \check{\phi}_A(x)$ with an equality at $x = 0$. Hence $u(t, x) \leq \check{\phi}_A(x) + \psi(t)$ around $(\bar{t}, 0)$. By (H2), (H3) and (H4), this implies that, for $h > 0$ and $|x|$ small,

$$u(\bar{t}, x) = S(h, u(\bar{t} - h, \cdot))(x) \leq S(h, \check{\phi}_A)(x) + \psi(\bar{t} - h).$$

Using Step 3, we infer that

$$\psi(\bar{t}) = u(\bar{t}, 0) \leq -Ah + \psi(\bar{t} - h).$$

Hence

$$\psi'(\bar{t}) + A \leq 0.$$

As $A < \bar{A}$ is arbitrary, we can conclude that $\psi'(\bar{t}) + \bar{A} \leq 0$ and thus that (8) holds.

In conclusion, $u = S(u_0)$ is a viscosity solution to (1), which proves that $S = S^{\bar{A}}$. \square

3. IDENTIFICATION OF THE CL

As in the previous part, we fix two smooth, Lipschitz and uniformly concave Hamiltonians $H^l : [0, R^l] \rightarrow \mathbb{R}$ and $H^r : [0, R^r] \rightarrow \mathbb{R}$, with $H^l(R^l) = H^r(R^r) = H^l(0) = H^r(0) = 0$ (for some $R^l, R^r > 0$) and set

$$H(x, p) = H^l(p)\mathbf{1}_{\{x < 0\}} + H^r(p)\mathbf{1}_{\{x > 0\}}.$$

We now denote by L^∞ the set of measurable maps ρ such that $\rho(x) \in [0, R^l]$ a.e. in $(-\infty, 0)$ and $\rho(x) \in [0, R^r]$ a.e. in $(0, \infty)$. We endow the set L^∞ with the topology of the L^1_{loc} convergence. We denote by $\bar{S}^{CL, l}$ (resp. $\bar{S}^{CL, r}$) the classical semi-group of conservation law in $(0, \infty) \times \mathbb{R}$ associated with the flux H^l (resp. H^r).

Our aim is to identify any semi-group S^{CL} on L^∞ satisfying the following properties:

- (H1) $S^{CL} : [0, \infty) \times L^\infty \rightarrow L^\infty$ is a continuous homogeneous semi-group,
- (H2) S^{CL} satisfies the L^1 contraction property: for any $u_0^1, u_0^2 \in L^\infty \cap L^1(\mathbb{R})$,

$$\|S^{CL}(t, u_0^1) - S^{CL}(t, u_0^2)\|_{L^1(\mathbb{R})} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R})},$$
- (H3) S^{CL} preserves mass: if $\rho_0 \in L^\infty \cap L^1(\mathbb{R})$, then $\int_{\mathbb{R}} S^{CL}(t, u_0) = \int_{\mathbb{R}} u_0$ for any $t \geq 0$,
- (H4) S^{CL} has a finite speed of propagation: there exists $C_0 > 0$ such that,

if $u_0^1 = u_0^2$ on $[a, b]$, then $S^{CL}(t, u_0^1) = S^{CL}(t, u_0^2)$ on $[a + C_0 t, b - C_0 t]$ for $t \in [0, C_0^{-1}(b - a)]$,
- (H5) S^{CL} coincides locally with $\bar{S}^{CL, l/r}$ away from $x = 0$: for any $u_0 \in L^\infty$ and $x \neq 0$,

$$S^{CL}(t, u_0)(y) = \begin{cases} \bar{S}^{CL, l}(t, u_0)(y) & \text{if } x < 0 \\ \bar{S}^{CL, r}(t, u_0)(y) & \text{if } x > 0 \end{cases} \quad \text{for any } (t, y) \text{ with } |y - x| < |x| - C_0 t,$$
- (H6) S^{CL} is scale invariant: for any $\epsilon, t > 0$ and $u_0 \in L^\infty$, $S^{CL}(t/\epsilon, u_0(\cdot/\epsilon))(x/\epsilon) = S^{CL}(t, u_0)(x)$.

Theorem 3.1. *Under the conditions above, there exists $\bar{A} \in [0, A_{\max}]$ such that $S^{CL} = S^{CL, \bar{A}}$, where $A_{\max} = \min\{\max H^l, \max H^r\}$ and $S^{CL, \bar{A}}$ is the conservation law semi-group with flux-limiter \bar{A} , i.e., for a given initial condition $\rho_0 \in L^\infty$, the entropy solution to*

$$\begin{cases} \partial_t \rho + \partial_x (H(x, \rho)) = 0 & \text{in } (0, \infty) \times (\mathbb{R} \setminus \{0\}) \\ (\rho(t, 0^-), \rho(t, 0^+)) \in \mathcal{G}_{\bar{A}} & \text{a.e. in } (0, \infty) \\ \rho(0, x) = \rho_0(x) & \text{in } \mathbb{R} \end{cases} \quad (10)$$

where the germ $\mathcal{G}_{\bar{A}}$ is given by

$$\mathcal{G}_{\bar{A}} = \{(q^-, q^+) \in [0, R^l] \times [0, R^r], H^l(q^-) = H^r(q^+) = \min\{\bar{A}, H^{l,+}(q^-), H^{r,-}(q^+)\}\}.$$

Above $(H^{l/r})^+$ (respectively $(H^{l/r})^-$) denotes the smallest nondecreasing (resp. nonincreasing) map above $H^{l/r}$. The notion of \mathcal{G} -entropy solution for (10) is introduced in [2]: it is a Kruzhkov entropy solution in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$, its trace at $t = 0$ is ρ_0 and its trace at $x = 0$ belongs to $\mathcal{G}_{\bar{A}}$ (the existence of a strong trace being given by [13, 16]). As $\mathcal{G}_{\bar{A}}$ is a maximal, L^1 -dissipative and complete germ (see [6]—see also [2] for the definition of these notions, which are not explicitly used here), the solution to (10) exists and is unique (see [2]).

We recall that, because of Assumption (H3), Assumption (H2) is equivalent to a comparison principle for S^{CL} : if $\rho_{0,1}, \rho_{0,2} \in L^\infty$ and $\rho_{0,1} \leq \rho_{0,2}$, then $S^{CL}(\rho_{0,1}) \leq S^{CL}(\rho_{0,2})$ (see [8]).

Throughout the proof, we set $S = S^{CL}$, $S^A = S^{CL, A}$ and $\bar{S}^{l/r} = \bar{S}^{CL, l/r}$ for simplicity. We also assume without loss of generality that $C_0 \geq \|H^l\|_\infty$, so that the CL semi-groups $\bar{S}^{l/r}$ also have a finite speed of propagation C_0 .

Our first step is a standard consequence of the mass preservation property:

Lemma 3.2. *Let $\rho_0 \in L^\infty$ and $\rho = S(\rho_0)$. Then ρ satisfies a kind of Rankine-Hugoniot condition at $x = 0$:*

$$H^l(\rho(t, 0^-)) = H^r(\rho(t, 0^+)) \quad \text{for a.e. } t \geq 0.$$

Proof. As the property we aim to prove is local near $x = 0$ and S satisfies a finite speed of propagation, we can assume that ρ_0 has a compact support. Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a smooth nonnegative test function equal to 1 on $\mathbb{R} \setminus (-2, 2)$ and vanishing on $[-1, 1]$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \psi(x/\epsilon)$. As ρ is a solution of the CL in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, we have, for any $t \geq 0$,

$$\int_{\mathbb{R}} \rho(t) \psi_\epsilon + \int_0^t \int_{\mathbb{R}} H(x, \rho) \psi_{\epsilon, x} = \int_{\mathbb{R}} \rho_0 \psi_\epsilon$$

We let $\epsilon \rightarrow 0$ to find

$$\int_{\mathbb{R}} \rho(t) + \int_0^t (H^r(\rho(s, 0^+)) - H^l(\rho(s, 0^-))) ds = \int_{\mathbb{R}} \rho_0,$$

which proves the claim thanks to (H3). \square

The next remark makes the link between the CL and the HJ equation.

Lemma 3.3. *Let $\rho_0 \in L^\infty$ have a compact support and assume that ρ is an entropy solution of the CL in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ with initial condition ρ_0 . Let $u(t, x) := \int_{-\infty}^x \rho(t, y) dy$. Then u is a viscosity solution to the HJ equation*

$$\begin{cases} \partial_t u + H(x, \partial_x u) = 0 & \text{in } (0, \infty) \times (\mathbb{R} \setminus \{0\}) \\ u(0, x) = u_0(x) & \text{in } \mathbb{R} \end{cases} \quad (11)$$

where $u_0(x) := \int_{-\infty}^0 \rho_0(y) dy$. If, in addition, there exists $(q^-, q^+) \in [0, R^l] \times [0, R^r]$ such that

$$(\rho(t, 0^-), \rho(t, 0^+)) = (q^-, q^+) \quad \text{for a.e. } t \geq 0, \quad (12)$$

then u satisfies also the boundary condition

$$u(t, 0) = -tA + u_0(0) \quad \text{in } (0, \infty) \quad (13)$$

where $A := H^l(q^-) = H^r(q^+)$.

Finally, for any $\rho_0 \in L^\infty$, if $\rho := S(\rho_0)$ satisfies (12), there exists $u_0 \in Lip$ such that $\rho = \partial_x u$ where u is the unique viscosity solution to the boundary value problem (11)-(13).

Proof. The fact that u is a viscosity solution in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ comes from assumption (H5): Indeed, fix $t > 0$ and $x \neq 0$. Let $w(s, \cdot) = \bar{S}^{HJ}(s, u(t, \cdot))$. Then it is known (see, for instance, [7] for a local version of the statement) that $\partial_x w$ is an entropy solution of the CL with initial condition $\partial_x u(t, \cdot) = \rho(t, \cdot)$. As, by (H5), ρ is also an entropy solution in the triangle

$$\Delta := \{(s, y), s \in (t, t - C_0^{-1}|x|), y \in (x - |x| + C_0(s - t), x + |x| - C_0(s - t))\},$$

we infer by the uniqueness of the entropy solution that $\partial_x w(s, \cdot) = \rho(s, \cdot) = \partial_x u(s, \cdot)$ in Δ . On the other hand, for any test function φ with a compact support in Δ , we have, by the definition of u and since w is a solution of the HJ equation while ρ solves the CL,

$$\iint_{\Delta} u \varphi_t = \iint_{\Delta} \rho \tilde{\varphi}_t = - \iint_{\Delta} H(x, \rho) \tilde{\varphi}_x = \iint_{\Delta} H(x, \rho) \varphi = \iint_{\Delta} H(x, \partial_x w) \varphi = - \iint_{\Delta} w_t \varphi.$$

where we have set $\tilde{\varphi}(s, y) = \int_y^\infty \varphi(s, x) dx$. This proves that $u_t = w_t$ in Δ , and thus that $u = w$ in Δ since $u(t, \cdot) = w(t, \cdot)$. Hence u is a viscosity solution to the HJ equation away from $x = 0$. In addition, by the continuity of $t \rightarrow \rho(t)$ in L^1_{loc} , u is continuous up to time $t = 0$ and $u(0, x) = u_0(x)$.

Let us now check the boundary condition at $x = 0$ under the addition assumption (12). Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a smooth nonnegative test function equal to 1 on $\mathbb{R} \setminus (-2, 2)$ and vanishing on $[-1, 1]$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \psi(x/\epsilon)$. As ρ is a weak solution to the CL in $\mathbb{R}_+ \times (-\infty, 0)$, we have

$$\int_{-\infty}^0 \rho(t) \psi_\epsilon - \int_0^t \int_{-\infty}^0 H(x, \rho) (\psi_\epsilon)_x = \int_{-\infty}^0 \rho(0) \psi_\epsilon$$

We let $\epsilon \rightarrow 0$. Then $\int_{-\infty}^0 \rho(t) \psi_\epsilon$ tends to $u(t, 0)$, $\int_{-\infty}^0 \rho(0) \psi_\epsilon$ tends to $u_0(0)$ while $\int_0^t \int_{-\infty}^0 H(x, \rho) (\psi_\epsilon)_x$ tends to $-\int_0^t H(x, \rho(s, 0^-)) ds$. Assumption (12) then implies the boundary condition $u(t, 0) = -tA + u_0(0)$ for $t \geq 0$.

For the last part of the proof, we proceed by approximation: given $M > 0$, let $\rho_{0, M} = \rho_0 \mathbf{1}_{[-M, M]}$, $\rho_M = S(\rho_{0, M})$ and let u_M be built as above. Note that ρ_M tends to ρ in L^1_{loc} as $M \rightarrow \infty$ by the continuity of the semi-group. Then, for any $T > 0$ and M large enough, ρ_M satisfies (12) on $[0, T]$, so

that $\tilde{u}_M := u_M - u_M(0, 0)$ solves (11)-(13) on the time interval $[0, T]$ and satisfies $\rho_M = \partial_x \tilde{u}_M$. By compactness, (\tilde{u}_M) converges, as $M \rightarrow \infty$ and up to a subsequence, to some solution u with the required properties on the time interval $[0, \infty)$ since T was arbitrary. \square

The next remark is about blow-up limits of solutions at points $(t, 0)$. It is a central consequence of the existence of strong traces for solutions of CL in the sense of Vasseur [16] and Panov [13] and of strong trace for the gradient of solution of HJ equation in the sense of Monneau [12].

Lemma 3.4. *Let $\rho_0 \in L^\infty$ and, for $\bar{t} > 0$ and $\epsilon > 0$, set $\rho_\epsilon^\epsilon(t, x) = S(\bar{t} + \epsilon t, \rho_0)(\epsilon x)$. Then for a.e. $\bar{t} > 0$ and as $\epsilon \rightarrow 0^+$, the sequence ρ_ϵ^ϵ converges in L^1_{loc} to a constant (in time) function $\bar{\rho}_\epsilon(x) = q_\epsilon^- \mathbf{1}_{\{x < 0\}} + q_\epsilon^+ \mathbf{1}_{\{x > 0\}}$, where $(q_\epsilon^-, q_\epsilon^+) = (S(\bar{t}, \rho_0)(0^-), S(\bar{t}, \rho_0)(0^+))$, which is a stationary solution for S : $S(t, \bar{\rho}_\epsilon) = \bar{\rho}_\epsilon$ for any $t \geq 0$.*

Proof. As the result holds in a neighborhood of $x = 0$, we can assume without loss of generality that ρ_0 has a compact support. Let us set $u(t, x) := \int_{-\infty}^x \rho(t, y) dy$. We know from Lemma 3.3 that u is a viscosity solution to the HJ equation in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$. Let \bar{t} be such that the Lipschitz function $u(t, 0)$ has a derivative at \bar{t} . Following [12], u then admits a full half-derivative at $(\bar{t}, 0^\pm)$: namely, there exists q^\pm and $q_t \in \mathbb{R}$ such that

$$u^\epsilon(t, x) = \epsilon^{-1} (u(\bar{t} + \epsilon t, \epsilon x) - u(\bar{t}, 0)) \rightarrow u^0(t, x) := tq_t + xq^- \mathbf{1}_{x < 0} + xq^+ \mathbf{1}_{x > 0} \quad \text{as } \epsilon \rightarrow 0^+,$$

the convergence being locally uniform in $\mathbb{R} \times (-\infty, 0)$ and in $\mathbb{R} \times (0, \infty)$. Let now $\rho_\epsilon^\epsilon(t, x) = S(\bar{t} + \epsilon t, \rho_0)(\epsilon x)$. By the semi-group and the scaling properties, ρ^ϵ is a solution for S : $S(t, \rho_\epsilon^\epsilon(0, \cdot)) = \rho_\epsilon^\epsilon(t, \cdot)$. As ρ^ϵ is a bounded solution to the CL with a uniformly concave flux H in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, the family (ρ^ϵ) is bounded in $BV_{loc}(\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}))$ and therefore relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$. Let (ϵ_n) be such that (ρ^{ϵ_n}) converges to some $\bar{\rho}$ in L^1_{loc} . We claim that

$$\bar{\rho}(t, x) = q^- \mathbf{1}_{x < 0} + q^+ \mathbf{1}_{x > 0}. \quad (14)$$

Note that the claim implies the convergence of the whole sequence (ρ^ϵ) to $\bar{\rho}$. To prove the claim we note that $\partial_x u^\epsilon = \rho^\epsilon$. As u^ϵ converges to u^0 , we infer that $\rho^\epsilon = \partial_x u^\epsilon$ converges to $\partial_x u^0 = q^- \mathbf{1}_{x < 0} + q^+ \mathbf{1}_{x > 0}$ in L^∞ -weak-*. But ρ^ϵ converges to $\bar{\rho}$ in L^1_{loc} , so that (14) holds. As ρ^ϵ is a solution for S , $\bar{\rho}$ is also a solution for S . From now on we set $(q_\epsilon^-, q_\epsilon^+) = (q^-, q^+)$ to emphasize the dependence in \bar{t} . Recall that $(q_\epsilon^-, q_\epsilon^+)$ exists for a.e. $t \geq 0$.

Let us finally check that $(q_\epsilon^-, q_\epsilon^+) = (S(\bar{t}, \rho_0)(0^-), S(\bar{t}, \rho_0)(0^+))$ for a.e. \bar{t} . Set $\rho := S(\cdot, \rho_0)$. By the strong trace property (see [13, 16]), for any $b > 0$,

$$\text{ess-lim}_{x \rightarrow 0^+} \int_0^b |\rho(t, x) - \rho(t, 0^+)| dt = 0.$$

Hence (by a simple change of variable)

$$\lim_{\epsilon \rightarrow 0} \int_0^1 \int_0^b \int_0^1 |\rho(t + \epsilon s, \epsilon x) - \rho(t + \epsilon s, 0^+)| dx dt ds = 0.$$

By the L^1 convergence of ρ_ϵ^ϵ for a.e. \bar{t} to q_ϵ^\pm in $\mathbb{R} \times (0, \infty)$ and the L^1 convergence of $\rho(\cdot + \epsilon s, 0^+)$ to $\rho(\cdot, 0^+)$, we infer that

$$\int_0^1 \int_0^b \int_0^1 |q_t^+ - \rho(t, 0^+)| dx dt ds = 0,$$

from which we conclude that $q_t^+ = \rho(t, 0^+)$ for a.e. t . The proof of the a.e. equality $q_t^+ = S(\bar{t}, \rho_0)(0^+)$ is symmetric. \square

Let us now introduce a few notation. Given $A \in [0, A_{\max}]$, let $p_A^{r/l, -}$ and $p_A^{r/l, +}$ be respectively the smallest and the largest solutions to $H^{r/l}(p) = A$. We define the maps

$$\check{\psi}_A(x) = p_A^{l, -} \mathbf{1}_{x \leq 0} + p_A^{r, +} \mathbf{1}_{x \geq 0}, \quad \hat{\psi}_A(x) = p_A^{l, +} \mathbf{1}_{x \leq 0} + p_A^{r, -} \mathbf{1}_{x \geq 0}. \quad (15)$$

Lemma 3.5. *Assume that there exists $\bar{A} \in [0, A_{\max}]$ such that $\hat{\psi}_{\bar{A}}$ is a stationary solution for S : $S(t, \hat{\psi}_{\bar{A}}) = \hat{\psi}_{\bar{A}}$ for any $t \geq 0$. Then, for any $\alpha \in [0, A_{\max}] \setminus \{\bar{A}\}$ and any $\beta \in (\bar{A}, A_{\max}]$, $\hat{\psi}_\alpha$ and $\check{\psi}_\beta$ are not stationary solutions for S .*

Proof. We start with a preliminary remark. Fix $\alpha, \beta \in [0, A_{\max}]$ with $\alpha \neq \beta$. We claim that $\hat{\psi}_\alpha$ and $\hat{\psi}_\beta$ cannot be two stationary solutions for S . To fix the ideas, we suppose that $\alpha < \beta$ and argue by contradiction, assuming that $\hat{\psi}_\alpha$ and $\hat{\psi}_\beta$ are two stationary solutions for S .

For $A \in [0, A_{\max}]$, let us set $\hat{\phi}_A(x) = p_A^{l,+} x \mathbf{1}_{x < 0} + p_A^{r,-} x \mathbf{1}_{x > 0}$. Note that $\hat{\psi}_A = \partial_x \hat{\phi}_A$. For $\epsilon > 0$, let us set

$$u^\epsilon(t, x) = \max\{\hat{\phi}_\alpha(x) - \alpha(t + \epsilon), \hat{\phi}_\beta(x) - \beta(t + \epsilon)\}.$$

Note that u^ϵ is a viscosity solution to the HJ equation in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ as the maximum of two solutions, the Hamiltonian being concave. Hence $\rho^\epsilon = \partial_x u^\epsilon$ is an entropic solution of the CL in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$. On the other hand, as $\alpha < \beta$, there exists $\eta > 0$ such that $u^\epsilon(t, x) = \hat{\phi}_\alpha(x) - \alpha(t + \epsilon)$ in $\mathbb{R}_+ \times (-\eta, \eta)$. Thus $\rho^\epsilon = \hat{\psi}_\alpha$ in $(0, \infty) \times (-\eta, \eta)$. As $\hat{\psi}_\alpha$ is a solution for S , while ρ^ϵ is a solution to the CL in $(0, \infty) \times (\mathbb{R} \setminus \{0\})$, ρ^ϵ is also a solution for S thanks to the finite speed of propagation property. Let us set $\rho := \partial_x u$ with

$$u(t, x) = \max\{\hat{\phi}_\alpha(x) - \alpha t, \hat{\phi}_\beta(x) - \beta t\}.$$

Then ρ^ϵ tends to ρ in L^1_{loc} , which implies that ρ is also a solution for S by continuity of the semi-group. As $\rho(0) = \hat{\psi}_\beta$, we have therefore $S(t, \hat{\psi}_\beta) = \rho$. On the other hand, $\rho(t)$ equals $\hat{\psi}_\alpha$ in a neighborhood of $x = 0$ for $t > 0$: this shows that $\hat{\phi}_\beta$ cannot be a stationary solution for S and proves the claim.

We now come back to the proof of the Lemma. As by assumption $\hat{\psi}_{\bar{A}}$ is a stationary solution for S , the claim above shows that $\hat{\phi}_\alpha$ cannot be a stationary solution for S for any $\alpha \in [0, A_{\max}] \setminus \{\bar{A}\}$.

Let now $\beta \in (\bar{A}, A_{\max}]$. Replacing $\hat{\psi}_\alpha$ by $\hat{\psi}_{\bar{A}}$ and $\hat{\psi}_\beta$ by $\check{\psi}_\beta$ in the proof of the claim, we can show exactly in the same way as above that $\check{\psi}_\beta$ cannot be a stationary solution for S . \square

We are now ready to build the flux limiter \bar{A} . Let us set $q^\pm = S(1, \hat{\psi}_{A_{\max}})(0^\pm)$, where $\hat{\psi}_{A_{\max}}$ is defined in (15). By Lemma 3.2, $H^l(q^-) = H^r(q^+)$. We set $\bar{A} = H^l(q^-) = H^r(q^+)$.

Lemma 3.6. *We have $(q^-, q^+) = (p_{\bar{A}}^{l,+}, p_{\bar{A}}^{r,-})$ and $\hat{\psi}_{\bar{A}}$ is a stationary solution for S . Moreover, for any $\alpha \in (\bar{A}, A_{\max}]$, the maps $x \rightarrow p_\alpha^{l,+} \mathbf{1}_{x < 0} + p_\alpha^{r,+} \mathbf{1}_{x > 0}$ and $x \rightarrow p_\alpha^{l,-} \mathbf{1}_{x < 0} + p_\alpha^{r,-} \mathbf{1}_{x > 0}$ cannot be a stationary solution for S .*

Proof. By Lemma 3.3, we know that, if we set $\rho = S(\hat{\psi}_{A_{\max}})$, then there exists $u_0 \in Lip$ such that $\rho = \partial_x u$, where u is the unique viscosity solution to (11) satisfying the boundary condition (13) with A replaced by \bar{A} . One easily checks, as Step 3 in the proof of Theorem 2.1, that

$$u(t, x) = \max\{u_0(x) - A_{\max} t, \hat{\phi}_{\bar{A}}(x) - \bar{A} t\}.$$

As $u(t, x) = \hat{\phi}_{\bar{A}}(x) - \bar{A} t$ in a neighborhood of $x = 0$, we infer that

$$(\rho(t, 0^-), \rho(t, 0^+)) = (\partial_x \hat{\phi}_{\bar{A}}(0^-), \partial_x \hat{\phi}_{\bar{A}}(0^+)) = (p_{\bar{A}}^{l,+}, p_{\bar{A}}^{r,-}).$$

Moreover Lemma 3.4 states that $\hat{\psi}_{\bar{A}}$ is a stationary solution for S .

Let now $\alpha \in (\bar{A}, A_{\max}]$. Assume that the map $\tilde{\rho}(x) := p_\alpha^{l,+} \mathbf{1}_{x < 0} + p_\alpha^{l,+} \mathbf{1}_{x > 0}$ is a stationary solution for S . As $\tilde{\rho} \geq \hat{\psi}_{A_{\max}}$, we infer by comparison that $S(t, \tilde{\rho})(x) \geq S(t, \hat{\psi}_{A_{\max}})(x) = \rho(t, x)$ a.e.. Thus $S(1, \tilde{\rho})(0^-) \geq p_{\bar{A}}^{l,+}$. But, as by assumption $\tilde{\rho}$ is a stationary solution, we have $p_\alpha^{l,+} = S(1, \tilde{\rho})(0^-) \geq p_{\bar{A}}^{l,+}$, which contradicts the assumption $\alpha > \bar{A}$. In the same way, one easily checks that the map $x \rightarrow p_\alpha^{l,-} \mathbf{1}_{x < 0} + p_\alpha^{r,-} \mathbf{1}_{x > 0}$ cannot be a stationary solution for S if $\alpha \in (\bar{A}, A_{\max}]$. \square

Proof of Theorem 3.1. Recall that $\bar{A} = H^l(q^-) = H^r(q^+)$ where $q^\pm = S(1, \hat{\psi}_0)(0^\pm)$. Let us prove that $S = S^{\bar{A}}$. Fix $\rho_0 \in L^\infty$ and $\rho(t, x) = S(t, \rho_0)(x)$. We know that ρ is an entropy solution to the CL in $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ and that ρ satisfies the initial condition. Let us check that $(\rho(\bar{t}, 0^-), \rho(\bar{t}, 0^+)) \in \mathcal{G}_{\bar{A}}$ a.e.

Indeed, for a.e. $\bar{t} > 0$, the trace $(q^-, q^+) = (\rho(\bar{t}, 0^-), \rho(\bar{t}, 0^+))$ exists and the map $s : (t, x) \rightarrow q^- \mathbf{1}_{\{x < 0\}} + q^+ \mathbf{1}_{\{x > 0\}}$ is a stationary solution for S (Lemma 3.4). Let us assume for a while that $(q^-, q^+) \notin \mathcal{G}_{\bar{A}}$. By Lemma 3.2, we have $H^l(q^-) = H^r(q^+)$. Let $A = H^l(q^-) = H^r(q^+)$. If $A > \bar{A}$, then either $s(x) = p_\alpha^{l,+} \mathbf{1}_{x < 0} + p_\alpha^{l,+} \mathbf{1}_{x > 0}$, or $s(x) = p_\alpha^{l,-} \mathbf{1}_{x < 0} + p_\alpha^{r,-} \mathbf{1}_{x > 0}$, or $s = \hat{\psi}_A$, or $s = \check{\psi}_A$. The first two cases are excluded by Lemma 3.6 while the last two ones are excluded by Lemma 3.5. Thus $A \leq \bar{A}$. As $(q^-, q^+) \notin \mathcal{G}_{\bar{A}}$, we necessarily have $A < \bar{A}$, $q^- = p_A^{l,+}$ and $q^+ = p_A^{r,-}$. Thus $s = \hat{\psi}_A$. But Lemma 3.5 excludes also this case. Hence $(q^-, q^+) \in \mathcal{G}_{\bar{A}}$, which concludes the proof. \square

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