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# A simple test to distinguish between deterministic and stochastic trends in non-stationary time series

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## Abstract

For non-stationary times series, the main difficult is to identify the nature of non-stationarity. In macroeconomic and financial modelling, there are two kinds of non-stationarity. The first one is due to the presence of a deterministic trend and generally modeled by a  $d$ -order polynomial of time. The second is due to the presence of a stochastic trend which is modeled by integrating a stationary process up to a suitable order  $d$ . In this paper we propose a simple test to decide between these two kinds of trend. Simulations show that the test has a good empirical size and power. The test is finally applied to some real macroeconomic and financial datasets.

*Keywords:* Economic modelling, Hypothesis testing, Statistical test

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# 1 Introduction

For non stationary times series, deterministic and stochastic trends were suggested by Nelson and Plosser(1982) to model such non stationary evolution in time. In macroeconomic modelling, there are two competing theories. Following the Monetarist theory of the business cycle, Friedman (1968), Lucas Jr (1972) and Lucas Jr (1975), the Gross Domestic Product (GDP) is characterised by stationary movements around a deterministic trend (low frequencies are dominant), the monetary shocks are the main sources of growth and have transient effects, therefore the GDP returns to its natural path. Following the Real Business Cycle, Kydland and Prescott (1982), Long Jr and Plosser (1983), Hénin (1989b), Hénin (1989a) and Hairault and Hénin (1995), the GDP is an accumulation of random permanent shocks generated at each observation period, i.e. the GDP is a stochastic trend governed by permanent productivity shocks (technology, economic policy, demographics, ) which are dominant sources of economic fluctuations.

In finance, Black and Scholes (1973) suggested that financial assets can be modelled by a geometric Brownian motion. This implies that the logarithm of financial asset is a random walk (or unit root process) and hence the non-stationarity is due to a stochastic trend. The assumption of Black and Scholes (1973) implies also that the random walk is governed by a gaussian white noise, but in practice many authors have pointed out the heteroskedastic behaviour of returns, therefore the normality assumption of returns does not hold. Fortunately, Engle (1982) and Bollerslev (1986) have proposed ARCH and GARCH specification to returns. Such specification does not alter the martingale properties of the financial assets and hence does not contradict the market efficiency theory. Many authors proposed to test the presence of unit root, the popular tests are the Augmented test of Dickey and Fuller (1979) (noted here by ADF), the test of Kwiatkowski et al. (1992) (known as KPSS), and the test of Ouliaris et al. (1989) (noted here as OPP). However these tests look for the

presence of unit root around a linear trend (this is the case of ADF and KPSS), OPP have extended the ADF test to test unit root around a  $d$ -order polynomial trend ( $d = 1, \dots, 5$ ). Boutahar and Royer-Carenzi (2024) provided a procedure, based on KPSS and OPP tests with a polynomial trends with degrees  $d = 1, \dots, 5$ , they also pointed out that the Dickey-Fuller test detects the presence of spurious unit root for series with a quadratic deterministic trend.

In this paper we propose a new test to decide between a deterministic trend and a stochastic one. For a given time series  $(Y_t), 1 \leq t \leq n$ , the paper develops the test of the null hypothesis

$$H_0 : Y_t = a_0 + a_1t + \dots + a_d t^d + Z_t \quad (1)$$

against the alternative

$$H_1 : (1 - L)^d Y_t = a_d + Z_t \quad (2)$$

where  $L$  is a back shift operator,  $LY_t = Y_{t-1}$ , and  $(Z_t)$  is assumed to be an invertible moving average process,  $MA(q)$ :

$$Z_t = B(L)\varepsilon_t, \quad B(z) = 1 + b_1z + \dots + b_qz^q, \quad (3)$$

$d$  and  $q$  are assumed to be known, and  $(\varepsilon_t)$  is an i.i.d. sequence of random variables with zero mean and variance  $\sigma^2$ .

In section 2, we define the test statistic and study the limiting distribution under the null of deterministic trend and its consistency against the alternative of stochastic trend. In section 3, we perform some simulations by computing the empirical size and power of the proposed test. We propose a strategy to decide the kind of trend (deterministic or stochastic) when  $(d, q)$  are unknown; and we study also other kinds of deterministic trend

to illustrate how our strategy works. In section 4 we apply our test to Gross Domestic Product and to the Standard & Poor's 500 index datasets.

### Remarks

1. Contrary to the papers cited above, we do not impose any restriction on the degree  $d$  of the polynomial for deterministic trend or the integration order  $d$  of the stochastic trend.
2. The assumption (3) on the process  $(Z_t)$  seems to be more restrictive, however it is known that every stationary process can be approximated by an  $MA(q)$  for  $q$  large enough, for short memory time series  $q$  is small, but for long memory times series we need  $q$  to be large (see Boutahar (2009) and the references therein to decide if a given time series is short or long memory).
3. In practice the degrees  $d$  and  $q$  are unknown, in section 3 we will define empirical estimators of them.

## 2 The new test statistic

To define the test statistic we consider the differenced time series

$$X_t = (1 - L)^d Y_t,$$

and its empirical autocorrelation function  $\hat{\rho}$ :

$$\hat{\rho}(j) = \frac{\hat{\gamma}(j)}{\hat{\gamma}(0)}, \quad \hat{\gamma}(j) = \frac{1}{n} \sum_{t=1}^{n-|j|} (X_{t+|j|} - \bar{X})(X_t - \bar{X}), \quad \bar{X} = \frac{1}{n} \sum_{t=1}^n X_t.$$

The test statistic is given by

$$T_n = \sqrt{n} * \left( \sum_{j=1}^{d+q} \hat{\rho}(j) + \frac{1}{2} \right), \tag{4}$$

## 2.1 The limiting distribution of $T_n$ under the null hypothesis

**Theorem 1.** *Under the null hypothesis  $H_0$  we have*

$$T_n \xrightarrow{\mathcal{L}} T = \sum_{j=1}^{d+q} v_j \quad (5)$$

where  $V = {}^t(v_1, \dots, v_{d+q})$  is a  $(d+q)$  random gaussian vector with 0 mean and a matrix of variance-covariance  $W$  with the  $(i,j)$ -th term given by:

$$\begin{aligned} w_{i,j} &= \sum_{k=1}^{\infty} (\rho(k+i) + \rho(k-i) - \rho(i)\rho(k)) (\rho(k+j) + \rho(k-j) - \rho(j)\rho(k)), \\ \rho(j) &= 0, \quad \forall |j| > d+q. \end{aligned} \quad (6)$$

$\xrightarrow{\mathcal{L}}$  denotes the convergence in law and  ${}^tX$  denotes the transpose of the vector (or matrix)  $X$ .

For the proof, we need the following lemma

Let  $\gamma(k)$  (resp.  $\rho(k)$ ) denote the autocovariance (resp. autocorrelation) function of the process  $(X_t)$

$$\gamma(k) = E((X_{t+k} - \mu)(X_t - \mu)), \quad \mu = E(X_t), \quad \rho(k) = \frac{\gamma(k)}{\gamma(0)}.$$

**Lemma 2.** *Under the null hypothesis  $H_0$  and (3) we have*

$$\sum_{j=1}^{q+d} \rho(j) = -\frac{1}{2} \quad (7)$$

Proof of the lemma.

Let  $f(\lambda)$  denote the spectral density of  $(X_t)$ , it's well known that the link between the spectral density and the autocovariance function is given by

$$\begin{aligned} \gamma(k) &= \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda, \quad \text{for all } k \in \mathbb{Z} \\ f(\lambda) &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-ik\lambda} \gamma(k), \quad \text{for all } \lambda \in [-\pi, \pi]. \end{aligned}$$

Now since  $X_t = a_d + (1 - L)^d B(L)\varepsilon_t$ , it follows that  $(X_t)$  is a stationary and non invertible  $MA(d + q)$  with the spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} |1 - e^{i\lambda}|^{2d} |B(e^{i\lambda})|^2,$$

hence

$$f(0) = 0 = \frac{1}{2\pi} \sum_{k=-(d+q)}^{d+q} \gamma(k),$$

consequently

$$0 = \gamma(0) + 2 \sum_{k=1}^{d+q} \gamma(k)$$

and then we obtain (7).

Proof of the theorem.

From theorem 7.2.1 of Brockwell and Davis (1991) we have that

$$\sqrt{n} \text{ }^t(\hat{\rho}(1) - \rho(1), \dots, \hat{\rho}(d + q) - \rho(d + q)) \xrightarrow{\mathcal{L}} V. \quad (8)$$

Let  $E_1 = \text{ }^t(1, \dots, 1)$ ,  $\dim E_1 = d + q$ .

By a continuous mapping theorem it follows that

$$\begin{aligned} \sqrt{n} (\hat{\rho}(1) - \rho(1), \dots, \hat{\rho}(d + q) - \rho(d + q)) E_1 &\xrightarrow{\mathcal{L}} \text{ }^t V E_1; \\ &= \sum_{j=1}^{d+q} v_j \\ &= T \end{aligned}$$

The desired conclusion holds since by the above lemma we have that

$$\sqrt{n} (\hat{\rho}(1) - \rho(1), \dots, \hat{\rho}(d + q) - \rho(d + q)) E_1 = T_n.$$

## 2.2 The consistency of $T_n$ against the alternative $H_1$

**Theorem 3.** *Under the alternative hypothesis  $H_1$  we have*

$$|T_n| \xrightarrow{P} +\infty. \quad (9)$$

$\xrightarrow{P}$  denotes the convergence in probability.

Proof. Under the alternative hypothesis  $H_1$  the process  $X_t$  is an invertible  $MA(q)$ , therefore

$$\begin{aligned} \sum_{j=1}^{d+q} \rho(j) + \frac{1}{2} &= \sum_{j=1}^q \rho(j) + \frac{1}{2} \\ &\neq 0. \end{aligned} \tag{10}$$

$T_n$  can be written as

$$\begin{aligned} T_n &= \sqrt{n} * \left( \sum_{j=1}^{q+d} \hat{\rho}(j) - \rho(j) \right) + \sqrt{n} * \left( \sum_{j=1}^{q+d} \rho(j) + \frac{1}{2} \right) \\ &= T_{1,n} + T_{2,n}. \end{aligned} \tag{11}$$

Moreover

$$T_{1,n} \xrightarrow{\mathcal{L}} T = \sum_{j=1}^{d+q} v_j \tag{12}$$

where  $V = {}^t(v_1, \dots, v_{d+q})$  is a  $(d+q)$  random gaussian vector with 0 mean and a matrix of variance-covariance  $W$  with the  $(i, j)$ -th term given by:

$$w_{i,j} = \sum_{k=1}^{\infty} (\rho(k+i) + \rho(k-i) - \rho(i)\rho(k)) (\rho(k+j) + \rho(k-j) - \rho(j)\rho(k)),$$

$\rho(j) = 0, \forall |j| > q$ . The desired conclusion holds since from (10) we have that

$$|T_{2,n}| \longrightarrow +\infty \text{ as } n \rightarrow +\infty.$$

### 2.2.1 Particular case

In this section, we explore the case of linear trend, we test the null hypothesis

$$H_0 : Y_t = a_0 + a_1 t + Z_t \tag{13}$$

against the alternative

$$H_1 : (1 - L)Y_t = a_1 + Z_t, \tag{14}$$



and we suppose that the process  $Z_t$  is an *i.i.d.*( $0, \sigma^2$ ). Such situation can happened when after differencing  $Y_t$ , the the time series  $X_t = (1 - L)Y_t$  is a white noise, this usually the case for some financial time series. Autocorrelation tests (Box and Pierce (1970), Ljung and Box (1978) can be used to decide if  $X_t$  is a white noise or not).

Under the null hypothesis  $H_0$ ,  $X_t = a_1 + Z_t - Z_{t-1}$  in a non-invertible  $MA(1)$ , then we have  $\rho(1) = -\frac{1}{2}$ ,  $\rho(j) = 0$ ,  $\forall |j| > 1$ , therefore from (8) we get

$$\sqrt{n} * \left( \hat{\rho}(1) + \frac{1}{2} \right) \xrightarrow{\mathcal{L}} N(0, w_{1,1})$$

$$w_{1,1} = \sum_{k=1}^{\infty} (\rho(k+1) + \rho(k-1) - \rho(1)\rho(k))^2 = \frac{1}{2},$$

the test statistic is given by  $T_n = \sqrt{n} * \left( \hat{\rho}(1) + \frac{1}{2} \right)$ , and satisfies under  $H_0$ :

$$\sqrt{2} * T_n \xrightarrow{\mathcal{L}} N(0, 1). \quad (15)$$

Under the alternative  $H_1$ ,  $X_t = a_1 + Z_t$  is an i.i.d sequence, hence  $\rho(j) = 0$ ,  $\forall |j| \geq 1$ .

$$\begin{aligned} T_n &= \sqrt{n} * \left( \hat{\rho}(1) + \frac{1}{2} \right) \\ &= \sqrt{n}\hat{\rho}(1) + \frac{\sqrt{n}}{2} \\ &\xrightarrow{P} +\infty \end{aligned}$$

since  $\sqrt{n}\hat{\rho}(1) \xrightarrow{\mathcal{L}} N(0, 1)$ .

Note that in this case the limiting distribution of  $T_n$  is free of parameters, however if  $Z_t$  is an  $MA(q)$ , this limiting distribution depends on the correlation function of the process  $X_t$  and the exact quantiles of this limiting distribution cannot be computed. We will discuss on the next section how to resolve this problem by using the classical method of plug in (i.e. by replacing  $\rho(j)$  by  $\hat{\rho}(j)$ ) and by using Monte Carlo simulations.

### 3 Simulation study

First we present the Monte Carlo procedure to compute empirical quantiles of the limiting distribution of  $T_n$ .

Let  $W^{1/2}$  be an arbitrary square root of the matrix  $W$  given by (6):  $W^{1/2}W^{t/2} = W$ . From (8) it follows that

$$\hat{\rho}(j) \xrightarrow{P} \rho(j), \quad \forall 1 \leq j \leq d+q,$$

moreover the infinite summations in terms  $w_{i,j}$  are reduced to

$$w_{i,j} = \sum_{k=1}^{d+q+\max(i,j)} (\rho(k+i) + \rho(k-i) - \rho(i)\rho(k)) (\rho(k+j) + \rho(k-j) - \rho(j)\rho(k)),$$

which implies that

$$\hat{W} \xrightarrow{P} W$$

where  $\hat{W}$  is obtained from  $W$  by replacing  $\rho(j)$  by  $\hat{\rho}(j)$ .

It follows that the asymptotic distribution of  $\sqrt{n}^t (\hat{\rho}(1) - \rho(1), \dots, \hat{\rho}(d+q) - \rho(d+q))$  is the same as  $\hat{W}^{1/2}N$  where  $N$  is a Gaussian random vector ( $\dim N = d+q$ ) with 0 mean and variance the identity matrix  $I_{d+q}$ .

Hence the test statistic  $T_n = \sqrt{n} (\hat{\rho}(1) - \rho(1), \dots, \hat{\rho}(d+q) - \rho(d+q)) E_1$  has the same asymptotic distribution as  ${}^t E_1 \hat{W}^{1/2}N$ ,  ${}^t E_1 = (1, \dots, 1)$ , consequently we compute the empirical quantiles of the limiting distribution of  $T_n$  as follows:

We draw  $M$  realisations of a Gaussian random vector with 0 mean and variance the identity matrix  $I_{d+q}$ ,  $N_m^*$ ,  $m = 1, \dots, M$ , we compute

$$T_m^* = {}^t E_1 \hat{W}^{1/2} N_m^*, \quad m = 1, \dots, M.$$

For a fixed nominal size  $\alpha$ , the empirical quantile of order  $\alpha$  of the limiting distribution of  $T_n$  is given by

$$\hat{q}_\alpha = T_{([\alpha M])}^*$$

$[x]$  denotes the integer part of  $x$ , and  $T_{(i)}^*$  are the order statistics of  $T^*$ , i.e.

$$T_{(1)}^* \leq T_{(2)}^* \leq \dots \leq T_{(M)}^*.$$

The critical region of the test statistic  $T_n$  is then given by

$$] - \infty, \hat{q}_{\alpha/2}] \cup [\hat{q}_{1-\alpha/2}, +\infty[ \quad (16)$$

In order to evaluate the finite-sample performance of our test  $T_n$  we run Monte-Carlo simulations for several models. For each model we run  $R = 10000$  realisations and for four sample sizes,  $n = 50, 100, 200$  and  $500$ . We compute the empirical size and empirical power for three nominal levels  $\alpha = 1\%, 5\%$  and  $10\%$ . To compute the empirical quantiles of the limiting distribution of  $T_n$  we simulate  $M = 10000$  realisations of  $T_m^*$ .

We consider the two models (1) and (2) and the four cases:

Case 1:  $Z_t$  in an i.i.d  $N(0, 1)$  and  $d$  known.

Case 2:  $Z_t$  in an  $MA(q)$ ,  $(d, q)$  known.

Case 3:  $Z_t$  in an  $MA(q)$ ,  $q$  unknown, and  $d$  known.

Case 4:  $Z_t$  in an  $MA(q)$ ,  $(d, q)$  unknown.

### 1. Case 1, with $q = 0, d = 1$

We consider the following Data Generating Process(DGP):

$$Y_t = 1 + 0.05t + Z_t, 1 \leq t \leq n \quad (17)$$

to study the empirical size and the following DGP

$$(1 - L)Y_t = 0.05 + Z_t, 1 \leq t \leq n \quad (18)$$

to study the empirical power,  $Z_t = \varepsilon_t \rightsquigarrow N(0, 1)$ .

The DGP corresponds to the particular specification discussed in paragraph 2.2.1. In the

simulations  $(d, q) = (1, 0)$  is assumed to be known, hence the test statistic is given by  $T_n = \sqrt{n} * (\hat{\rho}(1) + \frac{1}{2})$ .

Recall that in this case we do not need to compute empirical quantiles of limiting distribution of  $T_n$ .

Table 1: Empirical size (left) and power (right) of  $T_n$ . Case 1

$\alpha$	$n$	50	100	200	500	$\alpha$	$n$	50	100	200	500
1		1.48	1.23	1.07	0.97	1		100	100	100	100
5		5.66	5.39	5.37	5	5		100	100	100	100
10		10.40	10.31	10.14	10.18	10		100	100	100	100

From Table 1, we see that the test statistic  $T_n$  has a good size and power even for small sample size.

## 2. Case 2, $d = 1, q = 1$

The DGPs are the same as (17) and (18) with:

$$Z_t = \varepsilon_t + 0.8\varepsilon_{t-1}, \varepsilon_t \rightsquigarrow N(0, 1).$$

In the simulations  $(d, q) = (1, 1)$  is assumed to be known, hence the test statistic is given by  $T_n = \sqrt{n} * \left( \sum_{j=1}^2 \hat{\rho}(j) + \frac{1}{2} \right)$ .

Table 2: Empirical size (left) and power (right) of  $T_n$ . Case 2

$\alpha$	$n$	50	100	200	500	$\alpha$	$n$	50	100	200	500
1		0.04	0.22	0.51	0.84	1		33.29	74.97	99.89	100
5		0.99	2.96	3.71	4.30	5		49.28	88.39	100	100
10		4.85	7.76	8.45	9.21	10		58.49	92.78	100	100

From Table 2, if the differenced times series is correlated, then the test  $T_n$  has a size and power distortion for small size sample. However such bias disappears as the sample size increases.

### 3. Case 3, $d = 1$ , $q$ unknown

The DGPs are the same as in the preceding case (Case 2), and  $q$  is assumed to be unknown. In this case, we need to define an empirical estimator for  $q$ . Note that under the DGP (17) the process  $X_t = (1 - L)Y_t$  is a stationary  $MA(q + 1)$ ; therefore  $\rho(j) = 0, \forall |j| > q + 1$ . By applying (8), it follows that an empirical estimator of  $q$  can be given by

$$\hat{q}_n = \arg \max_{1 \leq k \leq qmax} \left\{ k, \hat{\rho}(k) \notin \left[ -\frac{1}{\sqrt{n}} u_{\alpha/2} \sqrt{\hat{v}_k}, \frac{1}{\sqrt{n}} u_{\alpha/2} \sqrt{\hat{v}_k} \right] \right\} - 1 \quad (19)$$

where  $\hat{v}_k = 1 + 2 * \hat{\rho}^2(1) + \dots + 2 * \hat{\rho}^2(k)$ ,  $\hat{\rho}(k)$  is the empirical autocorelation of the process  $X_t$ ,  $u_\alpha$  is the quantile of order  $\alpha$  of the standard normal distribution  $N(0, 1)$ , and  $qmax$  is a given large integer, a suitable choice may be  $qmax = [n^\gamma], 0 < \gamma < 1$ , where  $n$  is the sample size.

In the simulations  $q$  is assumed to be unknown, hence the test statistic is given by  $T_n = \sqrt{n} * \left( \sum_{j=1}^{\hat{q}_n+1} \hat{\rho}(j) + \frac{1}{2} \right)$ , where  $\hat{q}_n$  is given by (19).

Table 3: Empirical size (left) and power (right) of  $T_n$ . Case 3

$\alpha$	$n$	50	100	200	500	$\alpha$	$n$	50	100	200	500
1		5.66	0.09	0.25	0.33	1		92.27	97.87	99.98	100
5		7.43	2.01	2.65	3.4	5		93.19	98.99	100	100
10,		10.4	9.96	6.95	7.7	10		95	99.57	100	100

From Table 3, we observe that when we need to estimate the order  $q$  then the test statistic  $T_n$  has a size and power distortion, the bias in the size is more severe than the one of the power. However such biases disappear for  $n$  large enough.

#### 4. Case 4, $(d, q)$ unknown

This case is the more realistic one, which is often encountered in practice. The classical procedure in Box et al. (2015) time series modelling is to identify first the parameter  $d$ . The well-known approach is to apply a test to check stationarity, take differences if the time series is non stationary and repeat the procedure until stationarity, i.e. the empirical estimator of  $d$  is given by:

**Algorithm 1**

*Step 0* :  $d = 0, Y_t^{(0)} = Y_t$

*Step 1*: Apply the KPSS test to  $Y_t^{(d)}$

*Step 2*: If the null is rejected  $d = d + 1, Y_t^{(d)} = (1 - L)Y_t^{(d-1)},$

*go to the Step 1,*

*else end.*

The empirical estimator is simply the number of differentiations needed to achieve stationarity of  $(Y_t)$ .

**Remark.** Note that the process  $X_t$

$$X_t = (1 - L)^d Y_t = \begin{cases} a_d + (1 - L)^d Z_t \text{ under } H_0, (1) \\ a_d + Z_t \text{ under } H_1, (2) \end{cases}$$

is stationary, hence application of the KPSS test helps us to achieve stationarity, but no information is obtained about the kind of the trend.

To estimate the moving average order  $q$  of  $Z_t$ , let  $q_X$  denotes the order of the process  $X_t$  which may be  $d + q$  (resp.  $q$ ) under  $H_0$  (resp. under  $H_1$ ). To estimate  $q_X$  we use the following estimator

$$\hat{q}_X = \arg \max_{1 \leq k \leq q_{max}} \left\{ k, \hat{\rho}(k) \notin \left[ -\frac{1}{\sqrt{n}} u_{\alpha/2} \sqrt{\hat{v}_k}, \frac{1}{\sqrt{n}} u_{\alpha/2} \sqrt{\hat{v}_k} \right] \right\} \quad (20)$$

where  $\hat{v}_k = 1 + 2 * \hat{\rho}^2(1) + \dots + 2 * \hat{\rho}^2(k)$ ,  $\hat{\rho}(k)$  is the empirical autocorrelation of the process  $X_t = (1 - L)^{\hat{d}_n} Y_t$ ,  $\hat{d}_n$  is computed by the use of **Algorithm 1** and  $u_\alpha$  is the quantile of order  $\alpha$  of the standard normal distribution  $N(0, 1)$ .

Since the test statistic will be computed under the null  $H_0$ , an estimator of the order  $q$  of  $Z_t$  will be given by

$$\hat{q}_n = \hat{q}_X - \hat{d}_n \quad (21)$$

To compute the test statistic we adopt the following strategy:

**Strategy:**

*Step 1: Apply the **Algorithm 1** to time series  $Y_t$  to compute an estimator  $\hat{d}_n$  of  $d$*

*Step 2: Apply (21) to time series  $X_t = (1 - L)^{\hat{d}_n} Y_t$  to compute an estimator  $\hat{q}_n$  of  $q$*

*Step 3: Compute the test statistic  $T_n = \sqrt{n} * \left( \sum_{j=1}^{\hat{d}_n + \hat{q}_n} \hat{\rho}(j) + \frac{1}{2} \right)$ , where  $\hat{\rho}(j)$  is the autocorrelation function of  $X_t$ .*

We consider the following Data Generating Process(DGP):

$$Y_t = 1 + 0.05t + 0.02t^2 + Z_t, 1 \leq t \leq n \quad (22)$$

to study the empirical size and the following DGP

$$(1 - L)^2 Y_t = 0.05 + Z_t \quad (23)$$

to study the empirical power,

$$Z_t = \varepsilon_t + 0.8\varepsilon_{t-1}, \varepsilon_t \rightsquigarrow N(0, 1). \quad (24)$$

Table 4: Empirical size (left) and power (right) of  $T_n$ . Case 4

$\alpha$	$n$	50	100	200	500	$\alpha$	$n$	50	100	200	500
1		0.71	5.9	12.70	0.92	1		91.23	85.51	90.39	93.71
5		11.11	29.96	23.72	4.29	5		92.31	87.90	92.84	93.98
10		28.70	47.26	28.54	8.52	10		92.78	89.65	93.78	94.40

From Table 4, we observe the same behaviour of  $T_n$  as in the preceding case when we need to estimate  $(d, q)$ .

### 3.0.1 Extension to non-polynomial trend

In this subsection we will study the empirical size of the test statistic  $T_n$  when under the null hypothesis the deterministic trend is not a polynomial of time. The null hypothesis is

$$H_0 : Y_t = m(t) + Z_t, 1 \leq t \leq n \quad (25)$$

( $Z_t$ ) is assumed to be an invertible moving average process,  $MA(q)$ , with  $q$  unknown:

$$Z_t = B(L)\varepsilon_t, B(z) = 1 + b_1z + \dots + b_qz^q, \quad (26)$$

where  $m(t)$  is an arbitrary function of time  $t$ . If  $m(t)$  is infinitely differentiable with bounded derivatives on  $t = 0$ , then a Taylor expansion of  $m(t)$  is given by

$$m(t) = \sum_{j=0}^{\infty} a_j t^j, a_j = \frac{1}{j!} \frac{d^j m(0)}{dt^j}.$$

Since

$$a_n \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

the function  $m(t)$  can be well approximated by a finite polynomial

$$P_d(t) = a_0 + a_1 t + \dots + a_d t^d, \text{ for } d \text{ large enough}$$



To illustrate how our strategy can be extended to a more general deterministic trend, we consider the two following DGPs under the null hypothesis

$$Y_t = 1 + \log(1 + t) + Z_t, 1 \leq t \leq n \quad (27)$$

$$Y_t = 1 + e^{t/100} + Z_t, 1 \leq t \leq n \quad (28)$$

where  $Z_t$  is given by (24). The linear trend moves between these two trends since  $\log(1+t) \leq t \leq e^{t/100}$  for large  $t$ .

Table 5: Empirical size (left: DGP (27)) and (right: DGP (28)) of  $T_n$

$\alpha$	$n$	50	100	200	500	$\alpha$	$n$	50	100	200	500
1		5.66	0.05	0.33	0.43	1		5.29	0.07	0.28	0.84
5		7.28	2.07	4.44	4.86	5		7.21	2.22	3.61	3.85
10		10.10	7.66	10.17	9.98	10		9.88	7.35	9.07	7.83

Table 5 shows that if the deterministic trend is not a polynomial of time, then  $T_n$  has a size distortion but the bias disappears as the sample size increases. Simulations not reported in this paper indicate that the number of differentiations needed to obtain stationarity is smaller for the DGP (27) than (28). Moreover the number  $d$  of differentiations needed to achieve stationarity of  $(Y_t)$  increases with the sample size  $n$ .

## 4 Application to real data

For a given non stationary real data  $Y_t, 1 \leq t \leq n$ , we apply the strategy presented in the preceding section to decide if the trend is deterministic or stochastic.

## 1) Macroeconomic data

We consider the logarithm of the annual  $GDP$  (given on trillion of dollars) of USA (data source : <https://data.worldbank.org>) on the period 1960-2023, the time series is plotted in figure 1. We obtain  $\hat{d}_n = 2$ , i.e. we need to differentiate two times to achieve stationarity of  $GDP$ . Hence the process

$$X_t = (1 - L)^2 \log(GDP)_t = \begin{cases} \text{is an } MA(2 + q) \text{ under } H_0, (1) \\ \text{is an } MA(q) \text{ under } H_1, (2) \end{cases}$$

If we denote by  $q_X$  the order of the process  $X_t$  then by using (20) we obtain the estimator  $\hat{q}_X = 1$ . This in turn indicates that the  $\log(GDP)$  is generated by a stochastic trend, since under the null of deterministic trend we have that  $q_X \geq 2$ .

Hence our finding is in accordance with several previous studies, for example Nelson and Plosser (1982) showed that the GDP is an ARIMA(0,1,1), based on historical data 1909-1977, however we find that the GDP is an ARIMA(0,2,1), based on a recent historical data 1960-2023, more precisely we obtain the following model

$$(1 - L)^2 \log(GDP)_t = Z_t, \quad Z_t = \varepsilon_t - 0.6885\varepsilon_{t-1},$$

$\varepsilon_t$  is a white non-gaussian noise with mean 0.000645 and variance 0.0005.

## 2) Financial data

We consider the Standard and Poor's ( $SP500$ ) financial time series data, i.e. the daily closing from 2000 to December 2023 (currency in USD). The data was obtained from Yahoo Finance. The time series is plotted in figure 1. We obtain  $\hat{d}_n = 1$ , i.e. we need to differentiate only once to achieve stationarity of  $SP500$ . Hence the process  $X_t$  satisfies

$$X_t = (1 - L) \log(SP500)_t = \begin{cases} \text{is an } MA(1 + q) \text{ under } H_0, (1) \\ \text{is an } MA(q) \text{ under } H_1, (2) \end{cases}$$

We obtain  $\hat{q}_X = 1$ , as the order of the process  $X_t$  which implies, by using (21), that the order of the process  $Z_t$  is  $\hat{q}_n = 0$  (i.e. the same as the DGP in case 1 in the simulation study).

By applying our test, we obtain  $T_n = 30.91$  and from (15) we obtain  $P - value = 0$ . Consequently the null hypothesis of deterministic trend is strongly rejected, therefore the  $\log(SP500)$  is generated by a stochastic trend, and our conclusion is in accordance with market efficiency theory.

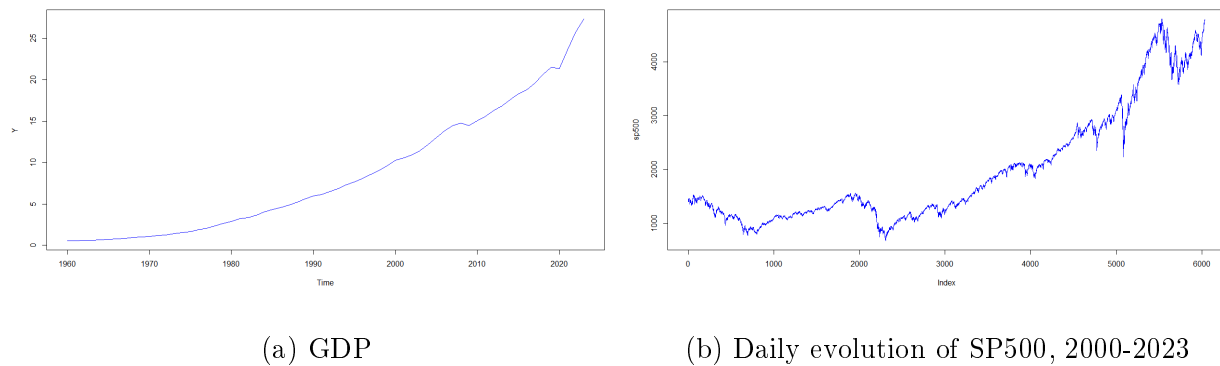


Figure 1: plots of GDP and SP500

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