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Muriel Boulakia, Maya de Buhan, Tiphaine Delaunay, Sébastien Imperiale, Philippe Moireau. Solving inverse source wave problem from Carleman estimates to observer design. 2024. hal-04788439

**HAL Id: hal-04788439**

**<https://hal.science/hal-04788439v1>**

Preprint submitted on 18 Nov 2024

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# Solving inverse source wave problem – from Carleman estimates to observer design

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**Code availability:**  
Freefem++ numerical  
illustrations are available on  
request

**Funding:** This work was  
supported by ANR  
Projet-ANR-19-CE48-0004  
ODISSE

**Competing interests:**  
The author declare no  
competing interests.

## Abstract

In this work, we are interested by the identification in a wave equation of a space dependent source term multiplied by a known time and space dependent function, from internal *velocity* or *field* measurements. The first part of the work consists in proving stability inequalities associated with this inverse problem from adapted Carleman estimates. Then, we present a sequential reconstruction strategy which is proved to be equivalent to the minimization of a cost functional with Tikhonov regularization. Based on the obtained stability estimates, the reconstruction error is evaluated with respect to the noise intensity. Finally, the proposed method is illustrated with numerical simulations, both in the case of regular source terms and of piecewise constant source terms.

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Inverse problems related to wave equation phenomena are widely used in engineering (structural health monitoring [49, 61]), environmental sciences (seismic exploration [42]) and life sciences (tomography [31], elastography [30], pulse wave propagation [60]). In a very general view, these inverse problems can be summarized into three categories of reconstruction problems that are strongly intertwined. On one hand, state reconstruction aims to register the wave solution, or equivalently to reconstruct the initial conditions of the observed wave propagation. On the other hand, parameter identification takes into account situations where the model is partially uncertain and attempts to reconstruct the material properties of the physical domain of propagation. Between these two estimation problems, source term reconstruction is devoted to the identification of external loading on the system, whether through volume sources or boundary conditions. In essence, the source reconstruction problem can be considered as an intermediate problem between state estimation and parameter identification, since it is fundamentally an identification problem, even if it remains a linear inverse problem like in state estimation. Moreover, many identification problems are solved from a sequence of linearized subproblems [37] each corresponding to a source term inversion. At the same time, source reconstruction problems are often analyzed thanks to stability estimates based themselves on observability inequalities developed for the reconstruction of initial conditions [59].

The present work naturally falls within this framework by attempting to generalize the strategy proposed in [59] to a more general class of source term. More precisely, the work [59] deals with the reconstruction from local *field* measurements of a space-dependent source term multiplied by a known time-dependent function. Hence, the theoretical results obtained in that work relied heavily on the variable separation in the source term. Here, we propose to investigate a more general case by reconstructing a space-dependent source term multiplied by a known space- and time-dependent function. Therefore, while in [59] the reconstruction stability was based on a classical observability estimate in combination with a Volterra inversion, here the general structure of the source leads us to prove a general stability estimate derived from a Carleman estimate [1, 16].

Carleman estimates provide a general tool for proving stability results for problems with single measurements [5]. Without claiming to be exhaustive, we can quote the following works for parabolic problems [11, 28, 39, 56], for hyperbolic problems [11, 17, 19, 22, 23, 24, 25, 26, 27, 56] and in particular for wave equations [14, 20, 32, 40, 43]. In this paper, we establish two new Carleman estimates for different semi-norms of the internal measurements. Our first estimate which involves measurements of the spatial gradient of the field will be well suited to *velocity* measurements whereas our second estimate which involves measurements of the Laplacian of a time primitive function of the field will be well suited to *field* measurements. Moreover, the stability estimates derived from the two Carleman estimates will highlight the degree of ill-posedness of the related inverse problem (in the sense of Definition 1.1) and as a consequence different regularization strategies will be used to bridge the gap in regularity.

With regard to the reconstruction strategy, we have further developed the sequential approach proposed in [35, 59]. Minimizing a least squares criterion [2, 3] is certainly the most widely used approach to reconstruct source terms from internal measurements. This has led to the so-called 4D-Var approach to data assimilation [6, 7, 36] or the full-wave inversion (FWI) strategy [52]. Let us also mention that other numerical reconstruction methods based on the minimization of a functional convexified through the use of Carleman weights have been proposed, in [18, 56, 58] and then in [46, 50, 55]. As an alternative, the sequential approaches, also

known as observer design [4], aim at formulating a reconstruction strategy that uses the measurements as they become available, thus defining a model that uses the available measurements through a feedback to converge asymptotically to the target system and source term. By relying on a dynamic programming principle, we show that a sequential strategy can be proved to correspond exactly to the minimization of the least squares criterion [57, 59]. Therefore, the reconstruction is obtained in only one forward simulation under the assumption that we can compute and store the underlying Riccati operator involved in the feedback term. Note that in [50], the authors recover a zero order potential of a wave propagation problem. Our approach is not directly applicable to this setting. However, assuming that the zero order potential is not large, one could use asymptotic series, to reduce the problem to a sequence of source recovery problem. The question of convergence of this sequence of inverse problems would be a delicate but interesting question.

This paper is structured as follows. In the first section, we present the inverse problem we are interested in. Then, in Section 2, we state and prove Carleman estimates, which are based on the work of [47] and involve non classical internal measurement terms which are designed for our inverse problem. These Carleman estimates are applied to establish stability properties of the inverse source problem. In Section 3, we then propose a minimization strategy based on an adapted Tikhonov regularization and we derive from the stability inequalities estimates of the reconstruction error for noisy measurements. We then present a reconstruction method based on an optimal observer which corresponds to the minimization as presented in [59]. Finally, Section 4 is dedicated to the presentation of some numerical results. In particular, we consider the case where a piecewise constant source term has to be reconstructed and present an iterative procedure relying on the use of adaptive basis.

## 1 Problem statement

### 1.1 Model

Let  $T > 0$  and  $\Omega$  be a bounded smooth enough, connected open domain in  $\mathbb{R}^d$ ,  $d \geq 1$ . We consider the following wave equation

$$\begin{cases} \partial_{tt}u(x,t) - \Delta u(x,t) = \sigma(x,t)\theta(x), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \\ \partial_t u(x,0) = v_0(x), & x \in \Omega. \end{cases} \quad (1)$$

Here, the right hand side is the product of a given function  $\sigma$  which is assumed to belong to  $H^1(0,T;L^\infty(\Omega))$  and an unknown function  $\theta$  which is assumed to depend only on the space variable  $x$ . Moreover,  $(u_0, v_0)$  is an initial condition given in  $\mathcal{Z} = H_0^1(\Omega) \times L^2(\Omega)$ .

If we introduce the notation  $z = (u, v)^\top$  with  $v = \partial_t u$ , we can rewrite (1) as a first-order system:

$$\begin{cases} \dot{z} = Az + B\theta, & \text{in } (0,T), \\ z(0) = z_0, \end{cases} \quad (2)$$

where

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \sigma(x,t)\text{Id} \end{pmatrix}, \quad (3)$$

and the operator  $A$  is an unbounded skew-adjoint operator from  $\mathcal{D}(A)$  into  $\mathcal{Z}$  defined by

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(\Delta_0) \times H_0^1(\Omega), \quad (4)$$



with  $\Delta_0$  the Laplacian operator with homogeneous Dirichlet boundary conditions. Since the operator  $A$  is maximal dissipative, it is a generator of a  $C^0$ -semigroup  $(e^{tA})_{t \geq 0}$  on  $\mathcal{Z}$ .

For the sequel, we also introduce the definition of the *sensitivity*, i.e. the family  $(L(t))_{t \geq 0}$  of bounded operators, so that, for a given  $\theta \in L^2(\Omega)$ , the application  $t \mapsto L(t)\theta$  is solution in  $C^0([0, T], \mathcal{Z})$  of

$$\begin{cases} \dot{z} = Az + B\theta, \\ z(0) = 0. \end{cases} \quad (5)$$

Then, the Duhamel's formula for the solution of (1) can be written  $z(t) = e^{tA}z_0 + L(t)\theta$  for all  $t \in [0, T]$ .

## 1.2 Observation operator

In this study, we assume that, to reconstruct the function  $\theta$ , we have access to *velocity* or *field* measurements on the solution of (1). These measurements are available on a subdomain  $\omega$  that can not be chosen arbitrarily.

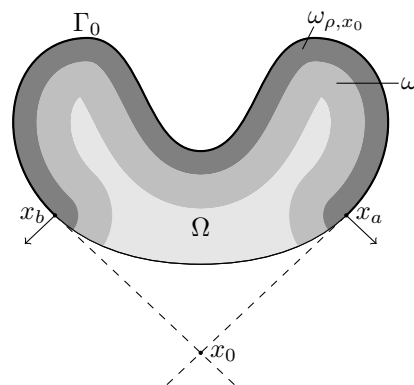
First, let us introduce a part of the boundary on which the observation domain have to lean on. For every  $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$ , we define

$$\Gamma_0 := \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\},$$

where  $\nu$  is the outward unit normal to  $\partial\Omega$ . Then, for every  $\rho > 0$ , we define

$$\omega_{\rho, x_0} := \Omega \cap \bigcup_{x \in \Gamma_0} B(x, \rho). \quad (6)$$

We refer to Figure 1 for an illustration.



**Figure 1.** An example of domain  $\omega_{\rho, x_0}$  defined in (6).

We assume that the observation domain  $\omega$  is an open and nonempty subset of  $\Omega$  such that the boundaries  $\partial\omega$  and  $\partial(\Omega \setminus \omega)$  are Lipschitz, and that the *multiplier condition* is satisfied: there exists  $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$  and  $\rho > 0$  such that

$$\omega_{\rho, x_0} \subset \omega. \quad (7)$$

We also assume that the observation time  $T$  satisfies the *time condition*:

$$T > T_0 := \sup_{x \in \bar{\Omega}} |x - x_0|. \quad (8)$$

As observation, we consider measurements of  $u$ , solution of (1), in the domain  $\omega$  during the time interval  $(0, T)$ , or any of its time derivative. In fact, we will see that measuring  $u|_{\omega}$  or  $\partial_t u|_{\omega}$  will introduce major differences in the proposed inversion method due to the fact that  $u$  and  $\partial_t u$  play different roles in the first-order formulation (2). By contrast, when measuring  $\partial_{tt} u$  (or any high order time

derivative), we can, without loss of generality, get back to the case where  $\partial_t u|_\omega$  is observed by simply integrating

$$\partial_t u|_\omega(t) = \partial_t u|_\omega(0) + \int_0^t \partial_{tt} u|_\omega(\tau) \, d\tau.$$

Therefore, in this article, we will always separate our study between the *field* measurements configuration where  $u|_\omega$  is measured and the *velocity* measurements configuration where  $\partial_t u|_\omega$  is measured. For each configuration, let us define the observation operator  $C$  usually introduced in control and observation theory – see for instance [38] and references therein.

In the case of *velocity* measurements, the observation operator is given by

$$C = \begin{pmatrix} 0 & I_\omega \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \tag{9}$$

where  $\mathcal{Y} = L^2(\omega)$  and  $I_\omega$  is the restriction operator in  $\mathcal{L}(L^2(\Omega), L^2(\omega))$ . Notice that the adjoint associated with  $C$  is given by

$$C^* = \begin{pmatrix} 0 & \mathbb{1}_\omega \end{pmatrix}^\top,$$

where  $\mathbb{1}_\omega : L^2(\omega) \rightarrow L^2(\Omega)$  is the extension operator by 0 in  $\Omega \setminus \omega$ .

In the case of *field* measurements, the observation operator is given by

$$C = \begin{pmatrix} I_\omega & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \tag{10}$$

where  $\mathcal{Y} = L^2(\omega)$  and  $I_\omega$  is the restriction operator in  $\mathcal{L}(H_0^1(\Omega), L^2(\omega))$ . Let us remark that, since in practice, the measurements are subject to error, we do not restrict the observation space to  $H^1(\omega)$  which would have been the natural space associated with the solution of (2). The adjoint associated with  $C$  is given by

$$C^* = \begin{pmatrix} F_\omega & 0 \end{pmatrix}^\top$$

where  $F_\omega \in \mathcal{L}(L^2(\omega), H_0^1(\Omega))$  is given by: for all  $\phi \in L^2(\omega)$ ,  $F_\omega \phi$  is the solution of

$$\begin{cases} -\Delta \psi = \chi_\omega \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where  $\chi_\omega$  is the characteristic function of  $\omega$ .

### 1.3 Inverse problem setting

We consider an actual trajectory  $\check{u}$ , modeled as a mild solution  $\check{z}$  of (2) associated with given  $(u_0, v_0)$  and  $\sigma$ , and an unknown  $\check{\theta} \in H_0^1(\Omega)$ . Since the initial data are known and since the inverse problem that we will address is linear, we can reduce ourselves to the case where the initial conditions are null. Therefore, we consider the following system

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(x, t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = 0, & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \tag{12}$$

and its first-order formulation

$$\begin{cases} \dot{\check{z}} = A\check{z} + B\check{\theta}, & \text{in } (0, T), \\ \check{z}(0) = 0, \end{cases} \tag{13}$$

where  $B$  and  $A$  are respectively defined by (3) and (4). We want to reconstruct  $\check{\theta}$  and, to do so, we assume that we have access to measurements modeled by the

observation operator defined by (9) or (10). We denote by  $y^\delta \in L^2(0, T; L^2(\omega))$  the available measurement which is assumed to be perturbed by a measurement error of amplitude  $\delta$ , namely

$$\exists \delta > 0 \text{ such that } \|y^\delta - C\check{z}\|_{L^2(0, T; L^2(\omega))} \leq \delta\sqrt{T}.$$

Therefore, recovering  $\check{\theta}$  from  $y^\delta$  consists in inverting the linear *input-output* operator

$$\Psi_T : \begin{cases} L^2(\Omega) \rightarrow L^2(0, T; L^2(\omega)), \\ \theta \mapsto (t \mapsto CL(t)\theta). \end{cases} \quad (14)$$

We immediately note that  $\Psi_T$  is bounded, since, for both definitions (9) and (10),  $C$  is a bounded operator from  $\mathcal{Z}$  to  $L^2(\omega)$ .

Inspired by the static inverse problem community, we propose a definition of the degree of ill-posedness of an inverse problem associated with our evolution problem.

**Definition 1.1.** *Let  $s, r \in \mathbb{N}$  be given. We consider an input-output operator*

$$\Psi_T : H^s(\Omega) \rightarrow L^2(0, T; H^r(\omega)).$$

*If there exist  $n, m \in \mathbb{N}$  satisfying  $n \leq s$  and  $r \leq m$  and  $c > 0$  such that, for all  $\theta \in H^s(\Omega)$ , we have*

$$\|\Psi_T\theta\|_{L^2(0, T; H^m(\omega))} \geq c\|\theta\|_{H^n(\Omega)},$$

*then we say that the problem of inverting  $\Psi_T$  is ill-posed of degree  $m - r + s - n$ .*

We will show in Section 2.2 that the problem of inverting the input-output operator  $\Psi_T$  given by (14) corresponds to a mildly ill-posed problem of degree 1 in the case of *velocity* measurements and of degree 2 in the case of *field* measurements (see [15] for instance). These properties will be related to stability estimates associated with our inverse problem which are stated in Theorems 2.3 and 2.4. derived from the Carleman estimates given in Section 2.1. In Section 3, we will propose a regularization strategy to invert  $\Psi_T$  using observer methods.

## 2 Stability inequalities associated with our inverse source problem

### 2.1 Carleman estimates

In this section, we state two different Carleman estimates. These estimates will be used in Subsection 2.2 to prove stability estimates associated with our inverse source problem. To do so, we will follow the Bukhgeim-Klibanov method first presented in [5] and further developed in [11, 56] and apply our Carleman estimates to the time derivative of the solution of (12). In what follows, we will rely on the classical techniques used to prove Carleman estimates for the wave equation initially proposed by [10] and then modified and used in [13, 20, 22, 24, 40, 47] and also presented in [56, Theorem 2.7.1].

Before stating our results, let us introduce the adequate Carleman weights. For  $\beta \in (0, 1)$  and  $\lambda > 0$ , we define, for all  $(x, t) \in \Omega \times (-T, T)$

$$\psi(x, t) = |x - x_0|^2 - \beta t^2 + c_0 \text{ and } \phi(x, t) = e^{\lambda\psi(x, t)}, \quad (15)$$

where  $c_0 > 0$  is chosen such that  $\psi \geq 1$  in  $\Omega \times (-T, T)$ .

The theorem which follows corresponds to our first Carleman inequality. In this inequality, the right hand side involve the  $L^2((-T, T) \times \omega)$  weighted norms of the field  $w$  and its spatial gradient  $\nabla w$ . Let us notice that, in this theorem and in the next one, there is no condition on the final time  $T$  that can be arbitrary small. This is due to the fact that we have the restrictive conditions  $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$  in  $\Omega$ .

**Theorem 2.1.** *We assume that  $\omega$  satisfies (7). Let  $\phi$  be the weight function defined by (15). Then, for every  $\beta \in (0, 1)$ , there exist  $\lambda > 0$ ,  $s_0 > 0$  and a positive constant  $c$  such that, for all  $s \geq s_0$ , we have,*

$$\begin{aligned} & s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ & \leq c \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + cs^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla w|^2 + s^2|w|^2) dx dt, \end{aligned} \quad (16)$$

for all  $w \in L^2(-T, T; H_0^1(\Omega)) \cap H^1(-T, T; L^2(\Omega))$  such that  $Pw = \partial_{tt}w - \Delta w \in L^2(\Omega \times (-T, T))$  and  $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$  in  $\Omega$ .

To prove this result, we adapt the proof of Theorem 2.1 in [47] which states a Carleman estimate involving boundary measurements. Therefore, we will not give all the details but mainly focus on the original part of the proof that corresponds to the derivation of our different observation terms.

In this section, we write  $a \lesssim b$  if there exists a positive constant  $c > 0$  independent of  $s$  such that  $a \leq cb$ .

*Proof.* In this proof, all the computations will be done for smooth functions and lastly a density argument allows to extend the result to the regularity given in the theorem.

The beginning of the proof closely follows the one of Theorem 2.1 in [47]. More precisely, we start from equation (2.13) in [47] that we rewrite here with the difference that we do not highlight the dependence of the inequality with respect to  $\lambda$ : there exists  $s_0 > 0$  such that, for all  $s > s_0$ , any function  $z$  in  $C^2(\overline{\Omega} \times [-T, T])$  such that  $z(\cdot, 0) = 0$  in  $\Omega$ ,  $z(\cdot, \pm T) = \partial_t z(\cdot, \pm T) = 0$  in  $\Omega$  and  $z = 0$  on  $\partial\Omega \times [-T, T]$  satisfies

$$\begin{aligned} & s \int_{-T}^T \int_{\Omega} (|\partial_t z|^2 + |\nabla z|^2 + s^2|z|^2) dx dt + \int_{-T}^T \int_{\Omega} |P_1 z|^2 dx dt \\ & \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} z|^2 dx dt + s \int_{-T}^T \int_{\Gamma_0} |\partial_\nu z|^2 d\sigma dt, \end{aligned} \quad (17)$$

where

$$P_1 z = \partial_{tt} z - \Delta z + s^2 \lambda^2 \phi^2 z (|\partial_t \psi|^2 - |\nabla \psi|^2), \quad (18)$$

and

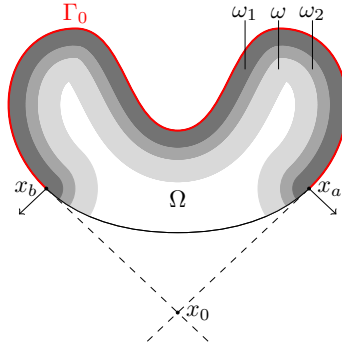
$$\begin{aligned} P_{conj} z &= e^{s\phi} P(e^{-s\phi} z) \\ &= \partial_{tt} z - 2s\lambda\phi(\partial_t z \partial_t \psi - \nabla z \cdot \nabla \psi) + s^2 \lambda^2 \phi^2 z (|\partial_t \psi|^2 - |\nabla \psi|^2) \\ &\quad - \Delta z - s\lambda\phi z (\partial_{tt} \psi - \Delta \psi) - s\lambda^2 z (|\partial_t \psi|^2 - |\nabla \psi|^2) \phi. \end{aligned} \quad (19)$$

Let  $w$  be given in  $C^2(\overline{\Omega} \times [-T, T])$  such that  $w = 0$  on  $\partial\Omega \times [-T, T]$ ,  $w(\cdot, 0) = 0$  in  $\Omega$ ,  $w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$  in  $\Omega$ . We define  $v$  by the classical change of variables  $v = e^{s\phi} w$  in  $\Omega \times (-T, T)$ .

We will apply inequality (17) to  $z = \chi v$  where  $\chi \in C^\infty(\overline{\Omega})$  is a cut-off function such that  $0 \leq \chi \leq 1$  and

$$\chi(x) = \begin{cases} 0, & \text{if } x \in \omega_1, \\ 1, & \text{if } x \in \Omega \setminus \omega_2 \end{cases} \quad (20)$$

where  $\omega_1 = \omega_{\rho_1, x_0}$  and  $\omega_2 = \omega_{\rho_2, x_0}$  are defined by (6) with  $\rho_1 < \rho_2 < \rho$  and illustrated in Figure 2.



**Figure 2.** Examples of domains  $\omega$ ,  $\omega_1$  and  $\omega_2$ .

We notice that

$$\begin{aligned}
 \partial_t z &= \chi \partial_t v, & \partial_{tt} z &= \chi \partial_{tt} v, & \nabla z &= \nabla \chi v + \chi \nabla v, \\
 \Delta z &= \Delta \chi v + 2 \nabla \chi \cdot \nabla v + \chi \Delta v, \\
 P_1 z &= \chi P_1 v - \Delta \chi v - 2 \nabla \chi \cdot \nabla v, \\
 P_{conj} z &= \chi P_{conj} v + 2s\lambda\phi \nabla \chi \cdot \nabla \psi v - \Delta \chi v - 2 \nabla \chi \cdot \nabla v.
 \end{aligned} \tag{21}$$

Using that  $v = z$  on  $\Omega \setminus \omega_2$ , we obtain

$$\begin{aligned}
 & s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\
 &= s \int_{-T}^T \int_{\Omega \setminus \omega_2} (|\partial_t z|^2 + |\nabla z|^2 + s^2 |z|^2) \, dx \, dt + \int_{-T}^T \int_{\Omega \setminus \omega_2} |P_1 z|^2 \, dx \, dt \\
 & \quad + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 \, dx \, dt.
 \end{aligned}$$

Therefore, applying inequality (17) to  $z$ , we get that, for  $s \geq s_0$ ,

$$\begin{aligned}
 & s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\
 & \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} z|^2 \, dx \, dt + s \int_{-T}^T \int_{\Gamma_0} |\partial_\nu z|^2 \, d\sigma \, dt \\
 & \quad + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 \, dx \, dt.
 \end{aligned}$$

Eventually, since  $\partial_\nu z = \partial_\nu \chi v + \chi \partial_\nu v = 0$  on  $\Gamma_0$  and using (21), we get

$$\begin{aligned}
 & s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\
 & \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 \, dx \, dt + s \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt \\
 & \quad + \int_{-T}^T \int_{\omega_2} |P_1 v|^2 \, dx \, dt.
 \end{aligned} \tag{22}$$

Next, using the definitions (18) and (19), we observe that

$$|P_1 v|^2 \lesssim |P_{conj} v|^2 + s^2 (|\partial_t v|^2 + |\nabla v|^2 + |v|^2),$$

thus, for  $s$  large enough, we can replace the last term in (22) to obtain

$$\begin{aligned}
 & s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt + \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\
 & \lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 \, dx \, dt + s^2 \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s |v|^2) \, dx \, dt.
 \end{aligned} \tag{23}$$

Let us now add a term in  $\partial_t v(0)$  in the left-hand side. According to the definition (18) of  $P_1$ , an integration by parts in space gives:

$$\begin{aligned} \int_{-T}^0 \int_{\Omega} P_1 v \partial_t v \, dx \, dt &= \frac{1}{2} \int_{-T}^0 \int_{\Omega} \partial_t (|\partial_t v|^2) \, dx \, dt + \frac{1}{2} \int_{-T}^0 \int_{\Omega} \partial_t (|\nabla v|^2) \, dx \, dt \\ &\quad + \frac{s^2 \lambda^2}{2} \int_{-T}^0 \int_{\Omega} \phi^2 (|\partial_t \psi|^2 - |\nabla \psi|^2) \partial_t (|v|^2) \, dx \, dt, \end{aligned}$$

and since  $v(\cdot, 0) = 0$  in  $\Omega$ ,  $v(\cdot, -T) = \partial_t v(\cdot, -T) = 0$  in  $\Omega$ ,

$$\begin{aligned} \int_{-T}^0 \int_{\Omega} P_1 v \partial_t v \, dx \, dt &= \frac{1}{2} \int_{\Omega} |\partial_t v(0)|^2 \, dx \\ &\quad - \frac{s^2 \lambda^2}{2} \int_{-T}^0 \int_{\Omega} \partial_t (\phi^2 (|\partial_t \psi|^2 - |\nabla \psi|^2)) |v|^2 \, dx \, dt. \end{aligned}$$

Rearranging the terms and multiplying by  $s^{1/2}$ , we get

$$\begin{aligned} \frac{s^{1/2}}{2} \int_{\Omega} |\partial_t v(0)|^2 \, dx &= s^{1/2} \int_{-T}^0 \int_{\Omega} P_1 v \partial_t v \, dx \, dt \\ &\quad + \frac{s^{5/2} \lambda^2}{2} \int_{-T}^0 \int_{\Omega} \partial_t (\phi^2 (|\partial_t \psi|^2 - |\nabla \psi|^2)) |v|^2 \, dx \, dt. \end{aligned}$$

Then, by Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned} s^{1/2} \int_{\Omega} |\partial_t v(0)|^2 \, dx &\lesssim \int_{-T}^0 \int_{\Omega} |P_1 v|^2 \, dx \, dt \\ &\quad + s \int_{-T}^0 \int_{\Omega} |\partial_t v|^2 \, dx \, dt + s^{5/2} \int_{-T}^0 \int_{\Omega} |v|^2 \, dx \, dt. \end{aligned}$$

Therefore, extending the integrals between 0 and  $T$  in the right-hand side, we have

$$\begin{aligned} s^{1/2} \int_{\Omega} |\partial_t v(0)|^2 \, dx &\lesssim \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\ &\quad + s \int_{-T}^T \int_{\Omega} |\partial_t v|^2 \, dx \, dt + s^{5/2} \int_{-T}^T \int_{\Omega} |v|^2 \, dx \, dt. \quad (24) \end{aligned}$$

Injecting this inequality in (22), we obtain that, for  $s \geq s_0$ ,

$$\begin{aligned} s^{1/2} \int_{\Omega} |\partial_t v(0)|^2 \, dx + s \int_{-T}^T \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + s^2 |v|^2) \, dx \, dt &+ \int_{-T}^T \int_{\Omega} |P_1 v|^2 \, dx \, dt \\ &\lesssim \int_{-T}^T \int_{\Omega} |P_{conj} v|^2 \, dx \, dt + s^2 \int_{-T}^T \int_{\omega_2} (|\partial_t v|^2 + |\nabla v|^2 + s |v|^2) \, dx \, dt. \end{aligned}$$

Next, we come back to the initial function  $w = e^{-s\phi} v$ . After classical computations that we do not detail here (we refer for instance to [47]), we get

$$\begin{aligned} s^{1/2} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 \, dx + s \int_{-T}^T \int_{\Omega} e^{2s\phi} (|\partial_t w|^2 + |\nabla w|^2 + s^2 |w|^2) \, dx \, dt \\ &\lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt \\ &\quad + s^2 \int_{-T}^T \int_{\omega_2} e^{2s\phi} (|\partial_t w|^2 + |\nabla w|^2 + s^2 |w|^2) \, dx \, dt. \quad (25) \end{aligned}$$

To conclude the proof, it remains to remove the term in  $\partial_t w$  in the right hand side. To do so, we define a function  $\rho \in C^\infty(\Omega)$  such that

$$\rho(x) = \begin{cases} 1, & \text{if } x \in \omega_2, \\ 0, & \text{if } x \in \Omega \setminus \omega, \end{cases}$$

and observe that, from the definition of  $P$ , we obviously have

$$\int_{-T}^T \int_{\omega} e^{2s\phi} \rho (\partial_{tt} w - \Delta w - Pw) w \, dx \, dt = 0. \quad (26)$$

Let us consider separately the three integrals coming from this equality. For the first term we have, thanks to two integrations by parts in time,

$$\begin{aligned} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_{tt} w w \, dx \, dt &= - \int_{-T}^T \int_{\omega} \partial_t w \partial_t (e^{2s\phi} \rho w) \, dx \, dt \\ &= - \int_{-T}^T \int_{\omega} |\partial_t w|^2 e^{2s\phi} \rho \, dx \, dt - s \int_{-T}^T \int_{\omega} \partial_t (|w|^2) e^{2s\phi} \partial_t \phi \rho \, dx \, dt \\ &= - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt + 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t \phi|^2 |w|^2 \rho \, dx \, dt \\ &\quad + s \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_{tt} \phi \rho |w|^2 \, dx \, dt. \end{aligned} \quad (27)$$

For the second term in (26) we have, using Green formula

$$\begin{aligned} &- \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta w w \, dx \, dt \\ &= \int_{-T}^T \int_{\omega} \nabla (e^{2s\phi} \rho w) \cdot \nabla w \, dx \, dt \\ &= \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla w|^2 \, dx \, dt + s \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla (w^2) \cdot \nabla \phi \rho \, dx \, dt \\ &\quad + \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla (w^2) \cdot \nabla \rho \, dx \, dt \\ &= \int_{-T}^T \int_{\omega} e^{2s\phi} |\nabla w|^2 \rho \, dx \, dt - s \int_{-T}^T \int_{\omega} e^{2s\phi} \Delta \phi \rho |w|^2 \, dx \, dt \\ &\quad - 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla \phi|^2 |w|^2 \, dx \, dt - \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \Delta \rho |w|^2 \, dx \, dt \\ &\quad - 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \nabla \rho \cdot \nabla \phi |w|^2 \, dx \, dt. \end{aligned} \quad (28)$$

Using (27) and (28) in (26), we thus obtain that

$$\begin{aligned} &\int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\nabla w|^2 \, dx \, dt \\ &= \int_{-T}^T \int_{\omega} e^{2s\phi} \left( s \partial_{tt} \phi \rho + 2s^2 |\partial_t \phi|^2 \rho - \frac{1}{2} \Delta \rho - 2s \nabla \rho \cdot \nabla \phi - s \Delta \phi \rho \right. \\ &\quad \left. - 2s^2 |\nabla \phi|^2 \rho \right) |w|^2 \, dx \, dt - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho Pw w \, dx \, dt. \end{aligned} \quad (29)$$

From (29) and by definition of  $\rho$ , we deduce the following estimate,

$$\begin{aligned} \int_{-T}^T \int_{\omega_2} e^{2s\phi} |\partial_t w|^2 \, dx \, dt &\lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt \\ &\quad + \int_{-T}^T \int_{\omega} (|\nabla w|^2 + s^2 |w|^2) \, dx \, dt. \end{aligned} \quad (30)$$

Using this inequality in the right-hand side of (25), we finally get (16).  $\square$



We now establish a second Carleman estimate. In this inequality, the measurement terms in the right hand side involve the  $L^2((-T, T) \times \omega)$  weighted norms of a time primitive function  $W$  of the field and its Laplacian  $\Delta W$ .

**Theorem 2.2.** *We assume that  $\omega$  satisfies (7). Let  $\phi$  be the weight function defined by (15). Then, for every  $\beta \in (0, 1)$ , there exist  $\lambda > 0$ ,  $s_0 > 0$  and  $c > 0$  such that for all  $s \geq s_0$  and for all  $w \in L^2(-T, T; H_0^1(\Omega)) \cap H^1(-T, T; L^2(\Omega))$  such that  $Pw = \partial_{tt}w - \Delta w \in L^2(\Omega \times (-T, T))$ ,  $W \in L^2(-T, T; H^2(\omega))$  and  $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$  in  $\Omega$ , we have*

$$\begin{aligned} & s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ & \leq c \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + cs^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4|W|^2) dx dt \end{aligned} \quad (31)$$

where  $W$  is given by  $W(t) = \int_0^t w(\tau) d\tau$ .

*Proof.* In this proof, as in the previous proof, all the computations will be done for smooth functions and lastly a density argument allows to extend the result to the regularity given in the theorem. We restart from the equation (29) obtained in the proof of Theorem 2.1. On one hand, we can deduce from (29) that

$$\begin{aligned} \int_{-T}^T \int_{\omega_2} e^{2s\phi} |\nabla w|^2 dx dt & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt \\ & \quad + \int_{-T}^T \int_{\omega} (|\partial_t w|^2 + s^2|w|^2) dx dt. \end{aligned} \quad (32)$$

Using this inequality in the right-hand side of (25), we get

$$\begin{aligned} & s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx + \int_{-T}^T \int_{\Omega} e^{2s\phi} (s|\nabla w|^2 + s|\partial_t w|^2 + s^3|w|^2) dx dt \\ & \lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2|w|^2) dx dt. \end{aligned} \quad (33)$$

On the other hand, using the fact that  $w = \partial_t W$  (since  $W$  is a time primitive of  $w$ ), the second term of (26) can also be integrated by parts in time to obtain

$$\begin{aligned} & - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta w w dx dt \\ & = - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \Delta (\partial_t W) w dx dt \\ & = \int_{-T}^T \int_{\omega} \partial_t (e^{2s\phi} \rho w) \Delta W dx dt - \left[ \int_{\omega} e^{2s\phi} \rho w \Delta W dx \right]_{-T}^T \\ & = 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t \phi w \Delta W dx dt + \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t w \Delta W dx dt. \end{aligned}$$

Using this equality and (27) in (26), we get

$$\begin{aligned} & \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 dx dt \\ & = s \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_{tt} \phi \rho |w|^2 dx dt + 2s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t \phi|^2 |w|^2 \rho dx dt \\ & \quad + 2s \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t \phi w \Delta W dx dt + \int_{-T}^T \int_{\omega} e^{2s\phi} \rho \partial_t w \Delta W dx dt \\ & \quad - \int_{-T}^T \int_{\omega} e^{2s\phi} \rho Pw w dx dt. \end{aligned}$$

Using Young inequality for the last three terms, we obtain

$$\begin{aligned}
& \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt \\
& \lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} \left( s |\partial_{tt} \phi| \rho + 2s^2 |\partial_t \phi|^2 \rho + s^2 \rho |\partial_t \phi| + \frac{s^2 \rho}{2} \right) |w|^2 \, dx \, dt \\
& \quad + \int_{-T}^T \int_{\omega} e^{2s\phi} \left( \frac{\rho}{2} + \rho |\partial_t \phi| \right) |\Delta W|^2 \, dx \, dt + \frac{1}{2s^2} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |Pw|^2 \, dx \, dt \\
& \quad + \frac{1}{2} \int_{-T}^T \int_{\omega} e^{2s\phi} \rho |\partial_t w|^2 \, dx \, dt. \tag{34}
\end{aligned}$$

We absorb the last term on the right side of (34) by the left-hand side. Then, for  $s$  large enough, we get

$$\begin{aligned}
\int_{-T}^T \int_{\omega} e^{2s\phi} |\partial_t w|^2 \, dx \, dt & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt \\
& \quad + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^2 |w|^2) \, dx \, dt. \tag{35}
\end{aligned}$$

Let us now compute

$$\begin{aligned}
& s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} |w|^2 \, dx \, dt \\
& = s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} w \partial_t W \, dx \, dt \\
& = -2s^3 \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_t \phi w W \, dx \, dt - s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} \partial_t w W \, dx \, dt \\
& \lesssim \frac{s^4}{\epsilon} \int_{-T}^T \int_{\omega} e^{2s\phi} |W|^2 \, dx \, dt + \epsilon \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt,
\end{aligned}$$

for any  $\epsilon > 0$ . Starting from (35), we use the above inequality to obtain

$$\begin{aligned}
& \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt \\
& \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^2 |w|^2) \, dx \, dt \\
& \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 \, dx \, dt + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + \frac{s^4}{\epsilon} |W|^2) \, dx \, dt \\
& \quad + \epsilon \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt. \tag{36}
\end{aligned}$$

Taking  $\epsilon$  small enough, we deduce that

$$\begin{aligned}
\int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t w|^2 + s^2 |w|^2) \, dx \, dt & \lesssim \frac{1}{s^2} \int_{-T}^T \int_{\omega} e^{2s\phi} |Pw|^2 \, dx \, dt \\
& \quad + \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4 |W|^2) \, dx \, dt. \tag{37}
\end{aligned}$$

To conclude, we use the inequality (33) and apply inequality (37) to replace the *observation terms* in the right-hand side.  $\square$

## 2.2 Stability inequalities associated with the inverse source problem

In what follows, we will state two stability estimates for our inverse source problem. The first stability estimate stated in Theorem 2.3 will be adapted to *velocity* measurements whereas the second stability estimate stated in Theorem 2.4 will be adapted to *field* measurements. Following the classical Bukhgeim-Klibanov method introduced in [5, 11, 56], these stability estimates will come as consequences of the Carleman estimates stated in Section 2.1. Let us also mention that Lipschitz stability estimates have also been obtained for coefficient inverse problems for hyperbolic equations in [58], in [56, section 3.6] and in [55].

In this section, we assume that the known part of the right hand side  $\sigma$  satisfies,

$$\sigma \in H^1(0, T; L^\infty(\Omega)) \text{ and } |\sigma(x, 0)| \geq \kappa > 0 \text{ a.e. in } \Omega. \quad (38)$$

### 2.2.1 Stability inequality associated with *velocity* measurements

**Theorem 2.3.** *We assume that  $\omega$  satisfies (7),  $T$  satisfies (8) and  $\sigma$  satisfies (38). Let  $u|_\theta$  be the solution of (12) associated with  $\theta \in L^2(\Omega)$ . There exists  $c > 0$  such that*

$$\int_{\Omega} |\theta|^2 dx \leq c \int_0^T \int_{\omega} (|\nabla \partial_t u|_\theta|^2 + |\partial_t u|_\theta|^2) dx dt. \quad (39)$$

*Proof.* Since the right-hand side  $\sigma\theta$  of (12) belongs to  $H^1(0, T; L^2(\Omega))$ , a classical existence result (we refer to [45] or [33]) allows to assert that  $u$  belongs to  $C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega))$ , and we have

$$\|u\|_{C^2([0, T]; L^2(\Omega))} + \|u\|_{C^1([0, T]; H^1(\Omega))} + \|u\|_{C^0([0, T]; H^2(\Omega))} \leq c \|\theta\|_{L^2(\Omega)}. \quad (40)$$

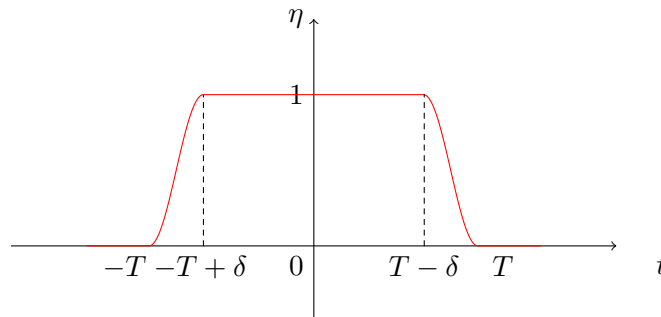
Following Bukhgeim-Klibanov method, we introduce  $v = \partial_t u$  that satisfies

$$\begin{cases} \partial_{tt} v(x, t) - \Delta v(x, t) = \partial_t \sigma(x, t) \theta(x), & (x, t) \in \Omega \times (0, T), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ \partial_t v(x, 0) = \sigma(x, 0) \theta(x), & x \in \Omega. \end{cases} \quad (41)$$

We extend the function  $v$  on  $(-T, 0)$  by setting  $v(\cdot, -t) = -v(\cdot, t)$ . Then, for  $\delta > 0$ , we introduce the cut-off function  $\eta \in C_c^\infty(\mathbb{R})$ , plotted in Figure 3, such that  $0 \leq \eta \leq 1$ ,

$$\eta(t) = \begin{cases} 1, & \text{if } -T + \delta \leq t \leq T - \delta, \\ 0, & \text{if } t \leq -T \text{ or } t \geq T, \end{cases} \quad (42)$$

and we set  $w = \eta v$  in  $\Omega \times (-T, T)$ . The function  $w$  satisfies  $w(\cdot, 0) = w(\cdot, \pm T) = \partial_t w(\cdot, \pm T) = 0$  in  $\Omega$ .



**Figure 3.** Cut-off function  $\eta$  defined in (42).

Therefore, since  $\omega$  satisfies (7), we can apply the Carleman estimate (16) of Theorem 2.1 to get,

$$\begin{aligned} s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx \\ \lesssim \int_{-T}^T \int_{\Omega} e^{2s\phi} |Pw|^2 dx dt + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\nabla w|^2 + s^2 |w|^2) dx dt. \end{aligned} \quad (43)$$

For the term in the left-hand side, we have  $\partial_t w(x, 0) = \eta(0) \partial_t v(x, 0) = \sigma(x, 0) \theta(x)$  where, by hypothesis,  $|\sigma(x, 0)| \geq \kappa > 0$  for  $x \in \Omega$ . Therefore, we have

$$s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\partial_t w(0)|^2 dx \geq \kappa^2 s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \quad (44)$$

For the first term in the right-hand side of (43), we notice that

$$Pw = \eta \partial_t \sigma \theta + 2 \partial_t \eta \partial_t v + \partial_{tt} \eta v. \quad (45)$$

On one hand, since  $\sigma \in H^1(0, T; L^\infty(\Omega))$  we directly have

$$\int_{-T}^T \int_{\Omega} e^{2s\phi} |\eta \partial_t \sigma \theta|^2 dx dt \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \quad (46)$$

On the other hand, the time derivatives of  $\eta$  vanish on  $[-T + \delta, T - \delta]$ . Moreover, since  $T$  satisfies the time condition (8), we can take  $\beta > 0$  such that

$$\frac{1}{T} \sup_{x \in \Omega} |x - x_0| < \sqrt{\beta} < 1. \quad (47)$$

In this case, if we take  $\delta$  sufficiently small, we have, by definition (15) of  $\psi$

$$\forall t \in [-T, -T + \delta] \cup [T - \delta, T], \forall x \in \Omega, \quad \psi(x, t) < c_0 < \psi(x, 0). \quad (48)$$

Therefore, using estimate (40), we have

$$\begin{aligned} \int_{-T}^T \int_{\Omega} e^{2s\phi} |2\partial_t \eta \partial_t v + \partial_{tt} \eta v|^2 dx dt &\lesssim e^{2s \exp(\lambda c_0)} \int_{-T}^T \int_{\Omega} (|\partial_{tt} u|^2 + |\partial_t u|^2) dx dt \\ &\lesssim e^{2s \exp(\lambda c_0)} \int_{\Omega} |\theta|^2 dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx. \end{aligned}$$

Hence, using this inequality, (44) and (46), estimate (43) leads to

$$\kappa^2 s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t \eta v|^2 + s^2 |\eta v|^2) dx dt.$$

Therefore, if  $s$  is chosen large enough, we can absorb the first term in the right-hand side by the term in the left-hand side. Finally we have

$$s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\partial_t \eta v|^2 + s^2 |\eta v|^2) dx dt \lesssim e^{2s \|\phi(0)\|_{C^0(\Omega)}} \int_{-T}^T \int_{\omega} |\nabla v|^2 + |v|^2 dx dt.$$

Since  $v = \partial_t u$  and  $v(0) = 0$  and since the integrals on  $(-T, T)$  are twice the integrals on  $(0, T)$ , we obtain (39).  $\square$

We deduce from the observability inequality (39) that the observations on  $\partial_t u$  must belong to  $L^2(0, T; H^1(\omega))$ , such that a stable reconstruction of  $\theta$  is possible with the  $L^2$ -norm. As a result, we are dealing with an ill-posed inverse problem of degree 1 in the sense of Definition 1.1.

### 2.2.2 Stability inequality associated with *field* measurements

Again, we follow the method introduced by Bukhgeim-Klibanov [5] and apply the Carleman estimate given by Theorem 2.2 to the time derivative of the solution.

**Theorem 2.4.** *We assume that  $\omega$  satisfies (7),  $T$  satisfies (8) and  $\sigma$  satisfies (38). Let  $u_{|\theta}$  be the solution of (12) associated with  $\theta \in L^2(\Omega)$ . There exists  $c > 0$  such that*

$$\int_{\Omega} |\theta|^2 dx \leq c \int_0^T \int_{\omega} |\Delta u_{|\theta}|^2 + |u_{|\theta}|^2 dx dt. \quad (49)$$

*Proof.* Let  $v$  be the solution of (41) and  $V$  its time primitive such that  $V(\cdot, 0) = 0$ . As in the proof of Theorem 2.3, we extend the function  $v$  on  $(-T, 0)$  by setting  $v(\cdot, -t) = -v(\cdot, t)$ . We set  $w = \eta v$  in  $\Omega \times (-T, T)$  where  $\eta$  satisfies (42) and  $W$  its time primitive. Reasoning exactly as in the proof of Theorem 2.3 and using the Carleman estimate (31) of Theorem 2.2, one can show that

$$\kappa^2 s^{\frac{1}{2}} \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx \lesssim \int_{\Omega} e^{2s\phi(0)} |\theta|^2 dx + s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta W|^2 + s^4 |W|^2) dx dt,$$

where, if  $s$  is chosen sufficiently large, we can absorb the first term in the right-hand side by the term in the left-hand side. Moreover, we notice that

$$W(x, t) = \int_0^t \eta(\tau) v(x, \tau) d\tau = \eta(t) V(x, t) - \int_0^t \eta'(\tau) V(x, \tau) d\tau.$$

So, since for all  $t \in (-T, T)$  and  $\tau$  such that  $|\tau| \leq |t|$ ,  $\phi(t) \leq \phi(\tau)$ , we have

$$\int_{-T}^T \int_{\omega} e^{2s\phi} |W|^2 dx dt \lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} |V|^2 dx dt,$$

and

$$\int_{-T}^T \int_{\omega} e^{2s\phi} |\Delta W|^2 dx dt \lesssim \int_{-T}^T \int_{\omega} e^{2s\phi} |\Delta V|^2 dx dt.$$

Therefore,

$$s^2 \int_{-T}^T \int_{\omega} e^{2s\phi} (|\Delta(\eta V)|^2 + s^4 |\eta V|^2) dx dt \lesssim e^{2s\|\phi(0)\|_{C^0(\Omega)}} \int_{-T}^T \int_{\omega} |\Delta V|^2 + |V|^2 dx dt.$$

Since  $v = \partial_t u$  and since the integrals on  $(-T, T)$  are twice the integrals on  $(0, T)$ , we obtain (49).  $\square$

Note that, in this case, we have an estimate of  $\theta$  with the  $L^2$ -norm if the observations on  $u$  belong to  $L^2(0, T; H^2(\omega))$ . Therefore, we are dealing with an ill-posed problem of degree 2.

## 3 From Tikhonov regularization strategy to observer design

### 3.1 Tikhonov regularization and error reconstruction with *velocity* measurements

We assume that a target trajectory  $\tilde{u}$  solution of (12) produced by a target source term parameter  $\tilde{\theta}$  has resulted in a set of measurements  $y^\delta$  where

$$\int_0^T \|y^\delta(t) - \partial_t \tilde{u}(t)\|_{L^2(\omega)}^2 dt \leq \delta^2 T, \quad (50)$$

for a parameter  $\delta > 0$ . We introduce the cost functional to minimize  $\mathcal{J}_T : H_0^1(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - \partial_t u_{|\theta}\|_{L^2(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{L^2(\omega)}^2 dt, \end{aligned} \quad (51)$$

where  $u_{|\theta}$  is the solution of (12) associated with the parameter  $\theta$ ,  $C$  is defined by (9) and  $\epsilon > 0$ . Since the functional is quadratic, the application  $\theta \mapsto I_\omega(\partial_t u_{|\theta})$  is continuous and  $\theta \mapsto \|\theta\|_{H_0^1(\Omega)}^2$  is continuous and coercive in  $H_0^1(\Omega)$ , we deduce the existence of a unique minimizer in  $H_0^1(\Omega)$  of  $\mathcal{J}_T$ . We denote it by  $\bar{\theta}_T := \underset{\theta \in H_0^1(\Omega)}{\operatorname{argmin}} \mathcal{J}_T$ .

We want to show a property on the reconstruction by quantifying the approximation between the target parameter  $\check{\theta}$  and the minimizer of the cost functional  $\bar{\theta}_T$  with respect to the noise level  $\delta$ . To do so, we introduce a regularization operator  $G_\omega^\alpha$  that will regularize the  $L^2(\omega)$  measurements in  $H_0^1(\omega)$ . Namely, for  $\alpha > 0$  small, we define  $G_\omega^\alpha \in \mathcal{L}(L^2(\omega), H_0^1(\omega))$  satisfying for all  $y \in L^2(\omega)$ ,  $G_\omega^\alpha y = y^\alpha$  where  $y^\alpha$  is solution of

$$\begin{cases} -\Delta y^\alpha + \frac{1}{\alpha^2} y^\alpha = \frac{1}{\alpha^2} y, & \text{in } \omega, \\ y^\alpha = 0, & \text{on } \partial\omega. \end{cases} \quad (52)$$

Let us look at some properties of this regularization operator. First, we give a stability property.

**Proposition 3.1.** *The regularization operator  $G_\omega^\alpha$  defined by (52) satisfies,*

$$\forall y \in L^2(\omega), \quad \|G_\omega^\alpha y\|_{H_0^1(\omega)} \leq \frac{1}{\sqrt{2\alpha}} \|y\|_{L^2(\omega)}.$$

*Proof.* We denote by  $y^\alpha = G_\omega^\alpha y$ . It satisfies the following variational formulation:

$$\int_\omega \nabla y^\alpha \nabla v \, dx + \frac{1}{\alpha^2} \int_\omega y^\alpha v \, dx = \frac{1}{\alpha^2} \int_\omega y v \, dx, \quad \forall v \in H_0^1(\omega).$$

Choosing the test function  $v = y^\alpha$  and using Young inequality, we directly get

$$\|\nabla y^\alpha\|_{L^2(\omega)}^2 \leq \frac{1}{2\alpha^2} \|y\|_{L^2(\omega)}^2.$$

□

This operator  $G_\omega^\alpha$  can be viewed as a regularization of the identity operator. We have the following approximation theorem.

**Proposition 3.2.** *The regularization operator  $G_\omega^\alpha$  defined by (52) satisfies,  $\forall y \in H_0^1(\omega)$  such that  $\Delta y \in L^2(\omega)$ ,*

$$\|(G_\omega^\alpha - Id)y\|_{H_0^1(\omega)} \leq \frac{\alpha}{\sqrt{2}} \|\Delta y\|_{L^2(\omega)}.$$

*Proof.* We denote  $y^\alpha = G_\omega^\alpha y$ . It satisfies the following variational formulation:

$$\int_\omega \nabla(y^\alpha - y) \nabla v \, dx + \frac{1}{\alpha^2} \int_\omega (y^\alpha - y)v \, dx = \int_\omega \Delta y v \, dx, \quad \forall v \in H_0^1(\omega).$$

Choosing the test function  $v = y^\alpha - y$  and using Young inequality, we directly get

$$\|\nabla(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 \leq \frac{\alpha^2}{2} \|\Delta y\|_{L^2(\omega)}^2 + \frac{1}{2\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2.$$

□

For further use, let us state a regularity result for the solution of (12).

**Proposition 3.3.** *We assume that  $\sigma \in H^2(0, T; W^{1, \infty}(\Omega))$  and  $\theta \in H_0^1(\Omega)$ . Let  $u_{|\theta}$  be the solution of (12) associated with  $\theta$ . Then  $u_{|\theta} \in C^1([0, T]; H^2(\Omega))$  and there exists  $c > 0$  depending on  $\sigma$  and  $T$  such that*

$$\|u_{|\theta}\|_{C^1([0, T]; H^2(\Omega))} \leq c(T, \sigma) \|\theta\|_{H_0^1(\Omega)}. \quad (53)$$

*Proof.* From classical existence result (we refer to [45] or [33]), one can show that,

$$\begin{aligned} \sigma\theta &\in L^2(0, T; L^2(\Omega)) \\ \Rightarrow \exists! u &\in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (12)}. \end{aligned} \quad (54)$$

Furthermore, let us consider the system (12) differentiated in time. Defining  $v = \partial_t u$ , we have

$$\begin{cases} \partial_{tt}v(x, t) - \Delta v(x, t) = \partial_t\sigma(x, t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = 0, & x \in \Omega, \\ \partial_tv(x, 0) = \sigma(x, 0)\theta(x), & x \in \Omega. \end{cases} \quad (55)$$

Similarly, for this problem, we have

$$\begin{aligned} (\sigma(0)\theta, \partial_t\sigma\theta) &\in L^2(\Omega) \times L^2(0, T; L^2(\Omega)) \\ \Rightarrow \exists! v &\in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (55)}. \end{aligned} \quad (56)$$

Let us now differentiate again the system (55) in time. Denoting  $w = \partial_tv$ , we have

$$\begin{cases} \partial_{tt}w(x, t) - \Delta w(x, t) = \partial_{tt}\sigma(x, t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ w(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ w(x, 0) = \sigma(x, 0)\theta(x), & x \in \Omega, \\ \partial_tw(x, 0) = \partial_t\sigma(x, 0)\theta(x), & x \in \Omega. \end{cases} \quad (57)$$

For this system, one has the following existence result,

$$\begin{aligned} (\sigma(0)\theta, \partial_t\sigma(0)\theta, \partial_{tt}\sigma\theta) &\in H_0^1(\Omega) \times L^2(\Omega) \times L^2(0, T; L^2(\Omega)), \\ \Rightarrow \exists! w &\in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega)) \text{ solution of (57)}. \end{aligned} \quad (58)$$

Gathering (54), (56) and (58) and using elliptic estimates, we deduce that the solution  $u$  of (12) belongs to  $C^1([0, T]; H^2(\Omega))$ . Moreover, using the estimates associated with the existence results, we have that (53) holds.  $\square$

The following theorem gives an estimate the reconstruction error  $\bar{\theta}_T - \check{\theta}$  with respect to the noise level in the case of *velocity* measurements. Before stating this result, let us define a cut-off function which is useful in the proof. Since  $\omega$  satisfies (7), there exists  $\tilde{\rho} > 0$  such that  $\omega_{\tilde{\rho}, x_0} \subset \omega$ . We define  $\omega_0 = \omega_{\rho_0, x_0}$  and  $\omega_1 = \omega_{\rho_1, x_0}$  for  $0 < \rho_0 < \rho_1 \leq \tilde{\rho}$ . Then we consider a cut-off function  $\eta \in C^3(\bar{\omega})$  which satisfies the following properties :

$$\begin{cases} \eta(x) = 1, & \text{if } x \in \omega_0, \\ \eta(x) = 0, & \text{if } x \in \omega \setminus \omega_1, \\ 0 \leq \eta(x) \leq 1 & \text{for every } x \in \bar{\omega}, \\ \nabla\eta \cdot \nu = 0 & \text{on } \partial\omega, \end{cases} \quad (59)$$

where  $\nu$  is the outward unit normal to  $\partial\omega$ . The existence of such a function is ensured since the domain  $\omega$  is sufficiently regular.



**Theorem 3.4.** *Let  $\omega$  satisfy (7) and  $T$  satisfy (8). We assume that  $\sigma$  satisfies (38),  $\sigma \in \mathbf{H}^2(0, T; \mathbf{W}^{1, \infty}(\Omega))$  and  $\check{\theta} \in \mathbf{H}_0^1(\Omega)$ . Moreover, we assume that there exists  $M > 0$  such that*

$$\|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \leq M. \quad (60)$$

*Then, there exists  $c > 0$  independent of  $M$  and  $\delta$  such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \leq c\sqrt{M}\sqrt{\delta}, \quad (61)$$

*where  $\bar{\theta}_T$  minimizes the cost functional (51) with  $\epsilon = \delta^2 M^{-2}$ .*

*Proof.* In this proof, we write  $a \lesssim b$  if there exists  $c > 0$  independent of  $\epsilon$ ,  $M$  and  $\delta$  such that  $a \leq cb$ .

First, using the prior on the parameter  $\check{\theta}$  and the estimate on the noise amplitude (50), we can bound the cost functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2 + \frac{1}{2} T \delta^2. \quad (62)$$

Since  $\omega_0$  itself satisfies (7), we can apply to  $\bar{\theta}_T - \check{\theta}$  the stability inequality given by Theorem 2.3 on  $\omega_0$ . We obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim \left( \int_0^T \|\partial_t u|_{(\bar{\theta}_T - \check{\theta})}\|_{\mathbf{H}^1(\omega_0)}^2 dt \right)^{\frac{1}{2}} = \left( \int_0^T \|\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}\|_{\mathbf{H}^1(\omega_0)}^2 dt \right)^{\frac{1}{2}}.$$

Therefore, using the cut-off function  $\eta \in C^3(\bar{\omega})$  which satisfies (59) and thanks to Poincaré inequality, we have

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left( \int_0^T \|\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}})\|_{\mathbf{H}^1(\omega_0)}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_0^T \|\eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}})\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

For a given  $\alpha > 0$ , we introduce the regularization operator  $G_\omega^\alpha$  defined by (52). By triangle inequality, we obtain

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left( \int_0^T \|(G_\omega^\alpha - \text{Id}) \left( \eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}) \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \|G_\omega^\alpha \left( \eta(\partial_t u|_{\bar{\theta}_T} - \partial_t u|_{\check{\theta}}) \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Adding and subtracting the noisy observations, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} &\lesssim \left( \int_0^T \|(G_\omega^\alpha - \text{Id}) \left( \eta \partial_t u|_{\bar{\theta}_T} \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \|(G_\omega^\alpha - \text{Id}) \left( \eta \partial_t u|_{\check{\theta}} \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \|G_\omega^\alpha \left( \eta(\partial_t u|_{\bar{\theta}_T} - y^\delta) \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \|G_\omega^\alpha \left( \eta(\partial_t u|_{\check{\theta}} - y^\delta) \right)\|_{\mathbf{H}_0^1(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

According to Proposition 3.3 and thanks to the regularity assumptions on  $\sigma$ ,  $\check{\theta}$  and  $\bar{\theta}_T$ , we have that  $u_{|\bar{\theta}_T}$  and  $u_{|\check{\theta}}$  belong to  $C^1([0, T]; H^2(\Omega))$ . So using Proposition 3.1 and Proposition 3.2, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim \alpha \left( \int_0^T \|\Delta(\eta \partial_t u_{|\bar{\theta}_T})\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} + \alpha \left( \int_0^T \|\Delta(\eta \partial_t u_{|\check{\theta}})\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\alpha} \left( \int_0^T \|\partial_t u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} + \frac{1}{\alpha} \left( \int_0^T \|\partial_t u_{|\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (63)$$

For the first two terms in the right-hand side, we notice that, for all  $v \in H_0^1(\Omega)$  such that  $\Delta v \in L^2(\Omega)$ , using Poincaré inequality and the definition of the cut-off function,

$$\begin{aligned} \|\Delta(\eta v)\|_{L^2(\omega)} &\leq \|v \Delta \eta + 2 \nabla v \nabla \eta + \eta \Delta v\|_{L^2(\omega)}, \\ &\lesssim \|v\|_{L^2(\omega)} \|\Delta \eta\|_{L^\infty(\omega)} + 2 \|\nabla v\|_{L^2(\omega)} \|\nabla \eta\|_{L^\infty(\omega)} + \|\eta\|_{L^\infty} \|\Delta v\|_{L^2(\omega)}, \\ &\lesssim \|\nabla v\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(\Omega)}, \end{aligned}$$

hence, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim \alpha \left( \int_0^T \|\Delta \partial_t u_{|\bar{\theta}_T}\|_{L^2(\omega)}^2 + \|\nabla \partial_t u_{|\bar{\theta}_T}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \alpha \left( \int_0^T \|\Delta \partial_t u_{|\check{\theta}}\|_{L^2(\omega)}^2 + \|\nabla \partial_t u_{|\check{\theta}}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\alpha} \left( \int_0^T \|\partial_t u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\alpha} \left( \int_0^T \|\partial_t u_{|\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Applying (53) for  $\theta = \bar{\theta}_T$  and  $\theta = \check{\theta}$  and using the definition of the cost functional (51) and error measurements amplitude (50), we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T \|\bar{\theta}_T\|_{H_0^1(\Omega)} + \alpha T \|\check{\theta}\|_{H_0^1(\Omega)} + \frac{1}{\alpha} (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Thanks to the prior on the parameter (60), we get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T (2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + \alpha T M + \frac{1}{\alpha} (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Using the bound of the cost-functional (62), we then have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \alpha T (M^2 + \epsilon^{-1} \delta^2 T)^{\frac{1}{2}} + \alpha T M + \frac{1}{\alpha} (\epsilon M^2 + \delta^2 T)^{\frac{1}{2}} + \frac{1}{\alpha} \delta \sqrt{T}.$$

Choosing  $\epsilon = \delta^2 M^{-2}$  and  $\alpha = \delta^{\frac{1}{2}} M^{-\frac{1}{2}}$ , we finally obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \delta^{\frac{1}{2}} M^{\frac{1}{2}} \left( T(1+T)^{\frac{1}{2}} + T + (1+T)^{\frac{1}{2}} + \sqrt{T} \right).$$

□

### 3.2 Tikhonov regularization and error reconstruction with *field* measurements

In this part, we assume that a target trajectory  $\check{u}$  solution of (12) created by a target source term parameter  $\check{\theta}$  has resulted in a set of measurements  $y^\delta$  where

$$\int_0^T \|y^\delta(t) - \check{u}(t)\|_{L^2(\omega)}^2 dt \leq \delta^2 T, \quad (64)$$

for a parameter  $\delta > 0$ . The cost functional to minimize  $\mathcal{J}_T : H_0^1(\Omega) \rightarrow \mathbb{R}$  in this context is defined by

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - u_{|\theta}\|_{L^2(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{L^2(\omega)}^2 dt, \end{aligned} \quad (65)$$

where  $u_{|\theta}$  is the solution of (12) associated with the parameter  $\theta$ ,  $C$  is defined by (10) and  $\epsilon > 0$ . As in the previous section, one can prove the existence of a unique minimizer in  $H_0^1(\Omega)$  of  $\mathcal{J}_T$ . We denote it by  $\bar{\theta}_T := \underset{\theta \in H_0^1(\Omega)}{\operatorname{argmin}} \mathcal{J}_T$ .

We then introduce a regularization operator  $H_\omega^\alpha$  that will regularize  $L^2(\omega)$  measurements in  $H^2(\omega)$ . More precisely, for  $\alpha > 0$  small, we define

$$H_\omega^\alpha \in \mathcal{L}(L^2(\omega), H_0^1(\omega) \cap H^2(\omega))$$

satisfying for all  $y \in L^2(\omega)$ ,  $H_\omega^\alpha y = y^\alpha$  where  $y^\alpha$  is solution of

$$\begin{cases} \Delta^2 y^\alpha + \frac{1}{\alpha^2} y^\alpha = \frac{1}{\alpha^2} y, & \text{in } \omega, \\ y^\alpha = \Delta y^\alpha = 0, & \text{on } \partial\omega, \end{cases} \quad (66)$$

Let us look at some properties of this regularization operator. First, we give a stability property.

**Proposition 3.5.** *The regularization operator  $H_\omega^\alpha$  defined by (66) satisfies,*

$$\forall y \in L^2(\omega), \quad \|\Delta H_\omega^\alpha y\|_{L^2(\omega)} \leq \frac{1}{\sqrt{2}\alpha} \|y\|_{L^2(\omega)}.$$

*Proof.* The function  $y^\alpha = H_\omega^\alpha y$  satisfies the following variational formulation

$$\int_\omega \Delta y^\alpha \Delta v dx + \frac{1}{\alpha^2} \int_\omega y^\alpha v dx = \frac{1}{\alpha^2} \int_\omega y v dx, \quad \forall v \in H_0^1(\omega) \cap H^2(\omega).$$

Choosing the test function  $v = y^\alpha$  and using Young inequality, we deduce that

$$\|\Delta y^\alpha\|_{L^2(\omega)}^2 \leq \frac{1}{2\alpha^2} \|y\|_{L^2(\omega)}^2.$$

□

As in the previous section, we can show an approximation theorem.

**Proposition 3.6.** *The regularization operator  $H_\omega^\alpha$  defined by (66) satisfies, for all  $y \in H_0^1(\omega) \cap H^2(\omega)$  with  $\nabla \Delta y \in L^2(\omega)^d$  and  $\Delta y = 0$  on  $\partial\omega$ ,*

$$\|\Delta(H_\omega^\alpha - Id)y\|_{L^2(\omega)} \leq c\sqrt{\alpha} \|\nabla \Delta y\|_{L^2(\omega)},$$

where  $c > 0$  is independent of  $\alpha$ .

*Proof.* We denote  $y^\alpha = H_\omega^\alpha y$ . The following variational formulation is satisfied: for all  $y$  as in the statement of the proposition and for all  $v \in H_0^1(\omega) \cap H^2(\omega)$

$$\int_\omega \Delta(y^\alpha - y) \Delta v \, dx + \frac{1}{\alpha^2} \int_\omega (y^\alpha - y) v \, dx = - \int_\omega \Delta y \Delta v \, dx.$$

Choosing the test function  $v = y^\alpha - y$  and using Green's formula and Cauchy–Schwarz inequality, we get

$$\|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 \leq \|\nabla \Delta y\|_{L^2(\omega)} \|\nabla(y^\alpha - y)\|_{L^2(\omega)}.$$

Using an interpolation property, we have

$$\|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 \leq c \|\nabla \Delta y\|_{L^2(\omega)} \|\Delta(y^\alpha - y)\|_{L^2(\omega)}^{\frac{1}{2}} \|y^\alpha - y\|_{L^2(\omega)}^{\frac{1}{2}},$$

where  $c > 0$  is a positive interpolation constant. Using Young inequality we obtain

$$\begin{aligned} \|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 &\leq \frac{c\alpha^{\frac{1}{2}}}{2} \|\nabla \Delta y\|_{L^2(\omega)} \|\Delta(y^\alpha - y)\|_{L^2(\omega)} \\ &\quad + \frac{c}{2\alpha^{\frac{1}{2}}} \|\nabla \Delta y\|_{L^2(\omega)} \|y^\alpha - y\|_{L^2(\omega)}. \end{aligned}$$

The use of Young inequality finally leads to

$$\begin{aligned} \|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{1}{\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 &\leq \frac{c^2\alpha}{4} \|\nabla \Delta y\|_{L^2(\omega)}^2 + \frac{1}{4} \|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 \\ &\quad + \frac{c^2\alpha}{4} \|\nabla \Delta y\|_{L^2(\omega)}^2 + \frac{1}{4\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2. \end{aligned}$$

Eventually, we have

$$\frac{3}{4} \|\Delta(y^\alpha - y)\|_{L^2(\omega)}^2 + \frac{3}{4\alpha^2} \|y^\alpha - y\|_{L^2(\omega)}^2 \leq \frac{c^2\alpha}{2} \|\nabla \Delta y\|_{L^2(\omega)}^2.$$

□

As before, we need a regularity result for the solution  $u$  of (12).

**Proposition 3.7.** *We assume that  $\sigma \in H^2(0, T; W^{1,\infty}(\Omega))$  and  $\theta \in H_0^1(\Omega)$ . Let  $u_{|\theta}$  be the solution of (12) associated with  $\theta$ . Then  $u_{|\theta} \in C^0([0, T]; H^2(\Omega))$ ,  $\nabla \Delta u_{|\theta} \in L^\infty(0, T; L^2(\Omega))^d$  and there exists  $c > 0$  depending on  $\sigma$  and  $T$  such that*

$$\sup_{t \in [0, T]} \|\nabla u_{|\theta}\|_{L^2(\Omega)} + \|\Delta u_{|\theta}\|_{L^2(\Omega)} + \|\nabla \Delta u_{|\theta}\|_{L^2(\Omega)} \leq c(T, \sigma) \|\theta\|_{H_0^1(\Omega)}. \quad (67)$$

*Proof.* On one hand, following the proof of Proposition 3.3, (54)-(58) hold. Hence,  $u$  solution of (12) belongs to  $C^2([0, T]; H_0^1(\Omega))$  and, in particular,

$$\sup_{t \in [0, T]} \|\nabla u_{|\theta}\|_{L^2(\Omega)} + \|\nabla \partial_{tt} u_{|\theta}\|_{L^2(\Omega)} + \|\partial_{tt} u_{|\theta}\|_{L^2(\Omega)} \leq c \|\theta\|_{H_0^1(\Omega)}, \quad (68)$$

for some  $c > 0$ . On the other hand, differentiating in space (12) we get, in the sense of distribution,

$$\nabla \partial_{tt} u_{|\theta} - \nabla \Delta u_{|\theta} = \sigma \nabla \theta + \theta \nabla \sigma,$$

which shows – using (68) – that  $\nabla \Delta u_{|\theta} \in L^\infty(0, T; L^2(\Omega))^d$  and (67) □

Let us now present the error reconstruction result in the case where *field* measurements are available.

**Theorem 3.8.** *Let  $\omega$  satisfy (7) and  $T$  satisfy (8). We assume that  $\sigma$  satisfies (38),  $\sigma \in \mathbf{H}^2(0, T; \mathbf{W}^{1, \infty}(\Omega))$  and  $\check{\theta} \in \mathbf{H}_0^1(\Omega)$ . Moreover, we assume that there exists  $M > 0$  such that*

$$\|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} \leq M. \quad (69)$$

*Then, there exists  $c > 0$  independent of  $M$  and  $\delta$  such that for  $\delta$  small*

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \leq cM^{2/3}\delta^{1/3}, \quad (70)$$

*where  $\bar{\theta}_T$  minimizes the cost functional (65) with  $\epsilon = \delta^2 M^{-2}$ .*

The proof of this theorem is very similar to that of Theorem 3.4.

*Proof.* In this proof, we write  $a \lesssim b$  if there exists a positive constant  $c > 0$  independent of  $\epsilon$ ,  $M$  and  $\delta$  such that  $a \leq cb$ .

First of all, we can bound the cost functional using the prior on the parameter  $\check{\theta}$  and the estimate on the noise amplitude (50),

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2}M^2 + \frac{1}{2}T\delta^2. \quad (71)$$

We introduce  $\omega_0$  and  $\omega_1$  as in the proof of Theorem 3.4. Using the stability inequality of Theorem 2.4 applied to  $\bar{\theta}_T - \check{\theta}$  on  $\omega_0$ , we get

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim \left( \int_0^T \|u_{|\bar{\theta}_T} - u_{|\check{\theta}}\|_{\mathbf{L}^2(\omega_0)}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|\Delta u_{|\bar{\theta}_T} - \Delta u_{|\check{\theta}}\|_{\mathbf{L}^2(\omega_0)}^2 dt \right)^{\frac{1}{2}}.$$

We artificially insert the noisy measurements  $y^\delta$  in the first term of the right-hand member. Since  $\omega_0 \subset \omega$ , we obtain

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim & \left( \int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|u_{|\check{\theta}} - y^\delta\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ & + \left( \int_0^T \|\Delta u_{|\bar{\theta}_T} - \Delta u_{|\check{\theta}}\|_{\mathbf{L}^2(\omega_0)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

The first term in the right-hand side is bounded by the cost-functional (65), the second term corresponds to the error measurements amplitude (64). In addition, using the cut-off function  $\eta \in C^3(\bar{\omega})$  which satisfies (59), we get

$$\|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + \delta\sqrt{T} + \left( \int_0^T \|\Delta(\eta u_{|\bar{\theta}_T} - \eta u_{|\check{\theta}})\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}}.$$

For some  $\alpha > 0$  given, introducing the regularization operator  $H_\omega^\alpha$  defined by (66) and using (71), we obtain

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim & (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} + \left( \int_0^T \|\Delta(H_\omega^\alpha(\eta u_{|\bar{\theta}_T} - \eta u_{|\check{\theta}}))\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ & + \left( \int_0^T \|\Delta((H_\omega^\alpha - \text{Id})(\eta u_{|\bar{\theta}_T} - \eta u_{|\check{\theta}}))\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Adding and subtracting the noisy observations, we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{\mathbf{L}^2(\Omega)} \lesssim & (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} + \left( \int_0^T \|\Delta(H_\omega^\alpha(\eta u_{|\bar{\theta}_T} - \eta y^\delta))\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ & + \left( \int_0^T \|\Delta(H_\omega^\alpha(\eta u_{|\check{\theta}} - \eta y^\delta))\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ & + \left( \int_0^T \|\Delta((H_\omega^\alpha - \text{Id})(\eta u_{|\bar{\theta}_T} - \eta u_{|\check{\theta}}))\|_{\mathbf{L}^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

According to Proposition 3.7,  $u_{|\bar{\theta}_T}$  and  $u_{|\check{\theta}}$  belong to  $C^0([0, T]; H^2(\Omega))$ ,  $\nabla \Delta u_{|\bar{\theta}_T}$  and  $\nabla \Delta u_{|\check{\theta}}$  belong to the space  $L^\infty(0, T; L^2(\Omega))^d$ . Therefore,  $\eta u_{|\bar{\theta}_T}$  and  $\eta u_{|\check{\theta}}$  belong to  $C^0(0, T; H_0^1(\omega) \cap H^2(\omega))$ , and the gradient of the laplacian of these functions is in  $L^\infty(0, T; L^2(\Omega))^d$ . Moreover by construction  $\Delta(\eta u_{|\bar{\theta}}) = \Delta(\eta u_{|\check{\theta}}) = 0$  on  $\partial\omega \times [0, T]$ . We can therefore use Propositions 3.5 and 3.6 to obtain

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} + \frac{1}{\alpha} \left( \int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &+ \frac{1}{\alpha} \left( \int_0^T \|u_{|\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} + \sqrt{\alpha} \left( \int_0^T \|\nabla \Delta(\eta u_{|\bar{\theta}_T})\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &+ \sqrt{\alpha} \left( \int_0^T \|\nabla \Delta(\eta u_{|\check{\theta}})\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (72)$$

Moreover, for all  $v \in H_0^1(\Omega) \cap H^2(\Omega)$  with  $\nabla \Delta v \in L^2(\Omega)$ , using Poincaré inequality and the definition of the cut-off function, we have

$$\|\nabla \Delta(\eta v)\|_{L^2(\omega)} \lesssim \|\nabla v\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(\Omega)} + \|\nabla \Delta v\|_{L^2(\Omega)}. \quad (73)$$

Using this property, (72) becomes

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} \\ &+ \frac{1}{\alpha} \left( \int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} + \frac{1}{\alpha} \left( \int_0^T \|u_{|\check{\theta}} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &+ \sqrt{\alpha} \left( \int_0^T \|\nabla \Delta u_{|\bar{\theta}_T}\|_{L^2(\omega)}^2 + \|\Delta u_{|\bar{\theta}_T}\|_{L^2(\omega)}^2 + \|\nabla u_{|\bar{\theta}_T}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}} \\ &+ \sqrt{\alpha} \left( \int_0^T \|\nabla \Delta u_{|\check{\theta}}\|_{L^2(\omega)}^2 + \|\Delta u_{|\check{\theta}}\|_{L^2(\omega)}^2 + \|\nabla u_{|\check{\theta}}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (74)$$

Applying (67) for  $\theta = \bar{\theta}_T$  and  $\theta = \check{\theta}$ , using the bound of the error measurements (64) and the definition (65) of  $\mathcal{J}_T$ , we obtain

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} + \frac{1}{\alpha} (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + \frac{1}{\alpha} \delta\sqrt{T} \\ &+ \sqrt{\alpha}\sqrt{T}\|\bar{\theta}_T\|_{H_0^1(\Omega)} + \sqrt{\alpha}\sqrt{T}\|\check{\theta}\|_{H_0^1(\Omega)}. \end{aligned} \quad (75)$$

Thanks to (69) and (71), we thus get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} &\lesssim (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} + \delta\sqrt{T} + \frac{1}{\alpha} (\epsilon M^2 + T\delta^2)^{\frac{1}{2}} \\ &+ \frac{1}{\alpha} \delta\sqrt{T} + \sqrt{\alpha}\sqrt{T} (M^2 + \epsilon^{-1}T\delta^2)^{\frac{1}{2}} + \sqrt{\alpha}\sqrt{T}M. \end{aligned}$$

Choosing  $\epsilon = \delta^2 M^{-2}$ , we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \lesssim \delta + \frac{\delta}{\alpha} + \sqrt{\alpha}M.$$

Choosing  $\alpha = \delta^{\frac{2}{3}} M^{-\frac{2}{3}}$  leads finally to (70).  $\square$

Note that, theoretically, in the case of *field* measurements, the power of the error measurement amplitude is lower than in the case of *velocity* measurements.

### 3.3 Observer design

To compute the minimizer of the cost functional (51), we use an approach presented in [59], namely we use a reduced-order Kalman filter where a Riccati operator is defined on the parameter space. In this section, we recall the main ideas of the proposed method, while details and general proofs are presented in [59]. The idea is first to work in the augmented space  $\mathbf{Z} = \mathcal{Z} \times \mathcal{P}$  where  $\mathcal{P}$  is the space of parameters. The dynamics in the augmented space reads

$$\underbrace{\frac{d}{dt} \begin{pmatrix} z \\ \theta \end{pmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} A & B(t) \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}(t)\mathbf{z}} \begin{pmatrix} z \\ \theta \end{pmatrix}, \quad \text{with} \quad \underbrace{\begin{pmatrix} z(0) \\ \theta(0) \end{pmatrix}}_{\mathbf{z}(0)} = \underbrace{\begin{pmatrix} 0 \\ \theta \end{pmatrix}}_{\boldsymbol{\zeta}}. \quad (76)$$

The least square minimization (either (51) or (65) depending on the definition of  $C$ ) rewritten with respect to the augmented variable reads

$$\mathcal{J}_T(\theta) = \frac{\epsilon}{2} \|\theta\|_{\mathbb{H}_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - \mathbf{C}z|_\theta\|_{L^2(\omega)}^2 dt, \quad (77)$$

with  $\mathbf{C} = (C \ 0)$  where  $C$  is defined by (9) or (10) according the considered case.

One way to find the minimizer of (77) is to compute its Frechet derivative and using the so called adjoint equation, to find the two-ends optimality system – as called in [8, 4]. Namely, we find the optimality condition

$$\bar{\boldsymbol{\zeta}}|_T := \begin{pmatrix} 0 \\ \bar{\theta} \end{pmatrix} = \mathbf{\Pi}_0 \bar{\mathbf{q}}_T(0),$$

where  $\bar{\theta}$  is the minimizer of (77) and  $\bar{\mathbf{q}}_T$  is associated with the optimal trajectory  $\bar{\mathbf{z}}_T = \mathbf{z}|_{\bar{\boldsymbol{\zeta}}|_T}$  through the two-ends system

$$\begin{cases} \dot{\bar{\mathbf{z}}}_T(t) = \mathbf{A}(t)\bar{\mathbf{z}}_T, & t \in (0, T), \\ \dot{\bar{\mathbf{q}}}_T(t) + \mathbf{A}(t)^* \bar{\mathbf{q}}_T = -\mathbf{C}^*(y^\delta(t) - \mathbf{C}\bar{\mathbf{z}}_T(t)), & t \in (0, T), \\ \bar{\mathbf{z}}_T(0) = \mathbf{\Pi}_0 \bar{\mathbf{q}}_T(0), \\ \bar{\mathbf{q}}_T(T) = 0, \end{cases} \quad (78)$$

$$\text{with } \mathbf{\Pi}_0 := \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon^{-1} \Delta_0^{-1} \end{pmatrix}.$$

To avoid calculating the two-ends system, which has the disadvantage of containing one dynamic forward in time and the second backward, we propose to introduce the so-called *Kalman observer* (or *Kalman estimator* in the stochastic community). Note that the Kalman observer decouples the two ends problem (78).

**Theorem 3.9.** *The Kalman estimator defined by the following dynamics*

$$\begin{cases} \dot{\hat{\mathbf{z}}} = \mathbf{A}(t)\hat{\mathbf{z}} + \mathbf{\Pi}(t)\mathbf{C}(t)^*(y^\delta(t) - \mathbf{C}(t)\hat{\mathbf{z}}(t)), & t \in (0, T), \\ \hat{\mathbf{z}}(0) = 0, \end{cases} \quad (79)$$

where  $\mathbf{\Pi}$  is the solution of the Riccati dynamics

$$\begin{cases} \dot{\mathbf{\Pi}}(t) = \mathbf{A}(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\mathbf{A}(t)^* - \mathbf{\Pi}(t)\mathbf{C}(t)^*\mathbf{C}(t)\mathbf{\Pi}(t), & \text{in } (0, T), \\ \mathbf{\Pi}(0) = \mathbf{\Pi}_0, \end{cases} \quad (80)$$

satisfies the fundamental identity

$$\forall t \in [0, T], \quad \bar{\mathbf{z}}_T(t) = \hat{\mathbf{z}}(t) + \mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t). \quad (81)$$



In particular, we observe that, taking  $T = t$ , we get that  $\hat{z}(t) = \bar{z}_t(t)$  for all  $t > 0$ . Therefore, at the final time, the Kalman observer reaches the trajectory associated with least-square estimator.

It should be noted that in our case of vanishing initial data and with our choice of  $\mathbf{II}_0$  it is possible to show that  $\mathbf{II}(t)$  is given by

$$\mathbf{II}(t) = \begin{pmatrix} L(t) \\ \text{Id} \end{pmatrix} \Lambda(t) (L^*(t) \text{ Id}), \quad (82)$$

where

$$\begin{cases} \dot{\Lambda} = -\Lambda L^* C^* C L \Lambda, & t > 0, \\ \Lambda(0) = \Lambda_0 := -\epsilon^{-1} \Delta_0^{-1}, \end{cases} \quad (83)$$

and, as a reminder,  $(L(t))_{t \geq 0}$  is the operator such that  $z_{|\theta}(t) = L(t)\theta$  is the solution of (5).

Therefore, we obtain the following optimal observer.

**Theorem 3.10.** *The unique minimizer  $\bar{\theta}_t$  of the cost functional (77) satisfies  $\bar{\theta}_t = \hat{\theta}(t)$  where  $\hat{\theta}$  is solution of the following dynamics*

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\hat{\Lambda}, & t > 0, \\ \dot{\hat{\theta}} = \Lambda L^* C^* (y^\delta - C\hat{z}), & t > 0, \\ \dot{\hat{\Lambda}} = -\Lambda L^* C^* C L \Lambda, & t > 0, \\ \hat{\theta}(0) = 0 \\ \hat{z}(0) = 0, \\ \hat{\Lambda}(0) = \Lambda_0 := -\epsilon^{-1} \Delta_0^{-1}, \end{cases} \quad (84)$$

where the operator  $(L(t))_{t \geq 0}$  is such that  $z_{|\theta}(t) = L(t)\theta$  is the solution of (5).

Note that the present estimator can be used for either *velocity* or *field* observations in  $L^2$  with the corresponding definition of the observation operator  $C$  and its adjoint  $C^*$ . Since this method is sequential, it has the advantage of not requiring multiple iterations to obtain the result. In this case, hence it is very efficient for parameters discretized in reasonable dimension spaces where the Riccati operator can be efficiently computed.

## 4 Numerical results

### 4.1 Discretization scheme

To start with, we will discretize in time and space the wave equation, and then, following a *discretize-then-optimize* strategy, we will apply to the discretized system the Tikhonov regularization method and observer design presented in Section 3. As in [59], in a classical way, we discretize the wave equation in time thanks to a mid-point scheme and we introduce a Lagrange finite element space  $\mathcal{V}_h$  for the space discretization of the field and velocity. Moreover, we introduce  $\mathcal{P}^h$  a finite-dimensional space which approximates  $\mathcal{P}$  and we consider  $\theta^h \in \mathcal{P}^h$ . Let  $N \in \mathbb{N}^*$  be given. We define the time step  $\tau$  by  $\tau = \frac{T}{N+1}$ . For  $n \leq N$ , at the  $(n+1)$ -th iteration,  $(u_{n+1}^h, v_{n+1}^h) \in \mathcal{V}_h \times \mathcal{V}_h$  is solution of  $u_0^h = 0, v_0^h = 0$  and

$$\begin{cases} \frac{u_{n+1}^h - u_n^h}{\tau} = \frac{v_{n+1}^h + v_n^h}{2}, \\ \frac{v_{n+1}^h - v_n^h}{\tau} - \Delta_0^h \frac{u_{n+1}^h + u_n^h}{2} = \sigma^h \left( (n + \frac{1}{2}) \tau \right) \theta^h, \end{cases} \quad (85)$$

where  $\Delta_0^h$  is the bounded linear operator in  $\mathcal{V}_h$  associated with the finite element discretization of the unbounded operator  $\Delta_0$  of  $L^2(\Omega)$  into  $L^2(\Omega)$ . Here,  $\sigma^h$  corresponds to the projection (or interpolation) of the function  $\sigma$  into the finite element space  $\mathcal{V}_h$ .

Let us also introduce the notation  $z_n^h = (u_n^h, v_n^h) \in \mathcal{Z}^h = \mathcal{V}_h \times \mathcal{V}_h$ . Then the first-order writing of this system is given by

$$z_{n+1|\zeta}^h = \Phi^{h,\tau} z_n^h + B_{n+1}^{h,\tau} \theta^h,$$

where

$$\Phi^{h,\tau} = \begin{pmatrix} \text{Id}_h & -\frac{\tau}{2} \text{Id}_h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}_h \end{pmatrix}^{-1} \begin{pmatrix} \text{Id}_h & \frac{\tau}{2} \text{Id}_h \\ \frac{\tau}{2} \Delta_0^h & \text{Id}_h \end{pmatrix}$$

and

$$B_{n+1}^{h,\tau} = \tau \begin{pmatrix} \text{Id}_h & -\frac{\tau}{2} \text{Id}_h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}_h \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \sigma^h((n + \frac{1}{2})\tau) \end{pmatrix}.$$

The time-and-space discretization of the cost functional (51) reads

$$\mathcal{J}_N(\zeta^h) = \frac{\epsilon}{2} \|\zeta^h\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{\tau}{2} \sum_{n=1}^N \|y_n^{\delta,h} - C^h z_{n|\zeta^h}^h\|_{\mathcal{Y}^h}^2, \quad (86)$$

associated with measurements  $y_n^{\delta,h}$  and an observation operator  $C^h \in \mathcal{L}(\mathcal{Z}^h, \mathcal{Y}^h)$  where  $\mathcal{Y}^h$  is the finite dimensional space the observations belong to.

We now state the following fundamental theorem which relates the minimizer of the cost functional to an observer. We refer to [59] for a proof of this result.

**Theorem 4.1.** *The minimizer  $\bar{\theta}_N^h$  of the cost functional (86) is equal to  $\hat{\theta}_N^h$  which is the last iteration of the recursive dynamics: for  $(\hat{z}_0^h, \hat{\theta}_0^h) = (0, 0)$  and  $(\hat{z}_n^h, \hat{\theta}_n^h)$  defined, for  $0 \leq n \leq N - 1$ , by*

$$\begin{cases} \hat{z}_{n+1}^h = \Phi^{h,\tau} \hat{z}_n^h + B_{n+1}^{h,\tau} \hat{\theta}_n^h, \\ \hat{\theta}_{n+1}^h = \hat{\theta}_n^h + \tau \Lambda_{n+1}^h L_{n+1}^{h*} C^* (y_{n+1}^{\delta,h} - C \hat{z}_{n+1}^h), \\ \hat{z}_{n+1}^h = \hat{z}_{n+1}^h + L_{n+1}^h (\hat{\theta}_{n+1}^h - \hat{\theta}_n^h) \end{cases} \quad (87)$$

where the operator  $(L_n^h)_{0 \leq n \leq N}$  is such that  $z_n^h|_{\theta} = L_n^h \theta$  is the solution of

$$\begin{cases} z_{n+1|\theta}^h = \Phi^{h,\tau} z_{n|\theta}^h + B_{n+1}^{h,\tau} \theta, & 0 \leq n \leq N - 1, \\ z_{0|\theta}^h = 0, \end{cases} \quad (88)$$

and the Riccati dynamics is given by  $\Lambda_{n+1}^h = (U_{n+1}^h)^{-1}$  where  $U_n^h$  satisfies

$$U_{n+1}^h = U_n^h + \tau L_{n+1}^{h*} C^{h*} C^h L_{n+1}^h, \quad 0 \leq n \leq N - 1. \quad (89)$$

with  $U_0^h = -\epsilon \Delta_0^h$ .

## 4.2 Wave propagation example

In what follows, we will illustrate the results obtained in the previous sections by testing the numerical reconstruction of a source term using an observer design. Moreover, we will illustrate the results given by Theorem 3.4 and depict the evolution of the error with respect to the noise of the available observations.

We consider the following wave equation

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(x, t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = \check{u}_0(x), & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (90)$$

where  $\check{u}_0$  is solution of

$$\begin{cases} -\Delta \check{u}_0(x) = 10, & x \in \Omega, \\ \check{u}_0(x) = 0, & x \in \partial\Omega, \end{cases} \quad (91)$$

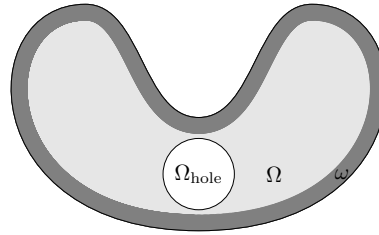
and  $\check{\theta}$  is solution of

$$\begin{cases} -\Delta \check{\theta}(x) = f, & x \in \Omega, \\ \check{\theta}(x) = 0, & x \in \partial\Omega, \end{cases} \quad (92)$$

with the right hand side  $f$  given by

$$f(x) = \begin{cases} 0, & \text{in } \Omega \setminus \Omega_{\text{hole}}, \\ 10, & \text{in } \Omega_{\text{hole}}. \end{cases}$$

The domain  $\Omega_{\text{hole}}$  is represented Figure 4.



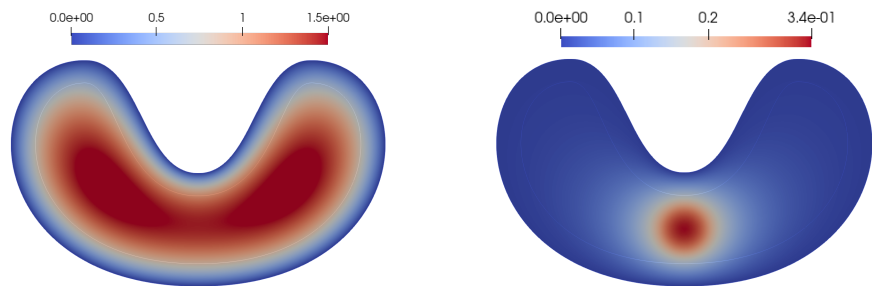
**Figure 4.** Representation of the domains  $\Omega$ ,  $\Omega_{\text{hole}}$  and the subdomain  $\omega$

The initial condition  $\check{u}_0$  and the target parameter  $\check{\theta}$  respectively defined by (91) and (92) are illustrated in Figure 5.

We also set

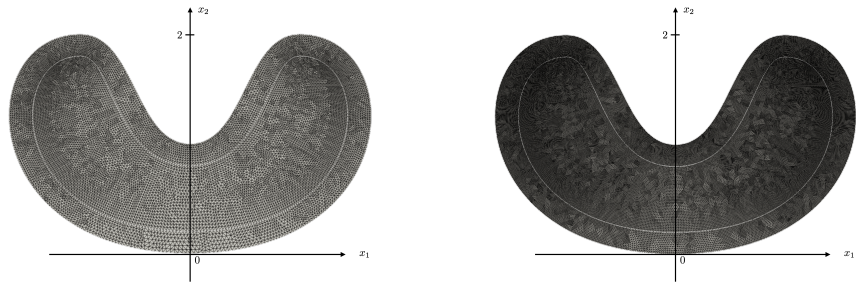
$$\sigma(x, t) = 1 + \exp(-20(x_2 - 0.9t + 0.4)^2), \quad (93)$$

where  $x_2$  represents the vertical component of  $x$ . By this way, this choice of  $\sigma$  corresponds to a persistent excitation and allows to mimic the propagation of a wave moving along the vertical axis from bottom to top. Moreover, we notice that  $\sigma$  is regular and bounded from below by a positive constant so that hypothesis (38) is satisfied.



**Figure 5.** Representation of  $\check{u}_0(x)$  (left) and  $\check{\theta}(x)$  (right)

In what follows, we will consider a  $\mathbb{P}_1$  finite element discretization with two different meshes: a mesh with 56193 nodes, that we will call the fine mesh and a mesh with 14169 nodes that we will call the coarse mesh. The two meshes are depicted in Figure 6. All the following numerical illustrations are implemented in Freefem++ [44].

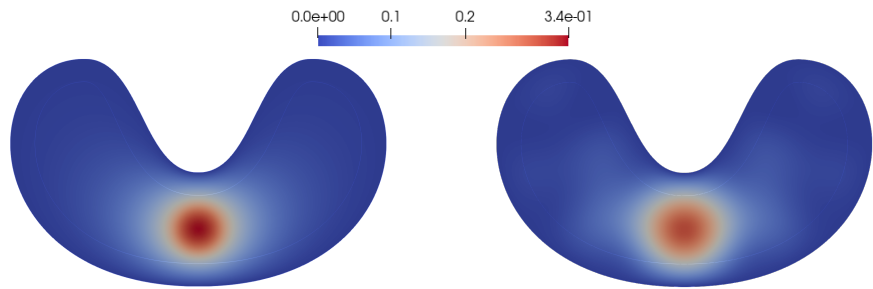


**Figure 6.** Representation of the coarse mesh with 14169 dofs (left) and of the fine mesh with 56193 dofs (right)

We approximate the target parameter  $\check{\theta}$  by the finite decomposition,

$$\check{\theta}^h := \sum_{i=0}^{N_{\text{mod}}} \check{\theta}_i^h \phi_i^h, \quad (94)$$

where  $(\phi_i^h)_{0 \leq i \leq N_{\text{mod}}}$  are the eigenvectors of  $\Delta_0^h$  associated with the  $N_{\text{mod}} + 1$  first eigenvalues. In order to have a relative error between  $\check{\theta}^h$  and  $\check{\theta}$  smaller than 10% with the fine mesh, we have set the number of modes to  $N_{\text{mod}} = 27$  (see Figure 7).



**Figure 7.** The target parameter  $\check{\theta}$  and its decomposition  $\check{\theta}^h$  given by with  $N_{\text{mod}} = 27$  modes

We represent in Figure 8, the time and space discretized known parameter  $\sigma^h((n + \frac{1}{2})\tau)$  (93) and the right-hand side  $\sigma^h((n + \frac{1}{2})\tau)\check{\theta}$  for the fine mesh.

To generate the measurements, we have run the direct problem on the fine mesh for a time window  $[0, 3]$  with a time step of  $\tau = 0.01$ . We illustrate an example of *velocity* measurements in Figure 9.

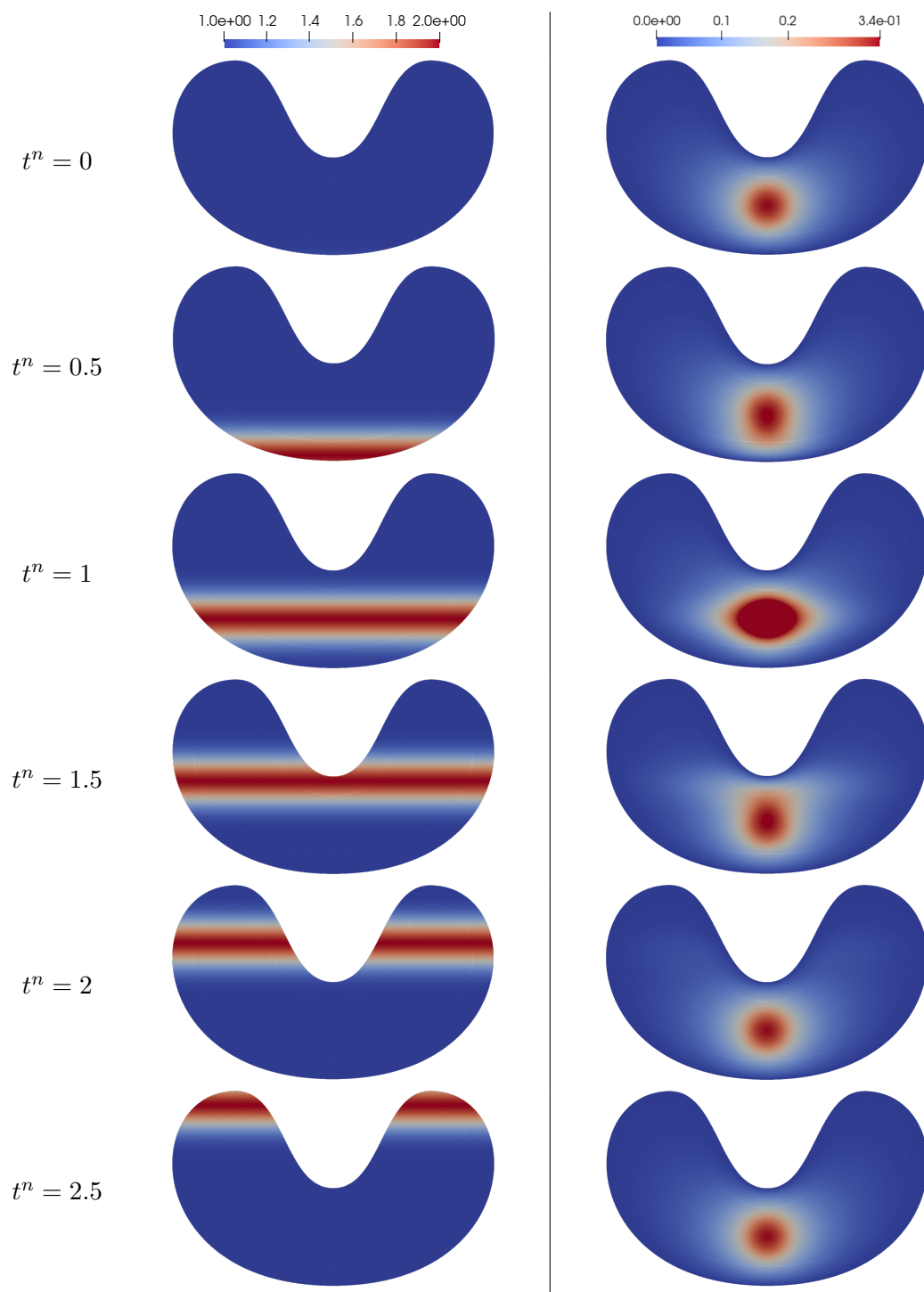
### 4.3 The case of noise-free *velocity* or *field* observations

We solve the inverse problem using scheme (87) implemented in Freefem++ under a time window of  $[0, 1]$  with a time step of  $\tau = 0.01$  and  $\epsilon = 0$ , using the meshes shown in Figure 6. Figure 10 shows the relative error norm between the observer and the target parameter approximation given by (94) over time, namely the above quantity

$$\|\hat{\theta}_n^h - \check{\theta}^h\|_{L^2(\Omega)} / \|\check{\theta}^h\|_{L^2(\Omega)}.$$

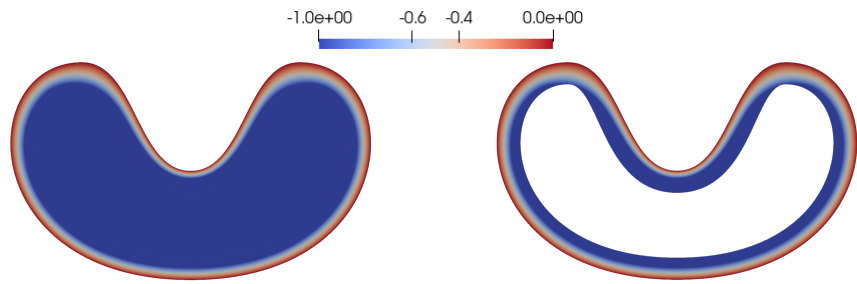
In the case of *velocity* measurements, we can see that the convergence when the inverse problem is solved on the same mesh as for the measurements is very good, with less than  $10^{-7}\%$  error at the final time, as shown in Figure 10 (right) and Figure 11 (right). If we compare Figures 7 (right) and 12 (right), we can see that the reconstructed source term is qualitatively similar to the target source term.

In the case of *field* measurements, we can see the same kind of results, with less than  $10^{-11}\%$  error at the final time, as illustrated in Figure 10 (left) and Figure 11 (left). In the same manner, if you compare Figures 7 (right) and 12 (left), we can

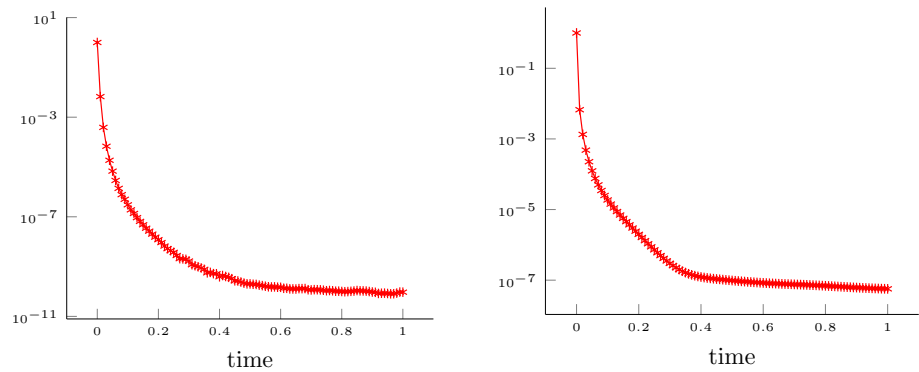


**Figure 8.** Representation of  $\sigma^h((n + \frac{1}{2})\tau)$  (left) and of  $\sigma^h((n + \frac{1}{2})\tau)\check{\theta}$  (right) at different times  $t^n$

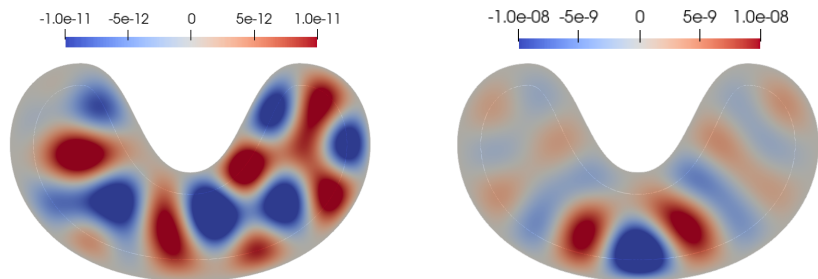
see that the reconstructed source term is qualitatively similar to the target source term.



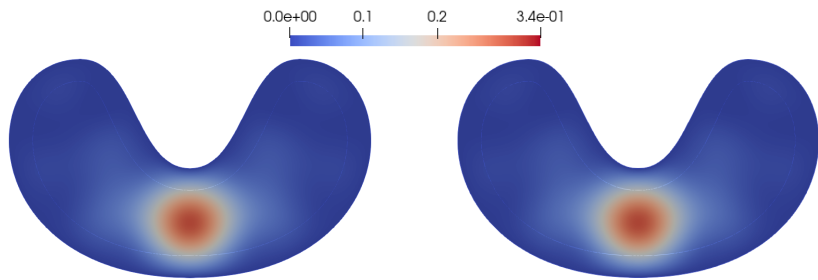
**Figure 9.** Solution  $\tilde{v}_n^h$  of the direct problem (left) and its corresponding measurements  $I_\omega(\tilde{v}_n^h)$  (right) at time  $t^n = 0.1$



**Figure 10.** Evolution of the relative error of the parameter reconstruction. The parameter, the *field* measurements (left) and the *velocity* measurements (right) are discretized on the fine mesh.



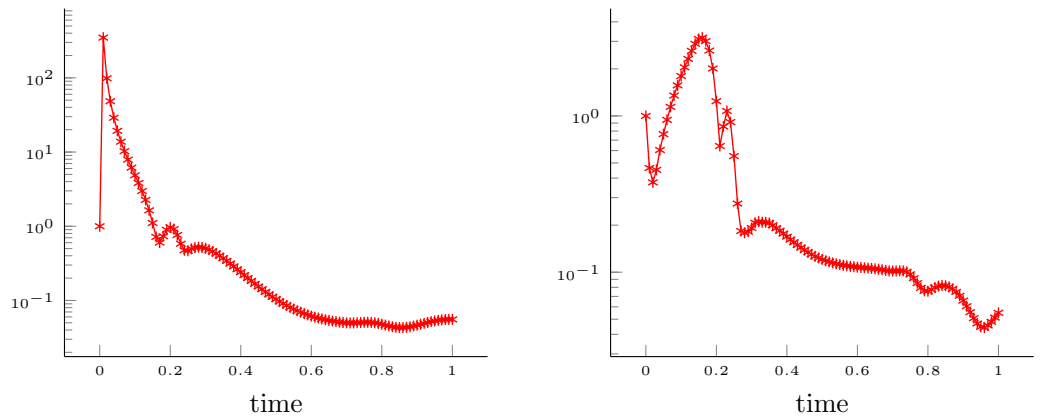
**Figure 11.** Error between  $\tilde{\theta}^h$  decomposed on the basis of  $\Delta_0^h$  and the observer  $\hat{\theta}_n^h$  on the fine mesh with *field* measurements (left) and *velocity* measurements (right)



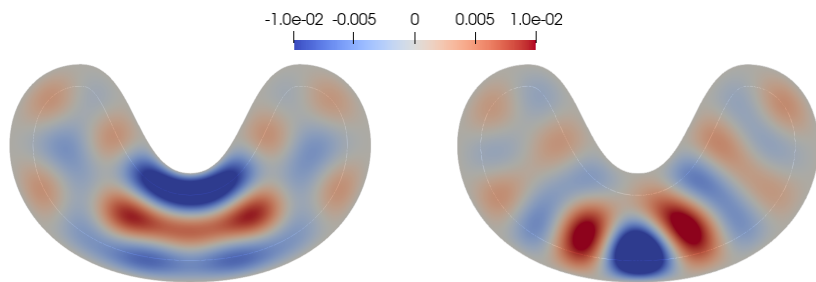
**Figure 12.** Observer  $\hat{\theta}_n^h$  on the fine mesh in the case of *field* measurements (left) and *velocity* measurements (right)

We now choose a coarser mesh for the space discretization of the parameters. Despite an increase of the error norm of the reconstruction close to the initial time, the relative error actually decreases from  $t^n = 0.2$ . As can be seen in Figure 13 and

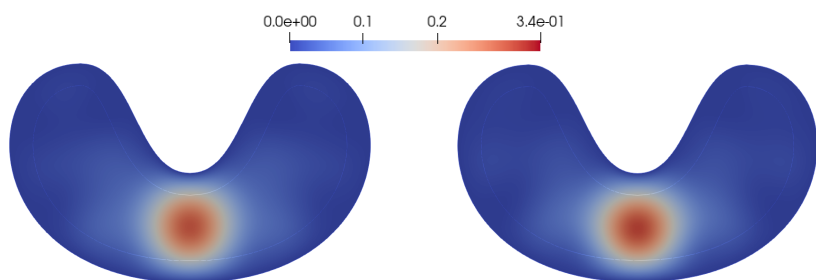
visually in Figure 14 the error at  $t^n = 1$  is about 5% both in the case of *velocity* measurements and in the case of *field* measurements. A visual comparison of Figures 7 (right) and 15 shows minimal qualitative differences between the observer at the final time and the target source term.



**Figure 13.** Evolution of the relative error of the parameter reconstruction. The parameter is discretized on the coarse mesh whereas the *field* measurements (left) and the *velocity* measurements (right) are discretized on the fine mesh.



**Figure 14.** Difference between  $\tilde{\theta}^h$  and the observer  $\hat{\theta}_n^h$  on the coarse mesh with *field* measurements (left) and *velocity* measurements (right)



**Figure 15.** Observer  $\hat{\theta}_n^h$  on the coarse mesh in the case of *field* measurements (left) and *velocity* measurements (right)

Although classical, hypothesis (38) is restrictive. Therefore another interesting experience is to see what happens when d (38) is fulfilled with  $\inf_{x \in \Omega} |\sigma(x, 0)|$  very low compared to 1. To this end, using velocity measurements, we test the convergence of the method on the fine mesh and in the absence of noise for  $\sigma$  given by

$$\sigma(x, t) = \exp(-20(x_2 - 0.9t + 0.4)^2). \quad (95)$$

Taking into account that  $x_2$  varies from 0 to 2 in the domain, the infimum of  $|\sigma(x, 0)|$  is of the order of  $10^{-18}$ , i.e., below machine precision when a 64 bits floating point representation is used. The corresponding source term is illustrated Figure 16.



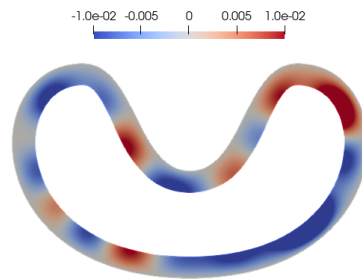
As can be seen in Figure 17, even if condition (38) is not satisfied, we can reconstruct the parameter we are looking for, with a relative error approaching  $10^{-7}$ .

#### 4.4 Noisy *velocity* observations

We now perturb the synthetic data  $(\tilde{y}_n^h)_{0 \leq n \leq N}$  of sampling rate  $T/N$  with a random artificial noise  $\chi_n^h$  such that  $\|\chi_n^h\|_{L^2(0,T;L^2(\omega))}^2 \leq \delta^2 T$ . To do so, we simulate an additive white Gaussian noise such that the magnitude of the error is a fixed percentage of the  $L^2$ -norm of the observations. Therefore, we define at each time  $n$ , the discretized noise data function as

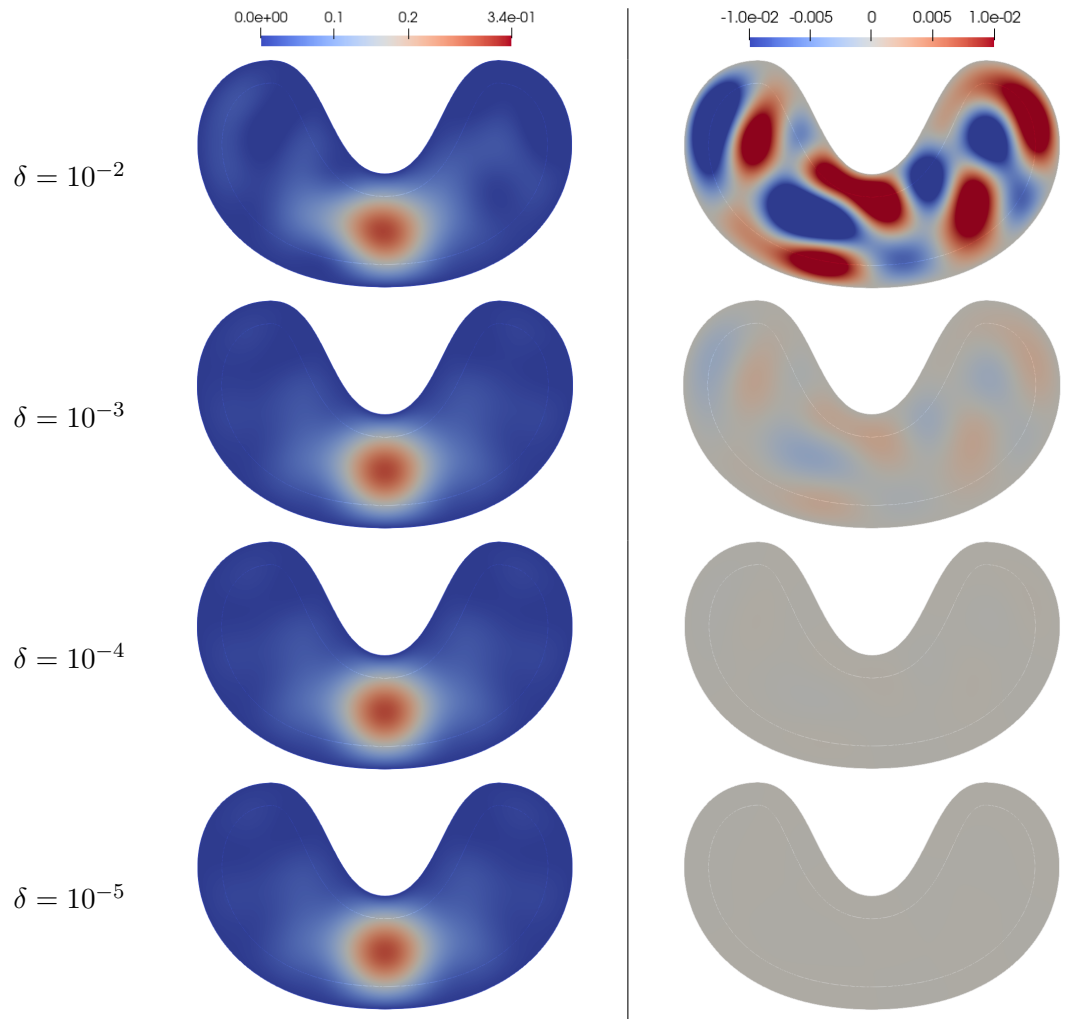
$$\chi_n^h = \frac{\delta}{\sqrt{TN_{mod}}} \sqrt{\tau \sum_{n=1}^N \|\tilde{y}_n^h\|_{L^2(\omega)}^2} \sum_{i=0}^{N_{mod}} (\xi_i)_n \phi_i^h, \quad (96)$$

where  $(\phi_i)_{0 \leq i \leq N_{mod}}$  are the  $N_{mod} + 1$  modes of  $\Delta_0^h$  and  $\xi$  are independent and identically distributed Gaussian variables in  $\mathcal{N}(0, 1)$ . Figure 18 shows an example of added noise.



**Figure 18.** Representation of  $\chi_n^h$  for  $\delta = 0.01$  at  $t^n = 0.1$

We solve the inverse problem using (87) on the fine mesh with time step  $\tau = 0.01$ . We use noisy data and take different values of  $\epsilon$ . Figure 19 illustrates that the reconstruction deteriorates as the noise increases. Figure 20 shows the relative error norm at time  $T = 1$  as a function of noise. We can see that the smaller the regularization, the better the reconstruction. In particular, for  $\epsilon = 10^{-6}$ , we obtain a slope of 1, which is better than the expected theoretical result (Theorem 3.4). This may be due to the fact that the observation domain is large, encompassing the entire boundary of the domain.



**Figure 19.** Representation of  $\hat{\theta}_n^h$  (left) and the error  $\tilde{\theta}^h - \hat{\theta}_n^h$  (right) at  $t^n = 1$  with different values of  $\delta$

#### 4.5 Towards the reconstruction of piecewise constant parameters

For the time being, we have proposed a method that suits well when the source term to reconstruct is smooth, *i.e.* it belongs to  $H_0^1(\Omega)$ . In the last section, we present possible alternatives to our method for less regular source terms by considering piecewise-constant source terms. In particular, we numerically assess how adaptive bases [41, 54] can be used in this framework. Adaptive bases have been used for inverse scattering problems [48, 51, 53]. For example, we can look for a property (e.g. shape) of the scatterer that can be related to a bounded inclusion. This is why, considering a piecewise constant source term, our inverse problem typically fits into this kind of method.

Until now, we have assumed that  $\tilde{\theta} \in H_0^1(\Omega)$  to obtain precise reconstruction estimates. However, as the  $L^2$ -norm of the source term appears in the obtained observability inequality, it seems appropriate to consider a classical Tikhonov regularization. Compared to (51) or to (65), we therefore introduce in the cost functional a  $L^2$ -Tikhonov regularization term by defining  $\mathcal{J}_T : L^2(\Omega) \rightarrow \mathbb{R}$  by: for all  $\theta \in L^2(\Omega)$ ,

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta - \theta_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - \partial_t u_{|\theta}\|_{L^2(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{L^2(\omega)}^2 dt. \end{aligned} \quad (97)$$

Therefore, the associated observer system (84) is slightly modified into

$$\begin{cases} \dot{\hat{\theta}} = \Lambda L^* C^* (y^\delta - C \hat{z}), & t > 0, \\ \dot{\Lambda} = -\Lambda L^* C^* C \Lambda, & t > 0, \\ \dot{\hat{z}} = A \hat{z} + B \hat{\theta} + L \dot{\hat{\theta}}, & t > 0, \\ \hat{\theta}(0) = \theta_0 \\ \hat{z}(0) = 0, \\ \Lambda(0) = \epsilon^{-1} I, \end{cases} \quad (98)$$

where  $\theta_0$  is a prior on the value of  $\check{\theta}$ ,  $I$  is the identity operator in  $L^2(\Omega)$ , the operator  $(L(t))_{t \geq 0}$  is such that  $z_{|\theta}(t) = L(t)\theta$  is the solution of (5) and  $C$  is defined by (9). We discretize this observer in time and space in the same way as in section 4.1. To take into account that the source term to be reconstructed may be discontinuous, we propose to use the adaptive basis proposed by [41] and often used in the literature [48, 53, 54]. The idea is to modify the basis of eigenfunctions on which we decompose  $\check{\theta}$  by considering

$$\check{\theta}^h := \sum_{k=1}^{\infty} \check{\theta}_k \phi_k, \quad (99)$$

where the family  $(\phi_k)_{k \in \mathbb{N}}$  is the set of eigenfunctions which satisfies the following elliptic eigenvalue problem

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\sqrt{|\nabla \theta_0|^2 + \nu^2}} \nabla \phi_k \right) = \mu_k \phi_k, & \text{in } \Omega, \\ \phi_k = 0, & \text{on } \partial\Omega, \end{cases} \quad (100)$$

for some given prior  $\theta_0$  and where  $\nu > 0$  is a small constant which ensures that the denominator is not null and  $\mu_k$  are the eigenvalues. This decomposition – used with an iterative reconstruction method – has proven to be remarkably effective to approximate piecewise constant functions.

Let us now explain how this adaptive basis can be used in our context. The inverse problem is solved iteratively: at a current step, the approximation of  $\check{\theta}^h$  – which has been reconstructed at the previous step – is taken as a prior in the functional (97). Problem (98) with the basis for the parameter being the basis obtained by solving (100) with the prior given by the previous step (only 10 modes are used). At the end of this step, we project the reconstructed parameter into a new adaptive basis (constructed using the prior just obtained) using again only 10 modes. Finally, the mesh is adapted to this projection.

In our numerical experiments, we have considered the following source term:

$$\check{\theta}(x) = \begin{cases} 1, & \text{in } \Omega_{\text{hole}}, \\ 0, & \text{in } \bar{\Omega} \setminus \Omega_{\text{hole}}, \end{cases} \quad (101)$$

as illustrated in Figure 21.

As in Section 4.2, the noise-free measurements are created in a first step by solving the direct problem on the fine mesh over a time window  $[0, 1]$  with a time step of  $\tau = 0.01$ . To solve the inverse problem we used as final time  $t^n = 1$  for each iteration with a time step of  $\tau = 0.01$ . In the first step, we used the coarse mesh and the source term is decomposed on the basis of the eigenvectors of the Laplace operator according to formula (94) with 30 modes. For the other iterations, only 10 modes are used, which ensures lower computational costs. Figure 22 depicts the observer  $\hat{\theta}_n^h$  obtained at the final time  $t^n = 1$  and the adapted mesh. We note that

during the first iteration we do not succeed in reconstructing the source term very accurately. In particular, the boundaries of the parameter are not captured. For the second iteration, we used the adaptive basis defined by (100). We observe that the mesh has fewer dofs (only 1834) and that the dofs are concentrated where the source term has large variations. The source term is already better reconstructed, especially at the boundary. In the last iteration, the mesh has even fewer dofs (1282). There are still differences with the target parameter, but the result is improved using the adaptive basis.

Let us conclude this section with a final remark. The  $L^2$ -norm is not the most appropriate norm for the source term  $\theta$ . One perspective would be to use a norm that takes into account the total variation. Indeed, this kind of norm is often used in image segmentation problems [9]. For instance, we can refer to the total variation–based image denoising model of [12] and its numerous extensions [21, 29, 34].

### Acknowledgment

Philippe Moireau would like to thank the Isaac Newton Institute for Mathematical Sciences for support (EPSRC grant number EP/R014604/1) and hospitality during the programme “The mathematical and statistical foundation of future data-driven engineering” when part of this work on this paper was undertaken.

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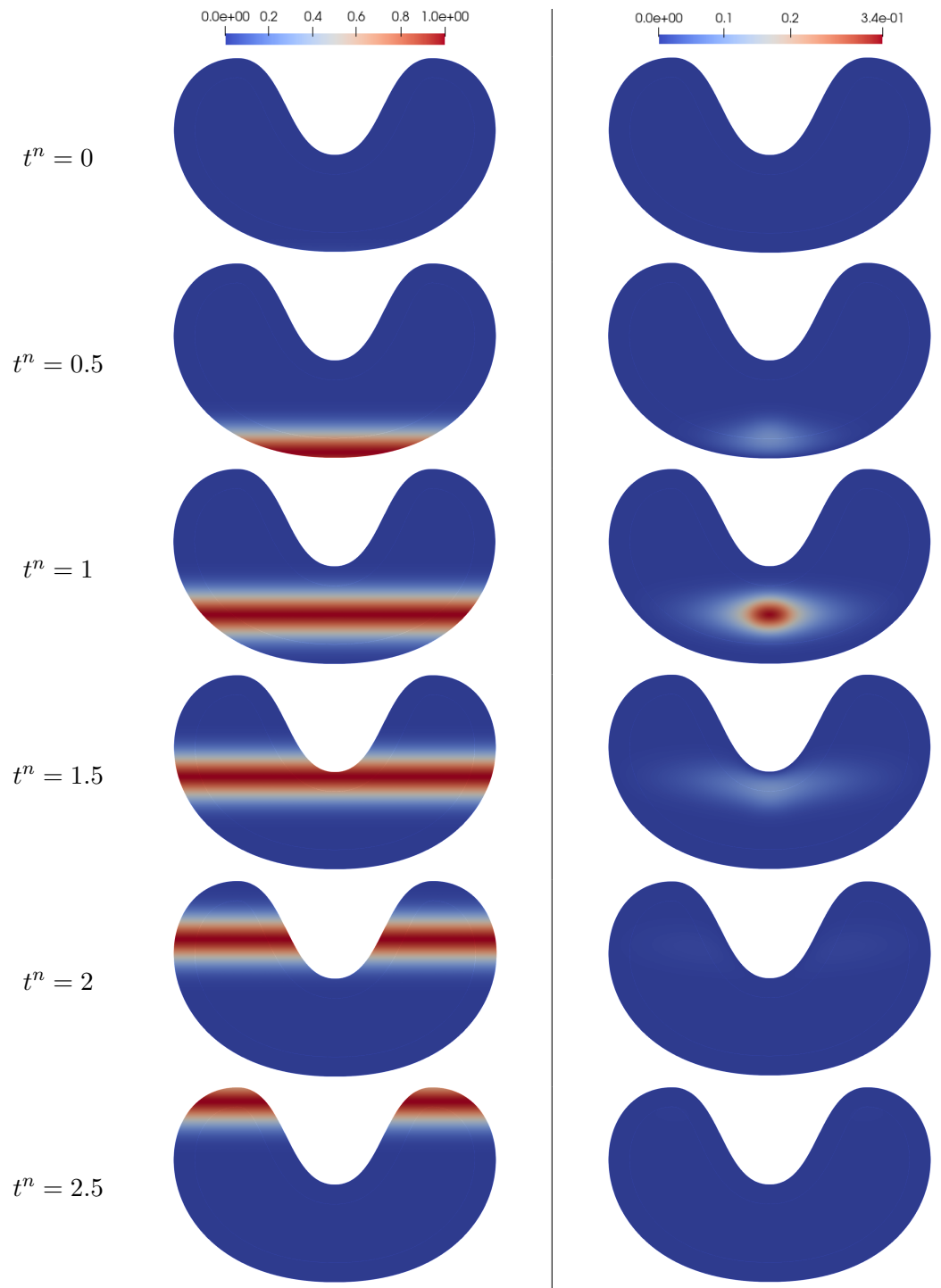
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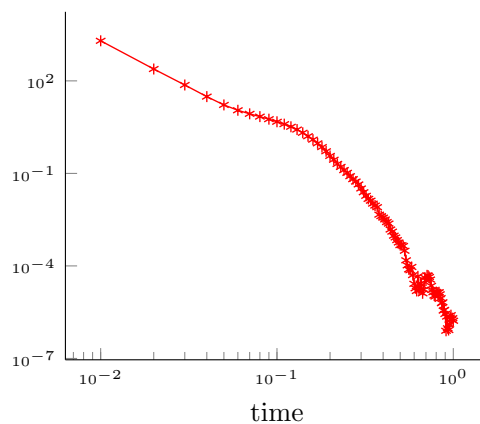


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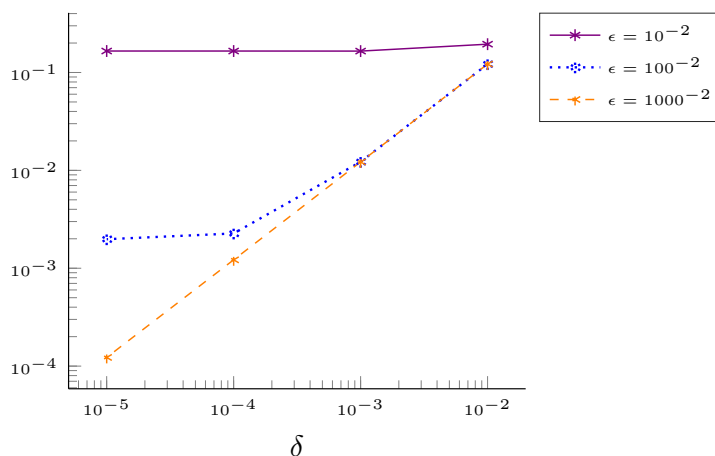


**Figure 16.** Representation of  $\sigma^h((n + \frac{1}{2})\tau)$  (left) and of  $\sigma^h((n + \frac{1}{2})\tau)\check{\theta}$  (right) at different times  $t^n$

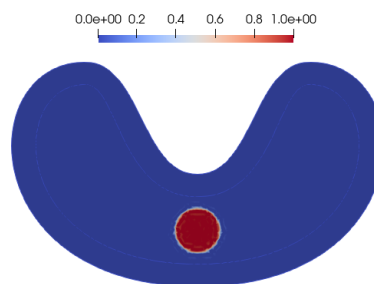




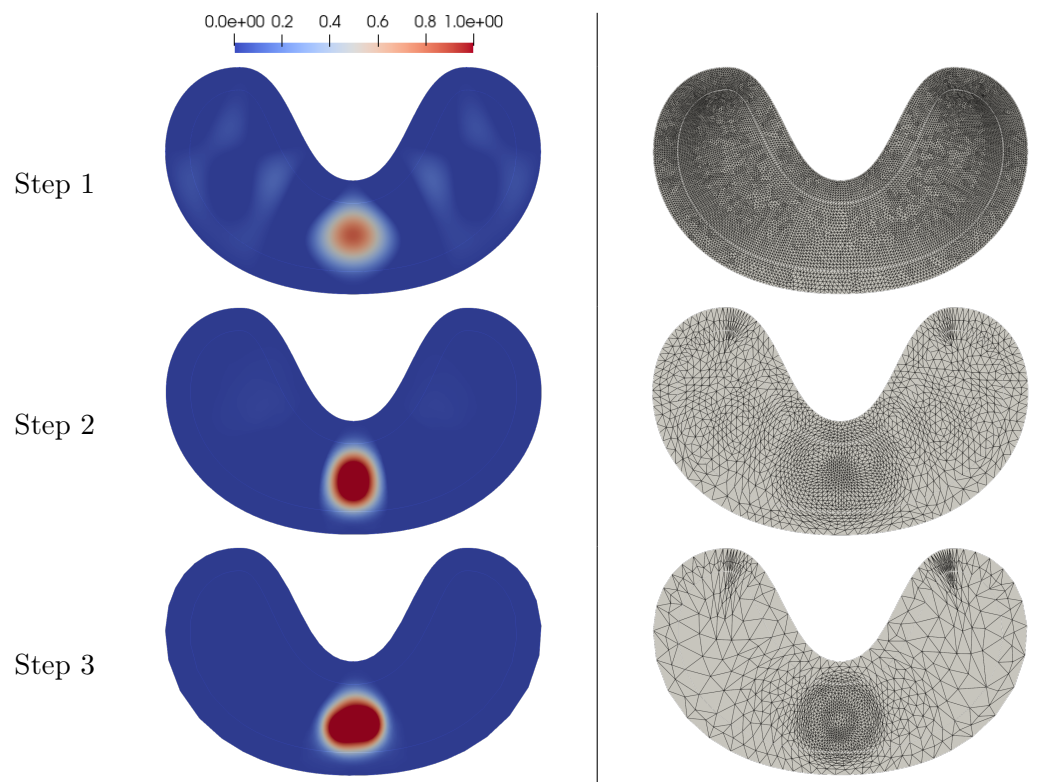
**Figure 17.** Evolution of the relative error of the parameter reconstruction. The parameter and the *velocity* measurements (right) are discretized on the fine mesh.



**Figure 20.** Representation of the relative error with respect to  $\delta$  in loglog scale



**Figure 21.** Representation of  $\check{\theta}$



**Figure 22.** Representation of  $\hat{\theta}_n^h$  (left) and its mesh (right) at  $t^n = 1$  at each step