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# Tessellations of an affine apartment by affine weight polytopes

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## Abstract

Let  $\mathbb{A}$  be a finite dimensional vector space and  $\Phi$  be a finite root system in  $\mathbb{A}$ . To this data is associated an affine poly-simplicial complex. Motivated by a forthcoming construction of connectified higher buildings, we study “affine weight polytopes” associated to these data. We prove that these polytopes tessellate  $\mathbb{A}$ . We also prove a kind of “mixed” tessellation, involving the affine weight polytopes and the poly-simplicial structure on  $\mathbb{A}$ .

## 1 Introduction

**Tessellation of  $\mathbb{A}$  by affine weight polytopes** Let  $\mathbb{A}$  be a finite dimensional vector space. A finite (reduced) root system in  $\mathbb{A}$  is a finite set  $\Phi$  of linear forms on  $\mathbb{A}$ , called roots, satisfying certain conditions (see [Bou81, VI, §1]). The associated affine (locally finite) hyperplane arrangement is  $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid (\alpha, k) \in \Phi \times \mathbb{Z}\}$ . The elements of  $\mathcal{H}$  are called affine walls. The hyperplane arrangement  $\mathcal{H}$  equips  $\mathbb{A}$  with the structure of a poly-simplicial complex, whose maximal poly-simplices - the alcoves - are the connected components of  $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$ . To each affine wall  $H$  is naturally associated a reflection  $s_H$ . Let  $W^{\text{aff}}$  be the affine Weyl group associated with  $(\mathbb{A}, \mathcal{H})$ , i.e.,  $W^{\text{aff}}$  is the group generated by the  $s_H$ , for  $H \in \mathcal{H}$ . This is naturally a Coxeter system and every closed alcove  $\overline{C}$  is a fundamental domain for the action of  $W^{\text{aff}}$  on  $\mathbb{A}$ . Fix a “fundamental alcove”  $C_f$  and pick one element  $\mathbf{b} = \mathbf{b}_{C_f}$  of  $C_f$ . Then for every alcove  $C$ , the set  $C \cap W^{\text{aff}}.\mathbf{b}$  contains exactly one element, that we denote  $\mathbf{b}_C$ . Denote by  $\text{vert}(\mathbb{A})$  the set of vertices of  $(\mathbb{A}, \mathcal{H})$ , i.e., the set of zero-dimensional faces of the associated poly-simplicial complex. For  $F$  a face of  $(\mathbb{A}, \mathcal{H})$ , we set

$$\mathcal{F}_F = \mathcal{F}_{\mathbf{b}, F} = \text{Int}_r(\text{conv}(W_F^{\text{aff}}.\mathbf{b}_C)),$$

where  $W_F^{\text{aff}}$  is the fixator of  $F$  in  $W^{\text{aff}}$ ,  $C$  is any alcove whose closure contains  $F$ ,  $\text{conv}$  denotes the convex hull, and given a subset  $E \subset \mathbb{A}$ ,  $\text{Int}_r(E)$  denotes the relative interior of  $E$  (that is the interior of  $E$  in the affine subspace spanned by  $E$ ). When  $F$  is 0-dimensional, i.e., when  $F = \{\lambda\}$  for some  $\lambda \in \text{vert}(\mathbb{A})$ , we usually write  $\mathcal{P}(\lambda)$  instead of  $\mathcal{F}_{\{\lambda\}}$ . Our first main result is the following (see Figures 3.6, 3.7, 3.8 and 3.9):

**Theorem 1.1.** (see Propositions 3.24, 3.27 and Theorem 3.28)

1. Let  $\lambda \in \text{vert}(\mathbb{A})$ . Then the assignment  $F \mapsto \mathcal{F}_F$  is a bijection between the set of faces of  $(\mathbb{A}, \mathcal{H})$  whose closure contains  $\lambda$  and the set of faces of  $\overline{\mathcal{P}(\lambda)}$ .
2. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ . Then we have  $\overline{\mathcal{F}_F} = \bigcap_{\lambda \in \text{vert}(F)} \overline{\mathcal{P}(\lambda)}$ , where  $\text{vert}(F)$  denotes the set of vertices of  $F$ .
3. Let  $\mathcal{F}$  denote the set of faces of  $(\mathbb{A}, \mathcal{H})$ . Then  $\mathbb{A} = \bigsqcup_{F \in \mathcal{F}} \mathcal{F}_F$ .

**Thickened tessellation of  $\mathbb{A}$**  We then study a variant of the tessellation of  $\mathbb{A}$  given by Theorem 1.1(3). Let  $\eta \in ]0, 1]$ . Let  $x \in \mathbb{A}$ . Let  $F = F_{\mathcal{P}}(x)$  be the face of  $(\mathbb{A}, \mathcal{H})$  such that  $\mathcal{F}_F$  contains  $x$  (the existence and uniqueness of such a face is provided by Theorem 1.1 (3)). Set  $F_{\eta}(x) = (1 - \eta)F + \eta x = \{(1 - \eta)y + \eta x \mid y \in F\}$ . For  $\lambda \in \text{vert}(\mathbb{A})$ , we denote by  $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}$  the homothety of  $\mathbb{A}$  with center  $\lambda$  and radius  $\eta$ . Then  $\overline{F_{\eta}(x)}$  is a ‘‘copy’’ of  $\overline{F}$  containing exactly one point of  $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$ , namely  $h_{\lambda, \eta}(x)$ , for each  $\lambda \in \text{vert}(F)$ . This is the convex hull of these points. Then we prove the following theorem:

**Theorem 1.2.** Let  $\eta \in ]0, 1]$ . Then  $\mathbb{A} = \bigsqcup_{x \in \mathbb{A}} \overline{F_{\eta}(x)}$ .

We deduce the following corollary, which is easier to interpret (see Figures 4.4 and 4.5):

**Corollary 1.3.** Let  $\eta \in ]0, 1]$ . Then we have

$$\mathbb{A} = \bigsqcup_{\lambda \in \text{vert}(\mathbb{A})} h_{\lambda, \eta}(\mathcal{P}(\lambda)) \sqcup \bigsqcup_{x \in \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \text{Fr}(\mathcal{P}(\lambda))} \overline{F_{\eta}(x)}.$$

Note that Theorem 1.2 and Corollary 1.3 provide some kinds of tessellations of  $\mathbb{A}$  involving homothetic images of the  $\mathcal{P}(\lambda)$  and homothetic images of the faces of  $(\mathbb{A}, \mathcal{H})$ .

The polytopes  $\overline{\mathcal{P}(\lambda)}$  that we study already appear in the litterature, in various contexts of representation theory. If  $\lambda \in \mathbb{A}$ , then we can write  $\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{\lambda}^0}$ , where  $\overline{\mathcal{P}_{\lambda}^0} = \text{conv}\{w \cdot \mathbf{b}' \mid w \in W_{\lambda}^v\}$ , where  $W_{\lambda}^v$  is the finite Weyl group associated with the root system  $\Phi_{\lambda} = \{\alpha \in \Phi \mid \alpha(\lambda) \in \mathbb{Z}\}$  and  $\mathbf{b}'$  is a regular element of  $\mathbb{A}$  (i.e.,  $w \cdot \mathbf{b}' \neq \mathbf{b}'$ , for every  $w \in W_{\lambda}^v$ ). The polytope  $\overline{\mathcal{P}_{\lambda}^0}$  is called the **weight polytope** and is studied in [Vin91] and [BJK23]. Its intersection with a closed vectorial chamber of  $\mathbb{A}$  is studied in [BGH23].

**Motivation from the theory of higher buildings** Let  $\mathbb{K}$  be a field equipped with a valuation  $\omega : \mathbb{K}^* \rightarrow \mathbb{Z}^n$ , for some  $n \in \mathbb{Z}_{\geq 1}$ , where  $\mathbb{Z}^n$  is equipped with the lexicographic order. For example, let  $\mathbb{k}$  be a field and set  $\mathbb{K} = \mathbb{k}(t_1, \dots, t_n)$ , which is contained in  $\mathbb{k}((t_n))((t_{n-1})) \dots ((t_1))$ , with

$$\omega \left( \sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} t_1^{i_1} \dots t_n^{i_n} \right) = \min\{(i_1, \dots, i_n) \in \mathbb{Z}^n \mid a_{i_1, \dots, i_n} \neq 0\},$$

for  $\sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} t_1^{i_1} \dots t_n^{i_n} \in \mathbb{K}$ . Such fields naturally appear as successive completions of function fields of  $n$ -dimensional algebraic varieties.

We would like to study groups of the form  $G = \mathbf{G}(\mathbb{K})$ , for  $\mathbf{G}$  a split reductive group, as well as certain arithmetic subgroups of them (for instance their presentations, certain (co)homological groups or their (co)homological dimension). When  $n = 1$ , then  $G$  acts on a classical Bruhat-Tits building which is connected and contractible (see [BT72] and [BT84]). We would like to use buildings techniques to study  $G$ , when  $n \geq 2$ . In this case

Bennett and Parshin independently defined analogues of Bruhat-Tits buildings in the case where  $\mathbf{G}$  is  $\mathrm{SL}_m$  or  $\mathrm{PGL}_m$ , for  $m \in \mathbb{Z}_{\geq 2}$  (see [Ben94] and [Par94]), using lattices of  $\mathbb{K}^m$ . In [Ben94], Bennett develops an axiomatic definition of  $\Lambda$ -buildings (where  $\Lambda = \omega(\mathbb{K}^*)$ ,  $\Lambda = \mathbb{Z}^n$  in our case). In [HIL20], the three last named authors prove that  $G$  acts on a  $\mathbb{Z}^n$ -building  $\mathcal{I}$ , generalizing (in the quasi-split case) Bruhat-Tits construction via “parahoric” subgroups.

When  $n \geq 2$ ,  $\mathcal{I}$  is a bit like a set of Russian dolls:  $\mathcal{I}$  is a Bruhat-Tits building whose vertices are  $\mathbb{Z}^{n-1}$ -buildings. A problem is that  $\mathcal{I}$  is not connected since the latter  $\mathbb{Z}^{n-1}$ -buildings are connected components of  $\mathcal{I}$ . As the connectedness and contractibility of Bruhat-Tits buildings are important ingredients of the classical theory, we would like to embed the  $\mathbb{Z}^n$ -building  $\mathcal{I}$  in a “connectified building  $\hat{\mathcal{I}}$  or  $\hat{\mathcal{I}}^\eta$ , for  $\eta \in ]0, 1[$ , on which  $G$  would act and which would have better topological properties. We hope that our tessellations are a first step in this direction.

**Structure of the paper** The paper is organized as follows. In section 2, we study the polytopes  $\mathcal{P}(\lambda)$  (or more precisely translates of these polytopes) one by one. We describe the set of faces and study the orthogonal projection on such a polytope.

In section 3, we study the polytopes  $\overline{\mathcal{P}(\lambda)}$  and prove that they tessellate  $\mathbb{A}$ .

In section 4, we prove Theorem 1.2.

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## 2 Vectorial Weyl polytopes

Let  $\mathbb{A}$  be a finite dimensional vector space and  $\Phi$  be a finite root system in  $\mathbb{A}$ . This equips  $\mathbb{A}$  with the structure of a poly-simplicial complex. The aim of this paper is to study tessellations of  $\mathbb{A}$  by polytopes  $\overline{\mathcal{P}(\lambda)}$  associated with vertices  $\lambda$  of  $\mathbb{A}$ . Before studying the layout of the different polytopes, we begin by studying the polytopes one by one.

In this section, we study the vectorial weight polytope  $\overline{\mathcal{P}^0} = \overline{\mathcal{P}_\Psi^0}$  associated with a finite root system  $\Psi$ . In the applications,  $\Psi$  will be a root system of the form  $\Phi_\lambda = \{\alpha \in \Phi \mid \alpha(\lambda) \in \mathbb{Z}\}$ , for  $\lambda$  a vertex of  $(\mathbb{A}, \Phi)$  and then  $\overline{\mathcal{P}(\lambda)}$  will be a translate of  $\overline{\mathcal{P}_{\Phi_\lambda}^0}$ . To define this polytope, we choose a regular point of  $\mathbb{A}$  and we take the convex hull of the orbit of this point under the action of the finite Weyl group of  $\Psi$ .

In subsection 2.1, we study general facts on polyhedra.

In subsection 2.2, we define the vectorial weight polytope  $\overline{\mathcal{P}^0}$ .

In subsection 2.3, we describe the faces of  $\overline{\mathcal{P}^0}$ .

In subsection 2.4, we study the restriction to certain affine subspaces of  $\mathbb{A}$  of the orthogonal projection on  $\overline{\mathcal{P}^0}$ .

### 2.1 Preliminary results on polyhedra and polytopes

In this section,  $\mathbb{A}$  is a finite dimensional real vector space. We equip  $\mathbb{A}$  with a scalar product  $\langle \cdot \mid \cdot \rangle$  (for the moment,  $\langle \cdot \mid \cdot \rangle$  is arbitrary, but in most of the applications, it will be invariant under the action of the vectorial or affine Weyl groups defined later). If  $E$  is a subset of  $\mathbb{A}$ , we denote by  $\overset{\circ}{E}$  its interior. If  $E$  is non-empty, its **support**  $\text{supp}(E)$  is the smallest affine subspace of  $\mathbb{A}$  containing  $E$ . Its **relative interior**, denoted  $\text{Int}_r(E)$ , is then the interior of  $E$  in its support. In particular, if  $E$  is convex and closed, we have  $E = \overline{\text{Int}_r(E)}$ . When  $E$  is an affine subspace of  $\mathbb{A}$ , its **direction**, denoted  $\text{dir}(E)$ , is the vector subspace  $\{x - x' \mid x, x' \in E\}$  of  $\mathbb{A}$ .

We mainly follow the terminology of [BG09]. A (closed) **polyhedron** is the intersection of finitely many closed half-spaces of  $\mathbb{A}$ . A (closed) **polytope** is a bounded polyhedron. Let  $\mathcal{Q}$  be a polyhedron of  $\mathbb{A}$ . A **support hyperplane** of  $\mathcal{Q}$  is a hyperplane  $H$  such that  $\mathcal{Q}$  is contained in one of the closed half-spaces delimited by  $H$  and such that  $\mathcal{Q} \cap H \neq \emptyset$ . A **closed face** of  $\mathcal{Q}$  is either  $\mathcal{Q}$  or a set of the form  $H \cap \mathcal{Q}$ , where  $H$  is a support hyperplane. Contrary to [BG09], we do not consider  $\emptyset$  as a face of  $\mathcal{Q}$ . A **closed facet** of  $\mathcal{Q}$  is a face of codimension 1 in the support of  $\mathcal{Q}$ . By [BG09, Theorem 1.10], every proper closed face of  $\mathcal{Q}$  is an intersection of closed facets. We denote by  $\text{Facet}(\mathcal{Q})$  the set of closed facets of  $\mathcal{Q}$ . An **open face** or simply a **face** of  $\mathcal{Q}$  is the relative interior of a closed face of  $\mathcal{Q}$ . Note that in [BG09], what we call ‘‘closed faces’’ are simply called ‘‘faces’’ and the ‘‘open faces’’ are the relative interior of faces: we make this choice in order to be coherent with the usual definition of faces in an apartment (see subsections 2.2 and 3.2). If  $F_1, F_2$  are open faces of a polyhedron for which  $\overline{F_1} \cap \overline{F_2}$  is non-empty, we set  $F_1 \wedge F_2 = \text{Int}_r(\overline{F_1} \cap \overline{F_2})$ . It is a face of  $\mathcal{Q}$  by [BG09, Proposition 1.8 and Theorem 1.10].

#### 2.1.1 Orthogonal projection on polytopes

If  $E$  is a convex closed subset of  $\mathbb{A}$ , we denote by  $\pi_E : \mathbb{A} \rightarrow E$  the orthogonal projection on  $E$ : for every  $x \in \mathbb{A}$ ,  $\pi_E(x)$  is the unique point  $y_x$  of  $E$  satisfying  $|x - y_x| = \min\{|z - x| \mid z \in E\}$ . Moreover, if  $y_x \in E$ , then we have

$$(y_x = \pi_E(x)) \iff (\langle x - y_x \mid y - y_x \rangle \leq 0, \forall y \in E), \quad (2.1)$$

by [HUL93, III Theorem 3.1.1].

In this preliminary subsection, we study the projection  $\pi_{\mathcal{Q}}$ , for  $\mathcal{Q}$  a polyhedron of  $\mathbb{A}$ . In particular, Proposition 2.3 provides a description of its fibers.

For  $x \in \mathbb{A}$  and  $r \in \mathbb{R}_{\geq 0}$ , we denote by  $B(x, r)$  the open ball of center  $x$  and radius  $r$ .

**Lemma 2.1.** *Let  $\mathcal{Q}$  be a polyhedron of  $\mathbb{A}$  with non-empty interior. Let  $x \in \mathcal{Q}$  and let  $\text{Facet}_x(\mathcal{Q})$  be the set of closed facets of  $\mathcal{Q}$  containing  $x$ . For  $\overline{\mathcal{F}} \in \text{Facet}_x(\mathcal{Q})$  with relative interior  $\mathcal{F}$ , we denote by  $\mathcal{D}_{\overline{\mathcal{F}}}$  the closed half-space of  $\mathbb{A}$  delimited by  $\overline{\mathcal{F}}$  and containing  $\mathcal{Q}$ . Let  $\tilde{\mathcal{Q}} = \bigcap_{\overline{\mathcal{F}} \in \text{Facet}_x(\mathcal{Q})} \mathcal{D}_{\overline{\mathcal{F}}}$ . Then there exists  $r > 0$  such that  $B(x, r) \cap \mathcal{Q} = B(x, r) \cap \tilde{\mathcal{Q}}$ .*

*Proof.* We have  $\mathcal{Q} \subset \mathcal{D}_{\overline{\mathcal{F}}}$  for every  $\overline{\mathcal{F}} \in \text{Facet}_x(\mathcal{Q})$  and thus  $\mathcal{Q} \subset \tilde{\mathcal{Q}}$ .

For  $\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q}) \setminus \text{Facet}_x(\mathcal{Q})$ , choose  $r_{\overline{\mathcal{F}}} > 0$  such that  $B(x, r_{\overline{\mathcal{F}}}) \cap \overline{\mathcal{F}} = \emptyset$ . Let  $r = \min\{r_{\overline{\mathcal{F}}} \mid \overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q}) \setminus \text{Facet}_x(\mathcal{Q})\}$  (we set  $r = 1$  if  $\text{Facet}(\mathcal{Q}) = \text{Facet}_x(\mathcal{Q})$ ). Let  $y \in \tilde{\mathcal{Q}} \cap B(x, r)$ . Assume by contradiction that  $y \notin \mathcal{Q}$ . Let  $z \in \tilde{\mathcal{Q}} \cap B(x, r)$ . Then  $[z, y]$  meets  $\text{Fr}(\mathcal{Q})$  in a point  $y'$ . By [BG09, Corollary 1.7], we have  $\text{Fr}(\mathcal{Q}) = \bigcup_{\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q})} \overline{\mathcal{F}}$ . We have  $y' \notin \bigcup_{\overline{\mathcal{F}} \in \text{Facet}_x(\mathcal{Q})} \overline{\mathcal{F}}$  and thus  $y' \in \bigcup_{\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q}) \setminus \text{Facet}_x(\mathcal{Q})} \overline{\mathcal{F}}$ . Consequently  $y' \notin B(x, r)$ : a contradiction. Therefore  $\tilde{\mathcal{Q}} \cap B(x, r) \subset \mathcal{Q}$ , and the lemma follows.  $\square$

Let  $\mathcal{Q}$  be a polyhedron. If  $\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q})$ , we denote by  $\delta_{\overline{\mathcal{F}}}$  the (closed) ray of  $\text{dir}(\text{supp}(\overline{\mathcal{F}}))$ , which is orthogonal to the direction of the support of  $\overline{\mathcal{F}}$  and such that for all  $x \in \overline{\mathcal{F}}$ ,  $(x + \delta_{\overline{\mathcal{F}}}) \cap \mathcal{Q} = \{x\}$ . If  $\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q})$ , we denote by  $\mathcal{D}_{\overline{\mathcal{F}}}$  the half-space of  $\mathbb{A}$  containing  $\mathcal{Q}$  and delimited by the hyperplane containing  $\overline{\mathcal{F}}$ . We have

$$\mathcal{D}_{\overline{\mathcal{F}}} = \{x \in \mathbb{A} \mid \langle x - a \mid z \rangle \leq 0, \forall z \in \delta_{\overline{\mathcal{F}}}\}, \quad (2.2)$$

for any  $a \in \overline{\mathcal{F}}$ .

**Lemma 2.2.** *Let  $\mathcal{Q}$  be a polyhedron with non-empty interior. Let  $x \in \mathbb{A} \setminus \mathcal{Q}$  and  $J = \{\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q}) \mid \pi_{\mathcal{Q}}(x) \in \overline{\mathcal{F}}\}$ . Then  $\pi_{\mathcal{Q}}(x) = \pi_{\tilde{\mathcal{Q}}}(x)$ , where  $\tilde{\mathcal{Q}} = \bigcap_{\overline{\mathcal{F}} \in J} \mathcal{D}_{\overline{\mathcal{F}}}$ .*

*Proof.* By Lemma 2.1, there exists  $r > 0$  such that  $\mathcal{Q} \cap B(\pi_{\mathcal{Q}}(x), r) = \tilde{\mathcal{Q}} \cap B(\pi_{\mathcal{Q}}(x), r)$ . We have  $d(x, y) \geq d(x, \pi_{\mathcal{Q}}(x))$  for every  $y \in \tilde{\mathcal{Q}} \cap B(\pi_{\mathcal{Q}}(x), r)$  and thus  $d(x, \cdot)|_{\tilde{\mathcal{Q}}}$  admits a local minimum at  $\pi_{\mathcal{Q}}(x)$ . But as  $d(x, \cdot)|_{\tilde{\mathcal{Q}}}$  is a convex function, by [HUL93, IV 1.3 (c)], this minimum is actually a global minimum, which proves that  $\pi_{\mathcal{Q}}(x) = \pi_{\tilde{\mathcal{Q}}}(x)$ .  $\square$

**Proposition 2.3.** *Let  $\mathcal{Q}$  be a polyhedron with non-empty interior and let  $\pi_{\mathcal{Q}}$  be the orthogonal projection on  $\mathcal{Q}$ . Let  $x \in \text{Fr}(\mathcal{Q})$  and  $J = \{\overline{\mathcal{F}} \in \text{Facet}(\mathcal{Q}) \mid x \in \overline{\mathcal{F}}\}$ . Then  $\pi_{\mathcal{Q}}^{-1}(\{x\}) = x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\overline{\mathcal{F}}}$ .*

*Proof.* Let  $\tilde{\mathcal{Q}} = \bigcap_{\overline{\mathcal{F}} \in J} \mathcal{D}_{\overline{\mathcal{F}}}$ . Let  $y \in x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\overline{\mathcal{F}}}$ . Then by Eq. (2.2), we have  $\langle y - x \mid z - x \rangle \geq 0$  for all  $z \in \tilde{\mathcal{Q}} \supset \mathcal{Q}$  and hence by Eq. (2.1), we have  $\pi_{\mathcal{Q}}(y) = x$ . Therefore

$$\pi_{\mathcal{Q}}^{-1}(\{x\}) \supset x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\overline{\mathcal{F}}}.$$

Let now  $y \in \pi_{\mathcal{Q}}^{-1}(\{x\})$ . Then by Lemma 2.2, we have  $\pi_{\mathcal{Q}}(y) = \pi_{\tilde{\mathcal{Q}}}(y) = x$ . For  $\overline{\mathcal{F}} \in J$ , choose  $z_{\overline{\mathcal{F}}} \in \delta_{\overline{\mathcal{F}}} \setminus \{0\}$  and respectively define  $f_{\overline{\mathcal{F}}} : \mathbb{A} \rightarrow \mathbb{R}$  and  $f : \mathbb{A} \rightarrow \mathbb{R}$  by  $f_{\overline{\mathcal{F}}}(z) = -\langle z \mid z_{\overline{\mathcal{F}}}\rangle$  and  $f(z) = \langle z \mid x - y \rangle$ , for  $z \in \mathbb{A}$ . Let  $z \in \mathbb{A}$  be such that  $f_{\overline{\mathcal{F}}}(z) \geq 0$ , for all  $\overline{\mathcal{F}} \in J$ . Let  $z' = z + x$ . Then we have  $f_{\overline{\mathcal{F}}}(z' - x) \geq 0$ , for all  $\overline{\mathcal{F}} \in J$  and thus by Eq. (2.2), we have  $z' \in \tilde{\mathcal{Q}}$ . By Eq. (2.1), we deduce  $f(z' - x) \geq 0$  and thus  $f(z) \geq 0$ . Using Farkas' Lemma, we deduce  $f \in \sum_{\overline{\mathcal{F}} \in J} \mathbb{R}_{\geq 0} f_{\overline{\mathcal{F}}}$ , or equivalently,  $y - x \in \sum_{\overline{\mathcal{F}} \in J} \mathbb{R}_{\geq 0} z_{\overline{\mathcal{F}}} = \sum_{\overline{\mathcal{F}} \in J} \delta_{\overline{\mathcal{F}}}$ , which proves the proposition.  $\square$

## 2.1.2 Extreme points of a polytope

Let  $\mathcal{Q}$  be a convex subset of  $\mathbb{A}$ . Recall (see [HUL12, Definition 2.3.1]) that a point  $M$  of  $\mathcal{Q}$  is called an **extreme point** of  $\mathcal{Q}$  if for every  $k \in \mathbb{Z}_{\geq 1}$ ,  $M_1, \dots, M_k \in \mathcal{Q}$ ,  $t_1, \dots, t_k \in ]0, 1]$  such that  $\sum_{i=1}^k t_i = 1$ , we have

$$\sum_{i=1}^k t_i M_i = M \Rightarrow M_1 = \dots = M_k = M.$$

We denote by  $\text{Extr}(\mathcal{Q})$  the set of extreme points of  $\mathcal{Q}$ .

**Lemma 2.4.** *Let  $H$  be a finite subgroup of the set of affine automorphisms of  $\mathbb{A}$ . Let  $\mathbf{b} \in \mathbb{A}$ . Then the set of extreme points of  $\text{conv}(H.\mathbf{b})$  is  $H.\mathbf{b}$ .*

*Proof.* Denote by  $E$  the set of extreme points of  $\text{conv}(H.\mathbf{b})$ . Then  $E$  is contained in  $H.\mathbf{b}$ . Then  $\text{conv}(H.\mathbf{b})$  is compact. By Krein-Milman theorem ([HUL12, Theorem 2.3.4]),  $\text{conv}(H.\mathbf{b})$  is the convex hull of  $E$ , and in particular,  $E$  is non-empty. As  $H$  stabilizes  $\text{conv}(H.\mathbf{b})$ ,  $E$  is  $H$ -invariant and thus  $E = H.\mathbf{b}$ .  $\square$

**Lemma 2.5.** *Let  $\mathcal{Q}$  be a closed polytope of  $\mathbb{A}$ .*

1. *Let  $\overline{\mathcal{F}}$  be a closed face of  $\mathcal{Q}$ . Then  $\text{Extr}(\overline{\mathcal{F}}) = \text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}$ .*
2. *Let  $(\overline{\mathcal{F}}_j)_{j \in J}$  be a family of closed faces of  $\mathcal{Q}$ . Then  $\text{Extr}(\bigcap_{j \in J} \overline{\mathcal{F}}_j) = \bigcap_{j \in J} \text{Extr}(\overline{\mathcal{F}}_j)$ .*

*Proof.* 1) By [BG09, Proposition 1.13], the set of closed faces of  $\mathcal{Q}$  (resp.  $\overline{\mathcal{F}}$ ) is the set of extreme sets of  $\mathcal{Q}$  (resp.  $\overline{\mathcal{F}}$ ) and thus the set of extreme points of  $\overline{\mathcal{F}}$  is the set of zero-dimensional faces of  $\overline{\mathcal{F}}$ . That is the set of zero-dimensional faces of  $\mathcal{Q}$ , that are contained in  $\overline{\mathcal{F}}$ , according to [BG09, Proposition 1.8]. Thus  $\text{Extr}(\overline{\mathcal{F}}) = \text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}$ .

2) Since the number of faces of  $\mathcal{Q}$  is finite (by [BG09, Theorem 1.10]), we may assume that  $J$  is finite. Then by [BG09, Theorem 1.10],  $\bigcap_{j \in J} \overline{\mathcal{F}}_j$  is a face of  $\mathcal{Q}$ . Then, by statement 1), we have  $\text{Extr}(\bigcap_{j \in J} \overline{\mathcal{F}}_j) = \text{Extr}(\mathcal{Q}) \cap \bigcap_{j \in J} \overline{\mathcal{F}}_j = \bigcap_{j \in J} (\text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}_j) = \bigcap_{j \in J} \text{Extr}(\overline{\mathcal{F}}_j)$ .  $\square$

## 2.2 Vectorial weight polytope

### 2.2.1 Vectorial root system

Let  $\mathbb{A}$  be a finite dimensional real vector space and denote by  $\mathbb{A}^* := \text{Hom}_{\mathbb{R}}(\mathbb{A}, \mathbb{R})$  its dual space. Let  $\Psi \subset \mathbb{A}^*$  be a reduced finite root system in  $\mathbb{A}$ , in the sense of [Bou81, VI, §1]. In particular,  $\Psi$  is a finite subset of  $\mathbb{A}^*$  generating  $\mathbb{A}^*$ . Elements of  $\Psi$  are called **roots** and to every  $\alpha \in \Psi$  is associated a unique **coroot**  $\alpha^\vee$ . We have  $\alpha(\alpha^\vee) = 2$  and the map  $\mathbb{A}^* \rightarrow \mathbb{A}^*$  defined by  $\beta \mapsto \beta - \beta(\alpha^\vee)\alpha$  stabilizes  $\Psi$ . One defines  $s_\alpha : \mathbb{A} \rightarrow \mathbb{A}$  by  $s_\alpha(x) = x - \alpha(x)\alpha^\vee$  for  $x \in \mathbb{A}$ . The set  $\Psi^\vee = \{\alpha^\vee \mid \alpha \in \Psi\} \subset \mathbb{A}$  is a root system in  $\mathbb{A}^*$  and for every  $\alpha \in \Psi$ ,  $\alpha(\Psi^\vee) \subset \mathbb{Z}$  and  $s_\alpha$  stabilizes  $\Psi^\vee$ .

Let  $\mathcal{H}_\Psi^0 = \mathcal{H}^0 = \{\alpha^{-1}(\{0\}) \mid \alpha \in \Psi\}$ . This is a locally finite hyperplane arrangement of  $\mathbb{A}$ . The elements of  $\mathcal{H}^0$  are called **walls** or **vectorial walls**. For  $x, y \in \mathbb{A}$ , we write  $x \sim_{\mathcal{H}^0} y$  if for every  $H \in \mathcal{H}^0$ , either  $(x, y \in H)$  or  $(x$  and  $y$  are strictly on the same side of  $H)$ . This is an equivalence relation on  $\mathbb{A}$ . Its classes are called the **faces** of  $(\mathbb{A}, \mathcal{H}^0)$  or simply the vectorial faces of  $\mathbb{A}$ . We denote by  $\mathcal{F}^0 = \mathcal{F}(\mathcal{H}^0)$  the set of faces of  $(\mathbb{A}, \mathcal{H}^0)$ . If  $E_1, E_2$  are two relatively open (i.e open in their support) convex subsets of  $\mathbb{A}$ , we say that  $E_1$  **dominates**  $E_2$  if  $\overline{E_1} \supset E_2$ . This equips the set of relatively open convex subsets of  $\mathbb{A}$  with an order. By restriction, this equips  $\mathcal{F}^0$  with an order called the **dominance order**.

A **vectorial chamber** (or **open vectorial chamber**) of  $(\mathbb{A}, \mathcal{H}^0)$  is a connected component of  $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}^0} H$ . The chambers are actually the faces which are not contained

in an element of  $\mathcal{H}^0$  or equivalently, the maximal faces for the dominance order. A **wall** of a vectorial chamber  $C^v$  is an element  $H$  of  $\mathcal{H}$  such that  $H \cap \overline{C^v}$  is a closed facet of  $\overline{C^v}$ .

Let  $C^v$  be a vectorial chamber of  $\Psi$ . The **associated basis** of  $\Psi$  is the set  $\Sigma_{C^v}$  of  $\alpha \in \Psi$  such that  $\alpha^{-1}(\{0\})$  is a wall of  $C^v$  and  $\alpha(C^v) > 0$ . Then by [Bou81, VI. 1.5 Théorème 2],  $\Sigma_{C^v}$  is a **basis** of  $\Psi$ , which means that  $(\alpha)_{\alpha \in \Sigma_{C^v}}$  is a basis of  $\mathbb{A}^*$  and every element of  $\Psi$  can be written as  $\sum_{\alpha \in \Sigma_{C^v}} n_\alpha \alpha$ , where  $(n_\alpha) \in (\mathbb{Z}_{\geq 0})^{\Sigma_{C^v}}$ , for some  $\epsilon \in \{-, +\}$ . We set  $\Psi_{\epsilon, C^v} = \Psi \cap \bigoplus_{\alpha \in \Sigma_{C^v}} \epsilon \mathbb{Z}_{\geq 0} \alpha$ , for  $\epsilon \in \{-, +\}$ .

The following proposition follows from [Bou81, §V 3.9 Proposition 5].

**Proposition 2.6.** *Let  $C^v$  be a vectorial chamber of  $(\mathbb{A}, \mathcal{H}^0)$ . Let  $\Sigma_{C^v}$  be the associated basis of  $\Psi$ . Let  $(\varpi_\alpha)_{\alpha \in \Sigma_{C^v}} \in \mathbb{A}^{\Sigma_{C^v}}$  be the dual basis of  $\Sigma_{C^v}$ . Then*

1.  $C^v = \bigoplus_{\alpha \in \Sigma_{C^v}} \mathbb{R}_{>0} \varpi_\alpha$ .
2. *The faces of  $C^v$  are exactly the  $\bigoplus_{\alpha \in E} \mathbb{R}_{>0} \varpi_\alpha$  such that  $E$  is a subset of  $\Sigma_{C^v}$ , where, by convention,  $\bigoplus_{\alpha \in \emptyset} \mathbb{R}_{>0} \varpi_\alpha = \{0\}$ .*

We choose a vectorial chamber  $C_f^v = C_{f, \Psi}^v$  of  $\mathbb{A}$ , that we call the **fundamental vectorial chamber**. Let  $\Sigma = \Sigma_{C_f^v}$ .

The **vectorial Weyl group** of  $\Psi$  is  $W_\Psi^v = \langle s_\alpha \mid \alpha \in \Psi \rangle \subset \text{GL}(\mathbb{A})$ . It is a finite group and  $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$  is a Coxeter system (see [BB05, 1.1] for a definition). By [Bou81, V §3],  $W_\Psi^v$  acts simply transitively on the set of vectorial chambers and  $\overline{C^v}$  is a fundamental domain for the action of  $W_\Psi^v$  on  $\mathbb{A}$ , for any vectorial chamber  $C^v$ . By [Bou81, VI 1.1 Proposition 3], there exists a  $W_\Psi^v$ -invariant scalar product  $\langle \cdot \mid \cdot \rangle$  on  $\mathbb{A}$ . We equip  $\mathbb{A}$  with this scalar product and regard  $\mathbb{A}$  as a Euclidean space. An element  $x$  of  $\mathbb{A}$  is **regular** (with respect to  $\mathcal{H}^0$ ) if it belongs to an open vectorial chamber of  $\mathbb{A}$ . Actually,  $x$  is regular if and only if  $w \mapsto w.x$  is injective on  $W_\Psi^v$ . If  $F^v \in \mathcal{F}(\mathcal{H}^0)$  is dominated by  $C_f^v$ , then its fixator in  $W_\Psi^v$  is

$$W_{F^v}^v = \langle s_\alpha \mid \alpha \in \Sigma, \alpha(F^v) = \{0\} \rangle, \quad (2.3)$$

according to [Bou81, V §3.3].

We sometimes assume that  $\Psi$  is **irreducible**. This means that  $\Psi$  cannot be written as a direct sum of roots systems, see [Bou81, VI §1.2]. When  $\Psi$  is reducible, we can write

$$\mathbb{A} = \bigoplus_{i=1}^{\ell} \mathbb{A}_i, \Psi = \bigsqcup_{i=1}^{\ell} \Psi_i \quad (2.4)$$

where  $\ell \in \mathbb{Z}_{\geq 2}$  and  $\Psi_i$  is an irreducible root system in  $\mathbb{A}_i$  for  $i \in \llbracket 1, \ell \rrbracket$ . We regard  $\Psi_i$  as a subset of  $\Psi$  by setting  $\alpha(\bigoplus_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}} \mathbb{A}_j) = \{0\}$  for  $\alpha \in \Psi_i$ . We denote by  $\mathcal{H}_i^0$  the hyperplane arrangement of  $(\mathbb{A}_i, \Psi_i)$ , for  $i \in \llbracket 1, \ell \rrbracket$ . Then every face  $F^v$  of  $\mathcal{F}^0$  can be written as  $F^v = F_1^v \times \dots \times F_\ell^v$ , where  $F_i^v \in \mathcal{F}^0(\mathcal{H}_i^0)$ , for  $i \in \llbracket 1, \ell \rrbracket$ .

**Cartan matrices** Let  $A = A(\Psi) = (\alpha(\beta^\vee))_{\alpha, \beta \in \Sigma}$ . This is the **Cartan matrix** of  $\Psi$ . Up to permutation of its coefficients, it is well-defined, independently of the choice of the basis  $\Sigma$  of  $\Psi$ , by [Bou81, VI, 1.5]. The Cartan matrix characterizes the root system up to isomorphism.

A matrix of the form  $A(\Phi)$ , for some root system  $\Phi$  is called a **Cartan matrix**.

The Cartan matrix  $A$  is called **decomposable** if there exists a decomposition  $\Sigma = \Sigma' \sqcup \Sigma''$ , with  $\Sigma'$  and  $\Sigma''$  non-empty, such that  $\alpha(\beta^\vee) = \beta(\alpha^\vee) = 0$ , for every  $(\alpha, \beta) \in \Sigma' \times \Sigma''$ . A Cartan matrix which is not decomposable is called **indecomposable**. Then  $\Psi$  is irreducible if and only if  $A$  is indecomposable, and under the notation of Eq. (2.4),  $A$  is the block diagonal matrix whose blocks are the  $A(\Psi_i)$ , for  $i \in \llbracket 1, \ell \rrbracket$ . One can use the Cartan matrices to classify finite roots systems, see [Kac94, Chapter 4] for example (where the Cartan matrices are the generalized Cartan matrices of finite type).



## 2.2.2 Vectorial weight polytope

**Definition 2.7.** We choose a point  $\mathbf{b} \in C_f^v$  and we set  $\overline{\mathcal{P}^0} = \overline{\mathcal{P}_{\mathbf{b}, \Psi}^0} = \overline{\mathcal{P}_{\mathbf{b}}^0} = \text{conv}(W_{\Psi}^v \cdot \mathbf{b})$ . We also set  $\mathcal{P}^0 = \text{Int}(\overline{\mathcal{P}^0})$ . Then  $\mathcal{P}^0$  (resp.  $\overline{\mathcal{P}^0}$ ) is the open (resp. closed) **vectorial Weyl polytope associated with  $\Psi$  and  $\mathbf{b}$** .

The closed vectorial Weyl polytope is compact. Note that by Lemma 2.8 below,  $\mathcal{P}^0$  is non-empty. This polytope naturally appears in different contexts. It is called the **weight polytope** and it appears for example in [Vin91] and [BJK23]. Its intersection with  $\overline{C_f^v}$  is studied in [BGH23].

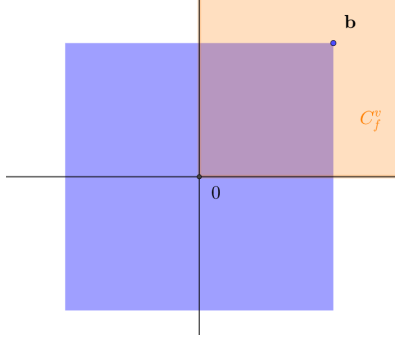


Figure 2.1: Polytope  $\mathcal{P}^0$  for the root system  $A_1 \times A_1$

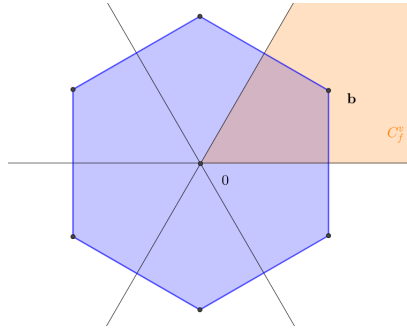


Figure 2.2: Polytope  $\mathcal{P}^0$  for the root system  $A_2$

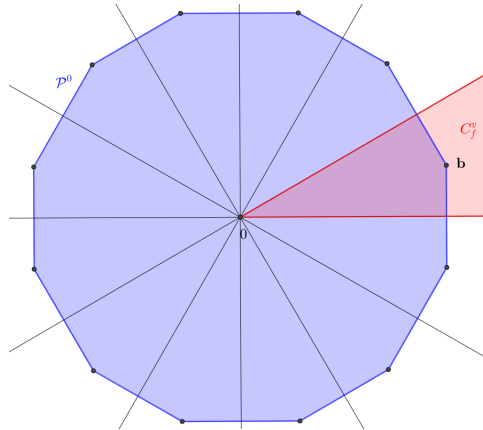


Figure 2.3: Polytope  $\mathcal{P}^0$  for the root system  $G_2$

**Lemma 2.8.** 1. Assume that  $\Psi$  is irreducible. Then for every  $x \in \mathbb{A} \setminus \{0\}$ ,  $\text{conv}(W_{\Psi}^v \cdot x)$  has non-empty interior.

2.  $\mathcal{P}^0$  is non-empty.

*Proof.* The support of  $W_\Psi^v \cdot x$  is  $W_\Psi^v$ -invariant and thus by [Bou81, VI §1.2 Corollaire], this support is  $\mathbb{A}$ , which proves 1).

By 1), if  $\Psi$  is irreducible, then  $\mathcal{P}^0$  is non-empty. Assume now that  $\Psi$  is reducible and keep the notation of Eq. (2.4). Write  $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_\ell)$ . Then  $\mathcal{P}_{\mathbf{b}, \Psi}^0 = \mathcal{P}_{\mathbf{b}_1, \Psi_1}^0 \times \dots \times \mathcal{P}_{\mathbf{b}_\ell, \Psi_\ell}^0$  and thus  $\mathcal{P}_{\mathbf{b}, \Psi}^0$  is non-empty.  $\square$

## 2.3 Description of the faces of the vectorial weight polytope

### 2.3.1 Faces of $\mathcal{P}^0$ meeting $C_f^v$

The main goal of this subsection is to prove Lemma 2.15, which describes the faces the of  $\mathcal{P}^0$  meeting  $C_f^v$ .

For  $J \subset \Sigma$ , one defines the **parabolic subgroup**  $W_J^v$  by  $W_J^v = W_{J, \Psi}^v = \langle s_\alpha \mid \alpha \in J \rangle \subset W_\Psi^v$ . Set  $Q_+^\vee = Q_{\mathbb{R}, +}^\vee = \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee$ ,  $Q_{J, +}^\vee = \bigoplus_{\alpha \in J} \mathbb{R}_{\geq 0} \alpha^\vee$ .

We equip  $W_\Psi^v$  with the Bruhat order (see [BB05, Chapter 2] for a definition).

**Lemma 2.9.** *Let  $w \in W_\Psi^v$  and  $\alpha \in \Sigma$ . Then  $ws_\alpha < w$  for the Bruhat order on  $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$  if and only if  $w \cdot \alpha \in \Psi_-$ .*

*Proof.* This follows from [Bou81, VI, §1 Proposition 17 (ii)] and [BB05, Corollary 1.4.4].  $\square$

**Lemma 2.10.** *Let  $w \in W_\Psi^v$  and  $J \subset \Sigma$ . The following assertions are equivalent:*

- (i) we have  $\mathbf{b} - w \cdot \mathbf{b} \in \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$ ,
- (ii) we have  $\mathbf{b} - w \cdot \mathbf{b} \in Q_{J, +}^\vee$ ,
- (iii) we have  $w \in W_J^v$ .

*Proof.* By [Bou81, Chapitre VI, §1 Proposition 18], if  $w \in W_J^v$ , then  $\mathbf{b} - w \cdot \mathbf{b} \in Q_{J, +}^\vee \subset \mathbf{b} - w \cdot \mathbf{b} \in \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$ . Hence (iii) implies (ii), which itself implies (i).

It remains to prove that (i) implies (iii). To do so, take  $w \in W_\Psi^v$  and assume that  $w \in W_\Psi^v \setminus W_J^v$ . Write  $w = s_{\alpha_k} \dots s_{\alpha_1}$ , with  $k = \ell_\Psi(w)$ ,  $\alpha_1, \dots, \alpha_k \in \Sigma$ , where  $\ell_\Psi$  denotes the length of the Coxeter system  $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$ . Then there exists  $j \in \llbracket 1, k \rrbracket$  such that  $\alpha_j \notin J$ . Take  $j$  minimal with this property. Then  $s_{\alpha_j} \cdot s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} = s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} - \alpha_j (s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b}) \cdot \alpha_j^\vee$ . As  $\mathbf{b}$  is regular, Lemma 2.9 implies  $s_{\alpha_j} \cdot s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} \in \mathbf{b} + \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee - \mathbb{R}_{>0} \alpha_j^\vee$ . By an induction using Lemma 2.9, we deduce

$$w \cdot \mathbf{b} \in \mathbf{b} + \left( \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee - \mathbb{R}_{>0} \alpha_j^\vee \right) - \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee.$$

In particular,  $w \cdot \mathbf{b} - \mathbf{b} \notin \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$ , which completes the proof of the lemma.  $\square$

**Lemma 2.11.** *Let  $\eta = \frac{1}{|\Sigma|} \min\{\alpha(\mathbf{b}) \mid \alpha \in \Sigma\}$ . Then for every  $(t_\alpha) \in ]0, \eta[^\Sigma$ ,  $\mathbf{b} - \sum_{\alpha \in \Sigma} t_\alpha \alpha^\vee \in \mathcal{P}^0$ .*

*Proof.* Let  $m = \min\{\alpha(\mathbf{b}) \mid \alpha \in \Sigma\}$ . Let  $\alpha \in \Sigma$ . Then  $\mathbf{b} - m\alpha^\vee \in [\mathbf{b}, s_\alpha \cdot \mathbf{b}] \subset \mathcal{P}^0$ . Let  $(t_\alpha) \in ]0, \eta[^\Sigma$ . For  $\alpha \in \Sigma$ , set  $t'_\alpha = \frac{1}{m} t_\alpha \in ]0, \frac{1}{|\Sigma|}[$ . Set  $t'_0 = 1 - \sum_{\alpha \in \Sigma} t'_\alpha \in ]0, 1[$ . Then

$$t'_0 \mathbf{b} + \sum_{\alpha \in \Sigma} t'_\alpha (\mathbf{b} - m\alpha^\vee) = \mathbf{b} - \sum_{\alpha \in \Sigma} t'_\alpha m\alpha^\vee = \mathbf{b} - \sum_{\alpha \in \Sigma} t_\alpha \alpha^\vee \in \overline{\mathcal{P}^0}.$$

As  $(\alpha^\vee)_{\alpha \in \Sigma}$  is a basis of  $\mathbb{A}$ , the map  $(\mathbb{R}_{>0})^\Sigma \rightarrow \mathbb{A}$  defined by  $(n_\alpha)_{\alpha \in \Sigma} \mapsto \sum_{\alpha \in \Sigma} n_\alpha \alpha^\vee$  is open, and the lemma follows.  $\square$

Let  $\mathcal{C}_\Psi = \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{>0} \alpha^\vee$ . By [Bou81, Chapitre VI, §1 Proposition 18], we have:

**Lemma 2.12.** *We have  $\mathcal{P}^0 \subset \mathbf{b} - \mathcal{C}_\Psi$ .*

For  $\beta \in \Sigma$ , set  $W_\beta^v = W_{\Sigma \setminus \{\beta\}, \Psi}^v$ . Then  $W_\beta^v$  is the fixator of the one dimensional face  $\overline{\mathcal{C}_f^v} \cap \bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$ , according to (2.3).

**Lemma 2.13.** *Let  $w \in W_\Psi^v$ . Then  $[w.\mathbf{b}, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$  if and only if  $w \in \bigcup_{\beta \in \Sigma} W_\beta^v$ .*

*Proof.* Let  $w \in W_\Psi^v \setminus \bigcup_{\beta \in \Sigma} W_\beta^v$ . Since  $w \in W_\Psi^v$ , Lemma 2.10 shows that  $u := w.\mathbf{b} - \mathbf{b} \in \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee$ . By using again Lemma 2.10, since  $w \notin \bigcup_{\beta \in \Sigma} W_\beta^v$ , we get that  $u \in \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{< 0} \alpha^\vee$ . By Lemma 2.11,  $\mathbf{b} + tu \in \mathcal{P}^0$  for  $t \in [0, 1]$  small enough. We have  $w.\mathbf{b} = \mathbf{b} + u$ . As  $\mathcal{P}^0$  is convex, either  $] \mathbf{b}, \mathbf{b} + u[ \subset \text{Fr}(\mathcal{P}^0)$  or  $] \mathbf{b}, \mathbf{b} + u[ \subset \mathcal{P}^0$  and thus  $] \mathbf{b}, \mathbf{b} + u[ = ] \mathbf{b}, w.\mathbf{b}[ \subset \mathcal{P}^0$ .

Let  $w \in \bigcup_{\beta \in \Sigma} W_\beta^v$ . Then by Lemma 2.10 and Lemma 2.12,  $[w.\mathbf{b}, \mathbf{b}] \subset \mathbb{A} \setminus (\mathbf{b} - \mathcal{C}_\Psi) \subset \mathbb{A} \setminus \mathcal{P}^0$ , which proves that  $[w.\mathbf{b}, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$ .  $\square$

For  $J \subset \Sigma$ , set  $\mathbb{A}_J = \bigcap_{\alpha \in J} \ker(\alpha)$ .

**Lemma 2.14.** *Let  $i, j \in I$ . Then  $\langle \varpi_i \mid \alpha_j^\vee \rangle = \frac{1}{2} \delta_{i,j} |\alpha_i^\vee|^2$ . In particular, for  $J \subset \Sigma$ , we have  $\mathbb{A}_J^\perp = \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$ .*

*Proof.* We have  $\langle \varpi_i \mid \alpha_i^\vee \rangle = \langle s_{\alpha_i}.\varpi_i \mid s_{\alpha_i}.\alpha_i^\vee \rangle = \langle \varpi_i - \alpha_i^\vee \mid -\alpha_i^\vee \rangle$ , so  $2\langle \varpi_i \mid \alpha_i^\vee \rangle = |\alpha_i^\vee|^2$ . If  $j \neq i$ , we have  $\langle \varpi_i \mid \alpha_j^\vee \rangle = \langle s_{\alpha_j}.\varpi_i \mid s_{\alpha_j}.\alpha_j^\vee \rangle = \langle \varpi_i \mid -\alpha_j^\vee \rangle$ , and hence  $\langle \varpi_i \mid \alpha_j^\vee \rangle = 0$ .  $\square$

Let  $\beta \in \Sigma$ . Set  $\mathcal{F}_\beta = \text{Int}_r(\text{conv}(W_\beta^v.\mathbf{b}))$ . Denote by  $H_\beta$  the affine hyperplane orthogonal to  $\bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$  and containing  $\mathbf{b}$ .

**Lemma 2.15.** *1. The facets of  $\mathcal{P}^0$  are the  $w.\mathcal{F}_\beta$  for  $w \in W_\Psi^v$  and  $\beta \in \Sigma$ .*

*2. For  $\beta \in \Sigma$ , the hyperplane containing  $\mathcal{F}_\beta$  is  $H_\beta = \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$ .*

*Proof.* As  $\mathcal{P}^0$  is  $W_\Psi^v$ -invariant,  $\text{Fr}(\mathcal{P}^0)$  is  $W_\Psi^v$ -invariant. Let  $\mathcal{F}$  be a facet of  $\mathcal{P}^0$ . Let  $E$  be the set of extreme points of  $\overline{\mathcal{F}}$ . By Lemma 2.4 and Lemma 2.5,  $E \subset W_\Psi^v.\mathbf{b}$  and thus replacing  $\mathcal{F}$  by  $w.\mathcal{F}$  for some  $w \in W_\Psi^v$  if needed, we may assume  $\mathbf{b} \in \overline{\mathcal{F}}$ .

Let  $x \in \overline{\mathcal{F}}$ . Then by Lemma 2.12, we can write  $x = \mathbf{b} - \sum_{\alpha \in \Sigma} n_\alpha \alpha^\vee$ , where  $(n_\alpha) \in (\mathbb{R}_{\geq 0})^\Sigma$ . Set

$$\text{supp}(x) = \{\alpha \in \Sigma \mid n_\alpha \neq 0\} \subset \Sigma.$$

Then for  $x, y \in \overline{\mathcal{F}}$ , we have  $\frac{1}{2}(x + y) \in \overline{\mathcal{F}}$  and  $\text{supp}(\frac{1}{2}(x + y)) = \text{supp}(x) \cup \text{supp}(y)$ . Therefore there exists  $x_0 \in \overline{\mathcal{F}}$  which has the largest support, i.e.,  $\text{supp}(x) \subseteq \text{supp}(x_0)$  for every  $x \in \overline{\mathcal{F}}$ .

We have  $[x_0, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$ . By Lemma 2.11 we deduce the existence of  $\beta \in \Sigma$  such that  $t\mathbf{b} + (1-t)x_0 \in \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$  for  $t \in [0, 1]$  large enough. But then  $x_0 \in \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$ . Therefore  $\text{supp}(x_0) \subset \Sigma \setminus \{\beta\}$  and  $\text{supp}(x) \subset \Sigma \setminus \{\beta\}$  for every  $x \in \mathcal{F}$ . In particular, for every extreme point  $Q$  of  $\overline{\mathcal{F}}$ , we have  $\text{supp}(Q) \subset \Sigma \setminus \{\beta\}$ . By Lemma 2.10, every extreme point of  $\overline{\mathcal{F}}$  is contained in  $W_\beta^v.\mathbf{b}$ . By Krein-Milman theorem we deduce  $\overline{\mathcal{F}} \subset \text{conv}(w.\mathbf{b} \mid w \in W_\beta^v) = \overline{\mathcal{F}_\beta}$ .

As  $\overline{\mathcal{P}^0} \cap H_\beta$  is a  $W_\beta^v$ -stable set containing  $\mathbf{b}$ ,  $\overline{\mathcal{P}^0} \cap H_\beta \supset \mathcal{F}_\beta$ . Moreover,  $\overline{\mathcal{P}^0} \cap H_\beta$  contains  $\{\mathbf{b}\} \cup \{\mathbf{b} - \alpha(\mathbf{b})\alpha^\vee \mid \alpha \in \Sigma \setminus \{\beta\}\}$ . As the affine subspace spanned by this set is  $H_\beta$ , and as  $\overline{\mathcal{F}_\beta} \subset \text{Fr}(\mathcal{P}^0)$  (by Lemma 2.13),  $\overline{\mathcal{P}^0} \cap H_\beta$  is a closed facet of  $\overline{\mathcal{P}^0}$ . By what we proved above we deduce that

$$\overline{\mathcal{P}^0} \cap H_\beta = \text{conv}(W_\beta^v.\mathbf{b}).$$

Thus, Statement (1) follows. Moreover, Statement (2) follows from Lemma 2.14.  $\square$

### 2.3.2 Arbitrary faces of $\mathcal{P}^0$

**Definition 2.16.** Let  $C^v$  be a vectorial chamber of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$ . Since  $\overline{C_f^v}$  is a fundamental domain for the action of  $W_\Psi^v$  on  $\mathbb{A}$ , there exists  $w \in W_\Psi^v$  be such that  $C^v = w.C_f^v$ . Then, we write:

$$\mathbf{b}_{C^v} = w.\mathbf{b}.$$

Note that  $\mathbf{b}_{C^v}$  does not depend on the choice of  $w$ , since any other  $w'$  as above has the form  $w' = ww_0$ , where  $w_0$  stabilizes  $C_f^v$ .

If  $F^v \in \mathcal{F}^0$ , we choose a vectorial chamber  $C^v$  dominating  $F^v$ , and we set:

$$\mathcal{F}_{F^v} = \text{Int}_r(\text{conv}(W_{F^v}^v.\mathbf{b}_{C^v})).$$

This does not depend on the choice of  $C^v$ , since  $W_{F^v}^v$  acts transitively on the set of vectorial chambers dominating  $F^v$ , according to [Bou81, Chapitre V, §4 Proposition 5].

The next two lemmas will be given without proof. They can be settled in a very similar way to their affine counterparts, which will be settled later, in Lemmas 3.17 and 3.18).

**Lemma 2.17.** Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  and let  $E$  be the *star* of  $F^v$ , i.e., the union of the faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  dominating  $F^v$ . Let  $C^v$  be a chamber of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  dominating  $F^v$  and  $\Sigma_{C^v}$  be the basis of  $\Psi$  associated with  $C^v$ . Let  $E'$  be the union of the faces of  $C^v$  dominating  $F^v$ . Let  $\Sigma_{F^v} = \{\alpha \in \Sigma_{C^v} \mid \alpha(F^v) = \{0\}\}$ . For  $\alpha \in \Sigma_{C^v}$ , set  $\mathring{D}_\alpha = \alpha^{-1}(\mathbb{R}_{>0})$ . Then

$$E = W_{F^v}^v.E' = \bigcap_{w \in W_{F^v}^v, \alpha \in \Sigma_{C^v} \setminus \Sigma_{F^v}} w.\mathring{D}_\alpha.$$

In particular,  $E$  is open and convex.

**Lemma 2.18.** Let  $F^v, F_1^v$  be faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  such that  $F_1^v$  dominates  $F^v$ . Let  $x \in F_1^v$  and  $y = \frac{1}{|W_{F^v}^v|} \sum_{w \in W_{F^v}^v} w.x$ . Then  $y \in F^v$  and  $y$  is the orthogonal projection of  $x$  on  $\text{vect}(F^v)$ .

**Lemma 2.19.** Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  which is not a chamber. Then  $\Phi_{F^v} = \{\alpha \in \Phi \mid \alpha(F^v) = \{0\}\}$  is a root system in  $\text{vect}(F^v)^\perp$ . Its Weyl group is  $W_{F^v} = \langle s_\alpha \mid \alpha \in \Phi_{F^v} \rangle$ .

*Proof.* Up to changing the choice of  $C_f^v$  if needed, we may assume that  $C_f^v$  dominates  $F^v$ . Let  $\Sigma_{F^v} = \Sigma \cap \Phi_{F^v}$ . We have  $\text{vect}(F^v) \subset \bigcap_{\alpha \in \Sigma_{F^v}} \ker(\alpha)$ . As  $\Sigma$  is a basis of  $\mathbb{A}^*$  and using a simple dimension argument based on Proposition 2.6, we deduce  $\text{vect}(F^v) = \bigcap_{\alpha \in \Sigma_{F^v}} \ker(\alpha)$ . By Lemma 2.14,  $\alpha^\vee \in \text{vect}(F^v)^\perp$  for every  $\alpha \in \Sigma_{F^v}$ . Therefore  $\Phi_{F^v}|_{\text{vect}(F^v)^\perp}$  satisfies [Bou81, VI §1.1 Définition 1].  $\square$

We equip  $\mathcal{F}^0$  with the dominance order  $\leq$ : for  $F_1^v, F_2^v \in \mathcal{F}^0$ , we have  $F_1^v \leq F_2^v$  if and only if  $F_1^v \subset \overline{F_2^v}$ .

A less precise version of the following proposition already appears in [Vin91, Proposition 3.2].

**Proposition 2.20.** (see Figure 2.4)

1. The assignment  $f : F^v \mapsto \mathcal{F}_{F^v}$  defines an isomorphism of ordered spaces between  $(\mathcal{F}^0, \geq)$  and the set of faces of  $(\mathcal{P}^0, \subset)$ .
2. Let  $(F_i^v)_{i \in [1, k]}$  be a finite family of faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$ . Then if  $\sum_{i=1}^k F_i^v \in \mathcal{F}^0$  (or equivalently if there exists  $F^v \in \mathcal{F}^0$  dominating  $\sum_{i=1}^k F_i^v$ ), we have  $\bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} = \overline{\mathcal{F}_{\sum_{i=1}^k F_i^v}}$ . Otherwise, we have  $\bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} = \emptyset$ .

3. For  $F^v \in \mathcal{F}^0$ , the set of extreme points of  $\overline{\mathcal{F}_{F^v}}$  is  $W_{F^v}^v \cdot \mathbf{b}_{C^v}$ , for any chamber  $C^v$  dominating  $F^v$ . The direction of  $\text{supp}(\mathcal{F}_{F^v})$  is  $(F^v)^\perp$  and in particular,  $\dim(\text{vect}(\mathcal{F}_{F^v})) = \text{codim}_{\mathbb{A}}(\text{vect}(F^v))$ .
4. Let  $\mathcal{F}$  be a face of  $\mathcal{P}^0$  and let  $y$  be the barycenter of the extreme points of  $\overline{\mathcal{F}}$ . Then  $\mathcal{F} = \mathcal{F}_{F^v}$ , where  $F^v$  is the unique face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  containing  $y$ , the point  $y$  is the orthogonal projection of  $\mathcal{F}$  on  $F^v$  and  $\overline{\mathcal{F}_{F^v}} \cap F^v = \{y\}$ . Moreover,  $\overline{\mathcal{F}_{F^v}}$  is contained in the star of  $F^v$ .

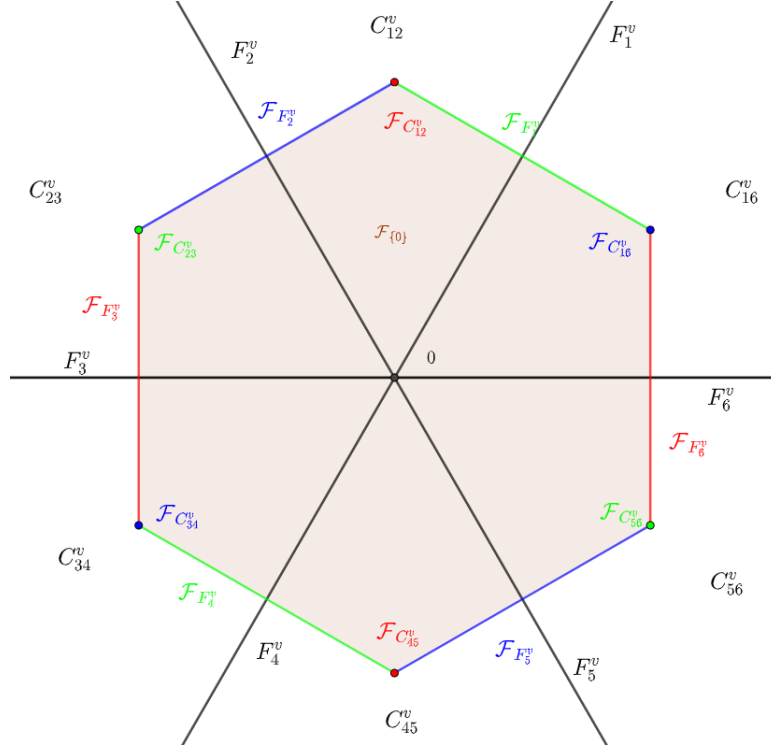


Figure 2.4: Illustration of Proposition 2.20 in the case of  $A_2$

*Proof. Step 1:* We first prove that the facets of  $\mathcal{P}^0$  are the  $\mathcal{F}_{F^v}$ , where the  $F^v$  are the one-dimensional faces of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ . To do so, let  $\beta \in \Sigma$ . Set  $F_{\beta}^v = \beta^{-1}(\mathbb{R}_{>0}) \cap \bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$ . Then the fixator of  $F_{\beta}^v$  in  $W_{\Psi}^v$  is  $W_{\beta}^v$  and thus  $\mathcal{F}_{F_{\beta}^v} = \mathcal{F}_{\beta}$ . If  $w \in W_{\Psi}^v$ , then

$$w \cdot \overline{\mathcal{F}_{F_{\beta}^v}} = \text{conv}(wW_{F_{\beta}^v}^v \cdot \mathbf{b}) = \text{conv}(wW_{F_{\beta}^v}^v w^{-1} \cdot w \cdot \mathbf{b}) = \text{conv}(W_{w \cdot F_{\beta}^v}^v w \cdot \mathbf{b}) = \overline{\mathcal{F}_{w \cdot F_{\beta}^v}}.$$

By Lemma 2.15, the facets of  $\mathcal{P}^0$  are the  $w \cdot \mathcal{F}_{\beta}$  such that  $w \in W_{\Psi}^v$  and  $\beta \in \Sigma$  so, by Proposition 2.6, the facets of  $\mathcal{P}^0$  are the  $\mathcal{F}_{F^v}$ , where the  $F^v$  are the one-dimensional faces of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ .

*Step 2:* We now prove the second assertion of 2): if  $(F_i^v)_{i \in [1, k]}$  is a family of faces of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  such that  $\overline{\mathcal{F}} := \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} \neq \emptyset$ , then  $\sum_{i=1}^k F_i^v \in \mathcal{F}^0$ . Indeed, take  $x \in \overline{\mathcal{F}}$ . Let  $F_x^v$  be the face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  containing  $x$ . Then by Lemma 2.17,  $F_x^v$  dominates  $\bigcup_{i=1}^k F_i^v$ . Therefore any chamber dominating  $F_x^v$  dominates  $\bigcup_{i=1}^k F_i^v$ .

*Step 3:* We now prove that the map  $f : F^v \mapsto \mathcal{F}_{F^v}$  is well-defined. To do so, let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  and  $F_1^v, \dots, F_k^v$  be the one dimensional faces of  $F^v$ . Then  $\overline{\mathcal{F}} := \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$  is a closed face of  $\mathcal{P}^0$  by [BG09, Theorem 1.10 and Proposition 1.8] and it is non-empty because it contains  $\overline{\mathcal{F}_{F^v}}$ . In particular, by Step 2, there exists a vectorial

chamber  $C^v$  dominating  $\bigcup_{i=1}^k F_i^v$ , and by Proposition 2.6, we get  $F^v = \sum_{i=1}^k F_i^v$ . We then have:

$$\begin{aligned}
\text{Extr}(\overline{\mathcal{F}}) &= \bigcap_{i=1}^k \text{Extr}(\overline{\mathcal{F}_{F_i^v}}) && \text{by Lemma 2.5} \\
&= \bigcap_{i=1}^k W_{F_i^v}^v \cdot \mathbf{b}_{C^v} && \text{by Lemma 2.4} \\
&= \left( \bigcap_{i=1}^k W_{F_i^v}^v \right) \cdot \mathbf{b}_{C^v} && \text{since } \mathbf{b}_{C^v} \text{ is regular} \\
&= W_{F^v}^v \cdot \mathbf{b}_{C^v}.
\end{aligned}$$

By Krein-Milman Theorem, we deduce  $\overline{\mathcal{F}} = \overline{\mathcal{F}_{F^v}}$ . Hence  $\mathcal{F}_{F^v} = \text{Int}_r(\overline{\mathcal{F}_{F^v}})$  is a face of  $\mathcal{P}^0$ .

*Step 4: We now prove that the map  $f : F^v \mapsto \mathcal{F}_{F^v}$  of statement 1) is surjective.* Indeed, let  $\mathcal{F}$  be a proper face of  $\mathcal{P}^0$ . Then by [BG09, Theorem 1.10], there exist  $k \in \mathbb{Z}_{\geq 1}$ , one dimensional vectorial faces  $F_1^v, \dots, F_k^v$  of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  such that  $\overline{\mathcal{F}} = \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$  and there exists a vectorial chamber  $C^v$  dominating  $\bigcup_{i=1}^k F_i^v$ . By the same computations as in Step 3, we have  $\overline{\mathcal{F}} = \overline{\mathcal{F}_{F^v}}$  with  $F^v := \sum_{i=1}^k F_i^v$ . Hence  $\mathcal{F} = \text{Int}_r(\overline{\mathcal{F}}) = \text{Int}_r(\overline{\mathcal{F}_{F^v}}) = \mathcal{F}_{F^v}$ .

*Step 5: We prove that  $f$  is non-decreasing.* Let  $F_1^v, F_2^v \in \mathcal{F}^0$  be such that  $F_1^v \subset \overline{F_2^v}$ . Then as  $W_\Psi^v$  acts by homeomorphisms on  $\mathbb{A}$ , we have  $W_{F_1^v}^v \supset W_{F_2^v}^v$  and any chamber dominating  $F_2^v$  dominates  $F_1^v$ . Therefore  $\overline{\mathcal{F}_{F_1^v}} \supset \overline{\mathcal{F}_{F_2^v}}$ .

*Step 6: We prove that, if  $F_1^v, F_2^v \in \mathcal{F}^0$  are such that  $\overline{\mathcal{F}_{F_1^v}} \supset \overline{\mathcal{F}_{F_2^v}}$ , then  $F_2^v$  dominates  $F_1^v$ .* Indeed, by Step 2, there exists a vectorial chamber  $C^v$  dominating  $F_1^v$  and  $F_2^v$ . By Lemma 2.5, we have  $\text{Extr}(\overline{\mathcal{F}_{F_1^v}}) = W_{F_1^v}^v \cdot \mathbf{b}_{C^v} \supset \text{Extr}(\overline{\mathcal{F}_{F_2^v}}) = W_{F_2^v}^v \cdot \mathbf{b}_{C^v}$ . As  $\mathbf{b}_{C^v}$  is regular, we deduce  $W_{F_1^v}^v \supset W_{F_2^v}^v$ . This implies  $F_1^v \subset \overline{F_2^v}$  by [Bou81, V §3.3 Proposition 1], Proposition 2.6 and [BB05, Proposition 2.4.1].

*Step 7: We prove assertion 1).* We have seen in Steps 4, 5 and 6 that  $f$  is surjective and non-decreasing, and that if  $f(F_1^v) \supset f(F_2^v)$ , then  $F_2^v$  dominates  $F_1^v$ . This implies that  $f$  is an isomorphism of ordered sets.

*Step 8: We finish the proof of 2).* Let  $(F_i^v)_{i \in [1, k]}$  be a family of faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  such that  $F^v := \sum_{i=1}^k F_i^v \in \mathcal{F}^0$ . Exactly the same arguments as in Step 3 show that  $\overline{\mathcal{F}_{F^v}} = \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$ .

*Step 9: We prove 3) and 4).* Indeed, let now  $\mathcal{F}$  be a face of  $\mathcal{P}^0$  and  $F^v \in \mathcal{F}^0$  be such that  $\mathcal{F} = \mathcal{F}_{F^v}$ . Then we proved in Lemma 2.4 that  $\text{Extr}(\overline{\mathcal{F}}) = W_{F^v}^v \cdot \mathbf{b}_{C^v}$ , where  $C^v$  is a vectorial chamber dominating  $F^v$ . Let  $y = \frac{1}{|W_{F^v}^v|} \sum_{w \in W_{F^v}^v} w \cdot \mathbf{b}_{C^v}$ . Then  $y$  is the barycenter of  $\text{Extr}(\overline{\mathcal{F}_{F^v}})$ . By Lemma 2.18, it belongs to  $F^v$ .

Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  and  $C^v$  be a chamber dominating  $F^v$ . Let  $x$  be the orthogonal projection of  $\mathbf{b}_{C^v}$  on  $\text{vect}(F^v)$ . Then  $x \in F^v$  by Lemma 2.18. We have  $W_{F^v}^v \cdot \mathbf{b}_{C^v} = x + W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$ . Therefore the direction of  $\text{supp}(\text{conv}(W_{F^v}^v \cdot \mathbf{b}_{C^v}))$  is the vector space spanned by  $W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$ . As  $\mathbf{b}_{C^v}$  is regular,  $|W_{F^v}^v \cdot \mathbf{b}_{C^v}| = |W_{F^v}^v| = |W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)|$ . Therefore  $\mathbf{b}_{C^v} - x$  is a regular element of  $(F^v)^\perp$  (for the root system  $\Phi_{F^v}$  defined in Lemma 2.19). Using Lemma 2.8, we deduce that  $W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$  spans  $(F^v)^\perp$ . Finally, the fact that  $\overline{\mathcal{F}_{F^v}}$  is contained in the star of  $F^v$  is a direct consequence of Lemma 2.17.  $\square$

**Corollary 2.21.** 1. Let  $F^v \in \mathcal{F}^0$ . Then  $\mathcal{F}_{F^v} = \overline{\mathcal{F}_{F^v}} \setminus \bigcup_{F_2^v \in \mathcal{F}^0, \overline{F_2^v} \supsetneq \overline{F^v}} \overline{\mathcal{F}_{F_2^v}}$ .

2. Let  $x \in \overline{\mathcal{P}^0}$ . Then there exists a unique face  $F_1^v$  of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  such that  $x \in \mathcal{F}_{F_1^v}$ .  
Moreover,  $\{F^v \in \mathcal{F}^0 \mid x \in \overline{\mathcal{F}_{F^v}}\} = \{F^v \in \mathcal{F}^0 \mid F^v \subset \overline{F_1^v}\}$ .

*Proof.* 1) Let  $F^v \in \mathcal{F}^0$ . Let  $\mathcal{F}_2$  be a face of  $\mathcal{F}_{F^v}$ . By [BG09, Proposition 1.8],  $\mathcal{F}_2$  is a face of  $\mathcal{P}^0$  and thus by Proposition 2.20, there exists  $F_2^v \in \mathcal{F}^0$  such that  $\mathcal{F}_2 = \mathcal{F}_{F_2^v}$  and  $F_2^v$  dominates  $F^v$ . We then conclude with [BG09, Corollary 1.7].

2) Let  $x \in \overline{\mathcal{P}^0}$ . By [BG09, Theorem 1.10 d)], there exists a unique face  $\mathcal{F}$  of  $\mathcal{P}^0$  containing  $x$  and  $\overline{\mathcal{F}}$  is the minimal closed face containing  $x$ . By Proposition 2.20, we can write  $\mathcal{F} = \mathcal{F}_{F_1^v}$  for some face  $F_1^v$  of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  and  $E = \{F^v \in \mathcal{F}^0 \mid x \in \overline{\mathcal{F}_{F^v}}\} = \{F_2^v \in \mathcal{F}^0 \mid \overline{F_2^v} \subset \overline{F_1^v}\}$ .  $\square$

**Corollary 2.22.** Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ . Let  $\pi = \pi_{\overline{\mathcal{P}^0}}$  and  $x \in \mathcal{F}_{F^v}$ . Then  $\pi^{-1}(\{x\}) = x + \overline{F^v}$ .

*Proof.* Let  $G^v$  be a one dimensional face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ . Then by Proposition 2.20, the direction of  $\text{supp}(\mathcal{F}_{G^v})$  is  $(G^v)^\perp$ . Hence we have  $\delta_{G^v} = \pm \overline{G^v}$ , with the notation of subsection 2.1. By Proposition 2.20,  $G^v \cap \mathcal{F}_{G^v} = \{z\}$ , for some  $z \in G^v$ . Therefore  $z - G^v$  contains 0 and thus it meets  $\mathcal{P}^0$ , which proves that  $\delta_{G^v} = \overline{G^v}$ .

Now let  $k$  be the dimension of the direction of  $F^v$  and let  $F_1^v, \dots, F_k^v$  be the one-dimensional faces of  $F^v$ , so that  $F^v = \sum_{i=1}^k F_i^v$ . Then by Corollary 2.21 2), the set of closed facets containing  $x$  is the set of closed facets  $\overline{\mathcal{F}_{G^v}}$  such that  $\overline{G^v} \subset \overline{F^v}$ : this is  $\{F_i^v \mid i \in [1, k]\}$ . By Proposition 2.3,  $\pi^{-1}(\{x\}) = x + \sum_{i=1}^k \overline{F_i^v} = x + \overline{F^v}$ .  $\square$

**Lemma 2.23.** Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  and  $\alpha \in \Phi$  be such that  $\alpha(F^v) > 0$ . Then  $\alpha(\overline{\mathcal{F}_{F^v}}) > 0$ .

*Proof.* Let  $x \in \mathcal{F}_{F^v}$  and  $F_1^v \in \mathcal{F}^0$  be such that  $x \in F_1^v$ . Then by Proposition 2.20 4),  $\overline{F_1^v} \supset F^v$ . As  $F_1^v$  is a face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ , we have  $\alpha(F_1^v) \in \{\mathbb{R}_{<0}, \{0\}, \mathbb{R}_{>0}\}$ , thus  $\alpha(\overline{F_1^v}) \in \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$ . As  $\alpha(F^v) = \mathbb{R}_{>0}$ , we have  $\alpha(\overline{F_1^v}) = \mathbb{R}_{\geq 0}$  and  $\alpha(F_1^v) = \mathbb{R}_{>0}$ , which proves the lemma.  $\square$

## 2.4 Projections restricted to “parabolic” affine subspaces

In this subsection, we study the restriction of the orthogonal projection  $\pi_{\overline{\mathcal{P}^0}}$  to certain affine subspaces parallel to the faces of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ . We prove that under some conditions, subsets of these subspaces are stabilized by the projection (see Proposition 2.33). The proof is a bit intricate, and one of the main tools consists in introducing a suitable auxiliary set  $\tilde{\mathcal{P}}$  contained in the affine subspaces we want to study and such that the orthogonal projections on  $\overline{\mathcal{P}^0}$  and on  $\tilde{\mathcal{P}}$  coincide on large subsets of those affine subspaces.

For  $F^v$  a one dimensional face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ , we denote by  $H_{F^v}$  the hyperplane of  $\mathbb{A}$  containing  $\overline{\mathcal{F}_{F^v}}$  and by  $\mathcal{D}_{F^v}$  the half-space delimited by  $H_{F^v}$  and containing  $\mathcal{P}^0$ . We write  $\pi$  instead of  $\pi_{\overline{\mathcal{P}^0}}$ . By Proposition 2.20 (3), if  $z \in \mathcal{F}_{F^v}$ , then we have:

$$\mathcal{D}_{F^v} = \{y \in \mathbb{A} \mid \langle y - z \mid x \rangle \leq 0, \forall x \in F^v\}. \quad (2.5)$$

**Lemma 2.24.** 1. Let  $x \in \mathbb{A}$  and  $F^v \in \mathcal{F}^0$  be such that  $x \in F^v$ . Then  $\pi(x) \in F^v$  and for every  $\tilde{F}^v \in \mathcal{F}^0$  such that  $\pi(x) \in \overline{\mathcal{F}_{\tilde{F}^v}}$ , we have  $\tilde{F}^v \subset \overline{F^v}$ .

2. Let  $F^v \in \mathcal{F}^0$ . Then  $\pi^{-1}(F^v) \subset F^v$ .

*Proof.* 1) Let  $F_1^v \in \mathcal{F}^0$  be such that  $\pi(x) \in \mathcal{F}_{F_1^v}$ . By Corollary 2.22,  $x - \pi(x) \in \overline{F_1^v}$ . Let  $F_2^v \in \mathcal{F}^0$  be such that  $\pi(x) \in F_2^v$ . By Proposition 2.20, we have  $\overline{F_2^v} \supset F_1^v$ . Therefore  $x \in \overline{F_1^v} + F_2^v \subset \overline{F_2^v} + F_2^v \subset F_2^v$ . As the vectorial faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  form a partition of  $\mathbb{A}$ , we deduce  $F_2^v = F^v$ , which proves that  $\pi(x) \in F^v$ . Moreover, by Corollary 2.21, for every  $\tilde{F}^v$  such that  $\pi(x) \in \overline{\mathcal{F}_{\tilde{F}^v}}$ , we have  $\tilde{F}^v \subset \overline{F_1^v} \subset \overline{F_2^v} = \overline{F^v}$ .

2) Let  $x \in F^v \cap \overline{\mathcal{P}^0}$  and  $y \in \pi^{-1}(x)$ . Let  $F_1^v \in \mathcal{F}^0$  be such that  $y \in F_1^v$ . Then by 1),  $x \in F_1^v$  and as the faces of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  form a partition of  $\mathbb{A}$ , we deduce that  $F_1^v = F^v$ .  $\square$

We index the basis  $\Sigma$  of  $\Psi$  associated with  $C_f^v$  by a finite set  $I$ , i.e., we write  $\Sigma = \{\alpha_i \mid i \in I\}$ , where  $|I| = |\Sigma|$ . Let  $(\varpi_i) \in \mathbb{A}^I$  be the dual basis of  $(\alpha_i)_{i \in I}$ , i.e., a basis satisfying  $\alpha_i(\varpi_j) = \delta_{i,j}$  for all  $i, j \in I$ . For  $i \in I$ , we set

$$\mathcal{D}_i = \{x \in \mathbb{A} \mid \langle x - \mathbf{b} \mid \varpi_i \rangle \leq 0\}$$

and

$$\mathcal{D}_i^0 = \{x \in \mathbb{A} \mid \langle x \mid \varpi_i \rangle \leq 0\}$$

(we have  $\mathcal{D}_i = \mathcal{D}_{\mathbb{R}_{>0}\varpi_i}$  for the notation of Eq. (2.5), by Lemma 2.14). For  $J \subset I$ , we set  $V_J = \bigoplus_{j \in J} \mathbb{R}\varpi_j$ .

**Lemma 2.25.** *Let  $V$  be a finite dimensional vector space,  $k \in \mathbb{Z}_{>0}$  and  $f_1, \dots, f_k \in V^*$  be a free family. For  $i \in \llbracket 1, k \rrbracket$ , set  $E_i = f_i^{-1}(\mathbb{R}_{\geq 0})$  and  $L_i = E_i \cap \bigcap_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} f_j^{-1}(\{0\})$ . Then*

$$\bigcap_{i=1}^k E_i = \text{conv}\left(\bigcup_{i=1}^k L_i\right).$$

*Proof.* Let  $(f_i)_{i \in \mathcal{B}}$  be a basis of  $V^*$ , where  $\mathcal{B}$  is a set containing  $\llbracket 1, k \rrbracket$ . For  $i \in \mathcal{B}$ , let  $e_i \in V$  be such that  $f_j(e_i) = \delta_{i,j}$ , for  $j \in \mathcal{B}$ . Then  $\bigcap_{i=1}^k E_i = \bigoplus_{i=1}^k \mathbb{R}_{\geq 0}e_i \oplus V'$ , where  $V' = \bigoplus_{i \in \mathcal{B} \setminus \llbracket 1, k \rrbracket} \mathbb{R}e_i$ . Let  $i \in \llbracket 1, k \rrbracket$ . Then  $L_i = \mathbb{R}_{\geq 0}e_i \oplus V'$ , which proves the result.  $\square$

**Lemma 2.26.** *Let  $J \subset I$ . For any  $x \in (\bigoplus_{j \in J} \mathbb{R}\alpha_j^\vee)$ , we have:*

$$(\alpha_j(x) \geq 0, \forall j \in J) \Rightarrow x \in \bigoplus_{j \in J} \mathbb{R}_{\geq 0}\alpha_j^\vee.$$

*Proof.* For  $K \subset J$ , denote by  $A_K$  the Cartan matrix  $A_K = (\alpha_i(\alpha_j^\vee))_{i,j \in K}$ . Write  $J = J_1 \sqcup \dots \sqcup J_k$ , with  $A_{J_i}$  indecomposable for every  $i \in \llbracket 1, k \rrbracket$ . Let  $x \in \bigoplus_{j \in J} \mathbb{R}\alpha_j^\vee$ . Write  $x = \sum_{i=1}^k x_i$ , with  $x_i \in \bigoplus_{\ell \in J_i} \mathbb{R}\alpha_\ell^\vee$ , for  $i \in \llbracket 1, k \rrbracket$ . Assume that  $\alpha_j(x) \geq 0$  for all  $j \in J$ . Let  $i \in \llbracket 1, k \rrbracket$ . Then  $\alpha_j(x) = \alpha_j(x_i)$  for all  $j \in J_i$ . By [Kac94, Theorem 4.3 (Fin)] applied to the transpose of  $A_{J_i}$ , we have  $x_i \in \bigoplus_{j \in J_i} \mathbb{R}_{\geq 0}\alpha_j^\vee$ , and the lemma follows.  $\square$

For  $i \in I$ , we set  $\mathcal{F}_i = \mathcal{F}_{\mathbb{R}_{>0}\varpi_i}$ , with the notation of Definition 2.16. The  $\mathcal{F}_i$ , for  $i \in I$ , are the open facets of  $\mathcal{P}^0$  meeting  $C_f^v$ .

**Lemma 2.27.** *Let  $J \subset I$ . Then we have*

$$V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 \subset \bigoplus_{i \in I} \mathbb{R}_{\leq 0}\alpha_i^\vee \subset \bigcap_{i \in I} \mathcal{D}_i^0.$$

*Proof.* If  $J$  is empty, the result is clear so we assume  $J$  non-empty. For  $j \in J$ , denote by  $H_j^0$  the hyperplane delimiting  $\mathcal{D}_j^0$ . We have:

$$H_j^0 = \{x \in \mathbb{A} \mid \langle x \mid \varpi_j \rangle = 0\}$$



and by Lemma 2.14,  $H_j^0 = \bigoplus_{k \in I \setminus \{j\}} \mathbb{R} \alpha_k^\vee$ . Set  $L_j = V_J \cap \bigcap_{k \in J \setminus \{j\}} H_k^0 \cap \mathcal{D}_j^0$ . As  $(\varpi_k)_{k \in J}$  is a free family of  $\mathbb{A}$ , the family  $(x \mapsto \langle x \mid \varpi_k \rangle)_{k \in J}$  is a free family of  $V_J^*$  and thus by Lemma 2.25,

$$V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 = \text{conv}(L_j \mid j \in J). \quad (2.6)$$

Let  $j \in J$ . Let  $V_j' = \bigoplus_{k \in (I \setminus J) \cup \{j\}} \mathbb{R} \alpha_k^\vee$ . As  $(\alpha_i(\alpha_{i'}^\vee))_{i, i' \in (I \setminus J) \cup \{j\}}$  is a Cartan matrix (by [Kac94, Lemma 4.4] for example), it is invertible and  $(\alpha_k|_{V_j'})_{k \in (I \setminus J) \cup \{j\}}$  is a free family of  $(V_j')^*$ . Therefore there exists  $x_j \in V_j'$  such that  $\alpha_k(x_j) = -\delta_{j,k}$ , for  $k \in (I \setminus J) \cup \{j\}$ . By Lemma 2.26, we have  $x_j \in -\bigoplus_{k \in (I \setminus J) \cup \{j\}} \mathbb{R}_{\geq 0} \alpha_k^\vee$ . By Lemma 2.14, we deduce  $\langle \varpi_j \mid x_j \rangle \leq 0$  and  $x_j \in H_k^0$  for all  $k \in J \setminus \{j\}$ . Moreover,  $x_j \in V_J$  and thus  $x_j \in L_j$ . For dimension reasons,  $L_j = \mathbb{R}_{\geq 0} x_j$  and hence by Eq. (2.6), we have  $V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 \subset \bigoplus_{i \in I} \mathbb{R}_{\leq 0} \alpha_i^\vee$ . We conclude with Lemma 2.14.  $\square$

As in §2.1, we write  $\bigwedge_{j \in J} \mathcal{F}_j = \text{Int}_r(\bigcap_{j \in J} \overline{\mathcal{F}_j})$ , for  $J \subset I$ . By Proposition 2.20, we have:

$$\bigwedge_{j \in J} \mathcal{F}_j = \mathcal{F}_{\sum_{j \in J} \mathbb{R}_{> 0} \varpi_j}. \quad (2.7)$$

**Lemma 2.28.** *Let  $J \subset I$ . Let  $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$ . Let  $z \in \bigwedge_{j \in J} \mathcal{F}_j$ . Then for all  $i \in I \setminus J$ ,  $\overline{\mathcal{P}^0} \cap (z + V_J) \subset \mathring{\mathcal{D}}_i$ .*

*Proof.* We have  $\overline{\mathcal{P}^0} \subset \bigcap_{i \in I} \mathcal{D}_i \subset \bigcap_{j \in J} \mathcal{D}_j$ . By Corollary 2.21 and Eq. (2.7), we have:

$$\{i \in I \mid z \in \overline{\mathcal{F}_i}\} = J. \quad (2.8)$$

By Eq. (2.5) we deduce  $\bigcap_{j \in J} \mathcal{D}_j = z + \bigcap_{j \in J} \mathcal{D}_j^0$ . Let  $x \in (z + V_J) \cap \overline{\mathcal{P}^0}$ . Then we can write  $x = z + v$ , with  $v \in V_J$ . We have  $x \in \bigcap_{j \in J} \mathcal{D}_j$  and thus  $v \in V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0$ . Let  $i \in I \setminus J$ . By Lemma 2.27, we have  $v \in \mathcal{D}_i^0$ . By Eq. (2.8),  $z \in \mathring{\mathcal{D}}_i$  and hence  $x = z + v \in \mathring{\mathcal{D}}_i + \mathcal{D}_i^0 = \mathring{\mathcal{D}}_i$ .  $\square$

Let  $J \subset I$  be non-empty. Let  $z \in \bigwedge_{j \in J} \mathcal{F}_j$  and  $\overline{\mathcal{P}} = \overline{\mathcal{P}_J(z)} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$ . For  $j \in J$  define  $f_j \in (V_J)^*$  by  $f_j(x) = \langle x \mid \varpi_j \rangle$ , for  $x \in V_J$ . Then  $(f_j)_{j \in J}$  is free. Let  $(a_j)_{j \in J} \subset V_J$  be a dual set of  $(f_j)_{j \in J}$ , i.e., a free set such that  $f_i(a_j) = \delta_{i,j}$ , for all  $i, j \in J$ . Then  $\overline{\mathcal{P}} = z - \sum_{j \in J} \mathbb{R}_{\geq 0} a_j$ . For  $J' \subset J$ , set  $\overline{\mathcal{F}_{J'}} = z - \sum_{j \in J'} \mathbb{R}_{\geq 0} a_j$ . The closed faces of  $\overline{\mathcal{P}}$  are the  $\overline{\mathcal{F}_{J'}}$ , for  $J' \subset J$ . In particular, the closed facets of  $\overline{\mathcal{P}}$  are the  $\overline{\mathcal{F}_j}$ ,  $j \in J$ , where  $\overline{\mathcal{F}_j} = \overline{\mathcal{F}_{J \setminus \{j\}}}$ , for  $j \in J$ . Let  $j \in J$ . The support of  $\overline{\mathcal{F}_j}$  is  $\tilde{H}_j = \{x \in (z + V_J) \mid \langle x - z \mid \varpi_j \rangle = 0\} = H_j \cap (z + V_J)$ , where  $H_j$  is the frontier of  $\mathcal{D}_j$ . For  $x \in \overline{\mathcal{F}_j}$ , we have  $(x + \mathbb{R}_{\geq 0} \varpi_j) \cap \overline{\mathcal{F}_j} = \{x\}$  and  $\varpi_j$  is orthogonal to the direction of  $\overline{\mathcal{F}_j}$ , so that  $\mathbb{R}_{\geq 0} \varpi_j = \delta_{\overline{\mathcal{F}_j}}$  for the notation of subsection 2.1 applied to the polyhedron  $\overline{\mathcal{P}} \subset z + V_J$ . Therefore by Proposition 2.3, we have:

$$\forall J' \subset J, \forall a \in \overline{\mathcal{F}_{J'}}, \pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \bigoplus_{j \in J \setminus \{j'\}} \mathbb{R}_{\geq 0} \varpi_j, \quad (2.9)$$

where  $\pi_{\overline{\mathcal{P}}} : z + V_J \rightarrow \overline{\mathcal{P}}$  is the orthogonal projection.

**Lemma 2.29.** *Let  $J \subset I$ . Let  $z \in \bigwedge_{j \in J} \mathcal{F}_j \cap \overline{\mathcal{C}_f^v}$ . Let  $\overline{\mathcal{P}} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$ . Then  $\overline{\mathcal{P}} \cap \overline{\mathcal{C}_f^v} = \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{\mathcal{C}_f^v}$ .*

*Proof.* We have  $\overline{\mathcal{P}^0} \subset \bigcap_{i \in I} \mathcal{D}_i \subset \bigcap_{i \in J} \mathcal{D}_i$ , and hence  $\overline{\mathcal{P}} \cap \overline{C_f^v} \supset \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{C_f^v}$ . Suppose that  $\overline{\mathcal{P}} \cap \overline{C_f^v} \supsetneq \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{C_f^v}$ . Let  $x \in (\overline{\mathcal{P}} \cap \overline{C_f^v}) \setminus \overline{\mathcal{P}^0}$ . Let  $\gamma : [0, 1] \rightarrow [x, z]$  be the affine parametrization such that  $\gamma(0) = x$  and  $\gamma(1) = z$ . Let  $t = \min\{s \in [0, 1] \mid \gamma(s) \in \overline{\mathcal{P}^0}\}$ . Then  $t > 0$  and  $\gamma(t) \in \text{Fr}(\overline{\mathcal{P}^0}) \cap \overline{C_f^v}$ .

Let  $F^v \in \mathcal{F}^0$  be such that  $\gamma(t) \in F^v$ . By Lemma 2.24,  $F^v \subset \overline{C_f^v}$ . Let  $J' = \{i \in I \mid \varpi_i \in \overline{F^v}\}$ . Then by Proposition 2.6, we have  $F^v = \bigoplus_{j \in J'} \mathbb{R}_{>0} \varpi_j$  and by Proposition 2.20, we have  $\mathcal{F}_{F^v} = \bigwedge_{j \in J'} \mathcal{F}_j = \text{Int}_r(\bigcap_{j \in J'} \overline{\mathcal{F}_j})$ . By Lemma 2.28, we have  $\gamma(t) \notin \overline{\mathcal{F}_j}$  for all  $j \in I \setminus J$  and thus  $J' \subset J$ .

By Corollary 2.21, the set of facets  $\mathcal{F}_{F_1^v}$  such that  $\gamma(t) \in \overline{\mathcal{F}_{F_1^v}}$  is

$$\{\mathcal{F}_{F_1^v} \mid F_1^v \subset \overline{F^v}\} = \{\mathcal{F}_{\mathbb{R}_{>0} \varpi_j} \mid j \in J'\} = \{\mathcal{F}_j \mid j \in J'\}.$$

Therefore by Lemma 2.1, there exists  $r > 0$  such that  $B(\gamma(t), r) \cap \overline{\mathcal{P}^0} = B(\gamma(t), r) \cap \bigcap_{j \in J'} \mathcal{D}_j$ . By assumption,  $z \in \bigcap_{j \in J'} \mathcal{D}_j$  and thus  $\gamma(t') \in \bigcap_{j \in J'} \mathcal{D}_j$ , for all  $t' \in [0, 1]$ . This proves that for  $t' \in [0, t]$  close enough to  $t$ ,  $\gamma(t') \in \overline{\mathcal{P}^0}$ , which contradicts our assumption.  $\square$

**Lemma 2.30.** *Let  $J \subset I$ . Let  $z \in \overline{C_f^v} \cap \bigwedge_{j \in J} \mathcal{F}_j$ . Let  $\overline{\mathcal{P}} = \overline{\mathcal{P}_J^0(z)} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$ . Let  $\pi_{\overline{\mathcal{P}}} : z + V_J \rightarrow \overline{\mathcal{P}}$  be the orthogonal projection on  $\overline{\mathcal{P}}$ . Then for all  $F_1^v \in \mathcal{F}^0$  such that  $F_1^v \subset \overline{C_f^v}$ , we have  $\pi_{\overline{\mathcal{P}}}((z + V_J) \cap F_1^v) \subset F_1^v$ .*

*Proof.* Let  $a \in \text{Fr}(\overline{\mathcal{P}})$ , where we regard  $\overline{\mathcal{P}^0}$  as a subset of  $z + V_J$ . In the notation of Eq. (2.9), let us write:

$$J_1 = \{j \in J \mid a \in \overline{\mathcal{F}_j}\}.$$

We first assume  $a \notin \overline{C_f^v}$  and we prove  $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \cap \overline{C_f^v} = \emptyset$ . We have  $]a, z] \cap \overline{C_f^v} = [b, z]$ , for some  $b \in \overline{C_f^v}$ . If for all  $i \in I$  such that  $\alpha_i(a) < 0$ , we had  $\alpha_i(b) > 0$ , then we would have  $b - \epsilon(b - a) \in \overline{C_f^v}$  for  $\epsilon \in \mathbb{R}_{>0}$  small enough. Therefore there exists  $i \in I$  such that  $\alpha_i(a) < 0$  and  $\alpha_i(b) = 0$ . We have  $b \in \overline{C_f^v} \cap \overline{\mathcal{P}}$  and thus  $b \in \overline{\mathcal{P}^0}$ , by Lemma 2.29. Let  $j \in J_1$ . Then  $b \in \overline{\mathcal{F}_j} \subset H_j$ , since  $z \in \bigwedge_{j \in J} \mathcal{F}_j$ . Therefore  $b \in H_j \cap \overline{\mathcal{P}^0} = \overline{\mathcal{F}_j} = \overline{\mathcal{F}_{\mathbb{R}_{>0} \varpi_j}}$ . Using Lemma 2.23, we deduce  $\alpha_j(b) > 0$  for all  $j \in J_1$ . In particular,  $i \notin J_1$ . By Eq. (2.9),  $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \sum_{j \in J_1} \mathbb{R}_{\geq 0} \varpi_j$  and thus  $\alpha_i(\pi_{\overline{\mathcal{P}}}^{-1}(\{a\})) = \{\alpha_i(a)\} \subset \mathbb{R}_{< 0}$ . Consequently  $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \cap \overline{C_f^v} = \emptyset$ . Therefore

$$\pi_{\overline{\mathcal{P}}}(\overline{C_f^v} \cap (z + V_J)) \subset \overline{C_f^v}.$$

Let now  $a \in \text{Fr}(\overline{\mathcal{P}}) \cap \overline{C_f^v}$ . Then by Lemma 2.29,  $\{j \in I \mid a \in \overline{\mathcal{F}_j}\} \supset J_1$ . Therefore by Eq. (2.9) and Corollary 2.22, we have  $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \sum_{j \in J_1} \mathbb{R}_{\geq 0} \varpi_j \subset \pi_{\overline{\mathcal{P}^0}}^{-1}(\{a\})$ . Using Lemma 2.24, we deduce  $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \subset F^v$ , if  $F^v$  is the face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$  containing  $a$ . Consequently, if  $x \in \overline{C_f^v} \cap (z + V_J)$ , then  $x$  and  $\pi_{\overline{\mathcal{P}}}(x)$  belong to the same face of  $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ .  $\square$

**Lemma 2.31.** *Let  $z \in \mathbb{A}$  and  $J \subset I$ . Set  $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$ . Write  $z = \sum_{i \in I} z_i \varpi_i$ , with  $(z_i) \in \mathbb{R}^I$ . Then:*

$$(z + V_J) \cap \overline{C_f^v} = \begin{cases} \emptyset & \text{if } \exists i \in I \setminus J \mid z_i < 0 \\ \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j & \text{otherwise} \end{cases}$$

If  $z_i \geq 0$  for all  $i \in I \setminus J$ , set  $F^v(z, J) = \sum_{i \in I \setminus J | z_i > 0} \mathbb{R}_{>0} \varpi_i + \sum_{j \in J} \mathbb{R}_{>0} \varpi_j$ . Then:

$$\begin{aligned} \overline{F^v(z, J)} &\supset (z + V_J) \cap \overline{C_f^v}, \\ (z + V_J) \cap \overline{F^v(z, J)} &= \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j = (z + V_J) \cap \overline{C_f^v}, \\ (z + V_J) \cap F^v(z, J) &= \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{>0} \varpi_j. \end{aligned}$$

**Lemma 2.32.** *Let  $J \subset I$ . Let  $z \in \overline{C_f^v} \cap \bigwedge_{j \in J} \mathcal{F}_j$ . Then for all  $y \in (z + V_J) \cap \overline{C_f^v}$ , we have  $\pi(y) \in (z + V_J)$ .*

*Proof.* Set  $\overline{\mathcal{P}} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$ . Let  $F^v = F^v(z, J)$ , with the notation of Lemma 2.31. Let  $x \in F^v \cap (z + V_J)$ . By Lemma 2.30, we have  $\pi_{\overline{\mathcal{P}}}(x) \in F^v$ . By Lemma 2.31, we can choose an open neighbourhood  $U$  of  $\pi_{\overline{\mathcal{P}}}(x)$  in  $z + V_J$  such that  $U \subset F^v$ . Then for all  $y \in U \cap \overline{\mathcal{P}}$ ,  $d(x, y) \geq d(x, \pi_{\overline{\mathcal{P}}}(x))$ . Let  $V^J = \bigoplus_{i \in I \setminus J} \mathbb{R} \alpha_i^\vee$ . Then  $V^J$  is orthogonal to  $V_J$  by Lemma 2.14 and  $U + V^J$  is an open subset of  $\mathbb{A}$ .

Let  $y \in (U + V^J) \cap \overline{\mathcal{P}^0}$ . Write  $y = u + v$ , with  $u \in U$  and  $v \in V^J$ . Then  $\langle u - \mathbf{b} \mid \varpi_j \rangle = \langle u + v - \mathbf{b} \mid \varpi_j \rangle = \langle y - \mathbf{b} \mid \varpi_j \rangle \leq 0$ , for all  $j \in J$  and thus  $u \in U \cap \overline{\mathcal{P}}$ . Moreover,

$$d(x, y)^2 = |y - x|^2 = |u + v - x|^2 = |u - x|^2 + |v|^2 \geq |u - x|^2 \geq d(x, \pi_{\overline{\mathcal{P}}}(x))^2.$$

Therefore the restriction to  $\overline{\mathcal{P}^0}$  of  $y \mapsto d(x, y)$  admits a local minimum at  $\pi_{\overline{\mathcal{P}}}(x)$ . As this map is convex (by [HUL12, B 1.3 (c)] for example), this minimum is global and thus  $\pi_{\overline{\mathcal{P}}}(x) = \pi_{\overline{\mathcal{P}^0}}(x)$ . Thus we proved:

$$\forall x \in F^v \cap (z + V_J), \pi(x) \in (z + V_J).$$

Let now  $x \in \overline{C_f^v} \cap (z + V_J)$ . Then, by Lemma 2.31, there exists  $(x_n) \in ((z + V_J) \cap F^v)^\mathbb{N}$  such that  $x_n \rightarrow x$ . Then  $\pi(x_n) \rightarrow \pi(x)$  (by [HUL12, A. (3.1.6)]). As  $\pi(x_n) \in z + V_J$  for all  $n \in \mathbb{N}$ , we deduce  $\pi(x) \in z + V_J$ , which completes the proof of the lemma.  $\square$

**Proposition 2.33.** *Let  $F^v$  be a face of  $(\mathbb{A}, \mathcal{H}_\Psi^0)$  and  $z \in \mathcal{F}_{F^v}$ . Let  $F_1^v$  be a face dominating  $z$ . In particular,  $F_1^v$  dominates  $F^v$  by Proposition 2.20. Let  $\pi$  be the orthogonal projection on  $\overline{\mathcal{P}^0}$ . Then:*

1.  $\pi((z + \text{vect}(F^v)) \cap \overline{F_1^v}) \subset (z + \text{vect}(F^v)) \cap \overline{F_1^v}$ .
2. Let  $y \in (z + \text{vect}(F^v)) \cap \overline{F_1^v}$  and let  $G^v \in \mathcal{F}^0$  be such that  $\pi(y) \in \mathcal{F}_{G^v}$ . Then  $G^v \subset \overline{F^v}$ .

*Proof.* 1) Up to changing the choice of the fundamental chamber  $C_f^v$ , we may assume  $F_1^v \subset \overline{C_f^v}$ . Let  $J = \{i \in I \mid \varpi_i \in \overline{F^v}\}$ . Then by Proposition 2.6,  $F^v = \bigoplus_{j \in J} \mathbb{R}_{>0} \varpi_j$  and by Proposition 2.20,  $z \in \bigwedge_{j \in J} \mathcal{F}_j$ . We have  $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$  and the result follows from Lemma 2.32 and Lemma 2.24.

2) Let  $y \in (z + \text{vect}(F^v)) \cap \overline{F_1^v}$ . Let  $G^v \in \mathcal{F}^0$  be such that  $\pi(y) \in \mathcal{F}_{G^v}$ . By Lemma 2.24,  $G^v \subset \overline{F_1^v}$ . In particular,  $G^v \subset \overline{C_f^v}$ , and hence, by Proposition 2.6,  $G^v = \sum_{j \in I | \varpi_j \in \overline{G^v}} \mathbb{R}_{>0} \varpi_j$ . Let now  $i \in I \setminus J$ . By 1) and Lemma 2.28,  $\pi(y) \notin \overline{\mathcal{F}_{\mathbb{R}_{>0} \varpi_i}}$ . Using Corollary 2.21 we deduce that  $\mathbb{R}_{>0} \varpi_i \not\subset \overline{G^v}$ . This being true for all  $i \in I \setminus J$ , we deduce that  $G^v \subset \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j = \overline{F^v}$ .  $\square$

### 3 Tessellation of $\mathbb{A}$ by the affine weight polytopes

Let  $\mathbb{A}$  be a finite dimensional vector space and let  $\Phi$  be a finite root system in  $\mathbb{A}$ . In this section, we study polytopes associated with the affine poly-simplicial complex defined from  $\Phi$ .

In subsection 3.1, we recall and prove general facts on the affine poly-simplicial complex associated with  $\Phi$ .

In subsection 3.2, we define and study the faces of the weight polytopes.

In subsection 3.3, we prove that the weight polytopes tessellate  $\mathbb{A}$ .

#### 3.1 Affine apartment

The main reference for this section is [Bou81, V]. Let  $\Phi$  be a (finite) root system in  $\mathbb{A}$ . Let  $\xi : \mathbb{A} \rightarrow \mathbb{R}$  be the constant map equal to 1 and  $\Phi^{\text{aff}} = \Phi + \mathbb{Z}\xi$ . Then by [Mac72, (2.1) Proposition],  $\Phi^{\text{aff}}$  is an **affine root system** in the sense of [Mac72, 2]. Let  $\mathcal{H} = \mathcal{H}_\Phi = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$  which equals  $\{\underline{\alpha}^{-1}(\{0\}) \mid \underline{\alpha} \in \Phi^{\text{aff}}\}$ . This is a locally finite hyperplane arrangement of  $\mathbb{A}$ , which means that every bounded subset  $E$  of  $\mathbb{A}$  meets finitely many hyperplanes. An element of  $\mathcal{H}$  is called a **wall** or an **affine wall**. For  $x, y \in \mathbb{A}$ , we write  $x \sim_{\mathcal{H}} y$  if for every  $H \in \mathcal{H}$ , either  $(x, y \in H)$  or  $(x$  and  $y$  are strictly on the same side of  $H)$ . This is an equivalence relation on  $\mathbb{A}$ . Its classes are called the **faces** of  $(\mathbb{A}, \mathcal{H})$  or simply the faces of  $\mathbb{A}$ . A **vertex** of  $\mathbb{A}$  is an element  $x$  of  $\mathbb{A}$  such that  $\{x\}$  is a face of  $(\mathbb{A}, \mathcal{H})$ . If  $F$  is a face of  $(\mathbb{A}, \mathcal{H})$ , then:

$$\forall \alpha \in \Phi, \exists n \in \mathbb{Z} \mid \alpha(F) \subset ]n, n + 1[ \text{ or } \alpha(F) = \{n\}. \quad (3.1)$$

An **alcove** (or **open alcove**) of  $(\mathbb{A}, \mathcal{H})$  is a connected component of  $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$ . The alcoves are actually the faces which are not contained in an element of  $\mathcal{H}$ . A **closed alcove** is a set of the form  $\overline{C}$ , for some alcove  $C$  of  $\mathbb{A}$ . If  $\overline{C}$  is a closed alcove of  $\mathbb{A}$ , its set of vertices, denoted  $\text{vert}(\overline{C})$ , is the set of vertices of  $\mathbb{A}$  which belong to  $\overline{C}$ .

We denote by  $\mathcal{F} = \mathcal{F}(\mathcal{H})$  the set of faces of  $(\mathbb{A}, \mathcal{H})$ . If  $F \in \mathcal{F}$ , we denote by  $\mathcal{F}_F = \mathcal{F}_F(\mathcal{H})$  the set of faces dominating  $F$  (i.e., the set of faces whose closure contains  $F$ ). When  $F = \{\lambda\}$ , for some vertex  $\lambda$ , we write  $\mathcal{F}_\lambda(\mathcal{H})$  instead of  $\mathcal{F}_{\{\lambda\}}(\mathcal{H})$ . We denote by  $\text{Alc}(\mathcal{H})$  the set of alcoves of  $(\mathbb{A}, \mathcal{H})$  and if  $F \in \mathcal{F}(\mathcal{H})$ , we denote by  $\text{Alc}_F(\mathcal{H})$  the set of alcoves of  $(\mathbb{A}, \mathcal{H})$  dominating  $F$ . The **affine poly-simplicial complex associated with  $\Phi$**  is  $\mathcal{F}(\mathcal{H})$ . When  $\Phi$  is irreducible, it is a simplicial complex.

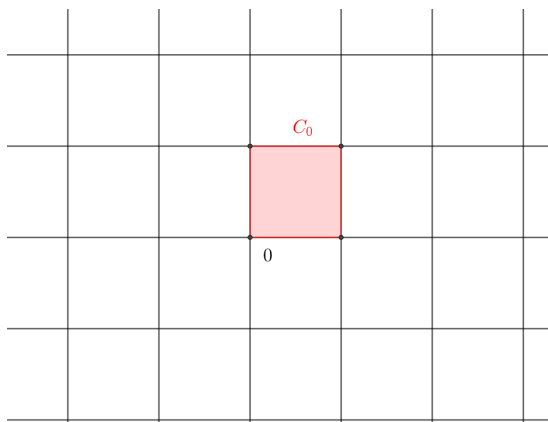


Figure 3.1: Affine apartment of type  $A_1 \times A_1$ . The alcoves are the squares.

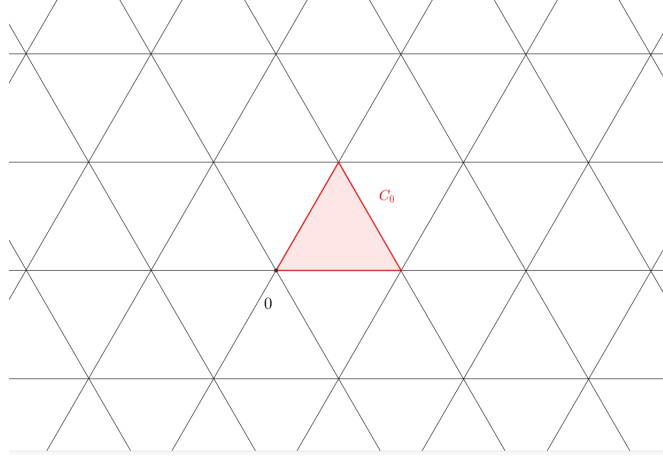


Figure 3.2: Affine apartment of type  $A_2$ . The alcoves are the equilateral triangles.

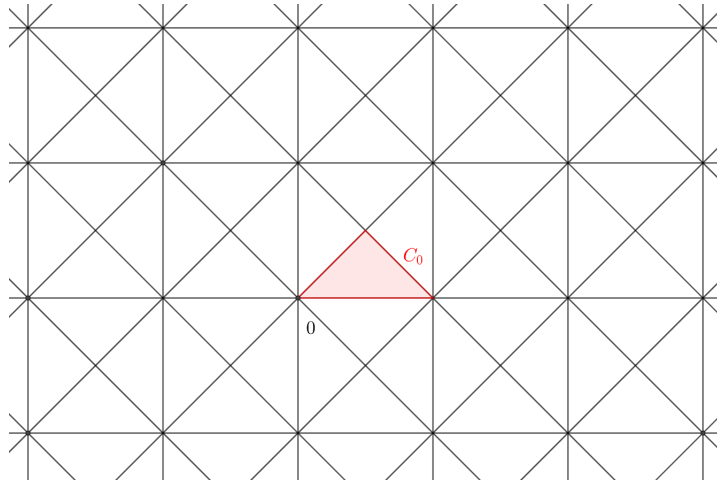


Figure 3.3: Affine apartment of type  $B_2$ . The alcoves are the right triangles.

### 3.1.1 Enclosure and closed faces of $(\mathbb{A}, \mathcal{H})$

**Definition 3.1.** A *half-apartment* of  $\mathbb{A}$  is a closed half-space of  $\mathbb{A}$  delimited by an element of  $\mathcal{H}$ . If  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$ , one sets  $D_{\alpha,k} = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$

Let  $\Omega \subset \mathbb{A}$ . Its **enclosure**  $\text{cl}(\Omega)$  is the intersection of all the half-apartments containing  $\Omega$ .

If  $\Omega$  is a subset of  $\mathbb{A}$  and  $\lambda \in \mathbb{A}$ , then the **germ of  $\Omega$  at  $\lambda$**  is the set  $\text{germ}_\lambda(\Omega)$  of subsets of  $\mathbb{A}$  of the form  $\Omega \cap E$  such that  $E$  is an open neighbourhood of  $\lambda$ . A subset  $\Omega'$  of  $\mathbb{A}$  is said to **contain** the germ  $\text{germ}_\lambda(\Omega)$  if it contains  $\Omega \cap E$  for some open neighbourhood  $E$  of  $\lambda$ . The enclosure of  $\text{germ}_\lambda(\Omega)$  is the intersection of the enclosures of the  $\Omega \cap E$  when  $E$  describes all the open neighbourhoods of  $\lambda$ .

A **closed face** of  $(\mathbb{A}, \mathcal{H})$  is a set of the form  $\overline{F}$ , for some  $F \in \mathcal{F}(\mathcal{H})$ .

**Lemma 3.2.** 1. Let  $F \in \mathcal{F}$ . For  $\underline{\alpha} \in \Phi^{\text{aff}}$ , set  $E_{\underline{\alpha}} = \{0\}$  if  $\underline{\alpha}(F) = \{0\}$ ,  $E_{\underline{\alpha}} = \mathbb{R}_{>0}$  if  $\underline{\alpha}(F) > 0$  and  $E_{\underline{\alpha}} = \mathbb{R}_{<0}$  if  $\underline{\alpha}(F) < 0$ . Then  $F = \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(E_{\underline{\alpha}})$  and  $\overline{F} = \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(\overline{E_{\underline{\alpha}}})$ . In particular,  $\overline{F}$  is enclosed.

2. Let  $F \in \mathcal{F}$  and  $x \in F$ . Then  $\text{cl}(x) = \overline{F}$ .

*Proof.* 1) The first equality follows from the definition of faces. As the elements of  $\Phi^{\text{aff}}$  are continuous maps from  $\mathbb{A}$  to  $\mathbb{R}$ , we have  $\overline{F} \subset \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(\overline{E_{\underline{\alpha}}})$ . Let  $x \in F$  and

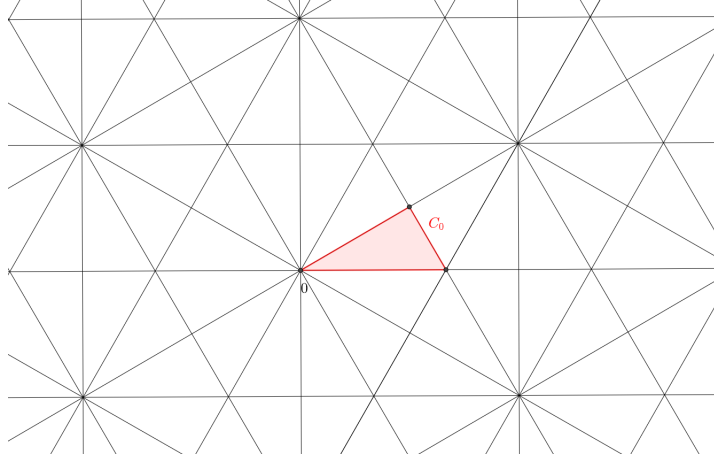


Figure 3.4: Affine apartment of type  $G_2$ . The alcoves are the right triangles.

$y \in \bigcap_{\alpha \in \Phi^{\text{aff}}} \alpha^{-1}(\overline{E_\alpha})$ . Then for  $z \in ]x, y[$  and  $\alpha \in \Phi^{\text{aff}}$ , we have  $\alpha(z) \in \overline{E_\alpha} + E_\alpha = E_\alpha$  and hence  $]x, y[ \subset F$ , which proves that  $y \in \overline{F}$  and concludes the proof of the second equality. The fact that  $\overline{F}$  is enclosed then follows from the fact that if  $\alpha \in \Phi^{\text{aff}}$ , then  $\alpha^{-1}(\{0\}) = \alpha^{-1}(\mathbb{R}_{\geq 0}) \cap \alpha^{-1}(\mathbb{R}_{\leq 0})$ .

We now prove 2). We have  $\text{cl}(x) \subset \text{cl}(F) = \overline{F}$ . By definition of the faces, every half-apartment containing  $x$  contains  $F$  and thus it contains  $\overline{F}$ . Therefore  $\text{cl}(x) \supset \overline{F}$ , which completes the proof of the lemma.  $\square$

**Proposition 3.3.** *Let  $E$  be a non-empty subset of  $\mathbb{A}$ . Then  $E$  is a closed face of  $(\mathbb{A}, \mathcal{H})$  if and only if the following two conditions are satisfied:*

1.  $E$  is enclosed,
2. for every  $\alpha \in \Phi^{\text{aff}}$ , we have either  $\alpha(E) \subset \mathbb{R}_{\geq 0}$  or  $\alpha(E) \subset \mathbb{R}_{\leq 0}$ .

*Proof.* Let  $F$  be a face of  $\mathcal{F}(\mathcal{H})$ . Then by Lemma 3.2,  $\overline{F}$  is enclosed. By definition of the faces, if  $\alpha \in \Phi^{\text{aff}}$ , we have  $\alpha(F) \subset \mathbb{R}_{> 0}$ ,  $\alpha(F) = \{0\}$  or  $\alpha(F) \subset \mathbb{R}_{< 0}$ . As  $\alpha(\overline{F}) \subset \overline{\alpha(F)}$  we deduce that  $\overline{F}$  satisfies 1. and 2.

Let now  $E$  be a non-empty subset satisfying 1. and 2. Then  $E$  is convex and closed. Let  $F = \text{Int}_r(E)$ . Then we have  $E = \overline{F}$ . Let  $S$  be the support of  $E$  and  $x, y \in F$ . Let  $\alpha \in \Phi^{\text{aff}}$ . If  $\alpha$  is constant on  $S$ , then  $\alpha(x) = \alpha(y)$ . Assume now that  $\alpha$  is non-constant on  $S$ . Let  $U_x, U_y$  be open subsets of  $S$  containing  $x$  and  $y$  respectively. Then by assumption  $\alpha(U_x)\alpha(U_y) \geq 0$  and hence  $\alpha(x)\alpha(y) > 0$ . Consequently,  $x \sim y$ , where  $\sim$  is the equivalence relation that we used to define the faces of  $(\mathbb{A}, \mathcal{H})$ . Consequently there exists  $F_1 \in \mathcal{F}$  such that  $F \subset F_1$ .

Using 1., we write  $\overline{F} = \bigcap_{\alpha \in \mathcal{E}} \alpha^{-1}(\mathbb{R}_{\geq 0})$ , where  $\mathcal{E} \subset \Phi^{\text{aff}}$ . Let  $x \in F$ . Let  $\alpha \in \mathcal{E}$ . Then  $\alpha(x) \geq 0$  and as  $\alpha$  has constant sign on  $F_1$ , we have  $\alpha(F_1) \geq 0$ . Consequently  $F_1 \subset \overline{F}$  and thus  $\overline{F_1} \subset \overline{F}$ . As  $F \subset F_1$ , we deduce  $\overline{F} = \overline{F_1} = E$ , and  $E$  is a closed face of  $(\mathbb{A}, \mathcal{H})$ .  $\square$

**Corollary 3.4.** *Let  $\overline{F_1}, \overline{F_2}$  be two closed faces of  $(\mathbb{A}, \mathcal{H})$ . Then either  $\overline{F_1} \cap \overline{F_2}$  is empty or  $\overline{F_1} \cap \overline{F_2}$  is a closed face of  $(\mathbb{A}, \mathcal{H})$ .*

Let  $C \in \text{Alc}(\mathcal{H})$ . A **wall of  $C$**  is a hyperplane  $H$  such that  $\text{supp}(\overline{C} \cap H) = H$ . A **panel of  $C$**  is an element of  $\mathcal{F}$  contained in  $\overline{C}$  and whose support is a wall of  $C$ .

One sets

$$\Sigma_C^{\text{aff}} := \{\alpha \in \Phi^{\text{aff}} \mid \alpha(C) > 0, \alpha^{-1}(\{0\}) \text{ is a wall of } C\}.$$

Assume that  $\Phi$  is irreducible. Then  $\Phi^{\text{aff}}$  is an irreducible affine root system by [Mac72, (4.7)] and  $\Sigma_C^{\text{aff}}$  is a basis of  $\Phi^{\text{aff}}$  in the sense of [Mac72, 4.6]. In particular,

$$\Phi^{\text{aff}} \subset \left( \sum_{\underline{\alpha} \in \Sigma_C^{\text{aff}}} \mathbb{Z}_{\geq 0} \underline{\alpha} \cup \sum_{\underline{\alpha} \in \Sigma_C^{\text{aff}}} (-\mathbb{Z}_{\geq 0}) \underline{\alpha} \right), \quad (3.2)$$

by [Mac72, (4.6) Proposition].

**Lemma 3.5.** 1. Let  $F \in \mathcal{F}(\mathcal{H})$ . Then there exists  $C \in \text{Alc}(\mathcal{H})$  such that  $\overline{F} \subset \overline{C}$ . Let  $\mathcal{M}_F = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F) > 0\}$ . Then

$$\overline{F} = \overline{C} \cap \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \underline{\alpha}^{-1}(\{0\}) = \bigcap_{\underline{\alpha} \in \mathcal{M}_F} \underline{\alpha}^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \underline{\alpha}^{-1}(\{0\}). \quad (3.3)$$

2. Let  $C \in \text{Alc}(\mathcal{H})$ . Let  $\underline{\alpha} \in \Sigma_C^{\text{aff}}$ . Then  $\overline{C} \cap \underline{\alpha}^{-1}(\{0\})$  is a closed panel of  $C$  and all the closed panels of  $C$  are of this form.

3. Let  $C \in \text{Alc}(\mathcal{H})$ . Then every closed face of  $(\mathbb{A}, \mathcal{H})$  strictly contained in  $\overline{C}$  can be written as an intersection of closed panels of  $\overline{C}$ .

*Proof.* 1) Let  $x \in F$ . Then by [Bou81, V §1.4 Proposition 6], there exists  $C \in \text{Alc}(\mathcal{H})$  such that  $x \in \overline{C}$ . But then by Lemma 3.2,  $\text{cl}(x) = \overline{F} \subset \overline{C} = \text{cl}(C)$ . Assume for the moment that  $\Phi$  is irreducible. With the same notation as in Lemma 3.2, we have

$$\overline{F} = \bigcap_{\underline{\beta} \in \Phi^{\text{aff}}} \underline{\beta}^{-1}(E_{\underline{\beta}}). \quad (3.4)$$

Let  $\underline{\beta} \in \Phi^{\text{aff}}$ . Then by Eq. (3.2), we can write  $\underline{\beta} = \epsilon \sum_{\underline{\alpha} \in \Sigma_C^{\text{aff}}} n_{\underline{\alpha}} \underline{\alpha}$ , with  $\epsilon \in \{-, +\}$  and  $(n_{\underline{\alpha}}) \in (\mathbb{Z}_{\geq 0})^{\Sigma_C^{\text{aff}}}$ . We have  $E_{\underline{\beta}} = \{0\}$  if and only if  $n_{\underline{\alpha}} = 0$ , for all  $\underline{\alpha} \in \mathcal{M}_F$ . Otherwise  $E_{\underline{\beta}} = \epsilon \mathbb{R}_{\geq 0}$ . In particular  $\underline{\beta}^{-1}(E_{\underline{\beta}}) \supset \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}}} \underline{\alpha}^{-1}(E_{\underline{\alpha}})$ . Therefore Eq. (3.4) becomes

$$\overline{F} = \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}}} \underline{\alpha}^{-1}(E_{\underline{\alpha}}) = \bigcap_{\underline{\alpha} \in \mathcal{M}_F} \underline{\alpha}^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \underline{\alpha}^{-1}(\{0\}). \quad (3.5)$$

When  $F = C$ , we have  $\mathcal{M}_C = \Sigma_C^{\text{aff}}$ , which proves Eq. (3.3), when  $\Phi$  is irreducible.

We now assume that  $\Phi$  is reducible. Using the notation of Eq. (2.4), with  $\Phi$  instead of  $\Psi$ , we write  $\mathbb{A} = \bigoplus_{i=1}^{\ell} \mathbb{A}_i$  and  $\Phi = \bigsqcup_{i=1}^{\ell} \Phi_i$ , where for each  $i \in \llbracket 1, \ell \rrbracket$ ,  $\Phi_i$  is an irreducible root system in  $\mathbb{A}_i$ . If  $i \in \llbracket 1, \ell \rrbracket$  and  $\underline{\alpha} \in \Phi_i^{\text{aff}}$ ,  $\underline{\alpha} : \mathbb{A}_i \rightarrow \mathbb{R}$ , we denote by  $\tilde{\underline{\alpha}}$  the extension of  $\underline{\alpha}$  to  $\mathbb{A}$ :  $\tilde{\underline{\alpha}}((x_1, \dots, x_i, \dots, x_{\ell})) = \underline{\alpha}(x_i)$ , for  $(x_1, \dots, x_{\ell}) \in \mathbb{A}$ . Write  $C = C_1 \times \dots \times C_{\ell}$  and  $F = F_1 \times \dots \times F_{\ell}$ . Then we have  $\Sigma_C^{\text{aff}} = \bigsqcup_{i=1}^{\ell} \{\tilde{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_{C_i}^{\text{aff}}\}$  and  $\mathcal{M}_F = \bigsqcup_{i=1}^{\ell} \{\tilde{\underline{\alpha}} \mid \underline{\alpha} \in \mathcal{M}_{F_i}\}$ . For  $i \in \llbracket 1, \ell \rrbracket$ , we have:

$$\bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_i}} \tilde{\underline{\alpha}}^{-1}(\{0\}) = \mathbb{A}_1 \times \dots \times \mathbb{A}_{i-1} \times \bigcap_{\underline{\alpha} \in \Sigma_{C_i}^{\text{aff}} \setminus \mathcal{M}_{F_i}} \underline{\alpha}^{-1}(\{0\}) \times \mathbb{A}_{i+1} \times \dots \times \mathbb{A}_{\ell}.$$

We thus deduce Eq. (3.3) from the irreducible case. This proves statement 1.

2) By Proposition 3.3, if  $\underline{\alpha} \in \Sigma_C^{\text{aff}}$ , then  $\underline{\alpha}^{-1}(\{0\}) \cap \overline{C} = \underline{\alpha}^{-1}(\mathbb{R}_{\geq 0}) \cap \underline{\alpha}^{-1}(\mathbb{R}_{\leq 0}) \cap \overline{C}$  is a closed face of  $(\mathbb{A}, \mathcal{H})$ . It is a panel by definition of  $\Sigma_C^{\text{aff}}$ . By 1. and for dimension reasons, all the closed panels are of this form, which proves statement 2.

3) Let  $F \in \mathcal{F}$  be such that  $F \subset \overline{C}$ . With the same notation as above, we have  $\overline{F} = \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} (\overline{C} \cap \underline{\alpha}^{-1}(\{0\}))$ . □

Let  $C \in \text{Alc}(\mathcal{H})$ . Then by Lemma 3.5 (applied with  $F = C$ ,  $\mathcal{M}_C = \Sigma_C^{\text{aff}}$ ),  $\overline{C}$  is a finite intersection of half-apartments. Therefore it is a polytope. Moreover the closed panels of  $C$  are exactly the facets of  $\overline{C}$ , when we regard it as a polytope, in the sense of [BG09, Definition 1.3]. By [BG09, Theorem 1.10 a),b)], Lemma 3.5 and Corollary 3.4, we deduce that the faces of  $\overline{C}$  regarded as a polytope are exactly the closed faces of  $(\mathbb{A}, \mathcal{H})$  which are contained in  $\overline{C}$  (except the empty set, that we excluded from the set of faces).

**Lemma 3.6.** *Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$ . Let  $E \subset \overline{C}$  be non-empty. Then  $\{F_1 \in \mathcal{F} \mid E \subset \overline{F_1}\}$  admits a minimal element  $F$  (for the dominance order). We have  $F \subset \overline{C}$ ,  $\text{cl}(E) = \overline{F}$  and if moreover,  $E$  is convex, then  $E$  meets  $F$ .*

*Proof.* By Proposition 3.3,  $\text{cl}(E)$  is a closed face of  $(\mathbb{A}, \mathcal{H})$ : there exists  $F \in \mathcal{F}$  such that  $\text{cl}(E) = \overline{F}$ . Moreover if  $F_1 \in \mathcal{F}$  contains  $E$ , then  $\overline{F_1} \supset \text{cl}(E) = \overline{F}$ .

Assume now that  $E$  is convex. Let  $\mathcal{E} = \{F_1 \in \mathcal{F} \mid \overline{F_1} \subset \overline{F}, F_1 \cap E \neq \emptyset\}$ . Let  $F_1, F_2 \in \mathcal{E}$ ,  $x_1 \in F_1$  and  $x_2 \in F_2$ . Let  $x_3 = \frac{1}{2}(x_1 + x_2) \in E \subset \overline{F}$ . Let  $F_3 \in \mathcal{F}$  be such that  $x_3 \in F_3$ . Then  $F_3 \in \mathcal{E}$  since  $x_3 \in \overline{F}$ .

Moreover,  $]x_1, x_2[ \cap \overline{F_3} \supset \{x_3\} \neq \emptyset$ . Let  $C$  be an alcove dominating  $F$ . By Eq. (3.3), we have  $\overline{F_3} = \bigcap_{\underline{\alpha} \in \mathcal{M}_{F_3}} \underline{\alpha}^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_3}} \underline{\alpha}^{-1}(\{0\})$ , with the notation of Lemma 3.5. Then if  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_3}$ , we have  $\underline{\alpha}(x_3) = 0$  and  $\underline{\alpha}(x_1), \underline{\alpha}(x_2) \geq 0$  so that  $\underline{\alpha}(x_1) = \underline{\alpha}(x_2) = 0$ . This proves that  $]x_1, x_2[ \subset \overline{F_3}$ . Moreover,  $\overline{F_3}$  is enclosed and thus by Lemma 3.2,  $\overline{F_3} \supset \overline{F_1} \cup \overline{F_2}$ . Hence  $\mathcal{E}$  admits a maximal element  $F_{\max}$  for the dominance order.

Let  $x \in E$  and  $F_x \in \mathcal{F}(\mathcal{H})$  be such that  $x \in F_x$ . Then  $F_x$  belongs to  $\mathcal{E}$  and thus  $x \in F_x \subset \overline{F_{\max}}$ . Hence  $E \subset \overline{F_{\max}}$ . By definition of  $\overline{F}$ , this implies that  $\overline{F} \subset \overline{F_{\max}}$  and hence  $\overline{F} = \overline{F_{\max}}$ . Therefore  $F$  meets  $E$ .  $\square$

### 3.1.2 Properties of the affine Weyl group

There exists a scalar product  $\langle \cdot \mid \cdot \rangle$  on  $\mathbb{A}$  which is invariant under the vectorial Weyl group  $W_{\Phi}^v$  of  $\Phi$ . We equip  $\mathbb{A}$  with such a scalar product. If  $H \in \mathcal{H}$ , we denote by  $s_H$  the orthogonal reflection of  $\mathbb{A}$  fixing  $H$ . Let  $C \in \text{Alc}(\mathcal{H})$ . A **wall** of  $C$  is an element of  $\mathcal{H}$  such that  $\text{supp}(\overline{C} \cap H) = H$ . The **affine Weyl group of  $(\mathbb{A}, \mathcal{H})$**  is the group  $W^{\text{aff}}$  of affine maps on  $\mathbb{A}$  generated by  $s_H$ , for  $H \in \mathcal{H}$ . If  $F \in \mathcal{F}(\mathcal{H})$ , we denote by  $W_F^{\text{aff}}$  the fixator of  $F$  in  $W^{\text{aff}}$ . Gathering the results of [Bou81, V §3.2, 3.3], we have the following theorem. We write it for the affine Weyl group and the faces of  $(\mathbb{A}, \mathcal{H})$ , but it is also true for the vectorial Weyl group and the vectorial faces, by the same reference. We refer to [BB05] for the definition of Coxeter groups.

**Theorem 3.7.** *1. Let  $C \in \text{Alc}(\mathcal{H})$  and  $S_C$  be the set of orthogonal reflections with respect to the walls of  $C$ . Then  $(W^{\text{aff}}, S_C)$  is a Coxeter group.*

*2. Let  $C \in \text{Alc}(\mathcal{H})$ . Then  $\overline{C}$  is a fundamental domain for the action of  $W^{\text{aff}}$  on  $\mathbb{A}$  and for every alcove  $C' \in \text{Alc}(\mathcal{H})$ , there exists a unique  $w \in W^{\text{aff}}$  such that  $w.C = C'$ .*

*3. Let  $F \in \mathcal{F}$  and  $C \in \text{Alc}_F(\mathcal{H})$ . Then  $W_F^{\text{aff}}$  is the parabolic subgroup of  $(W^{\text{aff}}, S_C)$  generated by the reflections with respect to the walls of  $C$  containing  $F$ . Moreover, if  $w \in W^{\text{aff}}$ , the following statements are equivalent:*

- a.  $w \in W_F^{\text{aff}}$ ,*
- b.  $w$  pointwise fixes  $\overline{F}$ ,*
- c.  $w.F$  meets  $F$ ,*
- d.  $w$  fixes a point of  $F$ .*

The lemma below also has a vectorial version.



**Lemma 3.8.** *Let  $F \in \mathcal{F}(\mathcal{H})$  and  $C \in \text{Alc}_F(\mathcal{H})$ . Then for every  $F' \in \mathcal{F}(\mathcal{H})$  such that  $F'$  dominates  $F$ , there exists  $w \in W_F^{\text{aff}}$  such that  $w.F' \subset \overline{C}$ . In particular,  $W_F^{\text{aff}}$  acts transitively on  $\text{Alc}_F(\mathcal{H})$ .*

*Equivalently, if  $E$  is the union of the faces dominating  $F$  and if  $E' = E \cap \overline{C}$ , then  $E'$  is a fundamental domain for the action of  $W_F^{\text{aff}}$  on  $E$ .*

*Proof.* Let  $C'$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  dominating  $F$ . Write  $C' = w.C$ , with  $w \in W^{\text{aff}}$ . Then  $w^{-1}.F \subset \overline{C}$ , so if  $x \in w^{-1}.F \subset \overline{C}$ , we have  $w.x \in \overline{C}$ , which proves that  $w.x = x$  by Theorem 3.7. Therefore  $w$  fixes  $w^{-1}.F$ , which proves that  $w$  fixes  $F$ . Now if  $F_1$  is a face of  $(\mathbb{A}, \mathcal{H})$  dominating  $F$  and  $C_1$  is an alcove dominating  $F_1$ , we can write  $C_1 = w_1.C$ , with  $w_1 \in W^{\text{aff}}$  fixing  $F$ . Then  $w_1^{-1}.F_1 \subset \overline{C}$ , which proves the lemma.  $\square$

**Lemma 3.9.** *Let  $C \in \text{Alc}(\mathcal{H})$  and  $F_1, F_2 \in \mathcal{F}(\mathcal{H})$  be such that  $F_1 \cup F_2 \subset \overline{C}$ . Then  $W_{F_1}^{\text{aff}} \subset W_{F_2}^{\text{aff}}$  (resp.  $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$ ) if and only if  $F_2 \subset \overline{F_1}$  (resp.  $F_1 = F_2$ ).*

*Proof.* By Theorem 3.7, if  $S_C$  denotes the set of orthogonal reflections with respect to the walls of  $C$ , then  $(W^{\text{aff}}, S_C)$  is a Coxeter group. Let  $i \in \{1, 2\}$ . Set  $\mathcal{M}_{F_i} = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F_i) > 0\}$ . We have  $W_{F_i}^{\text{aff}} = \langle s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_i} \rangle$ , by Theorem 3.7, where  $s_{\underline{\alpha}}$  denotes the reflection of  $W^{\text{aff}}$  fixing  $\underline{\alpha}^{-1}(\{0\})$ , for  $\underline{\alpha} \in \Phi^{\text{aff}}$ . By [BB05], if  $W_{F_1}^{\text{aff}} \subset W_{F_2}^{\text{aff}}$  (resp.  $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$ ), then we have  $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_1} \subset \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_2}$  (resp.  $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_1} = \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_2}$ ). Using Lemma 3.5 we deduce that  $\overline{F_2} \subset \overline{F_1}$  (resp.  $\overline{F_1} = \overline{F_2}$ ).

Conversely, as  $W^{\text{aff}}$  acts by homeomorphisms on  $\mathbb{A}$ , we have  $W_{F_i}^{\text{aff}} = W_{\overline{F_i}}^{\text{aff}}$  if  $i \in \{1, 2\}$  and thus if  $F_2 \subset \overline{F_1}$  (resp.  $\overline{F_1} = \overline{F_2}$ ), we have  $W_{F_2}^{\text{aff}} \supset W_{F_1}^{\text{aff}}$  (resp.  $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$ ).  $\square$

### 3.1.3 Properties of the faces and link between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}_\lambda^0)$

For  $x \in \mathbb{A}$ , we set  $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) \in \mathbb{Z}\}$ .

**Lemma 3.10.** *Let  $x \in \mathbb{A}$ . Then  $x \in \text{vert}(\mathbb{A})$  if and only if  $\bigcap_{\alpha \in \Phi_x} \ker(\alpha) = \{0\}$ .*

*Proof.* Let  $E = \bigcap_{\alpha \in \Phi_x} \ker(\alpha)$ . Let  $y \in E$ . Then for  $t \in ]0, 1[$  small enough, for every  $\alpha \in \Phi$ , if  $\alpha(x) \notin \mathbb{Z}$ , then  $\alpha(x + ty) \in ]\alpha(x), \alpha(x) + 1[$  and thus  $x \sim_{\mathcal{H}} x + ty$ . Therefore if  $\{x\}$  is a face of  $(\mathbb{A}, \mathcal{H})$ , then  $E = \{0\}$ . Conversely, assume  $E = \{0\}$ . Take  $y \in \mathbb{A} \setminus \{x\}$ . Then there exists  $\alpha \in \Phi_x$  such that  $\alpha(y) \neq \alpha(x)$ . Then  $y \notin \alpha^{-1}(\{\alpha(x)\})$  and as  $\alpha^{-1}(\{\alpha(x)\}) \in \mathcal{H}$ ,  $y \not\sim_H x$ . Thus the class of  $x$  is  $\{x\}$ .  $\square$

Let  $\lambda \in \text{vert}(\mathbb{A})$ . Then by [Bou81, Chapitre VI, §1 Proposition 4] and Lemma 3.10,  $\Phi_\lambda$  is a root system in  $\mathbb{A}^*$ , with dual root system  $\Phi_\lambda^\vee := \{\alpha^\vee \mid \alpha \in \Phi_\lambda\}$ . For  $\alpha \in \Phi_\lambda$  and  $x \in \mathbb{R}^*$ , if  $x\alpha \in \Phi_\lambda$ , then  $x\alpha \in \Phi$  and thus  $x \in \{-1, 1\}$ , since  $\Phi$  is reduced. Therefore  $\Phi_\lambda$  is a reduced root system in  $\mathbb{A}^*$ . If  $\alpha \in \Phi$ , we denote by  $s_\alpha \in W^v$  the associated reflection.

Let  $\mathcal{H}_\lambda^0 = \{\alpha^{-1}(\{0\}) \mid \alpha \in \Phi_\lambda\}$ . The connected components of  $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}_\lambda^0} H$  are the (vectorial) chambers of  $(\mathbb{A}, \Phi_\lambda)$ . If  $\lambda \in \text{vert}(\mathbb{A})$ , then we denote by  $\mathcal{F}_\lambda(\mathcal{H})$  the set of faces of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$  and we denote by  $\mathcal{F}(\mathcal{H}_\lambda^0)$  the set of faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$ .

**Lemma 3.11.** *Let  $\lambda \in \text{vert}(\mathbb{A})$  and  $F_1, F_2 \in \mathcal{F}_\lambda(\mathcal{H})$ . We assume that there exists a neighbourhood  $\Omega$  of  $\lambda$  such that  $F_1 \cap \Omega \subset \overline{F_2}$ . Then  $F_1 \subset \overline{F_2}$ .*

*Proof.* Let  $x \in \Omega \cap F_1$ . Let  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$  be such that  $D_{\alpha, k} \supset F_2$ . Then  $D_{\alpha, k} \supset \overline{F_2} \ni x$  and thus  $D_{\alpha, k} \supset \text{cl}(x) = \overline{F_1}$ . As  $\overline{F_2}$  is enclosed, we deduce that  $\overline{F_2} \supset F_1$ .  $\square$

**Proposition 3.12.** *Let  $\lambda \in \text{vert}(\mathbb{A})$ . Equip  $\mathcal{F}_\lambda(\mathcal{H})$  and  $\mathcal{F}(\mathcal{H}_\lambda^0)$  with the dominance order. Then the map  $\Theta : \mathcal{F}_\lambda(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H}_\lambda^0)$  defined by  $\Theta(F) = F^v(F) := \mathbb{R}_{>0}(F - \lambda)$ , for  $F \in \mathcal{F}_\lambda(\mathcal{H})$  is an isomorphism of partially ordered sets. Its inverse is the map  $\Xi : \mathcal{F}(\mathcal{H}_\lambda^0) \rightarrow \mathcal{F}_\lambda(\mathcal{H})$  which associates to an element  $F^v$  of  $\mathcal{F}(\mathcal{H}_\lambda^0)$  the unique face of  $\mathcal{F}(\mathcal{H})$  containing  $\text{germ}_\lambda(\lambda + F^v)$ . We have  $\Xi(F^v) = \text{cl}(\text{germ}_\lambda(\lambda + F^v))$  and  $\Xi(F^v) \subset \lambda + F^v$ .*

*Proof.* Set

$$\Omega = \Omega_\lambda := \bigcap_{\alpha \in \Phi_\lambda} \alpha^{-1}([\alpha(\lambda) - 1, \alpha(\lambda) + 1]) \cap \bigcap_{\alpha \in \Phi \setminus \Phi_\lambda} \alpha^{-1}([\alpha(\lambda)], [\alpha(\lambda)]].$$

Then  $\Omega$  is a convex open set containing  $\lambda$ .

Let  $F \in \mathcal{F}_\lambda(\mathcal{H})$  and  $x_1, x_2 \in \mathbb{R}_{>0}(F - \lambda)$ . Write  $x_1 = t_1(f_1 - \lambda)$ ,  $x_2 = t_2(f_2 - \lambda)$ , with  $t_1, t_2 \in \mathbb{R}_{>0}$  and  $f_1, f_2 \in F$ . Let  $\alpha \in \Phi_\lambda$ . Then the sign of  $\alpha - \alpha(\lambda)$  (which can take the values “ $< 0$ ”, “ $= 0$ ” or “ $> 0$ ”) is constant on  $\{f_1, f_2\}$  and thus  $\alpha(x_1)$  and  $\alpha(x_2)$  have the same sign. As this is true for every  $\alpha \in \Phi_\lambda$ , we deduce the existence of  $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$  such that  $\Theta(F) \subset F^v$ .

Let  $x_1 \in F^v$  and  $f \in F$ . Set  $x_2 = f - \lambda \in \Theta(F) \subset F^v$ . Let  $t_1 \in \mathbb{R}_{>0}$  be such that  $f_1 := x_1/t_1 + \lambda \in \Omega$ . Let  $\alpha \in \Phi \setminus \Phi_\lambda$ . By choice of  $\Omega$ , we have  $\alpha(f_1) \in ]\alpha(\lambda), [\alpha(\lambda)[[$ . By Eq. (3.1), we have  $\alpha(F) \subset ]\alpha(\lambda), [\alpha(\lambda)[[$  and thus  $\alpha - k$  has constant sign on  $\{f_1, f\}$ , for every  $k \in \mathbb{Z}$ . Let  $\alpha \in \Phi_\lambda$ . We have  $\alpha(F^v) \in \{\mathbb{R}_{<0}, \{0\}, \mathbb{R}_{>0}\}$ . If  $\alpha(F^v) = \{0\}$ , then  $\alpha(f_1) = \alpha(\lambda)$  and  $\alpha(x_2) = 0$ , so  $\alpha(f) = \alpha(\lambda)$ . Therefore  $\alpha + k$  has constant sign on  $\{f_1, f\}$ , for every  $k \in \mathbb{Z}$ . Assume now  $\alpha(F^v) = \mathbb{R}_{>0}$ . Then  $\alpha(f_1) > \alpha(\lambda)$  and thus  $\alpha(f_1) \in ]\alpha(\lambda), \alpha(\lambda) + 1[$ . We have  $\alpha(x_2) > 0$ , thus  $\alpha(f) > \alpha(\lambda)$ . By Eq. (3.1),  $\alpha(f) \in ]\alpha(\lambda), \alpha(\lambda) + 1[$ . Therefore  $\alpha + k$  has constant sign on  $\{f, f_1\}$ , for every  $k \in \mathbb{Z}$ . With the same reasoning when  $\alpha(F^v) = \mathbb{R}_{<0}$ , we deduce that  $f$  and  $f_1$  belong to the same face and thus  $f_1 \in F$ . Consequently  $x_1 = t_1(f_1 - \lambda) \in \Theta(F)$ . Therefore we proved  $\Theta(F) = F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$  and thus  $\Theta$  is well-defined.

Let now  $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$  and  $f_1, f_2 \in \Omega \cap (\lambda + F^v)$ . Let  $\alpha \in \Phi$ . Let  $x_1 = f_1 - \lambda$ ,  $x_2 = f_2 - \lambda \in F^v$ . If  $\alpha \notin \Phi_\lambda$ , then  $\alpha(f_1), \alpha(f_2) \in ]\alpha(\lambda), [\alpha(\lambda)[[$ . Therefore  $\alpha + k$  has constant sign on  $\{f_1, f_2\}$ , for every  $k \in \mathbb{Z}$ . Let now  $\alpha \in \Phi_\lambda$ . If  $\alpha(F^v) = \{0\}$ , then  $\alpha(f_1) = \alpha(f_2) = \alpha(\lambda)$  and thus  $\alpha + k$  has constant sign on  $\{f_1, f_2\}$  for every  $k \in \mathbb{Z}$ . If  $\alpha(F^v) = \mathbb{R}_{>0}$ , then  $\alpha(x_1), \alpha(x_2) > 0$  and thus  $\alpha(f_1), \alpha(f_2) \in ]\alpha(\lambda), \alpha(\lambda) + 1[$ , by definition of  $\Omega$ . With the same reasoning when  $\alpha(F^v) = \mathbb{R}_{<0}$ , we deduce that  $f_1$  and  $f_2$  belong to the same face  $F$  of  $\mathcal{F}_\lambda(\mathcal{H})$ . Then  $F$  is the unique face of  $\mathcal{F}_\lambda(\mathcal{H})$  containing  $(\lambda + F^v) \cap \Omega \supset \text{germ}_\lambda(\lambda + F^v)$ . Consequently,  $\Xi(F^v) = F$  is well-defined and we have

$$\Omega \cap (\lambda + F^v) \subset F = \Xi(F^v). \quad (3.6)$$

Moreover, for  $x \in F$ , we have  $\text{cl}(x) = \overline{F}$  and thus  $\text{cl}(\text{germ}_\lambda(\lambda + F^v)) = \overline{F}$ .

Let  $F \in \mathcal{F}_\lambda(\mathcal{H})$  and  $\tilde{F} = \Xi \circ \Theta(F)$ . Then  $\tilde{F} \supset \Omega \cap ]\lambda, f[$  for  $f \in F$ . Therefore  $\tilde{F} \cap F$  is non-empty and hence  $F = \tilde{F}$ , which proves that  $\Xi \circ \Theta = \text{Id}$ .

Let  $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ . Set  $\tilde{F}^v = \Theta \circ \Xi(F^v)$ . Let  $f \in F^v$  be such that  $\lambda + f \in \Omega$ . Then  $\lambda + f \in \Xi(F^v)$  by Eq. (3.6). Therefore  $\lambda + f - \lambda = f \in \tilde{F}^v$  and hence  $\tilde{F}^v \cap F^v \neq \emptyset$ . Consequently  $\tilde{F}^v = F^v$  and  $\Theta \circ \Xi = \text{Id}$ , which proves that  $\Theta$  is the inverse of  $\Xi$ .

Let  $F_1, F_2 \in \mathcal{F}_\lambda(\mathcal{H})$  be such that  $\overline{F_2} \supset F_1$ . Then  $\mathbb{R}_{>0}(F_2 - \lambda) = \mathbb{R}_{\geq 0}(\overline{F_2} - \lambda) \supset \mathbb{R}_{>0}(F_1 - \lambda)$  and thus  $\Theta$  is increasing.

Let  $F_1^v, F_2^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$  be such that  $\overline{F_2^v} \supset F_1^v$ . We have

$$\overline{\Xi(F_2^v)} \supset (\lambda + \overline{F_2^v}) \cap \Omega \supset (\lambda + F_1^v) \cap \Omega \supset \Xi(F_1^v) \cap \Omega,$$

by Eq. (3.6). Using Lemma 3.11, we deduce that  $\Xi(F_1^v) \subset \overline{\Xi(F_2^v)}$ , which proves that  $\Xi$  is increasing. Thus  $\Theta$  is an isomorphism.

Using Eq. (3.6) and the assetion following it, we get the last claim of the proposition.  $\square$

**Remark 3.13.** *Note that for dimension reasons, the bijection above sends alcoves on chambers.*

### 3.1.4 Star of a face

In the sequel, we write  $W_x^{\text{aff}} = W_{\{x\}}^{\text{aff}}$ , for the fixator of  $x \in \mathbb{A}$  in  $W^{\text{aff}}$ .

**Lemma 3.14.** *Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  and  $J \subset C$  be a non-empty set. Let  $F$  be the smallest face of  $C$  dominating  $J$  (this face is well-defined by Lemma 3.6). Then  $\bigcap_{x \in J} W_x^{\text{aff}} = W_F^{\text{aff}}$ .*

*Proof.* For every  $x \in J$ ,  $W_F^{\text{aff}} = W_{\overline{F}}^{\text{aff}}$  fixes  $x$  and thus  $W_F^{\text{aff}} \subset \bigcap_{x \in J} W_x^{\text{aff}}$ . Let  $w \in \bigcap_{x \in J} W_x^{\text{aff}}$ . By Lemma 3.6, there exists  $y \in \text{conv}(J) \cap F$ . Then  $w$  fixes  $y$ . By Theorem 3.7,  $W_y^{\text{aff}} = W_F^{\text{aff}}$ , which proves that  $w \in W_F^{\text{aff}}$  and completes the proof of the lemma.  $\square$

**Lemma 3.15.** *Assume that  $\Phi$  is irreducible. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  and let  $C$  be an alcove dominating  $F$ . Let  $\Sigma_C^{\text{aff}}$  be the basis associated with  $C$  and set  $\mathcal{M}_F = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F) > 0\}$ . Let  $x \in \mathbb{A}$ . Then there exists  $w \in W_F^{\text{aff}}$  such that  $\underline{\alpha}(w.x) \geq 0$  for every  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ .*

*Proof.* If  $F$  is an alcove,  $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$  is empty and there is nothing to prove. We now assume that  $F$  is not an alcove. Let  $\lambda \in \text{vert}(F)$ . Let  $F^v = \mathbb{R}_{>0}(F - \lambda)$  and  $C^v = \mathbb{R}_{>0}(C - \lambda)$ . By Proposition 3.12,  $F^v$  and  $C^v$  are faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$ . Let  $\Phi_{F^v} = \{\alpha \in \Phi_\lambda \mid \alpha(F^v) = \{0\}\}$  and let  $\Phi_{F^v,+} = \{\alpha \in \Phi_{F^v} \mid \alpha(C^v) > 0\}$ . By Lemma 2.19,  $\Phi_{F^v}$  is a root system in  $(\text{vect}(F^v))^\perp$ . By [Bou81, VI §1.7 Corollaire 1], there exists a chamber  $C_{F^v}^v$  of  $(\text{vect}(F^v))^\perp$  such that  $\Phi_{F^v,+} = \{\alpha \in \Phi_{F^v} \mid \alpha(C_{F^v}^v) > 0\}$ .

We can write  $x - \lambda = u + v$ , where  $u \in \text{vect}(F^v)$  and  $v \in \text{vect}(F^v)^\perp$ . Let  $W_{\Phi_{F^v}}^v = \langle s_\alpha \mid \alpha \in \Phi_{F^v} \rangle \subset W_{\Phi_{F^v}}^v$  be the vectorial Weyl group of  $\Phi_{F^v}$ . Then  $(\text{vect}(F^v))^\perp$  is the union of the closed vectorial chambers of  $\Phi_{F^v}$  and, since  $W_{\Phi_{F^v}}^v$  acts transitively on the set of vectorial chambers of  $\Phi_{F^v}$ , there exists  $w \in W_{\Phi_{F^v}}^v$  such that  $\alpha(w.v) \geq 0$ , for every  $\alpha \in \Phi_{F^v,+}$ . We then have  $\alpha(w.(x - \lambda)) \geq 0$ , for all  $\alpha \in \Phi_{F^v,+}$ , since  $W_{\Phi_{F^v}}^v$  pointwise fixes  $\text{vect}(F^v)$ .

We have  $t_\lambda W_{\Phi_{F^v}}^v t_{-\lambda} = W_F^{\text{aff}}$ , where  $t_\mu : \mathbb{A} \rightarrow \mathbb{A}$  is the translation by the vector  $\mu$ , for  $\mu \in \mathbb{A}$ . Let  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ . Set  $\underline{w} = t_\lambda w t_{-\lambda} \in W_F^{\text{aff}}$ . Write  $\underline{\alpha} = \alpha + k$ , where  $k \in \mathbb{Z}$ . Then  $\underline{\alpha}(\lambda) = 0$  and thus  $k = -\alpha(\lambda)$ . Moreover,  $\underline{\alpha}(C) > 0$  and thus  $\alpha \in \Phi_{F^v,+}$ . Then  $\underline{\alpha}(w.x) = \alpha(\lambda + w.(x - \lambda)) - \alpha(\lambda) = \alpha(w.(x - \lambda)) \geq 0$ . This being true for every  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ , the lemma follows.  $\square$

**Definition 3.16.** *Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ . The **star of  $F$**  is the union of the faces of  $(\mathbb{A}, \mathcal{H})$  dominating  $F$ .*

**Lemma 3.17.** *Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  and let  $E$  be the star of  $F$ . Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  dominating  $F$  and  $H_1, \dots, H_k \in \mathcal{H}$  be the walls of  $C$ . Up to renumbering, we may assume that  $H_1, \dots, H_{k'} \supset F$  and  $H_{k'+1}, \dots, H_k$  do not contain  $F$ , for some  $k' \in \llbracket 1, k \rrbracket$ . For  $i \in \llbracket 1, k \rrbracket$ , denote by  $\dot{D}_i$  the open half-space of  $\mathbb{A}$  delimited by  $H_i$  and containing  $C$ . Then  $E = \bigcap_{w \in W_F^{\text{aff}}, i \in \llbracket k'+1, k \rrbracket} w.\dot{D}_i$ . In particular,  $E$  is open and convex.*

*Proof.* Let  $E' = E \cap \overline{C}$ . Then by Lemma 3.8,  $E = W_F.E'$ .

We first assume that  $\Phi$  is irreducible. Let  $\Sigma_C^{\text{aff}}$  be the basis of  $\Phi^{\text{aff}}$  defined by  $C$ , i.e.,  $\Sigma_C^{\text{aff}}$  is the set of affine roots  $\underline{\alpha}$  such that  $\underline{\alpha}^{-1}(\{0\})$  is a wall of  $C$  and  $\underline{\alpha}(C) > 0$ . By Lemma 3.5, we have  $C = \{x \in \mathbb{A} \mid \underline{\alpha}(x) > 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$ . Let  $\mathcal{M}_F = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F) > 0\}$ . If  $F'$  is a face of  $C$ , then by Lemma 3.5, we can write  $F' = \{x \in \mathbb{A} \mid \underline{\alpha}(x) R_{\underline{\alpha}} 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$ , where  $(R_{\underline{\alpha}}) \in \{>, =\}^{\Sigma_C^{\text{aff}}}$ . Then  $\overline{F'} = \{x \in \mathbb{A} \mid \underline{\alpha}(x) \overline{R_{\underline{\alpha}}} 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$ , where  $\overline{R_{\underline{\alpha}}}$  is equal to “ $\geq$ ” if  $R_{\underline{\alpha}}$  is equal to “ $>$ ” and equal to “ $=$ ” if  $R_{\underline{\alpha}}$  is equal to “ $=$ ”. Therefore if  $\overline{F'} \supset F$ , the symbol  $R_{\underline{\alpha}}$  is equal to “ $>$ ” for every  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ . Therefore:

$$E' = \{x \in \mathbb{A} \mid \underline{\alpha}(x) \geq 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F, \underline{\alpha}(x) > 0, \forall \underline{\alpha} \in \mathcal{M}_F\}. \quad (3.7)$$

Let us prove that

$$E = \bigcap_{\underline{\alpha} \in W_F^{\text{aff}} \cdot \mathcal{M}_F} \underline{\alpha}^{-1}(\mathbb{R}_{>0}) := E_1. \quad (3.8)$$

If  $s_{\underline{\alpha}}$  denotes the orthogonal reflection with respect to  $\underline{\alpha}^{-1}(\{0\})$  for  $\underline{\alpha} \in \Phi^{\text{aff}}$ , then  $(W^{\text{aff}}, \{s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}}\})$  is a Coxeter group and  $W_F^{\text{aff}} = \langle s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F \rangle$ , according to Theorem 3.7 (3). Thus by [BB05, Corollaries 1.4.8 and 1.4.6] if  $w \in W_F^{\text{aff}}$  and  $\underline{\alpha} \in \mathcal{M}_F$ , we have  $\ell^{\text{aff}}(ws_{\underline{\alpha}}) = \ell^{\text{aff}}(w) + 1$ , where  $\ell^{\text{aff}}$  denotes the length on  $(W^{\text{aff}}, \{s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}}\})$ . By [Mac03, (2.2.8)], we deduce  $w \cdot \underline{\alpha} \in \Phi_+^{\text{aff}}$ , where  $\Phi_+^{\text{aff}} = \{\underline{\gamma} \in \Phi^{\text{aff}} \mid \underline{\gamma}(C) \geq 0\}$ . Since  $w \in \langle s_{\underline{\beta}} \mid \underline{\beta} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F \rangle$ , we can write  $w \cdot \underline{\alpha} = \underline{\alpha} + \sum_{\underline{\beta} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} n_{\underline{\beta}} \underline{\beta}$ , where the  $n_{\underline{\beta}}$  are non-negative integers (by [Mac72, (1.4) and 4.6 Proposition]). Therefore  $w \cdot \underline{\alpha}(E') > 0$ . Consequently  $E' \subset E_1$  and since  $E_1$  is  $W_F^{\text{aff}}$ -invariant, we deduce that  $E \subset E_1$ .

Let  $x \in E_1$ . Then by Lemma 3.15, there exists  $w \in W_F^{\text{aff}}$  such that  $\underline{\alpha}(w \cdot x) \geq 0$  for all  $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ . By definition of  $E_1$ , we then have  $w \cdot x \in E'$ , by Eq. (3.7) and thus  $x \in E$ , which proves Eq. (3.8), when  $\Phi$  is irreducible.

Assume now that  $\Phi$  is reducible. Using the notation of Eq. (2.4), with  $\Phi$  instead of  $\Psi$ , we write  $\mathbb{A} = \mathbb{A}_1 \times \dots \times \mathbb{A}_\ell$  and  $\Phi = \Phi_1 \sqcup \dots \sqcup \Phi_\ell$ , where  $\ell \in \mathbb{Z}_{\geq 2}$  and  $\Phi_i$  is an irreducible root system in  $\mathbb{A}_i$  for  $i \in \llbracket 1, \ell \rrbracket$ . Write  $C = C_1 \times \dots \times C_\ell$  and  $F = F_1 \times \dots \times F_\ell$ . Then with obvious notations, we have:

$$\begin{aligned} E &= (W_{F_1}^{\text{aff}} \times \dots \times W_{F_\ell}^{\text{aff}}) \cdot (E'_1 \times \dots \times E'_\ell) \\ &= (W_{F_1}^{\text{aff}} \cdot E'_1 \times \dots \times W_{F_\ell}^{\text{aff}} \cdot E'_\ell) \\ &= \bigcap_{i=1}^{\ell} \bigcap_{\underline{\alpha} \in W_{F_i}^{\text{aff}} \cdot \mathcal{M}_{F_i}} \underline{\alpha}^{-1}(\mathbb{R}_{>0}), \end{aligned}$$

which proves the lemma.  $\square$

**Lemma 3.18.** *Let  $F, F'$  be faces of  $(\mathbb{A}, \mathcal{H})$  such that  $F'$  dominates  $F$ . Let  $x \in F'$  and  $y = \frac{1}{|W_F^{\text{aff}}|} \sum_{w \in W_F^{\text{aff}}} w \cdot x$ . Then  $y \in F$  and  $y$  is the orthogonal projection of  $x$  on the support of  $F$ .*

*Proof.* By Lemma 3.17,  $y$  belongs to a face  $F''$  dominating  $F$ . We thus have  $W_{F''}^{\text{aff}} = W_{F''}^{\text{aff}} \subset W_F^{\text{aff}}$ . Moreover,  $W_{F''}^{\text{aff}}$  fixes  $y$  and thus by Theorem 3.7,  $W_F^{\text{aff}}$  fixes  $F''$ . Consequently  $W_F^{\text{aff}} = W_{F''}^{\text{aff}}$  and hence  $F = F''$ , by Lemma 3.9. In other words,  $y \in F$ .

Let  $z \in F$  and set  $t = \langle x - y \mid y - z \rangle$ . Then for  $w \in W_F^{\text{aff}}$ , we have  $t = \langle x - y \mid w^{-1} \cdot (y - z) \rangle = \langle w \cdot x - w \cdot y \mid y - z \rangle = \langle w \cdot x - y \mid y - z \rangle$ . Therefore

$$|W_F^{\text{aff}}|t = \sum_{w \in W_F^{\text{aff}}} \langle w \cdot x \mid y - z \rangle - |W_F^{\text{aff}}| \langle y \mid y - z \rangle = |W_F^{\text{aff}}| \langle y \mid y - z \rangle - |W_F^{\text{aff}}| \langle y \mid y - z \rangle = 0,$$

which proves the lemma.  $\square$

**Lemma 3.19.** *Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ ,  $\lambda \in \text{vert}(F)$  and  $E$  be the star of  $\lambda$ . Then  $F = E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$ .*

*Proof.* The inclusion  $F \subset E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$  is clear. Let  $x \in E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$  and let  $F_1$  be the face of  $(\mathbb{A}, \mathcal{H})$  containing  $x$ . Then  $F_1$  dominates  $\lambda$  and  $x - \lambda \in \mathbb{R}_{>0}(F_1 - \lambda) \cap \mathbb{R}_{>0}(F - \lambda)$ . As the faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  form a partition of  $\mathbb{A}$ , we deduce  $\mathbb{R}_{>0}(F_1 - \lambda) = \mathbb{R}_{>0}(F - \lambda)$  and thus  $F_1 = F$ , by Proposition 3.12. Therefore  $E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda)) \subset F$  and the lemma follows.  $\square$

## 3.2 Affine weight polytopes

### 3.2.1 Definition of affine weight polytopes

Let  $C_0$  be the fundamental alcove of  $\mathbb{A}$ , i.e., the alcove of  $\mathbb{A}$  based at 0 and contained in  $C_f^v$ . Let us fix  $\mathbf{b} \in C_0$ . Let  $C \in \text{Alc}(\mathcal{H})$ . Then there exists  $w \in W^{\text{aff}}$  such that  $C = w.C_0$ . We then set  $\mathbf{b}_C = w.\mathbf{b} \in C$ . The element  $w$  is uniquely determined by  $C$ , since  $W^{\text{aff}}$  acts simply transitively on the set of alcoves of  $(\mathbb{A}, \mathcal{H})$ . Thus  $\mathbf{b}_C$  is well-defined.

**Definition 3.20.** For  $\lambda \in \text{vert}(\mathbb{A})$ , we set  $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{P}_{\mathbf{b}, \Phi}(\lambda)} = \text{conv}(\{\mathbf{b}_C \mid C \in \text{Alc}_\lambda(\mathcal{H})\})$ . Then we call  $\overline{\mathcal{P}(\lambda)}$  the **closed affine Weyl polytope associated with  $\lambda$  (and  $\mathbf{b}$ )**. The **(open) affine Weyl polytope associated with  $\lambda$**  is the interior of  $\overline{\mathcal{P}(\lambda)}$ . By Lemma 3.8,

$$\overline{\mathcal{P}(\lambda)} = \text{conv}(W_\lambda^{\text{aff}}.\mathbf{b}_C) \quad (3.9)$$

for any  $C \in \text{Alc}_\lambda(\mathcal{H})$ .

Note that in order to avoid confusions we use roman letters to denote the faces of  $(\mathbb{A}, \mathcal{H})$  and curly letters for the faces of the  $\overline{\mathcal{P}(\lambda)}$ .

### 3.2.2 Description of the faces of the affine weight polytopes

Let  $\lambda \in \text{vert}(\mathbb{A})$ . We have  $W_{\Phi_\lambda}^v = \langle s_\alpha \mid \alpha \in \Phi_\lambda \rangle \subset W_\Phi^v$ . We denote by  $\mathbf{t}_\lambda$  the translation on  $\mathbb{A}$  by the vector  $\lambda$  (this is not an element of  $W^{\text{aff}}$  in general).

**Lemma 3.21.** Let  $\lambda \in \text{vert}(\mathbb{A})$ . Then  $W_\lambda^{\text{aff}} = \mathbf{t}_\lambda W_{\Phi_\lambda}^v \mathbf{t}_\lambda^{-1}$ .

*Proof.* Let  $\alpha \in \Phi_\lambda$ . Then  $\mathbf{t}_\lambda s_\alpha \mathbf{t}_\lambda^{-1}$  is the orthogonal reflection fixing  $\lambda + \alpha^{-1}(\{0\}) = \alpha^{-1}(\alpha(\lambda))$ . Therefore it belongs to  $W_\lambda^{\text{aff}}$ . By Theorem 3.7,  $W_\lambda^{\text{aff}}$  is generated by the reflections fixing the walls containing  $\lambda$ , which proves the lemma.  $\square$

Let  $\lambda \in \text{vert}(\mathbb{A})$ . Then  $\overline{\mathcal{P}(\lambda)} = \text{conv}(\mathbf{t}_\lambda W_{\Phi_\lambda}^v \mathbf{t}_\lambda^{-1}.\mathbf{b}_C) = \lambda + \text{conv}(W_{\Phi_\lambda}^v.(-\lambda + \mathbf{b}_C))$  for  $C \in \text{Alc}_\lambda(\mathcal{H})$ . Moreover,  $\alpha(\mathbf{b}_C) \notin \mathbb{Z}$ , for all  $\alpha \in \Phi$  and thus  $\alpha(-\lambda + \mathbf{b}_C) \neq 0$  for all  $\alpha \in \Phi_\lambda$ . Therefore, with the notation of subsection 2.2, we have:

$$\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{-\lambda + \mathbf{b}_C, \Phi_\lambda}^0}. \quad (3.10)$$

**Definition 3.22.** Let  $\lambda \in \text{vert}(\mathbb{A})$  and  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$ . We choose an alcove  $C$  dominating  $F$  and we set  $\mathcal{F}_F = \text{Int}_r(\text{conv}(W_F^{\text{aff}}.\mathbf{b}_C))$ . This is well-defined by Lemma 3.8.

Note that for  $\lambda \in \text{vert}(\mathbb{A})$ , we have  $\mathcal{F}_\lambda = \mathcal{P}(\lambda)$ .

**Lemma 3.23.** 1. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ . Then  $\overline{\mathcal{F}_F}$  is contained in the star of  $F$ . In particular, if  $\lambda \in \text{vert}(\mathbb{A})$ , then  $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{F}_{\{\lambda\}}}$  is contained in the star of  $\lambda$ .

2. Let  $B$  be a bounded subset of  $\mathbb{A}$ . Then  $\{\lambda \in \text{vert}(\mathbb{A}) \mid \overline{\mathcal{P}(\lambda)} \cap B \neq \emptyset\}$  is finite.

*Proof.* 1. Let  $E$  be the star of  $F$ . We have  $\overline{\mathcal{F}_F} = \text{conv}(W_F^{\text{aff}}.\mathbf{b}_C)$ , for any alcove  $C$  dominating  $F$ . As  $\mathbf{b}_C \in E$  and as  $E$  is stabilized by  $W_F^{\text{aff}}$ , we deduce  $W_F^{\text{aff}}.\mathbf{b}_C \subset E$ . By Lemma 3.17,  $E$  is convex and thus  $\overline{\mathcal{F}_F} \subset E$ .

2. As  $B$  is bounded, there exists a finite set of closed alcoves  $F$  such that  $B \subset \bigcup_{\overline{C} \in F} \overline{C}$ . Let  $F'$  be the set of closed alcoves containing a vertex of an element of  $F$ . Then by statement (1), if  $\lambda \in \text{vert}(\mathbb{A})$  is such that  $B \cap \overline{\mathcal{P}(\lambda)}$  is non-empty, then  $\lambda \in \bigcup_{\overline{C} \in F'} \overline{C} \cap \text{vert}(\mathbb{A})$ , which is finite. This proves statement (2).  $\square$

For a polytope  $\mathcal{Q}$  of  $\mathbb{A}$ , we denote by  $\text{Face}(\mathcal{Q})$  its set of faces. We equip  $\text{Face}(\mathcal{Q})$  with the dominance order  $\leq$ . Recall that, if  $E$  is an affine subspace of  $\mathbb{A}$ , its direction  $\text{dir}(E)$ , is the vector subspace  $\{x - x' \mid x, x' \in E\}$  of  $\mathbb{A}$ . (I prefer to recall the definition since it is far.)

**Proposition 3.24.** (see Figure 3.5) Let  $\lambda \in \text{vert}(\mathbb{A})$ . Let  $\mathcal{P}^0 = \mathcal{P}(\lambda) - \lambda$ .

1. The map  $\text{Face}(\overline{\mathcal{P}(\lambda)}) \rightarrow \text{Face}(\overline{\mathcal{P}^0})$  given by  $\mathcal{F} \mapsto \mathcal{F} - \lambda$  is an isomorphism of partially ordered sets.
2. Let  $C \in \text{Alc}_\lambda(\mathcal{H})$  and  $C^v = \mathbb{R}_{>0}(C - \lambda)$ . Set  $\mathbf{b}_{C^v}^0 = \mathbf{b}_C - \lambda$ . For each chamber  $C_1^v$  of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$ , set  $\mathbf{b}_{C_1^v}^0 = w \cdot \mathbf{b}_{C^v}^0$ , where  $w \in W_{\Phi_\lambda}^v$  satisfies  $C_1^v = w \cdot C^v$ . For  $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ , set  $\mathcal{F}_{F^v}^0 = \text{Int}_r(\text{conv}(W_{F^v, \Phi_\lambda}^v \cdot \mathbf{b}_{C_1^v}^0))$ , where  $C_1^v$  is a chamber of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  dominating  $F^v$ . Then for every  $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ , we have  $\mathcal{F}_{F^v}^0 + \lambda = \mathcal{F}_F$ , where  $F \in \mathcal{F}_\lambda(\mathcal{H})$  satisfies  $F^v = \mathbb{R}_{>0}(F - \lambda)$ .
3. The map  $F_1 \mapsto \mathcal{F}_{F_1}$  is an isomorphism between  $(\mathcal{F}_\lambda(\mathcal{H}), \leq)$  and  $(\text{Face}(\overline{\mathcal{P}(\lambda)}), \geq)$ . Moreover,  $\text{dir}(\text{supp}(\mathcal{F}_{F_1})) = (\text{dir}(\text{supp}(F_1)))^\perp$  and  $\dim(\text{supp}(\mathcal{F}_{F_1})) = \text{codim}_\mathbb{A}(F_1)$  for every  $F_1 \in \mathcal{F}_\lambda(\mathcal{H})$ .

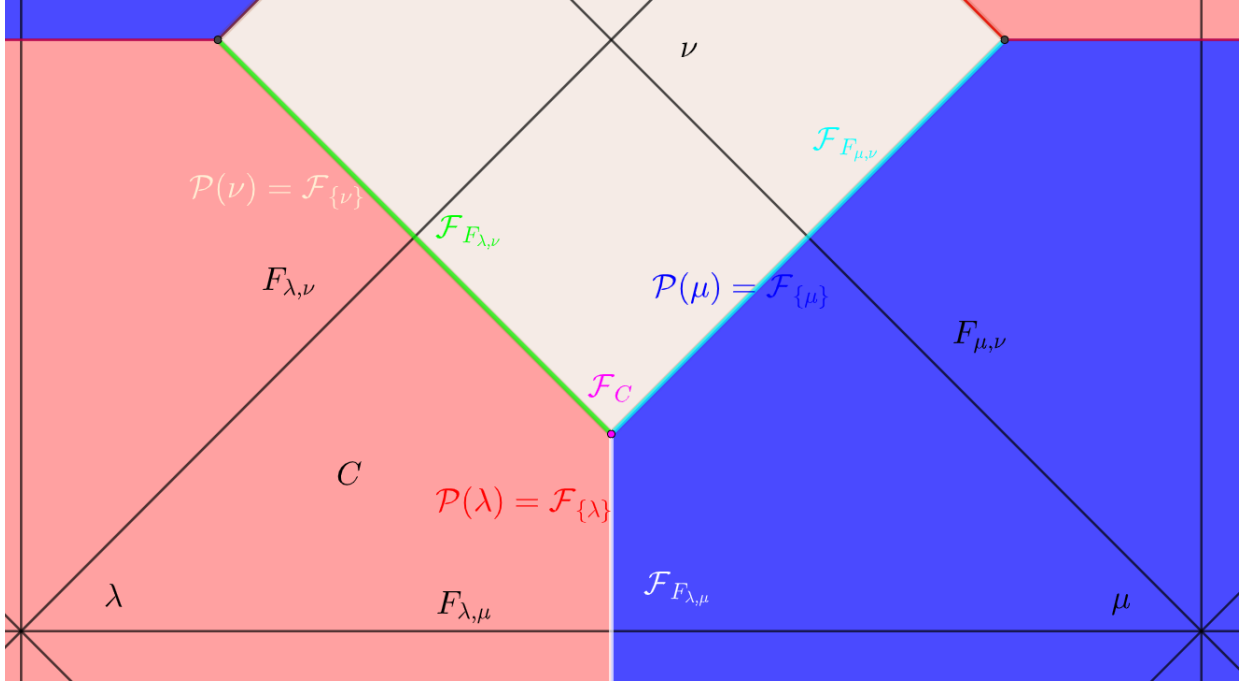


Figure 3.5: Illustration of Proposition 3.24: the case of  $B_2$ .

*Proof.* Statement (1) is clear.

Now, we prove statement (2). By Proposition 3.12,  $C^v$  is a chamber of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$ . By Eq. (3.10), we have  $\overline{\mathcal{P}(\lambda) - \lambda} = \overline{\mathcal{P}^0} = \overline{\mathcal{P}_{\mathbf{b}_{C^v, \Phi_\lambda}^0}^0} = \text{conv}(W_{\Phi_\lambda}^v \cdot \mathbf{b}_{C^v}^0)$ . Let  $F_1^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$  and let  $C_1^v$  be a chamber of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  dominating  $F_1^v$ . Let  $\mathcal{F} = \mathcal{F}_{F_1^v}^0 + \lambda$ . Let  $F_1, C_1 \in \mathcal{F}_\lambda(\mathbb{A})$  be such that  $F_1^v = \mathbb{R}_{>0}(F_1 - \lambda)$  and  $C_1^v = \mathbb{R}_{>0}(C_1 - \lambda)$ .

Using Lemma 3.8, we get  $C_1 = w_1 \cdot C$ , with  $w_1 \in W_\lambda^{\text{aff}}$ . Then we have:

$$w_1 \cdot C - \lambda = C_1 - \lambda = w_1 \cdot C - w_1 \cdot \lambda = \vec{w}_1 \cdot (C - \lambda),$$

where  $\vec{w}_1$  is the linear part of  $w_1$ . By Lemma 3.21,  $\vec{w}_1 \in W_{\Phi_\lambda}^v$ . Moreover we have:

$$\mathbb{R}_{>0} \vec{w}_1 \cdot (C - \lambda) = \vec{w}_1 \cdot C^v = \mathbb{R}_{>0}(C_1 - \lambda) = C_1^v.$$

Hence:

$$\overline{\mathcal{F}} = \mathbf{t}_\lambda \cdot \overline{\mathcal{F}_{F_1}^v} = \text{conv}(\mathbf{t}_\lambda \cdot W_{F_1^v, \Phi_\lambda}^v \cdot \mathbf{b}_{C^v}^0) = \text{conv}\left(W_{\mathbf{t}_\lambda \cdot F_1^v}^{\text{aff}} \cdot (\lambda + \mathbf{b}_{C_1^v}^0)\right) = \text{conv}(W_{F_1}^{\text{aff}} \cdot (\lambda + \mathbf{b}_{C_1^v}^0)),$$

where  $W_{F_1^v, \Phi_\lambda}^v$  is the fixator of  $F_1^v$  in  $W_{\Phi_\lambda}^v$ . Moreover,  $\mathbf{b}_{C_1} = w_1 \cdot \mathbf{b}_C = w_1 \cdot (\lambda + \mathbf{b}_{C^v}^0) = w_1 \cdot \lambda + \bar{w}_1 \cdot \mathbf{b}_{C^v}^0 = \lambda + \mathbf{b}_{C_1^v}^0$ , which proves that  $\overline{\mathcal{F}} = \text{conv}(W_{F_1}^{\text{aff}} \cdot \mathbf{b}_{C_1}) = \overline{\mathcal{F}_{F_1}}$ . Therefore  $\mathcal{F} = \text{Int}_r(\overline{\mathcal{F}_{F_1}}) = \mathcal{F}_{F_1}$ , which proves (2).

Finally, we prove statement (3). Indeed, by statements (1), (2) and by Propositions 3.12 and 2.20, the map  $F_1 \mapsto \mathcal{F}_{F_1}$  can be decomposed as

$$F_1 \mapsto \mathbb{R}_{>0}(F_1 - \lambda) \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1 - \lambda)}^0 \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1 - \lambda)}^0 + \lambda = \mathcal{F}_{F_1}$$

and then it is an isomorphism between  $(\mathcal{F}_\lambda(\mathcal{H}), \leq)$  and  $(\text{Face}(\overline{\mathcal{P}(\lambda)}), \geq)$ . The last assertion follows from Proposition 2.20 and statement (2).  $\square$

### 3.2.3 Faces and intersection of affine weight polytopes

Recall that, for any pair  $(\lambda, \mu) \in \mathbb{A}^2$ , we write  $[\lambda, \mu] = \{t\lambda + (1-t)\mu \mid t \in [0, 1]\}$  and  $] \lambda, \mu[ = \text{Int}_r([\lambda, \mu]) = \{t\lambda + (1-t)\mu \mid t \in ]0, 1[ \}$ .

**Lemma 3.25.** *Let  $\lambda, \mu \in \text{vert}(\mathbb{A})$  be adjacent. Then  $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$  is a closed facet of both  $\overline{\mathcal{P}(\lambda)}$  and  $\overline{\mathcal{P}(\mu)}$ . More precisely,*

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C),$$

where  $C$  is any alcove dominating  $] \lambda, \mu[$ .

*Proof.* As  $\lambda$  and  $\mu$  are adjacent,  $] \lambda, \mu[$  is a face of some alcove  $C$ . By Proposition 3.24,  $\overline{\mathcal{F}} := \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C)$  is a closed facet of both  $\overline{\mathcal{P}(\lambda)}$  and  $\overline{\mathcal{P}(\mu)}$ . It remains to prove that  $\lambda$  and  $\mu$  are on opposite sides of  $\overline{\mathcal{F}}$ . By Lemma 3.18,  $\lambda = \frac{1}{|W_\lambda^{\text{aff}}|} \sum_{w \in W_\lambda^{\text{aff}}} w \cdot \lambda \in \overline{\mathcal{P}(\lambda)}$  and similarly,  $\mu \in \overline{\mathcal{P}(\mu)}$ . By Lemma 3.18 applied with  $F = ] \lambda, \mu[$ , the orthogonal projection  $x$  of  $\mathbf{b}_C$  on the line containing  $] \lambda, \mu[$  belongs to  $] \lambda, \mu[ \cap \overline{\mathcal{F}}$ . The support of  $\overline{\mathcal{F}}$  is the hyperplane  $H = x + (\mathbb{R}(\mu - \lambda))^\perp$  by Proposition 3.24(3). Then  $\overline{\mathcal{P}(\lambda)}$  (resp.  $\overline{\mathcal{P}(\mu)}$ ) is contained in the half-space of  $\mathbb{A}$  delimited by  $H$  and containing  $\lambda$  (resp.  $\mu$ ), which proves the lemma.  $\square$

**Lemma 3.26.** *Let  $\lambda, \mu \in \text{vert}(\mathbb{A})$  and assume that there exists an alcove dominating both  $\lambda$  and  $\mu$ . Then  $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$  is a face of  $\overline{\mathcal{P}(\lambda)}$ . Moreover, if  $C$  is an alcove dominating  $\lambda$  and  $\mu$ , then  $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C) = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$ , where  $F$  is the smallest face dominating  $\{\lambda, \mu\}$  (whose existence is provided by Lemma 3.6).*

*Proof.* If  $\Phi$  is irreducible, then this is Lemma 3.25, so we assume that  $\Phi$  is reducible. We use the same notation as in Eq. (2.4), with  $\Phi$  instead of  $\Psi$ . Write  $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_\ell$  and  $\Phi = \bigsqcup_{i=1}^\ell \Phi_i$ . We have  $W^{\text{aff}} = W_{\Phi_1}^{\text{aff}} \times \dots \times W_{\Phi_\ell}^{\text{aff}}$ , where  $W_{\Phi_i}^{\text{aff}}$  is the affine Weyl group of  $(\mathbb{A}_i, \Phi_i)$ , for  $i \in [1, \ell]$ . Write  $\mathbf{b}_C = (\mathbf{b}_1, \dots, \mathbf{b}_\ell)$ . We have  $\text{vert}(\mathbb{A}) = \text{vert}(\mathbb{A}_1) \times \dots \times \text{vert}(\mathbb{A}_\ell)$  and if  $(\tau_1, \dots, \tau_\ell) \in \text{vert}(\mathbb{A})$ , we have  $\mathcal{P}_{\mathbf{b}, \Phi}((\tau_1, \dots, \tau_\ell)) = \mathcal{P}_{\mathbf{b}_1, \Phi_1}(\tau_1) \times \dots \times \mathcal{P}_{\mathbf{b}_\ell, \Phi_\ell}(\tau_\ell)$ . Write  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  and  $\mu = (\mu_1, \dots, \mu_\ell)$ . Then for all  $i \in [1, \ell]$ , either  $\lambda_i = \mu_i$ , or  $\lambda_i$  and  $\mu_i$  are adjacent. By Lemma 3.25, we get that:

$$\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)} \cap \overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)} = \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i)$$

and that this intersection is a facet of both  $\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)}$  and  $\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)}$ . We deduce that:

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \prod_{i=1}^\ell (\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)} \cap \overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)}) = \prod_{i=1}^\ell \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i)$$

is a face of both  $\mathcal{P}(\lambda)$  and  $\mathcal{P}(\mu)$ . The equalities

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C) = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$$

immediately follow from the facts that:

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \prod_{i=1}^{\ell} \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i) = \text{conv}\left(W_{\prod_{i=1}^{\ell} [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_C\right),$$

that  $\overline{F} = \prod_{i=1}^{\ell} [\lambda_i, \mu_i]$  and that  $W_{[\lambda, \mu]}^{\text{aff}} = W_F^{\text{aff}}$ .  $\square$

**Proposition 3.27.** 1. Let  $J$  be a non-empty subset of  $\text{vert}(\mathbb{A})$ . Then  $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$  is non-empty if and only if there exists an alcove of  $(\mathbb{A}, \mathcal{H})$  dominating  $J$ . Moreover, if  $F$  is the smallest face dominating  $J$  (which exists by Lemma 3.6), then  $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)} = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C) = \overline{\mathcal{F}_F} = \bigcap_{\lambda \in \text{vert}(F)} \overline{\mathcal{P}(\lambda)}$ , for any alcove  $C$  dominating  $F$ .

2. If  $F$  is a face of  $(\mathbb{A}, \mathcal{H})$ , then the set of extreme points of  $\overline{\mathcal{F}_F}$  is  $W_F^{\text{aff}} \cdot \mathbf{b}_C$ , for any alcove  $C$  dominating  $F$ . Moreover  $\mathcal{F}_F \cap F = \{x\}$ , where  $x$  is the barycenter of the extreme points of  $\mathcal{F}_F$ .

3. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  and  $x \in \mathcal{F}_F$ . Then  $\{\mu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\mu)}\} = \text{vert}(F)$ .

*Proof.* 1) Let  $J \subset \text{vert}(\mathbb{A})$  be non-empty. Assume that  $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$  is non-empty. Let  $x \in \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$  and let  $F$  be the face of  $(\mathbb{A}, \mathcal{H})$  containing  $x$ . Then by Lemma 3.23,  $F$  dominates  $J$  and, in particular,  $J$  is dominated by any alcove dominating  $F$ . Conversely, if  $J$  is dominated by a face  $F$  of  $(\mathbb{A}, \mathcal{H})$ , then  $W_F^{\text{aff}} \cdot \mathbf{b}_C \subset \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$ , for any alcove  $C$  dominating  $F$ .

Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  and  $J$  be a subset of  $\text{vert}(C)$  with cardinality at least 2. Let  $F$  be the smallest face of  $C$  dominating  $J$ , which exists by Lemma 3.6. Let  $\lambda \in J$ . Then

$$\overline{\mathcal{F}_J} := \bigcap_{\mu \in J} \overline{\mathcal{P}(\mu)} = \bigcap_{\mu \in J \setminus \{\lambda\}} (\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}).$$

By Lemma 3.26 and [BG09, Theorem 1.10],  $\overline{\mathcal{F}_J}$  is a face of  $\overline{\mathcal{P}(\lambda)}$ .

We have:

$$\begin{aligned} \text{Extr}(\mathcal{F}_J) &= \bigcap_{\mu \in J \setminus \{\lambda\}} \text{Extr}(\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}) && \text{by Lemma 2.5,} \\ &= \bigcap_{\mu \in J \setminus \{\lambda\}} W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C && \text{by Lemma 2.4,} \\ &= \left( \bigcap_{\mu \in J \setminus \{\lambda\}} W_{[\lambda, \mu]}^{\text{aff}} \right) \cdot \mathbf{b}_C && \text{since } \mathbf{b}_C \text{ is regular,} \\ &= W_F^{\text{aff}} \cdot \mathbf{b}_C && \text{by Lemma 3.14.} \end{aligned}$$

Thus by Krein-Milman theorem,  $\mathcal{F}_J = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$ , which proves statement (1).

2) Let  $F \in \mathcal{F}(\mathcal{H})$  and  $C \in \text{Alc}(\mathcal{H})$ , dominating  $F$ . By 1) and by Lemma 2.4, the set of extreme points of  $\overline{\mathcal{F}_F}$  is  $W_F^{\text{aff}} \cdot \mathbf{b}_C$ . Let now  $x = \frac{1}{|W_F^{\text{aff}}|} \sum_{w \in W_F^{\text{aff}}} w \cdot \mathbf{b}_C$ . Then  $x \in F$  according to Lemma 3.18. By Proposition 3.24, the direction of the support of  $F \cap \mathcal{F}_F$  is a singleton, which proves that  $\mathcal{F}_F \cap F = \{x\}$  and completes the proof of statement (2).

3) Let  $F \in \mathcal{F}(\mathcal{H})$  and  $x \in \mathcal{F}_F$ . Let  $\mu \in \text{vert}(\mathbb{A}) \setminus \text{vert}(F)$ . If  $\text{vert}(F) \cup \{\mu\}$  is not dominated by any alcove, then  $\mathcal{F}_F \cap \overline{\mathcal{P}(\mu)} = \emptyset$  according to statement (1) and thus



$x \notin \overline{\mathcal{P}(\mu)}$ . Assume that  $\mu \cup \text{vert}(F)$  is dominated by an alcove  $C$ . Then it follows from (1) that  $\overline{\mathcal{F}_F} \cap \overline{\mathcal{P}(\mu)} = \overline{\mathcal{F}_{F'}}$ , where  $F'$  is the smallest face dominating  $F \cup \{\mu\}$ , and by (2) that  $\mathcal{F}_{F'} \neq \mathcal{F}_F$ . Therefore  $\mathcal{F}_{F'}$  is a proper face of  $\mathcal{F}_F$  and by [BG09, Corollary 1.7],  $x \notin \mathcal{F}_{F'}$ , which proves that  $x \notin \overline{\mathcal{P}(\mu)}$ . Therefore  $\{\mu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\mu)}\} \subset \text{vert}(F)$ , and this is actually an equality by 1), which proves statement (3).  $\square$

### 3.3 Tessellation of $\mathbb{A}$ by the affine weight polytopes

We now prove that  $\mathbb{A}$  is tessellated by the  $\overline{\mathcal{P}(\lambda)}$ , for  $\lambda \in \text{vert}(\mathbb{A})$ , as stated in the introduction in Theorem 1.1.

**Theorem 3.28.** (see Figures 3.6, 3.7, 3.8 and 3.9)

Let  $\mathbb{A}$  be a finite dimensional vector space and  $\Phi$  be a finite root system in  $\mathbb{A}$ . Let  $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ , let  $\mathcal{F}$  be the set of faces of  $(\mathbb{A}, \mathcal{H})$  and let  $\text{vert}(\mathbb{A})$  be the set of vertices of  $(\mathbb{A}, \mathcal{H})$ . We choose a fundamental alcove  $C_0$  of  $(\mathbb{A}, \mathcal{H})$  and choose  $\mathbf{b} = \mathbf{b}_{C_0} \in C_0$ . If  $\lambda$  is a vertex of  $(\mathbb{A}, \mathcal{H})$ , we choose an alcove  $C_\lambda$  dominating  $\lambda$  and we set  $\mathbf{b}_{C_\lambda} = w \cdot \mathbf{b}$  where  $w$  is the element of the affine Weyl group  $W^{\text{aff}}$  of  $(\mathbb{A}, \mathcal{H})$  sending  $C_0$  to  $C$ . If  $F$  is a face of  $(\mathbb{A}, \mathcal{H})$ , we write  $W_F^{\text{aff}} := \text{Fix}_{W^{\text{aff}}}(F)$  and we set  $\mathcal{F}_F = \text{Int}_r(\text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C))$ , for any alcove  $C$  dominating  $F$ . Then  $\bigsqcup_{F \in \mathcal{F}} \mathcal{F}_F = \mathbb{A}$ . In particular, if we set  $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{F}_{\{\lambda\}}}$ , for  $\lambda \in \text{vert}(\mathbb{A})$ , then  $\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}$ .

*Proof.* Let  $E = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}$ . We prove that  $E = \mathbb{A}$  by proving that  $E$  is closed and dense in  $\mathbb{A}$ . Take  $(z_n) \in E^{\mathbb{N}}$  a converging sequence. Then  $(z_n)$  is bounded and by Lemma 3.23 (2), there exists a finite subset  $F \subset \text{vert}(\mathbb{A})$  such that  $\{z_n \mid n \in \mathbb{N}\} \subset \bigcup_{\mu \in F} \overline{\mathcal{P}(\mu)}$ . Then as the  $\overline{\mathcal{P}(\mu)}$  are closed,  $\lim z_n \in \bigcup_{\mu \in F} \overline{\mathcal{P}(\mu)} \subset E$ . Therefore  $E$  is closed.

Suppose that  $E \neq \mathbb{A}$  and fix  $x \in \mathbb{A} \setminus E$ . Let  $\lambda \in \text{vert}(\mathbb{A})$  and  $y \in \mathcal{P}(\lambda)$ . Let  $\gamma : [0, 1] \rightarrow [x, y]$  be the affine parametrization of  $[x, y]$  such that  $\gamma(0) = y$  and  $\gamma(1) = x$ . Let

$$t = \inf\{t' \in [0, 1] \mid \gamma(t') \notin E\}.$$

As  $E$  is closed,  $\gamma(t) \in E$ . Let  $\nu \in \text{vert}(\mathbb{A})$  be such that  $\gamma(t) \in \overline{\mathcal{P}(\nu)}$ . Then  $\gamma(t) \in \text{Fr}(\mathcal{P}(\nu))$  and thus there exists a facet  $\overline{\mathcal{F}}$  of  $\overline{\mathcal{P}(\nu)}$  containing  $\gamma(t)$ . By Propositions 3.24 and 3.27, there exists  $\nu' \in \text{vert}(\mathbb{A})$ , adjacent to  $\nu$  and such that  $\overline{\mathcal{P}(\nu)} \cap \overline{\mathcal{P}(\nu')} = \overline{\mathcal{F}}$ . If  $\gamma(t) \in \mathcal{F}$ , then by Lemma 2.1, there exists  $r \in \mathbb{R}_{>0}$  such that

$$B(\gamma(t), r) \cap \overline{\mathcal{P}(\nu)} = D \cap B(\gamma(t), r) \text{ and } B(\gamma(t), r) \cap \overline{\mathcal{P}(\nu')} = D' \cap B(\gamma(t), r),$$

where  $D$  and  $D'$  are the two closed half-spaces delimited by  $\text{supp}(\mathcal{F})$ . But then  $B(\gamma(t), r) \subset \overline{\mathcal{P}(\nu)} \cup \overline{\mathcal{P}(\nu')}$ , which is impossible by definition of  $t$ . Therefore

$$\gamma(t) \in \overline{\mathcal{F}} \setminus \mathcal{F}.$$

The facet  $\mathcal{F}$  is a polytope of  $\text{supp}(\mathcal{F})$  and thus its frontier is contained in a union  $\bigcup_{V \in F(\nu, \nu')} V$ , where  $F(\nu, \nu')$  is a finite set of affine subspaces of dimension at most  $\dim \mathbb{A} - 2$ . For  $\nu, \nu' \in \text{vert}(\mathbb{A})$ , set  $F(\nu, \nu') = \emptyset$  if  $\nu$  and  $\nu'$  are not adjacent. Then we proved:

$$\mathbb{A} \setminus E \subset \bigcup_{\nu, \nu' \in \text{vert}(\mathbb{A})} \bigcup_{V \in F(\nu, \nu')} \text{supp}(V \cup \{x\}).$$

As the right hand side of this formula is a countable union of proper affine subspaces of  $\mathbb{A}$ , it has Lebesgue measure 0. In particular,  $E$  is dense in  $\mathbb{A}$ . But as  $E$  is closed,  $E = \mathbb{A}$ . Thus we proved

$$\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}.$$

As for  $\lambda \in \text{vert}(\mathbb{A})$ ,  $\overline{\mathcal{P}(\lambda)}$  is the union of the  $\mathcal{F}_F$ , where  $F$  runs over the faces of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$  (see Proposition 3.24), we deduce that  $\mathbb{A} = \bigcup_{F \in \mathcal{F}} \mathcal{F}_F$ . Let  $F_1, F_2 \in \mathcal{F}$  be such that  $\mathcal{F}_{F_1} \cap \mathcal{F}_{F_2}$  is non-empty. Then  $\overline{\mathcal{F}_{F_1}} \cap \overline{\mathcal{F}_{F_2}} \neq \emptyset$  and thus by Proposition 3.27, there exists an alcove  $C$  dominating  $\text{vert}(F_1) \cup \text{vert}(F_2)$ , and  $\mathcal{F}_{F_1}, \mathcal{F}_{F_2}$  are faces of  $\overline{\mathcal{P}(\lambda)}$ , for any  $\lambda \in \text{vert}(C)$ , by Proposition 3.24 3). As the faces of  $\overline{\mathcal{P}(\lambda)}$  form a partition  $\overline{\mathcal{P}(\lambda)}$  (by [BG09, Theorem 1.10]), we deduce that  $F_1 = F_2$ , which concludes the proof of the theorem.  $\square$

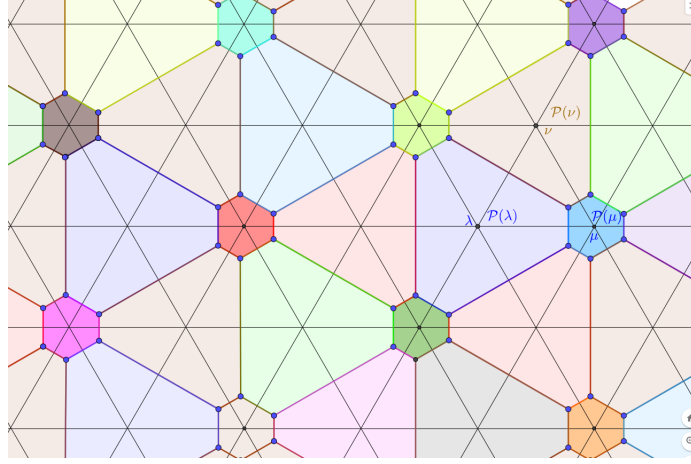


Figure 3.6: Illustration of Theorem 3.28: the case of  $A_2$  when  $\mathbf{b}$  is not the barycenter of the fundamental alcove.

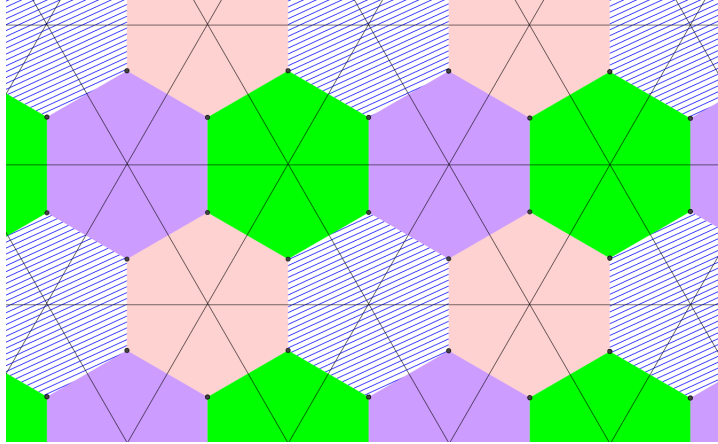


Figure 3.7: Illustration of Theorem 3.28: the case of  $A_2$  when  $\mathbf{b}$  is the barycenter of the fundamental alcove.

## 4 Thickened tessellation of $\mathbb{A}$

The aim of this section is as follows. Take  $\eta \in ]0, 1[$ . For  $\lambda \in \text{vert}(\mathbb{A})$ , denote by  $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}, h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda)$  the homothety centered at  $\lambda$  with ratio  $\eta$ . Let  $x \in h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$ . Write  $x = h_{\lambda, \eta}(x')$ , with  $x' \in \overline{\mathcal{P}(\lambda)}$ . Let  $F$  be the face of  $(\mathbb{A}, \mathcal{H})$  such that  $x' \in \mathcal{F}_F$ . Let  $\tilde{F}_\eta(x) = x + (1 - \eta)(\overline{F} - \lambda)$ . This is a copy of  $F$  which contains exactly one point of  $h_{\mu, \eta}(\overline{\mathcal{P}(\mu)})$ , for each vertex  $\mu$  of  $F$  (and which is the convex hull of these points). The main goal of this section is to prove Theorem 4.19, which

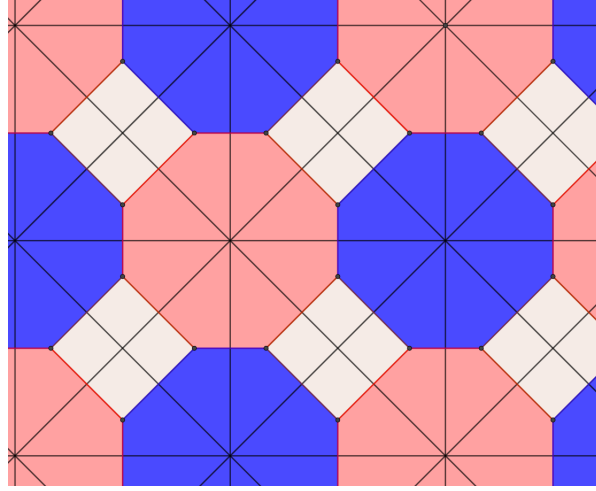


Figure 3.8: Illustration of Theorem 3.28: the case of  $B_2$ .

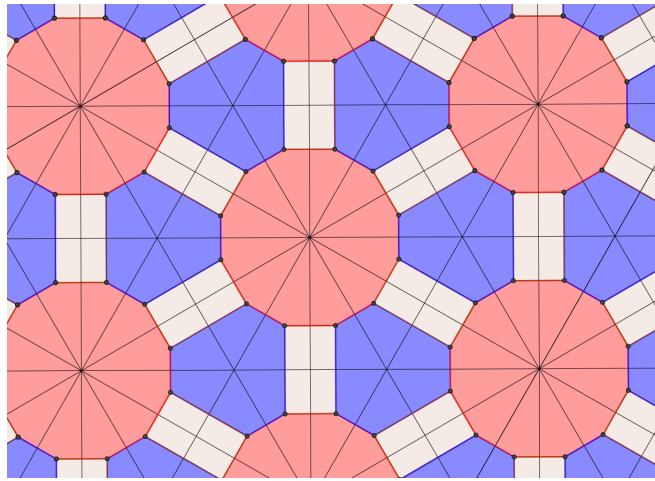


Figure 3.9: Illustration of Theorem 3.28: the case of  $G_2$ .

states that  $\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A}), x \in h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})} \overline{\tilde{F}_\eta(x)}$  (see Figures 4.1, 4.4 and 4.5). Since for all  $\mu \in \text{vert}(F)$ , we have  $\tilde{F}_\eta(h_{\mu, \eta}(x')) = \tilde{F}_\eta(x)$ , in order to describe  $\mathbb{A}$  as a disjoint union, we will introduce a new notation by replacing  $\overline{\tilde{F}_\eta(x)}$  by  $\overline{F_\eta(x')}$  (see Definition 4.13) We will then have  $\mathbb{A} = \bigsqcup_{x' \in \mathbb{A}} \overline{F_\eta(x')}$ .

We divide this section in two parts. Firstly, in subsection 4.1, we prove that if  $\lambda$  and  $\mu$  are two adjacent vertices of  $\mathbb{A}$ , then the projection of  $\overline{\mathcal{P}(\mu)}$  on  $\overline{\mathcal{P}(\lambda)}$  is contained in  $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$  (see Proposition 4.8). Then, in subsection 4.2, we prove Theorem 4.19.

## 4.1 Projection of a polytope on a neighbour polytope

If  $F$  is a face of  $(\mathbb{A}, \mathcal{H})$  and  $\lambda \in \text{vert}(F)$ , we denote by  $\text{vert}_\lambda(F)$  the set of vertices of  $F$  which are adjacent to  $\lambda$  (and different from  $\lambda$ ). If  $\lambda \in \text{vert}(\mathbb{A})$ , we denote by  $\pi_\lambda$  instead of  $\pi_{\overline{\mathcal{P}(\lambda)}}$  the orthogonal projection on  $\overline{\mathcal{P}(\lambda)}$ .

In this subsubsection, we prove that if  $\lambda$  and  $\mu$  are two vertices of  $\mathbb{A}$  contained in a common closed alcove, then  $\pi_\lambda(\overline{\mathcal{P}(\mu)}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$  (see Proposition 4.8).

**Lemma 4.1.** *1. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  and  $\lambda \in \text{vert}(F)$ . Then:*

$$\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{>0}(\mu - \lambda) = \sum_{\mu \in \text{vert}(F)} \mathbb{R}_{>0}(\mu - \lambda).$$

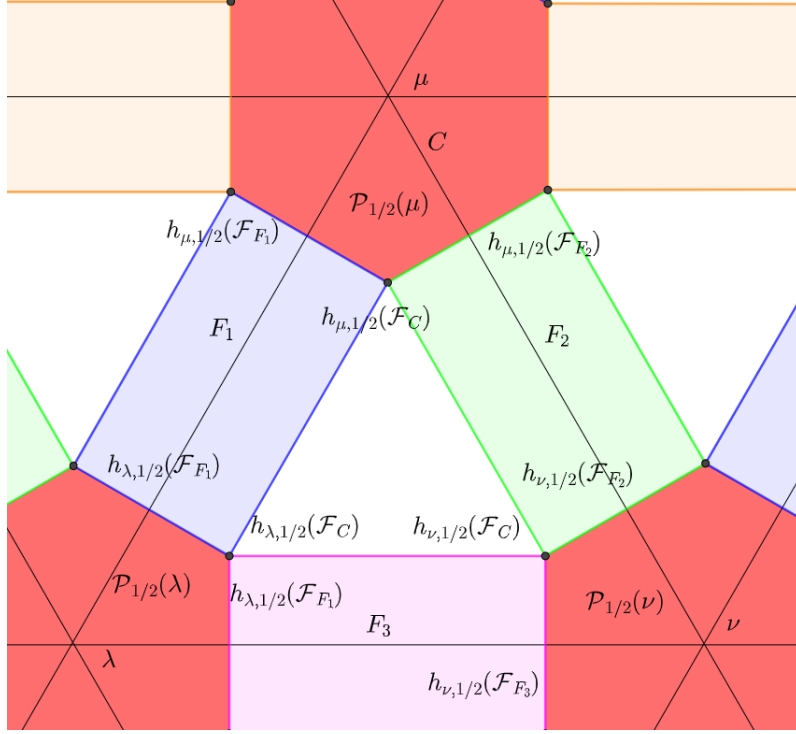


Figure 4.1: Illustration of Theorem 4.19 in the case of  $A_2$ . The represented alcove  $C$  is the equilateral triangle delimited by the black segments. The hexagons  $\mathcal{P}_{1/2}(\lambda)$ ,  $\mathcal{P}_{1/2}(\mu)$  and  $\mathcal{P}_{1/2}(\nu)$  are represented in red. The edges of  $C$  are the segments  $\overline{F_1}$ ,  $\overline{F_2}$  and  $\overline{F_3}$  and the vertices of  $C$  are  $\lambda$ ,  $\mu$  and  $\nu$ . The blue rectangle is the union of the  $\overline{\tilde{F}_{1/2}(x)}$  for  $x \in h_{\lambda,1/2}(\mathcal{F}_{F_1})$  (or equivalently  $x \in h_{\mu,1/2}(\mathcal{F}_{F_1})$ ). Similarly, the green and pink rectangles correspond to the  $\overline{\tilde{F}_{1/2}(x)}$  for  $x$  in  $h_{\mu,1/2}(\mathcal{F}_{F_2})$  and  $h_{\nu,1/2}(\mathcal{F}_{F_2})$  respectively. The face  $\overline{\mathcal{F}_C}$  is the (non represented) vertex  $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)}$ . The central white triangle is  $\overline{\tilde{F}_{1/2}(z)}$ , for  $z \in h_{\lambda,1/2}(\mathcal{F}_C)$  (or equivalently,  $z \in h_{\mu,1/2}(\mathcal{F}_C)$  or  $z \in h_{\nu,1/2}(\mathcal{F}_C)$ ).

2. Let  $\lambda \in \text{vert}(\mathbb{A})$  and  $F, F'$  be faces of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$ . Then  $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$  if and only if  $F$  is dominated by  $F'$ .

*Proof.* 1) The map  $\mu \mapsto ]\lambda, \mu[$  is a bijection from  $\text{vert}_\lambda(F)$  to the set of one dimensional faces of  $F$  dominating  $\lambda$ . By Proposition 2.6 and Proposition 3.12,  $\mathbb{R}_{>0}(F - \lambda)$  is the sum of its one-dimensional faces and thus we have  $\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{>0}(\mu - \lambda)$ . Using the notation of Eq. (2.4) (with  $\ell = 1$  if  $\Phi$  is irreducible), we write  $F = F_1 \times \dots \times F_\ell$  and  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ . We have  $\text{vert}_\lambda(F) = \bigsqcup_{i=1}^\ell \{(\lambda_1, \dots, \lambda_{i-1})\} \times \text{vert}_{\lambda_i}(F_i) \times \{(\lambda_{i+1}, \dots, \lambda_\ell)\}$  and  $\text{vert}(F) = \text{vert}(F_1) \times \dots \times \text{vert}(F_\ell)$  and the second equality of 1) follows.

2) If  $F$  is dominated by  $F'$ , then  $\text{vert}(F) \subset \overline{\text{vert}(F')}$  and thus  $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$ . Conversely, if  $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$ , then  $\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{\geq 0}(\mu - \lambda) \subset \sum_{\mu \in \text{vert}_\lambda(F')} \mathbb{R}_{\geq 0}(\mu - \lambda) = \overline{\mathbb{R}_{>0}(F' - \lambda)}$ , whence the result follows.  $\square$

**Lemma 4.2.** Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  and  $\lambda \in \text{vert}(F)$ . Let  $x \in \mathcal{F}_F$ . Then  $\pi_\lambda^{-1}(\{x\}) = x + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)$ .

*Proof.* By Proposition 3.24, the facets of  $\mathcal{P}(\lambda)$  are the  $\mathcal{F}_{F_1}$  for  $F_1 \in \mathcal{F}_\lambda(F)$  of dimension 1. If  $F_1 = ]\lambda, \mu[$  is such a face, with  $\mu \in \text{vert}(\mathbb{A})$ , then by Proposition 3.24, we have  $\delta_{\mathcal{F}_{F_1}} = \mathbb{R}_{\geq 0}(\mu - \lambda)$ , with the notation of subsection 2.1. Using Proposition 2.3, we deduce  $\pi_\lambda^{-1}(\{x\}) = x + \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{\geq 0}(\mu - \lambda)$ . Then, the result follows from Lemma 4.1.  $\square$

**Lemma 4.3.** *Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  and  $F$  be a proper subface of  $C$  which is not a vertex. Let  $x \in \mathcal{F}_F$ . Let  $\lambda, \mu \in \text{vert}(F)$  and  $\nu \in \text{vert}(C) \setminus \text{vert}(F)$ . Then  $\pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)} = \emptyset$ .*

*Proof.* Let  $F(\mu, \nu)$  be the smallest face of  $(\mathbb{A}, \mathcal{H})$  dominating both  $\mu$  and  $\nu$ . Such face exists according to Lemma 3.6. By Proposition 3.27, we have:

$$\overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)} = \bigcap_{\nu' \in \text{vert}_\mu(F(\mu, \nu))} \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu')}.$$

We have  $F(\mu, \nu) \not\subseteq \overline{F}$  and thus Lemma 4.1 implies the existence of  $\nu'$  belonging to  $\text{vert}_\mu(F(\mu, \nu)) \setminus \text{vert}(F)$ . Up to replacing  $\nu$  by  $\nu'$ , we may assume that  $\mu$  and  $\nu$  are adjacent.

Let  $\mathcal{D}(\mu, \nu)$  be the closed half-space delimited by  $\overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)}$  and containing  $\overline{\mathcal{P}(\mu)}$ . Let  $V = x + \sum_{\tau \in \text{vert}_\mu(F)} \mathbb{R}(\tau - \mu) = x + \text{vect}(F - \mu)$ . By Lemma 4.2, we have

$$\pi_\lambda^{-1}(\{x\}) = x + \mathbb{R}_{\geq 0}(\overline{F} - \lambda) \subset V.$$

Let  $C^v = \mathbb{R}_{>0}(C - \mu)$  and  $F^v = \mathbb{R}_{>0}(F - \mu)$ . Then  $C^v$  (resp.  $F^v$ ) is a chamber (resp. face) of  $(\mathbb{A}, \mathcal{H}_\mu^0)$  by Proposition 3.12. Let  $\Sigma$  be the basis of  $\Phi_\mu$  associated with  $C^v$ . Write  $\Sigma = \{\alpha_i, i \in I\}$  for some indexing set  $I$  satisfying  $|I| = |\Sigma|$ . Let  $(\varpi_i) \in (\mathbb{A}^*)^I$  be the dual basis of  $(\alpha_i)_{i \in I}$ . Let  $J = \{i \in I \mid \varpi_i \in \overline{F^v}\}$ . Then  $\mathbb{R}_{>0}(\nu - \mu)$  is a one dimensional face of  $C^v$  and thus  $\nu - \mu \in \mathbb{R}_{>0}\varpi_i$ , for some  $i \in I \setminus J$ . Then  $\mathcal{D}(\mu, \nu) - \mu$  is the half-space of  $\mathbb{A}$  containing  $\mathcal{P}_{\Phi_\mu, \mathbf{b}_C - \mu}^0$  and delimited by the support of  $W_{\mathbb{R}_{>0}\varpi_i}^v \cdot (\mathbf{b}_C - \mu)$ , where  $W_{\mathbb{R}_{>0}\varpi_i}^v$  is the fixator of  $\mathbb{R}_{>0}\varpi_i$  in  $W_{\Phi_\mu}^v$ . This half-space is denoted  $\mathcal{D}_i$  in Lemma 2.28. Therefore by Lemma 2.28, we have  $\mathcal{D}(\mu, \nu) - \mu \supset (V - \mu) \cap (\overline{\mathcal{P}(\mu)} - \mu)$  and hence  $\mathcal{D}(\mu, \nu) \supset V \cap \overline{\mathcal{P}(\mu)} \supset \pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu)}$ , which proves the lemma, since  $\mathcal{D}(\mu, \nu) \cap \overline{\mathcal{P}(\nu)}$  is empty.  $\square$

**Lemma 4.4.** *Let  $\lambda \in \text{vert}(\mathbb{A})$  and  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$ . Let  $x \in F$ . Then  $\pi_\lambda(x) \in F$ .*

*Proof.* We have  $x - \lambda \in \mathbb{R}_{>0}(F - \lambda)$ . Using Lemma 2.24, Eq. (3.10) and Proposition 3.12 we deduce  $\pi_{\overline{\mathcal{P}(\lambda)} - \lambda}(x - \lambda) = \pi_\lambda(x) - \lambda \in \mathbb{R}_{>0}(F - \lambda)$ . Consequently,  $\pi_\lambda(x) \in \lambda + \mathbb{R}_{>0}(F - \lambda)$ .

Let  $F'$  be the face of  $(\mathbb{A}, \mathcal{H})$  containing  $\pi_\lambda(x)$ . By Lemma 3.23,  $F'$  dominates  $\lambda$ . Therefore  $\pi_\lambda(x) \in F' \subset \lambda + \mathbb{R}_{>0}(F' - \lambda)$ . We deduce  $\mathbb{R}_{>0}(F - \lambda) \cap \mathbb{R}_{>0}(F' - \lambda) \neq \emptyset$ . As the faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  partition  $\mathbb{A}$ , we deduce  $\mathbb{R}_{>0}(F - \lambda) = \lambda + \mathbb{R}_{>0}(F' - \lambda)$  and thus  $F = F'$ , by Proposition 3.12.  $\square$

**Lemma 4.5.** *Let  $\lambda \in \text{vert}(\mathbb{A})$ ,  $C$  be an alcove of  $\mathbb{A}$  dominating  $\lambda$  and  $\mu \in \text{vert}(C) \setminus \{\lambda\}$ . Then  $\pi_\lambda(\overline{\mathcal{P}(\mu)} \cap \overline{C}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ .*

*Proof.* Let  $x \in \text{Fr}(\overline{\mathcal{P}(\lambda)}) \cap \overline{C}$ . Let  $F = F(x)$  be the face dominating  $\lambda$  and such that  $x \in \mathcal{F}_F$ . Then by Proposition 3.27,  $\{\nu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\nu)}\} = \text{vert}(F)$ . By contradiction, we make the following assumption:

$$\exists \mu' \in \text{vert}(C) \setminus \text{vert}(F) \mid (\pi_\lambda)^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu')} \cap \overline{C} \neq \emptyset. \quad (4.1)$$

Let  $\mu' \in \text{vert}(C) \setminus \text{vert}(F)$  be such that there exists  $y \in \pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu')} \cap \overline{C}$ . By Proposition 3.27,  $x \notin \overline{\mathcal{P}(\mu')}$  and thus  $x \neq y$ . Let  $\gamma : [0, 1] \rightarrow [x, y]$  be the affine parametrization such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . For  $t \in ]0, 1[$ , we have  $|y - \gamma(t)| < |y - x| = |y - \pi_\lambda(y)|$  and thus  $\gamma(t) \notin \overline{\mathcal{P}(\lambda)}$  for  $t \in ]0, 1[$ .

Let  $E = \{t \in [0, 1] \mid \exists \mu'' \in \text{vert}(C) \setminus \text{vert}(F), \gamma(t) \in \overline{\mathcal{P}(\mu'')}\}$ . Then  $E$  is non-empty by Eq. (4.1). Let  $t_0 = \inf E$ . Then  $\gamma(t_0) \in \overline{\mathcal{P}(\mu'')}$  for some  $\mu'' \in \text{vert}(C) \setminus \text{vert}(F)$ , since the

$\overline{\mathcal{P}(\tau)}$  are closed, for  $\tau \in \text{vert}(\mathbb{A})$ . Therefore  $t_0 = \min E$  and  $t_0 > 0$ . By Theorem 3.28 and Lemma 3.23, we have  $\gamma([0, t_0]) \subset \bigcup_{\nu \in \text{vert}(F)} \overline{\mathcal{P}(\nu)}$ . Let  $\nu \in \text{vert}(F)$  be such that for some  $\epsilon > 0$ , we have:

$$\gamma(t) \in \overline{\mathcal{P}(\nu)}, \quad \forall t \in ]t_0 - \epsilon, t_0[.$$

Then  $\gamma(t_0) \in \overline{\mathcal{P}(\nu)} \cap \overline{\mathcal{P}(\mu')}$ . By Lemma 4.2,  $\pi_\lambda^{-1}(\{x\})$  is convex and contains  $\gamma(t_0)$ . This contradicts Lemma 4.3. Therefore, Eq. (4.1) is wrong. In other words,

$$\forall x \in \overline{C} \cap \overline{\mathcal{P}(\lambda)}, (\pi_\lambda)^{-1}(\{x\}) \cap \overline{C} \subset \overline{C} \setminus \bigcup_{\mu' \in \text{vert}(C) \setminus \text{vert}(F(x))} \overline{\mathcal{P}(\mu')}. \quad (4.2)$$

Let now  $y \in \overline{\mathcal{P}(\mu)} \cap \overline{C}$ . Then by Lemma 4.4,  $x := \pi_\lambda(y) \in \overline{C}$ . Let  $F \in \mathcal{F}_\lambda$  be such that  $x \in \mathcal{F}_F$ . Then by Eq. (4.2),  $y \in \overline{C} \setminus \bigcup_{\mu' \in \text{vert}(C) \setminus \text{vert}(F)} \overline{\mathcal{P}(\mu')}$  and thus  $\mu \in \text{vert}(F)$ . Therefore  $\overline{\mathcal{P}(\mu)} \supset \overline{\mathcal{F}_F}$ , which completes the proof of the lemma.  $\square$

**Lemma 4.6.** *Let  $C_1 \in \text{Alc}(\mathcal{H})$  and  $\lambda, \mu \in \text{vert}(C_1)$ . Then:*

$$\bigcup_{C \in \text{Alc}_\mu(\mathcal{H})} \overline{C} \subset \bigcup_{C \in \text{Alc}_{\lambda, \mu}(\mathcal{H})} \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda),$$

where  $\text{Alc}_{\lambda, \mu}(\mathcal{H})$  is the set of alcoves of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$  and  $\mu$ .

*Proof.* Let  $E = \bigcup_{F \in \mathcal{F}_{\lambda, \mu}} \mathbb{R}_{> 0}(F - \lambda)$ , where  $\mathcal{F}_{\lambda, \mu}$  is the set of faces dominating  $\lambda$  and  $\mu$ . Then  $\lambda + \overline{E} \subset \lambda + \bigcup_{C \in \text{Alc}_{\lambda, \mu}(\mathcal{H})} \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$ . By Proposition 3.12 and Lemma 2.17, there exists  $X \subset \Phi_\lambda$  such that  $E = \bigcap_{\alpha \in X} \alpha^{-1}(\mathbb{R}_{> 0})$ . As  $E$  is a cone containing  $]0, \mu - \lambda[$ , it contains  $\mu - \lambda$ . Therefore  $\lambda + E$  is open and contains  $\mu$ . Consequently,  $\lambda + E$  contains a point of each alcove dominating  $\mu$ . Moreover  $\lambda + \overline{E} = \bigcap_{\alpha \in X} \alpha^{-1}(\mathbb{R}_{\geq \alpha(\lambda)})$  and thus it is enclosed, which proves that it contains  $\bigcup_{C \in \text{Alc}_\mu(\mathcal{H})} \overline{C}$ .  $\square$

**Lemma 4.7.** *Let  $C \in \text{Alc}(\mathcal{H})$  and  $\lambda \in \text{vert}(C)$ . Let  $E'$  be the union of the faces of  $C$  dominating  $\lambda$  and  $x \in E'$ . Let  $y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$ . Then  $[x, y] \cap E'$  contains a neighbourhood of  $x$  in  $[x, y]$ .*

*Proof.* Let  $E$  be the union of the faces of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$ . Then  $E$  is open by Lemma 3.17. Therefore there exists a face  $F$  dominating  $\lambda$  and  $\epsilon > 0$  such that  $F \supset ]x, x + \epsilon(y - x)[$ . We also have  $\lambda + \mathbb{R}_{> 0}(F - \lambda) \supset F$ . Moreover  $x, y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$  and thus it contains  $]x, x + \epsilon(y - x)[$ . The closed chamber  $\overline{C}^v := \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$  is the union of the faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  dominated by  $\overline{C}^v$ . Thus by Proposition 3.12, we have

$$\overline{C}^v = \bigsqcup_{F_1 \in \mathcal{F}_\lambda(C)} \mathbb{R}_{> 0}(F_1 - \lambda),$$

where  $\mathcal{F}_\lambda(C)$  is the set of subfaces of  $C$  dominating  $\lambda$ . Therefore there exists  $F_1 \in \mathcal{F}_\lambda(C)$  such that  $\mathbb{R}_{> 0}(F - \lambda) \cap \mathbb{R}_{> 0}(F_1 - \lambda)$  is non-empty, and as the faces of  $(\mathbb{A}, \mathcal{H}_\lambda^0)$  partition  $\mathbb{A}$ , we deduce that  $F = F_1$ . But  $F_1 \subset E'$  and the lemma follows.  $\square$

**Proposition 4.8.** *Let  $\lambda, \mu \in \text{vert}(\mathbb{A})$  be such that there exists an alcove of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$  and  $\mu$ . Then  $\pi_\lambda(\overline{\mathcal{P}(\mu)}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ .*

*Proof.* Let  $y \in \overline{\mathcal{P}(\mu)}$ . By Lemma 3.23 and Lemma 4.6, there exists an alcove  $C$  of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$  and  $\mu$  and such that  $y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$ . Let  $\mathcal{Q} = \overline{C} \cap \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ . Let  $x = \pi_\mathcal{Q}(y) \in \mathcal{Q}$ . By Lemmas 3.23 and 4.7, there exists  $z \in ]x, y] \cap \overline{C}$ . By Eq. (2.1), we have  $\pi_\mathcal{Q}(z) = \pi_\mathcal{Q}(y) = x$ .

By Lemmas 4.4 and 4.5, we have  $\pi_\lambda(z) \in \mathcal{Q}$  and by definition  $|a - z| \geq |a - \pi_\lambda(z)|$  for all  $a \in \overline{\mathcal{P}(\lambda)}$ . As  $\mathcal{Q} \subset \overline{\mathcal{P}(\lambda)}$ , we deduce  $\pi_\lambda(z) = \pi_\mathcal{Q}(z)$ . By Eq. (2.1), we have  $\pi_\lambda(z) = \pi_\lambda(y)$ . Therefore  $\pi_\lambda(y) \in \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ , which proves the proposition.  $\square$

## 4.2 Tessellation of apartments by thickened polytopes

Recall that, for  $\eta \in [0, 1]$  and  $\lambda \in \text{vert}(\mathbb{A})$ , we have the homothety  $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}$  given by  $h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda)$ , for  $x \in \mathbb{A}$ . We write  $\pi_{\lambda, \eta}$  instead of the orthogonal projection  $\pi_{h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})}$  on  $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$ .

Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda$ . Then by (3.10), we have  $\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{-\lambda + \mathbf{b}_C, \Phi_\lambda}^0}$ . Therefore

$$h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)}) = h_{\lambda, \eta}(\lambda) + \eta \text{conv}(W_{\Phi_\lambda}^v \cdot (-\lambda + \mathbf{b}_C)) = \lambda + \overline{\mathcal{P}_{\eta(-\lambda + \mathbf{b}_C), \Phi_\lambda}^0}. \quad (4.3)$$

**Remark 4.9.** Let  $F \in \mathcal{F}$  and  $a \in \mathcal{F}_F$ . Then similarly as in Lemma 4.2, we have

$$(\pi_{\lambda, \eta})^{-1}(h_{\lambda, \eta}(a)) = h_{\lambda, \eta}(a) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda).$$

**Lemma 4.10.** Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ ,  $\lambda \in \text{vert}(F)$  and  $\eta \in ]0, 1]$ . Then  $\pi_{\lambda, \eta}(F) \subset F$  and  $\pi_{\lambda, \eta}(\overline{F}) \subset \overline{F}$ .

*Proof.* Let  $x \in F$ . Then  $x \in \lambda + \mathbb{R}_{> 0}(F - \lambda)$ . By Lemma 2.24 and Eq. (4.3),  $\pi_{\lambda, \eta}(x) \in \lambda + \mathbb{R}_{> 0}(F - \lambda)$ . Let  $E$  be the star of  $\lambda$ . By Lemma 3.23, since  $h_{\lambda, \eta}$  stabilizes  $E$ , we have  $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)}) \subset E$ . Therefore by Lemma 3.19, we have  $\pi_{\lambda, \eta}(x) \in E \cap (\lambda + \mathbb{R}_{> 0}(F - \lambda)) = F$ . As  $\pi_{\lambda, \eta}$  is continuous (by [HUL12, A. (3.1.6)]), we deduce the result.  $\square$

**Proposition 4.11.** Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ . Then

$$F = \bigcap_{\lambda \in \text{vert}(F)} (\lambda + \mathbb{R}_{> 0}(F - \lambda)).$$

*Proof.* Let  $J = \text{vert}(F)$ . Set  $E = \bigcap_{\lambda \in J} (\lambda + \mathbb{R}_{> 0}(F - \lambda))$ . As  $F \subset \lambda + \mathbb{R}_{> 0}(F - \lambda)$  for  $\lambda \in J$ , we have  $F \subset E$ .

We now prove the converse inclusion. By Lemma 4.1, we have:

$$E = \bigcap_{\lambda \in J} \left( \lambda + \bigoplus_{\mu \in J} \mathbb{R}_{> 0}(\mu - \lambda) \right).$$

We first assume that  $\Phi$  is irreducible. Then by [Bou81, V 3.9 Proposition 8],  $C$  is a simplex and thus for every  $\lambda \in \text{vert}(C)$ ,  $(\mu - \lambda)_{\mu \in \text{vert}(C) \setminus \{\lambda\}}$  is a basis of  $\mathbb{A}$ . Fix  $\lambda \in J$ . Up to changing the 0 of the affine space  $\mathbb{A}$ , we may assume  $\lambda = 0$ . Let  $x \in E$ . Then we can write  $x = \sum_{\mu \in J \setminus \{0\}} t_\mu \mu$ , where  $(t_\mu) \in (\mathbb{R}_{> 0})^{J \setminus \{0\}}$ . Let  $\nu \in J$ . Then  $x \in \nu + \mathbb{R}_{> 0}(-\nu) + \bigoplus_{\mu \in J \setminus \{0\}} \mathbb{R}_{> 0}(\mu - \nu)$ , so we can write  $x = \nu + t'_0(-\nu) + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu(\mu - \nu)$  for some  $t'_0 \in \mathbb{R}$  and  $(t'_\mu) \in (\mathbb{R}_{> 0})^{J \setminus \{\nu\}}$ . Thus we have

$$(1 - t'_0 - \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu) \nu + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu \mu = \sum_{\mu \in J \setminus \{0\}} t_\mu \mu.$$

Therefore  $t'_\mu = t_\mu$  for  $\mu \in J \setminus \{0, \nu\}$ ,  $t_\nu < 1$  and  $\sum_{\mu \in J \setminus \{0\}} t_\mu = (1 - t'_0 - \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu) + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu < 1$ . This proves that  $E \subset \text{Int}_r(\text{conv}(J)) = F$  and thus when  $\Phi$  is irreducible, we have  $E = F$ .

We no longer assume the irreducibility of  $\Phi$ . In the notation of Eq. (2.4), with  $\Phi$  instead of  $\Psi$ , we write  $\mathbb{A} = \mathbb{A}_1 \times \dots \times \mathbb{A}_\ell$ . Let  $x = (x_i)_{i \in [1, \ell]} \in \mathbb{A}$  and  $i \in [1, \ell]$ . We have  $F = F_1 \times \dots \times F_\ell$ , where  $F_i \in \mathcal{F}(\mathbb{A}_i, \mathcal{H}_i)$  for  $i \in [1, \ell]$ .

We decompose  $J$  as  $J = J_1 \times \dots \times J_\ell$ , where  $J_i = \text{vert}(F_i)$  for  $i \in [1, \ell]$ . Let  $x \in E$ . Then:

$$\begin{aligned} x &\in \bigcap_{\lambda \in J_i} \left( \prod_{j=1}^{i-1} \mathbb{A}_j \times (\lambda + \sum_{\mu \in J_i \setminus \{\lambda\}} \mathbb{R}_{> 0}(\mu - \lambda)) \times \prod_{j=i+1}^{\ell} \mathbb{A}_j \right) \\ &= \prod_{j=1}^{i-1} \mathbb{A}_j \times \bigcap_{\mu \in J_i \setminus \{\lambda\}} \left( \lambda + \sum_{\mu \in J_i} \mathbb{R}_{> 0}(\mu - \lambda) \right) \times \prod_{j=i+1}^{\ell} \mathbb{A}_j \end{aligned}$$

By the first case we deduce  $x_i \in \text{Int}_r(\text{conv}(J_i))$ . Consequently  $x \in \prod_{i=1}^{\ell} \text{Int}_r(\text{conv}(J_i)) = \text{Int}_r(\text{conv}(J))$ , which completes the proof of the proposition.  $\square$

**Lemma 4.12.** *Let  $\mu \in \text{vert}(\mathbb{A})$ ,  $\eta \in [0, 1]$  and  $x \in \mathbb{A}$ . Let  $F \in \mathcal{F}(\mathcal{H})$  be such that  $\pi_\mu(x) \in \mathcal{F}_F$ . Let  $\lambda \in \text{vert}(F)$ . Then  $\pi_{\mu, \eta}(h_{\lambda, \eta}(x)) = h_{\mu, \eta}(\pi_\mu(x))$ .*

*Proof.* Let  $a = \pi_\mu(x)$ . By Lemma 4.2, we have  $x - a \in \mathbb{R}_{\geq 0}(\overline{F} - \mu)$ . We have:

$$h_{\lambda, \eta}(x) = h_{\mu, \eta}(a) + (1 - \eta)(\lambda - \mu) + \eta(x - a) \in h_{\mu, \eta}(a) + \mathbb{R}_{\geq 0}(\overline{F} - \mu),$$

and we conclude with Remark 4.9.  $\square$

**Definition 4.13.** *For  $x \in \mathbb{A}$ , we denote by  $F_{\mathcal{P}}(x)$  the face of  $(\mathbb{A}, \mathcal{H})$  such that  $x \in \mathcal{F}_{F_{\mathcal{P}}(x)}$ , which is well-defined by Theorem 3.28. For  $\eta \in [0, 1]$ , we set  $F_\eta(x) = \text{Int}_r(\text{conv}(\{h_{\lambda, \eta}(x) \mid \lambda \in \text{vert}(F_{\mathcal{P}}(x))\}))$  and  $\overline{F}_\eta(x) = \text{conv}(\{h_{\lambda, \eta}(x) \mid \lambda \in \text{vert}(F_{\mathcal{P}}(x))\})$  (see Figures 4.2 and 4.3). We have:*

$$\overline{F}_\eta(x) = \eta x + (1 - \eta)\text{conv}(\text{vert}(F_{\mathcal{P}}(x))) = \eta x + (1 - \eta)\overline{F_{\mathcal{P}}(x)}. \quad (4.4)$$

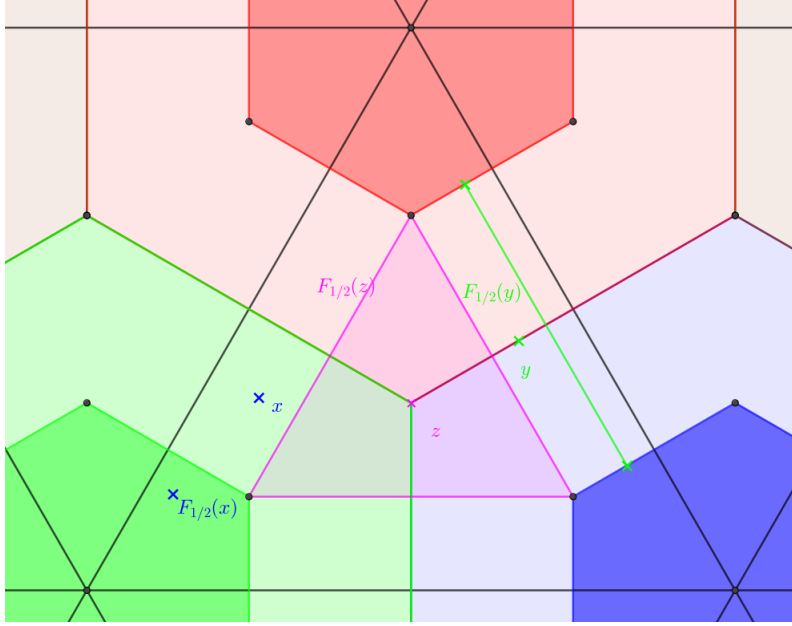


Figure 4.2: Pictures of  $F_\eta$  in the case of  $A_2$ .

**Lemma 4.14.** *Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  and  $x \in \overline{C}$ . Then:*

1. *for all  $y \in F_\eta(x)$ , for all  $\mu \in \text{vert}(C)$ , we have  $\pi_{\mu, \eta}(y) = h_{\mu, \eta}(\pi_\mu(x))$ ,*
2. *for all  $\mu \in \text{vert}(C)$ , for all  $y \in F_\eta(x)$ , we have  $\text{vert}(F_{\mathcal{P}}(x)) \subset \{\nu \in \text{vert}(C) \mid \pi_{\mu, \eta}(y) \in h_{\mu, \eta}(\overline{\mathcal{P}(\nu)})\}$  and this is an equality if and only if  $\mu \in \text{vert}(F_{\mathcal{P}}(x))$ .*

*Proof.* By Lemma 3.23, we have  $\text{vert}(F_{\mathcal{P}}(x)) \subset \text{vert}(C)$ . Let  $\mu \in \text{vert}(C)$ . By Propositions 3.27 and 4.8, we have

$$\pi_\mu(x) \in \bigcap_{\nu \in \text{vert}(F_{\mathcal{P}}(x)) \cup \{\mu\}} \overline{\mathcal{P}(\nu)}. \quad (4.5)$$

Let  $\lambda \in \text{vert}(F_{\mathcal{P}}(x))$ . By Proposition 3.27, we have  $\pi_\mu(x) \in \overline{\mathcal{F}_{F_{\mathcal{P}}(x)}}$ . Hence, if  $F'$  is the face such that  $\pi_\mu(x) \in \mathcal{F}_{F'}$ , we deduce that  $\text{vert}(F_{\mathcal{P}}(x)) \subset F'$ . By Lemma 4.12, we



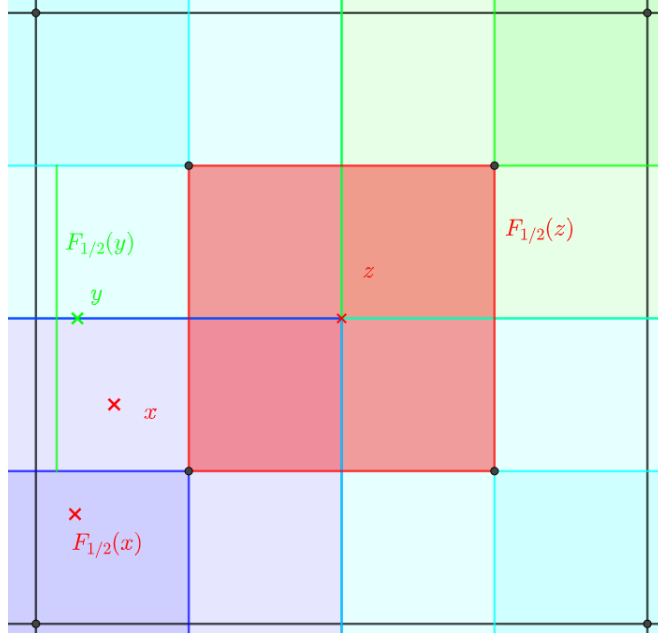


Figure 4.3: Pictures of  $F_\eta$  in the case of  $A_1 \times A_1$ .

get  $\pi_{\mu,\eta}(h_{\lambda,\eta}(x)) = h_{\mu,\eta}(\pi_\mu(x))$ . By Eq. (2.1),  $\pi_{\mu,\eta}^{-1}(h_{\mu,\eta}(\pi_\mu(x)))$  is convex. Therefore  $\pi_{\mu,\eta}(F_\eta(x)) = \{h_{\mu,\eta}(\pi_\mu(x))\}$ .

Let  $\mu \in \text{vert}(C)$ . Then

$$\begin{aligned} \{\nu \in \text{vert}(C) \mid \pi_{\mu,\eta}(y) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} &= \{\nu \in \text{vert}(C) \mid h_{\mu,\eta}(\pi_\mu(x)) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid \pi_\mu(x) \in \overline{\mathcal{P}(\nu)}\}. \end{aligned}$$

We conclude with Eq. (4.5), since  $\pi_\mu(x) = x$ , for  $\mu \in \text{vert}(F_{\mathcal{P}}(x))$ .  $\square$

**Lemma 4.15.** *Let  $F \in \mathcal{F}(\mathcal{H})$ . Let  $f \in \mathcal{F}_F$  and  $C$  be an alcove dominating  $f$ . Let  $\lambda \in \text{vert}(F)$  and  $x \in (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap \overline{C}$ . Then*

$$\pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)$$

and

$$\{\mu \in \text{vert}(\mathbb{A}) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\mu)})\} \subset \text{vert}(F).$$

*Proof.* We have

$$\begin{aligned} &h_{\lambda,\eta}(f) + \text{vect}(F - \lambda) \cap \overline{C} \\ &\subset (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap \left(\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda})\right) \\ &= h_{\lambda,\eta}(f + \text{vect}(F - \lambda)) \cap h_{\lambda,\eta}(\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda})) \\ &= h_{\lambda,\eta}((f + \text{vect}(F - \lambda)) \cap (\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda}))). \end{aligned}$$

By Lemma 4.12 applied to  $\mu = \lambda$ , we have  $\pi_{\lambda,\eta} \circ h_{\lambda,\eta} = h_{\lambda,\eta} \circ \pi_\lambda$ . Therefore if  $x \in (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap \overline{C}$ , we have

$$\pi_{\lambda,\eta}(x) = h_{\lambda,\eta}(\pi_\lambda(y)),$$

for some  $y \in (f + \text{vect}(F - \lambda)) \cap (\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda}))$ . Using Proposition 2.33, we have  $\pi_\lambda(y) \in f + \text{vect}(F - \lambda)$  and we deduce  $\pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)$ .

Now we have  $\pi_\lambda(y) = \pi_{\overline{\mathcal{P}(\lambda)}}(y) = \pi_{\overline{\mathcal{P}(\lambda)}-\lambda}(y-\lambda) + \lambda$ . By Eq. (3.10), we have  $\overline{\mathcal{P}(\lambda)} - \lambda = \overline{\mathcal{P}_{\mathbf{b}_{C-\lambda}}^0}$ . By Proposition 3.24 and Proposition 3.12, the map  $\mathcal{F}_{F_1} \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0$  is a bijection between the faces of  $\overline{\mathcal{P}(\lambda)}$  and those of  $\overline{\mathcal{P}(\lambda)} - \lambda$ . Hence we have  $\mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0 = \mathcal{F}_{F_1} - \lambda$ , for  $F_1 \in \mathcal{F}_\lambda(\mathcal{H})$ . By Proposition 2.33 (2),  $\pi_{\overline{\mathcal{P}(\lambda)}-\lambda}(y-\lambda) \in \mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0$ , for some subface  $F_1$  of  $F$  dominating  $\lambda$ . Therefore  $\pi_\lambda(y) \in \mathcal{F}_{F_1}$ , and we get:

$$\{\mu \in \text{vert}(\mathbb{A}) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\mu)})\} = \{\mu \in \text{vert}(\mathbb{A}) \mid \pi_\lambda(y) \in \overline{\mathcal{P}(\mu)}\} = \text{vert}(F_1),$$

which proves the lemma.  $\square$

**Lemma 4.16.** *Let  $\eta \in ]0, 1]$ . Let  $C \in \text{Alc}(\mathcal{H})$  and  $x \in \overline{C}$ . For  $\lambda \in \text{vert}(C)$ , set  $J_\lambda := J_{\lambda,\eta}(x) := \{\nu \in \text{vert}(C) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\nu)})\}$ . Let  $N_C(x) = N_C := \min\{|J_\mu| \mid \mu \in \text{vert}(C)\}$  and:*

$$J = \{\lambda \in \text{vert}(C) \mid |J_\lambda| = N_C\}.$$

*Then:*

1. *There exists a face  $F$  of  $(\mathbb{A}, \mathcal{H})$  such that  $J = \text{vert}(F)$ . There exists  $f_x \in \overline{C} \cap \mathcal{F}_F$  such that  $\pi_{\lambda,\eta}(x) = h_{\lambda,\eta}(f_x)$  for all  $\lambda \in J$ . More generally, for every  $\mu \in \text{vert}(C)$ , we have  $\pi_{\mu,\eta}(x) = h_{\mu,\eta}(\pi_\mu(f_x))$ .*
2. *For every  $\mu \in \text{vert}(C)$ , we have  $J_\mu \supset J$ ; in the particular case when  $\mu \in J$ , there is an equality  $J_\mu = J$ .*
3. *We have  $x \in \overline{F_\eta(f_x)}$ .*

*Proof.* Let  $\lambda_0 \in J$ . Let  $F$  be the face of  $(\mathbb{A}, \mathcal{H})$  dominating  $\lambda_0$  and such that  $\pi_{\lambda_0,\eta}(x) \in h_{\lambda_0,\eta}(\mathcal{F}_F)$ . By Lemma 4.10, we have  $\pi_{\lambda_0,\eta}(x) \in \overline{C}$ . Let  $F_1$  be the face of  $\overline{C}$  containing  $\pi_{\lambda_0,\eta}(x)$ . By Lemma 3.23, we have that  $\mathcal{F}_F$  is contained in the star  $E$  of  $F$ . As  $\lambda_0 \in \text{vert}(F)$  and  $\eta \in ]0, 1]$ , the set  $E$  is stabilized by  $h_{\lambda_0,\eta}$  and hence  $h_{\lambda_0,\eta}(\mathcal{F}_F) \subset E$ . Therefore  $F_1$  dominates  $F$ .

Write  $\pi_{\lambda_0,\eta}(x) = h_{\lambda_0,\eta}(f_x)$ , with  $f_x \in \mathcal{F}_F \cap \overline{C}$ . We have:

$$\begin{aligned} J_{\lambda_0} &= \{\nu \in \text{vert}(C) \mid \pi_{\lambda_0,\eta}(x) \in h_{\lambda_0,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid h_{\lambda_0,\eta}(f_x) \in h_{\lambda_0,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid f_x \in \overline{\mathcal{P}(\nu)}\} = \text{vert}(F), \end{aligned}$$

by Proposition 3.27. By Remark 4.9, we have

$$x \in h_{\lambda_0,\eta}(f_x) + \text{vect}(F - \lambda_0).$$

Let  $\mu \in \text{vert}(F)$ . Then

$$x \in h_{\mu,\eta}(f_x) + (1 - \eta)(\lambda_0 - \mu) + \text{vect}(F - \lambda_0) = h_{\mu,\eta}(f_x) + \text{vect}(F - \mu).$$

By Lemma 4.15, we have  $\pi_{\mu,\eta}(x) \in h_{\mu,\eta}(f_x) + \text{vect}(F - \mu)$  and

$$J_\mu = \{\nu \in \text{vert}(\mathbb{A}) \mid \pi_{\mu,\eta}(x) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} \subset \text{vert}(F) = J_{\lambda_0}.$$

By minimality of  $J_{\lambda_0}$ , we deduce

$$J_\mu = J_{\lambda_0} = \text{vert}(F) \subset J. \tag{4.6}$$

By the previous computations and by Proposition 3.27, we have  $h_{\mu,\eta}(f_x), \pi_{\mu,\eta}(x) \in h_{\mu,\eta}(\mathcal{F}_F) \cap (x + \text{vect}(F - \mu))$ . It follows from Proposition 3.24 that  $\text{dir}(\text{supp}(h_{\mu,\eta}(\mathcal{F}_F))) =$

$\text{dir}(\text{supp}(\mathcal{F}_F)) = \text{dir}(\text{supp}(F))^\perp$  and thus  $h_{\mu,\eta}(\mathcal{F}_F) \cap (x + \text{vect}(F - \mu))$  is either empty or a singleton. Therefore

$$h_{\mu,\eta}(f_x) = \pi_{\mu,\eta}(x).$$

Let  $\lambda \in \text{vert}(F)$ . By Lemma 4.2, we have

$$x \in \pi_{\lambda,\eta}(x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda) = h_{\lambda,\eta}(f_x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda).$$

Using Proposition 4.11, we deduce that  $x$  belongs to:

$$\begin{aligned} \bigcap_{\lambda \in \text{vert}(F)} (h_{\lambda,\eta}(f_x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)) &= \eta f_x + (1 - \eta) \bigcap_{\lambda \in \text{vert}(F)} (\lambda + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)) \\ &= \eta f_x + (1 - \eta)\overline{F}. \end{aligned}$$

Using (4.4) we deduce statement 3):

$$x \in \text{conv}(\{h_{\lambda,\eta}(f_x) \mid \lambda \in \text{vert}(F)\}). \quad (4.7)$$

Let  $\mu \in \text{vert}(C)$ . By Propositions 3.27 and 4.8, we have

$$\pi_\mu(f_x) \in \bigcap_{\nu \in \text{vert}(F) \cup \{\mu\}} \overline{\mathcal{P}(\nu)}. \quad (4.8)$$

Let  $F_2$  be the face of  $(\mathbb{A}, \mathcal{H})$  such that  $\pi_\mu(f_x) \in \mathcal{F}_{F_2}$ . Then by Proposition 3.27, we have  $\text{vert}(F_2) \supset \text{vert}(F)$ . Let  $\lambda \in \text{vert}(F)$ . By Lemma 4.12, we have  $\pi_{\mu,\eta}(h_{\lambda,\eta}(f_x)) = h_{\mu,\eta}(\pi_\mu(f_x))$ . By Eq. (2.1),  $(\pi_{\mu,\eta})^{-1}(h_{\mu,\eta}(\pi_\mu(f_x)))$  is convex. Using Eq. (4.7) we deduce

$$\pi_{\mu,\eta}(x) = h_{\mu,\eta}(\pi_\mu(f_x)).$$

Using Eq. (4.8) we deduce that  $J_\mu \supset \text{vert}(F) \cup \{\mu\} = J_{\lambda_0} \cup \{\mu\}$ . Consequently for  $\mu \in \text{vert}(C) \setminus J_{\lambda_0}$ , we have  $|J_\mu| \geq |J_{\lambda_0}| + 1 = N_C + 1$  and  $\mu \notin J$ . Therefore  $J \subset J_{\lambda_0}$  and hence by (4.6), we have

$$J = J_{\lambda_0} = \text{vert}(F),$$

which completes the proof of the lemma.  $\square$

**Lemma 4.17.** *Let  $C \in \text{Alc}(\mathcal{H})$  and  $x, x' \in \overline{C}$ . Then  $x = x'$  if and only if (for all  $\lambda \in \text{vert}(C)$ ,  $\pi_\lambda(x) = \pi_\lambda(x')$ ).*

*Proof.* Assume that  $\pi_\lambda(x) = \pi_\lambda(x')$  for every  $\lambda \in \text{vert}(C)$ . For  $\lambda \in \text{vert}(C)$ , set

$$J_\lambda = \{\mu \in \text{vert}(C) \mid \pi_\lambda(x) \in \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}\}.$$

Let  $J = \{\lambda \in \text{vert}(C) \mid |J_\lambda| = N_C(x)\}$ . By Lemma 4.16 applied with  $\eta = 1$ , we have  $\{x, x'\} \subset \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$ . Take  $\lambda \in J$ . Then  $\pi_\lambda(x) = x = \pi_\lambda(x') = x'$ .  $\square$

**Lemma 4.18.** *Let  $x \in \mathbb{A}$ ,  $\eta \in ]0, 1]$  and  $F$  be the face of  $(\mathbb{A}, \mathcal{H})$  containing  $x$ . Then  $\overline{F_\eta(x)} \subset F$ .*

*Proof.* Let  $F_1 \in \mathcal{F}(\mathcal{H})$  be such that  $x \in \mathcal{F}_{F_1}$ . Then by Lemma 3.23, we have  $F_1 \subset \overline{F}$ . For  $\lambda \in \text{vert}(F_1)$ , we thus have  $] \lambda, x] \subset F$  and  $h_{\lambda,\eta}(x) \in F$ . Consequently  $\overline{F_\eta(x)} = \text{conv}(\{h_{\lambda,\eta}(x) \mid \lambda \in \text{vert}(F_1)\}) \subset F$ .  $\square$

**Theorem 4.19.** *Let  $\mathbb{A}$  be a finite dimensional vector space and  $\Phi$  be a finite root system in  $\mathbb{A}$ . Let  $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ . We choose a fundamental alcove  $C_0$  of  $(\mathbb{A}, \mathcal{H})$  and choose  $\mathbf{b}_0 \in C_0$ . If  $\lambda$  is a vertex of  $(\mathbb{A}, \mathcal{H})$ , we choose an alcove  $C_\lambda$  dominating  $\lambda$  and we set  $\mathbf{b}_{C_\lambda} = w \cdot \mathbf{b}_0$  if  $w$  is the element of the affine Weyl group  $W^{\text{aff}}$  of  $(\mathbb{A}, \mathcal{H})$  sending  $C_0$  to  $C$ . We set  $\overline{\mathcal{P}(\lambda)} = \text{conv}(W_\lambda^{\text{aff}} \cdot \mathbf{b}_{C_\lambda})$ , where  $W_\lambda^{\text{aff}}$  is the fixator of  $\lambda$  in  $W^{\text{aff}}$ . If  $x \in \mathbb{A}$ , we define  $F_{\mathcal{P}}(x)$  as the unique face of  $(\mathbb{A}, \mathcal{H})$  such that  $x \in \mathcal{F}_{F_{\mathcal{P}}(x)} := \text{Int}_r(\text{conv}(W_{F_{\mathcal{P}}(x)}^{\text{aff}} \cdot \mathbf{b}_C))$ , where  $\mathbf{b}_C$  is an alcove dominating  $F_{\mathcal{P}}(x)$  and  $W_{F_{\mathcal{P}}(x)}^{\text{aff}}$  is the fixator of  $F_{\mathcal{P}}(x)$  in  $W^{\text{aff}}$ . If  $\eta \in ]0, 1]$ , we set  $F_\eta(x) = \eta x + (1 - \eta)F_{\mathcal{P}}(x)$ . Then we have:*

$$\forall \eta \in ]0, 1], \mathbb{A} = \bigsqcup_{x \in \mathbb{A}} \overline{F_\eta(x)}.$$

*Proof.* Let  $C$  be an alcove of  $(\mathbb{A}, \mathcal{H})$  and  $y \in \overline{C}$ . For  $\lambda \in \text{vert}(C)$ , set  $J_{\lambda, \eta}(y) = \{\mu \in \text{vert}(C) \mid \pi_{\lambda, \eta}(y) \in h_{\lambda, \eta}(\overline{\mathcal{P}(\mu)})\}$ . Let  $N_{C, \eta} = \min\{|J_{\mu, \eta}(y)| \mid \mu \in \text{vert}(C)\}$ , and

$$\tilde{J} = \{\lambda \in \text{vert}(C) \mid |J_{\lambda, \eta}(y)| = N_{C, \eta}\}.$$

By Lemma 4.16, there exists  $f_y \in \text{Int}_r\left(\bigcap_{\nu \in \tilde{J}} \overline{\mathcal{P}(\nu)}\right)$  such that  $y \in \overline{F_\eta(f_y)}$ . Therefore  $\bigcup_{x \in \overline{C}} \overline{F_\eta(x)} = \overline{C}$ .

Let  $x, x' \in \mathbb{A}$  be such that  $\overline{F_\eta(x)} \cap \overline{F_\eta(x')} \neq \emptyset$  and let  $y \in \overline{F_\eta(x)} \cap \overline{F_\eta(x')}$ . Let  $F$  and  $F'$  be the faces of  $(\mathbb{A}, \mathcal{H})$  containing  $x$  and  $x'$  respectively. Then by Lemma 4.18,  $y \in F \cap F'$ , which proves that  $F = F'$ . Let  $C$  be an alcove dominating  $F$ . By Lemma 4.14, we have  $\pi_{\mu, \eta}(y) = h_{\mu, \eta}(\pi_\mu(x)) = h_{\mu, \eta}(\pi_\mu(x'))$  for every  $\mu \in \text{vert}(C)$ . Therefore  $\pi_\mu(x) = \pi_\mu(x')$  for every  $\mu \in \text{vert}(C)$  and  $x = x'$  by Lemma 4.17.  $\square$

**Corollary 4.20.** *Let  $\eta \in ]0, 1]$ . We have:*

$$\mathbb{A} = \bigsqcup_{\lambda \in \text{vert}(\mathbb{A})} h_{\lambda, \eta}(\mathcal{P}(\lambda)) \sqcup \bigsqcup_{x \in \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \text{Fr}(\mathcal{P}(\lambda))} \overline{F_\eta(x)}.$$

**Remark 4.21.** *1. Note that when  $\eta = 0$ , we have  $F_0(\mathbf{b}_C) = \overline{C}$  for each alcove  $C$ . Therefore we still have  $\mathbb{A} = \bigcup_{x \in \mathbb{A}} F_0(x)$ , but this is no longer a disjoint union according to Lemma 3.25.*

*2. Let  $F$  be a face of  $(\mathbb{A}, \mathcal{H})$ . Let  $x \in \mathcal{F}_F$  and  $\eta \in ]0, 1[$ . Then for every vertex  $\lambda$  of  $F$ , we have  $F_\eta(x) = \eta x + (1 - \eta)F = h_{\lambda, \eta}(x) + (1 - \eta)(F - \lambda)$ . Indeed, we have  $h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda) = (1 - \eta)\lambda + \eta x$ .*

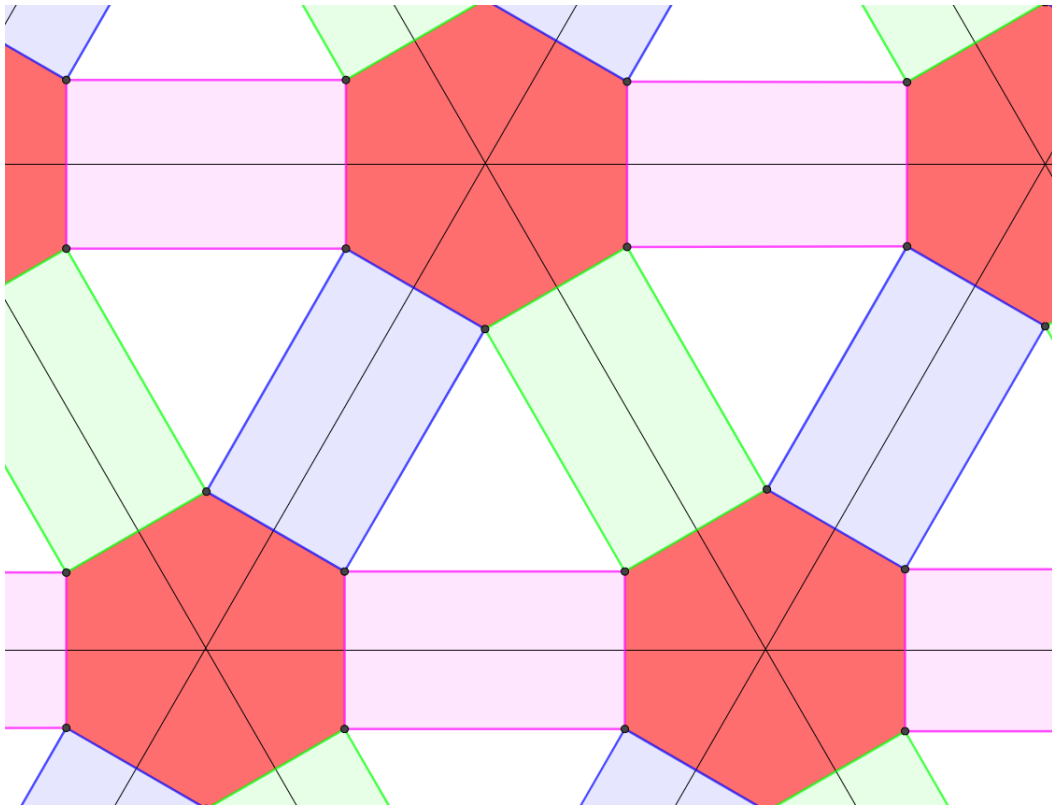


Figure 4.4: Illustration of Theorem 4.19 in the case of  $A_2$  and  $\eta = 1/2$ .

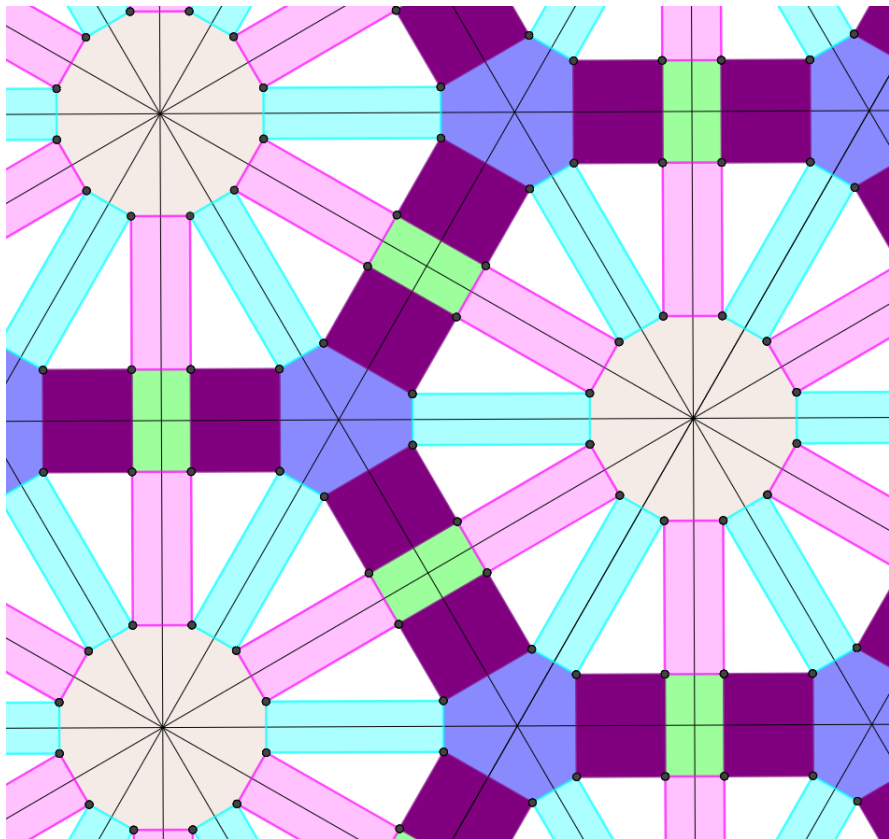


Figure 4.5: Illustration of Theorem 4.19 in the case of  $G_2$  and  $\eta = 1/2$ .

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