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Tessellations of an affine apartment by affine weight polytopes

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Abstract

Let \mathbb{A} be a finite dimensional vector space and Φ be a finite root system in \mathbb{A} . To this data is associated an affine poly-simplicial complex. Motivated by a forthcoming construction of connectified higher buildings, we study “affine weight polytopes” associated to these data. We prove that these polytopes tessellate \mathbb{A} . We also prove a kind of “mixed” tessellation, involving the affine weight polytopes and the poly-simplicial structure on \mathbb{A} .

1 Introduction

Tessellation of \mathbb{A} by affine weight polytopes Let \mathbb{A} be a finite dimensional vector space. A finite (reduced) root system in \mathbb{A} is a finite set Φ of linear forms on \mathbb{A} , called roots, satisfying certain conditions (see [Bou81, VI, §1]). The associated affine (locally finite) hyperplane arrangement is $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid (\alpha, k) \in \Phi \times \mathbb{Z}\}$. The elements of \mathcal{H} are called affine walls. The hyperplane arrangement \mathcal{H} equips \mathbb{A} with the structure of a poly-simplicial complex, whose maximal poly-simplices - the alcoves - are the connected components of $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$. To each affine wall H is naturally associated a reflection s_H . Let W^{aff} be the affine Weyl group associated with $(\mathbb{A}, \mathcal{H})$, i.e., W^{aff} is the group generated by the s_H , for $H \in \mathcal{H}$. This is naturally a Coxeter system and every closed alcove \overline{C} is a fundamental domain for the action of W^{aff} on \mathbb{A} . Fix a “fundamental alcove” C_f and pick one element $\mathbf{b} = \mathbf{b}_{C_f}$ of C_f . Then for every alcove C , the set $C \cap W^{\text{aff}} \cdot \mathbf{b}$ contains exactly one element, that we denote \mathbf{b}_C . Denote by $\text{vert}(\mathbb{A})$ the set of vertices of $(\mathbb{A}, \mathcal{H})$, i.e., the set of zero-dimensional faces of the associated poly-simplicial complex. For F a face of $(\mathbb{A}, \mathcal{H})$, we set

$$\mathcal{F}_F = \mathcal{F}_{\mathbf{b}, F} = \text{Int}_r(\text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)),$$

where W_F^{aff} is the fixator of F in W^{aff} , C is any alcove whose closure contains F , conv denotes the convex hull, and given a subset $E \subset \mathbb{A}$, $\text{Int}_r(E)$ denotes the relative interior of E (that is the interior of E in the affine subspace spanned by E). When F is 0-dimensional, i.e., when $F = \{\lambda\}$ for some $\lambda \in \text{vert}(\mathbb{A})$, we usually write $\mathcal{P}(\lambda)$ instead of $\mathcal{F}_{\{\lambda\}}$. Our first main result is the following (see Figures 3.6, 3.7, 3.8 and 3.9):

Theorem 1.1. (see Propositions 3.24, 3.27 and Theorem 3.28)

1. Let $\lambda \in \text{vert}(\mathbb{A})$. Then the assignment $F \mapsto \mathcal{F}_F$ is a bijection between the set of faces of $(\mathbb{A}, \mathcal{H})$ whose closure contains λ and the set of faces of $\overline{\mathcal{P}(\lambda)}$.
2. Let F be a face of $(\mathbb{A}, \mathcal{H})$. Then we have $\overline{\mathcal{F}_F} = \bigcap_{\lambda \in \text{vert}(F)} \overline{\mathcal{P}(\lambda)}$, where $\text{vert}(F)$ denotes the set of vertices of F .
3. Let \mathcal{F} denote the set of faces of $(\mathbb{A}, \mathcal{H})$. Then $\mathbb{A} = \bigsqcup_{F \in \mathcal{F}} \mathcal{F}_F$.

Thickened tessellation of \mathbb{A} We then study a variant of the tessellation of \mathbb{A} given by Theorem 1.1(3). Let $\eta \in]0, 1]$. Let $x \in \mathbb{A}$. Let $F = F_{\mathcal{P}}(x)$ be the face of $(\mathbb{A}, \mathcal{H})$ such that \mathcal{F}_F contains x (the existence and uniqueness of such a face is provided by Theorem 1.1 (3)). Set $F_{\eta}(x) = (1 - \eta)F + \eta x = \{(1 - \eta)y + \eta x \mid y \in F\}$. For $\lambda \in \text{vert}(\mathbb{A})$, we denote by $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}$ the homothety of \mathbb{A} with center λ and radius η . Then $\overline{F_{\eta}(x)}$ is a ‘‘copy’’ of \overline{F} containing exactly one point of $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$, namely $h_{\lambda, \eta}(x)$, for each $\lambda \in \text{vert}(F)$. This is the convex hull of these points. Then we prove the following theorem:

Theorem 1.2. Let $\eta \in]0, 1]$. Then $\mathbb{A} = \bigsqcup_{x \in \mathbb{A}} \overline{F_{\eta}(x)}$.

We deduce the following corollary, which is easier to interpret (see Figures 4.4 and 4.5):

Corollary 1.3. Let $\eta \in]0, 1]$. Then we have

$$\mathbb{A} = \bigsqcup_{\lambda \in \text{vert}(\mathbb{A})} h_{\lambda, \eta}(\mathcal{P}(\lambda)) \sqcup \bigsqcup_{x \in \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \text{Fr}(\mathcal{P}(\lambda))} \overline{F_{\eta}(x)}.$$

Note that Theorem 1.2 and Corollary 1.3 provide some kinds of tessellations of \mathbb{A} involving homothetic images of the $\mathcal{P}(\lambda)$ and homothetic images of the faces of $(\mathbb{A}, \mathcal{H})$.

The polytopes $\overline{\mathcal{P}(\lambda)}$ that we study already appear in the litterature, in various contexts of representation theory. If $\lambda \in \mathbb{A}$, then we can write $\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{\lambda}^0}$, where $\overline{\mathcal{P}_{\lambda}^0} = \text{conv}\{w \cdot \mathbf{b}' \mid w \in W_{\lambda}^v\}$, where W_{λ}^v is the finite Weyl group associated with the root system $\Phi_{\lambda} = \{\alpha \in \Phi \mid \alpha(\lambda) \in \mathbb{Z}\}$ and \mathbf{b}' is a regular element of \mathbb{A} (i.e., $w \cdot \mathbf{b}' \neq \mathbf{b}'$, for every $w \in W_{\lambda}^v$). The polytope $\overline{\mathcal{P}_{\lambda}^0}$ is called the **weight polytope** and is studied in [Vin91] and [BJK23]. Its intersection with a closed vectorial chamber of \mathbb{A} is studied in [BGH23].

Motivation from the theory of higher buildings Let \mathbb{K} be a field equipped with a valuation $\omega : \mathbb{K}^* \rightarrow \mathbb{Z}^n$, for some $n \in \mathbb{Z}_{\geq 1}$, where \mathbb{Z}^n is equipped with the lexicographic order. For example, let \mathbb{k} be a field and set $\mathbb{K} = \mathbb{k}(t_1, \dots, t_n)$, which is contained in $\mathbb{k}((t_n))((t_{n-1})) \dots ((t_1))$, with

$$\omega \left(\sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} t_1^{i_1} \dots t_n^{i_n} \right) = \min\{(i_1, \dots, i_n) \in \mathbb{Z}^n \mid a_{i_1, \dots, i_n} \neq 0\},$$

for $\sum_{(i_1, \dots, i_n) \in \mathbb{Z}^n} a_{i_1, \dots, i_n} t_1^{i_1} \dots t_n^{i_n} \in \mathbb{K}$. Such fields naturally appear as successive completions of function fields of n -dimensional algebraic varieties.

We would like to study groups of the form $G = \mathbf{G}(\mathbb{K})$, for \mathbf{G} a split reductive group, as well as certain arithmetic subgroups of them (for instance their presentations, certain (co)homological groups or their (co)homological dimension). When $n = 1$, then G acts on a classical Bruhat-Tits building which is connected and contractible (see [BT72] and [BT84]). We would like to use buildings techniques to study G , when $n \geq 2$. In this case

Bennett and Parshin independently defined analogues of Bruhat-Tits buildings in the case where \mathbf{G} is SL_m or PGL_m , for $m \in \mathbb{Z}_{\geq 2}$ (see [Ben94] and [Par94]), using lattices of \mathbb{K}^m . In [Ben94], Bennett develops an axiomatic definition of Λ -buildings (where $\Lambda = \omega(\mathbb{K}^*)$, $\Lambda = \mathbb{Z}^n$ in our case). In [HIL20], the three last named authors prove that G acts on a \mathbb{Z}^n -building \mathcal{I} , generalizing (in the quasi-split case) Bruhat-Tits construction via “parahoric” subgroups.

When $n \geq 2$, \mathcal{I} is a bit like a set of Russian dolls: \mathcal{I} is a Bruhat-Tits building whose vertices are \mathbb{Z}^{n-1} -buildings. A problem is that \mathcal{I} is not connected since the latter \mathbb{Z}^{n-1} -buildings are connected components of \mathcal{I} . As the connectedness and contractibility of Bruhat-Tits buildings are important ingredients of the classical theory, we would like to embed the \mathbb{Z}^n -building \mathcal{I} in a “connectified building $\hat{\mathcal{I}}$ or $\hat{\mathcal{I}}^\eta$, for $\eta \in]0, 1[$, on which G would act and which would have better topological properties. We hope that our tessellations are a first step in this direction.

Structure of the paper The paper is organized as follows. In section 2, we study the polytopes $\mathcal{P}(\lambda)$ (or more precisely translates of these polytopes) one by one. We describe the set of faces and study the orthogonal projection on such a polytope.

In section 3, we study the polytopes $\overline{\mathcal{P}(\lambda)}$ and prove that they tessellate \mathbb{A} .

In section 4, we prove Theorem 1.2.

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2 Vectorial Weyl polytopes

Let \mathbb{A} be a finite dimensional vector space and Φ be a finite root system in \mathbb{A} . This equips \mathbb{A} with the structure of a poly-simplicial complex. The aim of this paper is to study tessellations of \mathbb{A} by polytopes $\overline{\mathcal{P}(\lambda)}$ associated with vertices λ of \mathbb{A} . Before studying the layout of the different polytopes, we begin by studying the polytopes one by one.

In this section, we study the vectorial weight polytope $\overline{\mathcal{P}^0} = \overline{\mathcal{P}_\Psi^0}$ associated with a finite root system Ψ . In the applications, Ψ will be a root system of the form $\Phi_\lambda = \{\alpha \in \Phi \mid \alpha(\lambda) \in \mathbb{Z}\}$, for λ a vertex of (\mathbb{A}, Φ) and then $\overline{\mathcal{P}(\lambda)}$ will be a translate of $\overline{\mathcal{P}_{\Phi_\lambda}^0}$. To define this polytope, we choose a regular point of \mathbb{A} and we take the convex hull of the orbit of this point under the action of the finite Weyl group of Ψ .

In subsection 2.1, we study general facts on polyhedra.

In subsection 2.2, we define the vectorial weight polytope $\overline{\mathcal{P}^0}$.

In subsection 2.3, we describe the faces of $\overline{\mathcal{P}^0}$.

In subsection 2.4, we study the restriction to certain affine subspaces of \mathbb{A} of the orthogonal projection on $\overline{\mathcal{P}^0}$.

2.1 Preliminary results on polyhedra and polytopes

In this section, \mathbb{A} is a finite dimensional real vector space. We equip \mathbb{A} with a scalar product $\langle \cdot \mid \cdot \rangle$ (for the moment, $\langle \cdot \mid \cdot \rangle$ is arbitrary, but in most of the applications, it will be invariant under the action of the vectorial or affine Weyl groups defined later). If E is a subset of \mathbb{A} , we denote by $\overset{\circ}{E}$ its interior. If E is non-empty, its **support** $\text{supp}(E)$ is the smallest affine subspace of \mathbb{A} containing E . Its **relative interior**, denoted $\text{Int}_r(E)$, is then the interior of E in its support. In particular, if E is convex and closed, we have $E = \overline{\text{Int}_r(E)}$. When E is an affine subspace of \mathbb{A} , its **direction**, denoted $\text{dir}(E)$, is the vector subspace $\{x - x' \mid x, x' \in E\}$ of \mathbb{A} .

We mainly follow the terminology of [BG09]. A (closed) **polyhedron** is the intersection of finitely many closed half-spaces of \mathbb{A} . A (closed) **polytope** is a bounded polyhedron. Let \mathcal{Q} be a polyhedron of \mathbb{A} . A **support hyperplane** of \mathcal{Q} is a hyperplane H such that \mathcal{Q} is contained in one of the closed half-spaces delimited by H and such that $\mathcal{Q} \cap H \neq \emptyset$. A **closed face** of \mathcal{Q} is either \mathcal{Q} or a set of the form $H \cap \mathcal{Q}$, where H is a support hyperplane. Contrary to [BG09], we do not consider \emptyset as a face of \mathcal{Q} . A **closed facet** of \mathcal{Q} is a face of codimension 1 in the support of \mathcal{Q} . By [BG09, Theorem 1.10], every proper closed face of \mathcal{Q} is an intersection of closed facets. We denote by $\text{Facet}(\mathcal{Q})$ the set of closed facets of \mathcal{Q} . An **open face** or simply a **face** of \mathcal{Q} is the relative interior of a closed face of \mathcal{Q} . Note that in [BG09], what we call ‘‘closed faces’’ are simply called ‘‘faces’’ and the ‘‘open faces’’ are the relative interior of faces: we make this choice in order to be coherent with the usual definition of faces in an apartment (see subsections 2.2 and 3.2). If F_1, F_2 are open faces of a polyhedron for which $\overline{F_1} \cap \overline{F_2}$ is non-empty, we set $F_1 \wedge F_2 = \text{Int}_r(\overline{F_1} \cap \overline{F_2})$. It is a face of \mathcal{Q} by [BG09, Proposition 1.8 and Theorem 1.10].

2.1.1 Orthogonal projection on polytopes

If E is a convex closed subset of \mathbb{A} , we denote by $\pi_E : \mathbb{A} \rightarrow E$ the orthogonal projection on E : for every $x \in \mathbb{A}$, $\pi_E(x)$ is the unique point y_x of E satisfying $|x - y_x| = \min\{|z - x| \mid z \in E\}$. Moreover, if $y_x \in E$, then we have

$$(y_x = \pi_E(x)) \iff (\langle x - y_x \mid y - y_x \rangle \leq 0, \forall y \in E), \quad (2.1)$$

by [HUL93, III Theorem 3.1.1].

In this preliminary subsection, we study the projection π_Q , for Q a polyhedron of \mathbb{A} . In particular, Proposition 2.3 provides a description of its fibers.

For $x \in \mathbb{A}$ and $r \in \mathbb{R}_{\geq 0}$, we denote by $B(x, r)$ the open ball of center x and radius r .

Lemma 2.1. *Let Q be a polyhedron of \mathbb{A} with non-empty interior. Let $x \in Q$ and let $\text{Facet}_x(Q)$ be the set of closed facets of Q containing x . For $\overline{\mathcal{F}} \in \text{Facet}_x(Q)$ with relative interior \mathcal{F} , we denote by $\mathcal{D}_{\mathcal{F}}$ the closed half-space of \mathbb{A} delimited by \mathcal{F} and containing Q . Let $\tilde{Q} = \bigcap_{\overline{\mathcal{F}} \in \text{Facet}_x(Q)} \mathcal{D}_{\mathcal{F}}$. Then there exists $r > 0$ such that $B(x, r) \cap Q = B(x, r) \cap \tilde{Q}$.*

Proof. We have $Q \subset \mathcal{D}_{\mathcal{F}}$ for every $\overline{\mathcal{F}} \in \text{Facet}_x(Q)$ and thus $Q \subset \tilde{Q}$.

For $\overline{\mathcal{F}} \in \text{Facet}(Q) \setminus \text{Facet}_x(Q)$, choose $r_{\mathcal{F}} > 0$ such that $B(x, r_{\mathcal{F}}) \cap \overline{\mathcal{F}} = \emptyset$. Let $r = \min\{r_{\mathcal{F}} \mid \overline{\mathcal{F}} \in \text{Facet}(Q) \setminus \text{Facet}_x(Q)\}$ (we set $r = 1$ if $\text{Facet}(Q) = \text{Facet}_x(Q)$). Let $y \in \tilde{Q} \cap B(x, r)$. Assume by contradiction that $y \notin Q$. Let $z \in \tilde{Q} \cap B(x, r)$. Then $[z, y]$ meets $\text{Fr}(Q)$ in a point y' . By [BG09, Corollary 1.7], we have $\text{Fr}(Q) = \bigcup_{\overline{\mathcal{F}} \in \text{Facet}(Q)} \overline{\mathcal{F}}$. We have $y' \notin \bigcup_{\overline{\mathcal{F}} \in \text{Facet}_x(Q)} \overline{\mathcal{F}}$ and thus $y' \in \bigcup_{\overline{\mathcal{F}} \in \text{Facet}(Q) \setminus \text{Facet}_x(Q)} \overline{\mathcal{F}}$. Consequently $y' \notin B(x, r)$: a contradiction. Therefore $\tilde{Q} \cap B(x, r) \subset Q$, and the lemma follows. \square

Let Q be a polyhedron. If $\overline{\mathcal{F}} \in \text{Facet}(Q)$, we denote by $\delta_{\mathcal{F}}$ the (closed) ray of $\text{dir}(\text{supp}(Q))$, which is orthogonal to the direction of the support of $\overline{\mathcal{F}}$ and such that for all $x \in \overline{\mathcal{F}}$, $(x + \delta_{\mathcal{F}}) \cap Q = \{x\}$. If $\overline{\mathcal{F}} \in \text{Facet}(Q)$, we denote by $\mathcal{D}_{\mathcal{F}}$ the half-space of \mathbb{A} containing Q and delimited by the hyperplane containing \mathcal{F} . We have

$$\mathcal{D}_{\mathcal{F}} = \{x \in \mathbb{A} \mid \langle x - a \mid z \rangle \leq 0, \forall z \in \delta_{\mathcal{F}}\}, \quad (2.2)$$

for any $a \in \overline{\mathcal{F}}$.

Lemma 2.2. *Let Q be a polyhedron with non-empty interior. Let $x \in \mathbb{A} \setminus Q$ and $J = \{\overline{\mathcal{F}} \in \text{Facet}(Q) \mid \pi_Q(x) \in \overline{\mathcal{F}}\}$. Then $\pi_Q(x) = \pi_{\tilde{Q}}(x)$, where $\tilde{Q} = \bigcap_{\overline{\mathcal{F}} \in J} \mathcal{D}_{\mathcal{F}}$.*

Proof. By Lemma 2.1, there exists $r > 0$ such that $Q \cap B(\pi_Q(x), r) = \tilde{Q} \cap B(\pi_Q(x), r)$. We have $d(x, y) \geq d(x, \pi_Q(x))$ for every $y \in \tilde{Q} \cap B(\pi_Q(x), r)$ and thus $d(x, \cdot)|_{\tilde{Q}}$ admits a local minimum at $\pi_Q(x)$. But as $d(x, \cdot)|_{\tilde{Q}}$ is a convex function, by [HUL93, IV 1.3 (c)], this minimum is actually a global minimum, which proves that $\pi_Q(x) = \pi_{\tilde{Q}}(x)$. \square

Proposition 2.3. *Let Q be a polyhedron with non-empty interior and let π_Q be the orthogonal projection on Q . Let $x \in \text{Fr}(Q)$ and $J = \{\overline{\mathcal{F}} \in \text{Facet}(Q) \mid x \in \overline{\mathcal{F}}\}$. Then $\pi_Q^{-1}(\{x\}) = x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\mathcal{F}}$.*

Proof. Let $\tilde{Q} = \bigcap_{\overline{\mathcal{F}} \in J} \mathcal{D}_{\mathcal{F}}$. Let $y \in x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\mathcal{F}}$. Then by Eq. (2.2), we have $\langle y - x \mid z - x \rangle \geq 0$ for all $z \in \tilde{Q} \supset Q$ and hence by Eq. (2.1), we have $\pi_Q(y) = x$. Therefore

$$\pi_Q^{-1}(\{x\}) \supset x + \sum_{\overline{\mathcal{F}} \in J} \delta_{\mathcal{F}}.$$

Let now $y \in \pi_Q^{-1}(\{x\})$. Then by Lemma 2.2, we have $\pi_Q(y) = \pi_{\tilde{Q}}(y) = x$. For $\overline{\mathcal{F}} \in J$, choose $z_{\mathcal{F}} \in \delta_{\mathcal{F}} \setminus \{0\}$ and respectively define $f_{\mathcal{F}} : \mathbb{A} \rightarrow \mathbb{R}$ and $f : \mathbb{A} \rightarrow \mathbb{R}$ by $f_{\mathcal{F}}(z) = -\langle z \mid z_{\mathcal{F}} \rangle$ and $f(z) = \langle z \mid x - y \rangle$, for $z \in \mathbb{A}$. Let $z \in \mathbb{A}$ be such that $f_{\mathcal{F}}(z) \geq 0$, for all $\overline{\mathcal{F}} \in J$. Let $z' = z + x$. Then we have $f_{\mathcal{F}}(z' - x) \geq 0$, for all $\overline{\mathcal{F}} \in J$ and thus by Eq. (2.2), we have $z' \in \tilde{Q}$. By Eq. (2.1), we deduce $f(z' - x) \geq 0$ and thus $f(z) \geq 0$. Using Farkas' Lemma, we deduce $f \in \sum_{\overline{\mathcal{F}} \in J} \mathbb{R}_{\geq 0} f_{\mathcal{F}}$, or equivalently, $y - x \in \sum_{\overline{\mathcal{F}} \in J} \mathbb{R}_{\geq 0} z_{\mathcal{F}} = \sum_{\overline{\mathcal{F}} \in J} \delta_{\mathcal{F}}$, which proves the proposition. \square

2.1.2 Extreme points of a polytope

Let \mathcal{Q} be a convex subset of \mathbb{A} . Recall (see [HUL12, Definition 2.3.1]) that a point M of \mathcal{Q} is called an **extreme point** of \mathcal{Q} if for every $k \in \mathbb{Z}_{\geq 1}$, $M_1, \dots, M_k \in \mathcal{Q}$, $t_1, \dots, t_k \in]0, 1]$ such that $\sum_{i=1}^k t_i = 1$, we have

$$\sum_{i=1}^k t_i M_i = M \Rightarrow M_1 = \dots = M_k = M.$$

We denote by $\text{Extr}(\mathcal{Q})$ the set of extreme points of \mathcal{Q} .

Lemma 2.4. *Let H be a finite subgroup of the set of affine automorphisms of \mathbb{A} . Let $\mathbf{b} \in \mathbb{A}$. Then the set of extreme points of $\text{conv}(H.\mathbf{b})$ is $H.\mathbf{b}$.*

Proof. Denote by E the set of extreme points of $\text{conv}(H.\mathbf{b})$. Then E is contained in $H.\mathbf{b}$. Then $\text{conv}(H.\mathbf{b})$ is compact. By Krein-Milman theorem ([HUL12, Theorem 2.3.4]), $\text{conv}(H.\mathbf{b})$ is the convex hull of E , and in particular, E is non-empty. As H stabilizes $\text{conv}(H.\mathbf{b})$, E is H -invariant and thus $E = H.\mathbf{b}$. \square

Lemma 2.5. *Let \mathcal{Q} be a closed polytope of \mathbb{A} .*

1. *Let $\overline{\mathcal{F}}$ be a closed face of \mathcal{Q} . Then $\text{Extr}(\overline{\mathcal{F}}) = \text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}$.*
2. *Let $(\overline{\mathcal{F}}_j)_{j \in J}$ be a family of closed faces of \mathcal{Q} . Then $\text{Extr}(\bigcap_{j \in J} \overline{\mathcal{F}}_j) = \bigcap_{j \in J} \text{Extr}(\overline{\mathcal{F}}_j)$.*

Proof. 1) By [BG09, Proposition 1.13], the set of closed faces of \mathcal{Q} (resp. $\overline{\mathcal{F}}$) is the set of extreme sets of \mathcal{Q} (resp. $\overline{\mathcal{F}}$) and thus the set of extreme points of $\overline{\mathcal{F}}$ is the set of zero-dimensional faces of $\overline{\mathcal{F}}$. That is the set of zero-dimensional faces of \mathcal{Q} , that are contained in $\overline{\mathcal{F}}$, according to [BG09, Proposition 1.8]. Thus $\text{Extr}(\overline{\mathcal{F}}) = \text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}$.

2) Since the number of faces of \mathcal{Q} is finite (by [BG09, Theorem 1.10]), we may assume that J is finite. Then by [BG09, Theorem 1.10], $\bigcap_{j \in J} \overline{\mathcal{F}}_j$ is a face of \mathcal{Q} . Then, by statement 1), we have $\text{Extr}(\bigcap_{j \in J} \overline{\mathcal{F}}_j) = \text{Extr}(\mathcal{Q}) \cap \bigcap_{j \in J} \overline{\mathcal{F}}_j = \bigcap_{j \in J} (\text{Extr}(\mathcal{Q}) \cap \overline{\mathcal{F}}_j) = \bigcap_{j \in J} \text{Extr}(\overline{\mathcal{F}}_j)$. \square

2.2 Vectorial weight polytope

2.2.1 Vectorial root system

Let \mathbb{A} be a finite dimensional real vector space and denote by $\mathbb{A}^* := \text{Hom}_{\mathbb{R}}(\mathbb{A}, \mathbb{R})$ its dual space. Let $\Psi \subset \mathbb{A}^*$ be a reduced finite root system in \mathbb{A} , in the sense of [Bou81, VI, §1]. In particular, Ψ is a finite subset of \mathbb{A}^* generating \mathbb{A}^* . Elements of Ψ are called **roots** and to every $\alpha \in \Psi$ is associated a unique **coroot** α^\vee . We have $\alpha(\alpha^\vee) = 2$ and the map $\mathbb{A}^* \rightarrow \mathbb{A}^*$ defined by $\beta \mapsto \beta - \beta(\alpha^\vee)\alpha$ stabilizes Ψ . One defines $s_\alpha : \mathbb{A} \rightarrow \mathbb{A}$ by $s_\alpha(x) = x - \alpha(x)\alpha^\vee$ for $x \in \mathbb{A}$. The set $\Psi^\vee = \{\alpha^\vee \mid \alpha \in \Psi\} \subset \mathbb{A}$ is a root system in \mathbb{A}^* and for every $\alpha \in \Psi$, $\alpha(\Psi^\vee) \subset \mathbb{Z}$ and s_α stabilizes Ψ^\vee .

Let $\mathcal{H}_\Psi^0 = \mathcal{H}^0 = \{\alpha^{-1}(\{0\}) \mid \alpha \in \Psi\}$. This is a locally finite hyperplane arrangement of \mathbb{A} . The elements of \mathcal{H}^0 are called **walls** or **vectorial walls**. For $x, y \in \mathbb{A}$, we write $x \sim_{\mathcal{H}^0} y$ if for every $H \in \mathcal{H}^0$, either $(x, y \in H)$ or $(x$ and y are strictly on the same side of $H)$. This is an equivalence relation on \mathbb{A} . Its classes are called the **faces** of $(\mathbb{A}, \mathcal{H}^0)$ or simply the vectorial faces of \mathbb{A} . We denote by $\mathcal{F}^0 = \mathcal{F}(\mathcal{H}^0)$ the set of faces of $(\mathbb{A}, \mathcal{H}^0)$. If E_1, E_2 are two relatively open (i.e open in their support) convex subsets of \mathbb{A} , we say that E_1 **dominates** E_2 if $\overline{E_1} \supset E_2$. This equips the set of relatively open convex subsets of \mathbb{A} with an order. By restriction, this equips \mathcal{F}^0 with an order called the **dominance order**.

A **vectorial chamber** (or **open vectorial chamber**) of $(\mathbb{A}, \mathcal{H}^0)$ is a connected component of $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}^0} H$. The chambers are actually the faces which are not contained

in an element of \mathcal{H}^0 or equivalently, the maximal faces for the dominance order. A **wall** of a vectorial chamber C^v is an element H of \mathcal{H} such that $H \cap \overline{C^v}$ is a closed facet of $\overline{C^v}$.

Let C^v be a vectorial chamber of Ψ . The **associated basis** of Ψ is the set Σ_{C^v} of $\alpha \in \Psi$ such that $\alpha^{-1}(\{0\})$ is a wall of C^v and $\alpha(C^v) > 0$. Then by [Bou81, VI. 1.5 Théorème 2], Σ_{C^v} is a **basis** of Ψ , which means that $(\alpha)_{\alpha \in \Sigma_{C^v}}$ is a basis of \mathbb{A}^* and every element of Ψ can be written as $\sum_{\alpha \in \Sigma_{C^v}} n_\alpha \alpha$, where $(n_\alpha) \in (\mathbb{Z}_{\geq 0})^{\Sigma_{C^v}}$, for some $\epsilon \in \{-, +\}$. We set $\Psi_{\epsilon, C^v} = \Psi \cap \bigoplus_{\alpha \in \Sigma_{C^v}} \epsilon \mathbb{Z}_{\geq 0} \alpha$, for $\epsilon \in \{-, +\}$.

The following proposition follows from [Bou81, §V 3.9 Proposition 5].

Proposition 2.6. *Let C^v be a vectorial chamber of $(\mathbb{A}, \mathcal{H}^0)$. Let Σ_{C^v} be the associated basis of Ψ . Let $(\varpi_\alpha)_{\alpha \in \Sigma_{C^v}} \in \mathbb{A}^{\Sigma_{C^v}}$ be the dual basis of Σ_{C^v} . Then*

1. $C^v = \bigoplus_{\alpha \in \Sigma_{C^v}} \mathbb{R}_{>0} \varpi_\alpha$.
2. *The faces of C^v are exactly the $\bigoplus_{\alpha \in E} \mathbb{R}_{>0} \varpi_\alpha$ such that E is a subset of Σ_{C^v} , where, by convention, $\bigoplus_{\alpha \in \emptyset} \mathbb{R}_{>0} \varpi_\alpha = \{0\}$.*

We choose a vectorial chamber $C_f^v = C_{f, \Psi}^v$ of \mathbb{A} , that we call the **fundamental vectorial chamber**. Let $\Sigma = \Sigma_{C_f^v}$.

The **vectorial Weyl group** of Ψ is $W_\Psi^v = \langle s_\alpha \mid \alpha \in \Psi \rangle \subset \text{GL}(\mathbb{A})$. It is a finite group and $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$ is a Coxeter system (see [BB05, 1.1] for a definition). By [Bou81, V §3], W_Ψ^v acts simply transitively on the set of vectorial chambers and $\overline{C^v}$ is a fundamental domain for the action of W_Ψ^v on \mathbb{A} , for any vectorial chamber C^v . By [Bou81, VI 1.1 Proposition 3], there exists a W_Ψ^v -invariant scalar product $\langle \cdot \mid \cdot \rangle$ on \mathbb{A} . We equip \mathbb{A} with this scalar product and regard \mathbb{A} as a Euclidean space. An element x of \mathbb{A} is **regular** (with respect to \mathcal{H}^0) if it belongs to an open vectorial chamber of \mathbb{A} . Actually, x is regular if and only if $w \mapsto w.x$ is injective on W_Ψ^v . If $F^v \in \mathcal{F}(\mathcal{H}^0)$ is dominated by C_f^v , then its fixator in W_Ψ^v is

$$W_{F^v}^v = \langle s_\alpha \mid \alpha \in \Sigma, \alpha(F^v) = \{0\} \rangle, \quad (2.3)$$

according to [Bou81, V §3.3].

We sometimes assume that Ψ is **irreducible**. This means that Ψ cannot be written as a direct sum of roots systems, see [Bou81, VI §1.2]. When Ψ is reducible, we can write

$$\mathbb{A} = \bigoplus_{i=1}^{\ell} \mathbb{A}_i, \Psi = \bigsqcup_{i=1}^{\ell} \Psi_i \quad (2.4)$$

where $\ell \in \mathbb{Z}_{\geq 2}$ and Ψ_i is an irreducible root system in \mathbb{A}_i for $i \in \llbracket 1, \ell \rrbracket$. We regard Ψ_i as a subset of Ψ by setting $\alpha(\bigoplus_{j \in \llbracket 1, \ell \rrbracket \setminus \{i\}} \mathbb{A}_j) = \{0\}$ for $\alpha \in \Psi_i$. We denote by \mathcal{H}_i^0 the hyperplane arrangement of (\mathbb{A}_i, Ψ_i) , for $i \in \llbracket 1, \ell \rrbracket$. Then every face F^v of \mathcal{F}^0 can be written as $F^v = F_1^v \times \dots \times F_\ell^v$, where $F_i^v \in \mathcal{F}^0(\mathcal{H}_i^0)$, for $i \in \llbracket 1, \ell \rrbracket$.

Cartan matrices Let $A = A(\Psi) = (\alpha(\beta^\vee))_{\alpha, \beta \in \Sigma}$. This is the **Cartan matrix** of Ψ . Up to permutation of its coefficients, it is well-defined, independently of the choice of the basis Σ of Ψ , by [Bou81, VI, 1.5]. The Cartan matrix characterizes the root system up to isomorphism.

A matrix of the form $A(\Phi)$, for some root system Φ is called a **Cartan matrix**.

The Cartan matrix A is called **decomposable** if there exists a decomposition $\Sigma = \Sigma' \sqcup \Sigma''$, with Σ' and Σ'' non-empty, such that $\alpha(\beta^\vee) = \beta(\alpha^\vee) = 0$, for every $(\alpha, \beta) \in \Sigma' \times \Sigma''$. A Cartan matrix which is not decomposable is called **indecomposable**. Then Ψ is irreducible if and only if A is indecomposable, and under the notation of Eq. (2.4), A is the block diagonal matrix whose blocks are the $A(\Psi_i)$, for $i \in \llbracket 1, \ell \rrbracket$. One can use the Cartan matrices to classify finite roots systems, see [Kac94, Chapter 4] for example (where the Cartan matrices are the generalized Cartan matrices of finite type).

2.2.2 Vectorial weight polytope

Definition 2.7. We choose a point $\mathbf{b} \in C_f^v$ and we set $\overline{\mathcal{P}^0} = \overline{\mathcal{P}_{\mathbf{b}, \Psi}^0} = \overline{\mathcal{P}_{\mathbf{b}}^0} = \text{conv}(W_{\Psi}^v \cdot \mathbf{b})$. We also set $\mathcal{P}^0 = \text{Int}(\overline{\mathcal{P}^0})$. Then \mathcal{P}^0 (resp. $\overline{\mathcal{P}^0}$) is the open (resp. closed) **vectorial Weyl polytope associated with Ψ and \mathbf{b}** .

The closed vectorial Weyl polytope is compact. Note that by Lemma 2.8 below, \mathcal{P}^0 is non-empty. This polytope naturally appears in different contexts. It is called the **weight polytope** and it appears for example in [Vin91] and [BJK23]. Its intersection with $\overline{C_f^v}$ is studied in [BGH23].

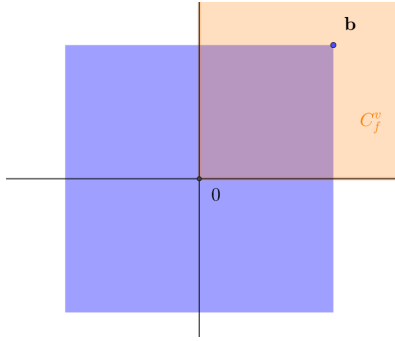


Figure 2.1: Polytope \mathcal{P}^0 for the root system $A_1 \times A_1$

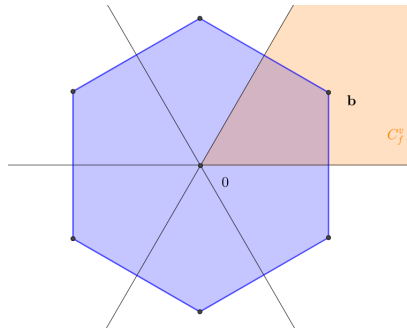


Figure 2.2: Polytope \mathcal{P}^0 for the root system A_2

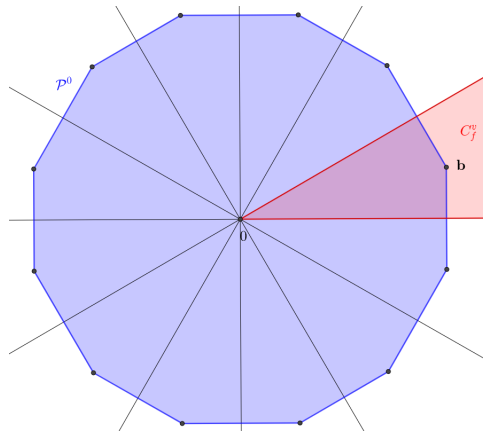


Figure 2.3: Polytope \mathcal{P}^0 for the root system G_2

Lemma 2.8. 1. Assume that Ψ is irreducible. Then for every $x \in \mathbb{A} \setminus \{0\}$, $\text{conv}(W_{\Psi}^v \cdot x)$ has non-empty interior.

2. \mathcal{P}^0 is non-empty.

Proof. The support of $W_\Psi^v \cdot x$ is W_Ψ^v -invariant and thus by [Bou81, VI §1.2 Corollaire], this support is \mathbb{A} , which proves 1).

By 1), if Ψ is irreducible, then \mathcal{P}^0 is non-empty. Assume now that Ψ is reducible and keep the notation of Eq. (2.4). Write $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_\ell)$. Then $\mathcal{P}_{\mathbf{b}, \Psi}^0 = \mathcal{P}_{\mathbf{b}_1, \Psi_1}^0 \times \dots \times \mathcal{P}_{\mathbf{b}_\ell, \Psi_\ell}^0$ and thus $\mathcal{P}_{\mathbf{b}, \Psi}^0$ is non-empty. \square

2.3 Description of the faces of the vectorial weight polytope

2.3.1 Faces of \mathcal{P}^0 meeting C_f^v

The main goal of this subsection is to prove Lemma 2.15, which describes the faces the of \mathcal{P}^0 meeting C_f^v .

For $J \subset \Sigma$, one defines the **parabolic subgroup** W_J^v by $W_J^v = W_{J, \Psi}^v = \langle s_\alpha \mid \alpha \in J \rangle \subset W_\Psi^v$. Set $Q_+^\vee = Q_{\mathbb{R}, +}^\vee = \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee$, $Q_{J, +}^\vee = \bigoplus_{\alpha \in J} \mathbb{R}_{\geq 0} \alpha^\vee$.

We equip W_Ψ^v with the Bruhat order (see [BB05, Chapter 2] for a definition).

Lemma 2.9. *Let $w \in W_\Psi^v$ and $\alpha \in \Sigma$. Then $ws_\alpha < w$ for the Bruhat order on $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$ if and only if $w \cdot \alpha \in \Psi_-$.*

Proof. This follows from [Bou81, VI, §1 Proposition 17 (ii)] and [BB05, Corollary 1.4.4]. \square

Lemma 2.10. *Let $w \in W_\Psi^v$ and $J \subset \Sigma$. The following assertions are equivalent:*

- (i) we have $\mathbf{b} - w \cdot \mathbf{b} \in \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$,
- (ii) we have $\mathbf{b} - w \cdot \mathbf{b} \in Q_{J, +}^\vee$,
- (iii) we have $w \in W_J^v$.

Proof. By [Bou81, Chapitre VI, §1 Proposition 18], if $w \in W_J^v$, then $\mathbf{b} - w \cdot \mathbf{b} \in Q_{J, +}^\vee \subset \mathbf{b} - w \cdot \mathbf{b} \in \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$. Hence (iii) implies (ii), which itself implies (i).

It remains to prove that (i) implies (iii). To do so, take $w \in W_\Psi^v$ and assume that $w \in W_\Psi^v \setminus W_J^v$. Write $w = s_{\alpha_k} \dots s_{\alpha_1}$, with $k = \ell_\Psi(w)$, $\alpha_1, \dots, \alpha_k \in \Sigma$, where ℓ_Ψ denotes the length of the Coxeter system $(W_\Psi^v, \{s_\alpha \mid \alpha \in \Sigma\})$. Then there exists $j \in \llbracket 1, k \rrbracket$ such that $\alpha_j \notin J$. Take j minimal with this property. Then $s_{\alpha_j} \cdot s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} = s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} - \alpha_j (s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b}) \cdot \alpha_j^\vee$. As \mathbf{b} is regular, Lemma 2.9 implies $s_{\alpha_j} \cdot s_{\alpha_{j-1}} \dots s_{\alpha_1} \cdot \mathbf{b} \in \mathbf{b} + \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee - \mathbb{R}_{>0} \alpha_j^\vee$. By an induction using Lemma 2.9, we deduce

$$w \cdot \mathbf{b} \in \mathbf{b} + \left(\bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee - \mathbb{R}_{>0} \alpha_j^\vee \right) - \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee.$$

In particular, $w \cdot \mathbf{b} - \mathbf{b} \notin \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$, which completes the proof of the lemma. \square

Lemma 2.11. *Let $\eta = \frac{1}{|\Sigma|} \min\{\alpha(\mathbf{b}) \mid \alpha \in \Sigma\}$. Then for every $(t_\alpha) \in]0, \eta[^\Sigma$, $\mathbf{b} - \sum_{\alpha \in \Sigma} t_\alpha \alpha^\vee \in \mathcal{P}^0$.*

Proof. Let $m = \min\{\alpha(\mathbf{b}) \mid \alpha \in \Sigma\}$. Let $\alpha \in \Sigma$. Then $\mathbf{b} - m\alpha^\vee \in [\mathbf{b}, s_\alpha \cdot \mathbf{b}] \subset \mathcal{P}^0$. Let $(t_\alpha) \in]0, \eta[^\Sigma$. For $\alpha \in \Sigma$, set $t'_\alpha = \frac{1}{m} t_\alpha \in]0, \frac{1}{|\Sigma|}[$. Set $t'_0 = 1 - \sum_{\alpha \in \Sigma} t'_\alpha \in]0, 1[$. Then

$$t'_0 \mathbf{b} + \sum_{\alpha \in \Sigma} t'_\alpha (\mathbf{b} - m\alpha^\vee) = \mathbf{b} - \sum_{\alpha \in \Sigma} t'_\alpha m\alpha^\vee = \mathbf{b} - \sum_{\alpha \in \Sigma} t_\alpha \alpha^\vee \in \overline{\mathcal{P}^0}.$$

As $(\alpha^\vee)_{\alpha \in \Sigma}$ is a basis of \mathbb{A} , the map $(\mathbb{R}_{>0})^\Sigma \rightarrow \mathbb{A}$ defined by $(n_\alpha)_{\alpha \in \Sigma} \mapsto \sum_{\alpha \in \Sigma} n_\alpha \alpha^\vee$ is open, and the lemma follows. \square

Let $\mathcal{C}_\Psi = \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{>0} \alpha^\vee$. By [Bou81, Chapitre VI, §1 Proposition 18], we have:

Lemma 2.12. *We have $\mathcal{P}^0 \subset \mathbf{b} - \mathcal{C}_\Psi$.*

For $\beta \in \Sigma$, set $W_\beta^v = W_{\Sigma \setminus \{\beta\}, \Psi}^v$. Then W_β^v is the fixator of the one dimensional face $\overline{\mathcal{C}_f^v} \cap \bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$, according to (2.3).

Lemma 2.13. *Let $w \in W_\Psi^v$. Then $[w.\mathbf{b}, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$ if and only if $w \in \bigcup_{\beta \in \Sigma} W_\beta^v$.*

Proof. Let $w \in W_\Psi^v \setminus \bigcup_{\beta \in \Sigma} W_\beta^v$. Since $w \in W_\Psi^v$, Lemma 2.10 shows that $u := w.\mathbf{b} - \mathbf{b} \in \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{\geq 0} \alpha^\vee$. By using again Lemma 2.10, since $w \notin \bigcup_{\beta \in \Sigma} W_\beta^v$, we get that $u \in \bigoplus_{\alpha \in \Sigma} \mathbb{R}_{< 0} \alpha^\vee$. By Lemma 2.11, $\mathbf{b} + tu \in \mathcal{P}^0$ for $t \in [0, 1]$ small enough. We have $w.\mathbf{b} = \mathbf{b} + u$. As \mathcal{P}^0 is convex, either $] \mathbf{b}, \mathbf{b} + u[\subset \text{Fr}(\mathcal{P}^0)$ or $] \mathbf{b}, \mathbf{b} + u[\subset \mathcal{P}^0$ and thus $] \mathbf{b}, \mathbf{b} + u[=] \mathbf{b}, w.\mathbf{b}[\subset \mathcal{P}^0$.

Let $w \in \bigcup_{\beta \in \Sigma} W_\beta^v$. Then by Lemma 2.10 and Lemma 2.12, $[w.\mathbf{b}, \mathbf{b}] \subset \mathbb{A} \setminus (\mathbf{b} - \mathcal{C}_\Psi) \subset \mathbb{A} \setminus \mathcal{P}^0$, which proves that $[w.\mathbf{b}, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$. \square

For $J \subset \Sigma$, set $\mathbb{A}_J = \bigcap_{\alpha \in J} \ker(\alpha)$.

Lemma 2.14. *Let $i, j \in I$. Then $\langle \varpi_i \mid \alpha_j^\vee \rangle = \frac{1}{2} \delta_{i,j} |\alpha_i^\vee|^2$. In particular, for $J \subset \Sigma$, we have $\mathbb{A}_J^\perp = \bigoplus_{\alpha \in J} \mathbb{R} \alpha^\vee$.*

Proof. We have $\langle \varpi_i \mid \alpha_i^\vee \rangle = \langle s_{\alpha_i}.\varpi_i \mid s_{\alpha_i}.\alpha_i^\vee \rangle = \langle \varpi_i - \alpha_i^\vee \mid -\alpha_i^\vee \rangle$, so $2\langle \varpi_i \mid \alpha_i^\vee \rangle = |\alpha_i^\vee|^2$. If $j \neq i$, we have $\langle \varpi_i \mid \alpha_j^\vee \rangle = \langle s_{\alpha_j}.\varpi_i \mid s_{\alpha_j}.\alpha_j^\vee \rangle = \langle \varpi_i \mid -\alpha_j^\vee \rangle$, and hence $\langle \varpi_i \mid \alpha_j^\vee \rangle = 0$. \square

Let $\beta \in \Sigma$. Set $\mathcal{F}_\beta = \text{Int}_r(\text{conv}(W_\beta^v.\mathbf{b}))$. Denote by H_β the affine hyperplane orthogonal to $\bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$ and containing \mathbf{b} .

Lemma 2.15. *1. The facets of \mathcal{P}^0 are the $w.\mathcal{F}_\beta$ for $w \in W_\Psi^v$ and $\beta \in \Sigma$.*

2. For $\beta \in \Sigma$, the hyperplane containing \mathcal{F}_β is $H_\beta = \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$.

Proof. As \mathcal{P}^0 is W_Ψ^v -invariant, $\text{Fr}(\mathcal{P}^0)$ is W_Ψ^v -invariant. Let \mathcal{F} be a facet of \mathcal{P}^0 . Let E be the set of extreme points of $\overline{\mathcal{F}}$. By Lemma 2.4 and Lemma 2.5, $E \subset W_\Psi^v.\mathbf{b}$ and thus replacing \mathcal{F} by $w.\mathcal{F}$ for some $w \in W_\Psi^v$ if needed, we may assume $\mathbf{b} \in \overline{\mathcal{F}}$.

Let $x \in \overline{\mathcal{F}}$. Then by Lemma 2.12, we can write $x = \mathbf{b} - \sum_{\alpha \in \Sigma} n_\alpha \alpha^\vee$, where $(n_\alpha) \in (\mathbb{R}_{\geq 0})^\Sigma$. Set

$$\text{supp}(x) = \{\alpha \in \Sigma \mid n_\alpha \neq 0\} \subset \Sigma.$$

Then for $x, y \in \overline{\mathcal{F}}$, we have $\frac{1}{2}(x + y) \in \overline{\mathcal{F}}$ and $\text{supp}(\frac{1}{2}(x + y)) = \text{supp}(x) \cup \text{supp}(y)$. Therefore there exists $x_0 \in \overline{\mathcal{F}}$ which has the largest support, i.e., $\text{supp}(x) \subseteq \text{supp}(x_0)$ for every $x \in \overline{\mathcal{F}}$.

We have $[x_0, \mathbf{b}] \subset \text{Fr}(\mathcal{P}^0)$. By Lemma 2.11 we deduce the existence of $\beta \in \Sigma$ such that $t\mathbf{b} + (1-t)x_0 \in \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$ for $t \in [0, 1]$ large enough. But then $x_0 \in \mathbf{b} + \bigoplus_{\alpha \in \Sigma \setminus \{\beta\}} \mathbb{R} \alpha^\vee$. Therefore $\text{supp}(x_0) \subset \Sigma \setminus \{\beta\}$ and $\text{supp}(x) \subset \Sigma \setminus \{\beta\}$ for every $x \in \mathcal{F}$. In particular, for every extreme point Q of $\overline{\mathcal{F}}$, we have $\text{supp}(Q) \subset \Sigma \setminus \{\beta\}$. By Lemma 2.10, every extreme point of $\overline{\mathcal{F}}$ is contained in $W_\beta^v.\mathbf{b}$. By Krein-Milman theorem we deduce $\overline{\mathcal{F}} \subset \text{conv}(w.\mathbf{b} \mid w \in W_\beta^v) = \overline{\mathcal{F}_\beta}$.

As $\overline{\mathcal{P}^0} \cap H_\beta$ is a W_β^v -stable set containing \mathbf{b} , $\overline{\mathcal{P}^0} \cap H_\beta \supset \mathcal{F}_\beta$. Moreover, $\overline{\mathcal{P}^0} \cap H_\beta$ contains $\{\mathbf{b}\} \cup \{\mathbf{b} - \alpha(\mathbf{b})\alpha^\vee \mid \alpha \in \Sigma \setminus \{\beta\}\}$. As the affine subspace spanned by this set is H_β , and as $\overline{\mathcal{F}_\beta} \subset \text{Fr}(\mathcal{P}^0)$ (by Lemma 2.13), $\overline{\mathcal{P}^0} \cap H_\beta$ is a closed facet of $\overline{\mathcal{P}^0}$. By what we proved above we deduce that

$$\overline{\mathcal{P}^0} \cap H_\beta = \text{conv}(W_\beta^v.\mathbf{b}).$$

Thus, Statement (1) follows. Moreover, Statement (2) follows from Lemma 2.14. \square

2.3.2 Arbitrary faces of \mathcal{P}^0

Definition 2.16. Let C^v be a vectorial chamber of $(\mathbb{A}, \mathcal{H}_\Psi^0)$. Since $\overline{C_f^v}$ is a fundamental domain for the action of W_Ψ^v on \mathbb{A} , there exists $w \in W_\Psi^v$ be such that $C^v = w.C_f^v$. Then, we write:

$$\mathbf{b}_{C^v} = w.\mathbf{b}.$$

Note that \mathbf{b}_{C^v} does not depend on the choice of w , since any other w' as above has the form $w' = ww_0$, where w_0 stabilizes C_f^v .

If $F^v \in \mathcal{F}^0$, we choose a vectorial chamber C^v dominating F^v , and we set:

$$\mathcal{F}_{F^v} = \text{Int}_r(\text{conv}(W_{F^v}^v.\mathbf{b}_{C^v})).$$

This does not depend on the choice of C^v , since $W_{F^v}^v$ acts transitively on the set of vectorial chambers dominating F^v , according to [Bou81, Chapitre V, §4 Proposition 5].

The next two lemmas will be given without proof. They can be settled in a very similar way to their affine counterparts, which will be settled later, in Lemmas 3.17 and 3.18).

Lemma 2.17. Let F^v be a face of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ and let E be the *star* of F^v , i.e., the union of the faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ dominating F^v . Let C^v be a chamber of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ dominating F^v and Σ_{C^v} be the basis of Ψ associated with C^v . Let E' be the union of the faces of C^v dominating F^v . Let $\Sigma_{F^v} = \{\alpha \in \Sigma_{C^v} \mid \alpha(F^v) = \{0\}\}$. For $\alpha \in \Sigma_{C^v}$, set $\mathring{D}_\alpha = \alpha^{-1}(\mathbb{R}_{>0})$. Then

$$E = W_{F^v}^v.E' = \bigcap_{w \in W_{F^v}^v, \alpha \in \Sigma_{C^v} \setminus \Sigma_{F^v}} w.\mathring{D}_\alpha.$$

In particular, E is open and convex.

Lemma 2.18. Let F^v, F_1^v be faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ such that F_1^v dominates F^v . Let $x \in F_1^v$ and $y = \frac{1}{|W_{F^v}^v|} \sum_{w \in W_{F^v}^v} w.x$. Then $y \in F^v$ and y is the orthogonal projection of x on $\text{vect}(F^v)$.

Lemma 2.19. Let F^v be a face of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ which is not a chamber. Then $\Phi_{F^v} = \{\alpha \in \Phi \mid \alpha(F^v) = \{0\}\}$ is a root system in $\text{vect}(F^v)^\perp$. Its Weyl group is $W_{F^v}^v = \langle s_\alpha \mid \alpha \in \Phi_{F^v} \rangle$.

Proof. Up to changing the choice of C_f^v if needed, we may assume that C_f^v dominates F^v . Let $\Sigma_{F^v} = \Sigma \cap \Phi_{F^v}$. We have $\text{vect}(F^v) \subset \bigcap_{\alpha \in \Sigma_{F^v}} \ker(\alpha)$. As Σ is a basis of \mathbb{A}^* and using a simple dimension argument based on Proposition 2.6, we deduce $\text{vect}(F^v) = \bigcap_{\alpha \in \Sigma_{F^v}} \ker(\alpha)$. By Lemma 2.14, $\alpha^\vee \in \text{vect}(F^v)^\perp$ for every $\alpha \in \Sigma_{F^v}$. Therefore $\Phi_{F^v}|_{\text{vect}(F^v)^\perp}$ satisfies [Bou81, VI §1.1 Définition 1]. \square

We equip \mathcal{F}^0 with the dominance order \leq : for $F_1^v, F_2^v \in \mathcal{F}^0$, we have $F_1^v \leq F_2^v$ if and only if $F_1^v \subset \overline{F_2^v}$.

A less precise version of the following proposition already appears in [Vin91, Proposition 3.2].

Proposition 2.20. (see Figure 2.4)

1. The assignment $f : F^v \mapsto \mathcal{F}_{F^v}$ defines an isomorphism of ordered spaces between (\mathcal{F}^0, \geq) and the set of faces of (\mathcal{P}^0, \subset) .
2. Let $(F_i^v)_{i \in [1, k]}$ be a finite family of faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$. Then if $\sum_{i=1}^k F_i^v \in \mathcal{F}^0$ (or equivalently if there exists $F^v \in \mathcal{F}^0$ dominating $\sum_{i=1}^k F_i^v$), we have $\bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} = \overline{\mathcal{F}_{\sum_{i=1}^k F_i^v}}$. Otherwise, we have $\bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} = \emptyset$.

3. For $F^v \in \mathcal{F}^0$, the set of extreme points of $\overline{\mathcal{F}_{F^v}}$ is $W_{F^v}^v \cdot \mathbf{b}_{C^v}$, for any chamber C^v dominating F^v . The direction of $\text{supp}(\mathcal{F}_{F^v})$ is $(F^v)^\perp$ and in particular, $\dim(\text{vect}(\mathcal{F}_{F^v})) = \text{codim}_{\mathbb{A}}(\text{vect}(F^v))$.
4. Let \mathcal{F} be a face of \mathcal{P}^0 and let y be the barycenter of the extreme points of $\overline{\mathcal{F}}$. Then $\mathcal{F} = \mathcal{F}_{F^v}$, where F^v is the unique face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ containing y , the point y is the orthogonal projection of \mathcal{F} on F^v and $\overline{\mathcal{F}_{F^v}} \cap F^v = \{y\}$. Moreover, $\overline{\mathcal{F}_{F^v}}$ is contained in the star of F^v .

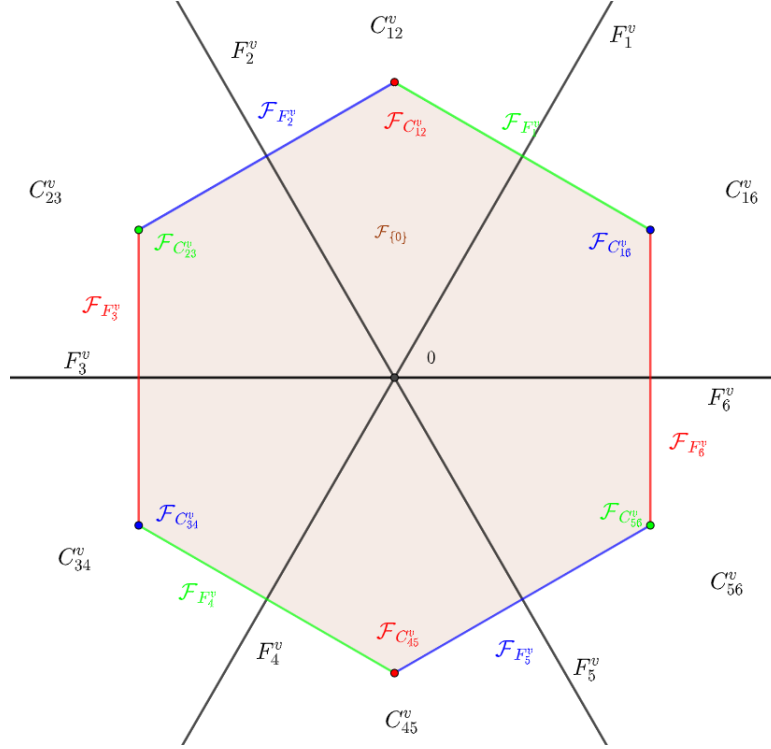


Figure 2.4: Illustration of Proposition 2.20 in the case of A_2

Proof. Step 1: We first prove that the facets of \mathcal{P}^0 are the \mathcal{F}_{F^v} , where the F^v are the one-dimensional faces of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$. To do so, let $\beta \in \Sigma$. Set $F_{\beta}^v = \beta^{-1}(\mathbb{R}_{>0}) \cap \bigcap_{\alpha \in \Sigma \setminus \{\beta\}} \ker(\alpha)$. Then the fixator of F_{β}^v in W_{Ψ}^v is W_{β}^v and thus $\mathcal{F}_{F_{\beta}^v} = \mathcal{F}_{\beta}$. If $w \in W_{\Psi}^v$, then

$$w \cdot \overline{\mathcal{F}_{F_{\beta}^v}} = \text{conv}(wW_{F_{\beta}^v}^v \cdot \mathbf{b}) = \text{conv}(wW_{F_{\beta}^v}^v w^{-1} \cdot w \cdot \mathbf{b}) = \text{conv}(W_{w \cdot F_{\beta}^v}^v w \cdot \mathbf{b}) = \overline{\mathcal{F}_{w \cdot F_{\beta}^v}}.$$

By Lemma 2.15, the facets of \mathcal{P}^0 are the $w \cdot \mathcal{F}_{\beta}$ such that $w \in W_{\Psi}^v$ and $\beta \in \Sigma$ so, by Proposition 2.6, the facets of \mathcal{P}^0 are the \mathcal{F}_{F^v} , where the F^v are the one-dimensional faces of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$.

Step 2: We now prove the second assertion of 2): if $(F_i^v)_{i \in [1, k]}$ is a family of faces of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ such that $\overline{\mathcal{F}} := \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}} \neq \emptyset$, then $\sum_{i=1}^k F_i^v \in \mathcal{F}^0$. Indeed, take $x \in \overline{\mathcal{F}}$. Let F_x^v be the face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ containing x . Then by Lemma 2.17, F_x^v dominates $\bigcup_{i=1}^k F_i^v$. Therefore any chamber dominating F_x^v dominates $\bigcup_{i=1}^k F_i^v$.

Step 3: We now prove that the map $f : F^v \mapsto \mathcal{F}_{F^v}$ is well-defined. To do so, let F^v be a face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ and F_1^v, \dots, F_k^v be the one dimensional faces of F^v . Then $\overline{\mathcal{F}} := \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$ is a closed face of \mathcal{P}^0 by [BG09, Theorem 1.10 and Proposition 1.8] and it is non-empty because it contains $\overline{\mathcal{F}_{F^v}}$. In particular, by Step 2, there exists a vectorial

chamber C^v dominating $\bigcup_{i=1}^k F_i^v$, and by Proposition 2.6, we get $F^v = \sum_{i=1}^k F_i^v$. We then have:

$$\begin{aligned}
\text{Extr}(\overline{\mathcal{F}}) &= \bigcap_{i=1}^k \text{Extr}(\overline{\mathcal{F}_{F_i^v}}) && \text{by Lemma 2.5} \\
&= \bigcap_{i=1}^k W_{F_i^v}^v \cdot \mathbf{b}_{C^v} && \text{by Lemma 2.4} \\
&= \left(\bigcap_{i=1}^k W_{F_i^v}^v \right) \cdot \mathbf{b}_{C^v} && \text{since } \mathbf{b}_{C^v} \text{ is regular} \\
&= W_{F^v}^v \cdot \mathbf{b}_{C^v}.
\end{aligned}$$

By Krein-Milman Theorem, we deduce $\overline{\mathcal{F}} = \overline{\mathcal{F}_{F^v}}$. Hence $\mathcal{F}_{F^v} = \text{Int}_r(\overline{\mathcal{F}_{F^v}})$ is a face of \mathcal{P}^0 .

Step 4: We now prove that the map $f : F^v \mapsto \mathcal{F}_{F^v}$ of statement 1) is surjective. Indeed, let \mathcal{F} be a proper face of \mathcal{P}^0 . Then by [BG09, Theorem 1.10], there exist $k \in \mathbb{Z}_{\geq 1}$, one dimensional vectorial faces F_1^v, \dots, F_k^v of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ such that $\overline{\mathcal{F}} = \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$ and there exists a vectorial chamber C^v dominating $\bigcup_{i=1}^k F_i^v$. By the same computations as in Step 3, we have $\overline{\mathcal{F}} = \overline{\mathcal{F}_{F^v}}$ with $F^v := \sum_{i=1}^k F_i^v$. Hence $\mathcal{F} = \text{Int}_r(\overline{\mathcal{F}}) = \text{Int}_r(\overline{\mathcal{F}_{F^v}}) = \mathcal{F}_{F^v}$.

Step 5: We prove that f is non-decreasing. Let $F_1^v, F_2^v \in \mathcal{F}^0$ be such that $F_1^v \subset \overline{F_2^v}$. Then as W_Ψ^v acts by homeomorphisms on \mathbb{A} , we have $W_{F_1^v}^v \supset W_{F_2^v}^v$ and any chamber dominating F_2^v dominates F_1^v . Therefore $\overline{\mathcal{F}_{F_1^v}} \supset \overline{\mathcal{F}_{F_2^v}}$.

Step 6: We prove that, if $F_1^v, F_2^v \in \mathcal{F}^0$ are such that $\overline{\mathcal{F}_{F_1^v}} \supset \overline{\mathcal{F}_{F_2^v}}$, then F_2^v dominates F_1^v . Indeed, by Step 2, there exists a vectorial chamber C^v dominating F_1^v and F_2^v . By Lemma 2.5, we have $\text{Extr}(\overline{\mathcal{F}_{F_1^v}}) = W_{F_1^v}^v \cdot \mathbf{b}_{C^v} \supset \text{Extr}(\overline{\mathcal{F}_{F_2^v}}) = W_{F_2^v}^v \cdot \mathbf{b}_{C^v}$. As \mathbf{b}_{C^v} is regular, we deduce $W_{F_1^v}^v \supset W_{F_2^v}^v$. This implies $F_1^v \subset \overline{F_2^v}$ by [Bou81, V §3.3 Proposition 1], Proposition 2.6 and [BB05, Proposition 2.4.1].

Step 7: We prove assertion 1). We have seen in Steps 4, 5 and 6 that f is surjective and non-decreasing, and that if $f(F_1^v) \supset f(F_2^v)$, then F_2^v dominates F_1^v . This implies that f is an isomorphism of ordered sets.

Step 8: We finish the proof of 2). Let $(F_i^v)_{i \in [1, k]}$ be a family of faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ such that $F^v := \sum_{i=1}^k F_i^v \in \mathcal{F}^0$. Exactly the same arguments as in Step 3 show that $\overline{\mathcal{F}_{F^v}} = \bigcap_{i=1}^k \overline{\mathcal{F}_{F_i^v}}$.

Step 9: We prove 3) and 4). Indeed, let now \mathcal{F} be a face of \mathcal{P}^0 and $F^v \in \mathcal{F}^0$ be such that $\mathcal{F} = \mathcal{F}_{F^v}$. Then we proved in Lemma 2.4 that $\text{Extr}(\overline{\mathcal{F}}) = W_{F^v}^v \cdot \mathbf{b}_{C^v}$, where C^v is a vectorial chamber dominating F^v . Let $y = \frac{1}{|W_{F^v}^v|} \sum_{w \in W_{F^v}^v} w \cdot \mathbf{b}_{C^v}$. Then y is the barycenter of $\text{Extr}(\overline{\mathcal{F}_{F^v}})$. By Lemma 2.18, it belongs to F^v .

Let F^v be a face of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ and C^v be a chamber dominating F^v . Let x be the orthogonal projection of \mathbf{b}_{C^v} on $\text{vect}(F^v)$. Then $x \in F^v$ by Lemma 2.18. We have $W_{F^v}^v \cdot \mathbf{b}_{C^v} = x + W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$. Therefore the direction of $\text{supp}(\text{conv}(W_{F^v}^v \cdot \mathbf{b}_{C^v}))$ is the vector space spanned by $W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$. As \mathbf{b}_{C^v} is regular, $|W_{F^v}^v \cdot \mathbf{b}_{C^v}| = |W_{F^v}^v| = |W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)|$. Therefore $\mathbf{b}_{C^v} - x$ is a regular element of $(F^v)^\perp$ (for the root system Φ_{F^v} defined in Lemma 2.19). Using Lemma 2.8, we deduce that $W_{F^v}^v \cdot (\mathbf{b}_{C^v} - x)$ spans $(F^v)^\perp$. Finally, the fact that $\overline{\mathcal{F}_{F^v}}$ is contained in the star of F^v is a direct consequence of Lemma 2.17. \square

Corollary 2.21. 1. Let $F^v \in \mathcal{F}^0$. Then $\mathcal{F}_{F^v} = \overline{\mathcal{F}_{F^v}} \setminus \bigcup_{F_2^v \in \mathcal{F}^0, \overline{F_2^v} \supsetneq \overline{F^v}} \overline{\mathcal{F}_{F_2^v}}$.

2. Let $x \in \overline{\mathcal{P}^0}$. Then there exists a unique face F_1^v of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ such that $x \in \mathcal{F}_{F_1^v}$. Moreover, $\{F^v \in \mathcal{F}^0 \mid x \in \overline{\mathcal{F}_{F^v}}\} = \{F^v \in \mathcal{F}^0 \mid F^v \subset \overline{F_1^v}\}$.

Proof. 1) Let $F^v \in \mathcal{F}^0$. Let \mathcal{F}_2 be a face of \mathcal{F}_{F^v} . By [BG09, Proposition 1.8], \mathcal{F}_2 is a face of \mathcal{P}^0 and thus by Proposition 2.20, there exists $F_2^v \in \mathcal{F}^0$ such that $\mathcal{F}_2 = \mathcal{F}_{F_2^v}$ and F_2^v dominates F^v . We then conclude with [BG09, Corollary 1.7].

2) Let $x \in \overline{\mathcal{P}^0}$. By [BG09, Theorem 1.10 d)], there exists a unique face \mathcal{F} of \mathcal{P}^0 containing x and $\overline{\mathcal{F}}$ is the minimal closed face containing x . By Proposition 2.20, we can write $\mathcal{F} = \mathcal{F}_{F_1^v}$ for some face F_1^v of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ and $E = \{F^v \in \mathcal{F}^0 \mid x \in \overline{\mathcal{F}_{F^v}}\} = \{F_2^v \in \mathcal{F}^0 \mid \overline{F_2^v} \subset \overline{F_1^v}\}$. \square

Corollary 2.22. Let F^v be a face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$. Let $\pi = \pi_{\overline{\mathcal{P}^0}}$ and $x \in \mathcal{F}_{F^v}$. Then $\pi^{-1}(\{x\}) = x + \overline{F^v}$.

Proof. Let G^v be a one dimensional face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$. Then by Proposition 2.20, the direction of $\text{supp}(\mathcal{F}_{G^v})$ is $(G^v)^{\perp}$. Hence we have $\delta_{G^v} = \pm \overline{G^v}$, with the notation of subsection 2.1. By Proposition 2.20, $G^v \cap \mathcal{F}_{G^v} = \{z\}$, for some $z \in G^v$. Therefore $z - G^v$ contains 0 and thus it meets \mathcal{P}^0 , which proves that $\delta_{G^v} = \overline{G^v}$.

Now let k be the dimension of the direction of F^v and let F_1^v, \dots, F_k^v be the one-dimensional faces of F^v , so that $F^v = \sum_{i=1}^k F_i^v$. Then by Corollary 2.21 2), the set of closed facets containing x is the set of closed facets $\overline{\mathcal{F}_{G^v}}$ such that $\overline{G^v} \subset \overline{F^v}$: this is $\{F_i^v \mid i \in [1, k]\}$. By Proposition 2.3, $\pi^{-1}(\{x\}) = x + \sum_{i=1}^k \overline{F_i^v} = x + \overline{F^v}$. \square

Lemma 2.23. Let F^v be a face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ and $\alpha \in \Phi$ be such that $\alpha(F^v) > 0$. Then $\alpha(\overline{\mathcal{F}_{F^v}}) > 0$.

Proof. Let $x \in \mathcal{F}_{F^v}$ and $F_1^v \in \mathcal{F}^0$ be such that $x \in F_1^v$. Then by Proposition 2.20 4), $\overline{F_1^v} \supset F^v$. As F_1^v is a face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$, we have $\alpha(F_1^v) \in \{\mathbb{R}_{<0}, \{0\}, \mathbb{R}_{>0}\}$, thus $\alpha(\overline{F_1^v}) \in \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. As $\alpha(F^v) = \mathbb{R}_{>0}$, we have $\alpha(\overline{F_1^v}) = \mathbb{R}_{\geq 0}$ and $\alpha(F_1^v) = \mathbb{R}_{>0}$, which proves the lemma. \square

2.4 Projections restricted to “parabolic” affine subspaces

In this subsection, we study the restriction of the orthogonal projection $\pi_{\overline{\mathcal{P}^0}}$ to certain affine subspaces parallel to the faces of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$. We prove that under some conditions, subsets of these subspaces are stabilized by the projection (see Proposition 2.33). The proof is a bit intricate, and one of the main tools consists in introducing a suitable auxiliary set $\tilde{\mathcal{P}}$ contained in the affine subspaces we want to study and such that the orthogonal projections on $\overline{\mathcal{P}^0}$ and on $\tilde{\mathcal{P}}$ coincide on large subsets of those affine subspaces.

For F^v a one dimensional face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$, we denote by H_{F^v} the hyperplane of \mathbb{A} containing $\overline{\mathcal{F}_{F^v}}$ and by \mathcal{D}_{F^v} the half-space delimited by H_{F^v} and containing \mathcal{P}^0 . We write π instead of $\pi_{\overline{\mathcal{P}^0}}$. By Proposition 2.20 (3), if $z \in \mathcal{F}_{F^v}$, then we have:

$$\mathcal{D}_{F^v} = \{y \in \mathbb{A} \mid \langle y - z \mid x \rangle \leq 0, \forall x \in F^v\}. \quad (2.5)$$

Lemma 2.24. 1. Let $x \in \mathbb{A}$ and $F^v \in \mathcal{F}^0$ be such that $x \in F^v$. Then $\pi(x) \in F^v$ and for every $\tilde{F}^v \in \mathcal{F}^0$ such that $\pi(x) \in \overline{\mathcal{F}_{\tilde{F}^v}}$, we have $\tilde{F}^v \subset \overline{F^v}$.

2. Let $F^v \in \mathcal{F}^0$. Then $\pi^{-1}(F^v) \subset F^v$.

Proof. 1) Let $F_1^v \in \mathcal{F}^0$ be such that $\pi(x) \in \mathcal{F}_{F_1^v}$. By Corollary 2.22, $x - \pi(x) \in \overline{F_1^v}$. Let $F_2^v \in \mathcal{F}^0$ be such that $\pi(x) \in F_2^v$. By Proposition 2.20, we have $\overline{F_2^v} \supset F_1^v$. Therefore $x \in \overline{F_1^v} + F_2^v \subset \overline{F_2^v} + F_2^v \subset F_2^v$. As the vectorial faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ form a partition of \mathbb{A} , we deduce $F_2^v = F^v$, which proves that $\pi(x) \in F^v$. Moreover, by Corollary 2.21, for every \tilde{F}^v such that $\pi(x) \in \overline{\mathcal{F}_{\tilde{F}^v}}$, we have $\tilde{F}^v \subset \overline{F_1^v} \subset \overline{F_2^v} = \overline{F^v}$.

2) Let $x \in F^v \cap \overline{\mathcal{P}^0}$ and $y \in \pi^{-1}(x)$. Let $F_1^v \in \mathcal{F}^0$ be such that $y \in F_1^v$. Then by 1), $x \in F_1^v$ and as the faces of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ form a partition of \mathbb{A} , we deduce that $F_1^v = F^v$. \square

We index the basis Σ of Ψ associated with C_f^v by a finite set I , i.e., we write $\Sigma = \{\alpha_i \mid i \in I\}$, where $|I| = |\Sigma|$. Let $(\varpi_i) \in \mathbb{A}^I$ be the dual basis of $(\alpha_i)_{i \in I}$, i.e., a basis satisfying $\alpha_i(\varpi_j) = \delta_{i,j}$ for all $i, j \in I$. For $i \in I$, we set

$$\mathcal{D}_i = \{x \in \mathbb{A} \mid \langle x - \mathbf{b} \mid \varpi_i \rangle \leq 0\}$$

and

$$\mathcal{D}_i^0 = \{x \in \mathbb{A} \mid \langle x \mid \varpi_i \rangle \leq 0\}$$

(we have $\mathcal{D}_i = \mathcal{D}_{\mathbb{R}_{>0}\varpi_i}$ for the notation of Eq. (2.5), by Lemma 2.14). For $J \subset I$, we set $V_J = \bigoplus_{j \in J} \mathbb{R}\varpi_j$.

Lemma 2.25. *Let V be a finite dimensional vector space, $k \in \mathbb{Z}_{>0}$ and $f_1, \dots, f_k \in V^*$ be a free family. For $i \in \llbracket 1, k \rrbracket$, set $E_i = f_i^{-1}(\mathbb{R}_{\geq 0})$ and $L_i = E_i \cap \bigcap_{j \in \llbracket 1, k \rrbracket \setminus \{i\}} f_j^{-1}(\{0\})$. Then*

$$\bigcap_{i=1}^k E_i = \text{conv}\left(\bigcup_{i=1}^k L_i\right).$$

Proof. Let $(f_i)_{i \in \mathcal{B}}$ be a basis of V^* , where \mathcal{B} is a set containing $\llbracket 1, k \rrbracket$. For $i \in \mathcal{B}$, let $e_i \in V$ be such that $f_j(e_i) = \delta_{i,j}$, for $j \in \mathcal{B}$. Then $\bigcap_{i=1}^k E_i = \bigoplus_{i=1}^k \mathbb{R}_{\geq 0}e_i \oplus V'$, where $V' = \bigoplus_{i \in \mathcal{B} \setminus \llbracket 1, k \rrbracket} \mathbb{R}e_i$. Let $i \in \llbracket 1, k \rrbracket$. Then $L_i = \mathbb{R}_{\geq 0}e_i \oplus V'$, which proves the result. \square

Lemma 2.26. *Let $J \subset I$. For any $x \in (\bigoplus_{j \in J} \mathbb{R}\alpha_j^\vee)$, we have:*

$$(\alpha_j(x) \geq 0, \forall j \in J) \Rightarrow x \in \bigoplus_{j \in J} \mathbb{R}_{\geq 0}\alpha_j^\vee.$$

Proof. For $K \subset J$, denote by A_K the Cartan matrix $A_K = (\alpha_i(\alpha_j^\vee))_{i,j \in K}$. Write $J = J_1 \sqcup \dots \sqcup J_k$, with A_{J_i} indecomposable for every $i \in \llbracket 1, k \rrbracket$. Let $x \in \bigoplus_{j \in J} \mathbb{R}\alpha_j^\vee$. Write $x = \sum_{i=1}^k x_i$, with $x_i \in \bigoplus_{\ell \in J_i} \mathbb{R}\alpha_\ell^\vee$, for $i \in \llbracket 1, k \rrbracket$. Assume that $\alpha_j(x) \geq 0$ for all $j \in J$. Let $i \in \llbracket 1, k \rrbracket$. Then $\alpha_j(x) = \alpha_j(x_i)$ for all $j \in J_i$. By [Kac94, Theorem 4.3 (Fin)] applied to the transpose of A_{J_i} , we have $x_i \in \bigoplus_{j \in J_i} \mathbb{R}_{\geq 0}\alpha_j^\vee$, and the lemma follows. \square

For $i \in I$, we set $\mathcal{F}_i = \mathcal{F}_{\mathbb{R}_{>0}\varpi_i}$, with the notation of Definition 2.16. The \mathcal{F}_i , for $i \in I$, are the open facets of \mathcal{P}^0 meeting C_f^v .

Lemma 2.27. *Let $J \subset I$. Then we have*

$$V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 \subset \bigoplus_{i \in I} \mathbb{R}_{\leq 0}\alpha_i^\vee \subset \bigcap_{i \in I} \mathcal{D}_i^0.$$

Proof. If J is empty, the result is clear so we assume J non-empty. For $j \in J$, denote by H_j^0 the hyperplane delimiting \mathcal{D}_j^0 . We have:

$$H_j^0 = \{x \in \mathbb{A} \mid \langle x \mid \varpi_j \rangle = 0\}$$

and by Lemma 2.14, $H_j^0 = \bigoplus_{k \in I \setminus \{j\}} \mathbb{R} \alpha_k^\vee$. Set $L_j = V_J \cap \bigcap_{k \in J \setminus \{j\}} H_k^0 \cap \mathcal{D}_j^0$. As $(\varpi_k)_{k \in J}$ is a free family of \mathbb{A} , the family $(x \mapsto \langle x \mid \varpi_k \rangle)_{k \in J}$ is a free family of V_J^* and thus by Lemma 2.25,

$$V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 = \text{conv}(L_j \mid j \in J). \quad (2.6)$$

Let $j \in J$. Let $V_j' = \bigoplus_{k \in (I \setminus J) \cup \{j\}} \mathbb{R} \alpha_k^\vee$. As $(\alpha_i(\alpha_{i'}^\vee))_{i, i' \in (I \setminus J) \cup \{j\}}$ is a Cartan matrix (by [Kac94, Lemma 4.4] for example), it is invertible and $(\alpha_k|_{V_j'})_{k \in (I \setminus J) \cup \{j\}}$ is a free family of $(V_j')^*$. Therefore there exists $x_j \in V_j'$ such that $\alpha_k(x_j) = -\delta_{j,k}$, for $k \in (I \setminus J) \cup \{j\}$. By Lemma 2.26, we have $x_j \in -\bigoplus_{k \in (I \setminus J) \cup \{j\}} \mathbb{R}_{\geq 0} \alpha_k^\vee$. By Lemma 2.14, we deduce $\langle \varpi_j \mid x_j \rangle \leq 0$ and $x_j \in H_k^0$ for all $k \in J \setminus \{j\}$. Moreover, $x_j \in V_J$ and thus $x_j \in L_j$. For dimension reasons, $L_j = \mathbb{R}_{\geq 0} x_j$ and hence by Eq. (2.6), we have $V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0 \subset \bigoplus_{i \in I} \mathbb{R}_{\leq 0} \alpha_i^\vee$. We conclude with Lemma 2.14. \square

As in §2.1, we write $\bigwedge_{j \in J} \mathcal{F}_j = \text{Int}_r(\bigcap_{j \in J} \overline{\mathcal{F}_j})$, for $J \subset I$. By Proposition 2.20, we have:

$$\bigwedge_{j \in J} \mathcal{F}_j = \mathcal{F}_{\sum_{j \in J} \mathbb{R}_{> 0} \varpi_j}. \quad (2.7)$$

Lemma 2.28. *Let $J \subset I$. Let $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$. Let $z \in \bigwedge_{j \in J} \mathcal{F}_j$. Then for all $i \in I \setminus J$, $\overline{\mathcal{P}^0} \cap (z + V_J) \subset \mathring{\mathcal{D}}_i$.*

Proof. We have $\overline{\mathcal{P}^0} \subset \bigcap_{i \in I} \mathcal{D}_i \subset \bigcap_{j \in J} \mathcal{D}_j$. By Corollary 2.21 and Eq. (2.7), we have:

$$\{i \in I \mid z \in \overline{\mathcal{F}_i}\} = J. \quad (2.8)$$

By Eq. (2.5) we deduce $\bigcap_{j \in J} \mathcal{D}_j = z + \bigcap_{j \in J} \mathcal{D}_j^0$. Let $x \in (z + V_J) \cap \overline{\mathcal{P}^0}$. Then we can write $x = z + v$, with $v \in V_J$. We have $x \in \bigcap_{j \in J} \mathcal{D}_j$ and thus $v \in V_J \cap \bigcap_{j \in J} \mathcal{D}_j^0$. Let $i \in I \setminus J$. By Lemma 2.27, we have $v \in \mathcal{D}_i^0$. By Eq. (2.8), $z \in \mathring{\mathcal{D}}_i$ and hence $x = z + v \in \mathring{\mathcal{D}}_i + \mathcal{D}_i^0 = \mathring{\mathcal{D}}_i$. \square

Let $J \subset I$ be non-empty. Let $z \in \bigwedge_{j \in J} \mathcal{F}_j$ and $\overline{\mathcal{P}} = \overline{\mathcal{P}_J(z)} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$. For $j \in J$ define $f_j \in (V_J)^*$ by $f_j(x) = \langle x \mid \varpi_j \rangle$, for $x \in V_J$. Then $(f_j)_{j \in J}$ is free. Let $(a_j)_{j \in J} \subset V_J$ be a dual set of $(f_j)_{j \in J}$, i.e., a free set such that $f_i(a_j) = \delta_{i,j}$, for all $i, j \in J$. Then $\overline{\mathcal{P}} = z - \sum_{j \in J} \mathbb{R}_{\geq 0} a_j$. For $J' \subset J$, set $\overline{\mathcal{F}_{J'}} = z - \sum_{j \in J'} \mathbb{R}_{\geq 0} a_j$. The closed faces of $\overline{\mathcal{P}}$ are the $\overline{\mathcal{F}_{J'}}$, for $J' \subset J$. In particular, the closed facets of $\overline{\mathcal{P}}$ are the $\overline{\mathcal{F}_j}$, $j \in J$, where $\overline{\mathcal{F}_j} = \overline{\mathcal{F}_{J \setminus \{j\}}}$, for $j \in J$. Let $j \in J$. The support of $\overline{\mathcal{F}_j}$ is $\tilde{H}_j = \{x \in (z + V_J) \mid \langle x - z \mid \varpi_j \rangle = 0\} = H_j \cap (z + V_J)$, where H_j is the frontier of \mathcal{D}_j . For $x \in \overline{\mathcal{F}_j}$, we have $(x + \mathbb{R}_{\geq 0} \varpi_j) \cap \overline{\mathcal{F}_j} = \{x\}$ and ϖ_j is orthogonal to the direction of $\overline{\mathcal{F}_j}$, so that $\mathbb{R}_{\geq 0} \varpi_j = \delta_{\overline{\mathcal{F}_j}}$ for the notation of subsection 2.1 applied to the polyhedron $\overline{\mathcal{P}} \subset z + V_J$. Therefore by Proposition 2.3, we have:

$$\forall J' \subset J, \forall a \in \overline{\mathcal{F}_{J'}}, \pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \bigoplus_{j \in J \setminus (J')} \mathbb{R}_{\geq 0} \varpi_j, \quad (2.9)$$

where $\pi_{\overline{\mathcal{P}}} : z + V_J \rightarrow \overline{\mathcal{P}}$ is the orthogonal projection.

Lemma 2.29. *Let $J \subset I$. Let $z \in \bigwedge_{j \in J} \mathcal{F}_j \cap \overline{\mathcal{C}_f^v}$. Let $\overline{\mathcal{P}} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$. Then $\overline{\mathcal{P}} \cap \overline{\mathcal{C}_f^v} = \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{\mathcal{C}_f^v}$.*

Proof. We have $\overline{\mathcal{P}^0} \subset \bigcap_{i \in I} \mathcal{D}_i \subset \bigcap_{i \in J} \mathcal{D}_i$, and hence $\overline{\mathcal{P}} \cap \overline{C_f^v} \supset \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{C_f^v}$. Suppose that $\overline{\mathcal{P}} \cap \overline{C_f^v} \supsetneq \overline{\mathcal{P}^0} \cap (z + V_J) \cap \overline{C_f^v}$. Let $x \in (\overline{\mathcal{P}} \cap \overline{C_f^v}) \setminus \overline{\mathcal{P}^0}$. Let $\gamma : [0, 1] \rightarrow [x, z]$ be the affine parametrization such that $\gamma(0) = x$ and $\gamma(1) = z$. Let $t = \min\{s \in [0, 1] \mid \gamma(s) \in \overline{\mathcal{P}^0}\}$. Then $t > 0$ and $\gamma(t) \in \text{Fr}(\overline{\mathcal{P}^0}) \cap \overline{C_f^v}$.

Let $F^v \in \mathcal{F}^0$ be such that $\gamma(t) \in \mathcal{F}_{F^v}$. By Lemma 2.24, $F^v \subset \overline{C_f^v}$. Let $J' = \{i \in I \mid \varpi_i \in \overline{F^v}\}$. Then by Proposition 2.6, we have $F^v = \bigoplus_{j \in J'} \mathbb{R}_{>0} \varpi_j$ and by Proposition 2.20, we have $\mathcal{F}_{F^v} = \bigwedge_{j \in J'} \mathcal{F}_j = \text{Int}_r(\bigcap_{j \in J'} \overline{\mathcal{F}_j})$. By Lemma 2.28, we have $\gamma(t) \notin \overline{\mathcal{F}_j}$ for all $j \in I \setminus J$ and thus $J' \subset J$.

By Corollary 2.21, the set of facets $\mathcal{F}_{F_1^v}$ such that $\gamma(t) \in \overline{\mathcal{F}_{F_1^v}}$ is

$$\{\mathcal{F}_{F_1^v} \mid F_1^v \subset \overline{F^v}\} = \{\mathcal{F}_{\mathbb{R}_{>0} \varpi_j} \mid j \in J'\} = \{\mathcal{F}_j \mid j \in J'\}.$$

Therefore by Lemma 2.1, there exists $r > 0$ such that $B(\gamma(t), r) \cap \overline{\mathcal{P}^0} = B(\gamma(t), r) \cap \bigcap_{j \in J'} \mathcal{D}_j$. By assumption, $z \in \bigcap_{j \in J'} \mathcal{D}_j$ and thus $\gamma(t') \in \bigcap_{j \in J'} \mathcal{D}_j$, for all $t' \in [0, 1]$. This proves that for $t' \in [0, t]$ close enough to t , $\gamma(t') \in \overline{\mathcal{P}^0}$, which contradicts our assumption. \square

Lemma 2.30. *Let $J \subset I$. Let $z \in \overline{C_f^v} \cap \bigwedge_{j \in J} \mathcal{F}_j$. Let $\overline{\mathcal{P}} = \overline{\mathcal{P}_J^0(z)} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$. Let $\pi_{\overline{\mathcal{P}}} : z + V_J \rightarrow \overline{\mathcal{P}}$ be the orthogonal projection on $\overline{\mathcal{P}}$. Then for all $F_1^v \in \mathcal{F}^0$ such that $F_1^v \subset \overline{C_f^v}$, we have $\pi_{\overline{\mathcal{P}}}((z + V_J) \cap F_1^v) \subset F_1^v$.*

Proof. Let $a \in \text{Fr}(\overline{\mathcal{P}})$, where we regard $\overline{\mathcal{P}^0}$ as a subset of $z + V_J$. In the notation of Eq. (2.9), let us write:

$$J_1 = \{j \in J \mid a \in \overline{\mathcal{F}_j}\}.$$

We first assume $a \notin \overline{C_f^v}$ and we prove $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \cap \overline{C_f^v} = \emptyset$. We have $]a, z] \cap \overline{C_f^v} = [b, z]$, for some $b \in \overline{C_f^v}$. If for all $i \in I$ such that $\alpha_i(a) < 0$, we had $\alpha_i(b) > 0$, then we would have $b - \epsilon(b - a) \in \overline{C_f^v}$ for $\epsilon \in \mathbb{R}_{>0}$ small enough. Therefore there exists $i \in I$ such that $\alpha_i(a) < 0$ and $\alpha_i(b) = 0$. We have $b \in \overline{C_f^v} \cap \overline{\mathcal{P}}$ and thus $b \in \overline{\mathcal{P}^0}$, by Lemma 2.29. Let $j \in J_1$. Then $b \in \overline{\mathcal{F}_j} \subset H_j$, since $z \in \bigwedge_{j \in J} \mathcal{F}_j$. Therefore $b \in H_j \cap \overline{\mathcal{P}^0} = \overline{\mathcal{F}_j} = \overline{\mathcal{F}_{\mathbb{R}_{>0} \varpi_j}}$. Using Lemma 2.23, we deduce $\alpha_j(b) > 0$ for all $j \in J_1$. In particular, $i \notin J_1$. By Eq. (2.9), $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \sum_{j \in J_1} \mathbb{R}_{\geq 0} \varpi_j$ and thus $\alpha_i(\pi_{\overline{\mathcal{P}}}^{-1}(\{a\})) = \{\alpha_i(a)\} \subset \mathbb{R}_{< 0}$. Consequently $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \cap \overline{C_f^v} = \emptyset$. Therefore

$$\pi_{\overline{\mathcal{P}}}(\overline{C_f^v} \cap (z + V_J)) \subset \overline{C_f^v}.$$

Let now $a \in \text{Fr}(\overline{\mathcal{P}}) \cap \overline{C_f^v}$. Then by Lemma 2.29, $\{j \in I \mid a \in \overline{\mathcal{F}_j}\} \supset J_1$. Therefore by Eq. (2.9) and Corollary 2.22, we have $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) = a + \sum_{j \in J_1} \mathbb{R}_{\geq 0} \varpi_j \subset \pi_{\overline{\mathcal{P}^0}}^{-1}(\{a\})$. Using Lemma 2.24, we deduce $\pi_{\overline{\mathcal{P}}}^{-1}(\{a\}) \subset F^v$, if F^v is the face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$ containing a . Consequently, if $x \in \overline{C_f^v} \cap (z + V_J)$, then x and $\pi_{\overline{\mathcal{P}}}(x)$ belong to the same face of $(\mathbb{A}, \mathcal{H}_{\Psi}^0)$. \square

Lemma 2.31. *Let $z \in \mathbb{A}$ and $J \subset I$. Set $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$. Write $z = \sum_{i \in I} z_i \varpi_i$, with $(z_i) \in \mathbb{R}^I$. Then:*

$$(z + V_J) \cap \overline{C_f^v} = \begin{cases} \emptyset & \text{if } \exists i \in I \setminus J \mid z_i < 0 \\ \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j & \text{otherwise} \end{cases}$$

If $z_i \geq 0$ for all $i \in I \setminus J$, set $F^v(z, J) = \sum_{i \in I \setminus J | z_i > 0} \mathbb{R}_{>0} \varpi_i + \sum_{j \in J} \mathbb{R}_{>0} \varpi_j$. Then:

$$\begin{aligned} \overline{F^v(z, J)} &\supset (z + V_J) \cap \overline{C_f^v}, \\ (z + V_J) \cap \overline{F^v(z, J)} &= \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j = (z + V_J) \cap \overline{C_f^v}, \\ (z + V_J) \cap F^v(z, J) &= \sum_{i \in I \setminus J} z_i \varpi_i + \sum_{j \in J} \mathbb{R}_{>0} \varpi_j. \end{aligned}$$

Lemma 2.32. *Let $J \subset I$. Let $z \in \overline{C_f^v} \cap \bigwedge_{j \in J} \mathcal{F}_j$. Then for all $y \in (z + V_J) \cap \overline{C_f^v}$, we have $\pi(y) \in (z + V_J)$.*

Proof. Set $\overline{\mathcal{P}} = (z + V_J) \cap \bigcap_{j \in J} \mathcal{D}_j$. Let $F^v = F^v(z, J)$, with the notation of Lemma 2.31. Let $x \in F^v \cap (z + V_J)$. By Lemma 2.30, we have $\pi_{\overline{\mathcal{P}}}(x) \in F^v$. By Lemma 2.31, we can choose an open neighbourhood U of $\pi_{\overline{\mathcal{P}}}(x)$ in $z + V_J$ such that $U \subset F^v$. Then for all $y \in U \cap \overline{\mathcal{P}}$, $d(x, y) \geq d(x, \pi_{\overline{\mathcal{P}}}(x))$. Let $V^J = \bigoplus_{i \in I \setminus J} \mathbb{R} \alpha_i^\vee$. Then V^J is orthogonal to V_J by Lemma 2.14 and $U + V^J$ is an open subset of \mathbb{A} .

Let $y \in (U + V^J) \cap \overline{\mathcal{P}^0}$. Write $y = u + v$, with $u \in U$ and $v \in V^J$. Then $\langle u - \mathbf{b} \mid \varpi_j \rangle = \langle u + v - \mathbf{b} \mid \varpi_j \rangle = \langle y - \mathbf{b} \mid \varpi_j \rangle \leq 0$, for all $j \in J$ and thus $u \in U \cap \overline{\mathcal{P}}$. Moreover,

$$d(x, y)^2 = |y - x|^2 = |u + v - x|^2 = |u - x|^2 + |v|^2 \geq |u - x|^2 \geq d(x, \pi_{\overline{\mathcal{P}}}(x))^2.$$

Therefore the restriction to $\overline{\mathcal{P}^0}$ of $y \mapsto d(x, y)$ admits a local minimum at $\pi_{\overline{\mathcal{P}}}(x)$. As this map is convex (by [HUL12, B 1.3 (c)] for example), this minimum is global and thus $\pi_{\overline{\mathcal{P}}}(x) = \pi_{\overline{\mathcal{P}^0}}(x)$. Thus we proved:

$$\forall x \in F^v \cap (z + V_J), \pi(x) \in (z + V_J).$$

Let now $x \in \overline{C_f^v} \cap (z + V_J)$. Then, by Lemma 2.31, there exists $(x_n) \in ((z + V_J) \cap F^v)^\mathbb{N}$ such that $x_n \rightarrow x$. Then $\pi(x_n) \rightarrow \pi(x)$ (by [HUL12, A. (3.1.6)]). As $\pi(x_n) \in z + V_J$ for all $n \in \mathbb{N}$, we deduce $\pi(x) \in z + V_J$, which completes the proof of the lemma. \square

Proposition 2.33. *Let F^v be a face of $(\mathbb{A}, \mathcal{H}_\Psi^0)$ and $z \in \mathcal{F}_{F^v}$. Let F_1^v be a face dominating z . In particular, F_1^v dominates F^v by Proposition 2.20. Let π be the orthogonal projection on $\overline{\mathcal{P}^0}$. Then:*

1. $\pi((z + \text{vect}(F^v)) \cap \overline{F_1^v}) \subset (z + \text{vect}(F^v)) \cap \overline{F_1^v}$.
2. Let $y \in (z + \text{vect}(F^v)) \cap \overline{F_1^v}$ and let $G^v \in \mathcal{F}^0$ be such that $\pi(y) \in \mathcal{F}_{G^v}$. Then $G^v \subset \overline{F^v}$.

Proof. 1) Up to changing the choice of the fundamental chamber C_f^v , we may assume $F_1^v \subset \overline{C_f^v}$. Let $J = \{i \in I \mid \varpi_i \in \overline{F^v}\}$. Then by Proposition 2.6, $F^v = \bigoplus_{j \in J} \mathbb{R}_{>0} \varpi_j$ and by Proposition 2.20, $z \in \bigwedge_{j \in J} \mathcal{F}_j$. We have $V_J = \bigoplus_{j \in J} \mathbb{R} \varpi_j$ and the result follows from Lemma 2.32 and Lemma 2.24.

2) Let $y \in (z + \text{vect}(F^v)) \cap \overline{F_1^v}$. Let $G^v \in \mathcal{F}^0$ be such that $\pi(y) \in \mathcal{F}_{G^v}$. By Lemma 2.24, $G^v \subset \overline{F_1^v}$. In particular, $G^v \subset \overline{C_f^v}$, and hence, by Proposition 2.6, $G^v = \sum_{j \in I | \varpi_j \in \overline{G^v}} \mathbb{R}_{>0} \varpi_j$. Let now $i \in I \setminus J$. By 1) and Lemma 2.28, $\pi(y) \notin \overline{\mathcal{F}_{\mathbb{R}_{>0} \varpi_i}}$. Using Corollary 2.21 we deduce that $\mathbb{R}_{>0} \varpi_i \not\subset \overline{G^v}$. This being true for all $i \in I \setminus J$, we deduce that $G^v \subset \sum_{j \in J} \mathbb{R}_{\geq 0} \varpi_j = \overline{F^v}$. \square

3 Tessellation of \mathbb{A} by the affine weight polytopes

Let \mathbb{A} be a finite dimensional vector space and let Φ be a finite root system in \mathbb{A} . In this section, we study polytopes associated with the affine poly-simplicial complex defined from Φ .

In subsection 3.1, we recall and prove general facts on the affine poly-simplicial complex associated with Φ .

In subsection 3.2, we define and study the faces of the weight polytopes.

In subsection 3.3, we prove that the weight polytopes tessellate \mathbb{A} .

3.1 Affine apartment

The main reference for this section is [Bou81, V]. Let Φ be a (finite) root system in \mathbb{A} . Let $\xi : \mathbb{A} \rightarrow \mathbb{R}$ be the constant map equal to 1 and $\Phi^{\text{aff}} = \Phi + \mathbb{Z}\xi$. Then by [Mac72, (2.1) Proposition], Φ^{aff} is an **affine root system** in the sense of [Mac72, 2]. Let $\mathcal{H} = \mathcal{H}_\Phi = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ which equals $\{\underline{\alpha}^{-1}(\{0\}) \mid \underline{\alpha} \in \Phi^{\text{aff}}\}$. This is a locally finite hyperplane arrangement of \mathbb{A} , which means that every bounded subset E of \mathbb{A} meets finitely many hyperplanes. An element of \mathcal{H} is called a **wall** or an **affine wall**. For $x, y \in \mathbb{A}$, we write $x \sim_{\mathcal{H}} y$ if for every $H \in \mathcal{H}$, either $(x, y \in H)$ or $(x$ and y are strictly on the same side of $H)$. This is an equivalence relation on \mathbb{A} . Its classes are called the **faces** of $(\mathbb{A}, \mathcal{H})$ or simply the faces of \mathbb{A} . A **vertex** of \mathbb{A} is an element x of \mathbb{A} such that $\{x\}$ is a face of $(\mathbb{A}, \mathcal{H})$. If F is a face of $(\mathbb{A}, \mathcal{H})$, then:

$$\forall \alpha \in \Phi, \exists n \in \mathbb{Z} \mid \alpha(F) \subset]n, n + 1[\text{ or } \alpha(F) = \{n\}. \quad (3.1)$$

An **alcove** (or **open alcove**) of $(\mathbb{A}, \mathcal{H})$ is a connected component of $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$. The alcoves are actually the faces which are not contained in an element of \mathcal{H} . A **closed alcove** is a set of the form \overline{C} , for some alcove C of \mathbb{A} . If \overline{C} is a closed alcove of \mathbb{A} , its set of vertices, denoted $\text{vert}(\overline{C})$, is the set of vertices of \mathbb{A} which belong to \overline{C} .

We denote by $\mathcal{F} = \mathcal{F}(\mathcal{H})$ the set of faces of $(\mathbb{A}, \mathcal{H})$. If $F \in \mathcal{F}$, we denote by $\mathcal{F}_F = \mathcal{F}_F(\mathcal{H})$ the set of faces dominating F (i.e., the set of faces whose closure contains F). When $F = \{\lambda\}$, for some vertex λ , we write $\mathcal{F}_\lambda(\mathcal{H})$ instead of $\mathcal{F}_{\{\lambda\}}(\mathcal{H})$. We denote by $\text{Alc}(\mathcal{H})$ the set of alcoves of $(\mathbb{A}, \mathcal{H})$ and if $F \in \mathcal{F}(\mathcal{H})$, we denote by $\text{Alc}_F(\mathcal{H})$ the set of alcoves of $(\mathbb{A}, \mathcal{H})$ dominating F . The **affine poly-simplicial complex associated with Φ** is $\mathcal{F}(\mathcal{H})$. When Φ is irreducible, it is a simplicial complex.

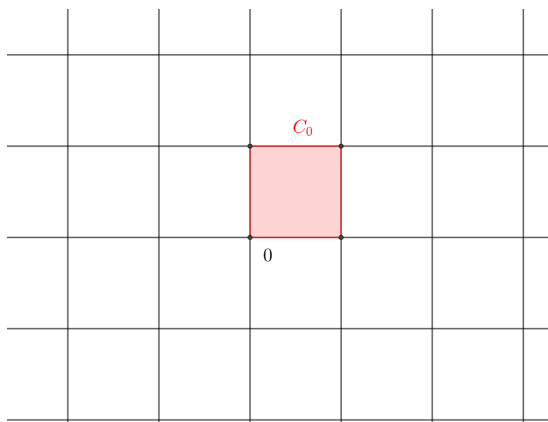


Figure 3.1: Affine apartment of type $A_1 \times A_1$. The alcoves are the squares.

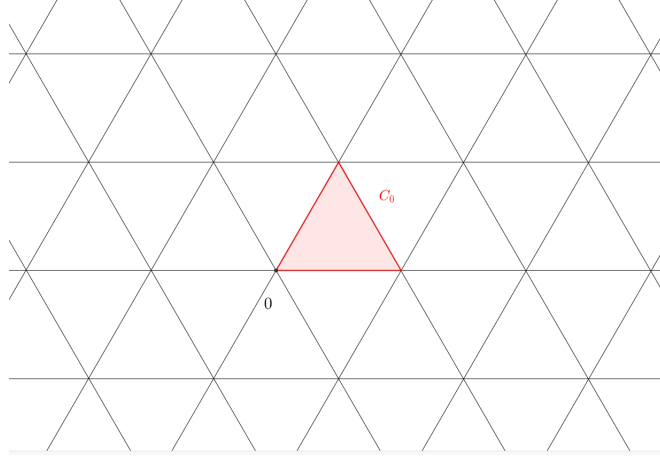


Figure 3.2: Affine apartment of type A_2 . The alcoves are the equilateral triangles.

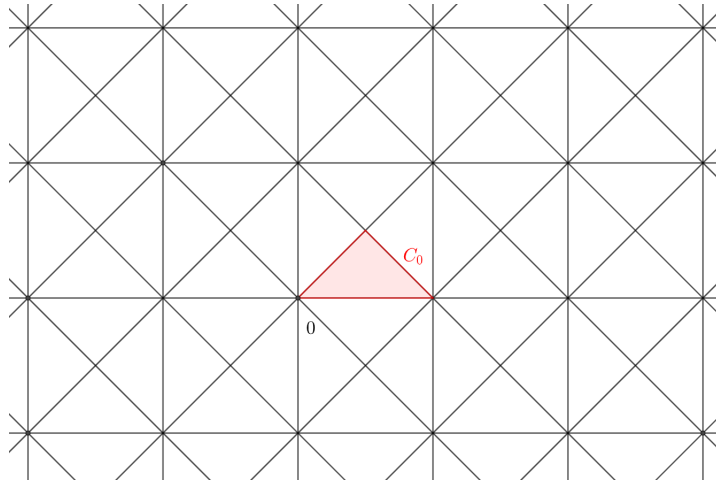


Figure 3.3: Affine apartment of type B_2 . The alcoves are the right triangles.

3.1.1 Enclosure and closed faces of $(\mathbb{A}, \mathcal{H})$

Definition 3.1. A *half-apartment* of \mathbb{A} is a closed half-space of \mathbb{A} delimited by an element of \mathcal{H} . If $\alpha \in \Phi$ and $k \in \mathbb{Z}$, one sets $D_{\alpha,k} = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$

Let $\Omega \subset \mathbb{A}$. Its **enclosure** $\text{cl}(\Omega)$ is the intersection of all the half-apartments containing Ω .

If Ω is a subset of \mathbb{A} and $\lambda \in \mathbb{A}$, then the **germ of Ω at λ** is the set $\text{germ}_\lambda(\Omega)$ of subsets of \mathbb{A} of the form $\Omega \cap E$ such that E is an open neighbourhood of λ . A subset Ω' of \mathbb{A} is said to **contain** the germ $\text{germ}_\lambda(\Omega)$ if it contains $\Omega \cap E$ for some open neighbourhood E of λ . The enclosure of $\text{germ}_\lambda(\Omega)$ is the intersection of the enclosures of the $\Omega \cap E$ when E describes all the open neighbourhoods of λ .

A **closed face of $(\mathbb{A}, \mathcal{H})$** is a set of the form \overline{F} , for some $F \in \mathcal{F}(\mathcal{H})$.

Lemma 3.2. 1. Let $F \in \mathcal{F}$. For $\underline{\alpha} \in \Phi^{\text{aff}}$, set $E_{\underline{\alpha}} = \{0\}$ if $\underline{\alpha}(F) = \{0\}$, $E_{\underline{\alpha}} = \mathbb{R}_{>0}$ if $\underline{\alpha}(F) > 0$ and $E_{\underline{\alpha}} = \mathbb{R}_{<0}$ if $\underline{\alpha}(F) < 0$. Then $F = \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(E_{\underline{\alpha}})$ and $\overline{F} = \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(\overline{E_{\underline{\alpha}}})$. In particular, \overline{F} is enclosed.

2. Let $F \in \mathcal{F}$ and $x \in F$. Then $\text{cl}(x) = \overline{F}$.

Proof. 1) The first equality follows from the definition of faces. As the elements of Φ^{aff} are continuous maps from \mathbb{A} to \mathbb{R} , we have $\overline{F} \subset \bigcap_{\underline{\alpha} \in \Phi^{\text{aff}}} \underline{\alpha}^{-1}(\overline{E_{\underline{\alpha}}})$. Let $x \in F$ and

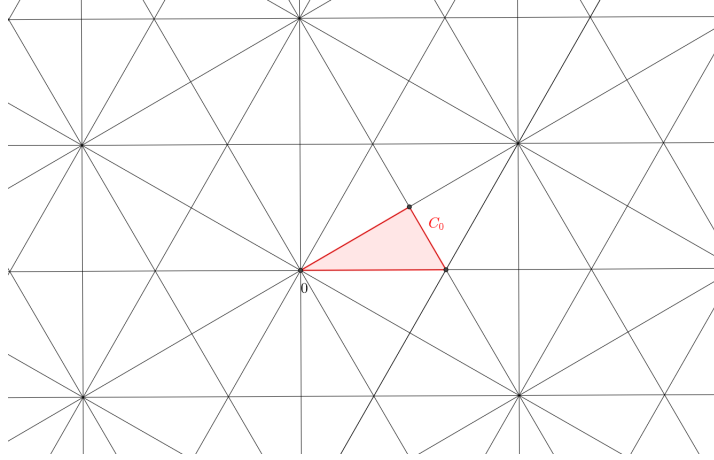


Figure 3.4: Affine apartment of type G_2 . The alcoves are the right triangles.

$y \in \bigcap_{\alpha \in \Phi^{\text{aff}}} \alpha^{-1}(\overline{E_\alpha})$. Then for $z \in]x, y[$ and $\alpha \in \Phi^{\text{aff}}$, we have $\alpha(z) \in \overline{E_\alpha} + E_\alpha = E_\alpha$ and hence $]x, y[\subset F$, which proves that $y \in \overline{F}$ and concludes the proof of the second equality. The fact that \overline{F} is enclosed then follows from the fact that if $\alpha \in \Phi^{\text{aff}}$, then $\alpha^{-1}(\{0\}) = \alpha^{-1}(\mathbb{R}_{\geq 0}) \cap \alpha^{-1}(\mathbb{R}_{\leq 0})$.

We now prove 2). We have $\text{cl}(x) \subset \text{cl}(F) = \overline{F}$. By definition of the faces, every half-apartment containing x contains F and thus it contains \overline{F} . Therefore $\text{cl}(x) \supset \overline{F}$, which completes the proof of the lemma. \square

Proposition 3.3. *Let E be a non-empty subset of \mathbb{A} . Then E is a closed face of $(\mathbb{A}, \mathcal{H})$ if and only if the following two conditions are satisfied:*

1. E is enclosed,
2. for every $\alpha \in \Phi^{\text{aff}}$, we have either $\alpha(E) \subset \mathbb{R}_{\geq 0}$ or $\alpha(E) \subset \mathbb{R}_{\leq 0}$.

Proof. Let F be a face of $\mathcal{F}(\mathcal{H})$. Then by Lemma 3.2, \overline{F} is enclosed. By definition of the faces, if $\alpha \in \Phi^{\text{aff}}$, we have $\alpha(F) \subset \mathbb{R}_{> 0}$, $\alpha(F) = \{0\}$ or $\alpha(F) \subset \mathbb{R}_{< 0}$. As $\alpha(\overline{F}) \subset \overline{\alpha(F)}$ we deduce that \overline{F} satisfies 1. and 2.

Let now E be a non-empty subset satisfying 1. and 2. Then E is convex and closed. Let $F = \text{Int}_r(E)$. Then we have $E = \overline{F}$. Let S be the support of E and $x, y \in F$. Let $\alpha \in \Phi^{\text{aff}}$. If α is constant on S , then $\alpha(x) = \alpha(y)$. Assume now that α is non-constant on S . Let U_x, U_y be open subsets of S containing x and y respectively. Then by assumption $\alpha(U_x)\alpha(U_y) \geq 0$ and hence $\alpha(x)\alpha(y) > 0$. Consequently, $x \sim y$, where \sim is the equivalence relation that we used to define the faces of $(\mathbb{A}, \mathcal{H})$. Consequently there exists $F_1 \in \mathcal{F}$ such that $F \subset F_1$.

Using 1., we write $\overline{F} = \bigcap_{\alpha \in \mathcal{E}} \alpha^{-1}(\mathbb{R}_{\geq 0})$, where $\mathcal{E} \subset \Phi^{\text{aff}}$. Let $x \in F$. Let $\alpha \in \mathcal{E}$. Then $\alpha(x) \geq 0$ and as α has constant sign on F_1 , we have $\alpha(F_1) \geq 0$. Consequently $F_1 \subset \overline{F}$ and thus $\overline{F_1} \subset \overline{F}$. As $F \subset F_1$, we deduce $\overline{F} = \overline{F_1} = E$, and E is a closed face of $(\mathbb{A}, \mathcal{H})$. \square

Corollary 3.4. *Let $\overline{F_1}, \overline{F_2}$ be two closed faces of $(\mathbb{A}, \mathcal{H})$. Then either $\overline{F_1} \cap \overline{F_2}$ is empty or $\overline{F_1} \cap \overline{F_2}$ is a closed face of $(\mathbb{A}, \mathcal{H})$.*

Let $C \in \text{Alc}(\mathcal{H})$. A **wall of C** is a hyperplane H such that $\text{supp}(\overline{C} \cap H) = H$. A **panel of C** is an element of \mathcal{F} contained in \overline{C} and whose support is a wall of C .

One sets

$$\Sigma_C^{\text{aff}} := \{\alpha \in \Phi^{\text{aff}} \mid \alpha(C) > 0, \alpha^{-1}(\{0\}) \text{ is a wall of } C\}.$$

Assume that Φ is irreducible. Then Φ^{aff} is an irreducible affine root system by [Mac72, (4.7)] and Σ_C^{aff} is a basis of Φ^{aff} in the sense of [Mac72, 4.6]. In particular,

$$\Phi^{\text{aff}} \subset \left(\sum_{\alpha \in \Sigma_C^{\text{aff}}} \mathbb{Z}_{\geq 0} \alpha \cup \sum_{\alpha \in \Sigma_C^{\text{aff}}} (-\mathbb{Z}_{\geq 0}) \alpha \right), \quad (3.2)$$

by [Mac72, (4.6) Proposition].

Lemma 3.5. 1. Let $F \in \mathcal{F}(\mathcal{H})$. Then there exists $C \in \text{Alc}(\mathcal{H})$ such that $\overline{F} \subset \overline{C}$. Let $\mathcal{M}_F = \{\alpha \in \Sigma_C^{\text{aff}} \mid \alpha(F) > 0\}$. Then

$$\overline{F} = \overline{C} \cap \bigcap_{\alpha \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \alpha^{-1}(\{0\}) = \bigcap_{\alpha \in \mathcal{M}_F} \alpha^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\alpha \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \alpha^{-1}(\{0\}). \quad (3.3)$$

2. Let $C \in \text{Alc}(\mathcal{H})$. Let $\alpha \in \Sigma_C^{\text{aff}}$. Then $\overline{C} \cap \alpha^{-1}(\{0\})$ is a closed panel of C and all the closed panels of C are of this form.

3. Let $C \in \text{Alc}(\mathcal{H})$. Then every closed face of $(\mathbb{A}, \mathcal{H})$ strictly contained in \overline{C} can be written as an intersection of closed panels of \overline{C} .

Proof. 1) Let $x \in F$. Then by [Bou81, V §1.4 Proposition 6], there exists $C \in \text{Alc}(\mathcal{H})$ such that $x \in \overline{C}$. But then by Lemma 3.2, $\text{cl}(x) = \overline{F} \subset \overline{C} = \text{cl}(C)$. Assume for the moment that Φ is irreducible. With the same notation as in Lemma 3.2, we have

$$\overline{F} = \bigcap_{\beta \in \Phi^{\text{aff}}} \beta^{-1}(E_{\beta}). \quad (3.4)$$

Let $\beta \in \Phi^{\text{aff}}$. Then by Eq. (3.2), we can write $\beta = \epsilon \sum_{\alpha \in \Sigma_C^{\text{aff}}} n_{\alpha} \alpha$, with $\epsilon \in \{-, +\}$ and $(n_{\alpha}) \in (\mathbb{Z}_{\geq 0})^{\Sigma_C^{\text{aff}}}$. We have $E_{\beta} = \{0\}$ if and only if $n_{\alpha} = 0$, for all $\alpha \in \mathcal{M}_F$. Otherwise $E_{\beta} = \epsilon \mathbb{R}_{\geq 0}$. In particular $\beta^{-1}(E_{\beta}) \supset \bigcap_{\alpha \in \Sigma_C^{\text{aff}}} \alpha^{-1}(E_{\alpha})$. Therefore Eq. (3.4) becomes

$$\overline{F} = \bigcap_{\alpha \in \Sigma_C^{\text{aff}}} \alpha^{-1}(E_{\alpha}) = \bigcap_{\alpha \in \mathcal{M}_F} \alpha^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\alpha \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} \alpha^{-1}(\{0\}). \quad (3.5)$$

When $F = C$, we have $\mathcal{M}_C = \Sigma_C^{\text{aff}}$, which proves Eq. (3.3), when Φ is irreducible.

We now assume that Φ is reducible. Using the notation of Eq. (2.4), with Φ instead of Ψ , we write $\mathbb{A} = \bigoplus_{i=1}^{\ell} \mathbb{A}_i$ and $\Phi = \bigsqcup_{i=1}^{\ell} \Phi_i$, where for each $i \in \llbracket 1, \ell \rrbracket$, Φ_i is an irreducible root system in \mathbb{A}_i . If $i \in \llbracket 1, \ell \rrbracket$ and $\alpha \in \Phi_i^{\text{aff}}$, $\alpha : \mathbb{A}_i \rightarrow \mathbb{R}$, we denote by $\tilde{\alpha}$ the extension of α to \mathbb{A} : $\tilde{\alpha}((x_1, \dots, x_i, \dots, x_{\ell})) = \alpha(x_i)$, for $(x_1, \dots, x_{\ell}) \in \mathbb{A}$. Write $C = C_1 \times \dots \times C_{\ell}$ and $F = F_1 \times \dots \times F_{\ell}$. Then we have $\Sigma_C^{\text{aff}} = \bigsqcup_{i=1}^{\ell} \{\tilde{\alpha} \mid \alpha \in \Sigma_{C_i}^{\text{aff}}\}$ and $\mathcal{M}_F = \bigsqcup_{i=1}^{\ell} \{\tilde{\alpha} \mid \alpha \in \mathcal{M}_{F_i}\}$. For $i \in \llbracket 1, \ell \rrbracket$, we have:

$$\bigcap_{\alpha \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_i}} \tilde{\alpha}^{-1}(\{0\}) = \mathbb{A}_1 \times \dots \times \mathbb{A}_{i-1} \times \bigcap_{\alpha \in \Sigma_{C_i}^{\text{aff}} \setminus \mathcal{M}_{F_i}} \alpha^{-1}(\{0\}) \times \mathbb{A}_{i+1} \times \dots \times \mathbb{A}_{\ell}.$$

We thus deduce Eq. (3.3) from the irreducible case. This proves statement 1.

2) By Proposition 3.3, if $\alpha \in \Sigma_C^{\text{aff}}$, then $\alpha^{-1}(\{0\}) \cap \overline{C} = \alpha^{-1}(\mathbb{R}_{\geq 0}) \cap \alpha^{-1}(\mathbb{R}_{\leq 0}) \cap \overline{C}$ is a closed face of $(\mathbb{A}, \mathcal{H})$. It is a panel by definition of Σ_C^{aff} . By 1. and for dimension reasons, all the closed panels are of this form, which proves statement 2.

3) Let $F \in \mathcal{F}$ be such that $F \subset \overline{C}$. With the same notation as above, we have $\overline{F} = \bigcap_{\alpha \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} (\overline{C} \cap \alpha^{-1}(\{0\}))$. □

Let $C \in \text{Alc}(\mathcal{H})$. Then by Lemma 3.5 (applied with $F = C$, $\mathcal{M}_C = \Sigma_C^{\text{aff}}$), \overline{C} is a finite intersection of half-apartments. Therefore it is a polytope. Moreover the closed panels of C are exactly the facets of \overline{C} , when we regard it as a polytope, in the sense of [BG09, Definition 1.3]. By [BG09, Theorem 1.10 a),b)], Lemma 3.5 and Corollary 3.4, we deduce that the faces of \overline{C} regarded as a polytope are exactly the closed faces of $(\mathbb{A}, \mathcal{H})$ which are contained in \overline{C} (except the empty set, that we excluded from the set of faces).

Lemma 3.6. *Let C be an alcove of $(\mathbb{A}, \mathcal{H})$. Let $E \subset \overline{C}$ be non-empty. Then $\{F_1 \in \mathcal{F} \mid E \subset \overline{F_1}\}$ admits a minimal element F (for the dominance order). We have $F \subset \overline{C}$, $\text{cl}(E) = \overline{F}$ and if moreover, E is convex, then E meets F .*

Proof. By Proposition 3.3, $\text{cl}(E)$ is a closed face of $(\mathbb{A}, \mathcal{H})$: there exists $F \in \mathcal{F}$ such that $\text{cl}(E) = \overline{F}$. Moreover if $F_1 \in \mathcal{F}$ contains E , then $\overline{F_1} \supset \text{cl}(E) = \overline{F}$.

Assume now that E is convex. Let $\mathcal{E} = \{F_1 \in \mathcal{F} \mid \overline{F_1} \subset \overline{F}, F_1 \cap E \neq \emptyset\}$. Let $F_1, F_2 \in \mathcal{E}$, $x_1 \in F_1$ and $x_2 \in F_2$. Let $x_3 = \frac{1}{2}(x_1 + x_2) \in E \subset \overline{F}$. Let $F_3 \in \mathcal{F}$ be such that $x_3 \in F_3$. Then $F_3 \in \mathcal{E}$ since $x_3 \in \overline{F}$.

Moreover, $]x_1, x_2[\cap \overline{F_3} \supset \{x_3\} \neq \emptyset$. Let C be an alcove dominating F . By Eq. (3.3), we have $\overline{F_3} = \bigcap_{\underline{\alpha} \in \mathcal{M}_{F_3}} \underline{\alpha}^{-1}(\mathbb{R}_{\geq 0}) \cap \bigcap_{\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_3}} \underline{\alpha}^{-1}(\{0\})$, with the notation of Lemma 3.5. Then if $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_3}$, we have $\underline{\alpha}(x_3) = 0$ and $\underline{\alpha}(x_1), \underline{\alpha}(x_2) \geq 0$ so that $\underline{\alpha}(x_1) = \underline{\alpha}(x_2) = 0$. This proves that $]x_1, x_2[\subset \overline{F_3}$. Moreover, $\overline{F_3}$ is enclosed and thus by Lemma 3.2, $\overline{F_3} \supset \overline{F_1} \cup \overline{F_2}$. Hence \mathcal{E} admits a maximal element F_{\max} for the dominance order.

Let $x \in E$ and $F_x \in \mathcal{F}(\mathcal{H})$ be such that $x \in F_x$. Then F_x belongs to \mathcal{E} and thus $x \in F_x \subset \overline{F_{\max}}$. Hence $E \subset \overline{F_{\max}}$. By definition of \overline{F} , this implies that $\overline{F} \subset \overline{F_{\max}}$ and hence $\overline{F} = \overline{F_{\max}}$. Therefore F meets E . \square

3.1.2 Properties of the affine Weyl group

There exists a scalar product $\langle \cdot \mid \cdot \rangle$ on \mathbb{A} which is invariant under the vectorial Weyl group W_{Φ}^v of Φ . We equip \mathbb{A} with such a scalar product. If $H \in \mathcal{H}$, we denote by s_H the orthogonal reflection of \mathbb{A} fixing H . Let $C \in \text{Alc}(\mathcal{H})$. A **wall** of C is an element of \mathcal{H} such that $\text{supp}(\overline{C} \cap H) = H$. The **affine Weyl group of $(\mathbb{A}, \mathcal{H})$** is the group W^{aff} of affine maps on \mathbb{A} generated by s_H , for $H \in \mathcal{H}$. If $F \in \mathcal{F}(\mathcal{H})$, we denote by W_F^{aff} the fixator of F in W^{aff} . Gathering the results of [Bou81, V §3.2, 3.3], we have the following theorem. We write it for the affine Weyl group and the faces of $(\mathbb{A}, \mathcal{H})$, but it is also true for the vectorial Weyl group and the vectorial faces, by the same reference. We refer to [BB05] for the definition of Coxeter groups.

Theorem 3.7. *1. Let $C \in \text{Alc}(\mathcal{H})$ and S_C be the set of orthogonal reflections with respect to the walls of C . Then (W^{aff}, S_C) is a Coxeter group.*

2. Let $C \in \text{Alc}(\mathcal{H})$. Then \overline{C} is a fundamental domain for the action of W^{aff} on \mathbb{A} and for every alcove $C' \in \text{Alc}(\mathcal{H})$, there exists a unique $w \in W^{\text{aff}}$ such that $w.C = C'$.

3. Let $F \in \mathcal{F}$ and $C \in \text{Alc}_F(\mathcal{H})$. Then W_F^{aff} is the parabolic subgroup of (W^{aff}, S_C) generated by the reflections with respect to the walls of C containing F . Moreover, if $w \in W^{\text{aff}}$, the following statements are equivalent:

- a. $w \in W_F^{\text{aff}}$,*
- b. w pointwise fixes \overline{F} ,*
- c. $w.F$ meets F ,*
- d. w fixes a point of F .*

The lemma below also has a vectorial version.

Lemma 3.8. *Let $F \in \mathcal{F}(\mathcal{H})$ and $C \in \text{Alc}_F(\mathcal{H})$. Then for every $F' \in \mathcal{F}(\mathcal{H})$ such that F' dominates F , there exists $w \in W_F^{\text{aff}}$ such that $w.F' \subset \overline{C}$. In particular, W_F^{aff} acts transitively on $\text{Alc}_F(\mathcal{H})$.*

Equivalently, if E is the union of the faces dominating F and if $E' = E \cap \overline{C}$, then E' is a fundamental domain for the action of W_F^{aff} on E .

Proof. Let C' be an alcove of $(\mathbb{A}, \mathcal{H})$ dominating F . Write $C' = w.C$, with $w \in W^{\text{aff}}$. Then $w^{-1}.F \subset \overline{C}$, so if $x \in w^{-1}.F \subset \overline{C}$, we have $w.x \in \overline{C}$, which proves that $w.x = x$ by Theorem 3.7. Therefore w fixes $w^{-1}.F$, which proves that w fixes F . Now if F_1 is a face of $(\mathbb{A}, \mathcal{H})$ dominating F and C_1 is an alcove dominating F_1 , we can write $C_1 = w_1.C$, with $w_1 \in W^{\text{aff}}$ fixing F . Then $w_1^{-1}.F_1 \subset \overline{C}$, which proves the lemma. \square

Lemma 3.9. *Let $C \in \text{Alc}(\mathcal{H})$ and $F_1, F_2 \in \mathcal{F}(\mathcal{H})$ be such that $F_1 \cup F_2 \subset \overline{C}$. Then $W_{F_1}^{\text{aff}} \subset W_{F_2}^{\text{aff}}$ (resp. $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$) if and only if $F_2 \subset \overline{F_1}$ (resp. $F_1 = F_2$).*

Proof. By Theorem 3.7, if S_C denotes the set of orthogonal reflections with respect to the walls of C , then (W^{aff}, S_C) is a Coxeter group. Let $i \in \{1, 2\}$. Set $\mathcal{M}_{F_i} = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F_i) > 0\}$. We have $W_{F_i}^{\text{aff}} = \langle s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_i} \rangle$, by Theorem 3.7, where $s_{\underline{\alpha}}$ denotes the reflection of W^{aff} fixing $\underline{\alpha}^{-1}(\{0\})$, for $\underline{\alpha} \in \Phi^{\text{aff}}$. By [BB05], if $W_{F_1}^{\text{aff}} \subset W_{F_2}^{\text{aff}}$ (resp. $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$), then we have $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_1} \subset \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_2}$ (resp. $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_1} = \Sigma_C^{\text{aff}} \setminus \mathcal{M}_{F_2}$). Using Lemma 3.5 we deduce that $\overline{F_2} \subset \overline{F_1}$ (resp. $\overline{F_1} = \overline{F_2}$).

Conversely, as W^{aff} acts by homeomorphisms on \mathbb{A} , we have $W_{F_i}^{\text{aff}} = W_{\overline{F_i}}^{\text{aff}}$ if $i \in \{1, 2\}$ and thus if $F_2 \subset \overline{F_1}$ (resp. $\overline{F_1} = \overline{F_2}$), we have $W_{F_2}^{\text{aff}} \supset W_{F_1}^{\text{aff}}$ (resp. $W_{F_1}^{\text{aff}} = W_{F_2}^{\text{aff}}$). \square

3.1.3 Properties of the faces and link between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}_\lambda^0)$

For $x \in \mathbb{A}$, we set $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) \in \mathbb{Z}\}$.

Lemma 3.10. *Let $x \in \mathbb{A}$. Then $x \in \text{vert}(\mathbb{A})$ if and only if $\bigcap_{\alpha \in \Phi_x} \ker(\alpha) = \{0\}$.*

Proof. Let $E = \bigcap_{\alpha \in \Phi_x} \ker(\alpha)$. Let $y \in E$. Then for $t \in]0, 1[$ small enough, for every $\alpha \in \Phi$, if $\alpha(x) \notin \mathbb{Z}$, then $\alpha(x + ty) \in]\alpha(x), \alpha(x) + 1[$ and thus $x \sim_{\mathcal{H}} x + ty$. Therefore if $\{x\}$ is a face of $(\mathbb{A}, \mathcal{H})$, then $E = \{0\}$. Conversely, assume $E = \{0\}$. Take $y \in \mathbb{A} \setminus \{x\}$. Then there exists $\alpha \in \Phi_x$ such that $\alpha(y) \neq \alpha(x)$. Then $y \notin \alpha^{-1}(\{\alpha(x)\})$ and as $\alpha^{-1}(\{\alpha(x)\}) \in \mathcal{H}$, $y \not\sim_H x$. Thus the class of x is $\{x\}$. \square

Let $\lambda \in \text{vert}(\mathbb{A})$. Then by [Bou81, Chapitre VI, §1 Proposition 4] and Lemma 3.10, Φ_λ is a root system in \mathbb{A}^* , with dual root system $\Phi_\lambda^\vee := \{\alpha^\vee \mid \alpha \in \Phi_\lambda\}$. For $\alpha \in \Phi_\lambda$ and $x \in \mathbb{R}^*$, if $x\alpha \in \Phi_\lambda$, then $x\alpha \in \Phi$ and thus $x \in \{-1, 1\}$, since Φ is reduced. Therefore Φ_λ is a reduced root system in \mathbb{A}^* . If $\alpha \in \Phi$, we denote by $s_\alpha \in W^v$ the associated reflection.

Let $\mathcal{H}_\lambda^0 = \{\alpha^{-1}(\{0\}) \mid \alpha \in \Phi_\lambda\}$. The connected components of $\mathbb{A} \setminus \bigcup_{H \in \mathcal{H}_\lambda^0} H$ are the (vectorial) chambers of $(\mathbb{A}, \Phi_\lambda)$. If $\lambda \in \text{vert}(\mathbb{A})$, then we denote by $\mathcal{F}_\lambda(\mathcal{H})$ the set of faces of $(\mathbb{A}, \mathcal{H})$ dominating λ and we denote by $\mathcal{F}(\mathcal{H}_\lambda^0)$ the set of faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$.

Lemma 3.11. *Let $\lambda \in \text{vert}(\mathbb{A})$ and $F_1, F_2 \in \mathcal{F}_\lambda(\mathcal{H})$. We assume that there exists a neighbourhood Ω of λ such that $F_1 \cap \Omega \subset \overline{F_2}$. Then $F_1 \subset \overline{F_2}$.*

Proof. Let $x \in \Omega \cap F_1$. Let $\alpha \in \Phi$ and $k \in \mathbb{Z}$ be such that $D_{\alpha, k} \supset F_2$. Then $D_{\alpha, k} \supset \overline{F_2} \ni x$ and thus $D_{\alpha, k} \supset \text{cl}(x) = \overline{F_1}$. As $\overline{F_2}$ is enclosed, we deduce that $\overline{F_2} \supset F_1$. \square

Proposition 3.12. *Let $\lambda \in \text{vert}(\mathbb{A})$. Equip $\mathcal{F}_\lambda(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}_\lambda^0)$ with the dominance order. Then the map $\Theta : \mathcal{F}_\lambda(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H}_\lambda^0)$ defined by $\Theta(F) = F^v(F) := \mathbb{R}_{>0}(F - \lambda)$, for $F \in \mathcal{F}_\lambda(\mathcal{H})$ is an isomorphism of partially ordered sets. Its inverse is the map $\Xi : \mathcal{F}(\mathcal{H}_\lambda^0) \rightarrow \mathcal{F}_\lambda(\mathcal{H})$ which associates to an element F^v of $\mathcal{F}(\mathcal{H}_\lambda^0)$ the unique face of $\mathcal{F}(\mathcal{H})$ containing $\text{germ}_\lambda(\lambda + F^v)$. We have $\Xi(F^v) = \text{cl}(\text{germ}_\lambda(\lambda + F^v))$ and $\Xi(F^v) \subset \lambda + F^v$.*

Proof. Set

$$\Omega = \Omega_\lambda := \bigcap_{\alpha \in \Phi_\lambda} \alpha^{-1}([\alpha(\lambda) - 1, \alpha(\lambda) + 1]) \cap \bigcap_{\alpha \in \Phi \setminus \Phi_\lambda} \alpha^{-1}([\alpha(\lambda)], [\alpha(\lambda)]].$$

Then Ω is a convex open set containing λ .

Let $F \in \mathcal{F}_\lambda(\mathcal{H})$ and $x_1, x_2 \in \mathbb{R}_{>0}(F - \lambda)$. Write $x_1 = t_1(f_1 - \lambda)$, $x_2 = t_2(f_2 - \lambda)$, with $t_1, t_2 \in \mathbb{R}_{>0}$ and $f_1, f_2 \in F$. Let $\alpha \in \Phi_\lambda$. Then the sign of $\alpha - \alpha(\lambda)$ (which can take the values “< 0”, “= 0” or “> 0”) is constant on $\{f_1, f_2\}$ and thus $\alpha(x_1)$ and $\alpha(x_2)$ have the same sign. As this is true for every $\alpha \in \Phi_\lambda$, we deduce the existence of $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ such that $\Theta(F) \subset F^v$.

Let $x_1 \in F^v$ and $f \in F$. Set $x_2 = f - \lambda \in \Theta(F) \subset F^v$. Let $t_1 \in \mathbb{R}_{>0}$ be such that $f_1 := x_1/t_1 + \lambda \in \Omega$. Let $\alpha \in \Phi \setminus \Phi_\lambda$. By choice of Ω , we have $\alpha(f_1) \in]\alpha(\lambda), [\alpha(\lambda)[[$. By Eq. (3.1), we have $\alpha(F) \subset]\alpha(\lambda), [\alpha(\lambda)[[$ and thus $\alpha - k$ has constant sign on $\{f_1, f\}$, for every $k \in \mathbb{Z}$. Let $\alpha \in \Phi_\lambda$. We have $\alpha(F^v) \in \{\mathbb{R}_{<0}, \{0\}, \mathbb{R}_{>0}\}$. If $\alpha(F^v) = \{0\}$, then $\alpha(f_1) = \alpha(\lambda)$ and $\alpha(x_2) = 0$, so $\alpha(f) = \alpha(\lambda)$. Therefore $\alpha + k$ has constant sign on $\{f_1, f\}$, for every $k \in \mathbb{Z}$. Assume now $\alpha(F^v) = \mathbb{R}_{>0}$. Then $\alpha(f_1) > \alpha(\lambda)$ and thus $\alpha(f_1) \in]\alpha(\lambda), \alpha(\lambda) + 1[$. We have $\alpha(x_2) > 0$, thus $\alpha(f) > \alpha(\lambda)$. By Eq. (3.1), $\alpha(f) \in]\alpha(\lambda), \alpha(\lambda) + 1[$. Therefore $\alpha + k$ has constant sign on $\{f, f_1\}$, for every $k \in \mathbb{Z}$. With the same reasoning when $\alpha(F^v) = \mathbb{R}_{<0}$, we deduce that f and f_1 belong to the same face and thus $f_1 \in F$. Consequently $x_1 = t_1(f_1 - \lambda) \in \Theta(F)$. Therefore we proved $\Theta(F) = F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ and thus Θ is well-defined.

Let now $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ and $f_1, f_2 \in \Omega \cap (\lambda + F^v)$. Let $\alpha \in \Phi$. Let $x_1 = f_1 - \lambda$, $x_2 = f_2 - \lambda \in F^v$. If $\alpha \notin \Phi_\lambda$, then $\alpha(f_1), \alpha(f_2) \in]\alpha(\lambda), [\alpha(\lambda)[[$. Therefore $\alpha + k$ has constant sign on $\{f_1, f_2\}$, for every $k \in \mathbb{Z}$. Let now $\alpha \in \Phi_\lambda$. If $\alpha(F^v) = \{0\}$, then $\alpha(f_1) = \alpha(f_2) = \alpha(\lambda)$ and thus $\alpha + k$ has constant sign on $\{f_1, f_2\}$ for every $k \in \mathbb{Z}$. If $\alpha(F^v) = \mathbb{R}_{>0}$, then $\alpha(x_1), \alpha(x_2) > 0$ and thus $\alpha(f_1), \alpha(f_2) \in]\alpha(\lambda), \alpha(\lambda) + 1[$, by definition of Ω . With the same reasoning when $\alpha(F^v) = \mathbb{R}_{<0}$, we deduce that f_1 and f_2 belong to the same face F of $\mathcal{F}_\lambda(\mathcal{H})$. Then F is the unique face of $\mathcal{F}_\lambda(\mathcal{H})$ containing $(\lambda + F^v) \cap \Omega \supset \text{germ}_\lambda(\lambda + F^v)$. Consequently, $\Xi(F^v) = F$ is well-defined and we have

$$\Omega \cap (\lambda + F^v) \subset F = \Xi(F^v). \quad (3.6)$$

Moreover, for $x \in F$, we have $\text{cl}(x) = \overline{F}$ and thus $\text{cl}(\text{germ}_\lambda(\lambda + F^v)) = \overline{F}$.

Let $F \in \mathcal{F}_\lambda(\mathcal{H})$ and $\tilde{F} = \Xi \circ \Theta(F)$. Then $\tilde{F} \supset \Omega \cap]\lambda, f[$ for $f \in F$. Therefore $\tilde{F} \cap F$ is non-empty and hence $F = \tilde{F}$, which proves that $\Xi \circ \Theta = \text{Id}$.

Let $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$. Set $\tilde{F}^v = \Theta \circ \Xi(F^v)$. Let $f \in F^v$ be such that $\lambda + f \in \Omega$. Then $\lambda + f \in \Xi(F^v)$ by Eq. (3.6). Therefore $\lambda + f - \lambda = f \in \tilde{F}^v$ and hence $\tilde{F}^v \cap F^v \neq \emptyset$. Consequently $\tilde{F}^v = F^v$ and $\Theta \circ \Xi = \text{Id}$, which proves that Θ is the inverse of Ξ .

Let $F_1, F_2 \in \mathcal{F}_\lambda(\mathcal{H})$ be such that $\overline{F_2} \supset F_1$. Then $\mathbb{R}_{>0}(F_2 - \lambda) = \mathbb{R}_{\geq 0}(\overline{F_2} - \lambda) \supset \mathbb{R}_{>0}(F_1 - \lambda)$ and thus Θ is increasing.

Let $F_1^v, F_2^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ be such that $\overline{F_2^v} \supset F_1^v$. We have

$$\overline{\Xi(F_2^v)} \supset (\lambda + \overline{F_2^v}) \cap \Omega \supset (\lambda + F_1^v) \cap \Omega \supset \Xi(F_1^v) \cap \Omega,$$

by Eq. (3.6). Using Lemma 3.11, we deduce that $\Xi(F_1^v) \subset \overline{\Xi(F_2^v)}$, which proves that Ξ is increasing. Thus Θ is an isomorphism.

Using Eq. (3.6) and the assertion following it, we get the last claim of the proposition. \square

Remark 3.13. *Note that for dimension reasons, the bijection above sends alcoves on chambers.*

3.1.4 Star of a face

In the sequel, we write $W_x^{\text{aff}} = W_{\{x\}}^{\text{aff}}$, for the fixator of $x \in \mathbb{A}$ in W^{aff} .

Lemma 3.14. *Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ and $J \subset C$ be a non-empty set. Let F be the smallest face of C dominating J (this face is well-defined by Lemma 3.6). Then $\bigcap_{x \in J} W_x^{\text{aff}} = W_F^{\text{aff}}$.*

Proof. For every $x \in J$, $W_F^{\text{aff}} = W_{\overline{F}}^{\text{aff}}$ fixes x and thus $W_F^{\text{aff}} \subset \bigcap_{x \in J} W_x^{\text{aff}}$. Let $w \in \bigcap_{x \in J} W_x^{\text{aff}}$. By Lemma 3.6, there exists $y \in \text{conv}(J) \cap F$. Then w fixes y . By Theorem 3.7, $W_y^{\text{aff}} = W_F^{\text{aff}}$, which proves that $w \in W_F^{\text{aff}}$ and completes the proof of the lemma. \square

Lemma 3.15. *Assume that Φ is irreducible. Let F be a face of $(\mathbb{A}, \mathcal{H})$ and let C be an alcove dominating F . Let Σ_C^{aff} be the basis associated with C and set $\mathcal{M}_F = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F) > 0\}$. Let $x \in \mathbb{A}$. Then there exists $w \in W_F^{\text{aff}}$ such that $\underline{\alpha}(w.x) \geq 0$ for every $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$.*

Proof. If F is an alcove, $\Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$ is empty and there is nothing to prove. We now assume that F is not an alcove. Let $\lambda \in \text{vert}(F)$. Let $F^v = \mathbb{R}_{>0}(F - \lambda)$ and $C^v = \mathbb{R}_{>0}(C - \lambda)$. By Proposition 3.12, F^v and C^v are faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$. Let $\Phi_{F^v} = \{\alpha \in \Phi_\lambda \mid \alpha(F^v) = \{0\}\}$ and let $\Phi_{F^v,+} = \{\alpha \in \Phi_{F^v} \mid \alpha(C^v) > 0\}$. By Lemma 2.19, Φ_{F^v} is a root system in $(\text{vect}(F^v))^\perp$. By [Bou81, VI §1.7 Corollaire 1], there exists a chamber $C_{F^v}^v$ of $(\text{vect}(F^v))^\perp$ such that $\Phi_{F^v,+} = \{\alpha \in \Phi_{F^v} \mid \alpha(C_{F^v}^v) > 0\}$.

We can write $x - \lambda = u + v$, where $u \in \text{vect}(F^v)$ and $v \in \text{vect}(F^v)^\perp$. Let $W_{\Phi_{F^v}}^v = \langle s_\alpha \mid \alpha \in \Phi_{F^v} \rangle \subset W_{\Phi_{F^v}}^v$ be the vectorial Weyl group of Φ_{F^v} . Then $(\text{vect}(F^v))^\perp$ is the union of the closed vectorial chambers of Φ_{F^v} and, since $W_{\Phi_{F^v}}^v$ acts transitively on the set of vectorial chambers of Φ_{F^v} , there exists $w \in W_{\Phi_{F^v}}^v$ such that $\alpha(w.v) \geq 0$, for every $\alpha \in \Phi_{F^v,+}$. We then have $\alpha(w.(x - \lambda)) \geq 0$, for all $\alpha \in \Phi_{F^v,+}$, since $W_{\Phi_{F^v}}^v$ pointwise fixes $\text{vect}(F^v)$.

We have $t_\lambda W_{\Phi_{F^v}}^v t_{-\lambda} = W_F^{\text{aff}}$, where $t_\mu : \mathbb{A} \rightarrow \mathbb{A}$ is the translation by the vector μ , for $\mu \in \mathbb{A}$. Let $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$. Set $\underline{w} = t_\lambda w t_{-\lambda} \in W_F^{\text{aff}}$. Write $\underline{\alpha} = \alpha + k$, where $k \in \mathbb{Z}$. Then $\underline{\alpha}(\lambda) = 0$ and thus $k = -\alpha(\lambda)$. Moreover, $\underline{\alpha}(C) > 0$ and thus $\alpha \in \Phi_{F^v,+}$. Then $\underline{\alpha}(w.x) = \alpha(\lambda + w.(x - \lambda)) - \alpha(\lambda) = \alpha(w.(x - \lambda)) \geq 0$. This being true for every $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$, the lemma follows. \square

Definition 3.16. *Let F be a face of $(\mathbb{A}, \mathcal{H})$. The **star of F** is the union of the faces of $(\mathbb{A}, \mathcal{H})$ dominating F .*

Lemma 3.17. *Let F be a face of $(\mathbb{A}, \mathcal{H})$ and let E be the star of F . Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ dominating F and $H_1, \dots, H_k \in \mathcal{H}$ be the walls of C . Up to renumbering, we may assume that $H_1, \dots, H_{k'} \supset F$ and $H_{k'+1}, \dots, H_k$ do not contain F , for some $k' \in \llbracket 1, k \rrbracket$. For $i \in \llbracket 1, k \rrbracket$, denote by \dot{D}_i the open half-space of \mathbb{A} delimited by H_i and containing C . Then $E = \bigcap_{w \in W_F^{\text{aff}}, i \in \llbracket k'+1, k \rrbracket} w.\dot{D}_i$. In particular, E is open and convex.*

Proof. Let $E' = E \cap \overline{C}$. Then by Lemma 3.8, $E = W_F.E'$.

We first assume that Φ is irreducible. Let Σ_C^{aff} be the basis of Φ^{aff} defined by C , i.e., Σ_C^{aff} is the set of affine roots $\underline{\alpha}$ such that $\underline{\alpha}^{-1}(\{0\})$ is a wall of C and $\underline{\alpha}(C) > 0$. By Lemma 3.5, we have $C = \{x \in \mathbb{A} \mid \underline{\alpha}(x) > 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$. Let $\mathcal{M}_F = \{\underline{\alpha} \in \Sigma_C^{\text{aff}} \mid \underline{\alpha}(F) > 0\}$. If F' is a face of C , then by Lemma 3.5, we can write $F' = \{x \in \mathbb{A} \mid \underline{\alpha}(x) R_{\underline{\alpha}} 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$, where $(R_{\underline{\alpha}}) \in \{>, =\}^{\Sigma_C^{\text{aff}}}$. Then $\overline{F'} = \{x \in \mathbb{A} \mid \underline{\alpha}(x) \overline{R_{\underline{\alpha}}} 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}}\}$, where $\overline{R_{\underline{\alpha}}}$ is equal to “ \geq ” if $R_{\underline{\alpha}}$ is equal to “ $>$ ” and equal to “ $=$ ” if $R_{\underline{\alpha}}$ is equal to “ $=$ ”. Therefore if $\overline{F'} \supset F$, the symbol $R_{\underline{\alpha}}$ is equal to “ $>$ ” for every $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$. Therefore:

$$E' = \{x \in \mathbb{A} \mid \underline{\alpha}(x) \geq 0, \forall \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F, \underline{\alpha}(x) > 0, \forall \underline{\alpha} \in \mathcal{M}_F\}. \quad (3.7)$$

Let us prove that

$$E = \bigcap_{\underline{\alpha} \in W_F^{\text{aff}} \cdot \mathcal{M}_F} \underline{\alpha}^{-1}(\mathbb{R}_{>0}) := E_1. \quad (3.8)$$

If $s_{\underline{\alpha}}$ denotes the orthogonal reflection with respect to $\underline{\alpha}^{-1}(\{0\})$ for $\underline{\alpha} \in \Phi^{\text{aff}}$, then $(W^{\text{aff}}, \{s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}}\})$ is a Coxeter group and $W_F^{\text{aff}} = \langle s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F \rangle$, according to Theorem 3.7 (3). Thus by [BB05, Corollaries 1.4.8 and 1.4.6] if $w \in W_F^{\text{aff}}$ and $\underline{\alpha} \in \mathcal{M}_F$, we have $\ell^{\text{aff}}(ws_{\underline{\alpha}}) = \ell^{\text{aff}}(w) + 1$, where ℓ^{aff} denotes the length on $(W^{\text{aff}}, \{s_{\underline{\alpha}} \mid \underline{\alpha} \in \Sigma_C^{\text{aff}}\})$. By [Mac03, (2.2.8)], we deduce $w \cdot \underline{\alpha} \in \Phi_+^{\text{aff}}$, where $\Phi_+^{\text{aff}} = \{\underline{\gamma} \in \Phi^{\text{aff}} \mid \underline{\gamma}(C) \geq 0\}$. Since $w \in \langle s_{\underline{\beta}} \mid \underline{\beta} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F \rangle$, we can write $w \cdot \underline{\alpha} = \underline{\alpha} + \sum_{\underline{\beta} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F} n_{\underline{\beta}} \underline{\beta}$, where the $n_{\underline{\beta}}$ are non-negative integers (by [Mac72, (1.4) and 4.6 Proposition]). Therefore $w \cdot \underline{\alpha}(E') > 0$. Consequently $E' \subset E_1$ and since E_1 is W_F^{aff} -invariant, we deduce that $E \subset E_1$.

Let $x \in E_1$. Then by Lemma 3.15, there exists $w \in W_F^{\text{aff}}$ such that $\underline{\alpha}(w \cdot x) \geq 0$ for all $\underline{\alpha} \in \Sigma_C^{\text{aff}} \setminus \mathcal{M}_F$. By definition of E_1 , we then have $w \cdot x \in E'$, by Eq. (3.7) and thus $x \in E$, which proves Eq. (3.8), when Φ is irreducible.

Assume now that Φ is reducible. Using the notation of Eq. (2.4), with Φ instead of Ψ , we write $\mathbb{A} = \mathbb{A}_1 \times \dots \times \mathbb{A}_\ell$ and $\Phi = \Phi_1 \sqcup \dots \sqcup \Phi_\ell$, where $\ell \in \mathbb{Z}_{\geq 2}$ and Φ_i is an irreducible root system in \mathbb{A}_i for $i \in \llbracket 1, \ell \rrbracket$. Write $C = C_1 \times \dots \times C_\ell$ and $F = F_1 \times \dots \times F_\ell$. Then with obvious notations, we have:

$$\begin{aligned} E &= (W_{F_1}^{\text{aff}} \times \dots \times W_{F_\ell}^{\text{aff}}) \cdot (E'_1 \times \dots \times E'_\ell) \\ &= (W_{F_1}^{\text{aff}} \cdot E'_1 \times \dots \times W_{F_\ell}^{\text{aff}} \cdot E'_\ell) \\ &= \bigcap_{i=1}^{\ell} \bigcap_{\underline{\alpha} \in W_{F_i}^{\text{aff}} \cdot \mathcal{M}_{F_i}} \underline{\alpha}^{-1}(\mathbb{R}_{>0}), \end{aligned}$$

which proves the lemma. \square

Lemma 3.18. *Let F, F' be faces of $(\mathbb{A}, \mathcal{H})$ such that F' dominates F . Let $x \in F'$ and $y = \frac{1}{|W_F^{\text{aff}}|} \sum_{w \in W_F^{\text{aff}}} w \cdot x$. Then $y \in F$ and y is the orthogonal projection of x on the support of F .*

Proof. By Lemma 3.17, y belongs to a face F'' dominating F . We thus have $W_{F''}^{\text{aff}} = W_{F''}^{\text{aff}} \subset W_F^{\text{aff}}$. Moreover, $W_{F''}^{\text{aff}}$ fixes y and thus by Theorem 3.7, W_F^{aff} fixes F'' . Consequently $W_F^{\text{aff}} = W_{F''}^{\text{aff}}$ and hence $F = F''$, by Lemma 3.9. In other words, $y \in F$.

Let $z \in F$ and set $t = \langle x - y \mid y - z \rangle$. Then for $w \in W_F^{\text{aff}}$, we have $t = \langle x - y \mid w^{-1} \cdot (y - z) \rangle = \langle w \cdot x - w \cdot y \mid y - z \rangle = \langle w \cdot x - y \mid y - z \rangle$. Therefore

$$|W_F^{\text{aff}}|t = \sum_{w \in W_F^{\text{aff}}} \langle w \cdot x \mid y - z \rangle - |W_F^{\text{aff}}| \langle y \mid y - z \rangle = |W_F^{\text{aff}}| \langle y \mid y - z \rangle - |W_F^{\text{aff}}| \langle y \mid y - z \rangle = 0,$$

which proves the lemma. \square

Lemma 3.19. *Let F be a face of $(\mathbb{A}, \mathcal{H})$, $\lambda \in \text{vert}(F)$ and E be the star of λ . Then $F = E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$.*

Proof. The inclusion $F \subset E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$ is clear. Let $x \in E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda))$ and let F_1 be the face of $(\mathbb{A}, \mathcal{H})$ containing x . Then F_1 dominates λ and $x - \lambda \in \mathbb{R}_{>0}(F_1 - \lambda) \cap \mathbb{R}_{>0}(F - \lambda)$. As the faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ form a partition of \mathbb{A} , we deduce $\mathbb{R}_{>0}(F_1 - \lambda) = \mathbb{R}_{>0}(F - \lambda)$ and thus $F_1 = F$, by Proposition 3.12. Therefore $E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda)) \subset F$ and the lemma follows. \square

3.2 Affine weight polytopes

3.2.1 Definition of affine weight polytopes

Let C_0 be the fundamental alcove of \mathbb{A} , i.e., the alcove of \mathbb{A} based at 0 and contained in C_f^v . Let us fix $\mathbf{b} \in C_0$. Let $C \in \text{Alc}(\mathcal{H})$. Then there exists $w \in W^{\text{aff}}$ such that $C = w.C_0$. We then set $\mathbf{b}_C = w.\mathbf{b} \in C$. The element w is uniquely determined by C , since W^{aff} acts simply transitively on the set of alcoves of $(\mathbb{A}, \mathcal{H})$. Thus \mathbf{b}_C is well-defined.

Definition 3.20. For $\lambda \in \text{vert}(\mathbb{A})$, we set $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{P}_{\mathbf{b}, \Phi}(\lambda)} = \text{conv}(\{\mathbf{b}_C \mid C \in \text{Alc}_\lambda(\mathcal{H})\})$. Then we call $\overline{\mathcal{P}(\lambda)}$ the **closed affine Weyl polytope associated with λ (and \mathbf{b})**. The **(open) affine Weyl polytope associated with λ** is the interior of $\overline{\mathcal{P}(\lambda)}$. By Lemma 3.8,

$$\overline{\mathcal{P}(\lambda)} = \text{conv}(W_\lambda^{\text{aff}}.\mathbf{b}_C) \quad (3.9)$$

for any $C \in \text{Alc}_\lambda(\mathcal{H})$.

Note that in order to avoid confusions we use roman letters to denote the faces of $(\mathbb{A}, \mathcal{H})$ and curly letters for the faces of the $\overline{\mathcal{P}(\lambda)}$.

3.2.2 Description of the faces of the affine weight polytopes

Let $\lambda \in \text{vert}(\mathbb{A})$. We have $W_{\Phi_\lambda}^v = \langle s_\alpha \mid \alpha \in \Phi_\lambda \rangle \subset W_\Phi^v$. We denote by \mathbf{t}_λ the translation on \mathbb{A} by the vector λ (this is not an element of W^{aff} in general).

Lemma 3.21. Let $\lambda \in \text{vert}(\mathbb{A})$. Then $W_\lambda^{\text{aff}} = \mathbf{t}_\lambda W_{\Phi_\lambda}^v \mathbf{t}_\lambda^{-1}$.

Proof. Let $\alpha \in \Phi_\lambda$. Then $\mathbf{t}_\lambda s_\alpha \mathbf{t}_\lambda^{-1}$ is the orthogonal reflection fixing $\lambda + \alpha^{-1}(\{0\}) = \alpha^{-1}(\alpha(\lambda))$. Therefore it belongs to W_λ^{aff} . By Theorem 3.7, W_λ^{aff} is generated by the reflections fixing the walls containing λ , which proves the lemma. \square

Let $\lambda \in \text{vert}(\mathbb{A})$. Then $\overline{\mathcal{P}(\lambda)} = \text{conv}(\mathbf{t}_\lambda W_{\Phi_\lambda}^v \mathbf{t}_\lambda^{-1}.\mathbf{b}_C) = \lambda + \text{conv}(W_{\Phi_\lambda}^v.(-\lambda + \mathbf{b}_C))$ for $C \in \text{Alc}_\lambda(\mathcal{H})$. Moreover, $\alpha(\mathbf{b}_C) \notin \mathbb{Z}$, for all $\alpha \in \Phi$ and thus $\alpha(-\lambda + \mathbf{b}_C) \neq 0$ for all $\alpha \in \Phi_\lambda$. Therefore, with the notation of subsection 2.2, we have:

$$\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{-\lambda + \mathbf{b}_C, \Phi_\lambda}^0}. \quad (3.10)$$

Definition 3.22. Let $\lambda \in \text{vert}(\mathbb{A})$ and F be a face of $(\mathbb{A}, \mathcal{H})$ dominating λ . We choose an alcove C dominating F and we set $\mathcal{F}_F = \text{Int}_r(\text{conv}(W_F^{\text{aff}}.\mathbf{b}_C))$. This is well-defined by Lemma 3.8.

Note that for $\lambda \in \text{vert}(\mathbb{A})$, we have $\mathcal{F}_\lambda = \mathcal{P}(\lambda)$.

Lemma 3.23. 1. Let F be a face of $(\mathbb{A}, \mathcal{H})$. Then $\overline{\mathcal{F}_F}$ is contained in the star of F . In particular, if $\lambda \in \text{vert}(\mathbb{A})$, then $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{F}_{\{\lambda\}}}$ is contained in the star of λ .

2. Let B be a bounded subset of \mathbb{A} . Then $\{\lambda \in \text{vert}(\mathbb{A}) \mid \overline{\mathcal{P}(\lambda)} \cap B \neq \emptyset\}$ is finite.

Proof. 1. Let E be the star of F . We have $\overline{\mathcal{F}_F} = \text{conv}(W_F^{\text{aff}}.\mathbf{b}_C)$, for any alcove C dominating F . As $\mathbf{b}_C \in E$ and as E is stabilized by W_F^{aff} , we deduce $W_F^{\text{aff}}.\mathbf{b}_C \subset E$. By Lemma 3.17, E is convex and thus $\overline{\mathcal{F}_F} \subset E$.

2. As B is bounded, there exists a finite set of closed alcoves F such that $B \subset \bigcup_{\overline{C} \in F} \overline{C}$. Let F' be the set of closed alcoves containing a vertex of an element of F . Then by statement (1), if $\lambda \in \text{vert}(\mathbb{A})$ is such that $B \cap \overline{\mathcal{P}(\lambda)}$ is non-empty, then $\lambda \in \bigcup_{\overline{C} \in F'} \overline{C} \cap \text{vert}(\mathbb{A})$, which is finite. This proves statement (2). \square

For a polytope \mathcal{Q} of \mathbb{A} , we denote by $\text{Face}(\mathcal{Q})$ its set of faces. We equip $\text{Face}(\mathcal{Q})$ with the dominance order \leq . Recall that, if E is an affine subspace of \mathbb{A} , its direction $\text{dir}(E)$, is the vector subspace $\{x - x' \mid x, x' \in E\}$ of \mathbb{A} . (I prefer to recall the definition since it is far.)

Proposition 3.24. (see Figure 3.5) Let $\lambda \in \text{vert}(\mathbb{A})$. Let $\mathcal{P}^0 = \mathcal{P}(\lambda) - \lambda$.

1. The map $\text{Face}(\overline{\mathcal{P}(\lambda)}) \rightarrow \text{Face}(\overline{\mathcal{P}^0})$ given by $\mathcal{F} \mapsto \mathcal{F} - \lambda$ is an isomorphism of partially ordered sets.
2. Let $C \in \text{Alc}_\lambda(\mathcal{H})$ and $C^v = \mathbb{R}_{>0}(C - \lambda)$. Set $\mathbf{b}_{C^v}^0 = \mathbf{b}_C - \lambda$. For each chamber C_1^v of $(\mathbb{A}, \mathcal{H}_\lambda^0)$, set $\mathbf{b}_{C_1^v}^0 = w \cdot \mathbf{b}_{C^v}^0$, where $w \in W_{\Phi_\lambda}^v$ satisfies $C_1^v = w \cdot C^v$. For $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$, set $\mathcal{F}_{F^v}^0 = \text{Int}_r(\text{conv}(W_{F^v, \Phi_\lambda}^v \cdot \mathbf{b}_{C_1^v}^0))$, where C_1^v is a chamber of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ dominating F^v . Then for every $F^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$, we have $\mathcal{F}_{F^v}^0 + \lambda = \mathcal{F}_F$, where $F \in \mathcal{F}_\lambda(\mathcal{H})$ satisfies $F^v = \mathbb{R}_{>0}(F - \lambda)$.
3. The map $F_1 \mapsto \mathcal{F}_{F_1}$ is an isomorphism between $(\mathcal{F}_\lambda(\mathcal{H}), \leq)$ and $(\text{Face}(\overline{\mathcal{P}(\lambda)}), \geq)$. Moreover, $\text{dir}(\text{supp}(\mathcal{F}_{F_1})) = (\text{dir}(\text{supp}(F_1)))^\perp$ and $\dim(\text{supp}(\mathcal{F}_{F_1})) = \text{codim}_\mathbb{A}(F_1)$ for every $F_1 \in \mathcal{F}_\lambda(\mathcal{H})$.

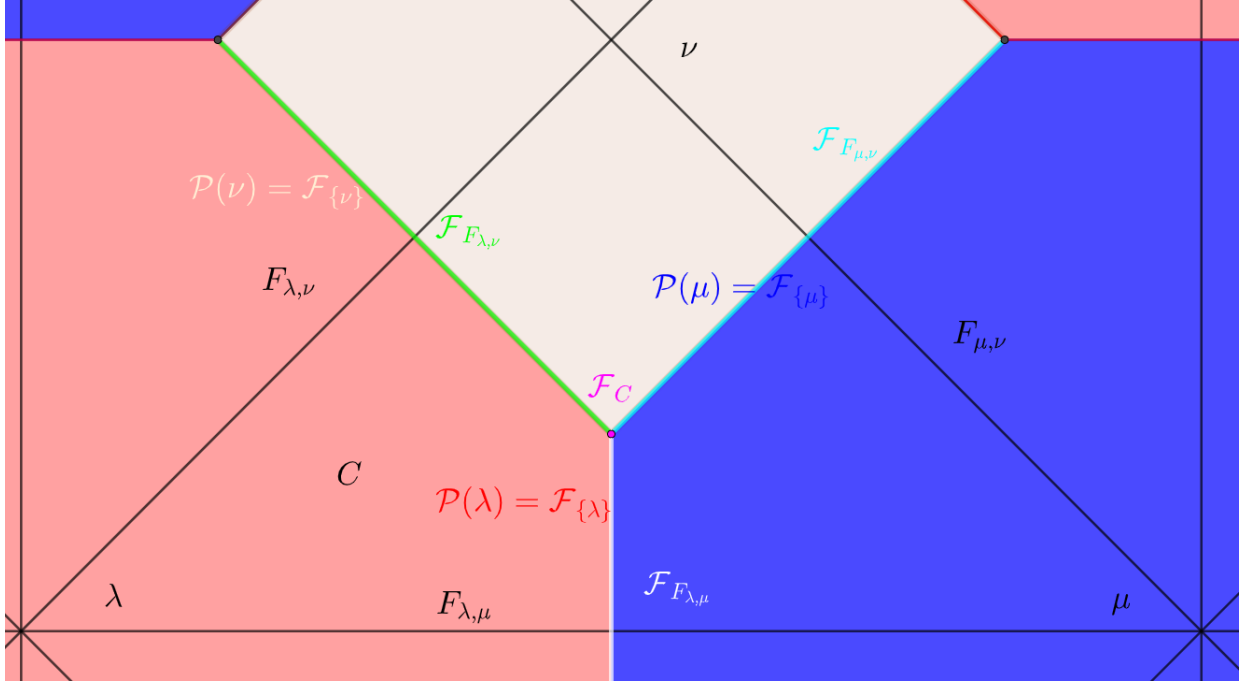


Figure 3.5: Illustration of Proposition 3.24: the case of B_2 .

Proof. Statement (1) is clear.

Now, we prove statement (2). By Proposition 3.12, C^v is a chamber of $(\mathbb{A}, \mathcal{H}_\lambda^0)$. By Eq. (3.10), we have $\overline{\mathcal{P}(\lambda) - \lambda} = \overline{\mathcal{P}^0} = \overline{\mathcal{P}_{\mathbf{b}_{C^v, \Phi_\lambda}^0}^0} = \text{conv}(W_{\Phi_\lambda}^v \cdot \mathbf{b}_{C^v}^0)$. Let $F_1^v \in \mathcal{F}(\mathcal{H}_\lambda^0)$ and let C_1^v be a chamber of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ dominating F_1^v . Let $\mathcal{F} = \mathcal{F}_{F_1^v}^0 + \lambda$. Let $F_1, C_1 \in \mathcal{F}_\lambda(\mathbb{A})$ be such that $F_1^v = \mathbb{R}_{>0}(F_1 - \lambda)$ and $C_1^v = \mathbb{R}_{>0}(C_1 - \lambda)$.

Using Lemma 3.8, we get $C_1 = w_1 \cdot C$, with $w_1 \in W_\lambda^{\text{aff}}$. Then we have:

$$w_1 \cdot C - \lambda = C_1 - \lambda = w_1 \cdot C - w_1 \cdot \lambda = \vec{w}_1 \cdot (C - \lambda),$$

where \vec{w}_1 is the linear part of w_1 . By Lemma 3.21, $\vec{w}_1 \in W_{\Phi_\lambda}^v$. Moreover we have:

$$\mathbb{R}_{>0} \vec{w}_1 \cdot (C - \lambda) = \vec{w}_1 \cdot C^v = \mathbb{R}_{>0}(C_1 - \lambda) = C_1^v.$$

Hence:

$$\overline{\mathcal{F}} = \mathbf{t}_\lambda \cdot \overline{\mathcal{F}_{F_1}^v} = \text{conv}(\mathbf{t}_\lambda \cdot W_{F_1, \Phi_\lambda}^v \cdot \mathbf{b}_{C^v}^0) = \text{conv}\left(W_{\mathbf{t}_\lambda \cdot F_1}^{\text{aff}} \cdot (\lambda + \mathbf{b}_{C_1}^0)\right) = \text{conv}(W_{F_1}^{\text{aff}} \cdot (\lambda + \mathbf{b}_{C^v}^0)),$$

where W_{F_1, Φ_λ}^v is the fixator of F_1^v in $W_{\Phi_\lambda}^v$. Moreover, $\mathbf{b}_{C_1} = w_1 \cdot \mathbf{b}_C = w_1 \cdot (\lambda + \mathbf{b}_{C^v}^0) = w_1 \cdot \lambda + \bar{w}_1 \cdot \mathbf{b}_{C^v}^0 = \lambda + \mathbf{b}_{C_1}^0$, which proves that $\overline{\mathcal{F}} = \text{conv}(W_{F_1}^{\text{aff}} \cdot \mathbf{b}_{C_1}) = \overline{\mathcal{F}_{F_1}}$. Therefore $\mathcal{F} = \text{Int}_r(\overline{\mathcal{F}_{F_1}}) = \mathcal{F}_{F_1}$, which proves (2).

Finally, we prove statement (3). Indeed, by statements (1), (2) and by Propositions 3.12 and 2.20, the map $F_1 \mapsto \mathcal{F}_{F_1}$ can be decomposed as

$$F_1 \mapsto \mathbb{R}_{>0}(F_1 - \lambda) \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1 - \lambda)}^0 \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1 - \lambda)}^0 + \lambda = \mathcal{F}_{F_1}$$

and then it is an isomorphism between $(\mathcal{F}_\lambda(\mathcal{H}), \leq)$ and $(\text{Face}(\overline{\mathcal{P}(\lambda)}), \geq)$. The last assertion follows from Proposition 2.20 and statement (2). \square

3.2.3 Faces and intersection of affine weight polytopes

Recall that, for any pair $(\lambda, \mu) \in \mathbb{A}^2$, we write $[\lambda, \mu] = \{t\lambda + (1-t)\mu \mid t \in [0, 1]\}$ and $] \lambda, \mu[= \text{Int}_r([\lambda, \mu]) = \{t\lambda + (1-t)\mu \mid t \in]0, 1[\}$.

Lemma 3.25. *Let $\lambda, \mu \in \text{vert}(\mathbb{A})$ be adjacent. Then $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ is a closed facet of both $\overline{\mathcal{P}(\lambda)}$ and $\overline{\mathcal{P}(\mu)}$. More precisely,*

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C),$$

where C is any alcove dominating $] \lambda, \mu[$.

Proof. As λ and μ are adjacent, $] \lambda, \mu[$ is a face of some alcove C . By Proposition 3.24, $\overline{\mathcal{F}} := \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C)$ is a closed facet of both $\overline{\mathcal{P}(\lambda)}$ and $\overline{\mathcal{P}(\mu)}$. It remains to prove that λ and μ are on opposite sides of $\overline{\mathcal{F}}$. By Lemma 3.18, $\lambda = \frac{1}{|W_\lambda^{\text{aff}}|} \sum_{w \in W_\lambda^{\text{aff}}} w \cdot \lambda \in \overline{\mathcal{P}(\lambda)}$ and similarly, $\mu \in \overline{\mathcal{P}(\mu)}$. By Lemma 3.18 applied with $F =] \lambda, \mu[$, the orthogonal projection x of \mathbf{b}_C on the line containing $] \lambda, \mu[$ belongs to $] \lambda, \mu[\cap \overline{\mathcal{F}}$. The support of $\overline{\mathcal{F}}$ is the hyperplane $H = x + (\mathbb{R}(\mu - \lambda))^\perp$ by Proposition 3.24(3). Then $\overline{\mathcal{P}(\lambda)}$ (resp. $\overline{\mathcal{P}(\mu)}$) is contained in the half-space of \mathbb{A} delimited by H and containing λ (resp. μ), which proves the lemma. \square

Lemma 3.26. *Let $\lambda, \mu \in \text{vert}(\mathbb{A})$ and assume that there exists an alcove dominating both λ and μ . Then $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ is a face of $\overline{\mathcal{P}(\lambda)}$. Moreover, if C is an alcove dominating λ and μ , then $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C) = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$, where F is the smallest face dominating $\{\lambda, \mu\}$ (whose existence is provided by Lemma 3.6).*

Proof. If Φ is irreducible, then this is Lemma 3.25, so we assume that Φ is reducible. We use the same notation as in Eq. (2.4), with Φ instead of Ψ . Write $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_\ell$ and $\Phi = \bigsqcup_{i=1}^\ell \Phi_i$. We have $W^{\text{aff}} = W_{\Phi_1}^{\text{aff}} \times \dots \times W_{\Phi_\ell}^{\text{aff}}$, where $W_{\Phi_i}^{\text{aff}}$ is the affine Weyl group of (\mathbb{A}_i, Φ_i) , for $i \in [1, \ell]$. Write $\mathbf{b}_C = (\mathbf{b}_1, \dots, \mathbf{b}_\ell)$. We have $\text{vert}(\mathbb{A}) = \text{vert}(\mathbb{A}_1) \times \dots \times \text{vert}(\mathbb{A}_\ell)$ and if $(\tau_1, \dots, \tau_\ell) \in \text{vert}(\mathbb{A})$, we have $\mathcal{P}_{\mathbf{b}, \Phi}((\tau_1, \dots, \tau_\ell)) = \mathcal{P}_{\mathbf{b}_1, \Phi_1}(\tau_1) \times \dots \times \mathcal{P}_{\mathbf{b}_\ell, \Phi_\ell}(\tau_\ell)$. Write $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_\ell)$. Then for all $i \in [1, \ell]$, either $\lambda_i = \mu_i$, or λ_i and μ_i are adjacent. By Lemma 3.25, we get that:

$$\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)} \cap \overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)} = \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i)$$

and that this intersection is a facet of both $\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)}$ and $\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)}$. We deduce that:

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \prod_{i=1}^\ell \left(\overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\lambda_i)} \cap \overline{\mathcal{P}_{\mathbf{b}_i, \Phi_i}(\mu_i)} \right) = \prod_{i=1}^\ell \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i)$$

is a face of both $\mathcal{P}(\lambda)$ and $\mathcal{P}(\mu)$. The equalities

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \text{conv}(W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C) = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$$

immediately follow from the facts that:

$$\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} = \prod_{i=1}^{\ell} \text{conv}(W_{\Phi_i, [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_i) = \text{conv}\left(W_{\prod_{i=1}^{\ell} [\lambda_i, \mu_i]}^{\text{aff}} \cdot \mathbf{b}_C\right),$$

that $\overline{F} = \prod_{i=1}^{\ell} [\lambda_i, \mu_i]$ and that $W_{[\lambda, \mu]}^{\text{aff}} = W_F^{\text{aff}}$. \square

Proposition 3.27. 1. Let J be a non-empty subset of $\text{vert}(\mathbb{A})$. Then $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$ is non-empty if and only if there exists an alcove of $(\mathbb{A}, \mathcal{H})$ dominating J . Moreover, if F is the smallest face dominating J (which exists by Lemma 3.6), then $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)} = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C) = \overline{\mathcal{F}_F} = \bigcap_{\lambda \in \text{vert}(F)} \overline{\mathcal{P}(\lambda)}$, for any alcove C dominating F .

2. If F is a face of $(\mathbb{A}, \mathcal{H})$, then the set of extreme points of $\overline{\mathcal{F}_F}$ is $W_F^{\text{aff}} \cdot \mathbf{b}_C$, for any alcove C dominating F . Moreover $\mathcal{F}_F \cap F = \{x\}$, where x is the barycenter of the extreme points of \mathcal{F}_F .

3. Let F be a face of $(\mathbb{A}, \mathcal{H})$ and $x \in \mathcal{F}_F$. Then $\{\mu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\mu)}\} = \text{vert}(F)$.

Proof. 1) Let $J \subset \text{vert}(\mathbb{A})$ be non-empty. Assume that $\bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$ is non-empty. Let $x \in \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$ and let F be the face of $(\mathbb{A}, \mathcal{H})$ containing x . Then by Lemma 3.23, F dominates J and, in particular, J is dominated by any alcove dominating F . Conversely, if J is dominated by a face F of $(\mathbb{A}, \mathcal{H})$, then $W_F^{\text{aff}} \cdot \mathbf{b}_C \subset \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$, for any alcove C dominating F .

Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ and J be a subset of $\text{vert}(C)$ with cardinality at least 2. Let F be the smallest face of C dominating J , which exists by Lemma 3.6. Let $\lambda \in J$. Then

$$\overline{\mathcal{F}_J} := \bigcap_{\mu \in J} \overline{\mathcal{P}(\mu)} = \bigcap_{\mu \in J \setminus \{\lambda\}} (\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}).$$

By Lemma 3.26 and [BG09, Theorem 1.10], $\overline{\mathcal{F}_J}$ is a face of $\overline{\mathcal{P}(\lambda)}$.

We have:

$$\begin{aligned} \text{Extr}(\mathcal{F}_J) &= \bigcap_{\mu \in J \setminus \{\lambda\}} \text{Extr}(\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}) && \text{by Lemma 2.5,} \\ &= \bigcap_{\mu \in J \setminus \{\lambda\}} W_{[\lambda, \mu]}^{\text{aff}} \cdot \mathbf{b}_C && \text{by Lemma 2.4,} \\ &= \left(\bigcap_{\mu \in J \setminus \{\lambda\}} W_{[\lambda, \mu]}^{\text{aff}} \right) \cdot \mathbf{b}_C && \text{since } \mathbf{b}_C \text{ is regular,} \\ &= W_F^{\text{aff}} \cdot \mathbf{b}_C && \text{by Lemma 3.14.} \end{aligned}$$

Thus by Krein-Milman theorem, $\mathcal{F}_J = \text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C)$, which proves statement (1).

2) Let $F \in \mathcal{F}(\mathcal{H})$ and $C \in \text{Alc}(\mathcal{H})$, dominating F . By 1) and by Lemma 2.4, the set of extreme points of $\overline{\mathcal{F}_F}$ is $W_F^{\text{aff}} \cdot \mathbf{b}_C$. Let now $x = \frac{1}{|W_F^{\text{aff}}|} \sum_{w \in W_F^{\text{aff}}} w \cdot \mathbf{b}_C$. Then $x \in F$ according to Lemma 3.18. By Proposition 3.24, the direction of the support of $F \cap \mathcal{F}_F$ is a singleton, which proves that $\mathcal{F}_F \cap F = \{x\}$ and completes the proof of statement (2).

3) Let $F \in \mathcal{F}(\mathcal{H})$ and $x \in \mathcal{F}_F$. Let $\mu \in \text{vert}(\mathbb{A}) \setminus \text{vert}(F)$. If $\text{vert}(F) \cup \{\mu\}$ is not dominated by any alcove, then $\mathcal{F}_F \cap \overline{\mathcal{P}(\mu)} = \emptyset$ according to statement (1) and thus

$x \notin \overline{\mathcal{P}(\mu)}$. Assume that $\mu \cup \text{vert}(F)$ is dominated by an alcove C . Then it follows from (1) that $\overline{\mathcal{F}_F} \cap \overline{\mathcal{P}(\mu)} = \overline{\mathcal{F}_{F'}}$, where F' is the smallest face dominating $F \cup \{\mu\}$, and by (2) that $\mathcal{F}_{F'} \neq \mathcal{F}_F$. Therefore $\mathcal{F}_{F'}$ is a proper face of \mathcal{F}_F and by [BG09, Corollary 1.7], $x \notin \mathcal{F}_{F'}$, which proves that $x \notin \overline{\mathcal{P}(\mu)}$. Therefore $\{\mu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\mu)}\} \subset \text{vert}(F)$, and this is actually an equality by 1), which proves statement (3). \square

3.3 Tessellation of \mathbb{A} by the affine weight polytopes

We now prove that \mathbb{A} is tessellated by the $\overline{\mathcal{P}(\lambda)}$, for $\lambda \in \text{vert}(\mathbb{A})$, as stated in the introduction in Theorem 1.1.

Theorem 3.28. (see Figures 3.6, 3.7, 3.8 and 3.9)

Let \mathbb{A} be a finite dimensional vector space and Φ be a finite root system in \mathbb{A} . Let $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$, let \mathcal{F} be the set of faces of $(\mathbb{A}, \mathcal{H})$ and let $\text{vert}(\mathbb{A})$ be the set of vertices of $(\mathbb{A}, \mathcal{H})$. We choose a fundamental alcove C_0 of $(\mathbb{A}, \mathcal{H})$ and choose $\mathbf{b} = \mathbf{b}_{C_0} \in C_0$. If λ is a vertex of $(\mathbb{A}, \mathcal{H})$, we choose an alcove C_λ dominating λ and we set $\mathbf{b}_{C_\lambda} = w \cdot \mathbf{b}$ where w is the element of the affine Weyl group W^{aff} of $(\mathbb{A}, \mathcal{H})$ sending C_0 to C . If F is a face of $(\mathbb{A}, \mathcal{H})$, we write $W_F^{\text{aff}} := \text{Fix}_{W^{\text{aff}}}(F)$ and we set $\mathcal{F}_F = \text{Int}_r(\text{conv}(W_F^{\text{aff}} \cdot \mathbf{b}_C))$, for any alcove C dominating F . Then $\bigsqcup_{F \in \mathcal{F}} \mathcal{F}_F = \mathbb{A}$. In particular, if we set $\overline{\mathcal{P}(\lambda)} = \overline{\mathcal{F}_{\{\lambda\}}}$, for $\lambda \in \text{vert}(\mathbb{A})$, then $\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}$.

Proof. Let $E = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}$. We prove that $E = \mathbb{A}$ by proving that E is closed and dense in \mathbb{A} . Take $(z_n) \in E^{\mathbb{N}}$ a converging sequence. Then (z_n) is bounded and by Lemma 3.23 (2), there exists a finite subset $F \subset \text{vert}(\mathbb{A})$ such that $\{z_n \mid n \in \mathbb{N}\} \subset \bigcup_{\mu \in F} \overline{\mathcal{P}(\mu)}$. Then as the $\overline{\mathcal{P}(\mu)}$ are closed, $\lim z_n \in \bigcup_{\mu \in F} \overline{\mathcal{P}(\mu)} \subset E$. Therefore E is closed.

Suppose that $E \neq \mathbb{A}$ and fix $x \in \mathbb{A} \setminus E$. Let $\lambda \in \text{vert}(\mathbb{A})$ and $y \in \mathcal{P}(\lambda)$. Let $\gamma : [0, 1] \rightarrow [x, y]$ be the affine parametrization of $[x, y]$ such that $\gamma(0) = y$ and $\gamma(1) = x$. Let

$$t = \inf\{t' \in [0, 1] \mid \gamma(t') \notin E\}.$$

As E is closed, $\gamma(t) \in E$. Let $\nu \in \text{vert}(\mathbb{A})$ be such that $\gamma(t) \in \overline{\mathcal{P}(\nu)}$. Then $\gamma(t) \in \text{Fr}(\mathcal{P}(\nu))$ and thus there exists a facet $\overline{\mathcal{F}}$ of $\overline{\mathcal{P}(\nu)}$ containing $\gamma(t)$. By Propositions 3.24 and 3.27, there exists $\nu' \in \text{vert}(\mathbb{A})$, adjacent to ν and such that $\overline{\mathcal{P}(\nu)} \cap \overline{\mathcal{P}(\nu')} = \overline{\mathcal{F}}$. If $\gamma(t) \in \mathcal{F}$, then by Lemma 2.1, there exists $r \in \mathbb{R}_{>0}$ such that

$$B(\gamma(t), r) \cap \overline{\mathcal{P}(\nu)} = D \cap B(\gamma(t), r) \text{ and } B(\gamma(t), r) \cap \overline{\mathcal{P}(\nu')} = D' \cap B(\gamma(t), r),$$

where D and D' are the two closed half-spaces delimited by $\text{supp}(\mathcal{F})$. But then $B(\gamma(t), r) \subset \overline{\mathcal{P}(\nu)} \cup \overline{\mathcal{P}(\nu')}$, which is impossible by definition of t . Therefore

$$\gamma(t) \in \overline{\mathcal{F}} \setminus \mathcal{F}.$$

The facet \mathcal{F} is a polytope of $\text{supp}(\mathcal{F})$ and thus its frontier is contained in a union $\bigcup_{V \in F(\nu, \nu')} V$, where $F(\nu, \nu')$ is a finite set of affine subspaces of dimension at most $\dim \mathbb{A} - 2$. For $\nu, \nu' \in \text{vert}(\mathbb{A})$, set $F(\nu, \nu') = \emptyset$ if ν and ν' are not adjacent. Then we proved:

$$\mathbb{A} \setminus E \subset \bigcup_{\nu, \nu' \in \text{vert}(\mathbb{A})} \bigcup_{V \in F(\nu, \nu')} \text{supp}(V \cup \{x\}).$$

As the right hand side of this formula is a countable union of proper affine subspaces of \mathbb{A} , it has Lebesgue measure 0. In particular, E is dense in \mathbb{A} . But as E is closed, $E = \mathbb{A}$. Thus we proved

$$\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \overline{\mathcal{P}(\lambda)}.$$

As for $\lambda \in \text{vert}(\mathbb{A})$, $\overline{\mathcal{P}(\lambda)}$ is the union of the \mathcal{F}_F , where F runs over the faces of $(\mathbb{A}, \mathcal{H})$ dominating λ (see Proposition 3.24), we deduce that $\mathbb{A} = \bigcup_{F \in \mathcal{F}} \mathcal{F}_F$. Let $F_1, F_2 \in \mathcal{F}$ be such that $\mathcal{F}_{F_1} \cap \mathcal{F}_{F_2}$ is non-empty. Then $\overline{\mathcal{F}_{F_1}} \cap \overline{\mathcal{F}_{F_2}} \neq \emptyset$ and thus by Proposition 3.27, there exists an alcove C dominating $\text{vert}(F_1) \cup \text{vert}(F_2)$, and $\mathcal{F}_{F_1}, \mathcal{F}_{F_2}$ are faces of $\overline{\mathcal{P}(\lambda)}$, for any $\lambda \in \text{vert}(C)$, by Proposition 3.24 3). As the faces of $\overline{\mathcal{P}(\lambda)}$ form a partition $\overline{\mathcal{P}(\lambda)}$ (by [BG09, Theorem 1.10]), we deduce that $F_1 = F_2$, which concludes the proof of the theorem. \square

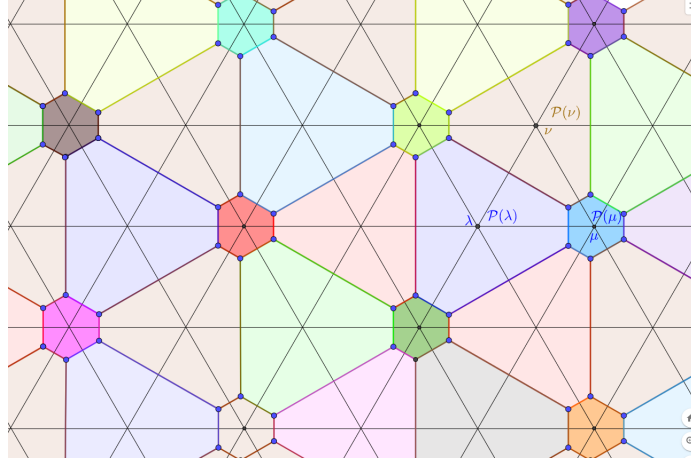


Figure 3.6: Illustration of Theorem 3.28: the case of A_2 when \mathbf{b} is not the barycenter of the fundamental alcove.

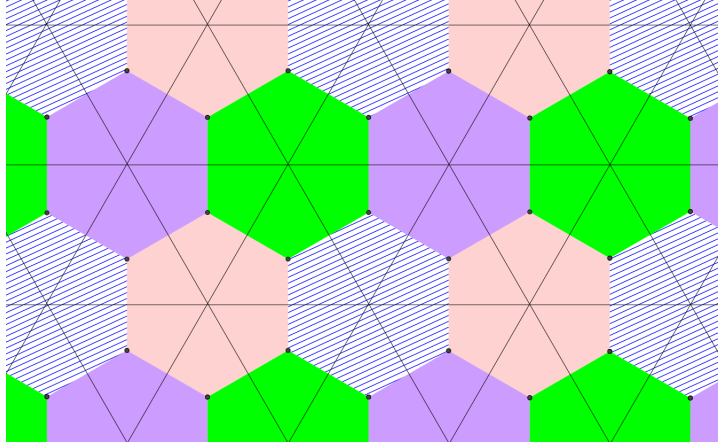


Figure 3.7: Illustration of Theorem 3.28: the case of A_2 when \mathbf{b} is the barycenter of the fundamental alcove.

4 Thickened tessellation of \mathbb{A}

The aim of this section is as follows. Take $\eta \in]0, 1[$. For $\lambda \in \text{vert}(\mathbb{A})$, denote by $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}, h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda)$ the homothety centered at λ with ratio η . Let $x \in h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$. Write $x = h_{\lambda, \eta}(x')$, with $x' \in \overline{\mathcal{P}(\lambda)}$. Let F be the face of $(\mathbb{A}, \mathcal{H})$ such that $x' \in \mathcal{F}_F$. Let $\tilde{F}_\eta(x) = x + (1 - \eta)(\overline{F} - \lambda)$. This is a copy of F which contains exactly one point of $h_{\mu, \eta}(\overline{\mathcal{P}(\mu)})$, for each vertex μ of F (and which is the convex hull of these points). The main goal of this section is to prove Theorem 4.19, which

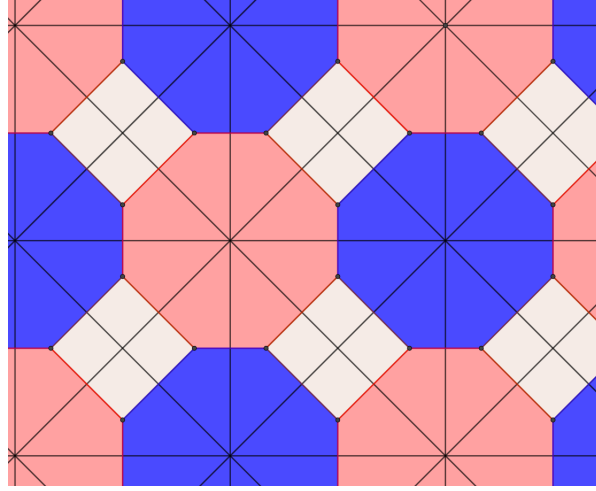


Figure 3.8: Illustration of Theorem 3.28: the case of B_2 .

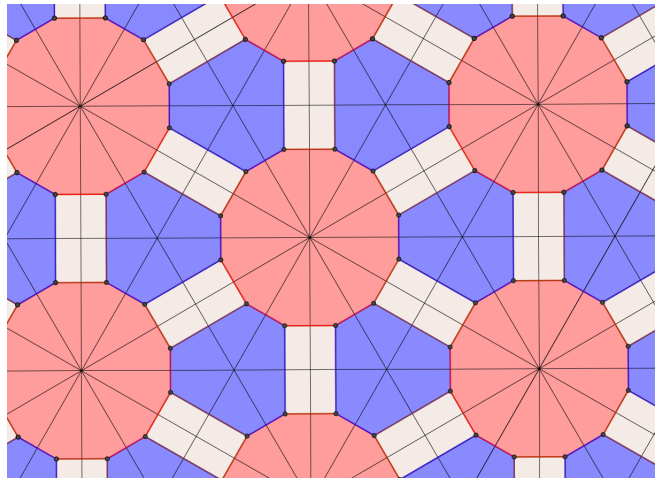


Figure 3.9: Illustration of Theorem 3.28: the case of G_2 .

states that $\mathbb{A} = \bigcup_{\lambda \in \text{vert}(\mathbb{A}), x \in h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})} \overline{\tilde{F}_\eta(x)}$ (see Figures 4.1, 4.4 and 4.5). Since for all $\mu \in \text{vert}(F)$, we have $\tilde{F}_\eta(h_{\mu, \eta}(x')) = \tilde{F}_\eta(x)$, in order to describe \mathbb{A} as a disjoint union, we will introduce a new notation by replacing $\overline{\tilde{F}_\eta(x)}$ by $\overline{F_\eta(x')}$ (see Definition 4.13) We will then have $\mathbb{A} = \bigsqcup_{x' \in \mathbb{A}} \overline{F_\eta(x')}$.

We divide this section in two parts. Firstly, in subsection 4.1, we prove that if λ and μ are two adjacent vertices of \mathbb{A} , then the projection of $\overline{\mathcal{P}(\mu)}$ on $\overline{\mathcal{P}(\lambda)}$ is contained in $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ (see Proposition 4.8). Then, in subsection 4.2, we prove Theorem 4.19.

4.1 Projection of a polytope on a neighbour polytope

If F is a face of $(\mathbb{A}, \mathcal{H})$ and $\lambda \in \text{vert}(F)$, we denote by $\text{vert}_\lambda(F)$ the set of vertices of F which are adjacent to λ (and different from λ). If $\lambda \in \text{vert}(\mathbb{A})$, we denote by π_λ instead of $\pi_{\overline{\mathcal{P}(\lambda)}}$ the orthogonal projection on $\overline{\mathcal{P}(\lambda)}$.

In this subsection, we prove that if λ and μ are two vertices of \mathbb{A} contained in a common closed alcove, then $\pi_\lambda(\overline{\mathcal{P}(\mu)}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$ (see Proposition 4.8).

Lemma 4.1. *1. Let F be a face of $(\mathbb{A}, \mathcal{H})$ and $\lambda \in \text{vert}(F)$. Then:*

$$\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{>0}(\mu - \lambda) = \sum_{\mu \in \text{vert}(F)} \mathbb{R}_{>0}(\mu - \lambda).$$

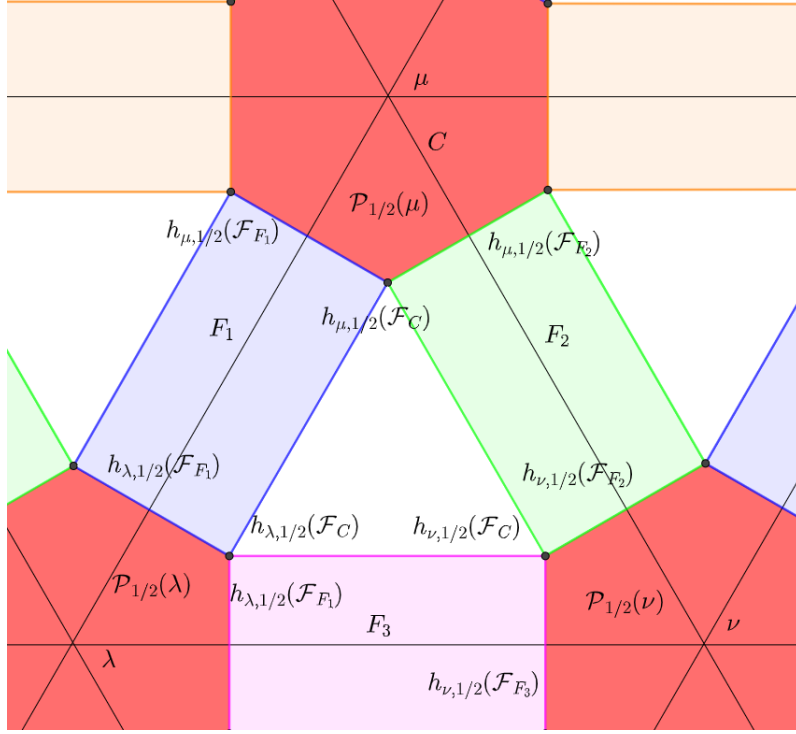


Figure 4.1: Illustration of Theorem 4.19 in the case of A_2 . The represented alcove C is the equilateral triangle delimited by the black segments. The hexagons $\mathcal{P}_{1/2}(\lambda)$, $\mathcal{P}_{1/2}(\mu)$ and $\mathcal{P}_{1/2}(\nu)$ are represented in red. The edges of C are the segments $\overline{F_1}$, $\overline{F_2}$ and $\overline{F_3}$ and the vertices of C are λ , μ and ν . The blue rectangle is the union of the $\overline{\tilde{F}_{1/2}(x)}$ for $x \in h_{\lambda,1/2}(\mathcal{F}_{F_1})$ (or equivalently $x \in h_{\mu,1/2}(\mathcal{F}_{F_1})$). Similarly, the green and pink rectangles correspond to the $\overline{\tilde{F}_{1/2}(x)}$ for x in $h_{\mu,1/2}(\mathcal{F}_{F_2})$ and $h_{\nu,1/2}(\mathcal{F}_{F_2})$ respectively. The face $\overline{\mathcal{F}_C}$ is the (non represented) vertex $\overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)}$. The central white triangle is $\overline{\tilde{F}_{1/2}(z)}$, for $z \in h_{\lambda,1/2}(\mathcal{F}_C)$ (or equivalently, $z \in h_{\mu,1/2}(\mathcal{F}_C)$ or $z \in h_{\nu,1/2}(\mathcal{F}_C)$).

2. Let $\lambda \in \text{vert}(\mathbb{A})$ and F, F' be faces of $(\mathbb{A}, \mathcal{H})$ dominating λ . Then $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$ if and only if F is dominated by F' .

Proof. 1) The map $\mu \mapsto]\lambda, \mu[$ is a bijection from $\text{vert}_\lambda(F)$ to the set of one dimensional faces of F dominating λ . By Proposition 2.6 and Proposition 3.12, $\mathbb{R}_{>0}(F - \lambda)$ is the sum of its one-dimensional faces and thus we have $\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{>0}(\mu - \lambda)$. Using the notation of Eq. (2.4) (with $\ell = 1$ if Φ is irreducible), we write $F = F_1 \times \dots \times F_\ell$ and $\lambda = (\lambda_1, \dots, \lambda_\ell)$. We have $\text{vert}_\lambda(F) = \bigsqcup_{i=1}^\ell \{(\lambda_1, \dots, \lambda_{i-1})\} \times \text{vert}_{\lambda_i}(F_i) \times \{(\lambda_{i+1}, \dots, \lambda_\ell)\}$ and $\text{vert}(F) = \text{vert}(F_1) \times \dots \times \text{vert}(F_\ell)$ and the second equality of 1) follows.

2) If F is dominated by F' , then $\text{vert}(F) \subset \text{vert}(F')$ and thus $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$. Conversely, if $\text{vert}_\lambda(F) \subset \text{vert}_\lambda(F')$, then $\mathbb{R}_{>0}(F - \lambda) = \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{\geq 0}(\mu - \lambda) \subset \sum_{\mu \in \text{vert}_\lambda(F')} \mathbb{R}_{\geq 0}(\mu - \lambda) = \overline{\mathbb{R}_{>0}(F' - \lambda)}$, whence the result follows. \square

Lemma 4.2. Let F be a face of $(\mathbb{A}, \mathcal{H})$ and $\lambda \in \text{vert}(F)$. Let $x \in \mathcal{F}_F$. Then $\pi_\lambda^{-1}(\{x\}) = x + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)$.

Proof. By Proposition 3.24, the facets of $\mathcal{P}(\lambda)$ are the \mathcal{F}_{F_1} for $F_1 \in \mathcal{F}_\lambda(F)$ of dimension 1. If $F_1 =]\lambda, \mu[$ is such a face, with $\mu \in \text{vert}(\mathbb{A})$, then by Proposition 3.24, we have $\delta_{\mathcal{F}_{F_1}} = \mathbb{R}_{\geq 0}(\mu - \lambda)$, with the notation of subsection 2.1. Using Proposition 2.3, we deduce $\pi_\lambda^{-1}(\{x\}) = x + \sum_{\mu \in \text{vert}_\lambda(F)} \mathbb{R}_{\geq 0}(\mu - \lambda)$. Then, the result follows from Lemma 4.1. \square

Lemma 4.3. *Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ and F be a proper subface of C which is not a vertex. Let $x \in \mathcal{F}_F$. Let $\lambda, \mu \in \text{vert}(F)$ and $\nu \in \text{vert}(C) \setminus \text{vert}(F)$. Then $\pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)} = \emptyset$.*

Proof. Let $F(\mu, \nu)$ be the smallest face of $(\mathbb{A}, \mathcal{H})$ dominating both μ and ν . Such face exists according to Lemma 3.6. By Proposition 3.27, we have:

$$\overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)} = \bigcap_{\nu' \in \text{vert}_\mu(F(\mu, \nu))} \overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu')}.$$

We have $F(\mu, \nu) \not\subseteq \overline{F}$ and thus Lemma 4.1 implies the existence of ν' belonging to $\text{vert}_\mu(F(\mu, \nu)) \setminus \text{vert}(F)$. Up to replacing ν by ν' , we may assume that μ and ν are adjacent.

Let $\mathcal{D}(\mu, \nu)$ be the closed half-space delimited by $\overline{\mathcal{P}(\mu)} \cap \overline{\mathcal{P}(\nu)}$ and containing $\overline{\mathcal{P}(\mu)}$. Let $V = x + \sum_{\tau \in \text{vert}_\mu(F)} \mathbb{R}(\tau - \mu) = x + \text{vect}(F - \mu)$. By Lemma 4.2, we have

$$\pi_\lambda^{-1}(\{x\}) = x + \mathbb{R}_{\geq 0}(\overline{F} - \lambda) \subset V.$$

Let $C^v = \mathbb{R}_{>0}(C - \mu)$ and $F^v = \mathbb{R}_{>0}(F - \mu)$. Then C^v (resp. F^v) is a chamber (resp. face) of $(\mathbb{A}, \mathcal{H}_\mu^0)$ by Proposition 3.12. Let Σ be the basis of Φ_μ associated with C^v . Write $\Sigma = \{\alpha_i, i \in I\}$ for some indexing set I satisfying $|I| = |\Sigma|$. Let $(\varpi_i) \in (\mathbb{A}^*)^I$ be the dual basis of $(\alpha_i)_{i \in I}$. Let $J = \{i \in I \mid \varpi_i \in \overline{F^v}\}$. Then $\mathbb{R}_{>0}(\nu - \mu)$ is a one dimensional face of C^v and thus $\nu - \mu \in \mathbb{R}_{>0}\varpi_i$, for some $i \in I \setminus J$. Then $\mathcal{D}(\mu, \nu) - \mu$ is the half-space of \mathbb{A} containing $\mathcal{P}_{\Phi_\mu, \mathbf{b}_C - \mu}^0$ and delimited by the support of $W_{\mathbb{R}_{>0}\varpi_i}^v \cdot (\mathbf{b}_C - \mu)$, where $W_{\mathbb{R}_{>0}\varpi_i}^v$ is the fixator of $\mathbb{R}_{>0}\varpi_i$ in $W_{\Phi_\mu}^v$. This half-space is denoted \mathcal{D}_i in Lemma 2.28. Therefore by Lemma 2.28, we have $\mathcal{D}(\mu, \nu) - \mu \supset (V - \mu) \cap (\overline{\mathcal{P}(\mu)} - \mu)$ and hence $\mathcal{D}(\mu, \nu) \supset V \cap \overline{\mathcal{P}(\mu)} \supset \pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu)}$, which proves the lemma, since $\mathcal{D}(\mu, \nu) \cap \overline{\mathcal{P}(\nu)}$ is empty. \square

Lemma 4.4. *Let $\lambda \in \text{vert}(\mathbb{A})$ and F be a face of $(\mathbb{A}, \mathcal{H})$ dominating λ . Let $x \in F$. Then $\pi_\lambda(x) \in F$.*

Proof. We have $x - \lambda \in \mathbb{R}_{>0}(F - \lambda)$. Using Lemma 2.24, Eq. (3.10) and Proposition 3.12 we deduce $\pi_{\overline{\mathcal{P}(\lambda)} - \lambda}(x - \lambda) = \pi_\lambda(x) - \lambda \in \mathbb{R}_{>0}(F - \lambda)$. Consequently, $\pi_\lambda(x) \in \lambda + \mathbb{R}_{>0}(F - \lambda)$.

Let F' be the face of $(\mathbb{A}, \mathcal{H})$ containing $\pi_\lambda(x)$. By Lemma 3.23, F' dominates λ . Therefore $\pi_\lambda(x) \in F' \subset \lambda + \mathbb{R}_{>0}(F' - \lambda)$. We deduce $\mathbb{R}_{>0}(F - \lambda) \cap \mathbb{R}_{>0}(F' - \lambda) \neq \emptyset$. As the faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ partition \mathbb{A} , we deduce $\mathbb{R}_{>0}(F - \lambda) = \lambda + \mathbb{R}_{>0}(F' - \lambda)$ and thus $F = F'$, by Proposition 3.12. \square

Lemma 4.5. *Let $\lambda \in \text{vert}(\mathbb{A})$, C be an alcove of \mathbb{A} dominating λ and $\mu \in \text{vert}(C) \setminus \{\lambda\}$. Then $\pi_\lambda(\overline{\mathcal{P}(\mu)} \cap \overline{C}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$.*

Proof. Let $x \in \text{Fr}(\overline{\mathcal{P}(\lambda)}) \cap \overline{C}$. Let $F = F(x)$ be the face dominating λ and such that $x \in \mathcal{F}_F$. Then by Proposition 3.27, $\{\nu \in \text{vert}(\mathbb{A}) \mid x \in \overline{\mathcal{P}(\nu)}\} = \text{vert}(F)$. By contradiction, we make the following assumption:

$$\exists \mu' \in \text{vert}(C) \setminus \text{vert}(F) \mid (\pi_\lambda)^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu')} \cap \overline{C} \neq \emptyset. \quad (4.1)$$

Let $\mu' \in \text{vert}(C) \setminus \text{vert}(F)$ be such that there exists $y \in \pi_\lambda^{-1}(\{x\}) \cap \overline{\mathcal{P}(\mu')} \cap \overline{C}$. By Proposition 3.27, $x \notin \overline{\mathcal{P}(\mu')}$ and thus $x \neq y$. Let $\gamma : [0, 1] \rightarrow [x, y]$ be the affine parametrization such that $\gamma(0) = x$ and $\gamma(1) = y$. For $t \in]0, 1[$, we have $|y - \gamma(t)| < |y - x| = |y - \pi_\lambda(y)|$ and thus $\gamma(t) \notin \overline{\mathcal{P}(\lambda)}$ for $t \in]0, 1[$.

Let $E = \{t \in [0, 1] \mid \exists \mu'' \in \text{vert}(C) \setminus \text{vert}(F), \gamma(t) \in \overline{\mathcal{P}(\mu'')}\}$. Then E is non-empty by Eq. (4.1). Let $t_0 = \inf E$. Then $\gamma(t_0) \in \overline{\mathcal{P}(\mu'')}$ for some $\mu'' \in \text{vert}(C) \setminus \text{vert}(F)$, since the

$\overline{\mathcal{P}(\tau)}$ are closed, for $\tau \in \text{vert}(\mathbb{A})$. Therefore $t_0 = \min E$ and $t_0 > 0$. By Theorem 3.28 and Lemma 3.23, we have $\gamma([0, t_0]) \subset \bigcup_{\nu \in \text{vert}(F)} \overline{\mathcal{P}(\nu)}$. Let $\nu \in \text{vert}(F)$ be such that for some $\epsilon > 0$, we have:

$$\gamma(t) \in \overline{\mathcal{P}(\nu)}, \quad \forall t \in]t_0 - \epsilon, t_0[.$$

Then $\gamma(t_0) \in \overline{\mathcal{P}(\nu)} \cap \overline{\mathcal{P}(\mu')}$. By Lemma 4.2, $\pi_\lambda^{-1}(\{x\})$ is convex and contains $\gamma(t_0)$. This contradicts Lemma 4.3. Therefore, Eq. (4.1) is wrong. In other words,

$$\forall x \in \overline{C} \cap \overline{\mathcal{P}(\lambda)}, (\pi_\lambda)^{-1}(\{x\}) \cap \overline{C} \subset \overline{C} \setminus \bigcup_{\mu' \in \text{vert}(C) \setminus \text{vert}(F(x))} \overline{\mathcal{P}(\mu')}. \quad (4.2)$$

Let now $y \in \overline{\mathcal{P}(\mu)} \cap \overline{C}$. Then by Lemma 4.4, $x := \pi_\lambda(y) \in \overline{C}$. Let $F \in \mathcal{F}_\lambda$ be such that $x \in \mathcal{F}_F$. Then by Eq. (4.2), $y \in \overline{C} \setminus \bigcup_{\mu' \in \text{vert}(C) \setminus \text{vert}(F)} \overline{\mathcal{P}(\mu')}$ and thus $\mu \in \text{vert}(F)$. Therefore $\overline{\mathcal{P}(\mu)} \supset \overline{\mathcal{F}_F}$, which completes the proof of the lemma. \square

Lemma 4.6. *Let $C_1 \in \text{Alc}(\mathcal{H})$ and $\lambda, \mu \in \text{vert}(C_1)$. Then:*

$$\bigcup_{C \in \text{Alc}_\mu(\mathcal{H})} \overline{C} \subset \bigcup_{C \in \text{Alc}_{\lambda, \mu}(\mathcal{H})} \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda),$$

where $\text{Alc}_{\lambda, \mu}(\mathcal{H})$ is the set of alcoves of $(\mathbb{A}, \mathcal{H})$ dominating λ and μ .

Proof. Let $E = \bigcup_{F \in \mathcal{F}_{\lambda, \mu}} \mathbb{R}_{> 0}(F - \lambda)$, where $\mathcal{F}_{\lambda, \mu}$ is the set of faces dominating λ and μ . Then $\lambda + \overline{E} \subset \lambda + \bigcup_{C \in \text{Alc}_{\lambda, \mu}(\mathcal{H})} \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$. By Proposition 3.12 and Lemma 2.17, there exists $X \subset \Phi_\lambda$ such that $E = \bigcap_{\alpha \in X} \alpha^{-1}(\mathbb{R}_{> 0})$. As E is a cone containing $]0, \mu - \lambda[$, it contains $\mu - \lambda$. Therefore $\lambda + E$ is open and contains μ . Consequently, $\lambda + E$ contains a point of each alcove dominating μ . Moreover $\lambda + \overline{E} = \bigcap_{\alpha \in X} \alpha^{-1}(\mathbb{R}_{\geq \alpha(\lambda)})$ and thus it is enclosed, which proves that it contains $\bigcup_{C \in \text{Alc}_\mu(\mathcal{H})} \overline{C}$. \square

Lemma 4.7. *Let $C \in \text{Alc}(\mathcal{H})$ and $\lambda \in \text{vert}(C)$. Let E' be the union of the faces of C dominating λ and $x \in E'$. Let $y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$. Then $[x, y] \cap E'$ contains a neighbourhood of x in $[x, y]$.*

Proof. Let E be the union of the faces of $(\mathbb{A}, \mathcal{H})$ dominating λ . Then E is open by Lemma 3.17. Therefore there exists a face F dominating λ and $\epsilon > 0$ such that $F \supset]x, x + \epsilon(y - x)[$. We also have $\lambda + \mathbb{R}_{> 0}(F - \lambda) \supset F$. Moreover $x, y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$ and thus it contains $]x, x + \epsilon(y - x)[$. The closed chamber $\overline{C}^v := \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$ is the union of the faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ dominated by \overline{C}^v . Thus by Proposition 3.12, we have

$$\overline{C}^v = \bigsqcup_{F_1 \in \mathcal{F}_\lambda(C)} \mathbb{R}_{> 0}(F_1 - \lambda),$$

where $\mathcal{F}_\lambda(C)$ is the set of subfaces of C dominating λ . Therefore there exists $F_1 \in \mathcal{F}_\lambda(C)$ such that $\mathbb{R}_{> 0}(F - \lambda) \cap \mathbb{R}_{> 0}(F_1 - \lambda)$ is non-empty, and as the faces of $(\mathbb{A}, \mathcal{H}_\lambda^0)$ partition \mathbb{A} , we deduce that $F = F_1$. But $F_1 \subset E'$ and the lemma follows. \square

Proposition 4.8. *Let $\lambda, \mu \in \text{vert}(\mathbb{A})$ be such that there exists an alcove of $(\mathbb{A}, \mathcal{H})$ dominating λ and μ . Then $\pi_\lambda(\overline{\mathcal{P}(\mu)}) \subset \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$.*

Proof. Let $y \in \overline{\mathcal{P}(\mu)}$. By Lemma 3.23 and Lemma 4.6, there exists an alcove C of $(\mathbb{A}, \mathcal{H})$ dominating λ and μ and such that $y \in \lambda + \mathbb{R}_{\geq 0}(\overline{C} - \lambda)$. Let $\mathcal{Q} = \overline{C} \cap \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$. Let $x = \pi_\mathcal{Q}(y) \in \mathcal{Q}$. By Lemmas 3.23 and 4.7, there exists $z \in]x, y] \cap \overline{C}$. By Eq. (2.1), we have $\pi_\mathcal{Q}(z) = \pi_\mathcal{Q}(y) = x$.

By Lemmas 4.4 and 4.5, we have $\pi_\lambda(z) \in \mathcal{Q}$ and by definition $|a - z| \geq |a - \pi_\lambda(z)|$ for all $a \in \overline{\mathcal{P}(\lambda)}$. As $\mathcal{Q} \subset \overline{\mathcal{P}(\lambda)}$, we deduce $\pi_\lambda(z) = \pi_\mathcal{Q}(z)$. By Eq. (2.1), we have $\pi_\lambda(z) = \pi_\lambda(y)$. Therefore $\pi_\lambda(y) \in \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}$, which proves the proposition. \square

4.2 Tessellation of apartments by thickened polytopes

Recall that, for $\eta \in [0, 1]$ and $\lambda \in \text{vert}(\mathbb{A})$, we have the homothety $h_{\lambda, \eta} : \mathbb{A} \rightarrow \mathbb{A}$ given by $h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda)$, for $x \in \mathbb{A}$. We write $\pi_{\lambda, \eta}$ instead of the orthogonal projection $\pi_{h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})}$ on $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)})$.

Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ dominating λ . Then by (3.10), we have $\overline{\mathcal{P}(\lambda)} = \lambda + \overline{\mathcal{P}_{-\lambda + \mathbf{b}_C, \Phi_\lambda}^0}$. Therefore

$$h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)}) = h_{\lambda, \eta}(\lambda) + \eta \text{conv}(W_{\Phi_\lambda}^v \cdot (-\lambda + \mathbf{b}_C)) = \lambda + \overline{\mathcal{P}_{\eta(-\lambda + \mathbf{b}_C), \Phi_\lambda}^0}. \quad (4.3)$$

Remark 4.9. Let $F \in \mathcal{F}$ and $a \in \mathcal{F}_F$. Then similarly as in Lemma 4.2, we have

$$(\pi_{\lambda, \eta})^{-1}(h_{\lambda, \eta}(a)) = h_{\lambda, \eta}(a) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda).$$

Lemma 4.10. Let F be a face of $(\mathbb{A}, \mathcal{H})$, $\lambda \in \text{vert}(F)$ and $\eta \in]0, 1]$. Then $\pi_{\lambda, \eta}(F) \subset F$ and $\pi_{\lambda, \eta}(\overline{F}) \subset \overline{F}$.

Proof. Let $x \in F$. Then $x \in \lambda + \mathbb{R}_{>0}(F - \lambda)$. By Lemma 2.24 and Eq. (4.3), $\pi_{\lambda, \eta}(x) \in \lambda + \mathbb{R}_{>0}(F - \lambda)$. Let E be the star of λ . By Lemma 3.23, since $h_{\lambda, \eta}$ stabilizes E , we have $h_{\lambda, \eta}(\overline{\mathcal{P}(\lambda)}) \subset E$. Therefore by Lemma 3.19, we have $\pi_{\lambda, \eta}(x) \in E \cap (\lambda + \mathbb{R}_{>0}(F - \lambda)) = F$. As $\pi_{\lambda, \eta}$ is continuous (by [HUL12, A. (3.1.6)]), we deduce the result. \square

Proposition 4.11. Let F be a face of $(\mathbb{A}, \mathcal{H})$. Then

$$F = \bigcap_{\lambda \in \text{vert}(F)} (\lambda + \mathbb{R}_{>0}(F - \lambda)).$$

Proof. Let $J = \text{vert}(F)$. Set $E = \bigcap_{\lambda \in J} (\lambda + \mathbb{R}_{>0}(F - \lambda))$. As $F \subset \lambda + \mathbb{R}_{>0}(F - \lambda)$ for $\lambda \in J$, we have $F \subset E$.

We now prove the converse inclusion. By Lemma 4.1, we have:

$$E = \bigcap_{\lambda \in J} \left(\lambda + \bigoplus_{\mu \in J} \mathbb{R}_{>0}(\mu - \lambda) \right).$$

We first assume that Φ is irreducible. Then by [Bou81, V 3.9 Proposition 8], C is a simplex and thus for every $\lambda \in \text{vert}(C)$, $(\mu - \lambda)_{\mu \in \text{vert}(C) \setminus \{\lambda\}}$ is a basis of \mathbb{A} . Fix $\lambda \in J$. Up to changing the 0 of the affine space \mathbb{A} , we may assume $\lambda = 0$. Let $x \in E$. Then we can write $x = \sum_{\mu \in J \setminus \{0\}} t_\mu \mu$, where $(t_\mu) \in (\mathbb{R}_{>0})^{J \setminus \{0\}}$. Let $\nu \in J$. Then $x \in \nu + \mathbb{R}_{>0}(-\nu) + \bigoplus_{\mu \in J \setminus \{0\}} \mathbb{R}_{>0}(\mu - \nu)$, so we can write $x = \nu + t'_0(-\nu) + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu(\mu - \nu)$ for some $t'_0 \in \mathbb{R}$ and $(t'_\mu) \in (\mathbb{R}_{>0})^{J \setminus \{\nu\}}$. Thus we have

$$(1 - t'_0 - \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu) \nu + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu \mu = \sum_{\mu \in J \setminus \{0\}} t_\mu \mu.$$

Therefore $t'_\mu = t_\mu$ for $\mu \in J \setminus \{0, \nu\}$, $t_\nu < 1$ and $\sum_{\mu \in J \setminus \{0\}} t_\mu = (1 - t'_0 - \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu) + \sum_{\mu \in J \setminus \{0, \nu\}} t'_\mu < 1$. This proves that $E \subset \text{Int}_r(\text{conv}(J)) = F$ and thus when Φ is irreducible, we have $E = F$.

We no longer assume the irreducibility of Φ . In the notation of Eq. (2.4), with Φ instead of Ψ , we write $\mathbb{A} = \mathbb{A}_1 \times \dots \times \mathbb{A}_\ell$. Let $x = (x_i)_{i \in [1, \ell]} \in \mathbb{A}$ and $i \in [1, \ell]$. We have $F = F_1 \times \dots \times F_\ell$, where $F_i \in \mathcal{F}(\mathbb{A}_i, \mathcal{H}_i)$ for $i \in [1, \ell]$.

We decompose J as $J = J_1 \times \dots \times J_\ell$, where $J_i = \text{vert}(F_i)$ for $i \in [1, \ell]$. Let $x \in E$. Then:

$$\begin{aligned} x &\in \bigcap_{\lambda \in J_i} \left(\prod_{j=1}^{i-1} \mathbb{A}_j \times (\lambda + \sum_{\mu \in J_i \setminus \{\lambda\}} \mathbb{R}_{>0}(\mu - \lambda)) \times \prod_{j=i+1}^{\ell} \mathbb{A}_j \right) \\ &= \prod_{j=1}^{i-1} \mathbb{A}_j \times \bigcap_{\mu \in J_i \setminus \{\lambda\}} \left(\lambda + \sum_{\mu \in J_i} \mathbb{R}_{>0}(\mu - \lambda) \right) \times \prod_{j=i+1}^{\ell} \mathbb{A}_j \end{aligned}$$

By the first case we deduce $x_i \in \text{Int}_r(\text{conv}(J_i))$. Consequently $x \in \prod_{i=1}^{\ell} \text{Int}_r(\text{conv}(J_i)) = \text{Int}_r(\text{conv}(J))$, which completes the proof of the proposition. \square

Lemma 4.12. *Let $\mu \in \text{vert}(\mathbb{A})$, $\eta \in [0, 1]$ and $x \in \mathbb{A}$. Let $F \in \mathcal{F}(\mathcal{H})$ be such that $\pi_\mu(x) \in \mathcal{F}_F$. Let $\lambda \in \text{vert}(F)$. Then $\pi_{\mu,\eta}(h_{\lambda,\eta}(x)) = h_{\mu,\eta}(\pi_\mu(x))$.*

Proof. Let $a = \pi_\mu(x)$. By Lemma 4.2, we have $x - a \in \mathbb{R}_{\geq 0}(\overline{F} - \mu)$. We have:

$$h_{\lambda,\eta}(x) = h_{\mu,\eta}(a) + (1 - \eta)(\lambda - \mu) + \eta(x - a) \in h_{\mu,\eta}(a) + \mathbb{R}_{\geq 0}(\overline{F} - \mu),$$

and we conclude with Remark 4.9. \square

Definition 4.13. *For $x \in \mathbb{A}$, we denote by $F_{\mathcal{P}}(x)$ the face of $(\mathbb{A}, \mathcal{H})$ such that $x \in \mathcal{F}_{F_{\mathcal{P}}(x)}$, which is well-defined by Theorem 3.28. For $\eta \in [0, 1]$, we set $F_\eta(x) = \text{Int}_r(\text{conv}(\{h_{\lambda,\eta}(x) \mid \lambda \in \text{vert}(F_{\mathcal{P}}(x))\}))$ and $\overline{F}_\eta(x) = \text{conv}(\{h_{\lambda,\eta}(x) \mid \lambda \in \text{vert}(F_{\mathcal{P}}(x))\})$ (see Figures 4.2 and 4.3). We have:*

$$\overline{F}_\eta(x) = \eta x + (1 - \eta)\text{conv}(\text{vert}(F_{\mathcal{P}}(x))) = \eta x + (1 - \eta)\overline{F_{\mathcal{P}}(x)}. \quad (4.4)$$

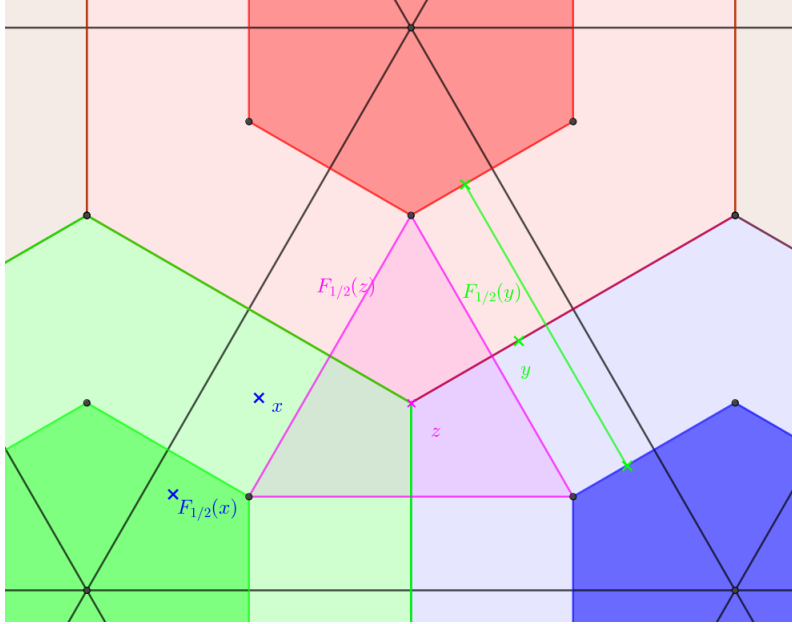


Figure 4.2: Pictures of F_η in the case of A_2 .

Lemma 4.14. *Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ and $x \in \overline{C}$. Then:*

1. *for all $y \in F_\eta(x)$, for all $\mu \in \text{vert}(C)$, we have $\pi_{\mu,\eta}(y) = h_{\mu,\eta}(\pi_\mu(x))$,*
2. *for all $\mu \in \text{vert}(C)$, for all $y \in F_\eta(x)$, we have $\text{vert}(F_{\mathcal{P}}(x)) \subset \{\nu \in \text{vert}(C) \mid \pi_{\mu,\eta}(y) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\}$ and this is an equality if and only if $\mu \in \text{vert}(F_{\mathcal{P}}(x))$.*

Proof. By Lemma 3.23, we have $\text{vert}(F_{\mathcal{P}}(x)) \subset \text{vert}(C)$. Let $\mu \in \text{vert}(C)$. By Propositions 3.27 and 4.8, we have

$$\pi_\mu(x) \in \bigcap_{\nu \in \text{vert}(F_{\mathcal{P}}(x)) \cup \{\mu\}} \overline{\mathcal{P}(\nu)}. \quad (4.5)$$

Let $\lambda \in \text{vert}(F_{\mathcal{P}}(x))$. By Proposition 3.27, we have $\pi_\mu(x) \in \overline{\mathcal{F}_{F_{\mathcal{P}}(x)}}$. Hence, if F' is the face such that $\pi_\mu(x) \in \mathcal{F}_{F'}$, we deduce that $\text{vert}(F_{\mathcal{P}}(x)) \subset F'$. By Lemma 4.12, we

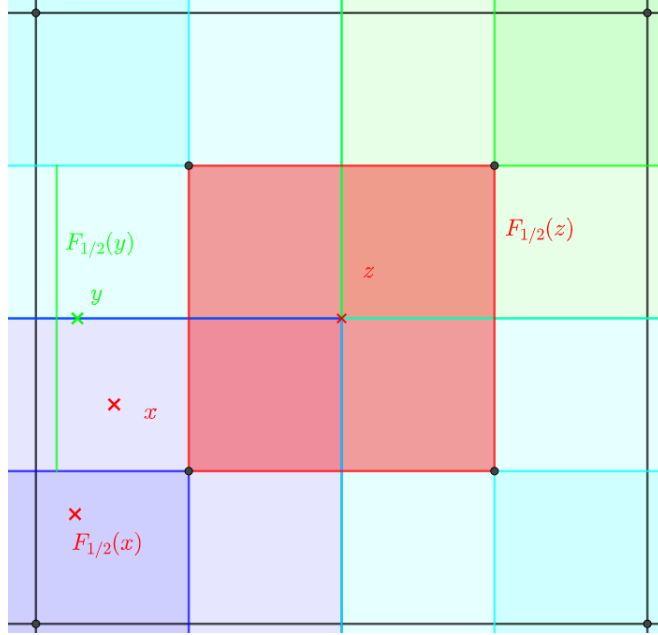


Figure 4.3: Pictures of F_η in the case of $A_1 \times A_1$.

get $\pi_{\mu,\eta}(h_{\lambda,\eta}(x)) = h_{\mu,\eta}(\pi_\mu(x))$. By Eq. (2.1), $\pi_{\mu,\eta}^{-1}(h_{\mu,\eta}(\pi_\mu(x)))$ is convex. Therefore $\pi_{\mu,\eta}(F_\eta(x)) = \{h_{\mu,\eta}(\pi_\mu(x))\}$.

Let $\mu \in \text{vert}(C)$. Then

$$\begin{aligned} \{\nu \in \text{vert}(C) \mid \pi_{\mu,\eta}(y) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} &= \{\nu \in \text{vert}(C) \mid h_{\mu,\eta}(\pi_\mu(x)) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid \pi_\mu(x) \in \overline{\mathcal{P}(\nu)}\}. \end{aligned}$$

We conclude with Eq. (4.5), since $\pi_\mu(x) = x$, for $\mu \in \text{vert}(F_{\mathcal{P}}(x))$. \square

Lemma 4.15. *Let $F \in \mathcal{F}(\mathcal{H})$. Let $f \in \mathcal{F}_F$ and C be an alcove dominating f . Let $\lambda \in \text{vert}(F)$ and $x \in (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap \overline{C}$. Then*

$$\pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)$$

and

$$\{\mu \in \text{vert}(\mathbb{A}) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\mu)})\} \subset \text{vert}(F).$$

Proof. We have

$$\begin{aligned} &h_{\lambda,\eta}(f) + \text{vect}(F - \lambda) \cap \overline{C} \\ &\subset (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap (\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda})) \\ &= h_{\lambda,\eta}(f + \text{vect}(F - \lambda)) \cap h_{\lambda,\eta}(\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda})) \\ &= h_{\lambda,\eta}((f + \text{vect}(F - \lambda)) \cap (\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda}))). \end{aligned}$$

By Lemma 4.12 applied to $\mu = \lambda$, we have $\pi_{\lambda,\eta} \circ h_{\lambda,\eta} = h_{\lambda,\eta} \circ \pi_\lambda$. Therefore if $x \in (h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)) \cap \overline{C}$, we have

$$\pi_{\lambda,\eta}(x) = h_{\lambda,\eta}(\pi_\lambda(y)),$$

for some $y \in (f + \text{vect}(F - \lambda)) \cap (\lambda + \mathbb{R}_{\geq 0}(\overline{C - \lambda}))$. Using Proposition 2.33, we have $\pi_\lambda(y) \in f + \text{vect}(F - \lambda)$ and we deduce $\pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(f) + \text{vect}(F - \lambda)$.

Now we have $\pi_\lambda(y) = \pi_{\overline{\mathcal{P}(\lambda)}}(y) = \pi_{\overline{\mathcal{P}(\lambda)} - \lambda}(y - \lambda) + \lambda$. By Eq. (3.10), we have $\overline{\mathcal{P}(\lambda)} - \lambda = \overline{\mathcal{P}_{\mathbf{b}_{C-\lambda}}^0}$. By Proposition 3.24 and Proposition 3.12, the map $\mathcal{F}_{F_1} \mapsto \mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0$ is a bijection between the faces of $\overline{\mathcal{P}(\lambda)}$ and those of $\overline{\mathcal{P}(\lambda)} - \lambda$. Hence we have $\mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0 = \mathcal{F}_{F_1} - \lambda$, for $F_1 \in \mathcal{F}_\lambda(\mathcal{H})$. By Proposition 2.33 (2), $\pi_{\overline{\mathcal{P}(\lambda)} - \lambda}(y - \lambda) \in \mathcal{F}_{\mathbb{R}_{>0}(F_1-\lambda)}^0$, for some subface F_1 of F dominating λ . Therefore $\pi_\lambda(y) \in \mathcal{F}_{F_1}$, and we get:

$$\{\mu \in \text{vert}(\mathbb{A}) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\mu)})\} = \{\mu \in \text{vert}(\mathbb{A}) \mid \pi_\lambda(y) \in \overline{\mathcal{P}(\mu)}\} = \text{vert}(F_1),$$

which proves the lemma. \square

Lemma 4.16. *Let $\eta \in]0, 1]$. Let $C \in \text{Alc}(\mathcal{H})$ and $x \in \overline{C}$. For $\lambda \in \text{vert}(C)$, set $J_\lambda := J_{\lambda,\eta}(x) := \{\nu \in \text{vert}(C) \mid \pi_{\lambda,\eta}(x) \in h_{\lambda,\eta}(\overline{\mathcal{P}(\nu)})\}$. Let $N_C(x) = N_C := \min\{|J_\mu| \mid \mu \in \text{vert}(C)\}$ and:*

$$J = \{\lambda \in \text{vert}(C) \mid |J_\lambda| = N_C\}.$$

Then:

1. *There exists a face F of $(\mathbb{A}, \mathcal{H})$ such that $J = \text{vert}(F)$. There exists $f_x \in \overline{C} \cap \mathcal{F}_F$ such that $\pi_{\lambda,\eta}(x) = h_{\lambda,\eta}(f_x)$ for all $\lambda \in J$. More generally, for every $\mu \in \text{vert}(C)$, we have $\pi_{\mu,\eta}(x) = h_{\mu,\eta}(\pi_\mu(f_x))$.*
2. *For every $\mu \in \text{vert}(C)$, we have $J_\mu \supset J$; in the particular case when $\mu \in J$, there is an equality $J_\mu = J$.*
3. *We have $x \in \overline{F_\eta(f_x)}$.*

Proof. Let $\lambda_0 \in J$. Let F be the face of $(\mathbb{A}, \mathcal{H})$ dominating λ_0 and such that $\pi_{\lambda_0,\eta}(x) \in h_{\lambda_0,\eta}(\mathcal{F}_F)$. By Lemma 4.10, we have $\pi_{\lambda_0,\eta}(x) \in \overline{C}$. Let F_1 be the face of \overline{C} containing $\pi_{\lambda_0,\eta}(x)$. By Lemma 3.23, we have that \mathcal{F}_F is contained in the star E of F . As $\lambda_0 \in \text{vert}(F)$ and $\eta \in]0, 1]$, the set E is stabilized by $h_{\lambda_0,\eta}$ and hence $h_{\lambda_0,\eta}(\mathcal{F}_F) \subset E$. Therefore F_1 dominates F .

Write $\pi_{\lambda_0,\eta}(x) = h_{\lambda_0,\eta}(f_x)$, with $f_x \in \mathcal{F}_F \cap \overline{C}$. We have:

$$\begin{aligned} J_{\lambda_0} &= \{\nu \in \text{vert}(C) \mid \pi_{\lambda_0,\eta}(x) \in h_{\lambda_0,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid h_{\lambda_0,\eta}(f_x) \in h_{\lambda_0,\eta}(\overline{\mathcal{P}(\nu)})\} \\ &= \{\nu \in \text{vert}(C) \mid f_x \in \overline{\mathcal{P}(\nu)}\} = \text{vert}(F), \end{aligned}$$

by Proposition 3.27. By Remark 4.9, we have

$$x \in h_{\lambda_0,\eta}(f_x) + \text{vect}(F - \lambda_0).$$

Let $\mu \in \text{vert}(F)$. Then

$$x \in h_{\mu,\eta}(f_x) + (1 - \eta)(\lambda_0 - \mu) + \text{vect}(F - \lambda_0) = h_{\mu,\eta}(f_x) + \text{vect}(F - \mu).$$

By Lemma 4.15, we have $\pi_{\mu,\eta}(x) \in h_{\mu,\eta}(f_x) + \text{vect}(F - \mu)$ and

$$J_\mu = \{\nu \in \text{vert}(\mathbb{A}) \mid \pi_{\mu,\eta}(x) \in h_{\mu,\eta}(\overline{\mathcal{P}(\nu)})\} \subset \text{vert}(F) = J_{\lambda_0}.$$

By minimality of J_{λ_0} , we deduce

$$J_\mu = J_{\lambda_0} = \text{vert}(F) \subset J. \tag{4.6}$$

By the previous computations and by Proposition 3.27, we have $h_{\mu,\eta}(f_x), \pi_{\mu,\eta}(x) \in h_{\mu,\eta}(\mathcal{F}_F) \cap (x + \text{vect}(F - \mu))$. It follows from Proposition 3.24 that $\text{dir}(\text{supp}(h_{\mu,\eta}(\mathcal{F}_F))) =$

$\text{dir}(\text{supp}(\mathcal{F}_F)) = \text{dir}(\text{supp}(F))^\perp$ and thus $h_{\mu,\eta}(\mathcal{F}_F) \cap (x + \text{vect}(F - \mu))$ is either empty or a singleton. Therefore

$$h_{\mu,\eta}(f_x) = \pi_{\mu,\eta}(x).$$

Let $\lambda \in \text{vert}(F)$. By Lemma 4.2, we have

$$x \in \pi_{\lambda,\eta}(x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda) = h_{\lambda,\eta}(f_x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda).$$

Using Proposition 4.11, we deduce that x belongs to:

$$\begin{aligned} \bigcap_{\lambda \in \text{vert}(F)} (h_{\lambda,\eta}(f_x) + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)) &= \eta f_x + (1 - \eta) \bigcap_{\lambda \in \text{vert}(F)} (\lambda + \mathbb{R}_{\geq 0}(\overline{F} - \lambda)) \\ &= \eta f_x + (1 - \eta) \overline{F}. \end{aligned}$$

Using (4.4) we deduce statement 3):

$$x \in \text{conv}(\{h_{\lambda,\eta}(f_x) \mid \lambda \in \text{vert}(F)\}). \quad (4.7)$$

Let $\mu \in \text{vert}(C)$. By Propositions 3.27 and 4.8, we have

$$\pi_\mu(f_x) \in \bigcap_{\nu \in \text{vert}(F) \cup \{\mu\}} \overline{\mathcal{P}(\nu)}. \quad (4.8)$$

Let F_2 be the face of $(\mathbb{A}, \mathcal{H})$ such that $\pi_\mu(f_x) \in \mathcal{F}_{F_2}$. Then by Proposition 3.27, we have $\text{vert}(F_2) \supset \text{vert}(F)$. Let $\lambda \in \text{vert}(F)$. By Lemma 4.12, we have $\pi_{\mu,\eta}(h_{\lambda,\eta}(f_x)) = h_{\mu,\eta}(\pi_\mu(f_x))$. By Eq. (2.1), $(\pi_{\mu,\eta})^{-1}(h_{\mu,\eta}(\pi_\mu(f_x)))$ is convex. Using Eq. (4.7) we deduce

$$\pi_{\mu,\eta}(x) = h_{\mu,\eta}(\pi_\mu(f_x)).$$

Using Eq. (4.8) we deduce that $J_\mu \supset \text{vert}(F) \cup \{\mu\} = J_{\lambda_0} \cup \{\mu\}$. Consequently for $\mu \in \text{vert}(C) \setminus J_{\lambda_0}$, we have $|J_\mu| \geq |J_{\lambda_0}| + 1 = N_C + 1$ and $\mu \notin J$. Therefore $J \subset J_{\lambda_0}$ and hence by (4.6), we have

$$J = J_{\lambda_0} = \text{vert}(F),$$

which completes the proof of the lemma. \square

Lemma 4.17. *Let $C \in \text{Alc}(\mathcal{H})$ and $x, x' \in \overline{C}$. Then $x = x'$ if and only if (for all $\lambda \in \text{vert}(C)$, $\pi_\lambda(x) = \pi_\lambda(x')$).*

Proof. Assume that $\pi_\lambda(x) = \pi_\lambda(x')$ for every $\lambda \in \text{vert}(C)$. For $\lambda \in \text{vert}(C)$, set

$$J_\lambda = \{\mu \in \text{vert}(C) \mid \pi_\lambda(x) \in \overline{\mathcal{P}(\lambda)} \cap \overline{\mathcal{P}(\mu)}\}.$$

Let $J = \{\lambda \in \text{vert}(C) \mid |J_\lambda| = N_C(x)\}$. By Lemma 4.16 applied with $\eta = 1$, we have $\{x, x'\} \subset \bigcap_{\lambda \in J} \overline{\mathcal{P}(\lambda)}$. Take $\lambda \in J$. Then $\pi_\lambda(x) = x = \pi_\lambda(x') = x'$. \square

Lemma 4.18. *Let $x \in \mathbb{A}$, $\eta \in]0, 1]$ and F be the face of $(\mathbb{A}, \mathcal{H})$ containing x . Then $\overline{F_\eta(x)} \subset F$.*

Proof. Let $F_1 \in \mathcal{F}(\mathcal{H})$ be such that $x \in \mathcal{F}_{F_1}$. Then by Lemma 3.23, we have $F_1 \subset \overline{F}$. For $\lambda \in \text{vert}(F_1)$, we thus have $]\lambda, x] \subset F$ and $h_{\lambda,\eta}(x) \in F$. Consequently $\overline{F_\eta(x)} = \text{conv}(\{h_{\lambda,\eta}(x) \mid \lambda \in \text{vert}(F_1)\}) \subset F$. \square

Theorem 4.19. *Let \mathbb{A} be a finite dimensional vector space and Φ be a finite root system in \mathbb{A} . Let $\mathcal{H} = \{\alpha^{-1}(\{k\}) \mid \alpha \in \Phi, k \in \mathbb{Z}\}$. We choose a fundamental alcove C_0 of $(\mathbb{A}, \mathcal{H})$ and choose $\mathbf{b}_0 \in C_0$. If λ is a vertex of $(\mathbb{A}, \mathcal{H})$, we choose an alcove C_λ dominating λ and we set $\mathbf{b}_{C_\lambda} = w \cdot \mathbf{b}_0$ if w is the element of the affine Weyl group W^{aff} of $(\mathbb{A}, \mathcal{H})$ sending C_0 to C . We set $\overline{\mathcal{P}(\lambda)} = \text{conv}(W_\lambda^{\text{aff}} \cdot \mathbf{b}_{C_\lambda})$, where W_λ^{aff} is the fixator of λ in W^{aff} . If $x \in \mathbb{A}$, we define $F_{\mathcal{P}}(x)$ as the unique face of $(\mathbb{A}, \mathcal{H})$ such that $x \in \mathcal{F}_{F_{\mathcal{P}}(x)} := \text{Int}_r(\text{conv}(W_{F_{\mathcal{P}}(x)}^{\text{aff}} \cdot \mathbf{b}_C))$, where \mathbf{b}_C is an alcove dominating $F_{\mathcal{P}}(x)$ and $W_{F_{\mathcal{P}}(x)}^{\text{aff}}$ is the fixator of $F_{\mathcal{P}}(x)$ in W^{aff} . If $\eta \in]0, 1]$, we set $F_\eta(x) = \eta x + (1 - \eta)F_{\mathcal{P}}(x)$. Then we have:*

$$\forall \eta \in]0, 1], \mathbb{A} = \bigsqcup_{x \in \mathbb{A}} \overline{F_\eta(x)}.$$

Proof. Let C be an alcove of $(\mathbb{A}, \mathcal{H})$ and $y \in \overline{C}$. For $\lambda \in \text{vert}(C)$, set $J_{\lambda, \eta}(y) = \{\mu \in \text{vert}(C) \mid \pi_{\lambda, \eta}(y) \in h_{\lambda, \eta}(\overline{\mathcal{P}(\mu)})\}$. Let $N_{C, \eta} = \min\{|J_{\mu, \eta}(y)| \mid \mu \in \text{vert}(C)\}$, and

$$\tilde{J} = \{\lambda \in \text{vert}(C) \mid |J_{\lambda, \eta}(y)| = N_{C, \eta}\}.$$

By Lemma 4.16, there exists $f_y \in \text{Int}_r\left(\bigcap_{\nu \in \tilde{J}} \overline{\mathcal{P}(\nu)}\right)$ such that $y \in \overline{F_\eta(f_y)}$. Therefore $\bigcup_{x \in \overline{C}} \overline{F_\eta(x)} = \overline{C}$.

Let $x, x' \in \mathbb{A}$ be such that $\overline{F_\eta(x)} \cap \overline{F_\eta(x')} \neq \emptyset$ and let $y \in \overline{F_\eta(x)} \cap \overline{F_\eta(x')}$. Let F and F' be the faces of $(\mathbb{A}, \mathcal{H})$ containing x and x' respectively. Then by Lemma 4.18, $y \in F \cap F'$, which proves that $F = F'$. Let C be an alcove dominating F . By Lemma 4.14, we have $\pi_{\mu, \eta}(y) = h_{\mu, \eta}(\pi_\mu(x)) = h_{\mu, \eta}(\pi_\mu(x'))$ for every $\mu \in \text{vert}(C)$. Therefore $\pi_\mu(x) = \pi_\mu(x')$ for every $\mu \in \text{vert}(C)$ and $x = x'$ by Lemma 4.17. \square

Corollary 4.20. *Let $\eta \in]0, 1]$. We have:*

$$\mathbb{A} = \bigsqcup_{\lambda \in \text{vert}(\mathbb{A})} h_{\lambda, \eta}(\mathcal{P}(\lambda)) \sqcup \bigsqcup_{x \in \bigcup_{\lambda \in \text{vert}(\mathbb{A})} \text{Fr}(\mathcal{P}(\lambda))} \overline{F_\eta(x)}.$$

Remark 4.21. *1. Note that when $\eta = 0$, we have $F_0(\mathbf{b}_C) = \overline{C}$ for each alcove C . Therefore we still have $\mathbb{A} = \bigcup_{x \in \mathbb{A}} F_0(x)$, but this is no longer a disjoint union according to Lemma 3.25.*

2. Let F be a face of $(\mathbb{A}, \mathcal{H})$. Let $x \in \mathcal{F}_F$ and $\eta \in]0, 1[$. Then for every vertex λ of F , we have $F_\eta(x) = \eta x + (1 - \eta)F = h_{\lambda, \eta}(x) + (1 - \eta)(F - \lambda)$. Indeed, we have $h_{\lambda, \eta}(x) = \lambda + \eta(x - \lambda) = (1 - \eta)\lambda + \eta x$.

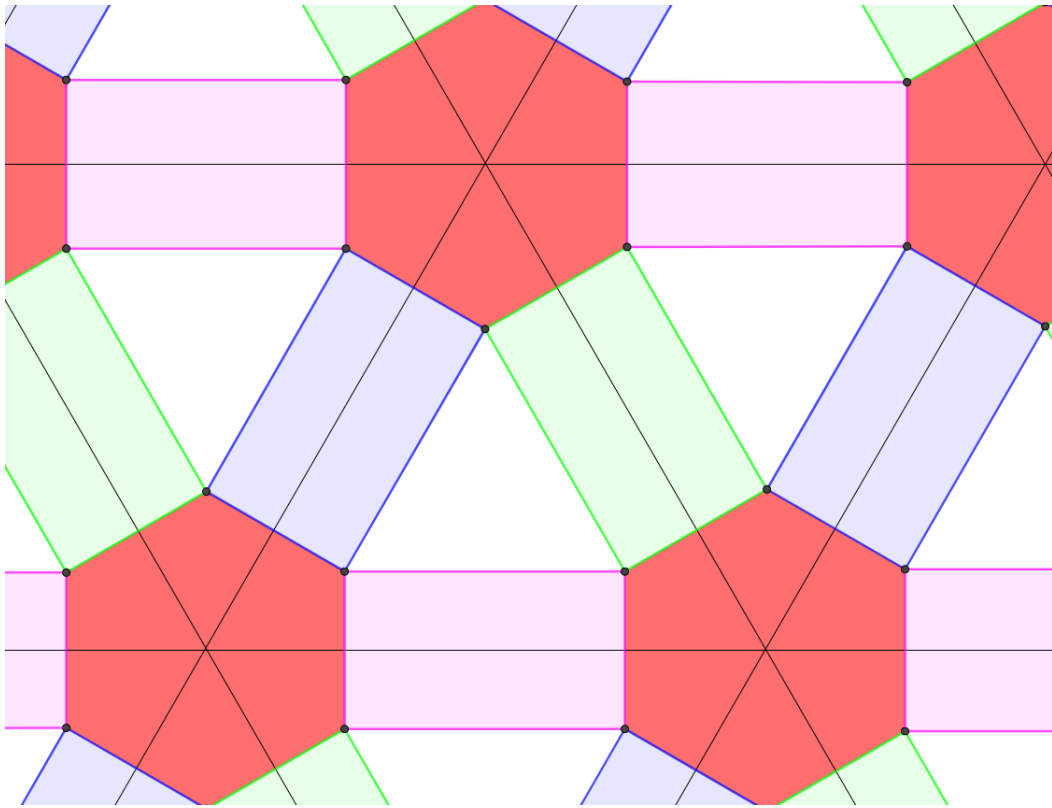


Figure 4.4: Illustration of Theorem 4.19 in the case of A_2 and $\eta = 1/2$.

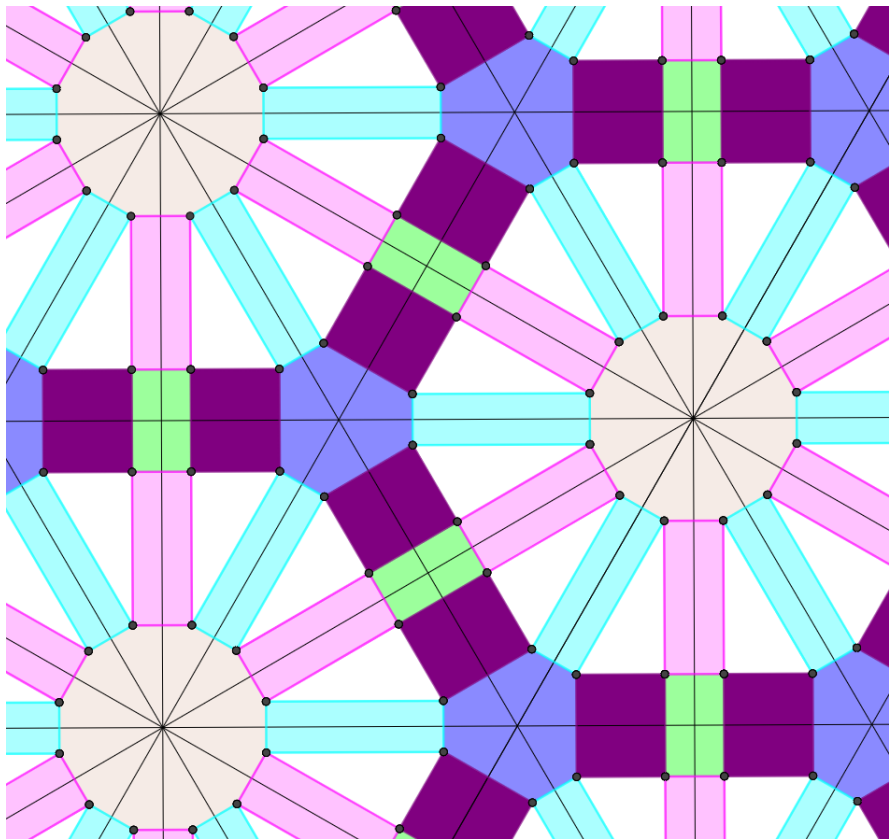


Figure 4.5: Illustration of Theorem 4.19 in the case of G_2 and $\eta = 1/2$.

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