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Hamiltonian Property Testing

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Abstract

Locality is a fundamental feature of many physical time evolutions. Assumptions on locality and related structural properties also underlie recently proposed procedures for learning an unknown Hamiltonian from access to the induced time evolution. However, no protocols to rigorously test whether an unknown Hamiltonian is in fact local were known. We investigate Hamiltonian locality testing as a property testing problem, where the task is to determine whether an unknown n -qubit Hamiltonian H is k -local or ε -far from all k -local Hamiltonians, given access to the time evolution along H . First, we emphasize the importance of the chosen distance measure: With respect to the operator norm, a worst-case distance measure, incoherent quantum locality testers require $\tilde{\Omega}(2^n)$ many time evolution queries and an expected total evolution time of $\tilde{\Omega}(2^n/\varepsilon)$, and even coherent testers need $\Omega(2^{n/2})$ many queries and $\Omega(2^{n/2}/\varepsilon)$ total evolution time. In contrast, when distances are measured according to the normalized Frobenius norm, corresponding to an average-case distance, we give a computationally efficient incoherent Hamiltonian locality testing algorithm with query complexity $\mathcal{O}(1/\varepsilon^4)$ and total evolution time $\mathcal{O}(1/\varepsilon^3)$, based on randomized measurements. In fact, our procedure can be used to simultaneously test a wide class of Hamiltonian properties beyond locality. Finally, we prove that learning a general Hamiltonian remains exponentially hard with this average-case distance, thereby establishing an exponential separation between Hamiltonian testing and learning. Our work initiates the study of property testing for quantum Hamiltonians, demonstrating that a broad class of Hamiltonian properties is efficiently testable even with limited quantum capabilities, and positioning Hamiltonian testing as an independent area of research alongside Hamiltonian learning.

1 Introduction

Time evolution of a quantum system according to the unitary group generated by some self-adjoint Hamiltonian is a crucial ingredient to quantum physics, going back to the early days of quantum mechanics [Sch26a; Sch26b; Sto32]. This makes extracting information about an unknown Hamiltonian from access to the corresponding time evolution a central task when it comes to understanding fundamental processes in physics. Moreover, given recent progress in experimental implementations of early quantum devices, the same task also gains technological relevance, for instance for benchmarking applications in quantum simulation and quantum computing.

In many physically relevant quantum systems, the Hamiltonian describing the time evolution is not an arbitrary self-adjoint operator but has additional structure. In particular, as physical processes often arise from local interactions, k -local Hamiltonians – Hamiltonians that can be written as a sum of terms that act non-trivially only on k out of the overall n many subsystems, often with $k = \mathcal{O}(\log(n))$ or $k = \mathcal{O}(1)$ – play an important role in modeling real-world systems. Additionally, some proposed quantum computing architectures also come with natural locality constraints, and are thus naturally described by (possibly even geometrically) local Hamiltonians. In addition to fundamental and practical relevance, locality also carries theoretical importance. For instance, a majority of recent works establishing rigorous guarantees for learning an unknown Hamiltonian from either its Gibbs state or from access to its time evolution (see Section 1.3 for an overview) relies on structural assumptions on the Hamiltonian that in particular require locality,

such as having a bounded-degree interaction graph. However, no protocols for testing whether an unknown Hamiltonian satisfies such assumptions were known.

In this work, we investigate the tasks of testing locality and more general properties of an unknown Hamiltonian in the framework of property testing. Concretely, given access to the time evolution according to an unknown Hamiltonian H , we aim to determine whether H is k -local or far from all k -local Hamiltonians. In fact, versions of this Hamiltonian locality testing task have already been proposed as interesting problems, albeit not studied, in [MW16; SY23]. We demonstrate that the feasibility of Hamiltonian locality testing crucially relies on how distances between Hamiltonians are measured. On the one hand, we establish hardness of locality testing with the distance measured by the operator norm. On the other hand, for the normalized Frobenius norm as distance measure, we give an efficient Hamiltonian locality tester. In fact, we show that our algorithm can be modified to efficiently test for a variety of Hamiltonian properties, specified by subsets of all possible Pauli strings. Finally, we highlight a crucial difference between Hamiltonian testing and learning: While we achieve efficient Hamiltonian property testing with respect to the normalized Frobenius norm, we show hardness of learning an arbitrary Hamiltonian with the same notion of distance.

1.1 Problem statement: Hamiltonian locality testing

Throughout, we consider the Pauli expansion $H = \sum_{P \in \mathbb{P}_n} \alpha_P P$ of an n -qubit Hamiltonian H , where $\mathbb{P}_n = \{\mathbb{I}, X, Y, Z\}^{\otimes n}$ is the set of n -qubit Paulis and where the $\alpha_P = \text{Tr}[HP]/2^n$ are the (real) Pauli basis coefficients of H . We call the Hamiltonian H k -local if $\alpha_P = 0$ holds for all $P \in \mathbb{P}_n$ with $|P| > k$. Here, $|P|$ denotes the weight of a Pauli string P , that is, the number of non-identity tensor factors in P . We phrase the problem of testing whether an unknown Hamiltonian is (at most) k -local as a property testing problem:

Definition 1.1 (Hamiltonian locality testing). *Given a locality parameter $1 \leq k \leq n$, a norm $\|\cdot\|$, and an accuracy parameter $\varepsilon \in (0, 1)$, the Hamiltonian k -locality testing problem, denoted as $\mathcal{T}_{\|\cdot\|}^{\text{loc}}(\varepsilon)$, is the following task: Given access to the time evolution according to an unknown Hamiltonian H , decide, with success probability $\geq 2/3$, whether*

(i) H is k -local, or

(ii) H is ε -far from being k -local, that is, $\|H - \tilde{H}\| \geq \varepsilon$ for all k -local Hamiltonians \tilde{H} .

If H satisfies neither (i) nor (ii), then any output of the tester is considered valid.

Here, we can use different norms $\|\cdot\|$ to measure the distance between two Hamiltonians. Motivated by operational interpretations as worst-case and average-case notions of distance, respectively, we focus on the Schatten ∞ -norm $\|\cdot\|_\infty$ (aka operator norm) and the normalized Schatten 2-norm $\frac{1}{\sqrt{2^n}} \|\cdot\|_2$ (aka normalized Frobenius norm).

In this work, we study how easy or hard it is to achieve Hamiltonian locality testing w.r.t. these different norms. On the one hand, we consider incoherent quantum algorithms, which can only perform measurements on single copies of time-evolved states, and which cannot interleave Hamiltonian time evolution with control operations. Here, we do, however, allow for adaptively chosen input states and measurements in the different experiments performed for testing. On the other hand, we also consider more general coherent quantum algorithms, describing the most general quantum experiments that can be performed when given access to an unknown Hamiltonian time evolution. Additionally, we may restrict the algorithms to have access to no or only few auxiliary qubits. See Section 2.4 for a detailed discussion of the kinds of quantum protocols that we consider. Our negative results cover both the incoherent and the coherent case. Our positive results are phrased in the framework of simple incoherent quantum testing algorithms. As such, they serve as examples of the power of simple-to-implement protocols for testing properties of a Hamiltonian, which may be feasible in short- or mid-term quantum devices.

1.2 Main results

An immediate approach towards solving the testing problem from Definition 1.1 is to (approximately) learn the unknown Hamiltonian to a sufficient accuracy and to then decide based on whether the learned hypothesis Hamiltonian has the desired property or not. Given that the unknown Hamiltonian H can be arbitrary,

instantiating this approach requires Hamiltonian learning protocols that work without any structural assumptions. Such procedures were recently proposed in [Car23; CW23]. However, as we discuss in more detail in Section 4.1, the naive “testing via learning” strategy with the protocols of [Car23; CW23] has an undesirable feature: When working with any norm except for the ℓ_∞ -norm on the coefficient vector $(\alpha_P)_P$, $\|\cdot\|_{\text{Pauli},\infty}$, it uses a number of queries to the unknown Hamiltonian and a total evolution time that both scale exponentially in n . This raises the question:

For which norms is it possible to efficiently test Hamiltonian locality?

We provide the first rigorous answers to this question. In our first result, we show that the locality testing task is hard with the Schatten ∞ -norm:

Theorem 1.2 (Hardness of Hamiltonian locality testing w.r.t. the operator norm – Informal). *For $k \leq \tilde{\mathcal{O}}(n)$, any ancilla-free, incoherent, adaptive quantum algorithm that solves the k -locality testing problem $\mathcal{T}_{\|\cdot\|_\infty}^{\text{loc}}(\varepsilon)$, even only under the additional promise that the unknown Hamiltonian H satisfies $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, has to make at least $N \geq \tilde{\Omega}(2^n)$ queries to the unknown Hamiltonian and has to use an expected total evolution time of at least $\mathbb{E}[T] \geq \tilde{\Omega}\left(\frac{2^n}{\varepsilon}\right)$. Even any coherent quantum algorithm achieving the same has to make at least $N \geq \Omega(2^{n/2})$ many queries and has to use a total evolution time of at least $T \geq \Omega\left(\frac{2^{n/2}}{\varepsilon}\right)$.*

Here and throughout, we use $\tilde{\Omega}$ to hide factors that are polylogarithmic in the leading order term. Theorem 1.2 says that testing k -locality is not possibly query-efficiently or with efficient evolution time when using the Schatten ∞ -norm as a distance measure. In fact, we show this for all Schatten p -norms, $p \geq 1$. With our next result, we demonstrate that this changes significantly when considering the normalized Schatten 2-norm instead. Then, we can achieve efficiency in terms of the number of queries, the total evolution time, and even with respect to the classical post-processing time. In particular, we give a procedure that tests whether an unknown Hamiltonian is k -local and that achieves this with a polynomial number of queries, a polynomial overall evolution time, and with polynomial classical post-processing time.

Theorem 1.3 (Efficient Hamiltonian locality testing w.r.t. normalized Frobenius norm – Informal). *Let $k \leq \tilde{\mathcal{O}}(n)$. When promised that the unknown Hamiltonian H satisfies $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, there is an ancilla-free, incoherent, non-adaptive quantum algorithm that solves the Hamiltonian k -locality testing problem $\mathcal{T}_{\frac{1}{\sqrt{2^n}}\|\cdot\|_2}^{\text{loc}}(\varepsilon)$ using $\mathcal{O}(\varepsilon^{-4})$ many queries to the unknown Hamiltonian, a total evolution time of $\mathcal{O}(\varepsilon^{-3})$, and a classical post-processing time of $\mathcal{O}\left(\frac{n^{k+3}}{\varepsilon^4}\right)$. Moreover, the testing algorithm uses only stabilizer states as inputs and stabilizer basis measurements at the output.*

In fact, while we state Theorem 1.3 in terms of locality testing, our procedure allows us to establish a more general Hamiltonian property testing result. In particular, we show: Let $S \subset \mathbb{P}_n$ be a subset with $|S| \leq \mathcal{O}(\text{poly}(n))$, let $\varepsilon \geq \Omega(1/\text{poly}(n))$. Then, we can efficiently test, even tolerantly, whether H consists only of Pauli terms in S or whether H is ε -far w.r.t. $\frac{1}{\sqrt{2^n}}\|\cdot\|_2$ from the set of all such Hamiltonians. More precisely, we can do so with $\mathcal{O}(1/\varepsilon^4) \leq \mathcal{O}(\text{poly}(n))$ many queries to H , with a total evolution time of $\mathcal{O}(1/\varepsilon^3) \leq \mathcal{O}(\text{poly}(n))$, and with a classical post-processing time of $\mathcal{O}(n^2|S|/\varepsilon^4) \leq \mathcal{O}(\text{poly}(n))$. Again, this testing procedure uses only stabilizer state inputs and stabilizer basis measurements at the output, both of which are efficiently implementable. Thus, by choosing a suitable set S , we can for example test whether an unknown Hamiltonian is exactly k -local, (at most or exactly) geometrically k -local, or whether it has a desired interaction graph.

Our algorithm achieving the guarantees in Theorem 1.3 and its extended version to more general Hamiltonian properties, specified by some subset S , are novel additions to the randomized measurement framework [Elb+22]. In particular, we inherit the “measure first, ask questions later” feature. That is, the data in our testing algorithm can be collected even without knowing the Hamiltonian property that is to be tested as long as an a priori bound on its size and the desired accuracy are known in advance; and once collected the data can be used to test multiple properties simultaneously. This then allows us to test properties such as whether a Hamiltonian has a sparse Pauli basis expansion or whether it is a low-intersection Hamiltonian.

In our final result, we highlight a large separation between Hamiltonian testing and learning. On the one hand, we have shown that normalizing the Frobenius norm makes it possible to efficiently test arbitrary

Hamiltonian properties. On the other hand, even after normalizing the distance measure, learning a general Hamiltonian from time evolution access remains hard:

Theorem 1.4 (Hardness of Hamiltonian learning w.r.t. normalized Frobenius norm – Informal). *Any (even coherent) quantum algorithm with a constant number of auxiliary qubits that, when given time evolution access to an arbitrary n -qubit Hamiltonian H , promised to satisfy $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, with success probability $\geq 2/3$, outputs (the classical description of) a Hamiltonian \hat{H} such that $\frac{1}{\sqrt{2^n}} \|H - \hat{H}\|_2 \leq \varepsilon$ has to make at least $\tilde{\Omega}(2^{2n})$ many queries to H . Any non-adaptive incoherent quantum algorithm achieving the same without auxiliary qubits has to use a total evolution time of at least $\tilde{\Omega}\left(\frac{2^{2n}}{\varepsilon}\right)$.*

Juxtaposing Theorem 1.3, and its extension to general properties, with Theorem 1.4, we see that the naive “testing via learning” approach fails for Hamiltonian property testing if we do not have prior promises on the structure of the unknown Hamiltonian. In fact, Hamiltonian locality testing, and even more general Hamiltonian property testing, is significantly easier than the infeasible task of general Hamiltonian learning, but requires approaches tailored specifically to testing.

1.3 Related work

Classical and quantum property testing. Since its origins [BLR93; RS96; GGR98], property testing has evolved into an important area of theoretical computer science, with connections to, among others, learning theory and probabilistically checkable proofs [Ron08; Gol17; BY22]. Among the plethora of classical property tasks, those of low-degree testing [Alo+03] and junta testing [Fis+04; Bla09] can be viewed as counterparts of quantum Hamiltonian locality testing. Similarly, testing for Fourier sparsity of a Boolean function [Gop+11] can be viewed as a classical version of testing whether a Hamiltonian has a sparse Pauli basis expansion. More recently, the field of quantum property testing [MW16] has emerged. It includes a long line of works on testing properties of classical objects from quantum data access [Deu85; DJ92; Sim97; AS07; Buh+08; Cha+10; ACL11; BHH11; HA11; AA18; Amb+; GL20]; investigations into testing properties of quantum states [HM13; OW15; HLM; Car+17; BO20; GNW21; SW22; Gre+23]; proofs of proximity for unitary properties [Dal+22]; unitary and channel versions of junta testing [CNY23; BY23]; unitary property testing more broadly [LW22; SY23]; and Hamiltonian symmetry testing [LW22].

Hamiltonian learning. One way to infer properties of a Hamiltonian clearly is to learn the coefficients of the Hamiltonian. There has been a lot of work on this topic, hence the references below should not be understood as a complete review of the field. For our purposes, there are two approaches to Hamiltonian learning. In the first, one has access to the unitary dynamics of the Hamiltonian of interest [SLP11; BAL19; Zub+21; HKT22; Wil+22; Yu+23; Car23; DOS23; Hua+23; CW23; Li+23; Möb+23; Sti+24; GCC24]. We are allowed to prepare appropriate initial states, choose how long the system should evolve, and perform measurements of our choosing afterwards. In the second, our aim is to learn the Hamiltonian from copies of Gibbs states, i.e., thermal equilibrium states [Ans+21; HKT22; RS23; Ono+23; Bak+23; GCC24]. For these algorithms to be efficient, it is usually assumed that the Hamiltonian to be learned is local. Our results can therefore be seen as complementary to the above protocols for Hamiltonian learning: we first determine the locality of the Hamiltonian using our results and then run an appropriate learning protocol.

Coherent vs. incoherent quantum learning. Quantum-enhanced learning algorithms can use advanced quantum processing, such as multi-copy measurements and coherent long-time evolutions with interleaving control operations, to achieve a quantum advantage over conventional algorithms. Recent work has investigated such advantages in learning and testing quantum states [BCL20; HKP21; Che+22; Hua+22; Aru+23; Che+23b; Faw+23a] as well as unitaries and quantum channels [ACQ22; Che+22; Hua+22; Car23; Che+23a; Ouf23; FOS23; Faw+23b], and in learning Hamiltonians [Hua+23; Li+23]. Our lower bounds constitute an addition to the toolkit developed in these prior works. Notably, the algorithms achieving our upper bounds use only simple quantum processing as is common in the randomized measurement paradigm [HKP20; Elb+22].

1.4 Techniques and proof overview

Let $d = 2^n$ be the dimension of an n -qubit system.

Hamiltonian locality testing lower bound. To prove the first Hamiltonian locality testing lower bound of Theorem 1.2, we identify an underlying Hamiltonian many-vs-one distinguishing problem that can be solved with high success probability by any successful locality tester, and then establish lower bounds for this distinguishing task. Concretely, we consider the following task: Given access to the time evolution along a Hamiltonian H that is promised to satisfy either (i) $H = 0$ or (ii) $H = \varepsilon(V |0\rangle\langle 0| V^\dagger - \mathbb{I}/d)$, where V is a Haar-random n -qubit unitary, decide whether (i) or (ii) is the case.

We show that Hamiltonian locality testing indeed suffices to solve this distinguishing problem via a concentration of measure argument. Namely, using the concentration of a Lipschitz function of Haar-random unitaries around its mean [MM13], we show that $\varepsilon(V |0\rangle\langle 0| V^\dagger - \mathbb{I}/d)$ with Haar-random V is $(\varepsilon/2)$ -far w.r.t. $\|\cdot\|_\infty$ from any fixed, traceless Hamiltonian K with $\|K\|_\infty \leq 1$ with probability $\geq 1 - \exp(-\Omega(2^n))$. As the set of k -local Hamiltonians admits an $\|\cdot\|_\infty$ -covering net \mathcal{H}_ε whose size satisfies $\log|\mathcal{H}_\varepsilon| \leq \min\{(k+1)(3n)^k, 4^n\}$, a union bound now implies that $\varepsilon(V |0\rangle\langle 0| V^\dagger - \mathbb{I}/d)$ with Haar-random V is simultaneously $(\varepsilon/4)$ -far, again w.r.t. $\|\cdot\|_\infty$, from all k -local Hamiltonians with high probability. Therefore, any high-probability algorithm for testing Hamiltonian k -locality to accuracy $\varepsilon/4$ in the operator norm also manages to distinguish between (i) $H = 0$ and (ii) $H = \varepsilon(V |0\rangle\langle 0| V^\dagger - \mathbb{I}/d)$ with Haar-random V with high success probability. In particular, any lower bound for this distinguishing task immediately implies a lower bound for Hamiltonian locality testing.

To establish such a lower bound, we follow [Faw+23b]. To explain the proof of the lower bound, we use the learning tree representation of [Che+22]. That is, we think of the possible outcomes of an adaptive distinguishing algorithm as leaves in a tree, where the observed measurement outcomes in each round determine how the learner moves from the root to a leaf. Viewed this way, Le Cam's two-point method implies that solving the distinguishing task with constant success probability requires the leaf distributions induced by the different hypothesis Hamiltonians to have at least a constant total variation (TV) distance. For technical reasons, we in fact consider a slightly different test: Distinguish between the channels (i) $\mathcal{U}_t = \text{id}$ or (ii) $\mathcal{U}_t(\cdot) = \alpha \text{id}(\cdot) + (1 - \alpha)e^{-itH}(\cdot)e^{itH}$ for a random H as above. We show that as long as $\alpha \leq \frac{1}{10N}$, the leaf distributions induced by these two different hypotheses still have at least a constant TV distance. Next, we use Pinsker's inequality to upper bound (the square of) this TV distance by the Kullback Leibler (KL) divergence, as the KL divergence is more amenable to decoupling the dependence between the random observations at different steps. Taking a mixture of the identity channel and the time evolution under the alternate hypothesis as in (ii) is important to make a second order Taylor expansion of the logarithm function that occurs in the KL divergence possible. However, this alone is not sufficient since the (expected) second order term in this expansion can diverge if the input states and measurement projectors are orthogonal. To address this issue, at each step $k \in [N]$, we distinguish between two types of paths of length k : those for which the overlap between the input state and the measurement operator (corresponding to the final node in the path) is too small, and those for which the overlap is not. When the overlap is too small, we use a simple lower bound on Born's probability under $\mathcal{U}_t(\cdot) = \alpha \text{id}(\cdot) + (1 - \alpha)e^{-itH}(\cdot)e^{itH}$ in terms of the probability under $\mathcal{U}_t = \text{id}$ (here again taking the mixture proves to be essential). When the overlap is not too small, we can Taylor expand the logarithm to second order and control the resulting term using Weingarten calculus. Overall, we achieve a TV distance upper bound of $\sum_{k=1}^N \mathcal{O}\left(\frac{\log(N)}{d} \cdot \mathbb{E}[\min(1, \varepsilon t_k)]\right)$, which when compared to the constant lower bound implies the claimed bounds on N , the number of queries, and on $\mathbb{E}[\sum_{k=1}^N t_k]$, the expected total evolution time.

Hamiltonian locality testing upper bound – Commuting case. For clarity of exposition, we begin with a simpler setting, focusing on commuting Hamiltonians consisting of terms from $\{\mathbb{I}, X\}^{\otimes n}$. That is, we assume that we can expand the Hamiltonian as $H = \sum_{P \in \{\mathbb{I}, X\}^{\otimes n}} \alpha_P P$. Our concrete task under consideration thus becomes: Given access to the time evolution along an unknown Hamiltonian $H = \sum_{P \in \{\mathbb{I}, X\}^{\otimes n}} \alpha_P P$ with $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, decide, with success probability $\geq 2/3$, whether H is k -local or ε -far from k -local in normalized Schatten 2-norm distance. The underlying idea for locality testing in this special case will carry over to the general setting.

Our algorithm for solving this property testing problem is as follows: We prepare the state $|0\rangle\langle 0|$ and let it evolve under the unknown Hamiltonian for time $t = \mathcal{O}(\varepsilon)$. At the end of this evolution, we perform a measurement in the computational basis $\{|i\rangle\langle i|\}_{i \in \{0,1\}^n}$. We repeat this procedure $N = \mathcal{O}(1/\text{poly}(\varepsilon))$ times. If at least one of the N rounds produces an n -bit string with at least $k+1$ non-zero entries, i.e., with Hamming weight $\geq k+1$, as measurement outcome, we conclude that the Hamiltonian H is ε -far from being k -local. Otherwise, we claim that H is k -local.

The proof of correctness for this algorithm has the following structure. Observe that $|j\rangle = X^j |0\rangle$ for any $j \in \{0,1\}^n$ and that $U_t = e^{itH} \approx \mathbb{I} + itH$ holds for short times t . Here, we write X^j to mean $X^{j_1} \otimes \dots \otimes X^{j_n}$, where $X^0 = \mathbb{I}$. Consequently, we have $\langle j|U_t|0\rangle \approx \delta_{j,0} + it\alpha_{X^j}$ for all $j \in \{0,1\}^n$. In particular, for any n -bit string j with weight $|j| > 0$,

$$|\langle j|U_t|0\rangle|^2 \approx t^2 |\alpha_{X^j}|^2.$$

If H is indeed k -local, then $\alpha_{X^j} = 0$ holds whenever $|j| > k$, and we find that

$$\sum_{j:|j|>k} |\langle j|U_t|0\rangle|^2 \approx 0,$$

so the probability that the algorithm falsely claims that H is far from k -local when it is indeed k -local is approximately zero.

Conversely, if the Hamiltonian H is ε -far from any k -local Hamiltonian in normalized Schatten 2-norm, this means that $\sum_{j:|j|>k} |\alpha_{X^j}|^2 \geq \varepsilon^2$ because of Parseval. We infer that

$$\sum_{j:|j|>k} |\langle j|U_t|0\rangle|^2 \gtrsim t^2 \varepsilon^2.$$

Note that we have to choose t small enough for the approximation to be correct. By repeating the algorithm $\mathcal{O}(t^{-2}\varepsilon^{-2}) = \mathcal{O}(\varepsilon^{-4})$ many times, we can increase the probability of the algorithm to be correct to a constant. To make the above reasoning precise, we use the Taylor expansion of e^{itH} in order to bound the error in the approximation $e^{itH} \approx \mathbb{I} + itH$.

Hamiltonian locality testing upper bound – General case. In the case of a general Hamiltonian H , we can no longer limit ourselves to preparing and measuring in the computational basis. Instead, we need to prepare and measure in several different bases. A convenient choice are some $d+1$ *mutually unbiased bases* (MUBs) \mathcal{B}_i , which are known to exist in our case since $d = 2^n$ is a prime power [WF89]. We write

$$\mathcal{B}_i = \{|\phi_{i,j}\rangle\}_{j \in \{1,\dots,d\}}, \quad 1 \leq i \leq d+1.$$

MUBs are known to be particularly suited for determining the state of a quantum system [WF89]. Moreover, MUBs were used for Pauli channel learning [FW20]. They can be explicitly constructed from covering the Pauli group with $d+1$ stabilizer groups that only have the identity element in common but are otherwise disjoint.

Motivated by our previous discussion, we consider for a Pauli operator P with weight $|P|$ the overlap $|\langle \phi_{i,\ell} | P | \phi_{i,j} \rangle|$ and find that it is either 0 or 1. This motivates the following algorithm: We choose $(i,j) \in [d] \times [d+1]$ uniformly at random and prepare the state $|\phi_{i,j}\rangle\langle \phi_{i,j}|$. We let it evolve under the unknown Hamiltonian for time $t = \mathcal{O}(\varepsilon)$. At the end of this evolution, we perform a measurement in the basis \mathcal{B}_i . We repeat this procedure $N = \mathcal{O}(1/\text{poly}(\varepsilon))$ times. If at least one of the N rounds produces an output ℓ such that all Pauli strings P with $|P| \leq k$ satisfy $|\langle \phi_{i,\ell} | P | \phi_{i,j} \rangle| = 0$ – which we denote by $|\phi_{i,\ell}\rangle \approx_k |\phi_{i,j}\rangle$, meaning that we detected a non-locality –, then we conclude that the Hamiltonian is ε -far from being local. Otherwise, we claim that H is k -local.

By construction, if $|\phi_{i,\ell}\rangle \approx_k |\phi_{i,j}\rangle$, then $|\langle \phi_{i,\ell} | H^m | \phi_{i,j} \rangle| = 0$ for $m \in \{0,1\}$. Using $e^{itH} \approx \mathbb{I} + itH$, we find that if the Hamiltonian is k -local, then in any single round, the probability of our procedure falsely detecting a non-locality is

$$\mathbb{P}(|\phi_{i,\ell}\rangle \approx_k |\phi_{i,j}\rangle) = \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} |\langle \phi_{i,\ell} | e^{itH} | \phi_{i,j} \rangle|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_k |\phi_{i,j}\rangle\}) \approx 0.$$

If the Hamiltonian is ϵ -far from being k -local, we can use the fact that $d + 1$ MUBs in dimension d form a 2-design [KR05] to infer the following upper bound on the probability of not detecting the non-locality in any single round:

$$\mathbb{P}(|\phi_{i,\ell}\rangle \sim_k |\phi_{i,j}\rangle) \leq \sum_{P:|P|\leq k} \frac{d}{d(d+1)} + \sum_{P:|P|\leq k} \frac{|\text{Tr}(Pe^{itH})|^2}{d(d+1)}. \quad (1)$$

We use $e^{itH} \approx \mathbb{I} + itH$ to conclude $\text{Tr}(Pe^{itH}) \approx itd\alpha_P$ if $P \neq \mathbb{I}$. Furthermore, expanding e^{itH} to second order, we can approximate

$$|\text{Tr}(e^{itH})|^2 \approx d^2 - d^2t^2 \sum_P |\alpha_P|^2.$$

Combining this with H being ϵ -far from k -local in normalized Frobenius norm, which means $\sum_{P:|P|>k} |\alpha_P|^2 \geq \epsilon^2$, the second term in Equation (1) can be upper bounded by $1 - t^2\epsilon^2$. The first term can be seen to quickly approach 0 as n grows. Hence,

$$\mathbb{P}(|\phi_{i,\ell}\rangle \sim_k |\phi_{i,j}\rangle) \lesssim 1 - \frac{t^2\epsilon^2}{2}.$$

Note that we again have to choose t small enough for the approximation to be correct. By repeating the algorithm $N = \mathcal{O}(t^{-2}\epsilon^{-2}) = \mathcal{O}(\epsilon^{-4})$ many times, we reduce the error probability to a small enough constant. Again, we use the Taylor expansion of e^{itH} to make the above reasoning precise. Choosing $t = \mathcal{O}(\epsilon)$ lets us control the higher order terms appearing in this expansion.

Hamiltonian learning lower bound. Similarly to the reasoning behind Theorem 1.2, we follow [Fl+12; Haa+17; BCL20; LN22; FOS23; Ouf23] and begin by identifying a distinguishing problem, whose existence we guarantee through a probabilistic argument, and that any successful general Hamiltonian learner can solve. We then establish lower bounds for that distinguishing task through information-theoretic arguments.

To set up our distinguishing problem, let $O = \text{diag}(+1, \dots, +1, -1, \dots, -1)$ be the diagonal $2^n \times 2^n$ matrix with half of the diagonal entries equal to $+1$ and the other half equal to -1 . We consider Hamiltonians H of the form $H = \epsilon UOU^\dagger$, where U is a Haar-random n -qubit unitary. Using well known expressions for the first and second moments of the Haar measure, we show that the expected square of the normalized Frobenius distance between two such Hamiltonians satisfies $\mathbb{E}_{U,V \sim \text{Haar}_n} [\frac{1}{2^n} \|H_U - H_V\|_2^2] = 2\epsilon^2$, and that the second moment of this quantity is bounded as $\mathbb{E}_{U,V \sim \text{Haar}_n} [\frac{1}{2^{2n}} \|H_U - H_V\|_2^4] \leq 6\epsilon^2$. Hölder's inequality then implies that the expected normalized Frobenius distance satisfies $\mathbb{E}_{U,V \sim \text{Haar}_n} [\frac{1}{\sqrt{2^n}} \|H_U - H_V\|_2] > 1.1\epsilon$. Combining this with a Lipschitz concentration argument, we conclude that $\frac{1}{\sqrt{2^n}} \|H_U - H_V\|_2 > \epsilon$ holds with probability $\geq 1 - \exp(-\Omega(2^{2n}))$. Therefore, by a union bound, there exists a set of $M = \exp(\Omega(4^n))$ unitaries U_x , $1 \leq x \leq M$, such that the Hamiltonians $H_x = \epsilon U_x O U_x^\dagger$ are pairwise ϵ -far apart w.r.t. $\frac{1}{\sqrt{2^n}} \|\cdot\|_2$. In particular, any algorithm that can learn an unknown Hamiltonian to average-case accuracy ϵ is able to distinguish between these M candidate Hamiltonians. Via Fano's inequality, this implies the mutual information lower bound $\mathcal{I}(X : Y) \geq \Omega(\log M) \geq \Omega(4^n)$, where $X \sim \text{Uniform}([M])$ and where the random variable Y describes the outcomes observed by the learner.

We then provide complementary mutual information upper bounds for two different scenarios. First, we consider procedures that use at most n_{aux} many auxiliary qubits. Then, the overall dimension of the Hilbert space encompassing all quantum systems throughout the N queries restricts the mutual information as $\mathcal{I}(X : Y) \leq \mathcal{O}(N \cdot (n + n_{\text{aux}}))$. Comparing this with the previous mutual information lower bound, we conclude that any such learner has to query the Hamiltonian time evolution at least $N \geq \Omega(4^n / (n + n_{\text{aux}}))$ many times. Note that this bound applies even to coherent learning algorithms, as long as the size of their auxiliary register is limited.

Second, for incoherent learners that use neither an auxiliary system nor adaptivity in their choice of experiments, we decompose the overall mutual information as $\mathcal{I}(X : Y) = \sum_{\ell=1}^N \mathcal{I}(X : Y_\ell)$, where the random variable Y_ℓ describes the measurement outcome that the learner observes in the ℓ^{th} experiment. That is, we have $\mathbb{P}[Y_\ell = y_\ell | X = x] = \lambda_{y_\ell} \langle \phi_{y_\ell}^\ell | e^{-it_\ell H_x} \rho_\ell e^{it_\ell H_x} | \phi_{y_\ell}^\ell \rangle$, where t_ℓ is the evolution time, ρ_ℓ is the input state, and the measurement is described by $\{\lambda_{y_\ell} | \phi_{y_\ell}^\ell \rangle \langle \phi_{y_\ell}^\ell | \}_{y_\ell}$. Using that $H_x^2 = \epsilon^2 \mathbb{I}$ and therefore $e^{itH_x} = \cos(t\epsilon)\mathbb{I} + i\sin(t\epsilon)U_x O U_x^\dagger$, we show via a careful analysis that $\mathcal{I}(X : Y_\ell) \leq \tilde{\mathcal{O}}(t_\ell \epsilon)$ and hence we obtain $\mathcal{I}(X : Y) \leq \mathcal{O}(\epsilon \sum_{\ell=1}^N t_\ell + \sqrt{\text{poly}(2^n)/M})$. Comparing this to our Fano-based mutual information

lower bound of $\mathcal{I}(X : Y) \geq \Omega(4^n)$ and recalling that M is doubly exponential in n , we obtain the lower bounds on the total evolution time and the query complexity stated in Theorem 1.4.

1.5 Directions for future work

Motivated by the question of how to test whether an unknown Hamiltonian is k -local, we proposed a framework for Hamiltonian property testing. Here, we considered different ways of measuring distances between Hamiltonians. With worst-case distance measures, exemplified by the operator norm, we showed that exponentially many queries as well as exponentially long time are required to solve the locality testing problem. In contrast, for the normalized Frobenius norm, leading to an average-case notion of distance, we gave a broadly applicable Hamiltonian testing algorithm that uses only single-copy measurements on short-time evolutions of certain randomized input states, and that is resource-efficient w.r.t. the number of queries, total evolution time, and computation time involved. Finally, still in the regime of average-case distances, we showed that learning is exponentially harder than testing.

There are several promising ways of extending our Hamiltonian property testing framework, for instance by changing the notions of distance and access. First, one may consider other physically motivated notions of distance, such as quantum Wasserstein distances [De +21] or distances relative to (some distribution over) a set of input states and a set of output observables of interest. The former would lead to Hamiltonian testing (and learning) that takes an underlying locality structure into account, whereas the latter would be reminiscent of classical shadows [HKP20; HCP23] and shadow tomography [Aar20; BO21]. Second, one may change the form of access to the unknown Hamiltonian from time evolution access to access to copies of a Gibbs state, an access model already well studied in Hamiltonian learning (compare Section 1.3). Also, exploring the possibility for quantitative improvements in our bounds seems important. For instance, coherent quantum algorithms may be able to achieve Heisenberg-limited scaling for testing, as they already have for Hamiltonian learning [Hua+23; Li+23] and unitary tomography [Haa+23; Zha+23]. Finally, we highlight another possible extension to the Hamiltonian testing framework proposed here: Whereas we focus on Hamiltonian testing, more general property testing questions for GKLS generators [Lin76; GKS76] from access to the generated quantum dynamical semigroups could be of interest.

Note added. After the first version of our work appeared on the arXiv, Francisco Escudero Gutiérrez shared with us his approach to tolerant Hamiltonian property testing and to local Hamiltonian learning based on entangled inputs [Esc24]. We are grateful for this exchange, which motivated us to improve upon the first version of our work, tightening the analysis underlying our upper bounds to remove the n -dependence in the total evolution time and number of experiments as well as to achieve a tolerant version of our tester.

2 Preliminaries

2.1 Notation and basic definitions

Let \log denote the natural logarithm and \log_2 the logarithm to base 2. For compactness, for $n \in \mathbb{N}$ we will abbreviate $[n] := \{1, \dots, n\}$. We write $\mathbf{1}(\mathcal{X})$ for the indicator function on the set or event \mathcal{X} . For a complex number $z \in \mathbb{C}$, we will denote by $\Re(z)$ its real part and $\Im(z)$ its imaginary part.

We will make extensive use of the Schatten p -norms $\|\cdot\|_p$, for $p \in \mathbb{N} \cup \{\infty\}$. They are defined for any $n \times m$ matrix X as

$$\|X\|_p := \text{Tr}[|X|^p]^{\frac{1}{p}} = \text{Tr}[(\sqrt{X^\dagger X})^p]^{\frac{1}{p}}.$$

Here, X^\dagger is the Hermitian conjugate of X . The case $p = \infty$ is the operator norm of X . The Schatten 2-norm is also known as the Frobenius norm, whereas the Schatten 1-norm also goes by the names of trace or nuclear norm. Here, we can employ different measures of distance between Hamiltonians. Moreover, we consider the normalized Schatten p -norms $\frac{1}{2^{n/p}}\|H\|_p$. And finally, we use $\|H\|_{\text{Pauli},p} \equiv (\sum_{P \in \mathbb{P}_n} |\alpha_P|^p)^{1/p}$ to denote the norm induced by the ℓ_p -norm of the coefficient vector of H . Note that $\frac{1}{\sqrt{2^n}}\|H\|_2 = \|H\|_{\text{Pauli},2}$ by Parseval's identity.

Depending on context, we will mean by a quantum state either a unit vector $|\psi\rangle \in \mathbb{C}^d$ for some appropriate dimension $d \in \mathbb{N}$ or a density matrix ρ , i.e., a positive-semidefinite $d \times d$ matrix of unit trace. A linear map

\mathcal{N} from $d_1 \times d_1$ to $d_2 \times d_2$ matrices will be called positive if it maps positive-semidefinite matrices to positive-semidefinite matrices. Moreover, it will be called completely positive if $\mathcal{N} \otimes \text{id}_n$ is positive for all n , where id_n is the identity map on an additional n -dimensional system. If a completely positive map is in addition trace-preserving, we will call it a quantum channel. For general background on quantum information theory, we refer the reader to a textbook such as [Wat18].

Finally, we will need some objects from classical information theory. Given two probability distributions P, Q on a finite alphabet \mathcal{Y} , their total variation distance is

$$\text{TV}(P, Q) := \frac{1}{2} \sum_{y \in \mathcal{Y}} |P(y) - Q(y)|.$$

Their Kullback-Leibler divergence is

$$\text{KL}(P||Q) := \sum_{y \in \mathcal{Y}} P(y) \log \left(\frac{P(y)}{Q(y)} \right)$$

if $\text{supp } P \subseteq \text{supp } Q$, where we define $0 \log 0 := 1$, and $+\infty$ otherwise.

Given two random variables X, Y on finite alphabets \mathcal{X}, \mathcal{Y} , respectively, and joint probability distribution P_{XY} on $\mathcal{X} \times \mathcal{Y}$ with marginals P_X on \mathcal{X} and P_Y on \mathcal{Y} , their mutual information is

$$\mathcal{I}(X : Y) := \text{KL}(P_{XY} || P_X \otimes P_Y).$$

2.2 Mutually unbiased bases of stabilizer states

We define the Pauli matrices in the standard way

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of n -strings formed by the Pauli operators together with the identity matrix will be written as $\mathbf{P}_n = \{\mathbb{I}, X, Y, Z\}^{\otimes n}$. Note that unlike the set $\{e^{i\theta\pi/2}\sigma_1 \otimes \cdots \otimes \sigma_n \mid \theta = 0, 1, 2, 3, \sigma_i \in \{\mathbb{I}, X, Y, Z\}\}$, which is the Pauli group, the set \mathbf{P}_n is not a group. We will also need the quotient group of the Pauli group with its centralizer. We will write this Abelian group as

$$\mathbf{P}_n = \left\{ e^{i\theta\pi/2} \sigma_1 \otimes \cdots \otimes \sigma_n \mid \theta = 0, 1, 2, 3, \sigma_i \in \{\mathbb{I}, X, Y, Z\} \right\} / \{\pm 1, \pm i\}.$$

There is a bijection between elements in \mathbf{P}_n and elements in \mathbb{P}_n , and we will write $P(g)$ for the element in \mathbb{P}_n corresponding to $g \in \mathbf{P}_n$.

We will make extensive use of the following known two lemmas, see e.g. [Ban+02; FW20; FOS23].

Lemma 2.1. \mathbf{P}_n can be covered by $d + 1$ stabilizer groups G_1, \dots, G_{d+1} satisfying for all $i \neq j$:

- $|G_i| = d$,
- $C_{G_i} = G_i$,
- $G_i \cap G_j = \{\mathbb{I}\}$.

Here, $C_{G_i} = \{g \in \mathbf{P}_n : [P(g), P(h)] = 0 \forall h \in G_i\}$. It can be thought of as the centralizer of G_i in the Pauli group, where we are only interested in elements with a fixed choice of sign.

From these stabilizer groups, we can construct sets of pure states: However, let us first note that $\{P(g) : g \in G_i\}$ are not groups. We can, however, turn them into groups: there are signs $\zeta_{i,g} \in \{\pm 1\}$ such that $\{\zeta_{i,g} P(g) : g \in G_i\}$ is an Abelian group, the stabilizer group. There are two ways to see this. Either, we start with a set of generators for G_i and keep track of the signs when multiplying the corresponding $P(g)$ in order to generate the group. Alternatively, we choose a common eigenstate $|\varphi\rangle$ of all the $P(g)$, $g \in G_i$, which exists since all the $P(g)$ commute and define $P(g)|\varphi\rangle = \zeta_{i,g}|\varphi\rangle$. In this case, $\{S(g) := \zeta_{i,g} P(g) : g \in G_i\}$ will be a stabilizer group for $|\varphi\rangle$.

Lemma 2.2. *Let $G \in \{G_1, \dots, G_{d+1}\}$. The set $\mathcal{M}_G = \{M_G^r = \frac{1}{d} \sum_{g \in G} (-1)^{p \circ r} S(p)\}_{r \in A_G}$ forms an orthonormal basis consisting of rank-1 stabilizer states.*

Here we use the notation $A_G = \mathbf{P}_n/G$ and $p \circ r = 0$ if $P(p)P(r) = P(r)P(p)$ and $p \circ r = 1$ if $P(p)P(r) = -P(r)P(p)$ (note that we could have put arbitrary signs in front of the $P(p)$, $P(r)$ without changing the value of $p \circ r$). To not overload the notation, we define $A_{G_i} = \{r_j^i\}_{j=1}^d$ and introduce the notation

$$|\phi_{i,j}\rangle\langle\phi_{i,j}| = M_{G_i}^{r_j^i} = \frac{1}{d} \sum_{p \in G_i} (-1)^{p \circ r_j^i} S(p) \quad (2)$$

Now we have $d+1$ mutually unbiased bases (MUBs) $\mathcal{B}_i = \{|\phi_{i,j}\rangle\}_{j=1}^d$ for $i = 1, \dots, d+1$. We will use the fact that MUBs of stabilizer states form a 2-design. 2-designs were used for the problem of testing the mixedness of states by [Yu21]. Stabilizer states and measurements were used for the problem of Pauli channel learning by [FW20].

Proposition 2.3 (Pauli group, MUB and 2-design). *The bases $\{\mathcal{B}_i = \{|\phi_{i,j}\rangle\}_{j=1}^d\}_{i=1}^{d+1}$ form an MUB and a 2-design. That is:*

$$\frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j=1}^d |\phi_{i,j}\rangle\langle\phi_{i,j}| \otimes |\phi_{i,j}\rangle\langle\phi_{i,j}| = \frac{\mathbb{I} + F}{d(d+1)}$$

where $F = \sum_{x,y} |xy\rangle\langle yx|$ is the flip operator.

Proof. One way to see this is to apply [KR05, Theorem 1]. We only need to prove that:

$$\sum_{i,j} \frac{1}{d(d+1)} \sum_{k,\ell} \frac{1}{d(d+1)} |\langle\phi_{i,j}|\phi_{k,\ell}\rangle|^4 = \int_{\text{Haar}} |\langle 0|\phi\rangle|^4 d\phi = \frac{2}{d(d+1)}.$$

This identity can be checked easily since $|\langle\phi_{i,j}|\phi_{k,\ell}\rangle| = \frac{1}{\sqrt{d}}$ if $i \neq k$ and $|\langle\phi_{i,j}|\phi_{i,\ell}\rangle| = \mathbf{1}(\{j = \ell\})$. Indeed, we can check $|\langle\phi_{i,j}|\phi_{k,\ell}\rangle| = \frac{1}{\sqrt{d}}$ if $i \neq k$:

$$\begin{aligned} |\langle\phi_{i,j}|\phi_{k,\ell}\rangle|^2 &= \text{Tr}(|\phi_{i,j}\rangle\langle\phi_{i,j}| |\phi_{k,\ell}\rangle\langle\phi_{k,\ell}|) \\ &= \text{Tr} \left(\frac{1}{d} \sum_{p \in G_i} (-1)^{p \circ r_j^i} S(p) \cdot \frac{1}{d} \sum_{p' \in G_k} (-1)^{p' \circ r_i^k} S(p') \right) \\ &= \frac{1}{d} \sum_{p \in G_i \cap G_k} (-1)^{p \circ r_j^i} (-1)^{p \circ r_i^k} = \frac{1}{d} \end{aligned}$$

since $G_i \cap G_k = \{\mathbb{I}\}$. Alternatively, we could have argued using [KR05, Theorem 3]. \square

Finally, the operation $p \circ q$ has a nice property that we'll need later, which has already been used, for example, in [FW20; FOS23].

Lemma 2.4. *Let $q \in \mathbf{P}_n$ and let G be any subgroup of \mathbf{P}_n . Then*

$$\frac{1}{|G|} \sum_{g \in G} (-1)^{p \circ q} = \mathbf{1}(q \in C_G).$$

2.3 Problem statement: Hamiltonian property testing

We consider n -qubit Hamiltonians H expanded in the Pauli basis, $H = \sum_{P \in \mathbf{P}_n} \alpha_P P$, with $\mathbf{P}_n = \{\mathbb{I}, X, Y, Z\}^{\otimes n}$ the set of n -qubit Paulis and with (real) coefficients $\alpha_P = \text{Tr}[HP]/2^n$. Throughout this work, the Hamiltonian properties of interest are characterized by a subset $S \subseteq \mathbf{P}_n$, and we say that H has property Π_S , write $H \in \Pi_S$, if $\alpha_P = 0$ for all $P \notin S$. For instance, the property of being (at most) k -local is characterized by the subset $S_{k\text{-loc}} = \{P \in \mathbf{P}_n : |P| \leq k\}$, where we use $|P|$ to denote the weight of P , that is, the number

of non-identity tensor factors. Other examples of properties that our framework encompasses are geometric locality or having a given interaction graph.

We now define Hamiltonian property testing of Π_S as the task of deciding, given access to the time evolution according to an unknown Hamiltonian H , whether H has property Π_S or whether H is far from all Hamiltonians that have property Π_S .

Problem 2.5 (Hamiltonian property testing). *Given a property Π_S associated to a subset $S \subseteq \mathbb{P}_n$, a norm $\|\cdot\|$, and an accuracy parameter $\varepsilon \in (0, 1)$, we denote by $\mathcal{T}_{\|\cdot\|}^{\Pi_S}(\varepsilon)$ the following Hamiltonian property testing problem: Given access to the time evolution according to an unknown Hamiltonian H , decide, with success probability $\geq 2/3$, whether*

(i) H has property Π_S , that is, $H \in \Pi_S$, or

(ii) H is ε -far from having property Π_S , that is, $\forall \tilde{H} \in \Pi_S: \left\| \|H - \tilde{H}\| \right\| \geq \varepsilon$.

If H satisfies neither (i) nor (ii), then any output of the tester is considered valid.

Remark 2.6. *A short discussion of the physical interpretation for the different norms used above is in order. Among the Schatten p -norms, we highlight the Schatten ∞ -norm (aka operator norm) as a worst-case distance. Namely, in Appendix A.1, we show that, at least for short evolution times, the Hamiltonian distance $\left\| \|H - \tilde{H}\| \right\|_\infty$ is tightly related to, among others, the worst-case output fidelity $\max_{|\psi\rangle} |\langle \psi | e^{-itH} e^{it\tilde{H}} | \psi \rangle|^2$ and the diamond norm distance between the unitary time evolution channels $e^{-itH}(\cdot)e^{itH}$ and $e^{-it\tilde{H}}(\cdot)e^{it\tilde{H}}$.*

Second, to illustrate the relevance of the normalization factor in normalized Schatten p -norms, we single out the normalized Frobenius norm and interpret it as an average-case distance measure. Here, we demonstrate in Appendix A.2 that $\frac{1}{\sqrt{2^n}} \left\| \|H - \tilde{H}\| \right\|_2$ is, again for short times, connected to the average output fidelity $\mathbb{E}_{|\psi\rangle \sim \text{Haar}_n} \left[|\langle \psi | e^{-itH} e^{it\tilde{H}} | \psi \rangle|^2 \right]$, the normalized Frobenius norm distance between the time evolutions e^{-itH} and $e^{-it\tilde{H}}$, and other average-case distance measures.

Finally, the norms $\|\cdot\|_{\text{Pauli},p}$, in particular for $p = 2, \infty$, have featured in recent work on Hamiltonian learning, see Section 1.3. They make intuitive sense for learning and testing tasks in which the interest is in estimating or validating properties of interaction strength parameters in a Hamiltonian.

The central goal of this work is to understand how easy or hard this Hamiltonian property testing problem is. More specifically, we are interested in the number of queries to the time evolution as well as in the total evolution time necessary and sufficient to solve the testing task. Additionally, we consider whether one can successfully approach Hamiltonian testing via Hamiltonian learning.

2.4 Types of strategies for Hamiltonian property testing

In order to solve the Hamiltonian property testing problem, we consider several different scenarios, depending on what kind of access to the quantum system and how many additional resources we grant the tester.

Incoherent strategies. We start by presenting *incoherent* strategies. That is, the tester can use the quantum system only once per step. Therefore, in round k of the overall procedure, the tester can prepare a quantum state ρ_k , let the time evolution $\mathcal{U}_t(\cdot) = e^{-itH}(\cdot)e^{itH}$ run for a chosen time t_k , and finally perform a measurement $\mathcal{M}_k = \left\{ \lambda_i^{(k)} \left| \phi_i^{(k)} \right\rangle \left\langle \phi_i^{(k)} \right| \right\}_{i \in \mathcal{I}_k}$, where \mathcal{I}_k is a set of measurement outcomes. Note that we could have taken a general positive operator-valued measure (POVM) here, but that without loss of generality, we can assume the POVM to consist of rank-one elements. If the input state, the duration of the time evolution, and the choice of measurement are allowed to depend on the outcomes of previous measurements, the incoherent strategy is *adaptive*, otherwise it is *non-adaptive*. We can also add ancilla qubits to the strategy by preparing the input state ρ_k on a $(d \times d_{\text{aux}})$ -dimensional system instead of restricting to dimension d , subsequently letting the time evolution act as $\mathcal{U}_{t_k} \otimes \text{id}$, where the identity acts on the ancilla qubits, and finally measuring both systems with \mathcal{M}_k . If we add the ancilla, we call the strategy *ancilla-assisted*, otherwise we will speak of *ancilla-free* strategies. If we allow an ancilla but require it to be of constant dimension, we speak of a *bounded-ancilla* strategy. See Figure 1 for an illustration of non-adaptive strategies and Figure 2 for an illustration of adaptive strategies.

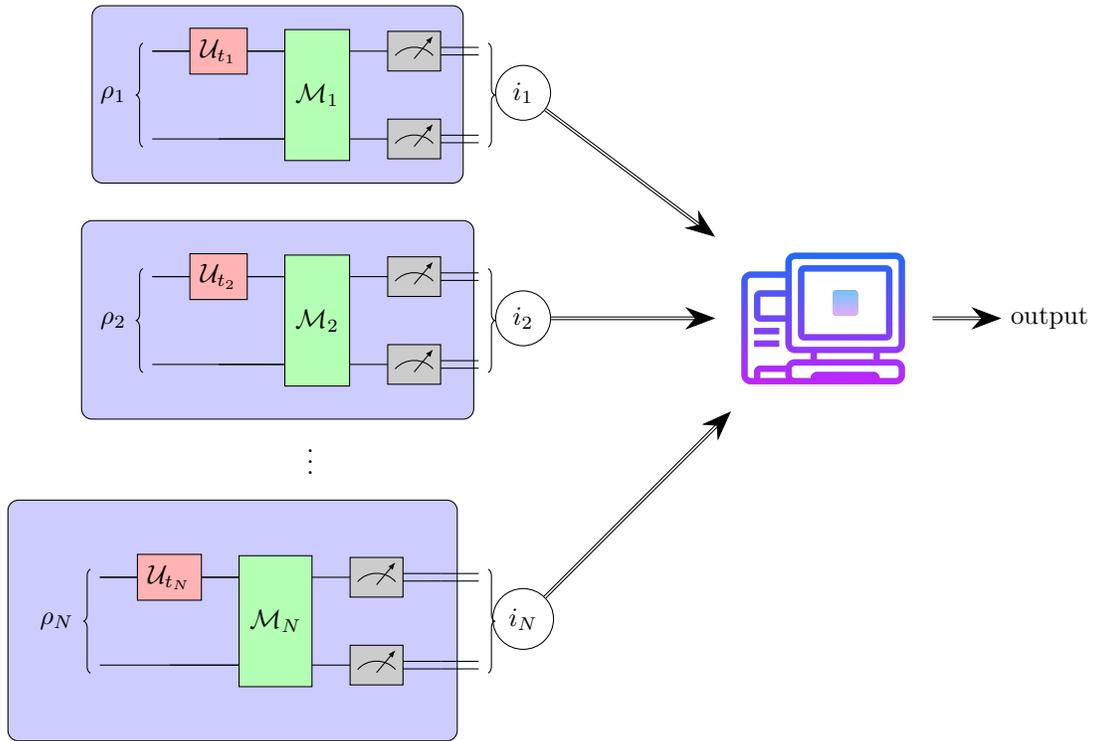


Figure 1: Illustration of a non-adaptive incoherent strategy for learning/testing properties of a Hamiltonian H from its time evolution channel $\mathcal{U}_t(\cdot) = e^{-itH}(\cdot)e^{itH}$. It is called ancilla-free if the auxiliary systems have dimension 1, otherwise it is called ancilla-assisted. The classical computer processes the observations (i_1, \dots, i_N) to distinguish between two hypotheses H_0/H_1 (in testing) or to produce an approximate Hamiltonian \hat{H} (in learning).

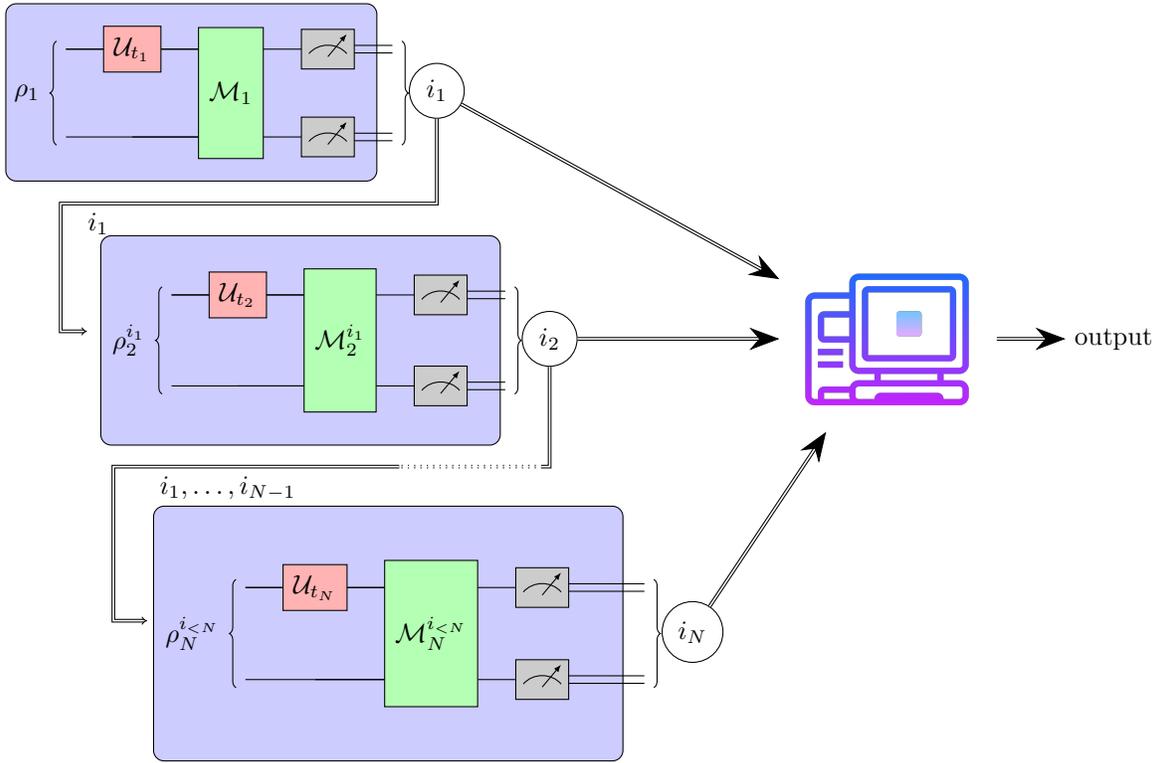


Figure 2: Illustration of an adaptive incoherent strategy for learning properties of a Hamiltonian H from its time evolution channel $\mathcal{U}_t(\cdot) = e^{-itH}(\cdot)e^{itH}$. It is called ancilla-free if the auxiliary systems have dimension 1, otherwise it is called ancilla-assisted. The classical computer processes the observations (i_1, \dots, i_N) to distinguish between two hypotheses H_0/H_1 (in testing) or to produce an approximate Hamiltonian \hat{H} (in learning).

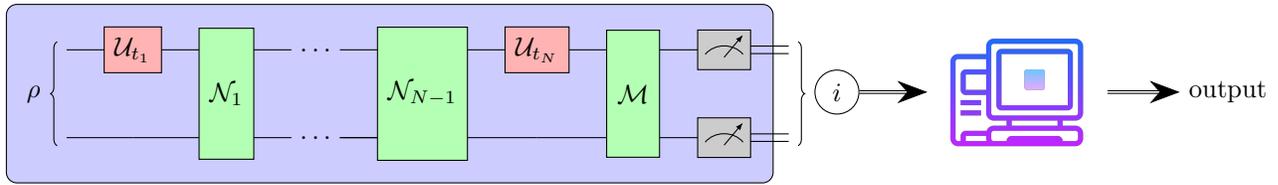


Figure 3: Illustration of a coherent strategy for learning properties of a Hamiltonian H from its time evolution channel $\mathcal{U}_t(\cdot) = e^{-itH}(\cdot)e^{itH}$. The classical computer processes the observation i to distinguish between two hypotheses H_0/H_1 (in testing) or to produce an approximate Hamiltonian \hat{H} (in learning).

Coherent strategies. The second type of strategies are the *coherent* strategies. Here, the tester is allowed to access the time-evolution multiple times before measuring, possibly interleaving these time evolutions with the application of quantum channels. Thus, the tester will prepare an initial quantum state ρ on a $(d \times d_{\text{aux}})$ -dimensional quantum system and then choose times t_1, \dots, t_N and quantum channels $\mathcal{N}_1, \dots, \mathcal{N}_{N-1}$ acting on quantum systems of dimension $d \times d_{\text{aux}}$ such that the output state before the measurement is

$$\rho^{\text{output}} = [\mathcal{U}_{t_N} \otimes \text{id}] \circ \mathcal{N}_{N-1} \circ [\mathcal{U}_{t_{N-1}} \otimes \text{id}] \circ \dots \circ \mathcal{N}_1 \circ [\mathcal{U}_{t_1} \otimes \text{id}](\rho),$$

where the identities act on the ancilla qubits. Finally, the tester measures ρ^{output} using a POVM acting on $(d \times d_{\text{aux}})$ -dimensional quantum systems. If $d_{\text{aux}} > 1$, we will say that the strategy is *ancilla-assisted*, otherwise that it is *ancilla-free*. If the ancilla has dimension bounded by $\text{const}^N \cdot d^{N-1}$, we speak of a *bounded-ancilla* strategy. Here, the dimension bound for the ancilla register is chosen such that the bounded-ancilla coherent setting generalizes the bounded-ancilla incoherent one. See Figure 3 for an illustration.

3 Lower bounds for Hamiltonian locality testing

In this section, we will give lower bounds for the Hamiltonian locality testing problem, thereby proving Theorem 1.2. We will first consider lower bounds for incoherent testing strategies in Section 3.1. Subsequently, we consider the more general coherent strategies in Section 3.2, for which we can only give weaker bounds.

3.1 Incoherent setting

In this section, we will prove the following hardness result for Hamiltonian locality testing with respect to unnormalized Schatten norms:

Theorem 3.1. *Let $n \geq \Omega(1)$, $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$ and $p \geq 1$. Suppose that $N \leq \exp(\mathcal{O}(n))/\varepsilon^{\mathcal{O}(1)}$. The problem $\mathcal{T}_{\|\cdot\|_p}^{\text{loc}}(\varepsilon)$, even under the additional promise that the unknown Hamiltonian H satisfies $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, requires an expected total evolution time of $\mathbb{E}\left[\sum_{k=1}^N t_k\right] = \Omega\left(\frac{2^n}{\varepsilon(n-\log \varepsilon)}\right)$ and a total number of independent experiments $N = \Omega\left(\frac{2^n}{n}\right)$ in the adaptive ancilla-free incoherent setting.*

In order to prove Theorem 3.1, we will consider the following test:

$$\mathcal{H}_0 : \mathcal{U}_t(\rho) = \text{id}(\rho) = e^{-it \cdot 0} \rho e^{it \cdot 0} \quad \text{vs} \quad \mathcal{H}_1 : \mathcal{U}_t(\rho) = e^{-itH} \rho e^{itH} \quad (3)$$

where $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ for $\eta > 0$ and $|v\rangle = V|0\rangle, V \sim \text{Haar}(d)$. We will need a couple of lemmas to determine the relation of this toy problem of distinguishing these two time evolutions to the problem $\mathcal{T}_{\|\cdot\|_p}^{\text{loc}}(\varepsilon)$ that we care about.

First, we show that the operator norm between H and any fixed traceless Hamiltonian (of bounded operator norm) is bounded below with high probability.

Lemma 3.2. *Let $d \geq 4$, $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ for $\eta > 0$ and $|v\rangle = V|0\rangle$, $V \sim \text{Haar}(d)$ and let K be a fixed traceless Hamiltonian such that $\|K\|_\infty \leq 1$. We have that*

$$\mathbb{P}\left(\|H - K\|_\infty \leq \frac{\eta}{2}\right) \leq \exp\left(-\frac{d}{1728}\right).$$

Proof. Consider the function:

$$f(V) = \langle v|K|v\rangle = \langle 0|V^\dagger K V|0\rangle.$$

We have $\mathbb{E}[f(V)] = \frac{\text{Tr}K}{d} = 0$. Moreover f is 2-Lipschitz with respect to the Euclidean norm. Indeed, let U and V be two unitaries and $|u\rangle = U|0\rangle$, $|v\rangle = V|0\rangle$. We can first use the triangle inequality and then the Cauchy-Schwarz inequality to show that:

$$\begin{aligned} |f(U) - f(V)| &= |\langle u|K|u\rangle - \langle v|K|v\rangle| \\ &\leq |\langle u - v|K|u\rangle| + |\langle v|K|u - v\rangle| \\ &\leq |\langle u - v|u - v\rangle|^{1/2} |\langle u|K^2|u\rangle|^{1/2} + |\langle u - v|u - v\rangle|^{1/2} |\langle v|K^2|v\rangle|^{1/2} \\ &\leq 2|\langle 0|(U - V)^\dagger(U - V)|0\rangle|^{1/2} \|K^2\|_\infty^{1/2} \\ &\leq 2\|U - V\|_2 \|K\|_\infty. \end{aligned}$$

Hence f concentrates around its mean [MM13, Corollary 17], for all $s > 0$:

$$\mathbb{P}(f(V) - \mathbb{E}[f(V)] \geq s) = \mathbb{P}(f(V) \geq s) \leq \exp\left(-\frac{ds^2}{48\|K\|_\infty^2}\right)$$

Therefore using $\langle v|(H - K)|v\rangle \leq \|H - K\|_\infty$ and $\|H\|_\infty \leq \eta$:

$$\begin{aligned} \mathbb{P}\left(\|H - K\|_\infty \leq \frac{\eta}{2}\right) &\leq \mathbb{P}\left(\langle v|(H - K)|v\rangle \leq \frac{\eta}{2}\right) \mathbf{1}\left(\left\{\|K\|_\infty \leq \frac{3\eta}{2}\right\}\right) \\ &= \mathbb{P}\left(\left(1 - \frac{1}{d}\right)\eta - f(V) \leq \frac{\eta}{2}\right) \mathbf{1}\left(\left\{\|K\|_\infty \leq \frac{3\eta}{2}\right\}\right) \\ &= \mathbb{P}\left(f(V) \geq \left(\frac{1}{2} - \frac{1}{d}\right)\eta\right) \mathbf{1}\left(\left\{\|K\|_\infty \leq \frac{3\eta}{2}\right\}\right) \\ &\leq \exp\left(-\frac{d\eta^2}{768\|K\|_\infty^2}\right) \mathbf{1}\left(\left\{\|K\|_\infty \leq \frac{3\eta}{2}\right\}\right) \\ &\leq \exp\left(-\frac{d}{1728}\right). \end{aligned}$$

Here, we used $1/d \leq 1/4$ in the second to last inequality. \square

Lemma 3.3. *Let $n \geq 2$, $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$ and $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ for $\eta > 0$ and $|v\rangle = V|0\rangle$, $V \sim \text{Haar}(d)$. Then, H is $(\eta/4)$ -far (in the operator norm) from all k -local traceless Hamiltonians of operator norm at most 1 with high probability.*

Remark 3.4. *Since for any $p \geq 1$ and any Hermitian operator X we have $\|X\|_p \geq \|X\|_\infty$, the (random) Hamiltonian H is $(\eta/4)$ -far in the p -norm from all k -local traceless Hamiltonians of operator norm at most 1. Therefore, we can focus on the case $p = \infty$.*

Proof of Lemma 3.3. We shall take \mathcal{H}_ε an ε -net in the space of k -local Hamiltonians for the ∞ -norm at the level of the coefficients. For each $K = \sum_{P \in \mathbb{P}_n, |P| \leq k} \alpha_P P \in \Pi_{S_{k-1\text{oc}}}$ such that $\|K\|_\infty \leq 1$ we have $\|\alpha\|_\infty = \max_P |\alpha_P| = \max_P \frac{\text{Tr}(PK)}{d} \leq \max_P \frac{\text{Tr}(|P|)\|K\|_\infty}{d} \leq 1$. For k -local Hamiltonians, many entries of α will be zero, namely all the ones that correspond to P such that $|P| > k$. In this case, we can construct an ε -net on the non-zero entries of cardinality at most $|\mathcal{H}_\varepsilon| \leq (1/\varepsilon)^{N_k}$ where $N_k = \sum_{s=0}^k \binom{n}{s} 3^s$. Now if

$K = \sum_{P \in \mathbb{P}_n: |P| \leq k} \alpha_P P \in \Pi_{S_{k-1\text{oc}}}$, then we can find a similar element $\tilde{K} = \sum_{P \in \mathbb{P}_n: |P| \leq k} \tilde{\alpha}_P P \in \mathcal{L}_k$ in our ε -net such that for all $|P| \leq k$ we have $|\alpha_P - \tilde{\alpha}_P| \leq \varepsilon$. Thus,

$$\|K - \tilde{K}\|_\infty \leq \|K - \tilde{K}\|_2 \leq \sqrt{d\varepsilon^2 N_k} \leq \frac{\eta}{4}$$

for $\varepsilon = \eta/(4\sqrt{dN_k})$. Now we apply the union bound and Lemma 3.2 to bound

$$\begin{aligned} \mathbb{P}_{V \sim \text{Haar}(d)} \left(\exists K \in \Pi_{S_{k-1\text{oc}}} : \|H - K\|_\infty \leq \frac{\eta}{4} \right) &\leq \mathbb{P}_{V \sim \text{Haar}(d)} \left(\exists K \in \mathcal{H}_\varepsilon : \|H - K\|_\infty \leq \frac{\eta}{2} \right) \\ &\leq \sum_{K \in \mathcal{H}_\varepsilon} \mathbb{P}_{V \sim \text{Haar}(d)} \left(\|H - K\|_\infty \leq \frac{\eta}{2} \right) \\ &\leq |\mathcal{H}_\varepsilon| \exp \left(-\frac{d}{1728} \right) \\ &= \left(\frac{1}{\varepsilon} \right)^{N_k} \exp \left(-\frac{d}{1728} \right) \\ &= \left(\frac{4\sqrt{dN_k}}{\eta} \right)^{N_k} \exp \left(-\frac{d}{1728} \right). \end{aligned}$$

Moreover, as $k \leq n/2$, we have the following simple upper bound on N_k :

$$N_k = \sum_{s=0}^k \binom{n}{s} 3^s \leq \min \left((k+1) \binom{n}{k} 3^k, 4^n \right) \leq \min \left((k+1)(3n)^k, 4^n \right).$$

Hence,

$$\left(\frac{4\sqrt{dN_k}}{\eta} \right)^{N_k} \exp \left(-\frac{d}{1728} \right) \leq \exp \left((k+1)(3n)^k \log \left(\frac{2^{3n}}{\eta} \right) - \frac{2^n}{1728} \right) \leq \exp(-\Omega(d))$$

if $n^2(3n)^k \leq c \cdot 2^n$ for a small constant c . This is valid, for instance, as long as $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$. Therefore, with high probability, $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ is not $(\eta/4)$ -close to any k -local Hamiltonian if $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$. \square

Now we proceed to prove Theorem 3.1. Our proof strategy is inspired by [Faw+23b].

Proof of Theorem 3.1. Let $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$, $U_{v,t} = e^{-i\eta t \cdot |v\rangle\langle v|}$ and let $\mathcal{U}_{v,t}$ be the unitary channel:

$$\mathcal{U}_{v,t}(\rho) = e^{-i\eta t \cdot (|v\rangle\langle v| - \mathbb{I}/d)} \rho e^{i\eta t \cdot (|v\rangle\langle v| - \mathbb{I}/d)} = e^{-i\eta t \cdot |v\rangle\langle v|} \rho e^{i\eta t \cdot |v\rangle\langle v|} = U_{v,t} \rho U_{v,t}^\dagger.$$

A 1/3-correct algorithm should distinguish between the identity channel and $\mathcal{U}_{v,t}$ with at least a probability 2/3 of success. In the incoherent setting, the tester can only choose an input ρ_k at each step k , the time evolution t_k and perform a measurement using the POVM $\mathcal{M}_k = \{\lambda_i^{(k)} |\phi_i^{(k)}\rangle\langle\phi_i^{(k)}|\}_{i \in \mathcal{I}_k}$ on the output quantum state $\mathcal{U}_{t_k}(\rho_k)$. These choices can depend on the previous observations, that is, the algorithm can be adaptive. Let $I_{\leq N} = (I_1, \dots, I_N)$ be the observations of this algorithm where N is a sufficient number of independent experiments to decide correctly with a probability at least 2/3.

Let P (resp Q) be the distribution of (I_1, \dots, I_N) under the null hypothesis \mathcal{H}_0 ($H = 0$) and the alternate hypothesis \mathcal{H}_1 ($H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$). The distribution of (I_1, \dots, I_N) under \mathcal{H}_0 is:

$$P := \left\{ \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \right\}_{i_1, \dots, i_N}.$$

Moreover, the distribution of (I_1, \dots, I_N) under \mathcal{H}_1 and conditioned on the choice of unitary V is:

$$Q_V := \left\{ \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | U_{v,t_k} \rho_k U_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle \right\}_{i_1, \dots, i_N}. \quad (4)$$

Let \mathcal{E} be the event that $\eta|V\rangle\langle 0|V^\dagger - \mathbb{I}/d$ is not $(\eta/4)$ -close to any k -local Hamiltonian. By Lemma 3.3, we have $\mathbb{P}(\mathcal{E}) \geq 1 - \exp(-\Omega(d))$. Under this event, every correct algorithm should be able to distinguish between P and Q_V so we can apply Le Cam's method [Le 73].

For technical reasons, we will use instead of Q_V the following parameterized distribution for $\alpha \in [0, 1]$:

$$Q_V^\alpha := \left\{ \prod_{k=1}^N \lambda_{i_k}^{(k)} \left\langle \phi_{i_k}^{(k)} \middle| (\alpha \rho_k + (1 - \alpha) U_{v,t_k} \rho_k U_{v,t_k}^\dagger) \middle| \phi_{i_k}^{(k)} \right\rangle \right\}_{i_1, \dots, i_N}.$$

So we need to generalize Le Cam's inequality for small α :

Lemma 3.5 (Generalized Le Cam). *Let $n \geq \Omega(1)$. Let $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$. We have for $\alpha \leq \frac{1}{10N}$:*

$$\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)] \geq \frac{1}{18}.$$

Proof. The proof can be found in Lemma C.1. □

Hence, as long as $\alpha \leq \frac{1}{10N}$ we can use Q_V^α instead of Q_V and have a similar Le Cam separation (albeit with a worse constant). From now on we will set $\alpha = \frac{1}{10N}$. The next step is to use Pinsker's inequality to move from the TV distance to the KL divergence, which is more suitable for studying the adaptive incoherent setting. By Jensen's and Pinsker's inequalities we have:

$$\frac{2}{18^2} \leq 2 \mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)]^2 \leq 2 \mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)^2] \leq \mathbb{E}_{V \sim \text{Haar}(d)} [\text{KL}(P \| Q_V^\alpha)].$$

The KL divergence can be expressed as follows:

$$\begin{aligned} \text{KL}(P \| Q_V^\alpha) &= \mathbb{E}_{i \sim P} (-\log) \left(\frac{Q_V^\alpha(i)}{P(i)} \right) \\ &= \mathbb{E}_{i \sim P} (-\log) \left(\prod_{k=1}^N \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| (\alpha \rho_k + (1 - \alpha) U_{v,t_k} \rho_k U_{v,t_k}^\dagger) \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \right) \\ &= \mathbb{E}_{i \sim P} \sum_{k=1}^N (-\log) \left(\alpha + (1 - \alpha) \frac{\left\langle \phi_{i_k}^{(k)} \middle| U_{v,t_k} \rho_k U_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right). \end{aligned}$$

Now, we will use a cut on $i \sim P$ as in [Faw+23b]. Note that here, since we do not have the freedom to choose a non-unitary channel, we choose $\alpha = \frac{1}{10N}$ rather than $\alpha = \frac{1}{2}$ as in [Faw+23b]. This comes with the cost of an additional logarithmic factor in the lower bound.

For $k \in [N]$ and $i_{\leq k} = (i_1, \dots, i_k)$, define the event $\mathcal{G}(k, i_{\leq k}) = \left\{ \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle \leq \frac{(1 - \cos(\eta t_k))}{d^2} \right\}$. We can distinguish whether the event \mathcal{G} is satisfied or not:

$$\begin{aligned} &\mathbb{E}_{V \sim \text{Haar}(d)} \text{KL}(P \| Q_V^\alpha) \\ &= \sum_{k=1}^N \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i_{\leq k}} (\mathbf{1}(\mathcal{G}(k, i_{\leq k})) + \mathbf{1}(\mathcal{G}^c(k, i_{\leq k}))) (-\log) \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| (\alpha \rho_k + (1 - \alpha) U_{v,t_k} \rho_k U_{v,t_k}^\dagger) \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right). \end{aligned}$$

Let us first analyze the setting when the event \mathcal{G} holds. Fix $k \in [N]$, observe that we have the inequality:

$$\begin{aligned} &(-\log) \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| (\alpha \rho_k + (1 - \alpha) U_{v,t_k} \rho_k U_{v,t_k}^\dagger) \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \\ &= (-\log) \left(\alpha + (1 - \alpha) \frac{\left\langle \phi_{i_k}^{(k)} \middle| U_{v,t_k} \rho_k U_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \leq \log \left(\frac{1}{\alpha} \right) \end{aligned}$$

The last inequality follows from the monotonicity of the logarithm. Then we can control the expectation under the event \mathcal{G} as follows:

$$\begin{aligned}
& \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i \leq k} \mathbf{1}(\mathcal{G}(k, i \leq k)) (-\log) \left(\frac{\langle \phi_{i_k}^{(k)} | (\alpha \rho_k + (1 - \alpha) U_{v, t_k} \rho_k U_{v, t_k}^\dagger) | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
& \leq \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i \leq k} \mathbf{1}(\mathcal{G}(k, i \leq k)) \log \left(\frac{1}{\alpha} \right) \\
& = \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i \leq k-1} \sum_{i_k} \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \mathbf{1}(\mathcal{G}(k, i \leq k)) \log \left(\frac{1}{\alpha} \right) \\
& \leq \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i \leq k-1} \sum_{i_k} \lambda_{i_k}^{(k)} \left(\frac{(1 - \cos(\eta t_k))}{d^2} \right) \mathbf{1}(\mathcal{G}(t, i \leq k)) \log \left(\frac{1}{\alpha} \right) \\
& \leq \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i \leq k-1} \sum_{i_k} \lambda_{i_k}^{(k)} \left(\frac{(1 - \cos(\eta t_k))}{d^2} \right) \log \left(\frac{1}{\alpha} \right) \\
& = \frac{\log(10N)}{d} \mathbb{E}_{i \leq k-1} (1 - \cos(\eta t_k))
\end{aligned} \tag{5}$$

where we used in the second inequality the fact that under \mathcal{G} we have $\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \leq \frac{(1 - \cos(\eta t_k))}{d^2}$ and in the last equality the fact $\sum_{i_k} \lambda_{i_k}^{(k)} = d$ which is an implication of the fact that $\mathcal{M}_k = \{\lambda_i^{(k)} | \phi_i^{(k)} \rangle \langle \phi_i^{(k)} | \}_{i \in \mathcal{I}_k}$ is a POVM. Next, under $\mathcal{G}^c(t, i \leq k)$, we will use instead the following inequality valid for all $x \in [\frac{1}{10N}, +\infty)$:

$$(-\log)(x) \leq -(x - 1) + 2 \log(10N)(x - 1)^2. \tag{6}$$

A simple proof can be found in Lemma C.2.

We apply the inequality (6) for

$$x = \frac{\langle \phi_{i_k}^{(k)} | (\alpha \rho_k + (1 - \alpha) U_{v, t_k} \rho_k U_{v, t_k}^\dagger) | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} = \alpha + (1 - \alpha) \frac{\langle \phi_{i_k}^{(k)} | U_v \rho_k U_v^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \geq \frac{1}{10N}.$$

Let $M_{v, t_k} = \mathbb{I} - U_{v, t_k} = -(e^{-i\eta t_k} - 1) |v\rangle \langle v|$ and $S_{v, t_k} = \mathbb{I} - \frac{1}{2} M_{v, t_k}$. The first term of the upper bound of Inequality (6) is

$$\begin{aligned}
-(x - 1) &= 1 - \frac{\langle \phi_{i_k}^{(k)} | (\alpha \rho_k + (1 - \alpha) U_{v, t_k} \rho_k U_{v, t_k}^\dagger) | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \\
&= (1 - \alpha) \frac{\langle \phi_{i_k}^{(k)} | M_{v, t_k} \rho_k S_{v, t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} + (1 - \alpha) \frac{\langle \phi_{i_k}^{(k)} | S_{v, t_k} \rho_k M_{v, t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \\
&= 2(1 - \alpha) \Re \frac{\langle \phi_{i_k}^{(k)} | M_{v, t_k} \rho_k S_{v, t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle},
\end{aligned} \tag{7}$$

and by using first the inequality $(x + y)^2 \leq 2(x^2 + y^2)$ and then the Cauchy Schwartz inequality applied for the vectors $\sqrt{\rho_k} | \phi_{i_k}^{(k)} \rangle$ and $\sqrt{\rho_k} M_{v, t_k}^\dagger | \phi_{i_k}^{(k)} \rangle$, we can upper bound the second term of Inequality (6) as

follows:

$$\begin{aligned}
(x-1)^2 &= \left(\frac{\langle \phi_{i_k}^{(k)} | (\alpha \rho_k + (1-\alpha) U_{v,t_k} \rho_k U_{v,t_k}^\dagger) | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} - 1 \right)^2 \\
&= (1-\alpha)^2 \left(\frac{2\Re \langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} - \frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k M_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right)^2 \\
&\leq 8(1-\alpha)^2 \left(\frac{|\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k | \phi_{i_k}^{(k)} \rangle|}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right)^2 + 2(1-\alpha)^2 \left(\frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k M_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right)^2 \\
&\leq 8(1-\alpha)^2 \left(\frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k M_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) + 2(1-\alpha)^2 \left(\frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k M_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right)^2. \quad (8)
\end{aligned}$$

Let us compute the expectation of Equation (7). Let M, S such that $M_{v,t_k} = VMV^\dagger$ and $S_{v,t_k} = VSV^\dagger$. Concretely

$$M = \left(\begin{array}{c|c} 1 - e^{-i\eta t_k} & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{0}_{d-1} \end{array} \right) \quad \text{and} \quad S = \left(\begin{array}{c|c} \frac{1}{2} + \frac{e^{-i\eta t_k}}{2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{I}_{d-1} \end{array} \right).$$

Note that $\text{Tr}(M) = (1 - e^{-i\eta t_k})$, $\text{Tr}(S) = d - \frac{1}{2} + \frac{e^{-i\eta t_k}}{2}$, $\text{Tr}(MS^\dagger) = i \sin(\eta t_k)$, $\text{Tr}(MS) = \frac{1 - e^{-2i\eta t_k}}{2}$ and $MM^\dagger = M + M^\dagger = M^\dagger M$. We have

$$\begin{aligned}
\Re(\text{Tr}(M)\text{Tr}(S)) &= \Re \left((1 - e^{-i\eta t_k}) \left(d - \frac{1}{2} + \frac{e^{-i\eta t_k}}{2} \right) \right) \\
&= (1 - \cos(\eta t_k)) \left(d - \frac{1}{2} + \frac{\cos(\eta t_k)}{2} \right) + \frac{1}{2} \sin^2(\eta t_k) \\
&\leq (1 - \cos(\eta t_k)) d + \frac{1}{2} \sin^2(\eta t_k).
\end{aligned}$$

Moreover,

$$\Re(\text{Tr}(M)\text{Tr}(S)) \geq -1/2 \sin^2(\eta t_k).$$

Hence,

$$|\Re(\text{Tr}(M)\text{Tr}(S))| \leq (1 - \cos(\eta t_k)) d + \frac{1}{2} \sin^2(\eta t_k).$$

So we have by Weingarten calculus [CS06] (see Appendix D for the most important facts used here):

$$\begin{aligned}
& \left| \mathbb{E}_{V \sim \text{Haar}(d)} \left(\Re \frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k S_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \right| \\
&= \left| \frac{\Re \mathbb{E}_{V \sim \text{Haar}(d)} \left(\langle \phi_{i_k}^{(k)} | V M V^\dagger \rho_k V S^\dagger V^\dagger | \phi_{i_k}^{(k)} \rangle \right)}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right| \\
&= \left| \frac{\Re \sum_{\alpha, \beta \in \mathfrak{S}_2} \text{Wg}(\alpha, \beta, d) \text{Tr}_\alpha(M, S^\dagger) \text{Tr}_{\beta(12)}(\rho_k, | \phi_{i_k}^{(k)} \rangle \langle \phi_{i_k}^{(k)} |)}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right| \\
&= \left| \frac{\Re \left(d \text{Tr}(M S^\dagger) + d \text{Tr}(M) \text{Tr}(S) \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle - \text{Tr}(M S^\dagger) \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle - \text{Tr}(M) \text{Tr}(S^\dagger) \right)}{d(d^2 - 1) \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right| \quad (9) \\
&= \left| \frac{1}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \left(\frac{\Re(\text{Tr}(M) \text{Tr}(S)) (d \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle - 1)}{d(d^2 - 1)} \right) \right| \\
&\leq \frac{1}{d(d^2 - 1) \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \left(\left[(1 - \cos(\eta t_k)) d + \frac{1}{2} \sin^2(\eta t_k) \right] (d \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle + 1) \right) \\
&\leq \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d} + \frac{(1 - \cos(\eta t_k))}{d^2 \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} + \frac{\sin^2(\eta t_k)}{d^2 \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right).
\end{aligned}$$

Recall the notation $\mathbb{E}_{i \leq t}(X(i_1, \dots, i_t)) = \sum_{i_1, \dots, i_t} \prod_{k=1}^t \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle X(i_1, \dots, i_t)$. If we take the expectation $\mathbb{E}_{i \leq t}$ under the event $\mathcal{G}^c(t, i_{\leq k})$, we obtain

$$\begin{aligned}
& \mathbb{E}_{V \sim \text{Haar}(d)} \mathbb{E}_{i_{\leq k}} \mathbf{1}(\mathcal{G}^c(k, i_k)) \left(\Re \frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k S_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
&\leq \mathbb{E}_{i_{\leq k}} \mathbf{1}(\mathcal{G}^c(k, i_k)) \left| \mathbb{E}_{V \sim \text{Haar}(d)} \left(\Re \frac{\langle \phi_{i_k}^{(k)} | M_{v,t_k} \rho_k S_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \right| \\
&\leq \mathbb{E}_{i_{\leq k}} \mathbf{1}(\mathcal{G}^c(k, i_k)) \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d} + \frac{(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d^2 \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
&\leq \mathbb{E}_{i_{\leq k}} \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d} + \frac{(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d^2 \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
&= \mathbb{E}_{i_{\leq k-1}} \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d} \right) + \mathbb{E}_{i_{\leq k-1}} \sum_{i_k} \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \cdot \mathcal{O} \left(\frac{(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d^2 \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
&= \mathbb{E}_{i_{\leq k-1}} \mathcal{O} \left(\frac{(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d} \right) \quad (10)
\end{aligned}$$

where we use $\sum_{i_k} \lambda_{i_k}^{(k)} = d$. We move to the expectation $\mathbb{E}_{i_{\leq k}}$ of the first term of Equation (8), it is non negative so we can safely remove the condition $\mathbf{1}(\mathcal{G}^c(k, i_k))$:

$$\begin{aligned}
& \mathbb{E}_V \mathbb{E}_{i_{\leq k}} \mathbf{1}(\mathcal{G}^c(k, i_k)) \frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \leq \mathbb{E}_V \mathbb{E}_{i_{\leq k}} \frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \\
& = \mathbb{E}_V \mathbb{E}_{i_{\leq k-1}} \sum_{i_k} \lambda_{i_k}^{(k)} \left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle \\
& = \mathbb{E}_{i_{\leq k-1}} \mathbb{E}_V \text{Tr}(M_{v,t_k} \rho_k M_{v,t_k}^\dagger) = \mathbb{E}_{i_{\leq k-1}} \mathbb{E}_V \text{Tr}((M_{v,t_k} + M_{v,t_k}^\dagger) \rho_k) \\
& = \mathbb{E}_{i_{\leq k-1}} \frac{2(1 - \cos(\eta t_k))}{d}
\end{aligned} \tag{11}$$

because $\mathbb{E}_V M_{v,t_k} = \frac{(1 - e^{-i\eta t_k})}{d} \mathbb{I}$.

Concerning the expectation of the second term of Equation (8), we apply again the Weingarten calculus (Lemma D.1) to have:

$$\begin{aligned}
& \mathbb{E}_{V \sim \text{Haar}(d)} \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right)^2 = \frac{\mathbb{E}_{V \sim \text{Haar}(d)} \left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle^2}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2} \\
& = \frac{\mathbb{E}_{V \sim \text{Haar}(d)} \text{Tr} \left(\left\langle \phi_{i_k}^{(k)} \middle| \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \middle| V M V^\dagger \rho_k V M^\dagger V^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \middle| V M V^\dagger \rho_k V M^\dagger V^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle \right)}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2} \\
& = \frac{1}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2} \sum_{\alpha, \beta \in \mathfrak{S}_4} \text{Wg}(\beta \alpha^{-1}, d) \text{Tr}_{\beta^{-1}}(M, M^\dagger, M, M^\dagger) \text{Tr}_{\alpha\gamma}(\rho_k, \left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle, \rho_k, \left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle).
\end{aligned}$$

Note that

$$\text{Tr}_{\alpha\gamma} \left[\rho_k, \left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle, \rho_k, \left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle \right] \in \left\{ 1, \text{Tr}(\rho_k^2), \left\langle \phi_{i_k}^{(k)} \middle| \rho_k^2 \middle| \phi_{i_k}^{(k)} \right\rangle, \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle, \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2 \right\},$$

$\text{Tr}(\rho_k^2) \leq 1$ and $\left\langle \phi_{i_k}^{(k)} \middle| \rho_k^2 \middle| \phi_{i_k}^{(k)} \right\rangle \leq \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle \leq 1$. Moreover, it is clear that since $\text{Tr}[M] \text{Tr}[M^\dagger] = \text{Tr}[MM^\dagger]$, we have $|\text{Tr}_\beta(M, M^\dagger, M, M^\dagger)| \leq \mathcal{O}((1 - \cos(\eta t_k))^2)$. In the case β is a 4 cycle we have $\text{Tr}(MM^\dagger MM^\dagger) = \text{Tr}(MMMM^\dagger M^\dagger) = \text{Tr}((M+M^\dagger)^2) = 4(1 - \cos(\eta t_k))^2$. Also, we know that for all $(\alpha, \beta) \in \mathfrak{S}_4^2$: $|\text{Wg}(\beta \alpha^{-1}, d)| \leq \frac{10}{d^4}$ [CS06] for $d \geq 4$, so

$$|\text{Wg}(\beta \alpha^{-1}, d) \text{Tr}_{\beta^{-1}}(M, M^\dagger, M, M^\dagger) \text{Tr}_{\alpha\gamma}(\left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle, \rho_k, \left\langle \phi_{i_k}^{(k)} \right\rangle \left\langle \phi_{i_k}^{(k)} \right\rangle, \rho_k)| \leq \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))^2}{d^4} \right).$$

Therefore we have:

$$\mathbb{E}_{V \sim \text{Haar}(d)} \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v,t_k} \rho_k M_{v,t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right)^2 \leq \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))^2}{d^4 \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2} \right).$$

Now if we take the expectation $\mathbb{E}_{i \leq k}$ under the event $\mathcal{G}^c(k, i \leq k) = \left\{ \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle > \frac{(1 - \cos(\eta t_k))}{d^2} \right\}$, we obtain:

$$\begin{aligned}
& \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v, t_k} \rho_k M_{v, t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right)^2 \\
& \leq \mathbb{E}_{i \leq k} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))^2}{d^4 \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \cdot \frac{1}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \\
& \leq \mathbb{E}_{i \leq k} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))^2}{d^4 \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \cdot \frac{d^2}{(1 - \cos(\eta t_k))} \right) \\
& = \mathbb{E}_{i \leq k-1} \sum_{i_k} \lambda_{i_k}^{(k)} \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d^2 \left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \\
& \leq \mathbb{E}_{i \leq k-1} \sum_{i_k} \lambda_{i_k}^{(k)} \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d^2} \right) = \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{d(1 - \cos(\eta t_k))}{d^2} \right) \\
& = \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{(1 - \cos(\eta t_k))}{d} \right) \tag{12}
\end{aligned}$$

where we use $\sum_{i_k} \lambda_{i_k}^{(k)} = d$. By adding up Equations (10), (11) and (12), we obtain:

$$\begin{aligned}
& \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) (-\log) \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| (\alpha \rho_k + (1 - \alpha) U_{v, t_k} \rho_k U_{v, t_k}^\dagger) \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \\
& \leq \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \left(\frac{2(1 - \alpha) \Re \left(\left\langle \phi_{i_k}^{(k)} \middle| M_{v, t_k} \rho_k S_{v, t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle \right)}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right) \\
& \quad + \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) \left(\frac{4 \log(10N) (1 - \alpha)^2 \left\langle \phi_{i_k}^{(k)} \middle| M_{v, t_k} \rho_k M_{v, t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle^2}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle^2} \right) \\
& \quad + \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} \mathbf{1}(\mathcal{G}^c(k, i \leq k)) 16 \log(10N) (1 - \alpha)^2 \left(\frac{\left\langle \phi_{i_k}^{(k)} \middle| M_{v, t_k} \rho_k M_{v, t_k}^\dagger \middle| \phi_{i_k}^{(k)} \right\rangle}{\left\langle \phi_{i_k}^{(k)} \middle| \rho_k \middle| \phi_{i_k}^{(k)} \right\rangle} \right)^2 \\
& = \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{\log(N) (1 - \cos(\eta t_k)) + \sin^2(\eta t_k)}{d} \right).
\end{aligned}$$

Therefore using this upper bound and the upper bound in Equation (5) we get an upper bound on the expected KL divergence:

$$\begin{aligned}
& \mathbb{E}_{V \sim \text{Haar}(d)} \text{KL}(P \| Q_V^\alpha) \\
&= \sum_{k=1}^N \mathbb{E}_{i \leq k} \mathbb{E}_{V \sim \text{Haar}(d)} (\mathbf{1}(\mathcal{G}(t, i \leq k)) + \mathbf{1}(\mathcal{G}^c(k, i \leq k))) (-\log) \left(\frac{\langle \phi_{i_k}^{(k)} | \mathcal{U}_{t_k}(\rho_k) | \phi_{i_k}^{(k)} \rangle}{\langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle} \right) \\
&\leq \sum_{k=1}^N \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{\log(N)(1 - \cos(\eta t_k))}{d} \right) + \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{\log(N)(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d} \right) \\
&= \sum_{k=1}^N \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{\log(N)(1 - \cos(\eta t_k) + \sin^2(\eta t_k))}{d} \right) \\
&\leq \sum_{k=1}^N \mathbb{E}_{i \leq k-1} \mathcal{O} \left(\frac{\log(N) \min(1, \eta t_k)}{d} \right)
\end{aligned}$$

where we used $1 - \cos(x) \leq \min(1, x)$ and $\sin^2(x) \leq \min(1, x)$ for $x \geq 0$. Finally since $\mathbb{E}_{V \sim \text{Haar}(d)} \text{KL}(P \| Q_V^\alpha) \geq \frac{2}{18^2}$ we deduce that:

$$\sum_{k=1}^N \mathbb{E}_{i \leq k-1} \min(1, \eta t_k) = \Omega \left(\frac{d}{\log(N)} \right).$$

In particular, we have $N \log(N) \geq \Omega(d)$ which implies that:

$$N = \Omega \left(\frac{d}{\log(d)} \right).$$

Finally, the expected total evolution time is lower bounded as follows:

$$\mathbb{E} \left[\sum_{k=1}^N t_k \right] = \Omega \left(\frac{d}{\eta \log(N)} \right).$$

We can set $\varepsilon = \eta/4$ and use our assumption that $N \leq \exp(\mathcal{O}(n))/\varepsilon^{\mathcal{O}(1)}$ to get the claimed expected total evolution time lower bound. \square

3.2 Coherent setting

In section, we will prove another hardness result in the more general coherent setting. However, we will pay for the greater generality with slightly weaker bounds.

Theorem 3.6. *Let $n \geq 2$, $k \leq \mathcal{O}\left(\frac{n}{\log(n)}\right)$, and $p \geq 1$. The problem $\mathcal{T}_{\|\cdot\|_p}^{\text{loc}}(\eta)$, even under the additional promise that the unknown Hamiltonian H satisfies $\text{Tr}[H] = 0$ and $\|H\|_\infty \leq 1$, requires a total evolution time of $\sum_{k=1}^N t_k = \Omega\left(\frac{2^{n/2}}{\varepsilon}\right)$ and a total number of independent experiments $N = \Omega(2^{n/2})$ in the ancilla-assisted coherent setting.*

In order to prove Theorem 3.6, we will again consider the following test:

$$\mathcal{H}_0 : \mathcal{U}_t(\rho) = \text{id}(\rho) = e^{-it \cdot 0} \rho e^{it \cdot 0} \quad \text{vs} \quad \mathcal{H}_1 : \mathcal{U}_t(\rho) = e^{-itH} \rho e^{itH} \tag{13}$$

where $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ for $\eta > 0$ and $|v\rangle = V|0\rangle$, $V \sim \text{Haar}(d)$.

Proof of Theorem 3.6. We use here the construction from Lemma 3.3. Let $H = \eta(|v\rangle\langle v| - \mathbb{I}/d)$ for $\eta > 0$ and $|v\rangle = V|0\rangle$, $V \sim \text{Haar}(d)$. H is $(\eta/4)$ -far (in the operator norm) from any k -local trace-less Hamiltonian of unit operator norm with high probability. Let \mathcal{E} be the event that H is $(\eta/4)$ -far (in the operator norm,

which we can again focus on as discussed in Remark 3.4) from any k -local trace-less Hamiltonian of unit operator norm. We have by Lemma 3.3:

$$\mathbb{P}_{V \sim \text{Haar}(d)} [\mathcal{E}^c] \leq \exp(-\Omega(d)).$$

In the sequential setting the tester can choose times t_1, \dots, t_N and any $(d \times d_{\text{aux}})$ -dimensional operations $\mathcal{N}_1, \dots, \mathcal{N}_{N-1}$, a $(d \times d_{\text{aux}})$ -dimensional input state ρ and a $(d \times d_{\text{aux}})$ -dimensional measurement device \mathcal{M} . Under the null hypothesis \mathcal{H}_0 , the map \mathcal{U}_t is always the identity so the output state (before the measurement) is:

$$\rho_0^{\text{output}}(\rho) = \mathcal{N}_{N-1} \circ \dots \circ \mathcal{N}_2 \circ \mathcal{N}_1(\rho).$$

In contrast, under the alternate hypothesis \mathcal{H}_1 , the map \mathcal{U}_t is close to the identity and the output state (before the measurement) is:

$$\rho_1^{\text{output}}(\rho) = [\mathcal{U}_{t_N} \otimes \text{id}] \circ \mathcal{N}_{N-1} \circ [\mathcal{U}_{t_{N-1}} \otimes \text{id}] \circ \dots \circ \mathcal{N}_2 \circ [\mathcal{U}_{t_2} \otimes \text{id}] \circ \mathcal{N}_1 \circ [\mathcal{U}_{t_1} \otimes \text{id}](\rho).$$

We will often drop the argument ρ for readability if it is not explicitly needed. On the one hand, using the correctness of the algorithm and the data processing inequality applied on the 1-norm we have:

$$\|\rho_0^{\text{output}} - \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [\rho_1^{\text{output}}]\|_1 \geq 2 \text{TV}(\text{Bern}(1/3) \| \text{Bern}(2/3)) = \frac{2}{3}.$$

On the other hand, we have by the triangle inequality

$$\begin{aligned} & \|\mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}] - \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [\rho_1^{\text{output}}]\|_1 \\ &= \frac{1}{\mathbb{P}(\mathcal{E})} \|\mathbb{P}(\mathcal{E}) \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}] - \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}} \mathbf{1}(\{\mathcal{E}\})]\|_1 \\ &= \frac{1}{\mathbb{P}(\mathcal{E})} \|\mathbb{P}(\mathcal{E}) \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}} \mathbf{1}(\{\mathcal{E}^c\})] - \mathbb{P}(\mathcal{E}^c) \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}} \mathbf{1}(\{\mathcal{E}\})]\|_1 \\ &\leq \|\mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}} \mathbf{1}(\{\mathcal{E}^c\})]\|_1 + \frac{\mathbb{P}(\mathcal{E}^c)}{\mathbb{P}(\mathcal{E})} \|\mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}} \mathbf{1}(\{\mathcal{E}\})]\|_1 \\ &\leq \mathbb{P}(\mathcal{E}^c) + \frac{\mathbb{P}(\mathcal{E}^c)}{\mathbb{P}(\mathcal{E})} \mathbb{P}(\mathcal{E}) \\ &\leq 2 \exp(-\Omega(d)) \end{aligned}$$

hence for $d = \Omega(1)$, by the triangle inequality:

$$\begin{aligned} & \|\rho_0^{\text{output}} - \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}]\|_1 \\ &\geq \|\rho_0^{\text{output}} - \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [\rho_1^{\text{output}}]\|_1 - \|\mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}] - \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [\rho_1^{\text{output}}]\|_1 \\ &\geq \frac{2}{3} - 2 \exp(-\Omega(d)) \geq \frac{1}{3}. \end{aligned}$$

Writing the input state as $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$, e.g., using its spectral decomposition, the triangle equality implies:

$$\sum_i \lambda_i \|\rho_0^{\text{output}}(|\phi_i\rangle\langle\phi_i|) - \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}(|\phi_i\rangle\langle\phi_i|)]\|_1 \geq \frac{1}{3}.$$

So there is a unit vector $|\phi\rangle = |\phi_i\rangle$ such that:

$$\|\rho_0^{\text{output}}(|\phi\rangle\langle\phi|) - \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}(|\phi\rangle\langle\phi|)]\|_1 \geq \frac{1}{3}.$$

We use the notation $U_t = e^{-itH}$ and the following Kraus representation of each channel $\mathcal{N}_t(\rho) = \sum_{k_i} A_{k_i} \rho A_{k_i}^\dagger$. Moreover, we will use the following shorthand notations for $\ell, m \in \mathbb{N}$, $\ell \leq m$:

$$\begin{aligned} \overrightarrow{\prod}_{i=\ell::m} X_i &:= X_\ell X_{\ell+1} \dots X_m, \\ \overleftarrow{\prod}_{i=\ell::m} X_i &:= X_m X_{m-1} \dots X_\ell. \end{aligned}$$

So we can write:

$$\begin{aligned} \rho_0^{\text{output}}(|\phi\rangle\langle\phi|) &= \sum_{k_1, \dots, k_{N-1}} \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} A_{k_i}^\dagger \right) \\ \rho_1^{\text{output}}(|\phi\rangle\langle\phi|) &= \sum_{k_1, \dots, k_{N-1}} (U_{t_N} \otimes \mathbb{I}) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i}(U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) (U_{t_N} \otimes \mathbb{I})^\dagger \end{aligned}$$

Hence, the triangle inequality implies

$$\begin{aligned} &\left\| \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}(|\phi\rangle\langle\phi|)] - \rho_0^{\text{output}}(|\phi\rangle\langle\phi|) \right\|_1 \\ &\leq \mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \left\| (U_{t_N} \otimes \mathbb{I}) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i}(U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) (U_{t_N} \otimes \mathbb{I})^\dagger \right. \\ &\quad \left. - \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} A_{k_i}^\dagger \right) \right\|_1. \end{aligned}$$

We can write the latter difference of states as a telescopic sum. A subsequent application of the triangle inequality yields:

$$\begin{aligned} &\mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \left\| (U_{t_N} \otimes \mathbb{I}) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i}(U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) (U_{t_N} \otimes \mathbb{I})^\dagger \right. \\ &\quad \left. - \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} A_{k_i}^\dagger \right) \right\|_1 \\ &\leq \mathbb{E}_V \sum_{s=1}^N \sum_{k_1, \dots, k_{N-1}} \left\| \left(\overleftarrow{\prod}_{s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}] \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \right\|_1 \end{aligned} \tag{A}$$

$$+ \mathbb{E}_V \sum_{s=1}^N \sum_{k_1, \dots, k_{N-1}} \left\| \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}]^\dagger \left(\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right) \right\|_1 \tag{B}$$

On the one hand, using $U_t - \mathbb{I} = (e^{-i\eta t} - 1) |v\rangle\langle v|$ we have

$$\begin{aligned}
& \left\| \left(\overleftarrow{\prod}_{s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}] \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \right\|_1 \\
&= |e^{-i\eta t_s} - 1| \left\| \left(\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right) (|v\rangle\langle v| \otimes \mathbb{I}) \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \right\|_1 \\
&\leq |e^{-i\eta t_s} - 1| \sqrt{\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} (U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle} \\
&\sqrt{\langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I}) \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] |\phi\rangle}
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Again, by using the Cauchy-Schwarz inequality for the sums over k_1, \dots, k_{N-1} and \mathbb{E}_V as well as using the Kraus identities $\sum_{k_i} A_{k_i}^\dagger A_{k_i} = \mathbb{I}$ in the last line we obtain:

$$\begin{aligned}
& \text{(A)} \\
&= \sum_{s=1}^N \mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \left\| \left(\overleftarrow{\prod}_{s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}] \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \right\|_1 \\
&\leq \sum_{s=1}^N \mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} |e^{-i\eta t_s} - 1| \sqrt{\langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} (U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle} \\
&\sqrt{\langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I}) \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] |\phi\rangle} \\
&\leq \sum_{s=1}^N |e^{-i\eta t_s} - 1| \sqrt{\mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \langle\phi| \left(\overrightarrow{\prod}_{i=1::N-1} (U_{t_i}^\dagger \otimes \mathbb{I}) A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} (U_{t_i} \otimes \mathbb{I}) \right) |\phi\rangle \left(\mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \right.} \\
&\left. \langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I}) \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] |\phi\rangle \right)^{\frac{1}{2}}} \\
&= \sum_{s=1}^N |e^{-i\eta t_s} - 1| \sqrt{\frac{1}{d}}
\end{aligned}$$

as $\mathbb{E}_{V \sim \text{Haar}(d)} [|v\rangle\langle v|] = \mathbb{E}_{V \sim \text{Haar}(d)} [V|0\rangle\langle 0|V^\dagger] = \frac{\text{Tr}(|0\rangle\langle 0|)\mathbb{I}}{d} = \frac{\mathbb{I}}{d}$. On the other hand, using $U_t - \mathbb{I} = (e^{-i\eta t} - 1) |v\rangle\langle v|$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
& \left\| \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}]^\dagger \left(\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right) \right\|_1 \\
&= |e^{i\eta t_s} - 1| \left\| \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) [|v\rangle\langle v| \otimes \mathbb{I}] \left(\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right) \right\|_1 \\
&\leq |e^{i\eta t_s} - 1| \sqrt{\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle} \\
&\sqrt{\langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] |\phi\rangle}.
\end{aligned}$$

Hence by using the Cauchy-Schwarz inequality for the sums over k_1, \dots, k_{N-1} and the expectation \mathbb{E}_V and by using the Kraus identities $\sum_{k_i} A_{k_i}^\dagger A_{k_i} = \mathbb{I}$ in the last line we obtain:

$$\begin{aligned}
& \text{(B)} \\
&= \sum_{s=1}^N \mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \left\| \left(\overleftarrow{\prod}_{i=1::N-1} A_{k_i} \right) |\phi\rangle\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) [(U_{t_s} - \mathbb{I}) \otimes \mathbb{I}]^\dagger \left(\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right) \right\|_1 \\
&\leq \sum_{s=1}^N \mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} |e^{i\eta t_s} - 1| \sqrt{\langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle} \\
&\quad \sqrt{\langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right] |\phi\rangle} \\
&\leq \sum_{s=1}^N |e^{i\eta t_s} - 1| \sqrt{\mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \langle\phi| \left(\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right) \left(\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right) |\phi\rangle \left(\mathbb{E}_V \sum_{k_1, \dots, k_{N-1}} \langle\phi| \left[\overrightarrow{\prod}_{i=1::s-1} A_{k_i}^\dagger \right] \right.} \\
&\quad \left. [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overrightarrow{\prod}_{i=s::N-1} A_{k_i}^\dagger (U_{t_{i+1}} \otimes \mathbb{I})^\dagger \right] \left[\overleftarrow{\prod}_{i=s::N-1} (U_{t_{i+1}} \otimes \mathbb{I}) A_{k_i} \right] [|v\rangle\langle v| \otimes \mathbb{I}] \left[\overleftarrow{\prod}_{i=1::s-1} A_{k_i} \right] |\phi\rangle \right)^{\frac{1}{2}}} \\
&= \sum_{s=1}^N |e^{i\eta t_s} - 1| \sqrt{\frac{1}{d}}.
\end{aligned}$$

Therefore, using $|e^{i\eta t} - 1| = \sqrt{(\cos(\eta t) - 1)^2 + \sin^2(\eta t)} = \sqrt{2(1 - \cos(\eta t))} \leq \min\{\sqrt{2}, \sqrt{2(\eta t)^2}\} = \min\{\sqrt{2}, \sqrt{2}\eta t\}$, we get the following upper bound:

$$\begin{aligned}
\left\| \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}(|\phi\rangle\langle\phi|)] - \rho_0^{\text{output}}(|\phi\rangle\langle\phi|) \right\|_1 &\leq \text{(A)} + \text{(B)} \\
&\leq 2 \sum_{s=1}^N |e^{i\eta t_s} - 1| \sqrt{\frac{1}{d}} \\
&\leq \frac{2}{\sqrt{d}} \sum_{s=1}^N \min\{2, \sqrt{2}\eta t_s\}
\end{aligned}$$

Finally, as $\left\| \mathbb{E}_{V \sim \text{Haar}(d)} [\rho_1^{\text{output}}(|\phi\rangle\langle\phi|)] - \rho_0^{\text{output}}(|\phi\rangle\langle\phi|) \right\|_1 \geq \frac{1}{3}$, we deduce that

$$\sum_{k=1}^N t_k \geq \frac{\sqrt{d}}{6\sqrt{2}\eta}$$

and

$$N \geq \frac{\sqrt{d}}{6\sqrt{2}}.$$

We can set $\varepsilon = \eta/4$ to finish the proof. \square

4 Upper bounds for Hamiltonian property testing

4.1 Upper bounds inherited from Hamiltonian learning

Before presenting our Hamiltonian property testing results, we discuss what the Hamiltonian learning results of [Car23; CW23] imply for testing. To the best of our knowledge, these are currently the only Hamiltonian learning algorithms from dynamics that work for arbitrary Hamiltonians without locality assumptions, and

thus the only ones immediately applicable to our locality testing scenario. Analyzing their performance as testers will further highlight the importance of the chosen norms and in particular demonstrate that a different approach is needed when taking the normalized Frobenius norm as distance measure.

[Car23; CW23] gave two different approaches – the former based on Pauli shadow tomography methods applied to the Choi state of the forward short-time evolution in combination with Chebyshev interpolation for polynomial derivative estimation, the latter using forward and backward short-time evolution and block-encodings to create pseudo-Choi states as well as (classical) shadow tomography tools – for learning an unknown Hamiltonian from query access. For our purposes, it is important to carefully consider the performance measure in their learning task. Namely, the procedures of both papers produce $\|\cdot\|_{\text{Pauli},p}$ approximations to the coefficient vector of an arbitrary unknown Hamiltonian. Concretely, [Car23, Theorem 1.3] achieves this for $p = \infty$. That is, their Hamiltonian learning algorithm produces estimates $\hat{\alpha}_P$, $P \in \{\mathbb{I}, X, Y, Z\}^{\otimes n} \setminus \{\mathbb{I}^{\otimes n}\}$, that satisfy $|\hat{\alpha}_P - \alpha_P| \leq \varepsilon$ simultaneously for all $P \in \{\mathbb{I}, X, Y, Z\}^{\otimes n} \setminus \{\mathbb{I}^{\otimes n}\}$, using $\tilde{\mathcal{O}}\left(\frac{n\|H\|_\infty^4}{\varepsilon^4}\right)$ queries to the Hamiltonian evolution, each for short time $t = \tilde{\mathcal{O}}\left(\frac{1}{\|H\|_\infty}\right)$, thus leading to a total evolution time of $\tilde{\mathcal{O}}\left(\frac{n\|H\|_\infty^3}{\varepsilon^4}\right)$. The guarantees in [CW23] are phrased for $p = 2$ and use a number of queries to the Hamiltonian evolution that scales linearly with the number of Pauli terms in the Hamiltonian. While not discussed explicitly in [CW23], this direct dependence on the number of terms can be removed when focusing on $p = \infty$.

As learning is a more demanding task than testing, the results of [Car23; CW23] immediately imply (even tolerant) Hamiltonian locality testers with the same query complexities and total evolution times as their learning procedures. However, there is an important caveat: This works only for the norm $\|\cdot\|_{\text{Pauli},\infty}$. When trying to solve a Hamiltonian property testing problem w.r.t. $\|\cdot\|_{\text{Pauli},p}$ for any $1 \leq p < \infty$, the only bounds that can be obtained immediately from [Car23; CW23] (via Hölder’s inequality) scale exponentially in n . This complication arises because in our testing task we do not want to make *any* assumptions on the unknown Hamiltonian, in particular it can have exponentially many Pauli terms. When considering unnormalized Schatten p -norms $\|\cdot\|_p$ on the level of the Hamiltonians, the situation is similarly bad if not worse, since naive attempts at controlling a $\|\cdot\|_p$ -difference even via the $\|\cdot\|_{\text{Pauli},1}$ -difference incur an additional exponential overhead due to $\|P\|_p = 2^{n/p}$ for all n -qubit Pauli strings P . Even normalizing the Schatten p -norms does not resolve this issue. For instance, by Parseval’s identity, $\frac{1}{\sqrt{2^n}}\|\cdot\|_2 = \|\cdot\|_{\text{Pauli},2}$, but we have observed above that the complexities of [Car23; CW23] scale exponentially for the case of $\|\cdot\|_{\text{Pauli},2}$ and arbitrary unknown Hamiltonians with potentially exponentially many terms.

In summary, while the results of [Car23; CW23] can in principle be used for Hamiltonian property testing, and even for the tolerant version thereof, they suffer from exponential query complexities and total evolution times for any of our norms of interest except for the weakest, $\|\cdot\|_{\text{Pauli},\infty}$. In particular, they do not give rise to query-efficient solutions for the operationally relevant norms $\|\cdot\|_\infty$ and $\frac{1}{\sqrt{2^n}}\|\cdot\|_2$. Additionally, their methods require potentially challenging-to-implement quantum capabilities (such as maximally entangled input states, access to both forward and backward time evolution, and/or entangled multi-copy measurements for shadows). Finally, neither of the two approaches achieves computational efficiency, even for $\|\cdot\|_{\text{Pauli},\infty}$. Thus, while relevant for Hamiltonian learning, we consider [Car23; CW23] insufficient for our Hamiltonian testing purposes and thus develop a new procedure that is tailored to the testing task at hand.

4.2 Upper bounds in the randomized measurement framework

In this section, we will prove a general theorem that shows that efficient property testing is possible with respect to the normalized Schatten-2 norm, from which Theorem 1.3 follows as a special case.

Definition 4.1 (Relation between states according to a property). *Let $|\phi\rangle$ and $|\psi\rangle$ be two unit vectors and $S \subset \mathbb{P}_n$. We say that $|\phi\rangle$ and $|\psi\rangle$ are equivalent under the property S if they are equal or $|\psi\rangle$ can be obtained from $|\phi\rangle$ by applying a Pauli operator in S . We denote this relation by \sim_S and its negation as $\not\sim_S$. Formally,*

$$\begin{aligned} |\phi\rangle \sim_S |\psi\rangle &\Leftrightarrow \exists \theta \in [0, 2\pi), \exists P \in S \cup \{\mathbb{I}\} : P|\phi\rangle = e^{i\theta}|\psi\rangle \\ &\Leftrightarrow \exists P \in S \cup \{\mathbb{I}\} : |\langle \phi | P | \psi \rangle| = 1. \end{aligned}$$

If $|\phi\rangle \not\sim_S |\psi\rangle$, we say that a violation of the property S is detected by the pair of unit vectors $(|\phi\rangle, |\psi\rangle)$.

Algorithm 1: Testing Properties for Hamiltonian Evolutions

Input : A Hamiltonian H , a property Π_S , and an accuracy parameter $\varepsilon \in (0, 1)$
Output: The null hypothesis H_0 or the alternate hypothesis H_1

- 1 $t \leftarrow \frac{\varepsilon}{6}$;
- 2 $N \leftarrow \left\lceil \frac{2 \log(3)}{t^2 \varepsilon^2} \right\rceil$;
- 3 **for** $s \leftarrow 1$ **to** N **do**
- 4 Sample $i_s \sim \text{Uniform}[d]$, $j_s \sim \text{Uniform}[d + 1]$;
- 5 Input state : $\rho_s = |\phi_{i_s, j_s}\rangle\langle\phi_{i_s, j_s}|$;
- 6 Evolve under H for time t ;
- 7 Measurement : $\mathcal{M}_s = \{|\phi_{i_s, \ell}\rangle\langle\phi_{i_s, \ell}|\}_{\ell}$ and observe $\ell_s \leftarrow \mathcal{M}_s(\mathcal{U}_t(\rho_s))$;
- 8 **if** $|\phi_{i_s, j_s}\rangle \approx_S |\phi_{i_s, \ell_s}\rangle$ **then**
- 9 **return** H_1 **and stop**
- 10 **end**
- 11 **end**
- 12 **return** H_0

With this definition in place, we can give the algorithm for testing the property Π_S as Algorithm 1. Recall that the $|\phi_{i,j}\rangle\langle\phi_{i,j}|$ are the MUBs constructed from stabilizer states defined in Equation (2). In Algorithm 1, the Hamiltonian is given as a black box that, given a time, runs the time evolution under the Hamiltonian for each input state provided and allows for any measurement at the end. The property Π_S is given as a list of strings specifying the Pauli operators in S .

Theorem 4.2. *Let $S \subset \mathbb{P}_n$ such that $|S \cup \{\mathbb{I}\}| \leq \frac{(2^n + 1)\varepsilon^4}{144}$, and let $\varepsilon \in (0, 1)$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_{\infty} \leq 1$. Algorithm 1 tests whether $H \in \Pi_S$ or $\frac{1}{\sqrt{2^n}}\|H - K\|_2 > \varepsilon$ for all Hamiltonians $K \in \Pi_S$ with probability at least $2/3$ using a total evolution time $\mathcal{O}(\frac{1}{\varepsilon^3})$, a total number of independent experiments $N = \mathcal{O}(\frac{1}{\varepsilon^4})$, and a total classical processing time $\mathcal{O}(\frac{n^2|S \cup \{\mathbb{I}\}|}{\varepsilon^4})$. Each experiment uses efficiently implementable states and measurements.*

In fact, as we argue in the proof of Theorem 4.2, the procedure uses only stabilizer state inputs and stabilizer basis measurements. Each of these can be realized with Clifford circuits and thus with at most $\mathcal{O}(\frac{n^2}{\log n})$ many Hadamard, phase, and controlled-NOT gates [AG04]. Thus, Algorithm 1 is efficient in terms of the number of experiments, the total evolution time, and the classical and quantum processing time.

Testing locality corresponds to the property $\Pi_{S_{k\text{-local}}}$, where $S_{k\text{-local}} = \{P \in \mathbb{P}_n : |P| \leq k\}$ satisfies $|S| = \sum_{s=0}^k \binom{n}{s} 3^s \leq (3n)^{k+1}$. For this special case, we obtain:

Corollary 4.3 (Testing locality). *Let $n \geq 2$ and $\varepsilon > 0$ be such that $(3n)^{k+1} \leq \frac{(2^n + 1)\varepsilon^4}{144}$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_{\infty} \leq 1$. Algorithm 1, when given the property $S = S_{k\text{-local}}$, tests whether H is k -local or $\frac{1}{\sqrt{d}}\|H - H_{\text{local}}\|_2 > \varepsilon$ for all k -local Hamiltonians H_{local} with probability at least $2/3$ using a total evolution time $\mathcal{O}(\frac{1}{\varepsilon^3})$, a total number of independent experiments $N = \mathcal{O}(\frac{1}{\varepsilon^4})$, and a total classical processing time $\mathcal{O}(\frac{(3n)^{k+3}}{\varepsilon^4})$.*

Remark 4.4 (Testing many properties). *In situations where we are interested in testing many properties at once or we are not confident about the exact property we want to test during the data acquisition phase, we should find a way to perform the Hamiltonian property testing with an arbitrarily small error probability. It turns out that changing the data processing (statistic/estimator) part of Algorithm 1 slightly solves this issue. Concretely, using the concentration of an estimator that compares the empirical number of violations, i.e., counting how many outcomes are measured such that $|\phi_{i_s, j_s}\rangle \approx_S |\phi_{i_s, \ell_s}\rangle$, with a threshold, we are able to achieve an error probability δ with a complexity that scales as $\log(1/\delta)$. Via a union bound and setting $\delta \mapsto \delta/M$, this allows us to test many properties at once with only an overhead that is logarithmic in M , number of properties. See Appendix B.1 for details.*

Remark 4.5 (Assumption on the set S). *The assumption on the set $S - |S \cup \{\mathbb{I}\}| \leq \frac{(2^n + 1)\varepsilon^4}{144}$ — for which we can prove a rigorous guarantee on the complexity of Algorithm 1 limits the range of properties we can test. If we are only interested in having efficient tests and thus $|S| \leq \mathcal{O}(\text{poly}(n))$, this assumption should always be satisfied. Nonetheless, we are able to extend the result of Theorem 4.2 to any set S using an additional number n_{aux} of ancilla qubits, where $n_{\text{aux}} = \left\lceil \log_2 \left(\frac{144 \cdot |S \cup \{\mathbb{I}\}|}{2^n \varepsilon^4} \right) \right\rceil$. See Appendix B.2 for details.*

Remark 4.6 (Tolerant testing). *In Theorem 4.2, the null hypothesis is that the unknown Hamiltonian H is itself an element of Π_S . In the spirit of tolerant property testing [PRR06], one may aim to weaken this to H merely being close to the set Π_S . In Appendix B.3, we extend our result to this tolerant Hamiltonian property testing scenario.*

Remark 4.7 (Assumption of independent and identically distributed (i.i.d.) input). *We assumed that the tester runs N i.i.d. unitary evolutions in order to decide the correct hypothesis. Note that the proof works even if the unitary evolutions $\mathcal{U}_1, \dots, \mathcal{U}_N$ are not identical, i.e., the Hamiltonians may be different as long as they remain in the same hypothesis class. However, the assumption of independence is crucial for the proof. Moreover, the naive application of the de Finetti theorem (on the Choi state, as in [Faw+24]) would require an overhead in the copy complexity that is exponential in the number of qubits.*

Remark 4.8 (Implications for testing w.r.t. other norms). *Theorem 4.2 is phrased in terms of $\frac{1}{\sqrt{2^n}} \|\cdot\|_2$. Recalling that $\frac{1}{\sqrt{2^n}} \|\cdot\|_2 = \|\cdot\|_{\text{Pauli},2}$ as well as the monotonicity $\|\cdot\|_{\text{Pauli},p} \leq \|\cdot\|_{\text{Pauli},q}$ for $1 \leq q \leq p \leq \infty$, we immediately see that the results of Theorem 4.2 also apply to Hamiltonian property testing with $\|\cdot\|_{\text{Pauli},p}$ for any $p \geq 2$ as distance measure. Similarly, as the normalized Schatten p -norms satisfy the monotonicity property $\frac{1}{2^{n/p}} \|\cdot\|_p \leq \frac{1}{2^{n/q}} \|\cdot\|_q$ for $1 \leq p \leq q \leq \infty$, the results of Theorem 4.2 immediately carry over to testing w.r.t. $\frac{1}{2^{n/p}} \|\cdot\|_p$ for any $1 \leq p \leq 2$.*

Proof of Theorem 4.2. We need to show that the error probability under the null and alternate hypotheses is at most $1/3$.

Error probability under the alternate hypothesis. Here we suppose that H is ε -far from Π_S . Let $\mathbf{i} = (i_1, \dots, i_N)$, $\mathbf{j} = (j_1, \dots, j_N)$ and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_N)$. The error probability is

$$\mathbb{E}_{\mathbf{i}, \mathbf{j}, \boldsymbol{\ell}} [\mathbb{P}_{H_1} [|\phi_{i_1, \ell_1}\rangle \sim_S |\phi_{i_1, j_1}\rangle, \dots, |\phi_{i_N, \ell_N}\rangle \sim_S |\phi_{i_N, j_N}\rangle]] = \mathbb{E}_{i_1, j_1, \ell_1} [\mathbb{P} (|\phi_{i_1, \ell_1}\rangle \sim_S |\phi_{i_1, j_1}\rangle)]^N.$$

We have

$$\begin{aligned} \mathbb{E}_{\mathbf{i}, \mathbf{j}, \boldsymbol{\ell}} [\mathbb{P} (|\phi_{i, \ell}\rangle \sim_S |\phi_{i, j}\rangle)] &= \frac{1}{d(d+1)} \sum_{i, j, \ell} |\langle \phi_{i, \ell} | e^{itH} | \phi_{i, j} \rangle|^2 \mathbf{1} (\{|\phi_{i, \ell}\rangle \sim_S |\phi_{i, j}\rangle\}) \\ &= \frac{1}{d(d+1)} \sum_{i, j, \ell} |\langle \phi_{i, \ell} | e^{itH} | \phi_{i, j} \rangle|^2 \mathbf{1} (\{\exists \theta, \exists P \in S \cup \{\mathbb{I}\} : e^{i\theta} |\phi_{i, \ell}\rangle = P |\phi_{i, j}\rangle\}) \\ &\leq \frac{1}{d(d+1)} \sum_{P \in S \cup \{\mathbb{I}\}} \sum_{i, j} |\langle \phi_{i, j} | P e^{itH} | \phi_{i, j} \rangle|^2 \sum_{\ell} \mathbf{1} (\{\exists \theta : e^{i\theta} |\phi_{i, \ell}\rangle = P |\phi_{i, j}\rangle\}) \\ &\stackrel{(a)}{\leq} \frac{1}{d(d+1)} \sum_{P \in S \cup \{\mathbb{I}\}} \sum_{i, j} |\langle \phi_{i, j} | P e^{itH} | \phi_{i, j} \rangle|^2 \\ &\stackrel{(b)}{=} \sum_{P \in S \cup \{\mathbb{I}\}} \frac{\text{Tr} (P e^{itH} e^{-itH} P^\dagger) + |\text{Tr} (P e^{itH})|^2}{d(d+1)} \\ &\stackrel{(c)}{=} \sum_{P \in S \cup \{\mathbb{I}\}} \frac{d}{d(d+1)} + \sum_{P \in S \cup \{\mathbb{I}\}} \frac{1}{d(d+1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr} (P H^m) \right|^2 \end{aligned}$$

where we used in (a) that $\sum_{\ell} \mathbf{1} (\{\exists \theta : e^{i\theta} |\phi_{i, \ell}\rangle = P |\phi_{i, j}\rangle\}) \leq 1$ because, if we have $\theta_1, \ell_1, \theta_2, \ell_2$ such that $e^{i\theta_1} |\phi_{i, \ell_1}\rangle = P |\phi_{i, j}\rangle = e^{i\theta_2} |\phi_{i, \ell_2}\rangle$, then $|\langle \phi_{i, \ell_1} | \phi_{i, \ell_2} \rangle| = 1$, but $\{|\phi_{i, \ell}\rangle\}_i$ is an orthonormal basis, so $\ell_1 = \ell_2$. In (b), we used the fact that $\{|\phi_{i, j}\rangle\}_{i, j}$ forms a 2-design.

The first term of (c) can be computed exactly:

$$\sum_{P \in S \cup \{\mathbb{I}\}} \frac{d}{d(d+1)} = \frac{|S \cup \{\mathbb{I}\}|}{d+1}.$$

For the second term of (c), we deal first with the case $P = \mathbb{I}$:

$$\begin{aligned} \frac{1}{d(d+1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 &= \frac{1}{d(d+1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(H^m) \right|^2 \\ &\leq \frac{1}{d(d+1)} \left| d - \frac{t^2}{2} \text{Tr}(H^2) + \sum_{m \geq 3} \frac{(it)^m}{m!} \text{Tr}(H^m) \right|^2 \\ &\leq \frac{1}{d(d+1)} (d^2 - dt^2 \text{Tr}(H^2) + 4d^2 t^4) \\ &= \frac{1}{(d+1)} \left(d - t^2 d \sum_P |\alpha_P|^2 + 4dt^4 \right). \end{aligned}$$

In the third step, we evaluated the squared absolute value $|z|^2 = \bar{z}z$ and bound the higher order terms, using that $|\text{Tr}(H^m)| \leq d \|H\|_\infty^m \leq d$ and $\sum_{m \geq 4} \frac{t^m}{m!} \text{Tr}(H^m) \leq 0.06 \cdot dt^4$ since $t \leq 1$. Note that third order terms vanish.

For the other cases in (c):

$$\begin{aligned} &\sum_{P \in S \setminus \{\mathbb{I}\}} \frac{1}{d(d+1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 \\ &= \frac{1}{d(d+1)} \sum_{P \in S \setminus \{\mathbb{I}\}} \left| (it)d\alpha_P + \sum_{m \geq 2} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 \\ &= \sum_{P \in S \setminus \{\mathbb{I}\}} \frac{|(it)d\alpha_P|^2}{d(d+1)} + \frac{1}{d(d+1)} \sum_{P \in S \setminus \{\mathbb{I}\}} \left| \sum_{m \geq 2} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 + 2\Re \sum_{P \in S \setminus \{\mathbb{I}\}} \frac{itd\alpha_P}{d(d+1)} \sum_{m \geq 2} \frac{(-it)^m}{m!} \text{Tr}(PH^m) \\ &\leq \sum_{P \in S \setminus \{\mathbb{I}\}} \frac{|(it)d\alpha_P|^2}{d(d+1)} + \frac{1}{d(d+1)} \sum_P \left| \sum_{m \geq 2} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 + \frac{2td}{d(d+1)} \sum_{m \geq 3} \frac{t^m}{m!} \sum_{P \in S \setminus \{\mathbb{I}\}} |\alpha_P| \cdot |\text{Tr}(PH^m)| \\ &\leq \sum_{P \in S \setminus \{\mathbb{I}\}} \frac{dt^2 \alpha_P^2}{d+1} + \frac{1}{d(d+1)} \sum_{P, m, m' \geq 2} \frac{(it)^m (-it)^{m'}}{m! m'} \text{Tr}(PH^m) \text{Tr}(PH^{m'}) + \frac{2t}{d+1} \sum_{m \geq 3} \frac{t^m}{m!} \sqrt{\sum_P \alpha_P^2 \sum_P |\text{Tr}(PH^m)|^2} \\ &\leq \frac{d}{d+1} t^2 \sum_{P \in S \setminus \{\mathbb{I}\}} \alpha_P^2 + \frac{1}{d+1} \sum_{m, m' \geq 2} \frac{(it)^m (-it)^{m'}}{m! m'} \text{Tr}(H^m H^{m'}) + \frac{2t}{d+1} \sum_{m \geq 3} \frac{t^m}{m!} \sqrt{d \text{Tr}(H^m H^m)} \\ &\leq \frac{d}{d+1} t^2 \sum_{P \in S \setminus \{\mathbb{I}\}} \alpha_P^2 + \frac{1}{d+1} \sum_{m, m' \geq 2} \frac{t^m}{m!} \cdot \frac{t^{m'}}{m'} \cdot d + \frac{2t}{d+1} \cdot t^3 \cdot \sqrt{d \cdot d} \\ &\leq \frac{d}{d+1} t^2 \left(\sum_{P \in S \setminus \{\mathbb{I}\}} \alpha_P^2 \right) + 3t^4, \end{aligned}$$

where we use $\sum_P \alpha_P^2 \leq 1$, $\frac{1}{d} \sum_P \text{Tr}(PA) \text{Tr}(PB) = \text{Tr}(AB)$, $|\text{Tr}(H^m)| \leq d \|H\|_\infty^m \leq d$, $\sum_{m \geq 2} \frac{t^m}{m!} = e^t - 1 - t \leq t^2$ and $\sum_{m \geq 3} \frac{t^m}{m!} = e^t - 1 - t - \frac{t^2}{2} \leq t^3$.

Therefore

$$\begin{aligned}
\mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \sim_S |\phi_{i,j}\rangle)] &= \frac{|S \cup \{\mathbb{I}\}|}{d+1} + \sum_{P \in S \cup \{\mathbb{I}\}} \frac{1}{d(d+1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 \\
&\leq \frac{|S \cup \{\mathbb{I}\}|}{d+1} + \frac{1}{d+1} \left(d - t^2 d \sum_{P \notin S} |\alpha_P|^2 + 7dt^4 \right) \\
&\leq 1 - t^2 \varepsilon^2 + \frac{|S \cup \{\mathbb{I}\}|}{d+1} + 7t^4 \\
&\leq 1 - \frac{t^2 \varepsilon^2}{2}
\end{aligned} \tag{14}$$

where we used

- For all $K \in \Pi_S$, $\frac{1}{\sqrt{d}} \|H - K\|_2 \geq \varepsilon$ so $\sum_{P \notin S} \alpha_P^2 \geq \varepsilon^2$ (choose $K = \sum_{P \in S} \frac{1}{d} \text{Tr}(PH)P$),
- $\frac{|S \cup \{\mathbb{I}\}|}{d+1} \leq \frac{t^2 \varepsilon^2}{4}$ and $7t^4 \leq \frac{t^2 \varepsilon^2}{4}$.

For $t = \frac{\varepsilon}{6}$, it is sufficient to add the condition that the size of the set S satisfies

$$|S \cup \{\mathbb{I}\}| \leq \frac{(d+1) \cdot \varepsilon^4}{144}.$$

We can choose the number of repetitions to be $N = \frac{2 \log(3)}{t^2 \varepsilon^2}$ so that:

$$\begin{aligned}
&\mathbb{E}_{i,j,\ell} [\mathbb{P}_{H_1} [|\phi_{i_1,\ell_1}\rangle \sim_S |\phi_{i_1,j_1}\rangle, \dots, |\phi_{i_N,\ell_N}\rangle \sim_S |\phi_{i_N,j_N}\rangle]] \\
&= \mathbb{E}_{i_1,j_1,\ell_1} [\mathbb{P} (|\phi_{i_1,\ell_1}\rangle \sim_S |\phi_{i_1,j_1}\rangle)]^N \leq \left(1 - \frac{t^2 \varepsilon^2}{2}\right)^N \leq \frac{1}{3}.
\end{aligned}$$

Error probability under the null hypothesis. Here we suppose that $H \in \Pi_S$. The error probability is

$$\mathbb{E}_{i,j,\ell} [\mathbb{P}_{H_1} [|\phi_{i_1,\ell_1}\rangle \approx_S |\phi_{i_1,j_1}\rangle \text{ or } \dots \text{ or } |\phi_{i_N,\ell_N}\rangle \approx_S |\phi_{i_N,j_N}\rangle]] \leq N \cdot \mathbb{E}_{i_1,j_1,\ell_1} [\mathbb{P} (|\phi_{i_1,\ell_1}\rangle \approx_S |\phi_{i_1,j_1}\rangle)].$$

Observe that under the event $\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}$ we have $\ell \neq j$ since $\mathbb{I} \in S \cup \{\mathbb{I}\}$. Also $|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle$ implies $\langle \phi_{i,\ell} | H | \phi_{i,j} \rangle = 0$ since $H \in \Pi_S$. To see this we can write $|\phi_{i,j}\rangle \langle \phi_{i,j}| = \frac{1}{d} \sum_{p \in G_i} (-1)^{r_j \circ p} S(p)$ where $r_j \in A_{G_i}$ then for $Q \in \mathbb{P}_n$ we have

$$\begin{aligned}
|\langle \phi_{i,\ell} | Q | \phi_{i,j} \rangle|^2 &= \frac{1}{d^2} \sum_{p_1, p_2 \in G_i} (-1)^{r_\ell^i \circ p_1 + r_j^i \circ p_2} \text{Tr}(S(p_1)QS(p_2)Q) \\
&= \frac{1}{d^2} \sum_{p_1, p_2 \in G_i} (-1)^{r_\ell^i \circ p_1 + r_j^i \circ p_2 + q \circ p_2} \text{Tr}(S(p_1)S(p_2)QQ) \\
&= \frac{1}{d} \sum_{p \in G_i} (-1)^{r_\ell^i \circ p + r_j^i \circ p + q \circ p} \\
&= \mathbf{1} (\{r_\ell^i r_j^i q \in G_i\}),
\end{aligned} \tag{15}$$

where we wrote q for the element in \mathbf{P}_n corresponding to $Q \in \mathbb{P}_n$. In the last line, we have used Lemma 2.4. Moreover, we used the fact $r \circ p + s \circ p = (rs) \circ p$ which can be seen from $(-1)^{(rs) \circ p} P(r)P(s)P(p) = P(p)P(r)P(s) = (-1)^{r \circ p} P(r)P(p)P(s) = (-1)^{r \circ p} (-1)^{s \circ p} P(r)P(s)P(p)$. Thus either $\langle \phi_{i,\ell} | Q | \phi_{i,j} \rangle = 0$ or $|\langle \phi_{i,\ell} | Q | \phi_{i,j} \rangle| = 1 \Leftrightarrow Q | \phi_{i,j} \rangle = e^{i\theta} | \phi_{i,\ell} \rangle$. Hence, under the event $\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}$ we have $\langle \phi_{i,\ell} | H^m | \phi_{i,j} \rangle =$

0 for $m = 0, 1$. Therefore:

$$\begin{aligned}
\mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle)] &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} |\langle \phi_{i,\ell} | e^{itH} |\phi_{i,j}\rangle|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}) \\
&= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \langle \phi_{i,\ell} | H^m |\phi_{i,j}\rangle \right|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}) \\
&\leq \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j,\ell} \left| \sum_{m \geq 2} \frac{(it)^m}{m!} \langle \phi_{i,\ell} | H^m |\phi_{i,j}\rangle \right|^2 \\
&= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j,\ell} \sum_{m,m' \geq 2} \frac{(it)^m}{m!} \cdot \frac{(-it)^{m'}}{m'} \langle \phi_{i,\ell} | H^m |\phi_{i,j}\rangle \langle \phi_{i,j} | H^{m'} |\phi_{i,\ell}\rangle \\
&= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{m,m' \geq 2} \frac{(it)^m}{m!} \cdot \frac{(-it)^{m'}}{m'} \text{Tr}(H^m H^{m'}) \\
&\leq \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{m,m' \geq 2} \frac{t^m}{m!} \cdot \frac{t^{m'}}{m'} |\text{Tr}(H^m H^{m'})| \\
&\leq \frac{1}{d(d+1)} \sum_{i=1}^{d+1} t^4 d = t^4,
\end{aligned}$$

where we used that $\{|\phi_{i,j}\rangle\}_j$ is an orthonormal basis, $|\text{Tr}(H^m)| \leq d$ and $\sum_{m \geq 2} \frac{(t)^m}{m!} \leq t^2$.

Hence, for $t \leq \frac{\varepsilon}{6}$ and $N = \left\lceil \frac{2 \log(3)}{t^2 \varepsilon^2} \right\rceil$ we have that

$$\begin{aligned}
&\mathbb{E}_{i,j,\ell} [\mathbb{P}_{H_1} [|\phi_{i_1,\ell_1}\rangle \approx_S |\phi_{i_1,j_1}\rangle \text{ or } \dots \text{ or } |\phi_{i_N,\ell_N}\rangle \approx_S |\phi_{i_N,j_N}\rangle]] \\
&\leq N \cdot \mathbb{E}_{i_1,j_1,\ell_1} [\mathbb{P}(|\phi_{i_1,\ell_1}\rangle \approx_S |\phi_{i_1,j_1}\rangle)] \leq \left\lceil \frac{2 \log(3)}{t^2 \varepsilon^2} \right\rceil t^4 \\
&\leq \frac{2 \log(3)}{\varepsilon^2} t^2 + t^4 \leq \frac{2 \log(3)}{36} + \frac{1}{6^4} < \frac{1}{3}.
\end{aligned}$$

Complexity

- Evolution time at each step $t = \frac{\varepsilon}{6}$,
- Number of independent experiments $N = \left\lceil \frac{2 \log(3)}{t^2 \varepsilon^2} \right\rceil = \left\lceil \frac{72 \log(3)}{\varepsilon^4} \right\rceil$,
- Total evolution time $Nt = \left\lceil \frac{72 \log(3)}{\varepsilon^4} \right\rceil \cdot \frac{\varepsilon}{6} \leq \frac{12 \log(3)}{\varepsilon^3} + \underbrace{\frac{\varepsilon}{6}}_{\leq 1/6}$,
- Total classical processing time: In each round $1 \leq s \leq N$, we check whether $|\langle \phi_{i_s,j_s} | Q | \phi_{i_s,\ell_s} \rangle| = 1$ for any $Q \in S \cup \{\mathbb{I}\}$. Via Equation (15), this reduces to checking whether $r_{\ell_s}^{i_s} r_{j_s}^{i_s} q \in G_{i_s}$. As G_{i_s} is a maximal Abelian subgroup, it equals its own commutator, so we can equivalently check whether $r_{\ell_s}^{i_s} r_{j_s}^{i_s} q$ commutes with all n generators of G_{i_s} . Each such commutation check can be performed in time $\mathcal{O}(n)$, see e.g. [SH14, Section 3]. This leads to a total classical processing time of $\mathcal{O}(Nn^2|S \cup \{\mathbb{I}\}|) = \mathcal{O}\left(\frac{n^2|S \cup \{\mathbb{I}\}|}{\varepsilon^4}\right)$.
- Complexity of state preparation and measurements: The input states used in our protocol are (randomly chosen) stabilizer states $|\phi_{i,j}\rangle$, each of which can be prepared using a Clifford circuit with

$\mathcal{O}(\frac{n^2}{\log n})$ many 2-qubit gates [AG04]. The measurements used in our protocol are w.r.t. some (randomly chosen) orthonormal basis of stabilizer states. Given any such a target basis, there always exists a Clifford unitary that maps the computational basis to the target basis. This can be seen via the surjective group homomorphism between n -qubit Cliffords and the symplectic automorphism group on $(2n)$ -bit strings [Haa16, Proposition II.14.], under which stabilizer groups correspond to (symplectically) isotropic subspaces, and then using the fact that every symplectic subspace can be obtained by applying a symplectic transformation to a span of a suitable number of canonical basis elements in \mathbb{F}_2^{2n} [Haa16, Proposition II.8]. Thus, each of our measurements can be implemented by a Clifford circuit with $\mathcal{O}(\frac{n^2}{\log n})$ many 2-qubit gates [AG04] followed by a computational basis measurement. \square

5 Lower bounds for learning a general Hamiltonian

In this section, we prove Theorem 1.4 on lower bounds for learning a general Hamiltonian in normalized Schatten 2-norm.

Theorem 5.1. *Let $n \geq 11$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_\infty \leq 1$.*

- *Any bounded-ancilla coherent algorithm that constructs \hat{H} such that $\frac{1}{\sqrt{d}}\|H - \hat{H}\|_2 \leq \varepsilon$ with probability at least $2/3$ has to use a number of independent experiments $N = \Omega\left(\frac{2^{2n}}{n}\right)$.*
- *Any non-adaptive ancilla-free incoherent algorithm that construct \hat{H} such that $\frac{1}{\sqrt{d}}\|H - \hat{H}\|_2 \leq \varepsilon$ with probability at least $2/3$ has to use a total evolution time $\Omega\left(\frac{2^{2n}}{\varepsilon}\right)$.*

We follow a standard strategy for proving lower bounds for learning problems [Fla+12; Haa+17; LN22; FOS23; Ouf23].

Proof. We use the following construction inspired by [BCL20]:

$$H_U = \varepsilon U O U^\dagger \quad \text{where} \quad U \sim \text{Haar}(d), \text{ and } O = \text{diag}(\underbrace{+1, \dots, +1}_{\frac{d}{2} \text{ times}}, \underbrace{-1, \dots, -1}_{\frac{d}{2} \text{ times}}).$$

Note that in particular $\text{Tr}[O] = 0$ by construction. For such a Hamiltonian we have $\|H\|_\infty \leq \varepsilon \|U\|_\infty \|O\|_\infty \|U^\dagger\|_\infty = \varepsilon \leq 1$. The expected distance between two independent Hamiltonians H_U and H_V satisfies:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{d} \|H_U - H_V\|_2^2 \right] &= \frac{\varepsilon^2}{d} \mathbb{E} [\text{Tr} [(U O U^\dagger)^2] + 2 \text{Tr} [U O U^\dagger V O V^\dagger] + \text{Tr} [(V O V^\dagger)^2]] \\ &= 2\varepsilon^2 + \frac{2\varepsilon^2}{d} \mathbb{E} [\text{Tr} [U O U^\dagger O]] \\ &= 2\varepsilon^2 + \frac{2\varepsilon^2}{d^2} (\text{Tr}[O])^2 \\ &= 2\varepsilon^2. \end{aligned}$$

Here, the second equality used invariance of the Haar measure, the third equality used $\mathbb{E} [U O U^\dagger] = \frac{\text{Tr}[O]}{d} \mathbb{I}$, and the last equality used $\text{Tr}[O] = 0$. Moreover for $d \geq 2$,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{d^2} \|H_U - H_V\|_2^4 \right] &= \frac{\varepsilon^4}{d^2} \mathbb{E} \left[(\text{Tr} [(U O U^\dagger)^2] + 2 \text{Tr} [U O U^\dagger V O V^\dagger] + \text{Tr} [(V O V^\dagger)^2])^2 \right] \\ &= \frac{\varepsilon^4}{d^2} \mathbb{E} \left[(2d + 2 \text{Tr}(U O U^\dagger V O V^\dagger))^2 \right] \\ &= 4\varepsilon^4 + \frac{4\varepsilon^4}{d} \mathbb{E} [\text{Tr} [U O U^\dagger O]] + \frac{4\varepsilon^4}{d^2} \mathbb{E} [(\text{Tr} [O U^\dagger O U])^2] \\ &\leq 4\varepsilon^4 + \frac{8\varepsilon^2}{d^2} \leq 6\varepsilon^2. \end{aligned}$$

Here, the third equality again used Haar invariance, while the fourth equality used $\mathbb{E}[UOU^\dagger] = \frac{\text{Tr}[O]}{d}\mathbb{1} = 0$ as well as (relying on [Mel23, Corollary 13])

$$\begin{aligned}
& \mathbb{E} \left[(\text{Tr} [(OU^\dagger OU)]^2) \right] \\
&= \text{Tr} \left[O^{\otimes 2} (\mathbb{E} [(U^\dagger OU)^{\otimes 2}]) \right] \\
&= \text{Tr} \left[O^{\otimes 2} \frac{1}{d^2 - 1} \left(\text{Tr}[O^{\otimes 2}] \mathbb{1} + F \text{Tr}[O^{\otimes 2} F] - \frac{1}{d} F \text{Tr}[O^{\otimes 2}] - \frac{1}{d} \text{Tr}[FO^{\otimes 2}] \mathbb{1} \right) \right] \\
&= \frac{1}{d^2 - 1} \left((\text{Tr}[O])^4 + (\text{Tr}[O^2])^2 - \frac{2}{d} (\text{Tr}[O^2]) (\text{Tr}[O])^2 \right) \\
&= \frac{d^2}{d^2 - 1} \\
&\leq 2,
\end{aligned}$$

with F the flip operator. So by Hölder's inequality:

$$\mathbb{E} \left[\frac{1}{\sqrt{d}} \|H_U - H_V\|_2 \right] \geq \sqrt{\frac{\mathbb{E} \left[\frac{1}{d} \|H_U - H_V\|_2^2 \right]^3}{\mathbb{E} \left[\frac{1}{d^2} \|H_U - H_V\|_2^4 \right]}} \geq \sqrt{\frac{(2\varepsilon^2)^3}{6\varepsilon^4}} > 1.1\varepsilon.$$

Observe that the function $f : (U, V) \mapsto \frac{1}{\sqrt{d}} \|H_U - H_V\|_2$ is Lipschitz:

$$\begin{aligned}
& |f(U, V) - f(U', V')| \\
&= \frac{\varepsilon}{\sqrt{d}} \left| \|UOU^\dagger - VOV^\dagger\|_2 - \|U'OU'^\dagger - V'OV'^\dagger\|_2 \right| \\
&\leq \frac{\varepsilon}{\sqrt{d}} \left| \|UOU^\dagger - VOV^\dagger - U'OU'^\dagger + V'OV'^\dagger\|_2 \right| \\
&\leq \frac{\varepsilon}{\sqrt{d}} \left(\|UOU^\dagger - U'OU'^\dagger\|_2 + \|U'OU'^\dagger - V'OV'^\dagger\|_2 + \|VOV^\dagger - V'OV'^\dagger\|_2 + \|VOV^\dagger - V'OV'^\dagger\|_2 \right) \\
&\leq \frac{\varepsilon}{\sqrt{d}} \left(\|U^\dagger - U'^\dagger\|_2 + \|U - U'\|_2 + \|V^\dagger - V'^\dagger\|_2 + \|V - V'\|_2 \right) \\
&= \frac{2\varepsilon}{\sqrt{d}} \left(\|U - U'\|_2 + \|V - V'\|_2 \right) \\
&\leq \frac{2\sqrt{2}\varepsilon}{\sqrt{d}} \|(U, V) - (U', V')\|_2,
\end{aligned}$$

where $\|(U, V) - (U', V')\|_2 := \sqrt{\|U - U'\|_2^2 + \|V - V'\|_2^2}$. Hence, by the concentration inequality of Lipschitz functions w.r.t. the Haar measure [MM13, Corollary 17]:

$$\mathbb{P}(f(U, V) - \mathbb{E}[f(U, V)] \leq -s) \leq \exp\left(-\frac{ds^2}{12\left(\frac{2\sqrt{2}\varepsilon}{\sqrt{d}}\right)^2}\right) = \exp\left(-\frac{s^2 d^2}{96\varepsilon^2}\right).$$

So, since $\mathbb{E}[f] \geq 1.1\varepsilon$:

$$\mathbb{P}(f(U, V) \leq \varepsilon) \leq \mathbb{P}(f(U, V) - \mathbb{E}[f(U, V)] \leq -0.1\varepsilon) \leq \exp\left(-\frac{\varepsilon^2 d^2}{9600\varepsilon^2}\right) = \exp\left(-\frac{d^2}{9600}\right).$$

Therefore, by iteratively picking independent Haar-random unitaries, we can construct a family $F = \{H_x = \varepsilon U_x O U_x^\dagger\}_{x \in [M]}$ where $M = \exp\left(\frac{d^2}{38400}\right)$ that is ε -separated with high probability. In fact, by the union

bound, we have that:

$$\begin{aligned}
\mathbb{P}(F \text{ is not } \varepsilon\text{-separated}) &= \mathbb{P}\left(\exists x \neq x' \in [M] : \frac{1}{\sqrt{d}}\|H_x - H_{x'}\|_2 < \varepsilon\right) \\
&\leq M^2 \mathbb{P}\left(\frac{1}{\sqrt{d}}\|H_1 - H_2\|_2 < \varepsilon\right) \\
&\leq \exp\left(\frac{d^2}{19200}\right) \cdot \exp\left(-\frac{d^2}{9600}\right) \\
&\leq \exp\left(-\frac{d^2}{19200}\right).
\end{aligned}$$

Let $X \sim \text{Uniform}[M]$ and Y be the observations of a correct algorithm. By Fano's inequality we have:

$$\mathcal{I}(X : Y) \geq (2/3) \log(M) - \log(2) \geq \frac{d^2}{38400} - \log(2)$$

Bounded-ancilla coherent algorithms. For algorithms that use auxiliary systems of dimension d_{aux} bounded by $c^N \cdot d^{N-1}$, the dimension of the output state is at most $c^N \cdot d^N$. So, a naive upper bound on the mutual information for an ancilla-bounded algorithm is given by

$$\mathcal{I}(X : Y) \leq 2 \log(c^N \cdot d^N) = 2N \log(c \cdot d).$$

Hence

$$N \geq \frac{d^2}{76800 \log(d \cdot c)} - \frac{\log(2)}{2 \log(d \cdot c)}.$$

Ancilla-free incoherent non-adaptive algorithms. In this setting, the algorithm selects a set of input states $\{\rho_\ell\}_{\ell \in [N]}$, time evolutions $\{t_\ell\}_{\ell \in [N]}$, and measurement devices $\{\mathcal{M}_\ell\}_{\ell \in [N]}$, which we assume without loss of generality to be of the form $\mathcal{M}_\ell = \{\lambda_{i_\ell} |\phi_{i_\ell}^\ell\rangle\langle\phi_{i_\ell}^\ell|\}_{i_\ell}$ for projectors $|\phi_{i_\ell}^\ell\rangle\langle\phi_{i_\ell}^\ell|$ and non-negative coefficients λ_{i_ℓ} satisfying $\sum_{i_\ell} \lambda_{i_\ell} = d$. At step $\ell \in [N]$, the input state is transmitted through the channel $\mathcal{U}_x^{t_\ell}(\rho) = e^{-it_\ell H_x} \rho e^{it_\ell H_x}$, and the output state is measured using the device \mathcal{M}_ℓ , resulting in the outcome I_ℓ .

For a non-adaptive algorithm, the observations $Y = (I_1, \dots, I_N)$ are independent, so the chain rule reads:

$$\mathcal{I}(X : Y) = \mathcal{I}(X : I_1, \dots, I_N) = \sum_{\ell=1}^N \mathcal{I}(X : I_\ell).$$

Fix $l \in [N]$ and recall the notation $\mathcal{U}_x^{t_\ell}(\rho) = e^{-it_\ell H_x} \rho e^{it_\ell H_x}$ where $H_x = \varepsilon U_x O U_x^\dagger$, we have the joint distribution of (X, I_ℓ) :

$$q(x, i_\ell) = \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle.$$

So the mutual information is:

$$\begin{aligned}
\mathcal{I}(X : I_\ell) &= \sum_{x, i_\ell} q(x, i_\ell) \log\left(\frac{q(x, i_\ell)}{q(x)q(i_\ell)}\right) \\
&= \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \log\left(\frac{\frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\frac{1}{M} \cdot \sum_y \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_y^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}\right) \\
&= \Sigma_1^\ell + \Sigma_2^\ell,
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1^\ell &= \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \log\left(\frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E}_{U \sim \text{Haar}(d)} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]}\right), \\
\Sigma_2^\ell &= \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \log\left(\frac{\mathbb{E}_{U \sim \text{Haar}(d)} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]}{\frac{1}{M} \sum_y \langle \phi_{i_\ell}^\ell | \mathcal{U}_y^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}\right),
\end{aligned}$$

and $\mathcal{U}_U^{t_\ell}(\rho) = e^{-it_\ell \varepsilon UOU^\dagger} \rho e^{it_\ell \varepsilon UOU^\dagger}$.

We have for $H = \varepsilon UOU^\dagger$, $e^{itH} = \cos(t\varepsilon)\mathbb{I} + i \sin(t\varepsilon)UOU^\dagger$ hence

$$\begin{aligned}
\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle &= \langle \phi_{i_\ell}^\ell | e^{-it_\ell H_x} \rho_\ell e^{it_\ell H_x} | \phi_{i_\ell}^\ell \rangle \\
&= \cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \sin^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle \\
&\quad + 2\Im \cos(t_\ell \varepsilon) \sin(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle, \\
\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle] &= \cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \sin^2(t_\ell \varepsilon) \mathbb{E} [\langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle] \\
&= \cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \sin^2(t_\ell \varepsilon) \left(\frac{d}{d^2-1} - \frac{d \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle}{d(d^2-1)} \right) \\
&\geq \cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \frac{\sin^2(t_\ell \varepsilon)}{d+1}, \\
\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle^2] &\leq \cos^4(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle^2 + \sin^4(t_\ell \varepsilon) \mathbb{E} [\langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle^2] \\
&\quad + 4 \cos^2(t_\ell \varepsilon) \sin^2(t_\ell \varepsilon) \mathbb{E} [\langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell | \phi_{i_\ell}^\ell \rangle \langle \phi_{i_\ell}^\ell | \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle] \\
&\quad + 2 \cos^2(t_\ell \varepsilon) \sin^2(t_\ell \varepsilon) \mathbb{E} [\langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle \langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle] \\
&\quad + 4\Im \cos(t_\ell \varepsilon) \sin^3(t_\ell \varepsilon) \mathbb{E} [\langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle \langle \phi_{i_\ell}^\ell | \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle] \\
&\leq \cos^4(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle^2 + \sin^4(t_\ell \varepsilon) \mathcal{O} \left(\frac{1}{d^2} \right) \\
&\quad + \cos^2(t_\ell \varepsilon) \sin^2(t_\ell \varepsilon) \mathcal{O} \left(\frac{\langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle}{d} \right). \tag{16}
\end{aligned}$$

For $\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]$, the calculations are similar as in Equation (9). To show Inequality (16), we used Weingarten calculus Appendix D and the remark that $\|O\|_\infty = 1$, $\text{Tr}(O) = 0$, $O^2 = \mathbb{I}$ and $|\text{Wg}(\pi, d)| \leq \mathcal{O}(\frac{1}{d^n})$ for $\pi \in \mathfrak{S}_n$ and $n = 2, 3, 4$.

In particular, we have

$$\begin{aligned}
0 &\leq \frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} \\
&\leq \frac{\cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \sin^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle + 2\Im \cos(t_\ell \varepsilon) \sin(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle}{\cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \frac{\sin^2(t_\ell \varepsilon)}{d+1}} \\
&\leq 1 + (d+1) + \sqrt{d+1} \leq 2d
\end{aligned}$$

for $d \geq 5$. Hence

$$\begin{aligned}
&\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \cdot \left| \frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} - 1 \right| \\
&= \frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} \cdot \left| \sin^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | UOU^\dagger \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle \right. \\
&\quad \left. + 2\Im \cos(t_\ell \varepsilon) \sin(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell UOU^\dagger | \phi_{i_\ell}^\ell \rangle - \frac{\sin^2(t_\ell \varepsilon)}{d+1} \right| \\
&\leq 6dt_\ell \varepsilon
\end{aligned}$$

Here, we used that $\sin^2(x) \leq x$ and that $\cos(x) \sin(x) \leq x$ for $x \geq 0$. So by taking the average over the outcome i_ℓ and assuming that $\sum_{\ell=1}^N t_\ell \leq \frac{d^2}{\varepsilon}$ (since otherwise we already have the statement that we set out to prove) we have

$$Y_x = \sum_{\ell=1}^N \sum_{i_\ell} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \cdot \left(\frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E} [\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} - 1 \right) \in [-6d^4, 6d^4].$$

Hence, by Hoeffding's inequality applied to the i.i.d. random variables $\{Y_x\}_{x \in [M]}$:

$$\mathbb{P} \left(\left| \frac{1}{M} \sum_x Y_x - \mathbb{E}[Y_U] \right| > \sqrt{\frac{12^2 d^8 \log(20)}{M}} \right) \leq 2 \exp \left(-\frac{M \cdot \frac{12^2 d^8}{M} \log(20)}{12^2 d^8} \right) = \frac{1}{10}.$$

Therefore with probability at least 9/10, for all $l \in [N]$, using the inequality $\log(x) \leq x - 1$, we have:

$$\begin{aligned} & \sum_{\ell=1}^N \Sigma_1^\ell - \sqrt{\frac{12^2 d^8 \log(20)}{M}} \\ &= \sum_{\ell=1}^N \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \log \left(\frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} \right) - \sqrt{\frac{12^2 d^8 \log(20)}{M}} \\ &\leq \frac{1}{M} \sum_x \sum_{\ell=1}^N \sum_{i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \left(\frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} - 1 \right) - \sqrt{\frac{12^2 d^8 \log(20)}{M}} \\ &\leq \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{\ell=1}^N \sum_{i_\ell} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_V^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \left(\frac{\langle \phi_{i_\ell}^\ell | \mathcal{U}_V^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle}{\mathbb{E}_{U \sim \text{Haar}(d)}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} - 1 \right) \right] \\ &\leq \sum_{\ell=1}^N \sum_{i_\ell} \lambda_{i_\ell} \frac{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle^2]}{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]} - 1 \\ &\leq \sum_{\ell=1}^N \sum_{i_\ell} \lambda_{i_\ell} \frac{\cos^4(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle^2 + \sin^4(t_\ell \varepsilon) \mathcal{O}\left(\frac{1}{d^2}\right) + \cos^2(t_\ell \varepsilon) \sin^2(t_\ell \varepsilon) \mathcal{O}\left(\frac{\langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle}{d}\right)}{\cos^2(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle + \frac{\sin^2(t_\ell \varepsilon)}{d+1}} - 1 \\ &\leq \sum_{\ell=1}^N \sum_{i_\ell} \lambda_{i_\ell} \cos^4(t_\ell \varepsilon) \langle \phi_{i_\ell}^\ell | \rho_\ell | \phi_{i_\ell}^\ell \rangle - 1 + \sum_{i_\ell} \lambda_{i_\ell} \sin^2(t_\ell \varepsilon) \mathcal{O}\left(\frac{1}{d}\right) + \sum_{i_\ell} \lambda_{i_\ell} \sin^2(t_\ell \varepsilon) \mathcal{O}\left(\frac{1}{d}\right) \\ &\leq \sum_{\ell=1}^N [\cos^4(t_\ell \varepsilon) - 1] + \mathcal{O}(\sin^2(t_\ell \varepsilon)) + \sqrt{\frac{12^2 d^8 \log(20)}{M}} \leq \sum_{\ell=1}^N \mathcal{O}(\min\{t_\ell^2 \varepsilon^2, 1\}) \\ &\leq \mathcal{O}\left(\sum_{\ell=1}^N t_\ell \varepsilon\right) \end{aligned} \tag{17}$$

where we use $\cos(x) \leq 1$, $\sin(x) \leq \min\{x, 1\}$. On the other hand, for the second sum Σ_2 , we can use again the inequality $\log(x) \leq x - 1$:

$$\begin{aligned} \Sigma_2^\ell &= \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \log \left(\frac{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]}{\frac{1}{M} \sum_y \langle \phi_{i_\ell}^\ell | \mathcal{U}_y^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle} \right) \\ &\leq \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \left(\frac{\mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle]}{\frac{1}{M} \sum_y \langle \phi_{i_\ell}^\ell | \mathcal{U}_y^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle} - 1 \right) \\ &= \sum_{i_\ell} \lambda_{i_\ell} \mathbb{E}[\langle \phi_{i_\ell}^\ell | \mathcal{U}_U^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle] - \sum_{x, i_\ell} \frac{1}{M} \lambda_{i_\ell} \langle \phi_{i_\ell}^\ell | \mathcal{U}_x^{t_\ell}(\rho_\ell) | \phi_{i_\ell}^\ell \rangle \\ &= \mathbb{E}[\text{Tr}(\mathcal{U}_U^{t_\ell}(\rho_\ell))] - \sum_x \frac{1}{M} \text{Tr}(\mathcal{U}_x^{t_\ell}(\rho_\ell)) \\ &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^2}{38400} - \log(2) &\leq \mathcal{I}(X : I_1, \dots, I_N) = \sum_{\ell=1}^N \mathcal{I}(X : I_\ell) = \sum_{\ell=1}^N (\Sigma_1^\ell + \Sigma_2^\ell) \\
&\leq \mathcal{O}\left(\sum_{\ell=1}^N t_\ell \varepsilon\right) + \sqrt{\frac{12^2 d^8 \log(20)}{M}} \\
&\leq \mathcal{O}\left(\sum_{\ell=1}^N t_\ell \varepsilon\right) + 1
\end{aligned}$$

for $n \geq 11$ since $M = \exp\left(\frac{d^2}{38400}\right)$. Finally:

$$\sum_{\ell=1}^N t_\ell \geq \Omega\left(\frac{d^2}{\varepsilon}\right).$$

□

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A Relating Hamiltonian distances to time evolution distances

In this appendix, we give operational interpretations for the operator norm distance between Hamiltonians as a worst-case distance and for the normalized Frobenius norm distance between Hamiltonians as an average-case distance. We do so by relating them to corresponding distances between the associated unitary evolutions in the limit of short times.

A.1 Operator norm distance between Hamiltonians

The operator norm distance between two Hamiltonians is connected to the following distance measures on the level of associated time evolution unitaries:

- (a) Worst-case fidelity between (pure) output states of the unitaries,
- (b) 1-to-1 norm distance between the unitary channels (i.e., worst-case 1-norm distance between output states over all input states without auxiliary system),
- (c) Diamond norm distance between unitary channels (i.e., worst-case 1-norm distance between output states over all input states with auxiliary system),
- (d) Operator norm distance between the unitaries up to a global phase,

$$\text{dist}_\infty(U, V) = \min_{\varphi \in [0, 2\pi)} \|U - e^{i\varphi}V\|_\infty.$$

We first recall the distance measures involved and the relations between them. Then, we demonstrate their connection to the operator norm distance between Hamiltonians.

First, we know that the 1-norm distance between two pure states is closely connected to their fidelity via $\|\langle \phi | \langle \phi | - |\psi\rangle\langle \psi | \|_1 = 2\sqrt{1 - |\langle \phi | \psi \rangle|^2}$, by the equality case in the Fuchs-van de Graaf inequalities [FV99]. This implies the following connection between the above distance measures (a) and (b):

$$\|\mathcal{U} - \mathcal{V}\|_{1 \rightarrow 1} = \max_{|\psi\rangle} \|U|\psi\rangle\langle \psi|U^\dagger - V|\psi\rangle\langle \psi|V^\dagger\|_1 = 2\sqrt{1 - \min_{|\psi\rangle} |\langle \psi | U^\dagger V |\psi \rangle|^2}.$$

Next, as we're dealing with unitary channels, the 1-to-1 and diamond norm distances coincide, $\|\mathcal{U} - \mathcal{V}\|_{1 \rightarrow 1} = \|\mathcal{U} - \mathcal{V}\|_\diamond$ (see [Wat18, Theorem 3.55]). Finally, from [Haa+23, Proposition 1.6], we know that for two unitaries U, V , the diamond norm distance between the associated channels \mathcal{U}, \mathcal{V} and the operator norm distance between the unitaries up to a global phase coincide up to multiplicative constants:

$$\frac{1}{2} \|\mathcal{U} - \mathcal{V}\|_\diamond \leq \text{dist}_\infty(U, V) \leq \|\mathcal{U} - \mathcal{V}\|_\diamond.$$

So, all four distance measures (a)-(d) above are equivalent.

To understand them on the level of Hamiltonians, let H, \tilde{H} be two n -qubit Hamiltonians with $\text{Tr}[H] = \text{Tr}[\tilde{H}]$. We write the associated unitary time evolutions as $U_t = e^{-itH}$ and $\tilde{U}_t = e^{-it\tilde{H}}$. Then, as $t \rightarrow 0$, we have for any input state $|\psi\rangle$:

$$\begin{aligned} |\langle \psi | U_t^\dagger \tilde{U}_t |\psi \rangle|^2 &= |\langle \psi | \mathbf{1} + itH - \frac{t^2}{2}H^2 - it\tilde{H} - \frac{t^2}{2}\tilde{H}^2 + t^2H\tilde{H} |\psi \rangle|^2 + \mathcal{O}(t^3) \\ &= \left(1 - \frac{t^2}{2} \langle \psi | (H - \tilde{H})^2 |\psi \rangle\right)^2 + t^2 \left(\langle \psi | (H - \tilde{H}) |\psi \rangle\right)^2 + \mathcal{O}(t^3) \\ &= 1 - t^2 \langle \psi | (H - \tilde{H})^2 |\psi \rangle + t^2 \left(\langle \psi | (H - \tilde{H}) |\psi \rangle\right)^2 + \mathcal{O}(t^3). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|\mathcal{U} - \mathcal{V}\|_{\diamond} &= \|\mathcal{U} - \mathcal{V}\|_{1 \rightarrow 1} \\
&= 2\sqrt{1 - \min_{|\psi\rangle} |\langle \psi | U_t^\dagger \tilde{U}_t | \psi \rangle|^2} \\
&= 2\sqrt{t^2 \max_{|\psi\rangle} \left(\langle \psi | (H - \tilde{H})^2 | \psi \rangle - \left(\langle \psi | (H - \tilde{H}) | \psi \rangle \right)^2 \right) + \mathcal{O}(t^3)} \\
&\begin{cases} \leq 2\sqrt{t^2 \max_{|\psi\rangle} \langle \psi | (H - \tilde{H})^2 | \psi \rangle + \mathcal{O}(t^3)} \\ \geq 2\sqrt{t^2 \left(\frac{1}{2}(\lambda_{\max}^2 + \lambda_{\min}^2) - \left(\frac{1}{2}(\lambda_{\max} + \lambda_{\min}) \right)^2 \right) + \mathcal{O}(t^3)} \end{cases} \tag{18} \\
&\begin{cases} = 2\sqrt{t^2 \|H - \tilde{H}\|_{\infty}^2 + \mathcal{O}(t^3)} \\ = 2\sqrt{t^2 \cdot \frac{(\lambda_{\max} - \lambda_{\min})^2}{4} + \mathcal{O}(t^3)} \end{cases} \\
&\begin{cases} = 2t\|H - \tilde{H}\|_{\infty} + \mathcal{O}(t^2) \\ \geq 2\sqrt{t^2 \cdot \frac{\max(\lambda_{\max}^2, \lambda_{\min}^2)}{4} + \mathcal{O}(t^3)} \end{cases} \\
&\begin{cases} = 2t\|H - \tilde{H}\|_{\infty} + \mathcal{O}(t^2) \\ = t\|H - \tilde{H}\|_{\infty} + \mathcal{O}(t^2) \end{cases} .
\end{aligned}$$

Here, we used $\lambda_{\max/\min}$ to denote the maximal and minimal eigenvalues of $H - \tilde{H}$. For the lower bound in Equation (18), we made the specific choice $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_{\max}\rangle + |\psi_{\min}\rangle)$, where $|\psi_{\max/\min}\rangle$ are the (orthogonal) eigenvectors of $H - \tilde{H}$ for the eigenvalues $\lambda_{\max/\min}$. In the second-to-last step, we used that $\text{Tr}[H - \tilde{H}] = 0$ implies that $\lambda_{\max} \geq 0$ and $\lambda_{\min} \leq 0$. Additionally, in the second-to-last as well as in the last step, we used the Taylor expansion $\sqrt{1+x} = 1 + \mathcal{O}(x)$ to get $\sqrt{t^2\|H - \tilde{H}\|_{\infty}^2 + \mathcal{O}(t^3)} = t\|H - \tilde{H}\|_{\infty}\sqrt{1 + \mathcal{O}(t)} = t\|H - \tilde{H}\|_{\infty}(1 + \mathcal{O}(t))$ and similarly $\sqrt{t^2 \cdot \frac{\lambda_{\max}^2}{4} + \mathcal{O}(t^3)} = t \cdot \frac{\lambda_{\max}}{2}\sqrt{1 + \mathcal{O}(t)} = t \cdot \frac{\lambda_{\max}}{2}(1 + \mathcal{O}(t))$. This derivation tells us that, for short times, we can understand the worst-case distances (a)-(d) via the operator norm distance between the two Hamiltonians underlying the unitary evolutions.

A.2 Normalized Frobenius norm distance between Hamiltonians

The normalized Frobenius norm distance between two Hamiltonians is connected to the following tightly related distance measures on the level of the associated time evolution unitaries:

- (a) Frobenius norm distance between (normalized) Choi states,
- (b) Normalized Frobenius norm distance between unitaries up to a global phase,
- (c) Average-case squared Frobenius norm distance over Haar-random input states,
- (d) Average-case squared trace norm distance over Haar-random input states,
- (e) Average-case fidelity between output states over Haar-random input states.

Note that due to our focus on unitary evolutions, pure input states lead to pure output states, so that the relevant average-case squared Frobenius and trace norm distances are related by constant factors. Thus, (c) and (d) are immediately related. And via the standard relation between the trace distance and the fidelity between pure states, we can immediately translate between (d) and (e). We now review the connections between (a), (b), and (c) established in prior work, and then relate (a) to the normalized Frobenius norm distance between Hamiltonians.

For two n -qubit channels \mathcal{N} and \mathcal{M} , we define $D(\mathcal{N}, \mathcal{M}) = \frac{1}{\sqrt{2}}\|C(\mathcal{N}) - C(\mathcal{M})\|_2$, where $C(\mathcal{N}) = (\mathcal{N} \otimes \text{id})(|\Omega\rangle\langle\Omega|)$ denotes the (normalized) Choi state obtained by applying the channel to a

canonical maximally entangled state. From [BY23, Proposition 15], we know: If \mathcal{N}, \mathcal{M} are unital, then

$$\mathbb{E}_{|\psi\rangle \sim \text{Haar}_n} \left[\|\mathcal{N}(|\psi\rangle\langle\psi|) - \mathcal{M}(|\psi\rangle\langle\psi|)\|_2^2 \right] = \frac{2^{n+1}}{2^n + 1} D(\mathcal{N}, \mathcal{M})^2.$$

Moreover, if we define, for two unitaries U, V ,

$$\text{dist}_2(U, V) = \frac{1}{\sqrt{2^n}} \min_{\varphi \in [0, 2\pi)} \|U - e^{i\varphi} V\|_2,$$

and if we denote the associated unitary channels by \mathcal{U} and \mathcal{V} , then [BY23, Lemma 14] tells us

$$\frac{1}{\sqrt{2}} \text{dist}_2(U, V) \leq D(\mathcal{U}, \mathcal{V}) \leq \text{dist}_2(U, V).$$

Now, consider two n -qubit Hamiltonians H, \tilde{H} with $\text{Tr}[H] = \text{Tr}[\tilde{H}]$, giving rise to unitary time evolutions $U_t = e^{-itH}$ and $\tilde{U}_t = e^{-it\tilde{H}}$. Then, as $t \rightarrow 0$, we have:

$$\begin{aligned} D(\mathcal{U}_t, \tilde{\mathcal{U}}_t) &= \frac{1}{\sqrt{2}} \left\| (U_t \otimes \mathbf{1}) \Omega(U_t^\dagger \otimes \mathbf{1}) - (\tilde{U}_t \otimes \mathbf{1}) \Omega(\tilde{U}_t^\dagger \otimes \mathbf{1}) \right\|_2 \\ &= \frac{1}{\sqrt{2}} \left\| \Omega - it[H \otimes \mathbf{1}, \Omega] - \left(\Omega - it[\tilde{H} \otimes \mathbf{1}, \Omega] \right) \right\|_2 + \mathcal{O}(t^2) \\ &= \frac{t}{\sqrt{2}} \left\| [(H - \tilde{H}) \otimes \mathbf{1}, \Omega] \right\|_2 + \mathcal{O}(t^2) \\ &= \frac{t}{\sqrt{2^n}} \|H - \tilde{H}\|_2 + \mathcal{O}(t^2). \end{aligned}$$

Thus, in the limit of short times, $\frac{1}{\sqrt{2^n}} \|H - \tilde{H}\|_2$ closely relates to $D(\mathcal{U}_t, \tilde{\mathcal{U}}_t)$ and thereby also to $\text{dist}(U_t, \tilde{U}_t)$ as well as to $\sqrt{\mathbb{E}_{|\psi\rangle \sim \text{Haar}_n} \left[\left\| U_t |\psi\rangle\langle\psi| U_t - \tilde{U}_t |\psi\rangle\langle\psi| \tilde{U}_t \right\|_2^2 \right]}$. This tells us that for short times, the different average-case distance measures (a)-(e) are connected to the normalized Frobenius distance between the underlying Hamiltonians.

B More general variants of Hamiltonian property testing

In this appendix, we discuss three variants and extensions of the Hamiltonian property testing procedure presented in Section 4.2. First, we demonstrate how a small modification to our original procedure allows us to test many properties simultaneously. Second, we show that we can test arbitrarily large properties when allowing the testing procedure to use auxiliary qubits. Third, we present a variant of the procedure that can be used for tolerant testing.

B.1 Testing many properties

If we want to test many properties S_1, \dots, S_M with the same data acquired, we need to change the algorithm decision rule. The new decision rule is

At step $x = 1, \dots, N$, let $\mathcal{E}_x = \{|\phi_{i_x, \ell_x}\rangle \sim_S |\phi_{i_x, j_x}\rangle\}$ and let $\bar{\mathcal{E}}_x$ be its complement. Answer the null hypothesis iff

$$\frac{1}{N} \sum_{x=1}^N \mathbf{1}(\{\bar{\mathcal{E}}_x\}) \leq \frac{3}{8} t^2 \varepsilon^2.$$

As we will see below, the error probability can be shown to be at most δ by Chernoff-Hoeffding's inequality [Hoe63] for the following set of parameters:

Algorithm 2: Testing Multiple Properties for Hamiltonian Evolutions

Input : A Hamiltonian H , a set of properties Π_{S_m} for $m \in \{1, \dots, M\}$, and an accuracy parameter $\varepsilon \in (0, 1)$

Output: The null hypothesis H_0^m or the alternate hypothesis H_1^m for each $m \in \{1, \dots, M\}$

- 1 $t \leftarrow \frac{\varepsilon}{6}$;
- 2 $N \leftarrow \left\lceil \frac{100 \log(M/\delta)}{t^2 \varepsilon^2} \right\rceil$;
- 3 **for** $s \leftarrow 1$ **to** N **do**
- 4 Sample $i_s \sim \text{Uniform}[d]$, $j_s \sim \text{Uniform}[d+1]$;
- 5 Input state : $\rho_s = |\phi_{i_s, j_s}\rangle\langle\phi_{i_s, j_s}|$;
- 6 Evolve under H for time t ;
- 7 Measurement : $\mathcal{M}_s = \{|\phi_{i_s, \ell}\rangle\langle\phi_{i_s, \ell}|\}_\ell$ and observe $\ell_s \leftarrow \mathcal{M}_s(\mathcal{U}_t(\rho_s))$;
- 8 **end**
- 9 **for** $m \leftarrow 1$ **to** M **do**
- 10 **if** $\frac{1}{N} \sum_{s=1}^N \mathbf{1}(\{|\phi_{i_s, \ell_s}\rangle\langle\phi_{i_s, \ell_s}\rangle \approx_{S_m} |\phi_{i_s, j_s}\rangle\langle\phi_{i_s, j_s}\rangle\}) \leq \frac{3}{8} t^2 \varepsilon^2$ **then**
- 11 | **return** H_0^m
- 12 | **else**
- 13 | **return** H_1^m
- 14 | **end**
- 15 **end**

- Evolution time at each step $t = \frac{\varepsilon}{6}$,
- Number of independent experiments $N = \left\lceil \frac{100 \log(1/\delta)}{t^2 \varepsilon^2} \right\rceil = \left\lceil \frac{60^2 \log(1/\delta)}{\varepsilon^4} \right\rceil$,
- Total evolution time $Nt = \left\lceil \frac{60^2 \log(1/\delta)}{\varepsilon^4} \right\rceil \cdot \frac{\varepsilon}{6} \leq \frac{600 \log(1/\delta)}{\varepsilon^3} + \underbrace{\frac{\varepsilon}{6}}_{\leq 1/6}$.

So, to test M properties we can take the error probability to be $\delta \rightarrow \frac{\delta}{M}$ and apply a union bound. The new complexity can be taken to be

- Evolution time at each step $t = \frac{\varepsilon}{6}$,
- Number of independent experiments $N = \left\lceil \frac{100 \log(M/\delta)}{t^2 \varepsilon^2} \right\rceil = \left\lceil \frac{60^2 \log(M/\delta)}{\varepsilon^4} \right\rceil$,
- Total evolution time

$$Nt = \left\lceil \frac{60^2 \log(M/\delta)}{\varepsilon^4} \right\rceil \cdot \frac{\varepsilon}{6} \leq \frac{600 \log(M/\delta)}{\varepsilon^3} + \underbrace{\frac{\varepsilon}{6}}_{\leq 1/6}.$$

The logarithmic scaling with M and $1/\delta$ is good, however, the dependency in ε might be sub-optimal.

With multiple properties, there is a null and alternate hypothesis for each property. The null hypothesis H_0^i at $i \in \{1, \dots, M\}$ becomes that $H \in \Pi_{S_i}$ and the alternate hypothesis H_1^i is that H is ε far from being in Π_{S_i} . The algorithm for the testing of multiple properties is therefore Algorithm 2.

Theorem B.1. *Let $S_i \subset \mathbb{P}_n$ such that $|S_i \cup \{\mathbb{I}\}| \leq \frac{(2^n + 1)\varepsilon^4}{144}$ for all $i \in \{1, \dots, M\}$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_\infty \leq 1$. For each $i \in \{1, \dots, M\}$, Algorithm 2 tests whether $H \in \Pi_{S_i}$ or whether $\frac{1}{\sqrt{2^n}} \|H - K\|_2 > \varepsilon$ for all Hamiltonians $K \in \Pi_{S_i}$. The algorithm succeeds with probability at least $1 - \delta$ and uses a total evolution time $\mathcal{O}\left(\frac{\log(M/\delta)}{\varepsilon^3}\right)$, a number of independent experiments $N = \mathcal{O}\left(\frac{\log(M/\delta)}{\varepsilon^4}\right)$, and a total classical processing time $\tilde{\mathcal{O}}\left(n^2 \sum_{i=1}^M \frac{|S_i \cup \{\mathbb{I}\}|}{\varepsilon^4}\right)$. Each experiment uses efficiently implementable states and measurements.*

As in Theorem 4.2, we in fact have stronger guarantees than just efficient implementability for the states and measurements. Also here, the input states are n -qubit stabilizer states and the output measurements are w.r.t. some n -qubit stabilizer state bases.

Proof Sketch. From the proof of correctness in Section 4.2 we have the following inequalities, conditioned on either the null or alternate hypothesis being true:

$$\begin{aligned}\mathbb{P}_{H_1}(\mathcal{E}) &= \mathbb{P}_{H_1}(|\phi_{i,\ell}\rangle \sim_S |\phi_{i,j}\rangle) \leq 1 - \frac{t^2 \varepsilon^2}{2}, \\ \mathbb{P}_{H_0}(\bar{\mathcal{E}}) &= \mathbb{P}_{H_0}(|\phi_{i,\ell}\rangle \not\sim_S |\phi_{i,j}\rangle) \leq t^4 \leq \frac{t^2 \varepsilon^2}{4},\end{aligned}$$

for $t = \frac{\varepsilon}{6}$ and $|S \cup \{\mathbb{I}\}| \leq \frac{(d+1)\varepsilon^4}{144}$. Now we have a linear behavior of the KL divergence:

$$\text{KL}\left(\frac{3}{8}t^2\varepsilon^2 \parallel \frac{1}{2}t^2\varepsilon^2\right), \text{KL}\left(\frac{3}{8}t^2\varepsilon^2 \parallel \frac{1}{4}t^2\varepsilon^2\right) \geq \frac{1}{100} \cdot t^2\varepsilon^2.$$

Here, we have used that

$$\begin{aligned}\text{KL}(p \parallel \alpha p) &\geq (-\log \alpha + \alpha - 1)p, \\ \text{KL}(\alpha p \parallel p) &\geq (\alpha \log \alpha + 1 - \alpha)p,\end{aligned}$$

for $\alpha \in (0, 1)$ and $p \in [0, 1]$, which can be inferred easily from $\log(1+x) \geq x/(1+x)$ for $x > -1$. Under the null hypothesis, since $\frac{3}{8}t^2\varepsilon^2 > \frac{1}{4}t^2\varepsilon^2 \geq \mathbb{P}_{H_0}[\bar{\mathcal{E}}]$, we can apply the Chernoff-Hoeffding inequality (see [Hoe63] or [Mul18, Theorem 2.1]):

$$\begin{aligned}\mathbb{P}_{H_0}\left[\frac{1}{N}\sum_{x=1}^N \mathbf{1}(\{\bar{\mathcal{E}}_x\}) > \frac{3}{8}t^2\varepsilon^2\right] &\leq \exp\left(-N \text{KL}\left(\frac{3}{8}t^2\varepsilon^2 \parallel \mathbb{P}_{H_0}[\bar{\mathcal{E}}]\right)\right) \\ &\leq \exp\left(-N \text{KL}\left(\frac{3}{8}t^2\varepsilon^2 \parallel \frac{1}{4}t^2\varepsilon^2\right)\right) \\ &\leq \exp\left(-N \cdot \frac{1}{100} \cdot t^2\varepsilon^2\right) \leq \delta\end{aligned}$$

for $N = \left\lceil \frac{100 \log(1/\delta)}{t^2\varepsilon^2} \right\rceil$. Here, we have used in the second inequality that $\alpha \mapsto \text{KL}(p \parallel \alpha p)$ is decreasing on $(0, 1)$ for any $p \in [0, 1]$, which can be verified by differentiating in α .

The alternate hypothesis case is similar. The bounds on the classical processing time and on the quantum circuit sizes required for state preparation and measurements follow as in the proof of Theorem 4.2. \square

Remark B.2. We highlight two applications of Theorem B.1. The first is Hamiltonian sparsity testing, that is, the task of testing whether an unknown Hamiltonian H has an at most k -sparse Pauli basis expansion or is ε -far w.r.t. $\frac{1}{\sqrt{d}} \|\cdot\|_2$ from all such Hamiltonians, where $k = \mathcal{O}(1)$. This can be embedded into the scenario of Theorem B.1 by testing $M = \binom{4^n - 1}{k} = \mathcal{O}(4^{nk})$ many properties of size k simultaneously, each corresponding to a possible set of at most k Pauli terms that appear with non-zero coefficients. As our bounds in Theorem B.1 scale logarithmically with the number of properties, Algorithm 2 solves this sparsity testing problem with an efficient number of queries and an efficient total evolution time. If $\frac{1}{\sqrt{2^n}} \|H\|_2 = 1$, meaning $\sum_{P \in \mathbb{P}_n} |\alpha_P|^2 = 1$, then Algorithm 2 can even be used to estimate the support of H . Namely, in this case, each of the M properties that gets assigned the corresponding null hypothesis in the second for-loop corresponds to a Pauli support such that the corresponding coefficients of H have a squared ℓ_2 -norm of $\geq 1 - \varepsilon^2$.

Second, we can use Theorem B.1 to test whether an unknown H is a low-intersection Hamiltonian. This property served as an important assumption in recent work on Hamiltonian learning, for instance in [HKT22; Hua+23]. We call a Hamiltonian $H = \sum_{P \in \mathbb{P}_n} \alpha_P P$ a (k, \mathfrak{d}) -intersection Hamiltonian if $|P| \leq k$ for all $P \in \mathbb{P}_n$ with $\alpha_P \neq 0$ and if $|\{Q \in \mathbb{P}_n \mid \alpha_Q \neq 0 \wedge \text{supp}(P) \cap \text{supp}(Q) \neq \emptyset\}| \leq \mathfrak{d}$ for all $P \in \mathbb{P}_n$ with $\alpha_P \neq 0$. Here, $\text{supp}(P) = \{1 \leq i \leq n \mid P_i \neq \mathbb{I}\}$ denotes the support of an n -qubit Pauli string. We speak of a low-intersection Hamiltonian if both $k, \mathfrak{d} = \mathcal{O}(1)$. Given fixed k and \mathfrak{d} , there are at most

Algorithm 3: Testing Arbitrary Properties for Hamiltonian Evolutions

Input : A Hamiltonian H , a property Π_S , and an accuracy parameter $\varepsilon \in (0, 1)$
Output: The null hypothesis H_0 or the alternate hypothesis H_1

- 1 $n_{\text{aux}} \leftarrow \left\lceil \log_2 \left(\frac{144 \cdot |S \cup \{\mathbb{I}\}|}{2^n \varepsilon^4} \right) \right\rceil$;
- 2 $S_{n_{\text{aux}}} \leftarrow (S \cup \{\mathbb{I}\}) \otimes \mathbb{I}_{2^{n_{\text{aux}}}}$;
- 3 $t \leftarrow \frac{\varepsilon}{6}$;
- 4 $N \leftarrow \left\lceil \frac{2 \log(3)}{t^2 \varepsilon^2} \right\rceil$;
- 5 **for** $s \leftarrow 1$ **to** N **do**
- 6 Sample $i_s \sim \text{Uniform}[2^{n+n_{\text{aux}}}]$, $j_s \sim \text{Uniform}[2^{n+n_{\text{aux}}} + 1]$;
- 7 Input state : $\rho_s = |\phi_{i_s, j_s}\rangle\langle\phi_{i_s, j_s}|$;
- 8 Evolve the first n qubits under H for time t ;
- 9 Measurement : $\mathcal{M}_s = \{|\phi_{i_s, \ell}\rangle\langle\phi_{i_s, \ell}|\}_{\ell}$ and observe $\ell_s \leftarrow \mathcal{M}_s[(\mathcal{U}_t \otimes \text{id}_{n_{\text{aux}}})(\rho_s)]$;
- 10 **if** $|\phi_{i_s, j_s}\rangle \approx_{S_{n_{\text{aux}}}} |\phi_{i_s, \ell_s}\rangle$ **then**
- 11 | **return** H_1 and **stop**
- 12 **end**
- 13 **end**
- 14 **return** H_0

$2^{\mathcal{O}(n^{\text{poly}(k, \mathfrak{d})})}$ many different dual interaction graphs (see [HKT22] for a definition) that a (k, \mathfrak{d}) -intersection Hamiltonian can have. (A loose bound that does not take \mathfrak{d} into account can be seen as follows: By the locality constraint, admissible dual interaction graphs have at most $\sum_{\ell=0}^k \binom{n}{\ell} 3^\ell \leq \mathcal{O}(n^{k+1})$ vertices and thus at most $\mathcal{O}(n^{2(k+1)})$ edges. As each edge can either be present or not, there are at most $2^{\mathcal{O}(n^{2(k+1)})}$ many admissible dual interaction graphs.) Therefore, by simultaneously testing all the size- $\mathcal{O}(n^{k+1})$ properties corresponding to different valid dual interaction graphs, Algorithm 2 can test whether an unknown H is a (k, \mathfrak{d}) -intersection Hamiltonian; and thanks to Theorem B.1, the query complexity and total evolution time for doing so can be bounded in terms of $\log 2^{\mathcal{O}(n^{\text{poly}(k, \mathfrak{d})})} = \mathcal{O}(n^{\text{poly}(k, \mathfrak{d})})$.

B.2 Testing arbitrarily large properties

In Theorem 4.2, we can test a property Π_S if the size of the set S satisfies:

$$|S \cup \{\mathbb{I}\}| \leq \frac{(d+1)\varepsilon^4}{144}.$$

To lift this assumption, we propose to use auxiliary systems. The idea is that if we can add an n_{aux} -qubit ancilla, then we can query the unitary

$$e^{itH} \otimes \mathbb{I}_{2^{n_{\text{aux}}}} = e^{itH \otimes \mathbb{I}_{2^{n_{\text{aux}}}}}.$$

So we can think of testing $H \otimes \mathbb{I}_{2^{n_{\text{aux}}}}$ instead of H , and the relevant property now is $S_{n_{\text{aux}}} = (S \cup \{\mathbb{I}\}) \otimes \mathbb{I}_{2^{n_{\text{aux}}}}$. This is not exactly true, because we cannot enforce that $H \otimes \mathbb{I}_{2^{n_{\text{aux}}}}$ is ε -far from $\Pi_{S_{n_{\text{aux}}}}$ given that H is ε -far from Π_S . However, in terms of distance, we only need that $\sum_{P \notin S} \alpha_P^2 \geq \varepsilon^2$ for our proof. To see this, let us bound $\mathbb{P}(\mathcal{E})$ under both the null and alternate hypothesis, where we write again $\mathcal{E} = \{|\phi_{i, \ell}\rangle \sim_S |\phi_{i, j}\rangle\}$ and $\bar{\mathcal{E}}$ for its complement.

First, under the null hypothesis, the main ingredient in the proof in Section 4.2 is the observation that if $H \in \Pi_S$ and $|\phi_{i, \ell}\rangle \approx_S |\phi_{i, j}\rangle$, then for all $j \neq \ell$ we have $\langle\phi_{i, \ell}| H^m |\phi_{i, j}\rangle = 0$ for $m = 0, 1$. This is not affected if we add ancilla as $|\phi_{i, \ell}\rangle \approx_{S_{n_{\text{aux}}}} |\phi_{i, j}\rangle \Rightarrow \forall P \in S_{n_{\text{aux}}} : |\langle\phi_{i, \ell}| P |\phi_{i, j}\rangle| \neq 1 \Rightarrow \forall P \in S \cup \{\mathbb{I}\} : |\langle\phi_{i, \ell}| P \otimes \mathbb{I}_{2^{n_{\text{aux}}}} |\phi_{i, j}\rangle| \neq 1$, thus $\langle\phi_{i, \ell}| (H \otimes \mathbb{I}_{2^{n_{\text{aux}}}})^m |\phi_{i, j}\rangle = \langle\phi_{i, \ell}| H^m \otimes \mathbb{I}_{2^{n_{\text{aux}}}} |\phi_{i, j}\rangle = 0$ for $m = 0, 1$. Hence, under the condition that the null hypothesis is true,

$$\mathbb{P}_{H_0}[\bar{\mathcal{E}}] \leq t^4,$$

by the same reasoning as before. Next, under the alternate hypothesis, we have (by the previous calculations):

$$\begin{aligned}
\mathbb{P}_{H_1}[\mathcal{E}] &\leq \sum_{P \in S_{n_{\text{aux}}} \cup \{\mathbb{I}\}} \frac{2^{n+n_{\text{aux}}}}{2^{n+n_{\text{aux}}}(2^{n+n_{\text{aux}}} + 1)} \\
&\quad + \sum_{P \in S_{n_{\text{aux}}} \cup \{\mathbb{I}\}} \frac{1}{2^{n+n_{\text{aux}}}(2^{n+n_{\text{aux}}} + 1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(P(H \otimes \mathbb{I}_{2^{n_{\text{aux}}}})^m) \right|^2 \\
&= \frac{|S \cup \{\mathbb{I}\}|}{(2^{n+n_{\text{aux}}} + 1)} + \sum_{P \in S \cup \{\mathbb{I}\}} \frac{1}{2^{n+n_{\text{aux}}}(2^{n+n_{\text{aux}}} + 1)} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) 2^{n_{\text{aux}}} \right|^2 \\
&\leq \frac{|S \cup \{\mathbb{I}\}|}{(2^{n+n_{\text{aux}}} + 1)} + \sum_{P \in S \cup \{\mathbb{I}\}} \frac{1}{2^{2n}} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2,
\end{aligned}$$

since $P \in S \Leftrightarrow P \otimes \mathbb{I}_{2^{n_{\text{aux}}}} \in S_{n_{\text{aux}}}$. Then we can continue as in the proof of Theorem 4.2 to have

$$\begin{aligned}
\mathbb{P}_{H_1}[\mathcal{E}] &\leq \frac{|S \cup \{\mathbb{I}\}|}{(2^{n+n_{\text{aux}}} + 1)} + \sum_{P \in S \cup \{\mathbb{I}\}} \frac{1}{2^{2n}} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \text{Tr}(PH^m) \right|^2 \\
&\leq 1 - t^2 \varepsilon^2 + \frac{|S \cup \{\mathbb{I}\}|}{(2^{n+n_{\text{aux}}} + 1)} + 7t^4.
\end{aligned}$$

Hence, if we choose $t = \frac{\varepsilon}{6}$, then we have $\mathbb{P}_{H_0}[\bar{\mathcal{E}}] \leq \frac{t^2 \varepsilon^2}{4}$. Thus, we have to fulfill the following in order to ensure $\mathbb{P}_{H_1}[\mathcal{E}] \leq 1 - \frac{t^2 \varepsilon^2}{2}$:

$$\frac{|S \cup \{\mathbb{I}\}|}{2^{n+n_{\text{aux}}}} \leq \frac{t^2 \varepsilon^2}{4} = \frac{\varepsilon^4}{144} \quad \text{and} \quad 7t^4 \leq \frac{t^2 \varepsilon^2}{4}.$$

The second inequality holds with our choice of t as $|S \cup \{\mathbb{I}\}| \geq 1$, and the first inequality holds if

$$n_{\text{aux}} \geq \log_2 \left(\frac{144 \cdot |S \cup \{\mathbb{I}\}|}{2^n \varepsilon^4} \right).$$

As we have always $|S \cup \{\mathbb{I}\}| \leq 2^{2n}$, a number of auxiliary qubits

$$n_{\text{aux}} \geq \log_2 \left(\frac{144 \cdot 2^n}{\varepsilon^4} \right) = n + \log_2 \left(\frac{144}{\varepsilon^4} \right)$$

is enough to ensure $\mathbb{P}_{H_1}[\mathcal{E}] \leq 1 - \frac{t^2 \varepsilon^2}{2}$. With this amount of ancilla qubits, we can test *any* set S , regardless of its size.

Since we have the inequalities

$$\mathbb{P}_{H_1}(\mathcal{E}) \leq 1 - \frac{t^2 \varepsilon^2}{2} \quad \text{and} \quad \mathbb{P}_{H_0}(\bar{\mathcal{E}}) \leq \frac{t^2 \varepsilon^2}{4} \quad \text{for} \quad t = \frac{\varepsilon}{6},$$

the query complexity and the total evolution time are the same as for the ancilla-free algorithm. The classical processing time changes compared to Theorem 4.2 in that we now need to check properties of $(n + n_{\text{aux}})$ -qubit Paulis. With the same arguments as in the proof of Theorem 4.2, we see that the relation $\sim_{S_{n_{\text{aux}}}}$ can be checked with $\mathcal{O}((n + n_{\text{aux}})^2 |S_{n_{\text{aux}}}|) = \tilde{\mathcal{O}}(n^2 |S \cup \{\mathbb{I}\}|)$ classical processing time. Here, we used $\tilde{\mathcal{O}}$ as a notational simplification that hides a factor $\text{polylog}(1/\varepsilon)$. Finally, the input states are now $(n + n_{\text{aux}})$ -qubit stabilizer states and the measurements are in some bases of $(n + n_{\text{aux}})$ -qubit stabilizer states, so the quantum circuit sizes can be bounded as before upon replacing n by $n + n_{\text{aux}}$, leading to quantum circuit complexities of $\mathcal{O}\left(\frac{(n+n_{\text{aux}})^2}{\log(n+n_{\text{aux}})}\right) = \tilde{\mathcal{O}}\left(\frac{n^2}{\log n}\right)$, where the $\tilde{\mathcal{O}}$ again hides factors that are polylogarithmic in $\frac{1}{\varepsilon}$.

Thus, we have shown the following theorem in this section:

Algorithm 4: Tolerantly Testing Properties for Hamiltonian Evolutions

Input : A Hamiltonian H , a property Π_S , and accuracy parameters $0 \leq \varepsilon_1 < \varepsilon_2 < 1$
Output: The null hypothesis H_0 or the alternate hypothesis H_1

- 1 $t \leftarrow \sqrt{\frac{1}{20}(\varepsilon_2^2 - \varepsilon_1^2)}$;
- 2 $N \leftarrow \left\lceil \frac{30 \log(3)(\varepsilon_2^2 + \varepsilon_1^2)}{t^2(\varepsilon_2^2 - \varepsilon_1^2)^2} \right\rceil$;
- 3 **for** $s \leftarrow 1$ **to** N **do**
- 4 Sample $i_s \sim \text{Uniform}[d]$, $j_s \sim \text{Uniform}[d + 1]$;
- 5 Input state : $\rho_s = |\phi_{i_s, j_s}\rangle\langle\phi_{i_s, j_s}|$;
- 6 Evolve under H for time t ;
- 7 Measurement : $\mathcal{M}_s = \{|\phi_{i_s, \ell}\rangle\langle\phi_{i_s, \ell}|\}_\ell$ and observe $\ell_s \leftarrow \mathcal{M}_s(\mathcal{U}_t(\rho_s))$;
- 8 **end**
- 9 **if** $\frac{1}{N} \sum_{s=1}^N \mathbf{1}(\{|\phi_{i_s, \ell_s}\rangle \approx_S |\phi_{i_s, j_s}\rangle\}) \leq \frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2)$ **then**
- 10 **return** H_0
- 11 **else**
- 12 **return** H_1
- 13 **end**

Theorem B.3. Let $S \subset \mathbb{P}_n$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_\infty \leq 1$. Using $n_{\text{aux}} = \left\lceil \log_2 \left(\frac{144 \cdot |S \cup \{\mathbb{I}\}|}{2^n \varepsilon^4} \right) \right\rceil$ ancilla qubits, Algorithm 3 tests whether $H \in \Pi_S$ or $\frac{1}{\sqrt{2^n}} \|H - K\|_2 > \varepsilon$ for all Hamiltonians $K \in \Pi_S$ with probability at least $2/3$ using a total evolution time $\mathcal{O}\left(\frac{1}{\varepsilon^3}\right)$, a total number of independent experiments $N = \mathcal{O}\left(\frac{1}{\varepsilon^4}\right)$, and a total classical processing time $\tilde{\mathcal{O}}\left(\frac{n^2 |S \cup \{\mathbb{I}\}|}{\varepsilon^4}\right)$. Each experiment uses efficiently implementable states and measurements on $\mathcal{O}(n + \log(1/\varepsilon))$ many qubits.

B.3 Tolerant Hamiltonian property testing

So far, we have focused on the standard setting of property testing. Now, we turn our attention to tolerant testing [PRR06]. In our Hamiltonian case, we formulate a tolerant property testing problem as follows:

Problem B.4 (Tolerant Hamiltonian property testing). Given a property Π_S associated to a subset $S \subseteq \mathbb{P}_n$, a norm $\|\cdot\|$, and two accuracy parameters $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, we denote by $\mathcal{T}_{\|\cdot\|}^{\Pi_S}(\varepsilon_1, \varepsilon_2)$ the following Hamiltonian property testing problem: Given access to the time evolution according to an unknown Hamiltonian H , decide, with success probability $\geq 2/3$, whether

(i) H is ε_1 -close to having property Π_S , that is, there exists $\tilde{H} \in \Pi_S$ such that $\left\| \|H - \tilde{H}\| \right\| \leq \varepsilon_1$, or

(ii) H is ε_2 -far from having property Π_S , that is, $\forall \tilde{H} \in \Pi_S: \left\| \|H - \tilde{H}\| \right\| \geq \varepsilon_2$.

If H satisfies neither (i) nor (ii), then any output of the tester is considered valid.

Clearly, the tolerant testing problem $\mathcal{T}_{\|\cdot\|}^{\Pi_S}(\varepsilon_1, \varepsilon_2)$ is at least as hard as $\mathcal{T}_{\|\cdot\|}^{\Pi_S}(\varepsilon_2)$, so the lower bounds of Section 3 for locality testing w.r.t. unnormalized Schatten p -norms carry over straightforwardly. More interestingly, in this section, we show, via a variant of the analysis from Section 4.2, that an analogue of Theorem 4.2 holds for Algorithm 4.

Theorem B.5. Let $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, and let $S \subset \mathbb{P}_n$ such that $|S \cup \{\mathbb{I}\}| \leq \frac{(2^n + 1)(\varepsilon_2^2 - \varepsilon_1^2)^2}{400}$. Suppose that the Hamiltonian H satisfies $\text{Tr}(H) = 0$ and $\|H\|_\infty \leq 1$. Algorithm 4 tests whether there exists $\tilde{H} \in \Pi_S$ with $\frac{1}{\sqrt{2^n}} \|H - \tilde{H}\|_2 \leq \varepsilon_1$ or $\frac{1}{\sqrt{2^n}} \|H - K\|_2 > \varepsilon_2$ for all Hamiltonians $K \in \Pi_S$ with probability at least $2/3$ using a total evolution time $\mathcal{O}\left(\frac{1}{(\varepsilon_2 - \varepsilon_1)^{2.5} \varepsilon_2^{0.5}}\right)$, a total number of independent experiments $N = \mathcal{O}\left(\frac{1}{(\varepsilon_2 - \varepsilon_1)^3 \varepsilon_2}\right)$, and a total classical processing time $\mathcal{O}\left(\frac{n^2 |S \cup \{\mathbb{I}\}|}{(\varepsilon_2 - \varepsilon_1)^3 \varepsilon_2}\right)$. Each experiment uses efficiently implementable states and measurements.

Proof Sketch. First, note that the error probability under the alternate hypothesis (i.e., in case H is ε_2 -far from having property Π_S) can be upper bounded with exactly the same reasoning as in the proof of Theorem 4.2 (Eq. 14) to obtain

$$\mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \sim_S |\phi_{i,j}\rangle)] \leq 1 - t^2 \varepsilon_2^2 + \frac{|S \cup \{\mathbb{I}\}|}{d+1} + 7t^4 \leq 1 - t^2 \varepsilon_2^2 + 8t^4$$

for $|S \cup \{\mathbb{I}\}| \leq (d+1)t^4$. So, we only have to adapt the analysis of the error probability under the null hypothesis.

Therefore, suppose that H is ε_1 -close to having property Π_S and let $\tilde{H} \in \Pi_S$ be such that $\frac{1}{\sqrt{2^n}} \|H - \tilde{H}\|_2 \leq \varepsilon_1$. Following the reasoning from the proof of Theorem 4.2, we have to upper bound the expression

$$\mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle)] = \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} |\langle \phi_{i,\ell} | e^{itH} | \phi_{i,j} \rangle|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}).$$

Using that under the event $\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}$, we have both $\langle \phi_{i,\ell} | \mathbb{I} | \phi_{i,j} \rangle = 0$ and $\langle \phi_{i,\ell} | \tilde{H} | \phi_{i,j} \rangle = 0$, we can expand the exponential series and obtain:

$$\begin{aligned} \mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle)] &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} |\langle \phi_{i,\ell} | e^{itH} | \phi_{i,j} \rangle|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}) \\ &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j \neq \ell} \left| \sum_{m \geq 0} \frac{(it)^m}{m!} \langle \phi_{i,\ell} | H^m | \phi_{i,j} \rangle \right|^2 \mathbf{1}(\{|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle\}) \\ &\leq \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j,\ell} \left| it \langle \phi_{i,\ell} | (H - \tilde{H}) | \phi_{i,j} \rangle + \sum_{m \geq 2} \frac{(it)^m}{m!} \langle \phi_{i,\ell} | H^m | \phi_{i,j} \rangle \right|^2 \\ &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j,\ell} t^2 \left| \langle \phi_{i,\ell} | (H - \tilde{H}) | \phi_{i,j} \rangle \right|^2 \end{aligned} \quad (19)$$

$$+ 2 \frac{1}{d(d+1)} \Re \sum_{i=1}^{d+1} \sum_{j,\ell} it \langle \phi_{i,\ell} | (H - \tilde{H}) | \phi_{i,j} \rangle \sum_{m \geq 2} \frac{(-it)^m}{m!} \langle \phi_{i,j} | H^m | \phi_{i,\ell} \rangle \quad (20)$$

$$+ \left| \sum_{m \geq 2} \frac{(it)^m}{m!} \langle \phi_{i,\ell} | H^m | \phi_{i,j} \rangle \right|^2. \quad (21)$$

The term in (21) has already been upper bounded by t^4 in the proof of Theorem 4.2. Now, we address the remaining two terms. First, note that

$$\begin{aligned} (19) &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} \sum_{j,\ell} t^2 \text{Tr}[\langle \phi_{i,\ell} | \langle \phi_{i,\ell} | (H - \tilde{H}) | \phi_{i,j} \rangle \langle \phi_{i,j} | (H - \tilde{H}) | \phi_{i,\ell} \rangle] \\ &= \frac{1}{d(d+1)} \sum_{i=1}^{d+1} t^2 \text{Tr}[(H - \tilde{H})^2] = t^2 \frac{\text{Tr}[(H - \tilde{H})^2]}{d} \leq t^2 \varepsilon_1^2, \end{aligned}$$

where the second step used that $\{|\phi_{i,j}\rangle\}_j$ forms an ONB for every i , and where the last step used that $\frac{1}{\sqrt{2^n}}\|H - \tilde{H}\|_2 \leq \varepsilon_1$. Now, we bound the final remaining term:

$$\begin{aligned}
(20) &= 2 \frac{1}{d(d+1)} \Re \sum_{i=1}^{d+1} \sum_{j,\ell} \sum_{m \geq 2} \frac{(-1)^m (it)^{m+1}}{m!} \text{Tr} \left[|\phi_{i,\ell}\rangle \langle \phi_{i,\ell}| (H - \tilde{H}) |\phi_{i,j}\rangle \langle \phi_{i,j}| H^m \right] \\
&= 2 \frac{1}{d} \Re \sum_{m \geq 2} \frac{(-1)^m (it)^{m+1}}{m!} \text{Tr} \left[(H - \tilde{H}) H^m \right] \\
&= 2 \frac{1}{d} \sum_{m \geq 2, m \text{ odd}} \frac{(-1)^m (it)^{m+1}}{m!} \text{Tr} \left[(H - \tilde{H}) H^m \right] \\
&= 2 \frac{1}{d} \sum_{m \geq 3, m \text{ even}} \frac{(-1)^{m-1} (it)^m}{(m-1)!} \text{Tr} \left[(H - \tilde{H}) H^{m-1} \right] \\
&\leq 3t^4.
\end{aligned}$$

where the second step used that $\{|\phi_{i,j}\rangle\}_j$ forms an ONB for every i ; the third step used that $H = H^\dagger$ and $\tilde{H} = \tilde{H}^\dagger$, so that $\text{Tr} \left[(H - \tilde{H}) H^m \right] \in \mathbb{R}$ for all m ; the fourth step used $|\text{Tr}[(H - \tilde{H}) H^{m-1}]| \leq \|H - \tilde{H}\|_1 \|H\|_\infty^{m-1} \leq \sqrt{d} \|H - \tilde{H}\|_2 \leq d$ and thus $\sum_{m \geq 4} \frac{t^m}{(m-1)!} |\text{Tr}[(H - \tilde{H}) H^{m-1}]| \leq 0.22 \cdot dt^4$ since $t \leq 1$.

Combining the upper bounds on (19), (20), and (21), we have shown that

$$\mathbb{E}_{i,j,\ell} [\mathbb{P}(|\phi_{i,\ell}\rangle \approx_S |\phi_{i,j}\rangle)] \leq t^2 \varepsilon_1^2 + 3t^4 + t^4 \leq t^2 \varepsilon_1^2 + 4t^4. \quad (22)$$

Following the logic laid out in Appendix B.1, we set $t^2 = \frac{1}{20}(\varepsilon_2^2 - \varepsilon_1^2)$ and the new decision rule is:

At step $s = 1, \dots, N$, let $\mathcal{E}_s = \{|\phi_{i_s, \ell_s}\rangle \sim_S |\phi_{i_s, j_s}\rangle\}$ and let $\bar{\mathcal{E}}_s$ be its complement. Answer the null hypothesis iff

$$\frac{1}{N} \sum_{s=1}^N \mathbf{1}(\{\bar{\mathcal{E}}_s\}) \leq \frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2).$$

To control the error probability under the null hypothesis, we apply the Chernoff-Hoeffding inequality [Hoe63]:

$$\begin{aligned}
\mathbb{P}_{H_0} \left[\frac{1}{N} \sum_{s=1}^N \mathbf{1}(\{\bar{\mathcal{E}}_s\}) > \frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \right] &\leq \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| \mathbb{P}_{H_0} [\bar{\mathcal{E}}] \right) \right) \\
&\leq \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| t^2 \varepsilon_1^2 + 4t^4 \right) \right) \\
&= \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| \frac{1}{5} t^2 (\varepsilon_2^2 + 4\varepsilon_1^2) \right) \right) \\
&= \exp \left(-N \cdot \frac{1}{10} \cdot t^2 \cdot \frac{(\varepsilon_1^2 - \varepsilon_2^2)^2}{2\varepsilon_2^2 + 3\varepsilon_1^2} \right) \\
&\leq \delta
\end{aligned}$$

where we used our choice of t and $\text{KL}(x||y) \geq \frac{1}{2x}(x-y)^2$ for $x \geq y$ and $N = \left\lceil \frac{30 \log(1/\delta)(\varepsilon_2^2 + \varepsilon_1^2)}{t^2(\varepsilon_2^2 - \varepsilon_1^2)^2} \right\rceil$.

The alternate hypothesis case is similar.

$$\begin{aligned}
\mathbb{P}_{H_1} \left[\frac{1}{N} \sum_{s=1}^N \mathbf{1}(\{\bar{\mathcal{E}}_s\}) \leq \frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \right] &\leq \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| \mathbb{P}_{H_1} [\bar{\mathcal{E}}] \right) \right) \\
&\leq \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| t^2 \varepsilon_2^2 - 8t^4 \right) \right) \\
&= \exp \left(-N \text{KL} \left(\frac{1}{5} t^2 (2\varepsilon_2^2 + 3\varepsilon_1^2) \middle\| \frac{1}{5} t^2 (3\varepsilon_2^2 + 2\varepsilon_1^2) \right) \right) \\
&= \exp \left(-N \cdot \frac{1}{10} \cdot t^2 \cdot \frac{(\varepsilon_1^2 - \varepsilon_2^2)^2}{3\varepsilon_2^2 + 2\varepsilon_1^2} \right) \\
&\leq \delta
\end{aligned}$$

where we used our choice of t and $\text{KL}(x\|y) \geq \frac{1}{2y}(x-y)^2$ for $x \leq y$ and $N = \left\lceil \frac{30 \log(1/\delta)(\varepsilon_2^2 + \varepsilon_1^2)}{t^2(\varepsilon_2^2 - \varepsilon_1^2)^2} \right\rceil = \mathcal{O} \left(\frac{\log(1/\delta)\varepsilon_2^2}{(\varepsilon_2^2 - \varepsilon_1^2)^3} \right) = \mathcal{O} \left(\frac{\log(1/\delta)}{(\varepsilon_2 - \varepsilon_1)^3 \varepsilon_2} \right)$. This leads to a total evolution time $Nt = \mathcal{O} \left(\frac{\log(1/\delta)\sqrt{(\varepsilon_2 - \varepsilon_1)(\varepsilon_2 + \varepsilon_1)}}{(\varepsilon_2 - \varepsilon_1)^3 \varepsilon_2} \right) = \mathcal{O} \left(\frac{\log(1/\delta)}{(\varepsilon_2 - \varepsilon_1)^{2.5} \varepsilon_2^{0.5}} \right)$. \square

C Deferred proofs

The following Lemma uses the notation from the proof of Theorem 3.1.

Lemma C.1 (Generalized Le Cam). *Let $n \geq \Omega(1)$. Let $k \leq \mathcal{O} \left(\frac{n}{\log(n)} \right)$. We have for $\alpha \leq \frac{1}{10N}$:*

$$\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)] \geq \frac{1}{18}.$$

We use Le Cam's method [Le 73]:

Proof of Lemma C.1. Let $x_k = \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle$ and $y_k = \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | U_{v,t_k} \rho_k U_{v,t_k}^\dagger | \phi_{i_k}^{(k)} \rangle$. Note that since $\text{Tr}(\rho_k) = 1$ we have $\sum_{i_k} x_k = \sum_{i_k} y_k = 1$.

Let \mathcal{E} be the event that $\eta(V|0\rangle\langle 0|V^\dagger - \mathbb{I}/d)$ is not $(\eta/4)$ -close to any k -local Hamiltonian. On the one hand, by the correctness of the algorithm and the data processing inequality, we have that:

$$\text{TV}(P, \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [Q_V]) \geq \text{TV}(\text{Bern}(1/3) \| \text{Bern}(2/3)) = \frac{1}{3}.$$

On the other hand, since we have $\mathbb{P}(\mathcal{E}) \geq 1 - \exp(-\Omega(d))$ by Lemma 3.2, we have by the triangle inequality:

$$\begin{aligned}
&\text{TV}(\mathbb{E}_{V \sim \text{Haar}(d)} [Q_V], \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [Q_V]) \\
&= \frac{1}{2} \sum_i \left| \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V(i)] - \mathbb{E}_{V \sim \text{Haar}(d)} \left[Q_V(i) \frac{\mathbf{1}(\{\mathcal{E}\})}{\mathbb{P}(\mathcal{E})} \right] \right| \\
&= \frac{1}{2} \frac{1}{\mathbb{P}(\mathcal{E})} \sum_i \left| \mathbb{P}(\mathcal{E}) \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V(i)] - \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V(i) \mathbf{1}(\{\mathcal{E}\})] \right| \\
&\leq \frac{1}{2} \frac{1}{\mathbb{P}(\mathcal{E})} \sum_i \left| \mathbb{P}(\mathcal{E}^c) \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V(i)] \mathbf{1}(\{\mathcal{E}\}) \right| + \frac{1}{2} \frac{1}{\mathbb{P}(\mathcal{E})} \sum_i \left| \mathbb{P}(\mathcal{E}) \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V(i) \mathbf{1}(\{\mathcal{E}^c\})] \right| \\
&\leq \frac{1}{2} \left(\frac{\mathbb{P}(\mathcal{E}^c) \mathbb{P}(\mathcal{E})}{\mathbb{P}(\mathcal{E})} + \mathbb{P}(\mathcal{E}^c) \right) \\
&\leq \exp(-\Omega(d)).
\end{aligned}$$

So by the triangle inequality

$$\begin{aligned}
\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V)] &\geq \text{TV}(P, \mathbb{E}_{V \sim \text{Haar}(d)} [Q_V]) \\
&\geq \text{TV}(P, \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [Q_V]) - \text{TV}(\mathbb{E}_{V \sim \text{Haar}(d)} [Q_V], \mathbb{E}_{V \sim \text{Haar}(d)|\mathcal{E}} [Q_V]) \\
&\geq \frac{1}{3} - \exp(-\Omega(d)) \geq \frac{2}{9}
\end{aligned}$$

for d (or, equivalently, n) larger than some constant.

The expected TV distance between P and Q_V^α can be expressed as follows:

$$\begin{aligned}
&2\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)] \\
&= \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | (\alpha \rho_k + (1-\alpha) U_{v, t_k} \rho_k U_{v, t_k}^\dagger) | \phi_{i_k}^{(k)} \rangle - \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \right| \right] \\
&= \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| \prod_{k=1}^N (\alpha x_k + (1-\alpha) y_k) - \prod_{k=1}^N (\alpha x_k + (1-\alpha) x_k) \right| \right] \\
&= \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| \sum_{S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k - \prod_{k \notin S} x_k \right) \right| \right] \\
&\geq \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| \sum_{S=\emptyset} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k - \prod_{k \notin S} x_k \right) \right| \right. \\
&\quad \left. - \sum_{i_1, \dots, i_N} \left| \sum_{\emptyset \neq S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k - \prod_{k \notin S} x_k \right) \right| \right]
\end{aligned}$$

When $S = \emptyset$, we recover the TV distance between P and Q up to a factor as follows:

$$\begin{aligned}
&\mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| \sum_{S=\emptyset} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k - \prod_{k \notin S} x_k \right) \right| \right] \\
&= \mathbb{E}_{V \sim \text{Haar}(d)} \left[\sum_{i_1, \dots, i_N} \left| (1-\alpha)^N \left(\prod_{k=1}^N y_k - \prod_{k=1}^N x_k \right) \right| \right] \\
&= \mathbb{E}_{V \sim \text{Haar}(d)} \left[(1-\alpha)^N \sum_{i_1, \dots, i_N} \left| \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | U_{v, t_k} \rho_k U_{v, t_k}^\dagger | \phi_{i_k}^{(k)} \rangle - \prod_{k=1}^N \lambda_{i_k}^{(k)} \langle \phi_{i_k}^{(k)} | \rho_k | \phi_{i_k}^{(k)} \rangle \right| \right] \\
&= 2(1-\alpha)^N \mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V)] \geq \frac{4(1-\alpha)^N}{9}.
\end{aligned}$$

When $S \neq \emptyset$, we can use the triangle inequality:

$$\begin{aligned}
& \sum_{i_1, \dots, i_N} \left| \sum_{\emptyset \neq S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k - \prod_{k \notin S} x_k \right) \right| \\
& \leq \sum_{i_1, \dots, i_N} \sum_{\emptyset \neq S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k + \prod_{k \notin S} x_k \right) \\
& = \sum_{\emptyset \neq S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \sum_{i_1, \dots, i_N} \prod_{k \in S} x_k \left(\prod_{k \notin S} y_k + \prod_{k \notin S} x_k \right) \\
& = \sum_{\emptyset \neq S \subset [N]} \alpha^{|S|} (1-\alpha)^{N-|S|} \prod_{k \in S} \sum_{i_k} x_k \left(\prod_{k \notin S} \sum_{i_k} y_k + \prod_{k \notin S} \sum_{i_k} x_k \right) \\
& = 2(1 - (1-\alpha)^N).
\end{aligned}$$

Therefore:

$$2\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)] \geq \frac{4(1-\alpha)^N}{9} - 2(1 - (1-\alpha)^N).$$

Let $\alpha = \frac{c}{N}$ where $0 \leq c \leq 1/10$ is a small constant. We have:

$$\frac{4}{9}(1-\alpha)^N - 2(1 - (1-\alpha)^N) = \frac{22}{9} \left(1 - \frac{c}{N}\right)^N - 2 \geq \frac{22}{9}(1-c) - 2 = \frac{4-22c}{9} \geq \frac{1}{9}.$$

Therefore

$$\mathbb{E}_{V \sim \text{Haar}(d)} [\text{TV}(P, Q_V^\alpha)] \geq \frac{1}{18}.$$

□

Lemma C.2. For all $x \in [\frac{1}{10N}, \infty)$,

$$(-\log)(x) \leq -(x-1) + 2\log(10N)(x-1)^2.$$

Proof. Let $f(x) = \log(x) - (x-1) + 2\log(10N)(x-1)^2$. We have for $c = \frac{1}{4\log(10N)} < 1$:

$$f'(x) = \frac{1}{x} - 1 + 4\log(10N)(x-1) = \frac{(1-x)}{cx} (c-x),$$

which is positive for $x \in (0, c) \cup (1, \infty)$ and negative for $c < x < 1$, $\lim_{x \rightarrow 0^+} f(x) = -\infty$ and $f(1) = 0$. Hence, there is a $0 < c' < c$ such that:

$$x \geq c' \iff f(x) \geq 0.$$

But we have

$$f\left(\frac{1}{10N}\right) = -\log(10N) - \frac{1}{10N} + 1 + 2\log(10N) \left(1 - \frac{1}{10N}\right)^2 \geq 1 - \frac{1}{10N} > 0.$$

Thus, we can take $c' < \frac{1}{10N}$ and for all $x \geq \frac{1}{10N} > c'$ we have $f(x) \geq 0$. □

D Weingarten Calculus

As we use a random Hamiltonian constructed from sampling a Haar-random unitary matrix in our lower bound proofs, we need some facts from Weingarten calculus in order to compute the corresponding expectation values with respect to the Haar measure. If π is a permutation of $[n]$, let $\text{Wg}(\pi, d)$ denote the Weingarten function of dimension d . The following lemma is useful for our results.

Lemma D.1 ([Gu13]). *Let U be a Haar-distributed unitary $(d \times d)$ -matrix and let $\{A_i, B_i\}_{i=1}^n$ be a sequence of complex $(d \times d)$ -matrices. We have the following formula for the expectation value:*

$$\begin{aligned} & \mathbb{E} [\text{Tr}(UB_1U^*A_1U \dots UB_nU^*A_n)] \\ &= \sum_{\alpha, \beta \in \mathfrak{S}_n} \text{Wg}(\beta\alpha^{-1}, d) \text{Tr}_{\beta^{-1}}(B_1, \dots, B_n) \text{Tr}_{\alpha\gamma_n}(A_1, \dots, A_n), \end{aligned}$$

where $\gamma_n = (12 \dots n)$ and $\text{Tr}_\sigma(M_1, \dots, M_n) = \prod_j \text{Tr}(\prod_{i \in C_j} M_i)$ for $\sigma = \prod_j C_j$ and C_j are cycles.

We will also need some values of Weingarten function:

Lemma D.2 ([CS06]). *The function $\text{Wg}(\pi, d)$ has the following values:*

- $\text{Wg}((1), d) = \frac{1}{d}$,
- $\text{Wg}((12), d) = \frac{-1}{d(d^2-1)}$,
- $\text{Wg}((1)(2), d) = \frac{1}{d^2-1}$,
- $\text{Wg}((123), d) = \frac{2}{d(d^2-1)(d^2-4)}$,
- $\text{Wg}((12)(3), d) = \frac{-1}{(d^2-1)(d^2-4)}$,
- $\text{Wg}((1)(2)(3), d) = \frac{d^2-2}{d(d^2-1)(d^2-4)}$,
- $\text{Wg}((1234), d) = -\frac{5}{d^7-14d^5+49d^3-36d}$,
- $\text{Wg}((12)(34), d) = \frac{d^2+6}{d^8-14d^6+49d^4-36d^2}$,
- $\text{Wg}((123)(4), d) = \frac{2d^2-3}{d^8-14d^6+49d^4-36d^2}$,
- $\text{Wg}((12)(3)(4), d) = -\frac{1}{d^5-10d^3+9d}$,
- $\text{Wg}((1)(2)(3)(4), d) = \frac{d^4-8d^2+6}{d^8-14d^6+49d^4-36d^2}$.