



**HAL**  
open science

# Stochastic homogenization of HJ equations: a differential game approach

Andrea Davini, Raimundo Saona, Bruno Ziliotto

► **To cite this version:**

Andrea Davini, Raimundo Saona, Bruno Ziliotto. Stochastic homogenization of HJ equations: a differential game approach. 2024. hal-04786300

**HAL Id: hal-04786300**

**<https://hal.science/hal-04786300v1>**

Preprint submitted on 15 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STOCHASTIC HOMOGENIZATION OF HJ EQUATIONS: A DIFFERENTIAL GAME APPROACH

ANDREA DAVINI, RAIMUNDO SAONA, AND BRUNO ZILIOOTTO

ABSTRACT. We prove stochastic homogenization for a class of non-convex and non-coercive first-order Hamilton-Jacobi equations in a finite-range of dependence environment for Hamiltonians that can be expressed by a max-min formula. We make use of the representation of the solution as a value function of a differential game to implement a game-theoretic approach to the homogenization problem.

## 1. INTRODUCTION

In this paper we study the asymptotic behavior, as  $\varepsilon \rightarrow 0^+$ , of the solutions of a stochastic Hamilton-Jacobi (HJ) equation of the form

$$(HJ_\varepsilon) \quad \partial_t u^\varepsilon + H\left(\frac{x}{\varepsilon}, D_x u^\varepsilon, \omega\right) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

for every fixed  $T > 0$ , where  $H: \mathbb{R}^d \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$  is a Lipschitz Hamiltonian which can be represented by a max-min formula. The dependence of the equation on the random environment  $(\Omega, \mathcal{F}, \mathbb{P})$  enters through the Hamiltonian  $H(x, p, \omega)$ , whose law is assumed to be *stationary*, i.e., invariant by space translation, and *ergodic*, i.e., any event that is invariant by space translations has either probability 0 or 1. Under the additional assumptions that the random variables  $(H(\cdot, p, \cdot))_{p \in \mathbb{R}^d}$  satisfy a finite range of dependence condition and that the underlined dynamics is oriented, we show homogenization for equation  $(HJ_\varepsilon)$ , see Theorem 2.2. Furthermore, by exploiting an argument taken from [27, Section 5], we are able to extend the homogenization result to a class of Lipschitz Hamiltonians that can be put in max-min form only locally in  $p$ , see Theorem 2.4. The full set of assumptions and the precise statements of our homogenization results are presented in Section 2. In what follows, we want to emphasize that the Hamiltonians we consider are non-coercive and non-convex in  $p$ .

The coercivity of  $H$  in the momentum is a condition which is often assumed in the homogenization theory of first-order HJ equations. Its role is to provide uniform  $L^\infty$ -bounds on the derivatives of the solutions to the equation  $(HJ_\varepsilon)$  and to an associated “cell” problem. The first homogenization results for equations of the form  $(HJ_\varepsilon)$  for coercive Hamiltonians were established in the periodic setting in the pioneering work [34] and later extended to the almost-periodic case in [32]. The generalization of these results in the stationary ergodic setting was established in [38, 36] under the additional assumption that the Hamiltonian is convex in  $p$ . By exploiting the metric character of first order HJ equations, homogenization has been extended to the case of quasiconvex Hamiltonians in [24, 9]. The question whether homogenization holds in the stationary ergodic setting for coercive Hamiltonians that are non-convex in the momentum remained an open problem for about fifteen years, until the third author provided in [40] the first counterexample to homogenization in dimensions greater than 1. Feldman and Souganidis generalized that example and showed in [28] that homogenization can fail for Hamiltonians of the form  $H(x, p, \omega) := G(p) + V(x, \omega)$  whenever  $G$  has a strict saddle point. This has shut the door to the possibility of having a general qualitative homogenization theory in the stationary ergodic setting in dimension  $d \geq 2$ , at least

without imposing further mixing conditions on the random environment. On the positive side, homogenization of equation  $(\text{HJ}_\varepsilon)$  for coercive and non-convex Hamiltonians of fairly general type has been established in dimension  $d = 1$  in [11, 29], and in any space dimension for Hamiltonians of the form  $H(x, p, \omega) = (|p|^2 - 1)^2 + V(x, \omega)$  in [10]. Further positive results in random environment satisfying a finite range of dependence condition have been obtained in [7] for Hamiltonians that are positively homogeneous of degree  $\alpha \geq 1$ , and for Hamiltonians with strictly star-shaped sublevel sets in [28]. Despite all these significant progresses, the general question of which equations of the form  $(\text{HJ}_\varepsilon)$  homogenize in the non-convex case remains largely uncharted.

When the coercivity condition of  $H$  in  $p$  is dropped, one loses control on the derivatives of the solutions of equation  $(\text{HJ}_\varepsilon)$  and of the associated “cell” problem, which are no longer Lipschitz continuous in general. As a consequence, homogenization of  $(\text{HJ}_\varepsilon)$  is known to fail even in the periodic case, regardless to the fact that the Hamiltonian is or is not convex in  $p$ , see for instance the introductions in [19, 20] and some examples in [15]. In this generality, supplementary conditions need to be assumed to compensate the lack of coercivity of the Hamiltonian. In the periodic and other compact settings, homogenization results in this vein have been obtained in [4, 5, 6, 3] and more recently in some convex situations in [13], for a class non-convex Hamiltonians in dimension  $d = 2$  in [19], and in other non-convex cases in [17, 15]. When  $H(x, p, \omega) := |p| + \langle V(x, \omega), p \rangle$ , equation  $(\text{HJ}_\varepsilon)$  is known in the literature (up to a sign change) as the *G-equation*, for which homogenization has been established, both in the periodic in [20, 39, 37], and in the stationary ergodic case in [35, 21], under a smallness condition on the divergence of  $V$ , but without imposing that  $|V| < 1$ , meaning that  $H$  is not assumed to be coercive in  $p$ .

This paper furnishes a new and fairly general class of non-convex and non-coercive Hamiltonians for which the equation  $(\text{HJ}_\varepsilon)$  homogenizes. In our first result, corresponding to Theorem 2.2, we prove homogenization for a class of non-convex and Lipschitz Hamiltonians that arises from Differential Game Theory. More specifically, we will consider Hamiltonians of the form

$$(H) \quad H(x, p, \omega) := \max_{b \in B} \min_{a \in A} \{-\ell(x, a, b, \omega) - \langle f(a, b), p \rangle\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega,$$

where the main assumptions are that the law of  $\ell$  has finite range of dependence, in the same spirit as [7, 28], and that there exists a direction  $e \in \mathbb{S}^{d-1}$  and  $\delta > 0$  such that

$$(f) \quad \langle f(a, b), e \rangle \geq \delta \quad \text{for all } a \in A, b \in B.$$

This latter condition is inspired by the joint work [30] of the third author, addressed to the study of the asymptotic of a discrete-time game on  $\mathbb{Z}^d$ . We refer the reader to [30, Section 4] for a discussion on the relation between these discrete games and the topic of stochastic homogenization. Interestingly, the assumption  $(f)$  precludes a Hamiltonian of the form  $(H)$  from being coercive, see Remark 2.3, and this is one significant originality that distinguishes our work from most of the contributions on the topic of stochastic homogenization.

Another important novelty relies on the proof technique. Indeed, thanks to the form  $(H)$  of the Hamiltonian, we can represent the solution of  $(\text{HJ}_\varepsilon)$  as the value function of a differential game, as explained in [27], and implement a game-theoretic approach. Such an approach has been rarely used in the context of homogenization of non-convex HJ equations (see e.g. [15] in the periodic setting), and, up to our knowledge, this is the first time that it is considered to prove a positive result in the stochastic case. By considering optimal strategies, generated paths, and the dynamic programming principle, we manage to show that the solutions of  $(\text{HJ}_\varepsilon)$  exhibit a concentration behavior asymptotically and that their mean satisfies an approximate subadditive inequality. The homogenization result is finally derived by exploiting the local Lipschitz character of the solutions of  $(\text{HJ}_\varepsilon)$ .

The probabilistic arguments that we use are related to the works [8, 28, 7], where the authors prove homogenization for several classes of first and second order nonconvex Hamilton-Jacobi equations. In order to do so, they consider an auxiliary stationary Hamilton-Jacobi equation, called the *metric problem* [8] (resp., [7, 28]), whose solutions can be interpreted as the minimal cost of going from one point of the space to another point (resp., to a planar surface). By analogy with techniques employed in first-passage percolation [2, 33], they combine Azuma's inequality with a subadditive argument to prove homogenization of the metric problem, and provide convergence rates and concentration estimates. Then, they resort to a PDE approach to relate the metric problem and the original Hamilton-Jacobi equation. Compared to their technique, our proof presents several key differences.

- (1) The concentration and subadditive inequalities techniques are applied to the value of a zero-sum differential game, which is a fairly different problem than the metric problem.
- (2) Our arguments rely mainly on a game-theoretic approach, by exploiting the monotonicity property (in the preferred direction  $e$ ) of the optimal trajectories, rather than on PDE arguments.
- (3) We deal with non-coercive Hamiltonians, while Hamiltonians in [8, 7, 28] are coercive.

This yields several difficulties, including that the space-Lipschitz constant of the solutions of  $(\text{HJ}_\varepsilon)$  are not uniformly bounded with respect to  $\varepsilon$ .

In our second result, corresponding to Theorem 2.4, we succeed to extend the homogenization result to a class of non-convex and non-coercive Lipschitz Hamiltonians that are not necessarily defined by a max-min formula as the ones considered above. This makes the game-theoretic approach even more notable, in the sense that we are able to deal with Hamiltonians that do not come *a priori* from a differential game. For such an extension, we exploit an argument introduced in [27, Section 5] to put these Hamiltonians in the form (H) when  $p$  is constrained within a ball  $B_R$ , but its use to prove a homogenization result in presence of a non-coercive Hamiltonian is not trivial and, as far as we know, new. The difficulty relies on the fact that, due to the lack of coercivity of the Hamiltonian, the Lipschitz constants in  $x$  of the solutions to  $(\text{HJ}_\varepsilon)$  are not uniformly bounded with respect to  $\varepsilon > 0$ , but explode with rate  $1/\varepsilon$ . From the game-theoretic viewpoint, this corresponds to a set of controls for Player 2 of the kind  $B_{R/\varepsilon}$ . In view of this, we had to tailor the proof of Theorem 2.2 in a form that is suited for this extension, by paying particular attention that the constants that come into play in the crucial estimates at the base of our arguments depend on parameters that can be controlled when  $\varepsilon$  goes to 0.

The paper is organized as follows. In Section 2 we present the notation, the standing assumptions and the statements of our homogenization results, namely Theorem 2.2 and Theorem 2.4. In Section 3 we present the reduction homogenization strategy we will follow to prove Theorem 2.2. Some proofs are deferred to Appendix B. In Section 4 we prove the probabilistic concentration result. Section 5 is devoted to the proofs of Theorem 2.2 and Theorem 2.4. Appendix A contains the deterministic PDE results, along with their proofs, that we use in the paper.

ACKNOWLEDGEMENTS. – AD is a member of the INdAM Research Group GNAMPA. He wishes to thank Guy Barles for a long and valuable email conversation occurred in the fall 2023 about possible generalizations of the comparison principle stated in Theorem A.1. AD is particularly grateful to Guy Barles for his careful reading of AD's attempts to generalize Theorem A.1 and for sharing his expertise on the matter, including the details to turn the argument sketched in [18, Remark 5.3] into a full proof. BZ is very grateful to Scott Armstrong and Pierre Cardaliaguet for all the enlightening discussions on homogenization theory. This work was supported by Sapienza Università di Roma - Research Funds 2018 and 2019, by the French Agence Nationale de la Recherche (ANR) under

reference ANR-21-CE40-0020 (CONVERGENCE project), and by the ERC CoG 863818 (ForM-SMArt) grant. It was partly done during a 1-year visit of BZ to the Center for Mathematical Modeling (CMM) at University of Chile in 2023, under the IRL program of CNRS.

## 2. ASSUMPTIONS AND MAIN RESULT

Throughout the paper, we will denote by  $d \in \mathbb{N}$  the dimension of the ambient space. We will denote either by  $B_r(x_0)$  or  $B(x_0, r)$  (respectively,  $\overline{B}_r(x_0)$  or  $\overline{B}(x_0, r)$ ) the open (resp., closed) ball in  $\mathbb{R}^d$  of radius  $r > 0$  centered at  $x_0 \in \mathbb{R}^d$ . When  $x_0 = 0$ , we will more simply write  $B_r$  (resp.,  $\overline{B}_r$ ). The symbol  $|\cdot|$  will denote the norm in  $\mathbb{R}^k$ , for any  $k \geq 1$ . We will write  $\varphi_n \rightrightarrows \varphi$  in  $E \subseteq \mathbb{R}^k$  to mean that the sequence of functions  $(\varphi_n)_n$  uniformly converge to  $\varphi$  on compact subsets of  $E$ . We will denote by  $C(X)$ ,  $UC(X)$ ,  $BUC(X)$ , and  $Lip(X)$  the space of continuous, uniformly continuous, bounded uniformly continuous, and Lipschitz continuous functions on a metric space  $X$ , respectively.

We will denote by  $(\Omega, \mathcal{F}, \mathbb{P})$  a probability space, where  $\mathbb{P}$  is a probability measure and  $\mathcal{F}$  is the  $\sigma$ -algebra of  $\mathbb{P}$ -measurable subsets of  $\Omega$ . We will assume that  $\mathbb{P}$  is *complete* in the usual measure theoretic sense. We will denote by  $\mathcal{B}(\mathbb{R}^k)$  the Borel  $\sigma$ -algebra on  $\mathbb{R}^k$  and equip the product space  $\mathbb{R}^d \times \Omega$  and  $\mathbb{R}^d \times A \times B \times \Omega$  with the product  $\sigma$ -algebras  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$  and  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$ , respectively.

We will assume that  $\mathbb{P}$  is invariant under the action of a one-parameter group  $(\tau_x)_{x \in \mathbb{R}^d}$  of transformations  $\tau_x: \Omega \rightarrow \Omega$ . More precisely, we assume that: the mapping  $(x, \omega) \mapsto \tau_x \omega$  from  $\mathbb{R}^d \times \Omega$  to  $\Omega$  is measurable;  $\tau_0 = id$ ;  $\tau_{x+y} = \tau_x \circ \tau_y$  for every  $x, y \in \mathbb{R}^d$ ; and  $\mathbb{P}(\tau_x(E)) = \mathbb{P}(E)$ , for every  $E \in \mathcal{F}$  and  $x \in \mathbb{R}^d$ . Lastly, we will assume that the action of  $(\tau_x)_{x \in \mathbb{R}^d}$  is *ergodic*, i.e., any measurable function  $\varphi: \Omega \rightarrow \mathbb{R}$  satisfying  $\mathbb{P}(\varphi(\tau_x \omega) = \varphi(\omega)) = 1$  for every fixed  $x \in \mathbb{R}^d$  is almost surely equal to a constant.

A random process  $f: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$  is said to be *stationary* with respect to  $(\tau_x)_{x \in \mathbb{R}^d}$  if  $f(x, \omega) = f(0, \tau_x \omega)$  for all  $(x, \omega) \in \mathbb{R}^d \times \Omega$ . Moreover, whenever the action of  $(\tau_x)_{x \in \mathbb{R}^d}$  is ergodic, we refer to  $f$  as a *stationary ergodic* process.

Let  $(X_i)_{i \in \mathcal{I}}$  be a (possibly uncountable) family of jointly measurable functions from  $\mathbb{R}^d \times \Omega$  to  $\mathbb{R}$ . We will say that the random variables  $(X_i)_{i \in \mathcal{I}}$  exhibit *long-range independence* (or, equivalently, have *finite range of dependence*) if there exists  $\rho > 0$  such that, for all pair of sets  $S, \widehat{S} \subseteq \mathbb{R}^d$  such that their Hausdorff distance  $d_H(S, \widehat{S}) > \rho$ , the generated  $\sigma$ -algebras  $\sigma(\{X_i(x, \cdot) : i \in \mathcal{I}, x \in S\})$  and  $\sigma(\{X_i(x, \cdot) : i \in \mathcal{I}, x \in \widehat{S}\})$  are independent, in symbols,

$$(FRD) \quad \sigma(\{X_i(x, \cdot) : i \in \mathcal{I}, x \in S\}) \perp \sigma(\{X_i(x, \cdot) : i \in \mathcal{I}, x \in \widehat{S}\}) \quad \text{whenever } d_H(S, \widehat{S}) > \rho.$$

In this paper, we will be concerned with the Hamilton-Jacobi equation of the form

$$(2.1) \quad \partial_t u + H(x, D_x u, \omega) = 0, \quad \text{in } (0, T) \times \mathbb{R}^d,$$

where the Hamiltonian  $H: \mathbb{R}^d \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$  is assumed to be stationary with respect to shifts in  $x$  variable, i.e.,  $H(x + y, p, \omega) = H(x, p, \tau_y \omega)$  for every  $x, y \in \mathbb{R}^d$ ,  $p \in \mathbb{R}^d$ , and  $\omega \in \Omega$ , and to belong to the class  $\mathcal{H}$  defined as follows.

**DEFINITION 2.1.** A function  $H: \mathbb{R}^d \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$  is said to be in the class  $\mathcal{H}$  if it is jointly measurable and it satisfies the following conditions, for some constant  $\beta > 0$ :

- (H1)  $|H(x, p, \omega)| \leq \beta(1 + |p|)$  for all  $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$ ;
- (H2)  $|H(x, p, \omega) - H(x, q, \omega)| \leq \beta|p - q|$  for all  $x, p, q \in \mathbb{R}^d$ ;
- (H3)  $|H(x, p, \omega) - H(y, p, \omega)| \leq \beta|x - y|$  for all  $x, y, p \in \mathbb{R}^d$ .

Assumptions (H1)-(H3) guarantee well-posedness in  $C([0, T] \times \mathbb{R}^d)$ , for every fixed  $T > 0$ , of the Cauchy problem associated with equation (2.1) when the initial datum is in  $UC(\mathbb{R}^d)$ . Furthermore,

the solutions are actually in  $UC([0, T] \times \mathbb{R}^d)$ . Solutions, subsolutions and supersolutions of (2.1) will be always understood in the viscosity sense, see [14, 16, 18, 22], and implicitly assumed continuous, if not otherwise specified.

The purpose of this paper is to prove a homogenization result for equation (2.1) for a subclass of stationary Hamiltonians belonging to  $\mathcal{H}$  that arise from Differential Game Theory and that can be expressed in the following max-min form:

$$(H) \quad H(x, p, \omega) := \max_{b \in B} \min_{a \in A} \{-\ell(x, a, b, \omega) - \langle f(a, b), p \rangle\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega.$$

Here  $A, B$  are compact subsets of  $\mathbb{R}^m$ , for some integer  $m$ , and the product space  $\mathbb{R}^d \times A \times B \times \Omega$  is equipped with the product  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{F}$ . The mapping  $f: A \times B \rightarrow \mathbb{R}^d$  is a continuous vector field, and the *running cost*  $\ell: \mathbb{R}^d \times A \times B \times \Omega \rightarrow \mathbb{R}$  is a jointly measurable function satisfying the following assumptions:

$$(\ell_1) \quad \ell(\cdot, \cdot, \cdot, \omega) \in \text{BUC}(\mathbb{R}^d \times A \times B) \quad \text{for every } \omega \in \Omega;$$

$$(\ell_2) \quad \text{there exists a constant } \text{Lip}(\ell) > 0 \text{ such that}$$

$$|\ell(x, a, b, \omega) - \ell(y, a, b, \omega)| \leq \text{Lip}(\ell) |x - y| \quad \text{for all } x, y \in \mathbb{R}^d, a \in A, b \in B \text{ and } \omega \in \Omega;$$

$$(\ell_3) \quad \ell \text{ is stationary with respect to } x, \text{ i.e.,}$$

$$\ell(x, a, b, \omega) = \ell(0, a, b, \tau_x \omega) \quad \text{for all } x \in \mathbb{R}^d, a \in A, b \in B \text{ and } \omega \in \Omega.$$

Throughout the paper, we will denote by  $\mathcal{H}_{dG}$  the subclass of Hamiltonians in  $\mathcal{H}$  that can be put in the form (H) with  $f$  and  $\ell$  satisfying assumptions (f) and  $(\ell_1)$ – $(\ell_2)$ , respectively. A Hamiltonian  $H$  belonging to  $\mathcal{H}_{dG}$  will be furthermore termed stationary to mean that assumption  $(\ell_3)$  is in force.

The specific form (H) of the Hamiltonian allows to represent solutions to equation (2.1) via suitable formulae issued from Differential Games, see Appendix A.3 for more details. In the sequel, we shall denote by  $\|\ell\|_\infty$  the  $L^\infty$ -norm of  $\ell$  on  $\mathbb{R}^d \times A \times B \times \Omega$ , which is finite due to  $(\ell_1)$ ,  $(\ell_3)$ , and the ergodicity assumption on  $\Omega$ .

To prove our homogenization result, we will assume the following additional crucial assumptions:

$$(\ell_4) \quad \text{(long-range independence) the random variables } (\ell(\cdot, a, b, \cdot))_{(a,b) \in A \times B} \text{ from } \mathbb{R}^d \times \Omega \text{ to } \mathbb{R} \text{ exhibit long-range independence, i.e., there exists } \rho > 0 \text{ such that (FRD) holds with } \mathcal{I} := A \times B \text{ and } X_i := \ell(\cdot, a, b, \cdot) \text{ where } i = (a, b);$$

$$(f) \quad \text{(oriented dynamics) the dynamics given by } f: A \times B \rightarrow \mathbb{R}^d \text{ is } \textit{oriented}, \text{ i.e., there exists } \delta > 0 \text{ and a direction } e \in \mathbb{S}^{d-1} \text{ such that}$$

$$\langle f(a, b), e \rangle \geq \delta \quad \text{for all } (a, b) \in A \times B.$$

Our main result reads as follows.

**THEOREM 2.2.** *Let  $H$  be a stationary Hamiltonian belonging to  $\mathcal{H}_{dG}$  and satisfying hypotheses  $(\ell_4)$  and (f). Then, the HJ equation  $(HJ_\varepsilon)$  homogenizes, i.e., there exists a continuous function  $\bar{H}: \mathbb{R}^d \rightarrow \mathbb{R}$ , called effective Hamiltonian, such that, for every uniformly continuous function  $g$  on  $\mathbb{R}$ , there exists a set  $\Omega_g$  of probability 1 such that, for every  $\omega \in \Omega_g$ , the solutions  $u^\varepsilon(\cdot, \cdot, \omega)$  of  $(HJ_\varepsilon)$  satisfying  $u^\varepsilon(0, \cdot, \omega) = g$  converges, locally uniformly on  $[0, T] \times \mathbb{R}^d$  as  $\varepsilon \rightarrow 0^+$ , to the unique solution  $\bar{u}$  of*

$$\begin{cases} \partial_t \bar{u} + \bar{H}(D_x \bar{u}) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ \bar{u}(0, \cdot) = g & \text{in } \mathbb{R}. \end{cases}$$

Furthermore,  $\bar{H}$  satisfies (H1) and (H2).

REMARK 2.3. We stress that a Hamiltonian of the form (H) with  $f$  satisfying condition (f) is never coercive. Indeed,

$$\lim_{t \rightarrow -\infty} H(x, te, \omega) = +\infty, \quad \lim_{t \rightarrow +\infty} H(x, te, \omega) = -\infty \quad \text{for every } (x, \omega) \in \mathbb{R}^d \times \Omega.$$

By exploiting suitable Lipschitz bounds for solutions to  $(\text{HJ}_\varepsilon)$  with Lipschitz initial data and a localization argument inspired by [27, Section 5], we are able to extend the homogenization result stated above to the subclass of Hamiltonians in  $\mathcal{H}$  described in the statement of the next theorem. This new subclass has nonempty intersection with  $\mathcal{H}_{dG}$ , but it is not fully contained in it.

THEOREM 2.4. *Let  $G$  be a stationary Hamiltonian belonging to  $\mathcal{H}$  and satisfying the following assumption:*

- (G1) *the random variables  $(G(\cdot, p, \cdot))_{p \in \mathbb{R}^d}$  from  $\mathbb{R}^d \times \Omega$  to  $\mathbb{R}$  exhibit long-rate independence, i.e., there exists  $\rho > 0$  such that (FRD) holds with  $\mathcal{I} := \mathbb{R}^d$  and  $X_i := H(\cdot, p, \cdot)$  where  $i = p$ .*

*Then, the homogenization result stated in Theorem 2.2 holds for any Hamiltonian  $H$  of the form  $H(x, p, \omega) := G(x, \pi(p), \omega) + \langle p, v \rangle$ , where  $v$  is a non-null vector in  $\mathbb{R}^d$  and  $\pi: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a linear map such that  $\pi(v) = 0$ .*

Examples of Hamiltonians  $G$  lying in  $\mathcal{H}$  and satisfying (G1) are the ones of the form  $G(x, p, \omega) := G_0(p) + V(x, \omega)$ , where  $G_0$  belongs to  $\mathcal{H}$  and  $V: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$  is a stationary function, globally bounded and Lipschitz on  $\mathbb{R}^d$ , which satisfies (FRD) with  $\mathcal{I} := \{0\}$  and  $X_0 := V$ .

### 3. REDUCTION ARGUMENTS FOR HOMOGENIZATION

In this section we describe the reduction strategy that we will follow to prove Theorem 2.2. The first step consists in noticing that, in order to prove homogenization for equation  $(\text{HJ}_\varepsilon)$ , it is enough to restrict to linear initial data instead of any  $g \in \text{UC}(\mathbb{R}^d)$ . The precise statement is the following.

THEOREM 3.1. *Let  $H$  satisfy hypotheses (H1)-(H3) and denote by  $\tilde{u}_\theta^\varepsilon$  the unique continuous solution of equation  $(\text{HJ}_\varepsilon)$  satisfying  $\tilde{u}_\theta^\varepsilon(0, x, \omega) = \langle \theta, x \rangle$  for all  $(x, \omega) \in \mathbb{R}^d \times \Omega$  and for every fixed  $\theta \in \mathbb{R}^d$  and  $\varepsilon > 0$ . Assume there exists a function  $\overline{H}: \mathbb{R}^d \rightarrow \mathbb{R}$  such that, for every  $\theta \in \mathbb{R}^d$ , the following convergence takes place for every  $\omega$  in a set  $\Omega_\theta$  of probability 1:*

$$(3.1) \quad \tilde{u}_\theta^\varepsilon(t, x, \omega) \rightrightarrows \langle \theta, x \rangle - t\overline{H}(\theta) \quad \text{in } [0, T) \times \mathbb{R}^d \text{ as } \varepsilon \rightarrow 0^+.$$

*Then,  $\overline{H}$  satisfies condition (H1)-(H2). Furthermore, for every fixed  $g \in \text{UC}(\mathbb{R}^d)$ , there exists a set  $\Omega_g$  of probability 1 such that the unique function  $u^\varepsilon(\cdot, \cdot, \omega) \in C([0, T) \times \mathbb{R}^d)$  which solves  $(\text{HJ}_\varepsilon)$  with initial condition  $u^\varepsilon(0, \cdot, \omega) = g$  in  $\mathbb{R}^d$  converges, locally uniformly in  $[0, T) \times \mathbb{R}^d$  as  $\varepsilon \rightarrow 0^+$ , to the unique solution  $\overline{u} \in C([0, T) \times \mathbb{R}^d)$  of*

$$(3.2) \quad \partial_t \overline{u} + \overline{H}(D_x \overline{u}) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

*with the initial condition  $\overline{u}(0, \cdot) = g$ , for every  $\omega \in \Omega_g$ .*

This reduction argument, which is of deterministic nature, was already contained in the pioneering work [34] on periodic homogenization, at least as far as first-order HJ equations are concerned. This holds, in fact, even in the case when equation  $(\text{HJ}_\varepsilon)$  presents an additional (vanishing) diffusive and possibly degenerate term. A proof of this can be found in [23] and is given for Hamiltonians that are coercive in the  $p$ -variable. Such a class does not include the kind of Hamiltonians we consider here, as pointed out in Remark 2.3. Yet, the extension follows by arguing as in [23], with the only difference that one has to use a different Comparison Principle, namely Theorem A.3, in place of [23, Proposition 2.4]. We refer the reader to Appendix B for the detailed argument.

Theorem 3.1 yields, in particular, that the effective Hamiltonian  $\overline{H}$  is identified by the following almost sure limit:

$$\overline{H}(\theta) := - \lim_{\varepsilon \rightarrow 0} \tilde{u}_\theta^\varepsilon(1, 0, \omega) \quad \text{for every fixed } \theta \in \mathbb{R}^d.$$

The second step in the reduction consists in observing that, in order to prove the local uniform convergence required to apply Theorem 3.1, it is enough to prove it for a fixed value of the time variable, that we chose equal to 1.

LEMMA 3.2. *Let  $\omega \in \Omega$  and  $\theta \in \mathbb{R}^d$  be fixed, and assume that*

$$(3.3) \quad \limsup_{\varepsilon \rightarrow 0^+} \sup_{y \in B_R} |\tilde{u}_\theta^\varepsilon(1, y, \omega) - \langle \theta, x \rangle + \overline{H}(\theta)| = 0 \quad \text{for every } R > 0.$$

Then, for every  $T > 0$ ,

$$(3.4) \quad \tilde{u}_\theta^\varepsilon(t, x, \omega) \rightrightarrows \langle \theta, x \rangle - t\overline{H}(\theta) \quad \text{in } [0, T] \times \mathbb{R}^d.$$

*Proof.* Since  $\omega$  will remain fixed throughout the proof, we will omit it from our notation. Let us fix  $\theta \in \mathbb{R}^d$ . We first take note of the following scaling relations

$$\tilde{u}_\theta^\varepsilon(t, x) = \varepsilon \tilde{u}_\theta^1(t/\varepsilon, x/\varepsilon) = t(\varepsilon/t) \tilde{u}_\theta^1(t/\varepsilon, x/\varepsilon) = t \tilde{u}_\theta^{\varepsilon/t}(1, x/t) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Fix  $T > 0$ . Then, for every fixed  $r \in (0, T)$ , we obtain

$$(3.5) \quad \begin{aligned} \sup_{r \leq t \leq T} \sup_{y \in B_R} |\tilde{u}_\theta^\varepsilon(t, y) - \langle \theta, y \rangle + t\overline{H}(\theta)| &= \sup_{r \leq t \leq T} \sup_{y \in B_R} |t(\tilde{u}_\theta^{\varepsilon/t}(1, y/t) - \langle \theta, y/t \rangle + \overline{H}(\theta))| \\ &\leq T \sup_{\varepsilon/T \leq \eta \leq \varepsilon/r} \sup_{z \in B_{R/r}} |\tilde{u}_\theta^\eta(1, z) - \langle \theta, z \rangle + \overline{H}(\theta)|. \end{aligned}$$

By (3.3), the right-hand side goes to 0 as  $\varepsilon \rightarrow 0^+$ . On the other hand, in view of Proposition A.6-(i) and of the fact that  $\tilde{u}_\theta^\varepsilon(0, x) = \langle \theta, x \rangle$  for all  $x \in \mathbb{R}^d$ , we have

$$\begin{aligned} \sup_{0 \leq t \leq r} \sup_{y \in \mathbb{R}^d} |\tilde{u}_\theta^\varepsilon(t, y) - \langle \theta, y \rangle + t\overline{H}(\theta)| &\leq r|\overline{H}(\theta)| + \sup_{0 \leq t \leq r} \sup_{y \in \mathbb{R}^d} |\tilde{u}_\theta^\varepsilon(t, y) - \langle \theta, y \rangle| \\ &\leq r(|\overline{H}(\theta)| + \|\ell\|_\infty + \|f\|_\infty). \end{aligned}$$

Assertion (3.4) follows from this and (3.5) by the arbitrariness of the choice of  $r \in (0, T)$ .  $\square$

In order to simplify some arguments, we find convenient to work with solutions with zero initial datum. We can always reduce to this case, without any loss of generality, by setting  $u_\theta^\varepsilon(t, x, \omega) := \tilde{u}_\theta^\varepsilon(t, x, \omega) - \langle \theta, x \rangle$  for all  $(t, x, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega$ . The function  $u_\theta^\varepsilon$  is the unique continuous function which solves equation (HJ $_\varepsilon$ ) with  $H_\theta := H(\cdot, \theta + \cdot, \omega)$  in place of  $H$  and which satisfies the initial condition  $u_\theta^\varepsilon(0, x, \omega) = 0$  for all  $(x, \omega) \in \mathbb{R}^d \times \Omega$ . Note that the Hamiltonian  $H_\theta$  is still given by the max-min formula (H) where  $\ell$  is replaced by  $\ell_\theta(x, a, b, \omega) := \ell(x, a, b, \omega) + \langle f(a, b), \theta \rangle$ . Furthermore,  $\ell_\theta$  satisfies the same conditions  $(\ell_1)$ – $(\ell_4)$ .

The last reduction remark consists in noticing that the following rescaling relation holds

$$u_\theta^\varepsilon(1, x, \omega) = \varepsilon u_\theta(1/\varepsilon, x/\varepsilon, \omega) \quad \text{for all } x \in \mathbb{R}^d \text{ and } \varepsilon > 0,$$

where we have denoted by  $u_\theta$  the function  $u_\theta^\varepsilon$  with  $\varepsilon = 1$ .

In the light of all this, the proof of Theorem 2.2 is thus reduced to show that, for every fixed  $\theta \in \mathbb{R}^d$ , there exists a set  $\Omega_\theta$  of probability 1 such that, for every  $\omega \in \Omega_\theta$ , we have

$$(3.6) \quad \limsup_{\varepsilon \rightarrow 0^+} \sup_{y \in B_R} |u_\theta^\varepsilon(1, y, \omega) + \overline{H}(\theta)| = \limsup_{t \rightarrow +\infty} \sup_{y \in B_{tR}} \left| \frac{u_\theta(t, y, \omega)}{t} + \overline{H}(\theta) \right| = 0 \quad \text{for all } R > 0,$$

for some deterministic function  $\overline{H}: \mathbb{R}^d \rightarrow \mathbb{R}$ .



## 4. PROBABILISTIC CONCENTRATION

In this section we shall prove, by making use of Azuma's martingale inequality, that  $u_\theta(t, 0, \cdot)$  is concentrated. For this, we will take advantage of the fact that  $u_\theta$  can be expressed via the Differential Game Theoretic formula (A.17) with initial datum  $g \equiv 0$  and running cost  $\ell_\theta(x, a, b, \omega) := \ell(x, a, b, \omega) + \langle f(a, b), \theta \rangle$ . Here is where assumptions (FRD) and (f) play a crucial role, by ensuring altogether that the solution of (HJ $_\varepsilon$ ) is robust with respect to local perturbations of the cost function  $\ell$ .

Throughout this section, we will weaken the conditions on the running cost  $\ell: \mathbb{R}^d \times A \times B \times \Omega \rightarrow \mathbb{R}$  and we will assume  $\ell$  to be only jointly measurable and such to satisfy conditions ( $\ell_1$ )-( $\ell_2$ ). In particular, no continuity and stationarity conditions with respect to  $x$  will be required.

We start by recalling a classic theorem on concentration of martingales, also known as Azuma's inequality.

LEMMA 4.1 (Concentration of martingales [12, 31]). *Let  $(X_n)_{n \in \mathbb{N}}$  be a martingale and  $(c_n)_{n \in \mathbb{N}}$  a real sequence such that, for all  $n \in \mathbb{N}$ ,  $|X_n - X_{n+1}| \leq c_n$  almost surely. Then, for all  $n \in \mathbb{N}$  and  $M > 0$ ,*

$$\mathbb{P}(|X_n - X_0| \geq M) \leq 2 \exp\left(\frac{-M^2}{2 \sum_{m=0}^{n-1} c_m^2}\right).$$

The probabilistic concentration result mentioned before is stated as follows.

PROPOSITION 4.2. *There exists a constant  $c = c(\rho, \delta, \text{Lip}(\ell), \|f\|_\infty) > 0$ , only depending on  $\rho$ ,  $\delta$ ,  $\text{Lip}(\ell)$  and  $\|f\|_\infty$ , such that, for all  $M > 0$  and  $t \geq 1$ ,*

$$\mathbb{P}\left(|u_\theta(t, 0, \cdot) - U_\theta(t)| \geq M\sqrt{t}\right) \leq \exp(-cM^2),$$

where  $U_\theta(t) := \mathbb{E}[u_\theta(t, 0, \cdot)]$  denotes the expectation of the random variable  $u_\theta(t, 0, \cdot)$ .

REMARK 4.3. We have tailored the proof of Proposition 4.2 in such a way that the constant  $c$  appearing in the statement does not depend on  $\|\ell\|_\infty$ . This is crucial in view of the extension of the homogenization result provided in Theorem 2.4.

To prove Proposition 4.2, we need a technical lemma first. The result is deterministic, hence the dependence on  $\omega$  will be omitted. It expresses that, thanks to the condition (f) on the dynamics, we can control the variation of the value function as we change the running cost on a strip that is orthogonal to the direction  $e$ .

LEMMA 4.4. *For any given pair  $R < \widehat{R}$  in  $\mathbb{R}$ , let us define the strip between  $R$  and  $\widehat{R}$  as follows:*

$$S_{R, \widehat{R}} := \{x : \langle x, e \rangle \in [R, \widehat{R}]\}.$$

*Let  $\ell, \widehat{\ell}: \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$  be Borel-measurable bounded running costs and let  $f: A \times B \rightarrow \mathbb{R}^d$  be a continuous vector field satisfying condition (f). For every fixed  $\theta \in \mathbb{R}^d$ , let us denote by  $u_\theta(t, x)$ ,  $\widehat{u}_\theta(t, x)$  the value functions defined via (A.17) with initial datum  $g \equiv 0$  and running cost  $\ell_\theta(x, a, b, \omega) := \ell(x, a, b, \omega) + \langle f(x, a, b), \theta \rangle$  and  $\widehat{\ell}_\theta(x, a, b, \omega) := \widehat{\ell}(x, a, b, \omega) + \langle f(x, a, b), \theta \rangle$ , respectively. If  $\ell = \widehat{\ell}$  on  $(\mathbb{R}^d \setminus S_{R, \widehat{R}}) \times A \times B$ , we have*

$$|u_\theta(t, x) - \widehat{u}_\theta(t, x)| \leq \frac{R - \widehat{R}}{\delta} \|\ell - \widehat{\ell}\|_\infty \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}^d.$$

We point out the generality of the previous statement: no continuity conditions on the running costs  $\ell, \widehat{\ell}$  are assumed in the above statement, and the functions  $u_\theta, \widehat{u}_\theta$  are still well defined.

*Proof.* Fix  $t > 0$  and  $x, \theta \in \mathbb{R}^d$ . Let  $R < \widehat{R}$  be arbitrary in  $\mathbb{R}$  and consider the strip  $S_{R, \widehat{R}}$ . Fix controls  $\alpha \in \Gamma(t)$  and  $b \in \mathbb{B}(t)$  and consider the solution  $y_x: [0, t] \rightarrow \mathbb{R}^d$  of the ODE

$$\begin{cases} \dot{y}_x(s) = f(\alpha[b](s), b(s)) & \text{in } [0, t] \\ y_x(0) = x. \end{cases}$$

From the orientation of the game, the map  $s \mapsto \langle y_x(s), e \rangle$  is strictly increasing in  $[0, t]$ . More precisely,

$$(4.1) \quad \frac{d}{ds} \langle y_x(s), e \rangle = \langle \dot{y}_x(s), e \rangle = \langle f(\alpha[b](s), b(s)), e \rangle \geq \delta \quad \text{for all } s \in [0, t].$$

If  $\langle y_x(0), e \rangle = \langle x, e \rangle \geq \widehat{R}$ , we derive that the curve  $y_x$  always lies in  $\mathbb{R}^d \setminus S_{R, \widehat{R}}$  and the assertion trivially follows since  $\ell = \widehat{\ell}$  on  $(\mathbb{R}^d \setminus S_{R, \widehat{R}}) \times A \times B$ . Let us then assume that  $\langle y_x(0), e \rangle < \widehat{R}$  and define two exit times  $t_1$  and  $t_2$  as follows:

$$t_1 := \inf \{s \in [0, t] : \langle y_x(s), e \rangle > R\}, \quad t_2 := \sup \left\{s \in [0, t] : R < \langle y_x(s), e \rangle < \widehat{R}\right\},$$

where we agree that  $t_1 = t_2 = t$  when the sets above are empty. Notice that  $t_2 - t_1 \leq (\widehat{R} - R)/\delta$ . Indeed, if  $t_2 - t_1 > 0$ , then  $t_1 < t$  and, by continuity of  $y_x$  and (4.1), we have  $\langle y_x(t_1), e \rangle = R$ . From (4.1), we infer

$$\widehat{R} - R \geq \langle y_x(t_2) - y_x(t_1), e \rangle = \int_{t_1}^{t_2} \langle f(\alpha[b](s), b(s)), e \rangle ds \geq (t_2 - t_1)\delta,$$

as it was claimed. Also, if  $t_2 < t$ , then, from (4.1), we have that  $\langle y_x(t), e \rangle > \widehat{R}$  for every  $t \in (t_2, t)$ . Consider deterministic running costs  $\ell, \widehat{\ell}: \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$  such that  $\ell = \widehat{\ell}$  on  $(\mathbb{R}^d \setminus S_{R, \widehat{R}}) \times A \times B$ . Then, in view of the previous remarks, we get

$$\begin{aligned} & \left| \int_0^t \left( \ell_\theta(y_x(t), \alpha(t), \beta(t)) - \widehat{\ell}_\theta(y_x(t), \alpha(t), \beta(t)) \right) ds \right| \\ &= \left| \int_{t_1}^{t_2} \left( \ell(y_x(t), \alpha(t), \beta(t)) - \widehat{\ell}(y_x(t), \alpha(t), \beta(t)) \right) ds \right| \\ &\leq (t_2 - t_1) \|\ell - \widehat{\ell}\|_\infty \leq \frac{\widehat{R} - R}{\delta} \|\ell - \widehat{\ell}\|_\infty. \end{aligned}$$

The assertion easily follows from this by arbitrariness of the choice of the control  $b \in \mathbb{B}(t)$  and of the strategy  $\alpha \in \Gamma(t)$ .  $\square$

In the proof of Proposition 4.2, we look at the conditional expectation of  $u_\theta(t, 0, \cdot)$  given the running cost in a neighborhood of zero. Taking a suitable sequence of increasing neighborhoods, we define the corresponding martingale and note that, by Lemma 4.4, it has bounded differences. Then, applying Azuma's inequality, see Lemma 4.1, we get the result.

*Proof of Proposition 4.2.* Fix  $t > 0$ . Recall the long-range independence parameter  $\rho$ . Denote  $n := \lceil t \rceil$  and  $C := \lceil \|f\|_\infty / \rho \rceil$ , where  $\lceil \cdot \rceil$  stands for the upper integer part. Note that, for all  $\alpha \in \Gamma(t)$  and  $b \in \mathbb{B}(t)$ , the solution  $y_0: [0, t] \rightarrow \mathbb{R}^d$  of the ODE

$$\begin{cases} \dot{y}_0(s) = f(\alpha[b](s), b(s)) & \text{in } [0, t] \\ y_0(0) = 0 \end{cases}$$

satisfies that  $|y_0(s)| \leq \rho C n$  for all  $s \in [0, t]$ . For  $r \in \{1, 2, \dots, Cn\}$ , let  $\mathcal{F}_r$  be the  $\sigma$ -algebra generated by the random variables  $\{\ell(y, a, b, \cdot) : a \in A, b \in B, \langle y, e \rangle \leq \rho r\}$ , and let  $\mathcal{F}_0$  be the trivial

$\sigma$ -algebra. We claim that, for all  $0 \leq r < Cn$ , we have that

$$(4.2) \quad |\mathbb{E}[u_\theta(t, 0, \cdot) | \mathcal{F}_{r+1}] - \mathbb{E}[u_\theta(t, 0, \cdot) | \mathcal{F}_r]| \leq \frac{8\rho^2}{\delta} \text{Lip}(\ell).$$

Indeed, recall that

$$S_{r, \widehat{r}} := \{y \in \mathbb{R}^d : \langle y, e \rangle \in [r, \widehat{r}]\}$$

and consider  $\widehat{\ell}$  defined by  $\widehat{\ell}(x, a, b, \omega) = \ell(x - 2\rho e, a, b, \omega)$  for  $(x, a, b, \omega) \in S_{\rho r, \rho(r+2)} \times A \times B \times \Omega$ , and  $\widehat{\ell}(x, a, b, \omega) = \ell(x, a, b, \omega)$  otherwise. Then, by Lemma 4.4, almost surely we have that

$$(4.3) \quad |u_\theta(t, 0, \omega) - \widehat{u}_\theta(t, 0, \omega)| \leq \frac{2\rho}{\delta} \text{Lip}(\ell) 2\rho = \frac{4\rho^2}{\delta} \text{Lip}(\ell).$$

where  $u_\theta$  (resp.  $\widehat{u}_\theta$ ) is the value function associated via (A.17) with the running cost  $\ell_\theta := \ell + \langle f(a, b), \theta \rangle$  (resp.  $\widehat{\ell}_\theta := \widehat{\ell} + \langle f(a, b), \theta \rangle$ ) and initial datum  $g \equiv 0$ . By definition,  $\widehat{u}_\theta$  is measurable with respect to the  $\sigma$ -field generated by  $\{\widehat{\ell}(x, a, b, \cdot) : a \in A, b \in B, x \in \mathbb{R}^d\}$ , which coincides with the  $\sigma$ -field generated by  $\mathcal{G} := \{\ell(x, a, b, \cdot) : a \in A, b \in B, x \in \mathbb{R}^d \setminus S_{\rho r, \rho(r+2)}\}$ . By long-range independence, the  $\sigma$ -field generated by  $\widehat{\mathcal{G}} := \{\ell(x, a, b, \cdot) : a \in A, b \in B, \langle x, e \rangle \geq \rho(r+2)\}$  is independent of the  $\sigma$ -field  $\mathcal{F}_{r+1}$ . Therefore,

$$\mathbb{E}[\widehat{\ell} | \mathcal{F}_{r+1}] = \mathbb{E}[\widehat{\ell} | \mathcal{F}_r],$$

and so we obtain that

$$(4.4) \quad \mathbb{E}[\widehat{u}_\theta(t, 0, \cdot) | \mathcal{F}_{r+1}] = \mathbb{E}[\widehat{u}_\theta(t, 0, \cdot) | \mathcal{F}_r].$$

Finally, using successively (4.3) and (4.4), we prove the claim:

$$\begin{aligned} |\mathbb{E}[u_\theta(t, 0, \cdot) | \mathcal{F}_{r+1}] - \mathbb{E}[u_\theta(t, 0, \cdot) | \mathcal{F}_r]| &\leq |\mathbb{E}[\widehat{u}_\theta(t, 0, \cdot) | \mathcal{F}_{r+1}] - \mathbb{E}[\widehat{u}_\theta(t, 0, \cdot) | \mathcal{F}_r]| + \frac{8\rho^2}{\delta} \text{Lip}(\ell) \\ &= \frac{8\rho^2}{\delta} \text{Lip}(\ell). \end{aligned}$$

For  $r \leq Cn$ , denote  $W_r := \mathbb{E}[u_\theta(t, 0, \cdot) | \mathcal{F}_r]$ . The process  $(W_r)_{r \leq Cn}$  is a martingale, and  $W_0 = \mathbb{E}[u_\theta(t, 0, \cdot)] = U_\theta(t)$ . Moreover, since  $u_\theta(t, 0, \cdot)$  is  $\mathcal{F}_{Cn}$ -measurable, we have that  $W_{Cn}(\cdot) = u_\theta(t, 0, \cdot)$ . By Azuma's inequality, see Lemma 4.1, applied to the martingale  $(W_\tau)_{\tau \leq Cn}$  and (4.2), for all  $M \geq 0$ ,

$$\mathbb{P}(|u_\theta(t, 0, \cdot) - U_\theta(t)| \geq M) = \mathbb{P}(|W_{Cn} - W_0| \geq M) \leq 2 \exp\left(-\frac{M^2}{2Cn \left(\frac{8\rho^2}{\delta} \text{Lip}(\ell)\right)^2}\right).$$

Therefore, there exists a constant  $c > 0$ , only depending on  $\rho, \delta, \text{Lip}(\ell)$ , and  $\|f\|_\infty$  (through  $C$ ), but not on  $\theta$ , such that, for all  $M > 0$  and  $t \geq 1$ , we have that

$$\mathbb{P}(|u_\theta(t, 0, \cdot) - U_\theta(t)| \geq M\sqrt{t}) \leq \exp(-cM^2),$$

as it was asserted.  $\square$

## 5. PROOF OF THE HOMOGENIZATION RESULTS

This section is devoted to the proof of our homogenization results stated in Theorems 2.2 and 2.4. We will denote by  $H$  a stationary Hamiltonian belonging to the class  $\mathcal{H}$ , and by  $u_\theta(\cdot, \cdot, \omega)$  the solution of the equation  $(\text{HJ}_\varepsilon)$  with  $\varepsilon = 1$ ,  $H_\theta := H(\cdot, \theta + \cdot, \omega)$  in place of  $H$ , and initial condition  $u_\theta(0, x, \omega) = 0$  for all  $(x, \omega) \in \mathbb{R}^d \times \Omega$ .

Note that, since the initial condition is zero, the function  $u_\theta$  is stationary, i.e., for all  $(t, x, \omega) \in [0, +\infty) \times \mathbb{R}^d \times \Omega$  and  $y \in \mathbb{R}^d$  we have  $u_\theta(t, x + y, \omega) = u_\theta(t, x, \tau_y \omega)$ ,  $\mathbb{P}$ -almost surely in  $\Omega$ .

We also recall that  $U_\theta(t)$  denotes the expectation of the random variable  $u_\theta(t, 0, \cdot)$ .

**5.1. Proof of Theorem 2.2.** In this subsection, we will furthermore assume that  $H$  belongs to the class  $\mathcal{H}_{dG}$ , so that  $u_\theta(t, x, \omega)$  can be represented via (A.17) with initial datum  $g \equiv 0$  and running cost  $\ell_\theta(x, a, b, \omega) := \ell(x, a, b, \omega) + \langle f(x, a, b), \theta \rangle$ . This allows us to make use of the results obtained in Section 4.

According to Section 3, the proof of Theorem 2.2 boils down to establishing the following result.

**PROPOSITION 5.1.** *For every fixed  $\theta \in \mathbb{R}^d$ , there exists a set  $\Omega_\theta$  of probability 1 such that, for every  $\omega \in \Omega_\theta$ , we have*

$$(5.1) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{y \in B_R} |u_\theta^\varepsilon(1, y, \omega) + \overline{H}(\theta)| = \limsup_{t \rightarrow +\infty} \sup_{y \in B_{tR}} \left| \frac{u_\theta(t, y, \omega)}{t} + \overline{H}(\theta) \right| = 0 \quad \text{for all } R > 0,$$

for some deterministic function  $\overline{H}: \mathbb{R}^d \rightarrow \mathbb{R}$ .

The proof of Proposition 5.1 is based on the following fact.

**PROPOSITION 5.2.** *For all  $\theta \in \mathbb{R}^d$ , we have that  $U_\theta(t)/t$  converges as  $t$  goes to infinity.*

The proof of Proposition 5.2 consists in showing that  $U_\theta(t)$ , satisfies an approximated subadditive inequality. We postpone this proof and proceed to show how to exploit this fact to prove Proposition 5.1. For this, we will also crucially exploit Proposition 4.2.

*Proof of Proposition 5.1.* According to Proposition 5.2, we define  $\overline{H}: \mathbb{R}^d \rightarrow \mathbb{R}$  by setting

$$\overline{H}(\theta) := - \lim_{t \rightarrow +\infty} \frac{U_\theta(t)}{t} = - \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[u_\theta(t, 0, \cdot)]}{t} \quad \text{for all } \theta \in \mathbb{R}^d.$$

Consider  $n \in \mathbb{N}$  and a discretization  $Z_n$  of  $B_{nR}$  consisting of at most  $(2n^3R)^d$  points such that any point of  $B_{nR}$  is  $(\sqrt{d}/n^2)$ -close to  $Z_n$ . We claim that, almost surely,

$$(5.2) \quad \limsup_{n \rightarrow +\infty} \sup_{z \in Z_n} \left| \frac{u_\theta(n, z, \omega)}{n} + \overline{H}(\theta) \right| = 0.$$

Indeed, consider  $M > 0$  arbitrary. Then, for  $n \in \mathbb{N}$ , we focus on the event

$$\sup_{z \in Z_n} \left| \frac{u_\theta(n, z, \omega)}{n} + \overline{H}(\theta) \right| \geq M.$$

Using the concentration given by Proposition 4.2 and the convergence stated in Proposition 5.2, we will prove that the probability of this event is summable. By Borel-Cantelli, we deduce that the limsup event has probability 0 and we conclude by noticing that the value of  $M$  is arbitrary. Using first the union bound, then Proposition 5.2, and last stationarity of  $u_\theta$ , we have that for  $n$  large enough,

$$\begin{aligned} \mathbb{P} \left( \sup_{z \in Z_n} \left| \frac{u_\theta(n, z, \cdot)}{n} + \overline{H}(\theta) \right| \geq M \right) &\leq \sum_{z \in Z_n} \mathbb{P} \left( \left| \frac{u_\theta(n, z, \cdot)}{n} + \overline{H}(\theta) \right| \geq M \right) \\ &\leq \sum_{z \in Z_n} \mathbb{P} \left( \left| \frac{u_\theta(n, z, \cdot)}{n} - \frac{U_\theta(n)}{n} \right| \geq \frac{M}{2} \right) \\ &= |Z_n| \mathbb{P} \left( \left| \frac{u_\theta(n, z, \cdot)}{n} - \frac{U_\theta(n)}{n} \right| \geq \frac{M}{2} \right). \end{aligned}$$

Note that, by the choice of  $Z_n$  and Proposition 4.2,

$$|Z_n| \mathbb{P} \left( \left| \frac{u_\theta(n, z, \cdot)}{n} - \frac{U_\theta(n)}{n} \right| \geq \frac{M}{2} \right) \leq (n^3R)^d \exp \left( -c \frac{M^2}{4} n \right).$$

Therefore, the sequence of events  $\left(\sup_{z \in Z_n} \left| \frac{u_\theta(n, z, \cdot)}{n} + \overline{H}(\theta) \right| \geq M\right)_{n \in \mathbb{N}}$  is summable. By Borel-Cantelli, we obtain that

$$\mathbb{P} \left( \limsup_{n \rightarrow \infty} \sup_{z \in Z_n} \left| \frac{u_\theta(n, z, \cdot)}{n} + \overline{H}(\theta) \right| \geq M \right) = 0,$$

which proves the claim since  $M > 0$  is arbitrary. Let us denote by  $\Omega_\theta$  the set of probability 1 for which (5.2) holds. By applying Theorem A.5 with  $H_\theta$  in place of  $H$  we derive that  $u_\theta(t, \cdot, \omega)$  is  $(\beta t + |\theta|)$ -Lipschitz in  $\mathbb{R}^d$  and  $u_\theta(\cdot, x, \omega)$  is  $\beta(1 + |\theta|)^2$ -Lipschitz in  $[0, +\infty)$ .<sup>1</sup> We infer

$$(5.3) \quad |u_\theta(t, x, \omega)| = |u_\theta(t, x, \omega) - u_\theta(0, x, \omega)| \leq \beta t(1 + |\theta|)^2 \quad \text{for all } (t, x, \omega) \in [0, +\infty) \times \mathbb{R}^d \times \Omega.$$

The latter implies

$$\begin{aligned} \sup_{y \in B_{tR}} \left| \frac{u_\theta(t, y, \omega)}{t} + \overline{H}(\theta) \right| &\leq \sup_{y \in B_{\lceil t \rceil R}} \left| \frac{u_\theta(t, y, \omega)}{t} + \overline{H}(\theta) \right| \\ &\leq \sup_{y \in B_{\lceil t \rceil R}} \left| \frac{u_\theta(\lceil t \rceil, y, \omega)}{t} + \overline{H}(\theta) \right| + \frac{\beta(1 + |\theta|)^2}{t} \\ &\leq \sup_{y \in B_{\lceil t \rceil R}} \left| \frac{u_\theta(\lceil t \rceil, y, \omega)}{\lceil t \rceil} + \overline{H}(\theta) \right| + 2 \frac{\beta(1 + |\theta|)^2}{t} \\ &\leq \sup_{z \in Z_{\lceil t \rceil}} \left| \frac{u_\theta(\lceil t \rceil, z, \omega)}{\lceil t \rceil} + \overline{H}(\theta) \right| + \frac{\sqrt{d}}{\lceil t \rceil} \text{Lip}(h) + 2 \frac{\beta(1 + |\theta|)^2}{t} \end{aligned}$$

We deduce that

$$\limsup_{t \rightarrow +\infty} \sup_{y \in B_{tR}} \left| \frac{u_\theta(t, y, \omega)}{t} + \overline{H}(\theta) \right| = 0. \quad \square$$

Let us turn back to the proof of Proposition 5.2.

*Proof of Proposition 5.2.* Fix  $\theta \in \mathbb{R}^d$  and  $t > 0$ . Denote  $\mathbf{B}(t) := \overline{B}(0, t \|f\|_\infty)$ . Note that, for all  $\alpha \in \Gamma(t)$  and  $b \in \mathbb{B}(t)$ , the solution of the ODE

$$\begin{cases} \dot{y}_0(s) = f(\alpha[b](s), b(s)) & \text{in } [0, t] \\ y_0(0) = 0 \end{cases}$$

is such that, for all  $s \in [0, t]$ , we have that  $y_0(s) \in \mathbf{B}(t)$ . By Proposition A.6 we know that the solution  $u_\theta(t, \cdot, \omega)$  is  $(\beta t + |\theta|)$ -Lipschitz in  $\mathbb{R}^d$ . We will discretize  $\mathbf{B}(t)$  accordingly. Consider a finite set  $Z$  of size  $\lceil 2 \|f\|_\infty (\beta t + |\theta|) t \rceil^{2d}$  such that any point of  $\mathbf{B}(t)$  is  $(\beta t + |\theta|)^{-1}$ -close to a point in  $Z$ . Using the Lipschitz property of  $u_\theta(t, \cdot, \omega)$ , the union bound, the fact that variables  $u_\theta(t, x, \omega)$  and  $u_\theta(t, 0, \omega)$  have the same distribution by stationarity and the concentration proven in Proposition 4.2, we get that, for all  $M > 0$ ,

$$\begin{aligned} \mathbb{P}(\exists x \in \mathbf{B}(t), |u_\theta(t, x, \cdot) - U_\theta(t)| \geq M) &\leq \mathbb{P}(\exists x \in Z, |u_\theta(t, x, \cdot) - U_\theta(t)| \geq M - 1) \\ &\leq \sum_{z \in Z} \mathbb{P}(|u_\theta(t, z, \cdot) - U_\theta(t)| \geq M - 1) \\ &= \sum_{z \in Z} \mathbb{P}(|u_\theta(t, 0, \cdot) - U_\theta(t)| \geq M - 1) \\ &\leq \lceil 2 \|f\|_\infty (\beta t + |\theta|) t \rceil^{2d} \exp(-c(M - 1)^2/t), \end{aligned}$$

<sup>1</sup>Since the Hamiltonian  $H_\theta$  satisfies (H1')-(H3') with  $\beta_1 := \beta(1 + |\theta|)$  and  $\beta_2 = \beta_3 := \beta$ .

where  $c$  is a constant depending on  $\rho, \delta, \text{Lip}(\ell)$ , and  $\|f\|_\infty$ , but not on  $\theta$ . Taking  $M_t := t^{3/4} + 1$  in place of  $M$  in the above inequality, we get that there exists a constant  $K$  only depending on  $\beta$  and  $|\theta|$  such that

$$\mathbb{P}(\exists x \in \mathbf{B}(t), |u_\theta(t, x, \cdot) - U_\theta(t)| \geq M_t) \leq K t^{4d} \exp(-ct^{1/2}).$$

Hence,

$$(5.4) \quad \mathbb{P}\left(\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot) \leq U_\theta(t) - M_t\right) \leq K t^{4d} \exp(-ct^{1/2}).$$

We now recall that  $|u_\theta(t, x, \omega)| \leq \beta t(1 + |\theta|)^2$ , see (5.3), in particular  $|U_\theta(t)| \leq \beta t(1 + |\theta|)^2$ . Then,

$$\begin{aligned} \mathbb{E}\left[\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot)\right] &\geq \mathbb{P}\left(\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot) \leq U_\theta(t) - M_t\right) (-\beta t(1 + |\theta|)^2) \\ &\quad + \mathbb{P}\left(\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot) > U_\theta(t) - M_t\right) (U_\theta(t) - M_t) \\ &= \mathbb{P}\left(\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot) \leq U_\theta(t) - M_t\right) (-\beta t(1 + |\theta|)^2 - U_\theta(t) + M_t) \\ &\quad + U_\theta(t) - M_t \\ &\geq U_\theta(t) - M_t - \mathbb{P}\left(\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot) \leq U_\theta(t) - M_t\right) 2\beta t(1 + |\theta|)^2. \end{aligned}$$

Combining with (5.4), we deduce that there exists a constant  $\widehat{K}$  depending on  $\beta, |\theta|, \rho, \delta, \text{Lip}(\ell)$  and  $\|f\|_\infty$  such that, for all  $t \geq 1$ ,

$$(5.5) \quad \mathbb{E}\left[\inf_{x \in \mathbf{B}(t)} u_\theta(t, x, \cdot)\right] \geq U_\theta(t) - \widehat{K}t^{3/4}.$$

We will now show that the sequence  $(-U_\theta(n))_{n \in \mathbb{N}}$  is almost subadditive and therefore it has a limit. Let  $n \geq 1$  and  $m \leq n$ . Recall that, by Theorems A.8 and A.9, we have the following formulae:

$$(5.6) \quad u_\theta(m, 0, \omega) = \sup_{\alpha \in \Gamma(m)} \inf_{b \in \mathbb{B}(m)} \left\{ \int_0^m \ell_\theta(y_0(s), \alpha[b](s), b(s), \omega) ds \right\},$$

and

$$(5.7) \quad u_\theta(m+n, 0, \omega) = \sup_{\alpha \in \Gamma(m)} \inf_{b \in \mathbb{B}(m)} \left\{ \int_0^m \ell_\theta(y_0(s), \alpha[b](s), b(s), \omega) ds + u_\theta(n, y_0(m), \omega) \right\}.$$

For each fixed strategy  $\alpha$ , pick a control  $b$  that approximates the infimum in (5.7) up to an error  $\delta > 0$ . By definition,

$$u_\theta(m+n, 0, \omega) + \delta \geq u_\theta(m, 0, \omega) + \inf_{x \in \mathbf{B}(m)} u_\theta(n, x, \omega).$$

By taking expectation, by the arbitrariness of  $\delta > 0$  and using the inequality in (5.5), we get

$$(5.8) \quad U_\theta(m+n) \geq U_\theta(m) + U_\theta(n) - \widehat{K}m^{3/4} \quad \text{for all } n \in \mathbb{N}.$$

Set  $a_n := -U_\theta(n)$  for all  $n \in \mathbb{N}$ , and  $z(h) := \widehat{K}h^{3/4}$  for all  $h \geq 1$ . By the previous inequality, the sequence  $(a_n)$  is *subadditive with an error term*  $z$ , i.e.,

$$a_{m+n} \leq a_m + a_n + z(m+n) \quad \text{for all } m, n \in \mathbb{N}.$$

Note that  $z$  is non-negative, non-decreasing and  $\int_1^{+\infty} z(h)/h^2 dh < +\infty$ . Furthermore, by Proposition A.6, the function  $U_\theta$  is Lipschitz in  $[0, +\infty)$ , in particular  $a_n/n$  is bounded since  $U_\theta(0) = 0$ . By [25, Theorem 23, page 162], the sequence  $(a_n/n)_{n \in \mathbb{N}}$  converges, hence  $(U_\theta(n)/n)_{n \in \mathbb{N}}$  converges. Since  $U_\theta$  is Lipschitz, we deduce that  $U_\theta(t)/t$  converges as  $t \rightarrow +\infty$ , as it was to be shown.  $\square$

REMARK 5.3. We remark for further use what we have actually shown with (5.8): there exists a constant  $\widehat{K}$ , only depending on  $\beta, |\theta|, \rho, \delta, \text{Lip}(\ell)$ , and  $\|f\|_\infty$ , such that

$$U_\theta(m+n) \geq U_\theta(m) + U_\theta(n) - \widehat{K}m^{3/4} \quad \text{for all } m, n \in \mathbb{N}.$$

5.2. **Proof of Theorem 2.4.** Let  $G$  be a stationary Hamiltonian satisfying conditions (G1) and (H1)–(H3), for some  $\beta > 0$ . According to [27, Lemma 5.1], for every fixed  $R > 0$  the Hamiltonian  $G$  can be represented as

$$G(x, p, \omega) = \max_{b \in B} \min_{a \in A} \left\{ -\ell(x, a, b, \omega) - \langle \hat{f}(a), p \rangle \right\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}^d \times \overline{B}_R \times \Omega,$$

where  $A := \overline{B}_1$ ,  $B := \overline{B}_R$ ,  $\ell(x, a, b) := -G(x, b, \omega) + \beta \langle a, b \rangle$  and  $\hat{f}(a) := -\beta a$ . We derive that

$$(5.9) \quad H(x, p, \omega) = \max_{b \in B} \min_{a \in A} \left\{ -\ell(x, a, b, \omega) - \langle f(a), p \rangle \right\} \quad \text{for all } (x, p, \omega) \in \mathbb{R}^d \times \overline{B}_R \times \Omega,$$

where  $f(a) := \pi^\top(\hat{f}(a)) + v$  and  $\pi^\top$  denotes the transpose of the linear map  $\pi$ . It is clear that  $f$  satisfies condition (f) with  $e := v/|v|$  and  $\delta := |v|$ , and  $\ell$  satisfies conditions  $(\ell_1)$ – $(\ell_4)$ . Furthermore,  $\|f\|_\infty \leq \beta$  and  $\text{Lip}(\ell) \leq \beta$ , independently on the choice of  $R > 0$  in the definition of the set  $B := \overline{B}_R$ . Now let us fix  $\theta \in \mathbb{R}^d$  and, for every fixed  $s \geq 1$ , define

$$(5.10) \quad H^s(x, p, \omega) := \max_{b \in B_s} \min_{a \in A} \left\{ -\ell(x, a, b, \omega) - \langle f(a), p \rangle \right\}, \quad (x, p, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega,$$

where  $B_s := \overline{B}_{D_s}$  with  $D := \beta + |\theta|$ . Let us denote by  $u_\theta^s(t, x, \omega)$  the solution of equation (HJ $_\varepsilon$ ) with  $\varepsilon = 1$  and  $H^s(\cdot, \theta + \cdot, \cdot)$  in place of  $H$  satisfying the initial condition  $u_\theta^s(0, x, \omega) = 0$  for all  $(x, \omega) \in \mathbb{R}^d \times \Omega$ . In view of Theorem A.5 and the fact that  $H(x, p, \omega) = H^s(x, p, \omega)$  on  $\mathbb{R}^d \times \overline{B}_{D_s} \times \Omega$ , we have that  $u_\theta^s(t, \cdot, \cdot) \equiv u_\theta(t, \cdot, \cdot)$  for all  $0 \leq t \leq s$ . By Proposition 4.2, there exists a constant  $c > 0$ , only depending on  $\beta, \rho$ , and  $\delta = |v|$ , such that, for all  $M > 0$  and  $t \geq 1$ ,

$$\mathbb{P} \left( |u_\theta(t, 0, \cdot) - U_\theta(t)| \geq M\sqrt{t} \right) = \mathbb{P} \left( |u_\theta^t(t, 0, \cdot) - U_\theta^t(t)| \geq M\sqrt{t} \right) \leq \exp(-cM^2).$$

Moreover, as underlined in Remark 5.3, there exists a constant  $\widehat{K}$ , only depending on  $\beta, \rho, \delta = |v|$ , and  $|\theta|$ , such that

$$U_\theta^s(m+n) \geq U_\theta^s(m) + U_\theta^s(n) - \widehat{K}m^{3/4} \quad \text{for all } n \in \mathbb{N}.$$

Applying this to  $s = m+n$  we get

$$U_\theta(m+n) \geq U_\theta(m) + U_\theta(n) - \widehat{K}m^{3/4} \quad \text{for all } n \in \mathbb{N}.$$

By arguing as in the last paragraph of the proof of Proposition 5.2, we conclude that  $U_\theta(t)/t$  converges. The result of Proposition 5.1 holds in the same way. We remark that its proof only relies on the validity of Proposition 4.2 and Proposition 5.2 and on the fact that  $H$  is a stationary Hamiltonian belonging to  $\mathcal{H}$ . This concludes the proof of Theorem 2.4.  $\square$

## APPENDIX A. PDE RESULTS

In this appendix we collect the PDE results used in the paper, which are of deterministic nature and which follow from the ones herein stated and proved by regarding at  $\omega$  as a fixed parameter. We will denote by  $H$  a continuous Hamiltonian defined on  $\mathbb{R}^d \times \mathbb{R}^d$  and satisfying further assumptions that will be specified case by case. Throughout this section, we will denote by  $\text{LSC}(X)$  and  $\text{USC}(X)$  the space of lower semicontinuous and upper semicontinuous real functions on a metric space  $X$ , respectively.

Let  $T > 0$  be fixed and consider the following evolutive HJ equation

$$(HJ) \quad \partial_t u + H(x, Du) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d.$$

We shall say that a function  $v \in \text{USC}((0, T) \times \mathbb{R}^d)$  is an (upper semicontinuous) *viscosity subsolution* of (HJ) if, for every  $\phi \in C^1((0, T) \times \mathbb{R}^d)$  such that  $v - \phi$  attains a local maximum at  $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ , we have

$$(A.1) \quad \partial_t \phi(t_0, x_0) + H(x_0, D_x \phi(t_0, x_0)) \leq 0.$$

Any such test function  $\phi$  will be called *supertangent* to  $v$  at  $(t_0, x_0)$ .

We shall say that  $w \in \text{LSC}((0, T) \times \mathbb{R}^d)$  is a (lower semicontinuous) *viscosity supersolution* of (HJ) if, for every  $\phi \in C^1((0, T) \times \mathbb{R}^d)$  such that  $w - \phi$  attains a local minimum at  $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$ , we have

$$(A.2) \quad \partial_t \phi(t_0, x_0) + H(x_0, D_x \phi(t_0, x_0)) \geq 0.$$

Any such test function  $\phi$  will be called *subtangent* to  $w$  at  $(t_0, x_0)$ . A continuous function on  $(0, T) \times \mathbb{R}^d$  is a *viscosity solution* of (HJ) if it is both a viscosity sub and supersolution. Solutions, subsolutions, and supersolutions will be always intended in the viscosity sense, hence the term *viscosity* will be omitted in the sequel.

**A.1. Comparison principles.** We start by stating and proving a comparison principle, which applies in particular to the case of bounded sub and supersolutions. The conditions assumed on  $H$  are far from being optimal, but general enough for our purposes. The proof is somewhat standard, we provide it below for the reader convenience.

**THEOREM A.1.** *Let  $H$  be a Hamiltonian satisfying the following assumptions, for some continuity modulus  $\omega$ :*

$$(H2') \quad |H(x, p) - H(x, q)| \leq \omega(|p - q|) \quad \text{for all } x, p, q \in \mathbb{R}^d.$$

$$(H4) \quad |H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)) \quad \text{for all } x, y, p \in \mathbb{R}^d,$$

*Let  $v \in \text{USC}([0, T] \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, T] \times \mathbb{R}^d)$  be, respectively, a bounded from above subsolution and a bounded from below supersolution of (HJ). Then, for all  $(t, x) \in (0, T) \times \mathbb{R}^d$ ,*

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)).$$

*Proof.* Since  $v$  is bounded from above, up to adding to  $v$  a suitable constant, we assume without any loss of generality that  $\sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) = 0$ . The assertion is thus reduced to proving that  $v \leq w$  in  $(0, T) \times \mathbb{R}^d$ . We proceed by contradiction. Assume that  $v > w$  at some point of  $(0, T) \times \mathbb{R}^d$ . Up to translations, we can assume without loss of generality that this point has the form  $(\bar{t}, 0)$  for some  $\bar{t} \in (0, T)$ . We will construct a point  $(x, y, p, q)$  where the continuity assumptions (H2')-(H4) do not hold, leading to a contradiction.

For every  $\eta, \varepsilon, b > 0$ , consider the auxiliary function  $\Phi: ([0, T] \times \mathbb{R}^d)^2 \rightarrow \mathbb{R}$  defined by

$$\Phi(t, x, s, y) := v(t, x) - w(s, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{(t - s)^2}{2\varepsilon} - \eta(\phi(x) + \phi(y)) - \frac{b}{T - t} - \frac{b}{T - s},$$

where  $\phi(x) := \sqrt{1 + |x|^2}$ .

Define  $\theta := v(\bar{t}, 0) - w(\bar{t}, 0) > 0$ . Then, since  $\bar{t} \in (0, T)$ , there exists  $\delta > 0$  small enough such that, for all  $\eta, b \in (0, \delta)$ ,

$$\Phi(\bar{t}, 0, \bar{t}, 0) = v(\bar{t}, 0) - w(\bar{t}, 0) - 2\eta\phi(0) - \frac{2b}{T - \bar{t}} > \frac{\theta}{2}.$$

Notice that

$$(A.3) \quad \Phi(t, x, s, y) \leq M - \eta(\phi(x) + \phi(y)) - \frac{b}{T - t} - \frac{b}{T - s} \quad \text{in } ([0, T] \times \mathbb{R}^d)^2$$

with  $M := (\|v^+\|_{L^\infty([0, T] \times \mathbb{R}^d)} + \|w^-\|_{L^\infty([0, T] \times \mathbb{R}^d)})$ , where we have denoted by  $v^+(x) := \max\{v(x), 0\}$  the positive part of  $v$  and by  $w^-(x) = \max\{-w(x), 0\}$  the negative part of  $w$ . We derive that, for



every  $\varepsilon > 0$  and  $\eta \in (0, \delta)$ , there exists  $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in ([0, T] \times \mathbb{R}^d)^2$ , which also depend on  $\eta$ , such that

$$(A.4) \quad \Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \sup_{([0, T] \times \mathbb{R}^d)^2} \Phi \geq \Phi(\bar{t}, 0, \bar{t}, 0) > \frac{\theta}{2}.$$

In view of (A.3) we infer

$$(A.5) \quad \eta(\phi(x_\varepsilon) + \phi(y_\varepsilon)) + \frac{b}{T - t_\varepsilon} + \frac{b}{T - s_\varepsilon} \leq \tilde{M} \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} + \frac{|t_\varepsilon - s_\varepsilon|}{\varepsilon} \leq \sqrt{\frac{2\tilde{M}}{\varepsilon}}$$

with  $\tilde{M} := M - \theta/2$ . From the first inequality in (A.5) we derive that exist constants  $T_b \in (0, T)$ , depending on  $b \in (0, \delta)$ , and  $\rho_\eta > 0$ , depending on  $\eta$ , such that  $t_\varepsilon, s_\varepsilon \in [0, T_b]$  and  $x_\varepsilon, y_\varepsilon \in B_{\rho_\eta}$ . For every fixed  $\eta \in (0, \delta)$ , from [22, Lemma 3.1] we derive that, up to subsequences,

$$(A.6) \quad \lim_{\varepsilon \rightarrow 0} (t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = (t_0, x_0, t_0, x_0) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0$$

for some  $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$  satisfying

$$(A.7) \quad v(t_0, x_0) - w(t_0, x_0) - 2\eta\phi(x_0) - \frac{2b}{T - t_0} = \sup_{(t, x) \in (0, T) \times \mathbb{R}^d} \Phi(t, x, t, x) > \frac{\theta}{2}.$$

In particular, such a point  $(t_0, x_0)$  actually lies in  $(0, T) \times \mathbb{R}^d$ , i.e.,  $t_0 > 0$ , since (A.7) implies  $v(t_0, x_0) - w(t_0, x_0) > 0$ . For every fixed  $\eta \in (0, \delta)$ , choose  $\varepsilon_\eta > 0$  small enough so that  $(t_\varepsilon, x_\varepsilon)$  and  $(s_\varepsilon, y_\varepsilon)$  both belong to  $(0, T) \times \mathbb{R}^d$  when  $\varepsilon \in (0, \varepsilon_\eta)$ . The function

$$\varphi(t, x) := w(s_\varepsilon, y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} + \eta(\phi(x) + \phi(y_\varepsilon)) + \frac{|t - s_\varepsilon|^2}{2\varepsilon} + \frac{b}{T - t}$$

is a supertangent to  $v(t, x)$  at the point  $(t_\varepsilon, x_\varepsilon)$  and  $v$  is a subsolution of (HJ), while the function

$$\psi(s, y) := v(t_\varepsilon, x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} - \eta(\phi(x_\varepsilon) + \phi(y)) - \frac{|t_\varepsilon - s|^2}{2\varepsilon} - \frac{b}{T - s}$$

is a subtangent to  $w(s, y)$  at the point  $(s_\varepsilon, y_\varepsilon)$  and  $w$  is a subsolution of (HJ). We infer

$$(A.8) \quad \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(x_\varepsilon, p_\varepsilon + \eta D\phi(x_\varepsilon)) \leq -c,$$

$$(A.9) \quad \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(y_\varepsilon, p_\varepsilon - \eta D\phi(y_\varepsilon)) \geq c,$$

where we have set

$$c := \frac{b}{T^2} \quad \text{and} \quad p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}.$$

By subtracting (A.8) from (A.9), we get, according to assumptions (H2')-(H4),

$$0 < 2c \leq H(y_\varepsilon, p_\varepsilon - \eta D\phi(y_\varepsilon)) - H(x_\varepsilon, p_\varepsilon + \eta D\phi(x_\varepsilon)) \leq \omega(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|)) + 2\omega(\eta).$$

Sending  $\varepsilon \rightarrow 0^+$ , we have that  $\omega(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|))$  vanishes in view of (A.6). Sending  $\eta \rightarrow 0^+$ , we have that  $\omega(\eta)$  vanishes. But this is a contradiction since  $0 < c$ .  $\square$

Next, we provide a result which is usually used to prove the finite speed of propagation of first order HJ equations, see for instance [18, Lemma 5.3].

**PROPOSITION A.2.** *Let  $H$  be a Hamiltonian satisfying (H4) and condition (H2), for some constant  $\beta > 0$ . Let  $v \in \text{USC}([0, T] \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, T] \times \mathbb{R}^d)$  be, respectively, a sub and a supersolution of (HJ). Then the function  $u := v - w$  is an upper semicontinuous subsolution to*

$$(A.10) \quad \partial_t u - \beta|Du| = 0 \quad \text{in } (0, T) \times \mathbb{R}^d.$$

*Proof.* Let  $\varphi \in C^1((0, T) \times \mathbb{R}^d)$  be supertangent to  $u$  in a point  $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$  and let us assume that  $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$  is a strict local maximum point of  $u - \varphi$ . Let us choose  $r > 0$  such that the open ball  $B := B_r((t_0, x_0))$  of radius  $r > 0$  centred at  $(t_0, x_0)$  is compactly contained in  $(0, T) \times \mathbb{R}^d$  and  $(t_0, x_0)$  is the only maximum point of  $u - \varphi$  in  $\overline{B}$ . Let us introduce the function

$$\Phi(t, x, s, y) := v(t, x) - w(s, y) - \frac{|x - y|^2}{2\varepsilon} - \frac{|t - s|^2}{2\varepsilon} - \varphi(t, x) \quad \text{for } (t, x, s, y) \in \overline{B} \times \overline{B}.$$

By uppersemicontinuity of  $\Phi$  and compactness of the domain, the maximum of  $\Phi$  on  $\overline{B} \times \overline{B}$  is attained at (at least) a point  $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in \overline{B} \times \overline{B}$ . In view of [22, Lemma 3.1], we infer that

$$(A.11) \quad \lim_{\varepsilon \rightarrow 0} (t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = (t_0, x_0, t_0, x_0) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0$$

Choose  $\varepsilon_0 > 0$  small enough so that  $(x_\varepsilon, t_\varepsilon), (y_\varepsilon, s_\varepsilon)$  belong to  $B$  for every  $\varepsilon \in (0, \varepsilon_0)$ . The function

$$\psi_1(t, x) := w(s_\varepsilon, y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} + \frac{|t - s_\varepsilon|^2}{2\varepsilon} + \varphi(t, x)$$

is a supertangent to  $v(t, x)$  at  $(t_\varepsilon, x_\varepsilon)$  and  $v$  is a subsolution to (HJ), hence

$$(A.12) \quad \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(x_\varepsilon, p_\varepsilon + D_x \varphi(t_\varepsilon, x_\varepsilon)) \leq 0,$$

where we have set  $p_\varepsilon := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$ . Analogously,

$$\psi_2(s, y) := v(t_\varepsilon, x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} - \frac{|t_\varepsilon - s|^2}{2\varepsilon} - \varphi(t_\varepsilon, x_\varepsilon)$$

is a subgradient to  $w(s, y)$  at the point  $(s_\varepsilon, y_\varepsilon)$  and  $w$  is a supersolution to (HJ), hence

$$(A.13) \quad \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + H(y_\varepsilon, p_\varepsilon) \geq 0,$$

By subtracting (A.13) from (A.12) and by taking into account conditions (H4)-(??), we get

$$\begin{aligned} 0 &\geq \partial_t \varphi(t_\varepsilon, x_\varepsilon) + H(x_\varepsilon, p_\varepsilon + D_x \varphi(t_\varepsilon, x_\varepsilon)) - H(y_\varepsilon, p_\varepsilon) \\ &\geq \partial_t \varphi(t_\varepsilon, x_\varepsilon) - \beta |D_x \varphi(t_\varepsilon, x_\varepsilon)| + H(x_\varepsilon, p_\varepsilon) - H(y_\varepsilon, p_\varepsilon) \\ &\geq \partial_t \varphi(t_\varepsilon, x_\varepsilon) - \beta |D_x \varphi(t_\varepsilon, x_\varepsilon)| - \omega(|x_\varepsilon - y_\varepsilon|(1 + |p_\varepsilon|)). \end{aligned}$$

Now we send  $\varepsilon \rightarrow 0^+$  to get, in view of (A.11),

$$0 \geq \partial_t \varphi(t_0, x_0) - \beta |D_x \varphi(t_0, x_0)|,$$

as it was to be shown.  $\square$

With the aid of the previous proposition, we can prove the following version of the Comparison Principle for unbounded sub and supersolutions.

**THEOREM A.3.** *Let  $H$  be a Hamiltonian satisfying assumptions (H2) and (H4). Let  $v \in \text{USC}([0, T] \times \mathbb{R}^d)$  and  $w \in \text{LSC}([0, T] \times \mathbb{R}^d)$  be, respectively, a sub and a supersolution of (HJ). Then,*

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) \quad \text{for every } (t, x) \in (0, T) \times \mathbb{R}^d.$$

*Proof.* Up to adding to  $v$  a suitable constant and to the trivial case  $\sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) = +\infty$ , we can assume, without any loss of generality, that  $\sup_{\mathbb{R}^d} (v(0, \cdot) - w(0, \cdot)) = 0$ . In view of Proposition A.2, the function  $u := v - w$  is an upper semicontinuous subsolution of (A.10) satisfying  $u(0, \cdot) \leq 0$  in  $\mathbb{R}^d$ . Let  $\Psi: \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^1$ , strictly increasing and bounded function satisfying  $\Psi(0) = 0$ . It is easily seen, due to the positive 1-homogeneity of equation (A.2), that the function  $(\Psi \circ u)(t, x)$  is a bounded, upper semicontinuous subsolution to (A.2). By applying Theorem A.1 we derive that  $(\Psi \circ u) \leq 0 = \Psi(0)$  in  $[0, T) \times \mathbb{R}^d$ . The assertion follows by the strict monotonicity of  $\Psi$ .  $\square$

**A.2. Existence results and Lipschitz estimates for solutions.** Throughout this subsection, we will assume that the (deterministic) Hamiltonian  $H$  satisfying the following assumptions for some constants  $\beta_1, \beta_2, \beta_3 > 0$ :

$$\begin{aligned} \text{(H1')} \quad & |H(x, p, \omega)| \leq \beta_1 (1 + |p|) && \text{for all } (x, p) \in \mathbb{R}^d \times \mathbb{R}^d; \\ \text{(H2')} \quad & |H(x, p, \omega) - H(x, q, \omega)| \leq \beta_2 |p - q| && \text{for all } x, p, q \in \mathbb{R}^d; \\ \text{(H3')} \quad & |H(x, p, \omega) - H(y, p, \omega)| \leq \beta_3 |x - y| && \text{for all } x, y, p \in \mathbb{R}^d. \end{aligned}$$

We begin with the following existence and uniqueness result, where the uniqueness part is guaranteed by Theorem A.3.

**THEOREM A.4.** *Let  $g \in \text{UC}(\mathbb{R}^d)$ . For every  $T > 0$ , the following problem*

$$\text{(HJP)} \quad \begin{cases} \partial_t u + H(x, Du) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ u(0, \cdot) = g(\cdot) & \text{on } \mathbb{R}^d \end{cases}$$

*has a unique viscosity solution in  $C([0, T] \times \mathbb{R}^d)$ . Furthermore, this solution belongs to  $\text{UC}([0, T] \times \mathbb{R}^d)$ .*

*Proof.* The case  $g \in \text{BUC}(\mathbb{R}^d)$  is proved in [18, Theorem 7.1]. Let us then assume  $g \in \text{UC}(\mathbb{R}^d)$ . Pick  $\psi \in C^{1,1}(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$  such that  $\|\psi - g\|_\infty \leq 1$ . In view of the previous step, the Cauchy problem (HJP) with  $\tilde{H}(x, p) := H(x, D\psi(x) + p)$  and  $g - \psi$  in place of  $H$  and  $g$ , respectively, admits a unique solution  $\tilde{u}(t, x)$  in  $C([0, T] \times \mathbb{R}^d)$ . Furthermore,  $\tilde{u}(t, x)$  in  $\text{UC}([0, T] \times \mathbb{R}^d)$ . We derive that  $u(t, x) := \tilde{u}(t, x) + \psi(x)$  belongs to  $\text{UC}([0, T] \times \mathbb{R}^d)$  and is a solution of the original Cauchy problem (HJP).  $\square$

We proceed to show suitable Lipschitz bounds for the solution of (HJP) when the initial datum is Lipschitz.

**THEOREM A.5.** *Let  $g \in \text{Lip}(\mathbb{R}^d)$  and let  $u$  be the unique continuous function in  $[0, +\infty) \times \mathbb{R}^d$  which solves the Cauchy problem (HJP) for every  $T > 0$ . Then  $u$  is Lipschitz in  $[0, T] \times \mathbb{R}^d$  for every  $T > 0$ . More precisely*

$$\|D_x u\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq (\beta_3 T + \text{Lip}(g)), \quad \|\partial_t u\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq \beta_1 (1 + \|Dg\|_\infty).$$

*Proof.* (i) Let us fix  $h \in \mathbb{R}^d$  and set

$$v_h(t, x) := u(t, x + h) - \beta_3 |h| t, \quad w_h(t, x) := u(t, x + h) + \beta_3 |h| t \quad \text{for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

By exploiting assumption (H2'), it is easily seen that  $v_h$  and  $w_h$  are, respectively, a viscosity sub and supersolution to (HJ). Indeed, the following inequalities hold in the viscosity sense:

$$\begin{aligned} \partial_t v_h + H(x, Dv_h) &= \partial_t u(t, x + h) - \beta_3 |h| + H(x, Du(t, x + h)) \\ &\leq \partial_t u(t, x + h) + H(x + h, Du(t, x + h)) \leq 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \end{aligned}$$

thus showing that  $v_h$  is a subsolution of (HJ). The assertion for  $w_h$  can be proved analogously. By the Comparison Principle, see Theorem A.3, we infer that, for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,

$$\begin{aligned} u(t, x) - v_h(t, x) &\leq u(0, x) - v_h(0, x) = g(x) - g(x + h) \leq \text{Lip}(g)|h| \\ w_h(t, x) - u(t, x) &\leq w_h(0, x) - u(0, x) = g(x + h) - g(x) \leq \text{Lip}(g)|h|, \end{aligned}$$

namely

$$\text{(A.14)} \quad |u(t, x + h) - u(t, x)| \leq (\beta_3 t + \text{Lip}(g)) |h| \quad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

thus showing the first assertion.

(ii) Let us first assume that  $g \in \text{Lip}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ . By assumption (H1') we have

$$\text{(A.15)} \quad |H(x, Dg(x))| \leq \beta_1 (1 + |Dg(x)|) \leq \beta_1 (1 + \|Dg\|_\infty) \quad \text{for all } x \in \mathbb{R}^d.$$

For notational simplicity, let us denote by  $M$  the most right-hand side term in the above inequality. It is easily seen that the functions

$$\underline{u}(t, x) := g(x) - Mt, \quad \bar{u}(t, x) := g(x) + Mt, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^d,$$

are, respectively, a sub and a supersolution of (HJP) for every  $T > 0$ . By the Comparison Principle, see Theorem A.3, we infer that  $\underline{u}(t, x) \leq u(t, x) \leq \bar{u}(t, x)$  for all  $(t, x) \in [0, +\infty) \times \mathbb{R}^d$ , i.e.,

$$\|u(t, \cdot) - g\|_\infty \leq Mt \quad \text{for all } t \geq 0.$$

By applying the Comparison Principle again we get

$$(A.16) \quad \|u(t+h, \cdot) - u(t, \cdot)\|_\infty \leq \|u(h, \cdot) - u(0, \cdot)\|_\infty \leq Mh = \beta_1 (1 + \|Dg\|_\infty) h \quad \text{for all } t, h \geq 0,$$

meaning that  $u$  is  $\beta_1 (1 + \|Dg\|_\infty)$ -Lipschitz in  $t$ .

The case  $g \in \text{Lip}(\mathbb{R}^d)$  can be obtained by approximation. Let us denote by  $g_n$  the mollification of  $g$  via a standard convolution kernel and by  $u_n$  the solution to the Cauchy problem (HJP) with  $g_n$  in place of  $g$ . Since  $\|Dg_n\|_\infty \leq \|Dg\|_\infty$  for every  $n \in \mathbb{N}$ , we deduce from what proved above that the functions  $u_n$  are  $\beta_1 (1 + \|Dg\|_\infty)$ -Lipschitz in  $t$  and  $(\beta_3 t + \text{Lip}(g))$ -Lipschitz in  $x$ , for every  $n \in \mathbb{N}$ . By the Ascoli–Arzelà Theorem and the stability of the notion of viscosity solution, we infer that the functions  $(u_n)_n$  are converging, locally uniformly in  $[0, +\infty) \times \mathbb{R}^d$ , to the solution  $u$  of the Cauchy problem (HJP) with initial datum  $g$ . We derive that  $u$  satisfies (A.16) as well, as it was to be shown.  $\square$

**A.3. Differential Games.** Throughout this subsection, we will work with a deterministic Hamiltonian  $H$  of the form

$$H(x, p) := \max_{b \in B} \min_{a \in A} \{-\ell(x, a, b) - \langle f(a, b), p \rangle\} \quad \text{for all } (x, p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where  $A, B$  are compact subsets of  $\mathbb{R}^m$ , for some integer  $m$ ,  $f: A \times B \rightarrow \mathbb{R}^d$  is continuous vector field, and the running cost  $\ell: \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$  is a bounded and continuous function satisfying assumption  $(\ell_2)$  appearing in Section 2. We shall denote by  $\|\ell\|_\infty$  the  $L^\infty$ -norm of  $\ell$  on  $\mathbb{R}^d \times A \times B$ . The Hamiltonian  $H$  clearly satisfies the properties (H1)–(H3) listed in Section 2.

Next, we provide a formula borrowed from Differential Game Theory to represent such solutions, see [27]. To this aim, we first observe that  $u(t, x)$  is a viscosity solution of the Cauchy problem (HJP) if and only  $\tilde{u}(t, x) := -u(T-t, x)$  is a viscosity solution of equation (HJ) with  $\check{H}(x, p) := H(x, -p)$  in place of  $H$  in the sense adopted in [27], see items (a)-(b) at the end of page 774, and satisfying the terminal condition  $\tilde{u}(T, x) = -g(x)$ , cf. [27, problem (HJ)]. Let us denote by

$$\mathbb{A}(T) := \{a: [0, T] \rightarrow A : a \text{ measurable}\}, \quad \mathbb{B}(T) := \{b: [0, T] \rightarrow B : b \text{ measurable}\}.$$

The sets  $A$  and  $B$  are to be regarded as action sets for Player 1 and 2, respectively. A nonanticipating strategy for Player 1 is a function  $\alpha: \mathbb{B}(T) \rightarrow \mathbb{A}(T)$  such that, for all  $b_1, b_2 \in \mathbb{B}(T)$  and  $\tau \in [0, T]$ ,

$$b_1(\cdot) = b_2(\cdot) \quad \text{in } [0, \tau] \quad \Rightarrow \quad \alpha[b_1](\cdot) = \alpha[b_2](\cdot) \quad \text{in } [0, \tau].$$

We will denote by  $\alpha \in \Gamma(T)$  the family of such nonanticipating strategies for Player 1. For every  $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ , let us set

$$(A.17) \quad v(t, x) := \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} \left\{ \int_0^t \ell(y_x(s), \alpha[b](s), b(s)) ds + g(y_x(t)) \right\},$$

where  $y_x: [0, t] \rightarrow \mathbb{R}^d$  is the solution of the ODE

$$(ODE) \quad \begin{cases} \dot{y}_x(s) = f(\alpha[b](s), b(s)) & \text{in } [0, t] \\ y_x(0) = x. \end{cases}$$

The function  $v$  defined by (A.17) is usually called *value function*.

PROPOSITION A.6.

(i) Let  $g \in W^{1,\infty}(\mathbb{R}^d)$ . Then  $v \in W^{1,\infty}([0, T] \times \mathbb{R}^d)$  for every fixed  $T > 0$ . More precisely, for every  $x, \hat{x} \in \mathbb{R}^d$  and  $t, \hat{t} \in (0, T)$ , we have

$$\begin{aligned} |v(t, x)| &\leq T \|\ell\|_\infty + \|g\|_\infty, \\ |v(t, x) - v(\hat{t}, \hat{x})| &\leq (\beta t + |\theta| + \text{Lip}(g)) |x - \hat{x}| + (\|\ell\|_\infty + \|f\|_\infty \text{Lip}(g)) |t - \hat{t}|; \end{aligned}$$

(ii) Let  $g \in \text{BUC}(\mathbb{R}^d)$ . Then  $v \in \text{BUC}([0, T] \times \mathbb{R}^d)$  for every fixed  $T > 0$ .

*Proof.* Item (i) follows directly from [27, Theorem 3.2] after observing that  $v(t, x) = -V(T - t, x)$ , where  $V$  is the function given by (A.17) with  $-\ell$  and  $-g$  in place of  $\ell$  and  $g$ , respectively. Let us prove (ii). Let  $(g_n)_n$  be a sequence of functions in  $W^{1,\infty}(\mathbb{R}^d)$  such that  $\|g_n - g\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0$  as  $n \rightarrow +\infty$ . For every fixed  $(t, x) \in (0, T) \times \mathbb{R}^d$ , any fixed strategy  $\alpha \in \Gamma(t)$  and every control  $b \in \mathbb{B}(t)$ , let us set

$$J[\alpha, b, g](t, x) := \int_0^t \ell(y_x(s), \alpha[b](s), b(s)) ds + g(y_x(t)),$$

where  $y_x: [0, t] \rightarrow \mathbb{R}^d$  is the solution of (ODE). We have

$$J[\alpha, b, g_n](t, x) - \|g_n - g\|_\infty \leq J[\alpha, b, g](t, x) \leq J[\alpha, b, g_n](t, x) + \|g_n - g\|_\infty.$$

By taking the infimum with respect to  $b \in \mathbb{B}(t)$  and then the supremum with respect to  $\alpha \in \Gamma(t)$ , we infer

$$\begin{aligned} \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} J[\alpha, \hat{b}, g_n](t, x) - \|g_n - g\|_\infty &\leq \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} J[\alpha, \hat{b}, g](t, x) \\ &\leq \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} J[\alpha, b, g_n](t, x) + \|g_n - g\|_\infty. \end{aligned}$$

This means that  $|v_n(t, x) - v(t, x)| \leq \|g_n - g\|_\infty$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , where  $v_n$  and  $v$  denote the value function associated to  $g_n$  and  $g$ , respectively. As a uniform limit of a sequence of equibounded Lipschitz functions, we conclude that  $v$  belongs to  $\text{BUC}([0, T] \times \mathbb{R}^d)$ .  $\square$

Via the same argument used in the proof of Proposition A.6-(i), we derive from [27, Theorem 3.1] the following fact, known as Dynamic Programming Principle.

**THEOREM A.7 (Dynamic Programming Principle).** *Let  $x \in \mathbb{R}^d$  and  $0 < \tau < T$  be fixed. Then,*

$$(A.18) \quad v(T, x) = \sup_{\alpha \in \Gamma(T-\tau)} \inf_{b \in \mathbb{B}(T-\tau)} \left\{ \int_0^{T-\tau} \ell(y_x(s), \alpha[b](s), b(s)) ds + v(\tau, y_x(T-\tau)) \right\}.$$

The following holds.

**THEOREM A.8.** *Let  $g \in \text{BUC}(\mathbb{R}^d)$ . For every fixed  $T > 0$ , the unique continuous solution of (HJP) is given by (A.17).*

*Proof.* When  $g$  is additionally assumed Lipschitz continuous, the assertion follows directly from [27, Theorem 4.1] via the same argument used in the proof of Proposition A.6-(i) and in view of what remarked at the beginning of this subsection. Let us assume  $g \in \text{BUC}(\mathbb{R}^d)$  and pick a sequence of functions  $g_n \in W^{1,\infty}(\mathbb{R}^d)$  such that  $\|g_n - g\|_{L^\infty(\mathbb{R}^d)} \rightarrow 0$  as  $n \rightarrow +\infty$ . Let us denote by  $v$  and  $v_n$  the value functions associated via (A.17) to  $g$  and  $g_n$ , respectively. Arguing as in the proof of Proposition A.6-(ii), we derive  $|v_n(t, x) - v(t, x)| \leq \|g_n - g\|_\infty$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ . From the previous step we know that  $v_n$  solves (HJP) with initial datum  $g_n$ . By the stability of the notion of viscosity solution, we conclude that  $v$  solves (HJP).  $\square$

We now extend the previous result to the case of initial data that are not necessarily bounded. The result is the following.

**THEOREM A.9.** *Let  $g \in \text{UC}(\mathbb{R}^d)$ . For every fixed  $T > 0$ , the unique viscosity solution in  $C([0, T] \times \mathbb{R}^d)$  of the Cauchy problem (HJP) is given by the representation formula (A.17).*

*Proof.* Let us pick  $\psi \in C^{1,1}(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$  such that  $\|\psi - g\|_\infty \leq 1$ . In view of Theorem A.8, the unique solution  $\tilde{u}(t, x)$  in  $C([0, T] \times \mathbb{R}^d)$  of the Cauchy problem (HJP) with  $\tilde{H}(x, p) := H(x, D\psi(x) + p)$  and  $g - \psi$  in place of  $H$  and  $g$ , respectively, is given by the formula (A.17) with  $\ell(y_x(s), \alpha[b](s), b(s)) + \langle f(\alpha[b](s), b(s)), D\psi(y_x(s)) \rangle$  and  $(g - \psi)(y_x(t))$  in place of  $g(y_x(t))$ . For every fixed strategy  $\alpha \in \Gamma(t)$  and every control  $b \in \mathbb{B}(t)$ , we have

$$\begin{aligned} & \int_0^t \left( \ell(y_x(s), \alpha[b](s), b(s)) + \langle f(\alpha[b](s), b(s)), D\psi(y_x(s)) \rangle \right) ds = \\ & \int_0^t \left( \ell(y_x(s), \alpha[b](s), b(s)) + \frac{d}{ds} \psi(y_x(s)) \right) ds = \int_0^t \ell(y_x(s), \alpha[b](s), b(s)) ds + \psi(y_x(t)) - \psi(x). \end{aligned}$$

We infer

$$\begin{aligned} \tilde{u}(t, x) &= \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} \left\{ \int_0^t \left( \ell(y_x(s), \alpha[b](s), b(s)) + \langle f(\alpha[b](s), b(s)), D\psi(y_x(s)) \rangle \right) ds + (g - \psi)(y_x(t)) \right\} \\ &= \sup_{\alpha \in \Gamma(t)} \inf_{b \in \mathbb{B}(t)} \left\{ \int_0^t \ell(y_x(s), \alpha[b](s), b(s)) ds + g(y_x(t)) \right\} - \psi(x). \end{aligned}$$

The conclusion readily follows after observing that the function  $u(t, x) := \tilde{u}(t, x) + \psi(x)$  is the continuous solution of the original Cauchy problem (HJP).  $\square$

## APPENDIX B. PROOF OF THEOREM 3.1

In this section we prove Theorem 3.1. The result is of deterministic nature, with  $\omega$  treated as a fixed parameter. We will therefore omit it from our notation. We will need the following fact.

**PROPOSITION B.1.** *Let us assume that all the hypotheses of Theorem 3.1 are in force. Let  $g \in \text{UC}(\mathbb{R}^d)$  and, for every  $\varepsilon > 0$ , let us denote by  $u^\varepsilon$  the unique function in  $C([0, T] \times \mathbb{R}^d)$  that solves  $(\text{HJ}_\varepsilon)$  subject to the initial condition  $u^\varepsilon(0, \cdot) = g$ . Set*

$$\begin{aligned} u^*(t, x) &:= \lim_{r \rightarrow 0} \sup \{ u^\varepsilon(s, y) : (s, y) \in (t - r, t + r) \times B_r(x), 0 < \varepsilon < r \}, \\ u_*(t, x) &:= \lim_{r \rightarrow 0} \inf \{ u^\varepsilon(s, y) : (s, y) \in (t - r, t + r) \times B_r(x), 0 < \varepsilon < r \}. \end{aligned}$$

*Let us assume that  $u^*$  and  $u_*$  are finite valued. Then*

- (i)  $u^* \in \text{USC}([0, T] \times \mathbb{R}^d)$  and it is a viscosity subsolution of (3.2);
- (ii)  $u_* \in \text{LSC}([0, T] \times \mathbb{R}^d)$  and it is a viscosity supersolution of (3.2).

We postpone the proof of Proposition B.1 and we first show how we employ it to prove Theorem 3.1.

*Proof of Theorem 3.1.* Let us prove that  $\overline{H}$  satisfies condition (H1). To this aim, we remark that the functions  $u^+$ ,  $u^-$  defined as

$$u^\pm(t, x) := \langle \theta, x \rangle \pm \beta(1 + |\theta|)t, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

are, respectively, a continuous super and subsolution of  $(\text{HJ}_\varepsilon)$  satisfying  $u^\pm(0, x) = \langle \theta, x \rangle$ , for every  $\varepsilon > 0$  in view of assumption (H1). By Theorem A.3 we infer  $u^- \leq \tilde{u}_\theta^\varepsilon \leq u^+$  in  $[0, T] \times \mathbb{R}^d$  for every  $\varepsilon > 0$ . Hence

$$|\overline{H}(\theta)| = \lim_{\varepsilon \rightarrow 0^+} |\tilde{u}_\theta^\varepsilon(1, 0)| \leq \beta(1 + |\theta|),$$

as it was to be shown. Let us now show that  $\overline{H}$  satisfies condition (H2). Fix  $\theta_1, \theta_2 \in \mathbb{R}^d$  and set

$$u_{\theta_2}^{\varepsilon, \pm}(t, x) := \tilde{u}_{\theta_2}^{\varepsilon}(t, x) + \langle \theta_1 - \theta_2, x \rangle \pm \beta |\theta_2 - \theta_1| t, \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

The function  $u_{\theta_2}^{\varepsilon, +}$  and  $u_{\theta_2}^{\varepsilon, -}$  are, respectively, a super and a subsolution of  $(\text{HJ}_\varepsilon)$ , in view of assumption (H2), which satisfy  $u_{\theta_2}^{\varepsilon, \pm}(0, x) = \langle \theta_1, x \rangle$ . By Theorem A.3 we derive that  $u_{\theta_2}^{\varepsilon, -} \leq \tilde{u}_{\theta_1}^{\varepsilon} \leq u_{\theta_2}^{\varepsilon, +}$  in  $[0, T] \times \mathbb{R}^d$ , hence

$$|\overline{H}(\theta_1) - \overline{H}(\theta_2)| = \lim_{\varepsilon \rightarrow 0^+} |\tilde{u}_{\theta_1}^{\varepsilon}(1, 0) - \tilde{u}_{\theta_2}^{\varepsilon}(1, 0)| \leq \beta |\theta_2 - \theta_1|.$$

We now proceed to prove the second part of the assertion. Let us first assume  $g \in C^1(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$ . Take a constant  $M$  large enough so that

$$M > \|H(x, Dg(x))\|_\infty.$$

Then the functions  $u_-(t, x) := g(x) - Mt$  and  $u_+(t, x) := g(x) + Mt$  are, respectively, a Lipschitz continuous sub and supersolution of  $(\text{HJ}_\varepsilon)$  for every  $\varepsilon > 0$ . By the Comparison Principle stated in Theorem A.3, we get  $u_- \leq u^\varepsilon \leq u_+$  in  $[0, T] \times \mathbb{R}^d$  for every  $\varepsilon > 0$ . By the definition of relaxed semilimits we infer

$$u_-(t, x) \leq u_*(t, x) \leq u^*(t, x) \leq u_+(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d,$$

in particular,  $u_*, u^*$  satisfy  $u_*(0, \cdot) = u^*(0, \cdot) = g$  on  $\mathbb{R}^d$ . By Proposition B.1, we know that  $u^*$  and  $u_*$  are, respectively, an upper semicontinuous subsolution and a lower semicontinuous supersolution of the effective equation (3.2). We can therefore apply Theorem A.3 again to obtain  $u^* \leq u_*$  on  $[0, T] \times \mathbb{R}^d$ . Since the opposite inequality holds by the definition of upper and lower relaxed semilimits, we conclude that the function

$$\bar{u}(t, x) := u_*(t, x) = u^*(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d$$

is the unique continuous viscosity solution of (3.2) such that  $\bar{u}(0, \cdot) = g$  on  $\mathbb{R}^d$ . Furthermore, by Theorem A.9, we also know that  $\bar{u}$  belongs to  $\text{UC}([0, T] \times \mathbb{R}^d)$ . The fact that the relaxed semilimits coincide implies that  $u^\varepsilon$  converge locally uniformly in  $[0, T] \times \mathbb{R}^d$  to  $\bar{u}$ , see for instance [1, Lemma 6.2, p. 80].

When the initial datum  $g$  is just uniformly continuous on  $[0, T] \times \mathbb{R}^d$ , the result easily follows from the above by approximating  $g$  with a sequence  $(g_n)_n$  of initial data belonging to  $C^1(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$ . Indeed, if we denote by  $u_n^\varepsilon$  and  $u^\varepsilon$  the solution of  $(\text{HJ}_\varepsilon)$  with initial datum  $g_n$  and  $g$ , respectively, we have, in view of Theorem A.3,

$$\|u_n^\varepsilon - u^\varepsilon\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq \|g_n - g\|_{L^\infty(\mathbb{R}^d)} \quad \text{for every } \varepsilon > 0.$$

The assertion follows from this by the stability of the notion of viscosity solution.  $\square$

*Proof of Proposition B.1.* The fact that  $u^*$  and  $u_*$  are upper and lower semicontinuous on  $[0, T] \times \mathbb{R}^d$  is an immediate consequence of their definition. Let us prove (i), i.e., that  $u^*$  is a subsolution of (3.2). The proof of (ii) is analogous.

We make use of Evans's perturbed test function method, see [26]. Let us assume, by contradiction, that  $u^*$  is not a subsolution of (3.2). Then there exists a function  $\phi \in C^1([0, T] \times \mathbb{R}^d)$  that is a strict supertangent of  $u^*$  at some point  $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$  and for which the subsolution test fails, i.e.,

$$(B.1) \quad \partial_t \phi(t_0, x_0) + \overline{H}(D_x \phi(t_0, x_0)) > 2\delta$$

for some  $\delta > 0$ . For  $r > 0$  define  $V_r := (t_0 - r, t_0 + r) \times B_r(x_0)$ . Choose  $r_0 > 0$  to be small enough so that  $V_{r_0}$  is compactly contained in  $[0, T] \times \mathbb{R}^d$  and  $u^* - \phi$  attains a strict local maximum at  $(t_0, x_0)$  in  $V_{r_0}$ . In particular, we have for every  $r \in (0, r_0)$

$$(B.2) \quad \max_{\partial V_r} (u^* - \phi) < \max_{\overline{V_r}} (u^* - \phi) = (u^* - \phi)(t_0, x_0).$$

Let us set  $\theta := D_x\phi(t_0, x_0)$  and for every  $\varepsilon > 0$  denote by  $\tilde{u}_\theta^\varepsilon$  the unique continuous function in  $[0, T] \times \mathbb{R}^d$  that solves  $(\text{HJ}_\varepsilon)$  subject to the initial condition  $\tilde{u}_\theta^\varepsilon(0, x) = \langle \theta, x \rangle$ . We claim that there is an  $r \in (0, r_0)$  such that the function

$$\phi^\varepsilon(t, x) := \phi(t, x) + \tilde{u}_\theta^\varepsilon(t, x) - (\langle \theta, x \rangle - t\overline{H}(\theta))$$

is a supersolution of  $(\text{HJ}_\varepsilon)$  in  $V_r$  for every  $\varepsilon > 0$  small enough. Indeed, by a direct computation we first get

$$(B.3) \quad \partial_t \phi^\varepsilon + H\left(\frac{x}{\varepsilon}, D_x \phi^\varepsilon\right) = \partial_t \phi + \overline{H}(\theta) + \partial_t \tilde{u}_\theta^\varepsilon + H\left(\frac{x}{\varepsilon}, D_x \tilde{u}_\theta^\varepsilon + D_x \phi - \theta\right)$$

in the viscosity sense in  $V_r$ . Using (B.1), the continuity of  $\overline{H}$  and the fact that  $\phi$  is of class  $C^1$ , we get that there is an  $r \in (0, r_0)$  such that for all sufficiently small  $\varepsilon > 0$  and all  $(t, x) \in V_r$

$$\partial_t \phi(t, x) + \overline{H}(\theta) > 2\delta.$$

Moreover, by taking into account (H2), we can further reduce  $r$  if necessary to get

$$H\left(\frac{x}{\varepsilon}, D\tilde{u}_\theta^\varepsilon + D_x \phi - \theta\right) > H\left(\frac{x}{\varepsilon}, D\tilde{u}_\theta^\varepsilon\right) - \delta \quad \text{in } V_r$$

in the viscosity sense. Plugging these relations into (B.3) and using the fact that  $\tilde{u}_\theta^\varepsilon$  is a solution of  $(\text{HJ}_\varepsilon)$ , we finally get

$$\partial_t \phi^\varepsilon + H\left(\frac{x}{\varepsilon}, D_x \phi^\varepsilon\right) > \delta + \partial_t \tilde{u}_\theta^\varepsilon + H\left(\frac{x}{\varepsilon}, D\tilde{u}_\theta^\varepsilon\right) = \delta > 0$$

in the viscosity sense in  $V_r$ , thus showing that  $\phi^\varepsilon$  is a supersolution of  $(\text{HJ}_\varepsilon)$  in  $V_r$ . Now we need a comparison principle for equation  $(\text{HJ}_\varepsilon)$  in  $V_r$  applied to  $\phi^\varepsilon$  and  $u^\varepsilon$  to infer that

$$\sup_{V_r}(u^\varepsilon - \phi^\varepsilon) \leq \max_{\partial V_r}(u^\varepsilon - \phi^\varepsilon).$$

Since condition (H2) is in force, the validity of this comparison principle is guaranteed by [22, Theorem 3.3 and Section 5.C]. Now notice that, by the assumption (3.1),  $\phi^\varepsilon \rightrightarrows \phi$  in  $\overline{V}_r$ . Taking the limsup of the above inequality as  $\varepsilon \rightarrow 0^+$  we obtain

$$\sup_{V_r}(u^* - \phi) \leq \limsup_{\varepsilon \rightarrow 0^+} \sup_{V_r}(u^\varepsilon - \phi^\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0^+} \max_{\partial V_r}(u^\varepsilon - \phi^\varepsilon) \leq \max_{\partial V_r}(u^* - \phi),$$

in contradiction with (B.2). This proves that  $u^*$  is a subsolution of (3.2).  $\square$

## REFERENCES

- [1] Y. ACHDOU, G. BARLES, H. ISHII, AND G. L. LITVINOV, *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, vol. 2074 of Lecture Notes in Mathematics, Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2013. Lecture Notes from the CIME Summer School held in Cetraro, August 29–September 3, 2011, Edited by Paola Loreti and Nicoletta Anna Tchou, Fondazione CIME/CIME Foundation Subseries.
- [2] K. S. ALEXANDER, *A note on some rates of convergence in first-passage percolation*, The Annals of Applied Probability, (1993), pp. 81–90.
- [3] O. ALVAREZ AND M. BARDI, *Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations*, Mem. Amer. Math. Soc., 204 (2010), pp. vi+77.
- [4] M. ARISAWA, *Ergodic problem for the Hamilton-Jacobi-Bellman equation. I. Existence of the ergodic attractor*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 14 (1997), pp. 415–438.
- [5] ———, *Ergodic problem for the Hamilton-Jacobi-Bellman equation. II*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 15 (1998), pp. 1–24.
- [6] M. ARISAWA AND P.-L. LIONS, *On ergodic stochastic control*, Comm. Partial Differential Equations, 23 (1998), pp. 2187–2217.
- [7] S. ARMSTRONG AND P. CARDALIAGUET, *Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions*, J. Eur. Math. Soc. (JEMS), 20 (2018), pp. 797–864.
- [8] S. N. ARMSTRONG, P. CARDALIAGUET, AND P. E. SOUGANIDIS, *Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations*, J. Amer. Math. Soc., 27 (2014), pp. 479–540.



- [9] S. N. ARMSTRONG AND P. E. SOUGANIDIS, *Stochastic homogenization of level-set convex Hamilton-Jacobi equations*, Int. Math. Res. Not. IMRN, 2013 (2013), pp. 3420–3449.
- [10] S. N. ARMSTRONG, H. V. TRAN, AND Y. YU, *Stochastic homogenization of a nonconvex Hamilton-Jacobi equation*, Calc. Var. Partial Differential Equations, 54 (2015), pp. 1507–1524.
- [11] ———, *Stochastic homogenization of nonconvex Hamilton-Jacobi equations in one space dimension*, J. Differential Equations, 261 (2016), pp. 2702–2737.
- [12] K. AZUMA, *Weighted Sums of Certain Dependent Random Variables*, Tohoku Mathematical Journal, 19 (1967).
- [13] M. BARDI, *Asymptotic properties of non-coercive hamiltonians with drift*, arXiv e-print (2024).
- [14] M. BARDI AND I. CAPUZZO-DOLCETTA, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.
- [15] M. BARDI AND G. TERRONE, *On the homogenization of some non-coercive Hamilton-Jacobi-Isaacs equations*, Commun. Pure Appl. Anal., 12 (2013), pp. 207–236.
- [16] G. BARLES, *Solutions de viscosité des équations de Hamilton-Jacobi*, vol. 17 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Paris, 1994.
- [17] G. BARLES, *Some homogenization results for non-coercive Hamilton-Jacobi equations*, Calc. Var. Partial Differential Equations, 30 (2007), pp. 449–466.
- [18] ———, *An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications*, in Hamilton-Jacobi equations: approximations, numerical analysis and applications, vol. 2074 of Lecture Notes in Math., Springer, Heidelberg, 2013, pp. 49–109.
- [19] P. CARDALIAGUET, *Ergodicity of Hamilton-Jacobi equations with a noncoercive nonconvex Hamiltonian in  $\mathbb{R}^2/\mathbb{Z}^2$* , Ann. Inst. H. Poincaré C Anal. Non Linéaire, 27 (2010), pp. 837–856.
- [20] P. CARDALIAGUET, J. NOLEN, AND P. E. SOUGANIDIS, *Homogenization and enhancement for the G-equation*, Arch. Ration. Mech. Anal., 199 (2011), pp. 527–561.
- [21] P. CARDALIAGUET AND P. E. SOUGANIDIS, *Homogenization and enhancement of the G-equation in random environments*, Comm. Pure Appl. Math., 66 (2013), pp. 1582–1628.
- [22] M. G. CRANDALL, H. ISHII, AND P.-L. LIONS, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.
- [23] A. DAVINI AND E. KOSYGINA, *Homogenization of viscous and non-viscous HJ equations: a remark and an application*, Calc. Var. Partial Differential Equations, 56 (2017), pp. Art. 95, 21.
- [24] A. DAVINI AND A. SICONOLFI, *Exact and approximate correctors for stochastic Hamiltonians: the 1-dimensional case*, Math. Ann., 345 (2009), pp. 749–782.
- [25] N. DE BRUIJN AND P. ERDÖS, *Some linear and some quadratic recursion formulas. ii*, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series A: Mathematical Sciences, 14 (1952), pp. 152–163.
- [26] L. C. EVANS, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), pp. 359–375.
- [27] L. C. EVANS AND P. E. SOUGANIDIS, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J., 33 (1984), pp. 773–797.
- [28] W. M. FELDMAN AND P. E. SOUGANIDIS, *Homogenization and non-homogenization of certain non-convex Hamilton-Jacobi equations*, J. Math. Pures Appl. (9), 108 (2017), pp. 751–782.
- [29] H. GAO, *Random homogenization of coercive Hamilton-Jacobi equations in 1d*, Calc. Var. Partial Differential Equations, 55 (2016), pp. Art. 30, 39.
- [30] G. GARNIER AND B. ZILIOOTTO, *Percolation games*, Mathematics of Operations Research, 48 (2023), pp. 2156–2166.
- [31] W. HOEFFDING, *Probability Inequalities for Sums of Bounded Random Variables*, Journal of the American Statistical Association, 58 (1963), pp. 13–30.
- [32] H. ISHII, *Almost periodic homogenization of Hamilton-Jacobi equations*, in International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), World Sci. Publ., River Edge, NJ, 2000, pp. 600–605.
- [33] H. KESTEN, *On the speed of convergence in first-passage percolation*, The Annals of Applied Probability, (1993), pp. 296–338.
- [34] P.-L. LIONS, G. PAPANICOLAOU, AND S. VARADHAN, *Homogenization of Hamilton-Jacobi equation*. unpublished preprint, circa 1987.
- [35] J. NOLEN AND A. NOVIKOV, *Homogenization of the G-equation with incompressible random drift in two dimensions*, Commun. Math. Sci., 9 (2011), pp. 561–582.
- [36] F. REZAKHANLOU AND J. E. TARVER, *Homogenization for stochastic Hamilton-Jacobi equations*, Arch. Ration. Mech. Anal., 151 (2000), pp. 277–309.
- [37] A. SICONOLFI, *Homogenization of the G-equation: a metric approach*, Partial Differ. Equ. Appl., 2 (2021), pp. Paper No. 50, 18.

- [38] P. E. SOUGANIDIS, *Stochastic homogenization of Hamilton-Jacobi equations and some applications*, *Asymptot. Anal.*, 20 (1999), pp. 1–11.
- [39] J. XIN AND Y. YU, *Periodic homogenization of the inviscid  $G$ -equation for incompressible flows*, *Commun. Math. Sci.*, 8 (2010), pp. 1067–1078.
- [40] B. ZILLOTTO, *Stochastic homogenization of nonconvex Hamilton-Jacobi equations: a counterexample*, *Comm. Pure Appl. Math.*, 70 (2017), pp. 1798–1809.