Hyperbolicity study of models for turbulent two-phase flows obtained from the variational principle

Emile Deleage

HYPERBOLICITY STUDY OF MODELS FOR TURBULENT TWO-PHASE FLOWS OBTAINED FROM THE VARIATIONAL PRINCIPLE

ÉMILE DELÉAGE[1,2]

Abstract. We present a class of hyperbolic systems modeling two-phase, two-velocity flows that can be obtained from Hamilton's principle of stationary action, under conventional constraints: conservation of the mass for each component. The governing equations imply the conservation of total momentum and total energy. The hyperbolicity is guaranteed for small relative velocities, i.e. close to equilibrium, by the presence of turbulence through a Reynolds stress tensor. This is the main novelty of the paper. The incorporation of this tensor in the equations is made by the addition of a turbulent term in the potential energy that appears in the Lagrangian of the model. Various forms of the turbulent term are proposed, and the ones leading to a hyperbolic system of equations are characterized by a general criterion. The presence of the Reynolds stress tensor is crucial for the hyperbolicity. Indeed, if it is removed from the Lagrangian, the resulting equations are not hyperbolic for small relative velocities.

1. Introduction and main results

Multi-phase flow modeling aims to describe the simultaneous flow of different materials (solid, liquid, gas). It has a wide range of applications in biology, geosciences and industry, and is an active area of research at the intersection between physics, mechanics and mathematics. In this study, we focus on the case of two phases interacting with each other. We derive new two-phase flow models from Hamilton's principle, and we show that the presence of turbulence guarantees the hyperbolicity of the resulting equations.

1.1. Different multiphase flow models and hyperbolicty criterion.

The multi-phase flow denomination can be used for very diverse phenomena occuring at various scales ([12]). Let us give some examples that can be found in nature. In geophysics, gravity driven flows such that rock, ice or snow avalanches and pyroclastic flows can be seen as multi-phase flows made of solid particles flowing in a surrounding gas ([8],[19]). Lava can also be modeled as a liquid-gas two phase flow (see the recent studies [24], [6]). In biology, the flow of human blood can be modeled as a suspension of red blood cells immersed in a viscous liquid (plasma) ([20], [11]). Multi-phase flow theory is also used at the planetary scale in climate science, to model for instance density stratified oceans, or ocean-atmosphere couplings (see [29]).

In order to model these flows with accuracy, it is necessary to use models that take into account the specifity of each situation. The kind of model used usually depends on key features of the multi-phase flow, such as the nature of each component and the topology of the mixture. The microscopic description tracks the motion and interactions of individual particles. It is well-adapted to the study of granular suspensions when the number of solid particles is small (see for instance [21]). When it is not possible to track each particle, a continuous description has to be used. There are two main kinds of continuous models for two-phase flows. When it is possible to identify distinct regions made of one phase and interacting with each other inside the flow, a separated two-phase system with interfaces can be used. This is the case for instance with the multi-layers models (see for instance

Date: November 15, 2024.

[1]). When the scale of these "one component regions" is much smaller than the typical scale of the motion, or when the two phases are dispersed, it is better to use an averaged description. In this kind of model, the two species are present at each space point, in a proportion given by volumes fractions. The dynamics is described by extended version of Euler or Navier-Stokes equations. Some models then use simplifying hypotheses in order to only keep one averaged velocity for all the phases (as in [8] for example) . In other cases, the full complexity of the dynamics is taken into account and the velocity of each phase is a distinct unknown of the model. For instance, this is the case of the class of models presented in [9] that includes the Baer-Nunziato model.

In the following, we will focus on this last approach, but the three approaches are linked. Indeed, a way to derive a fully dispersed model is to start from a microscopic or a separated one, and to apply an averaging process. Closure relations are then needed in order to express averages of nonlinear and interfacial terms. An introduction to averaging techniques can be found in [12]. For the link between separated and dispersed phase flow models, see also [13]. For more recent work on the topic, see [4] and [5].

We shall now give a brief presentation of an important mathematical feature of reversible multi-phase continuum models, which is the hyperbolicity property. The hyperbolicity property is an algebraic criterion that guarantees interesting behaviours when it is satisfied. From a mathematical point of view, it provides local well-posedness for the model. It means that, starting from a regular enough initial data, there exists a unique solution, which depends continuously on the initial data. From a physical point of view, the propagation of waves occurs at finite speed in hyperbolic systems, which is crucial for predictability and causality. For numerical purposes, hyperbolic systems can be solved by well-known and robust schemes. For an introduction to hyperbolic systems, see for instance the book [2].

A priori, the hyperbolicity property is not satisfied in every multi-phase flow model (see below). In the framework of multi-velocity models, the hyperbolicity is often equivalent to a condition on the relative velocities, i.e. on the difference between the velocity of two different phases. In [27] for instance, it is shown that the hyperbolicity of a large class of barotropic systems is equivalent to the condition $|v_1 - v_k| \neq c_k, k = 2, \dots, N$, where N is the number of different phases, v_i is the velocity of the phase i and c_i is the sound speed of the phase i.

In the following, we will show that the introduction of turbulence in multi-phase flow models enables to guarantee that the hyperbolicity property is satisfied. Furthermore, the resulting models can be obtained from a variational principle.

1.2. **Variational formulation of fluid mechanics.** Let us now describe the method that will be used in order to derive the multi-phase models. It is called Hamilton's stationary action principle. This principle offers a unified way to formulate the governing evolution equations of many physical systems. It enables to obtain the governing evolution equations from a scalar function called the Lagrangian. The Lagrangian L is a function of the trajectory of the particles and is defined as the difference between the kinetic and potential energies of the system. Once the Lagrangian of the system is specified, one can define the associated action, which is the time integral of the Lagrangian. The stationary action principle then states that the particles follow a trajectory for which the action is stationary, eventually with additional constraints on the motion. The equations stating that the action is stationary characterize the trajectory and are called Euler-Lagrange equations.

Let us give some examples in the context of continuous media. In this case, one often defines a Lagrangian density $\mathcal{L}$, which corresponds to a Lagrangian per unit of volume. The Lagrangian L is then given by the integral of the Lagrangian density $\mathcal{L}$ over the whole domain. For a compressible

barotropic fluid of density ρ and of velocity v, the Lagrangian density can be defined as

$$\mathcal{L} := \frac{1}{2}\rho|v|^2 - \rho\mathcal{E}(\rho).$$

The term $\rho|v|^2/2$ is the density of kinetic energy of the fluid, and the term $\rho\mathcal{E}(\rho)$ is called the density of internal energy. It is linked to the pressure of the fluid via the Maxwell relation $p(\rho) := \rho^2\mathcal{E}'(\rho)$. The trajectory, denoted $x(t, X)$, and the velocity are related by the following ordinary differential equation (ODE):

$$\partial_t x(t, X) = v(t, x(t, X)), \quad x(0, X) = X. \tag{1.1}$$

The capital letter X is used for the initial position of the particles, also called the Lagrangian coordinate. The letter x stands for the position of the particle in motion, also called the Eulerian coordinate. In order that $\mathcal{L}$ be a function of the trajectory only, one must specify how the density ρ depends on the trajectory. This is done by imposing the constraint of mass conservation, via the equation

$$\partial_t\rho + \operatorname{div}(\rho v) = 0. \tag{1.2}$$

With this constraint, one can apply the stationary action principle and obtain the corresponding Euler-Lagrange equation, which is an evolution equation for the velocity (see for instance [7] for detailed computations):

$$\partial_t v + (v \cdot \nabla)v + \frac{1}{\rho}\nabla p(\rho) = 0. \tag{1.3}$$

Equations (1.2) and (1.3) form together a well-known system called Euler compressible barotropic equations. This system is hyberbolic if and only if the pressure is an increasing function of the density, which amounts to require that the map $\rho \mapsto \rho\mathcal{E}(\rho)$ is convex. Other properties of this system are the conservation of momentum:

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + pI_d)^T = 0,$$

and of total energy:

$$\partial_t e + \operatorname{div}[(e + p)v] = 0, \quad \text{with} \quad e := \frac{1}{2}\rho|v|^2 + \rho\mathcal{E}(\rho)$$

for smooth solutions. Hamilton's stationary action principle provides an additional interpretation of these conservation laws. Indeed, Noether's Theorem (see [25]) states that conserved quantities correspond to symmetries of the Lagrangian. Here, the conservation of momentum comes from the spatial translation invariance of $\mathcal{L}$, and the conservation of energy is related to the invariance by time translation.

We now move on to the case of a two phase flow. The two phases will be denoted by the indices 1 and 2. The density, velocity and internal energy are respectively denoted by ρ_i, v_i and $\mathcal{E}_i$, $i = 1, 2$. Since the two velocities are different, it is also the case for the trajectories of the particles of the phase 1 and 2, denoted by x_i and defined as in (1.1) by

$$\partial_t x_i(t, X_i) = v_i(t, x_i(t, X_i)), \quad x_i(0, X_i) = X_i, \quad i = 1, 2.$$

The volume fraction of the phase i is denoted by α_i. The two volume fractions are linked by the constraint

$$\alpha_1 + \alpha_2 = 1, \tag{1.4}$$

which means that there is no vacuum. It is important to distinguish the true density ρ_i of each phase from the material density $\rho_{i,0} := \rho_i/\alpha_i$. The constraint of mass conservation only holds for the true densities:

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \quad i = 1, 2. \tag{1.5}$$

A possible choice to define the Lagrangian density of the mixture is to say that the kinetic (resp. internal) energy of the mixture is the sum of the kinetic (resp. internal) energy of each phase, weighted by the volume fractions:

$$\mathcal{L} = \frac{1}{2}\rho_1 |v_1|^2 + \frac{1}{2}\rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2 \mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right). \tag{1.6}$$

Physically, this means that there is no energy associated to the interactions between the two phases. Hamilton's principle under the constraints given by Equation (1.4) and Equations (1.5) yields a system of evolution equations for the variables $(\rho_1, v_1, \rho_2, v_2)$, for which the two pressures are equal $(p_1 = p_2)$ and the volume fractions are given functions of the densities: $\alpha_i = \alpha_i(\rho_1, \rho_2)$. This system is hyperbolic if and only if the relative velocity is greater than an explicit lower bound: the system is hyperbolic when

$$|v_2 - v_1|^2 > \rho c_w^2 \left[\left(\frac{\alpha_1}{\rho_1}\right)^{1/3} + \left(\frac{\alpha_2}{\rho_2}\right)^{1/3}\right]^3,$$

where c_w is the speed of sound of the mixture and $\rho := \rho_1 + \rho_2$ is the total density (see [14]). This lower bound for the relative velocity can be problematic in situation where the two velocities are close, e.g. near equilibrium. Indeed, when the criterion is not fulfilled, complex characteristics appear.

In [15], Hamilton's principle was again used together with the same Lagrangian (1.6) and the same constraints, and a different system was obtained, including an additional "lift" term. This counter-intuitive fact can be explain by the choice of the trajectories made by the author. Indeed, instead of defining the Lagrangian as a function of the trajectories of each phase x_1 and x_2, the mean velocity of the fluid $v := (\rho_1 v_1 + \rho_2 v_2)/\rho$ (with $\rho = \rho_1 + \rho_2$ the total density) was used to define a third trajectory $x(t, X)$, given by the ODE $\partial_t x(t, X) = v(t, x(t, X))$. The Lagrangian was then considered as a function of the trajectories $x(t, X)$ and $x_2(t, X_2)$. As a consequence, the evolution equation obtained for the velocity v_i differs by an additional lift term equal to $\operatorname{curl}(v_1) \wedge (v - v_i)$ $(i = 1, 2)$. The apparition of these lift terms is thus explained by the change of reference frame. Even if the obtained system is different, the hyperbolicity criterion remains unchanged for this modified model. In particular, the system is not hyperbolic when the norm of the relative velocity is small ([15],[14]).

One possible interpretation of this lack of hyperbolicity is that some interaction terms are missing and should be added in the Lagrangian. In [17], an extension of the kinetic energy by addition of a term of the form $(d\alpha_1/dt)^2$ modeling inertia enabled to obtain an unconditional hyperbolicity for a model of bubbly fluid (liquid-gas interaction). Allowing the Lagrangian to depend on derivatives of the unknowns is frequent in multi-fluid modeling. This is done for instance in Korteweg-type models, used in the context of liquid vapor mixture with a diffuse interface, and for which the Lagrangian depends on the gradient of the density (see e.g. [18]).

1.3. **Models studied and main results.** In this work, we will show that two phase flow models that are hyberbolic for small relative velocity can be obtained from Hamilton's stationary action principle by adding an additional term to the Lagrangian, corresponding to a turbulent interaction.

More precisely, we will consider the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{r}{2}\mathrm{Tr}(P),$$

where $r \in \{\rho_1, \rho_2, \rho\}$ is either the density of one of the two phases, or the total density, depending on the choice of the model, and P is the Reynolds stress tensor. P is a second order symmetric definite positive tensor whose evolution equation is prescribed as an additional constraint to the model and is given by

$$\partial_t P + (u\cdot\nabla)P + \frac{\partial u}{\partial x}P + P\left(\frac{\partial u}{\partial x}\right)^T = 0, \tag{1.7}$$

where the velocity u drives the evolution of the tensor P and is given by $u \in \{v_1, v_2, v\}$ depending on the choice of the model.

The Reynolds stress tensor appears in the context of turbulent flows when a Reynolds averaging is performed. The tensor P is then obtained from the second order correlations of the velocity (see [26]). Equation (1.7) is classical and appears whenever third order correlations are neglected ([23], [30]). It was shown in [28] that the same equation structure appears in the context of sheared shallow-water flows. In this context, the averaging operator is the depth-averaging. In the compressible barotropic case, Reynolds averaged equations form a system of hyperbolic equations (see [3]). In the previous study of the author [10], it was proved that this system of equations is symmetric hyperbolic if and only if the pressure is constant or the tensor P is scalar. The proof that Reynolds averaged equations can be obtained from Hamilton's stationary principle was presented in [16], in the case of a one phase flow.

Here, we aim to extend the Reynolds averaged equations to the case of a two phase flow. For two phases, there are various possibilities regarding the tensor P. Indeed, the turbulence can be carried either by one of the two phases, by the two phases, or by the mixture itself. We describe these different cases below.

Turbulence carried by the mixture velocity. Our first result concerns the case of a turbulence carried by the mean velocity. A possible interpretation is that the turbulence is caused by the mixture of the two phases. We thus first consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{\rho}{2}\mathrm{Tr}(P), \tag{1.8}$$

with convex internal energies, and assume that the evolution equation of P is given by

$$\partial_t P + (v\cdot\nabla)P + \frac{\partial v}{\partial x}P + P\left(\frac{\partial v}{\partial x}\right)^T = 0, \tag{1.9}$$

with v the mean velocity, associated to the conservation of total density:

$$\partial_t\rho + \mathrm{div}(\rho v) = 0$$

(this last equation being obtained by summing the conservation equation on ρ_1 and ρ_2). Assuming Equations (1.5) and (1.9) as constraints, the full system obtained by Hamilton's principle is given by (see Section 2 for the derivation):

$$\begin{cases} \partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, & i = 1,2, \quad (1.10a)\\ \partial_t v_i + (v_i\cdot\nabla v_i) + \mathrm{curl}(v_2)\wedge(v - v_i) + \nabla(U_i) + \dfrac{1}{\rho}\mathrm{div}(\rho P)^T = 0, & i = 1,2, \quad (1.10b)\\ \partial_t P + (v\cdot\nabla)P + \dfrac{\partial v}{\partial x}P + P\left(\dfrac{\partial v}{\partial x}\right)^T = 0. & \quad (1.10c) \end{cases}$$

The pressure of the system is defined by

$$p = p_1 = p_2, \qquad \text{with} \qquad p_i := \left(\frac{\rho_i}{\alpha_i}\right)^2 \mathcal{E}_i'\left(\frac{\rho_i}{\alpha_i}\right), \qquad i = 1, 2.$$

The relation $p_1 = p_2$ is the Euler-Lagrange equation associated to the variable α_1. It enables to express the volume fractions α_1 and $\alpha_2 = 1 - \alpha_1$ as functions of the densities ρ_1, ρ_2 only via the implicit functions theorem. The functions U_i that appear in (1.10b) can then be expressed as

$$U_i(\rho_1, \rho_2) := \left(\frac{\partial}{\partial \rho_i}\left[\rho_i \mathcal{E}_i\left(\frac{\rho_i}{\alpha_i}\right)\right]\right)\Bigg|_{\alpha_i = \alpha_i(\rho_1, \rho_2)} \qquad i = 1, 2. \tag{1.11}$$

As a consequence of the convexity of the internal energies $\rho_i \mathcal{E}_i\left(\frac{\rho_i}{\alpha_i}\right)$, the partial derivatives of the potentials U_i satisfy

$$a_1 := \frac{\partial U_1}{\partial \rho_1} > 0, \qquad b := \frac{\partial U_1}{\partial \rho_2} = \frac{\partial U_2}{\partial \rho_1} > 0, \qquad a_2 := \frac{\partial U_2}{\partial \rho_2} > 0, \tag{1.12}$$

as well as the additional relation

$$a_1 a_2 - b^2 = 0. \tag{1.13}$$

System (1.10) admits two additional conservation laws. For smooth solutions, the total momentum ρv is conserved:

$$\partial_t(\rho v) + \operatorname{div}(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + pId + \rho P)^T = 0, \tag{1.14}$$

as well as the total energy

$$\partial_t(e_1 + e_2) + \operatorname{div}(e_1 v_1 + e_2 v_2 + (\alpha_1 v_1 + \alpha_2 v_2)p + \rho P v)^T = 0. \tag{1.15}$$

The energy of each phase is given by

$$e_i := \frac{1}{2}\rho_i |v_i|^2 + \rho_i \mathcal{E}_i\left(\frac{\rho_i}{\alpha_i}\right) + \frac{1}{2}\rho_i \operatorname{Tr} P, \qquad i = 1, 2,$$

Once again, the variational formulation of System (1.10) enables to relate the conservation laws with symmetries of the Lagrangian density (1.8). Indeed, Noether's theorem states that the conservation of momentum (1.14) is associated to the invariance of (1.8) by spatial translations, while the conservation of energy (1.15) is associated to the invariance by time translation.

Note the presence of the lift term $\operatorname{curl}(v_2) \wedge (v - v_i)$ in the evolution equation of v_i. As explained before, this term appear because the Lagrangian was considered as a function of the two trajectories $x_1(t, X_1)$ associated to v_1 and $x(t, X)$ associated to the mean velocity v. The choice of the trajectory $x(t, X)$ is unevitable in the derivation of the equations in order to be able to express the constraint (1.9) in Lagrangian coordinates (see the detailed derivation in Section 2). Since the lift terms depend on the choice of the trajectories and are uncommon in the litterature, we also study a simpler version of System (1.10), where the lift terms are removed:

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, & i = 1, 2, \qquad (1.16a) \\ \partial_t v_i + (v_i \cdot \nabla v_i) + \nabla(U_i) + \dfrac{1}{\rho}\operatorname{div}(\rho P)^T = 0, & i = 1, 2, \qquad (1.16b) \\ \partial_t P + (v \cdot \nabla)P + \dfrac{\partial v}{\partial x} P + P\left(\dfrac{\partial v}{\partial x}\right)^T = 0. & \qquad (1.16c) \end{cases}$$

The hyperbolicity of Systems (1.10) and (1.16) is characterized by the following theorem:

Theorem 1. *(Hyperbolicity of two-phase flow with a turbulence carried by the mean velocity)*
Let us denote $\rho_{i,0} := \rho_i/\alpha_i$ the material density of each fluid, for $i = 1, 2$. Let also

$$\theta := \rho_1\rho_2\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 - \frac{1}{2}\left(\rho_1\sqrt{a_1} + \rho_2\sqrt{a_2}\right)^2, \tag{1.17}$$

a_1, a_2 being defined in (1.12)*. Then it holds that:*

- *In dimension $d = 1$, the lift terms of Equation* (1.10b) *vanish and System* (1.10) *is hyperbolic whenever $\rho_{1,0} \neq \rho_{2,0}$, for small relative velocity. In other words, there exists a constant $C = C(\rho_1, \rho_2, P) > 0$ such that* (1.10) *is hyperbolic whenever $|v_1 - v_2| < C$.*
- *When $d = 2$, suppose again that $\rho_{1,0} \neq \rho_{2,0}$. Also suppose that*

$$|v_1 - v_2| < C, \quad \text{and either} \quad \rho P > \theta I_d \quad \text{or} \quad \rho P < \theta I_d. \tag{1.18}$$

  Then both System (1.10) *and System* (1.16) *are hyperbolic.*
- *When $d = 3$, system* (1.16) *is hyperbolic under the assumptions $\rho_{1,0} \neq \rho_{2,0}$ and* (1.18)*. Under these two assumptions, System* (1.10) *is only weakly hyperbolic.*

In dimension 2 and 3, if the condition $\rho P > \theta Id$ or $\rho P < \theta Id$ is not satisfied, System (1.10) *and* (1.16) *are only weakly hyperbolic.*

The fact that the hyperbolicity of the system is improved after removing the lift terms is quite surprising and is a consequence of the presence of the Reynolds stress tensor. Indeed, as already explained in the end of paragraph 1.2, the hyperbolicity criterion is not affected by the addition or removal of lift terms for a two-phase flow without Reynolds stress tensor. See also Remark 2 below.

Turbulence carried by one of the two phases. We now suppose that the turbulence is only carried by one phase, say the phase 2. We thus consider the following Lagrangian :

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho_2\mathrm{Tr}(P_2). \tag{1.19}$$

We use the superscript 2 for the tensor P_2 to insist on the fact that it is carried by the phase two. As a consequence, we assume that the evolution of P_2 is given by

$$\partial_t P_2 + (v_2 \cdot \nabla)P_2 + \frac{\partial v_2}{\partial x}P_2 + P_2\left(\frac{\partial v_2}{\partial x}\right)^T = 0. \tag{1.20}$$

We also impose the conservation of the two densities:

$$\partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1, 2. \tag{1.21}$$

In order to lighten the presentation, the system obtained is given by Equation (4.4) of Section 4. The result concerning this system is the following:

Theorem 2. *(Hyperbolicity of two-phase flow with a turbulence carried by one phase only)*
The hyperbolicity of the system obtained by Hamilton's principle from the Lagrangian (1.19) *and the constraints* (1.20) *and* (1.21) *can be caracterized as follows:*

- *When $d = 1$, System* (4.4) *is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that* (4.4) *is hyperbolic when $|v_1 - v_2| \leq C$.*
- *When $d = 2, 3$ let*

$$\mu := \rho_1 a_1 - \frac{1}{2}\rho_2 a_2$$

  and assume that either $P_2 > \mu I_d$ or $P_2 < \mu I_d$. Then (4.4) *is hyperbolic for $|v_1 - v_2| \leq C$. If the condition $P_2 > \mu I_d$ or $P_2 < \mu I_d$ is not satisfied, System* (4.4) *is only weakly hyperbolic.*

Turbulence carried by each of the two phases. We also study the case of the mixture of two turbulent phases. In other words, each phase is endowed with a Reynolds stress tensor denoted P_i, $i = 1, 2$. The Lagrangian density is the following:

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho_1\mathrm{Tr}(P_1) - \frac{1}{2}\rho_2\mathrm{Tr}(P_2). \tag{1.22}$$

The densities are still conserved:

$$\partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1, 2.$$

The evolution equations for P_1, P_2 are given by

$$\partial_t P_i + (v_i \cdot \nabla)P_i + \frac{\partial v_i}{\partial x}P_i + P_i\left(\frac{\partial v_i}{\partial x}\right)^T = 0, \qquad i = 1, 2. \tag{1.23}$$

In order to lighten the presentation, the system obtained is given by Equation (4.6) of Section 4. The result concerning this system is the following:

Theorem 3. *(Hyperbolicity of the mixture of two turbulent phases)*
The hyperbolicity of the system obtained by Hamilton's principle from the Lagrangian (1.22) *under the constraints of mass conservation and* (1.23) *can be characterized as follows:*

- *When $d = 1$, System* (4.6) *is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that* (4.6) *is hyperbolic when $|v_1 - v_2| \leq C$.*
- *When $d = 2, 3$ assume that*

$$\frac{1}{3}P_1 \leq P_2 \leq 3P_1, \quad \text{or equivalently} \quad \frac{1}{3}P_2 \leq P_1 \leq 3P_2. \tag{1.24}$$

Then (4.6) *is hyperbolic whenever $|v_1 - v_2| \leq C$. If* (1.24) *is not satisfied,* (4.6) *is only weakly hyperbolic.*

General criterion of hyperbolicity for turbulent two-phase flows. We finally come back to the general Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{r}{2}\mathrm{Tr}(P),$$

The three previous results show that the turbulence $r\mathrm{Tr}P/2$ can be added in various ways to the potential energy of a two-phase flow in order to obtain a hyperbolic system of equations. The next and last result states that the three examples given previously are the only ones for which the resulting system of equations is hyperbolic.

Theorem 4. *(General criterion of hyperbolicity for turbulent two-phase flows)*
Consider the following general Lagrangian density:

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}r\mathrm{Tr}(P), \tag{1.25}$$

endowed with the constraints

$$\partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1, 2 \tag{1.26}$$

and

$$\partial_t P + (u \cdot \nabla)P + \frac{\partial u}{\partial x}P + P\left(\frac{\partial u}{\partial x}\right)^T = 0, \tag{1.27}$$

where

$$(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}.$$

Let us consider the system formed by Equations (1.26), (1.27) *and the Euler-Lagrange equations associated to the Lagrangian density* (1.25). *We denote by M the matrix of this system. Then, in dimension 1, all the eigenvalues of M are real when $v_1 = v_2$, whatever the choice of convex internal energies $\rho_i \mathcal{E}_i(\rho)$, $i = 1, 2$, if and only if the density r and the velocity u are compatible, i.e. the following equation is satisfied:*

$$\partial_t r + \operatorname{div}(ru) = 0.$$

Let us give two remarks to complete the results obtained.

Remark 1. When one of the two phases, say the phase 1, is made of solid particles, one often assumes that the material density of the phase 1, given by

$$\rho_{1,0} := \frac{\rho_1}{\alpha_1}$$

is a constant. This is the case for instance for a dry granular flow or for a granular suspension. Note that it is then straightforward to express the volume fractions as functions of the density ρ_1 :

$$\alpha_1 = \frac{\rho_1}{\rho_{1,0}}, \qquad \alpha_2 = 1 - \frac{\rho_1}{\rho_{1,0}}.$$

All the systems and hyperbolicity results obtained are still valid under the following modifications:

- There is no internal energy associated to the phase 1: $\mathcal{E}_1 = 0$.
- The pressure is defined by

$$p = p_1 = p_2 = \left(\frac{\rho_2}{\alpha_2}\right)^2 \mathcal{E}_2'\left(\frac{\rho_2}{\alpha_2}\right),$$

- The chemical potentials U_i are defined by

$$U_i(\rho_1, \rho_2) := \frac{\partial}{\partial \rho_i}\left[\rho_2 \mathcal{E}_2\left(\frac{\rho_2}{1 - \rho_1/\rho_{1,0}}\right)\right], \qquad i = 1, 2.$$

Remark 2. In dimension 2 or 3, the results obtained state that the hyperbolicity is valid under additional conditions on the spectrum of the Reynolds tensor (see (1.18) for instance). These additional conditions are necessary to prevent some eigenvalues of the systems to appear with a multiplicity greater than 1. They are called non resonance conditions. If they are not satisfied, the systems obtained are only weakly hyperbolic. The solution of weakly hyperbolic systems are known to be less regular than the ones of hyperbolic systems. In particular, phenomena like creation of vacuum or high concentration (delta shocks) can occur (see [22] for instance).

Recall that the Reynolds stress tensor is a symmetric, definite positive tensor. As a consequence, in some cases, these non resonance conditions are always satisfied. For instance, the condition $\rho P > \theta I_d$ or $\rho P < \theta I_d$ of Theorem 1 is automatically satisfied when $\theta < 0$, which is the case whenever a_1 is close to a_2. Similarly, the non resonance condition of Theorem 2 is automatically satisfied whenever $\mu < 0$.

Outline of the paper. The rest of this document is organised as follows. In Section 2, we give the derivation of the two-phase flow system with a mixture turbulence, System (1.10), from Hamilton's principle. In Section 3, we give a detailed study of the hyperbolicities of this system and its simplified version, System (1.16), in order to prove Theorem 1. In Section 4, we show the hyperbolicity of the two other models presented in the introduction (Theorems 2 and 3) and we establish the general hyperbolicity criterion stated in Theorem 4.

2. Derivation of the model from variational principle

Notations. The space dimension is denoted d and can be equal to 1, 2 or 3. We denote $\partial_i := \partial/\partial x_i$ the partial derivative of the variable x_i, for $1 \leq i \leq d$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a scalar function, we denote by $\nabla f \in \mathbb{R}^d$ the gradient of f, i.e. the vector field of components $\partial_i f$, $1 \leq i \leq d$.

If $Z = (Z_1, \dots, Z_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, the divergence of Z is the scalar function defined by

$$\mathrm{div}(Z) := \partial_1 Z_1 + \cdots + \partial_d Z_d.$$

We also denote by $\partial Z/\partial x$ the Jacobian matrix of Z, i.e. the matrix of coefficients $(\partial Z/\partial x)_{i,j} = \partial Z_i/\partial x_j$, for $1 \leq i, j \leq d$.

If $Z = (Z_i)_{1\leq i\leq d}$ and $Z' = (Z'_i)_{1\leq i\leq d}$ are two vector fields, we denote $Z \otimes Z'$ the second order tensor defined by $Z \otimes Z' := Z(Z')^T$, i.e. the matrix of coefficients $(Z \otimes Z')_{i,j} = Z_i Z'_j$, for $1 \leq i, j \leq d$.

If $A : \mathbb{R}^d \to M_d(\mathbb{R})$ is a second order tensor, we define the divergence of A as the line vector of $\mathbb{R}^d$ whose i-th component is given by the divergence of the i-th column of A.

We denote I_d the identity matrix. For any positive integer d, we denote $S_d^{++}(\mathbb{R})$ the set of symmetric definite positive matrices, i.e. the symmetric matrices of size d with a positive spectrum.

Derivation of System (1.10) **from variational principle.** We now give the derivation of System (1.10) from Hamilton's stationary action principle. Various arguments that are used in this proof can be found in other works. We refer to [16] for the derivation of Reynolds averaged equations via variational principle. Detailed computations of Euler-Lagrange equations in different reference frames were performed in [15]. Finally, we refer to [7] for an introduction to the application of Hamilton's principle to fluid mechanics.

We consider a two-phase flow with a Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{\rho}{2}\mathrm{Tr}(P), \tag{2.1}$$

where ρ_i (resp. v_i) is the density (resp. velocity) of the phase i, $i = 1, 2$. The two density are conserved :

$$\partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, \quad i = 1, 2, \tag{2.2}$$

which implies that the total density $\rho := \rho_1 + \rho_2$ is also conserved:

$$\partial_t\rho + \mathrm{div}(\rho v) = 0, \quad \text{with} \quad \rho v := \rho_1 v_1 + \rho_2 v_2.$$

α_1 and α_2 are the volume fractions of the phases 1 and 2 and verify the constraint

$$\alpha_1 + \alpha_2 = 1,$$

hence $\alpha_2 = 1 - \alpha_1$. The evolution of the Reynolds stress tensor is given by

$$\partial_t P + (v \cdot \nabla)P + \frac{\partial v}{\partial x}P + P\left(\frac{\partial v}{\partial x}\right)^T = 0. \tag{2.3}$$

We aim to show that Hamilton's principle applied to the Lagrangian (2.1) endowed with the two constraints (2.2) and (2.3) yields Equation (1.10b) as the Euler-Lagrange equation associated to the velocity v_i:

$$\partial_t v_i + (v_i \cdot \nabla v_i) + \mathrm{curl}(v_2) \wedge (v - v_i) + \nabla(U_i) + \frac{1}{\rho}\mathrm{div}(\rho P)^T = 0, \qquad i = 1, 2.$$

2.1. **Trajectories and Lagrangian quantities.** Let us suppose that the two-phase flow lies in a fixed open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. We denote by $X \in \Omega$ (or $X_i, i = 1, 2$) the initial position of an element of fluid at time $t = 0$. X is usually called the Lagrangian coordinate. The trajectory of an element of the phase 1, flowing at velocity v_1, that started at the point X_1 at time $t = 0$, is denoted by $x_1(t, X_1) \in \Omega$. It is defined by

$$\partial_t x_1(t, X_1) = v_1(t, x_1(t, X_1)), \qquad \text{with} \qquad x_1(0, X_1) = X_1.$$

We will assume that $x_1(t) : \Omega \to \Omega$ is a diffeomorphism for every $t \in [0, T]$, where $T > 0$ is fixed, and we will denote by $x_1(t)^{-1}$ its inverse. The trajectory x_1 enables to define Lagrangian quantities, i.e. quantities that depend on the Lagrangian coordinates. Three examples are the Lagrangian velocity v_1^l, the Lagrangian density ρ_1^l and the Lagrangian volume fraction α_1^l, defined by

$$v_1^l(t, X_1) := v_1(t, x_1(t, X_1)), \quad \rho_1^l(t, X_1) := \rho_1(t, x_1(t, X_1)), \quad \text{and} \quad \alpha_1^l(t, X_1) := \alpha_1(t, x_1(t, X_1)).$$

From the chain rule and the mass conservation, one then computes that

$$\partial_t \rho_1^l(t, X_1) = (\partial_t \rho_1 + v_1 \cdot \nabla \rho_1)(t, x_1(t, X_1)) = -[\rho_1 \mathrm{div}(v_1)](t, x_1(t, X_1)). \tag{2.4}$$

Using the identity $v_1 = v_1^l \circ x_1(t)^{-1}$ and the chain rule once again,

$$\begin{aligned} \mathrm{div}(v_1) =& \mathrm{Tr}\left(\frac{\partial v_1}{\partial x}\right) = \mathrm{Tr}\left[\left(\frac{\partial v_1^l}{\partial X_1} \circ x_1(t)^{-1}\right)\left(\frac{\partial x_1(t)^{-1}}{\partial x}\right)\right] = \mathrm{Tr}\left[\left(\frac{\partial v_1^l}{\partial X_1}\right)\left(\frac{\partial x_1}{\partial X_1}\right)^{-1}\right] \circ x_1(t)^{-1} \\ =& \frac{\partial_t J_1}{J_1} \circ x_1(t)^{-1}, \end{aligned}$$

where we defined

$$J_1 := \det\left(\frac{\partial x_1}{\partial X_1}\right)$$

the Jacobian of the trajectory x_1 and we used the identity

$$\frac{d}{ds}\left[\det A(s)\right] = \det A(s) \mathrm{Tr}\left[\frac{dA}{ds} A(s)^{-1}\right], \tag{2.5}$$

valid for any C^1 function $s \mapsto A(s)$ such that the square matrix $A(s)$ is invertible. Returning to (2.4), we obtain

$$\partial_t \rho_1^l = -\rho_1^l \frac{\partial_t J_1}{J_1}, \quad \text{i.e.} \quad \partial_t\left(\rho_1^l J_1\right) = 0, \quad \text{hence} \quad \rho_1^l = \frac{\rho_1|_{t=0}}{J_1} \tag{2.6}$$

(note that $x_1(0) = Id$, hence $J_1|_{t=0} = 1$). We deduce that the density ρ_1 is determined by its value at time $t = 0$ and by the trajectory x_1.

In a similar way, we define $x(t, X)$ the solution of

$$\partial_t x(t, X) = v(t, x(t, X)), \qquad x(0, X) = X.$$

In other words, $x(t, X)$ is the trajectory of a particle flowing at the mean velocity that started in position X at time 0. As before, one can define the Lagrangian mean velocity, the Lagrangian total density and the Lagrangian Reynolds tensor by

$$v^l(t, X) := v(t, x(t, X)), \qquad \rho^l(t, X) := \rho(t, x(t, X)) \quad \text{and} \quad P^l(t, X) := P(t, x(t, X)).$$

As in Equation (2.6), we deduce that

$$\rho^l = \frac{\rho^l|_{t=0}}{J}, \qquad \text{with} \qquad J := \det\left(\frac{\partial x}{\partial X}\right). \tag{2.7}$$

Furthermore, the chain rule yields that P^l solves an ordinary differential equation, given by

$$\partial_t P^l + \left(\frac{\partial v}{\partial x} \circ x(t)\right) P^l + P^l \left(\frac{\partial v}{\partial x} \circ x(t)\right)^T = 0.$$

Hence P is also uniquely determined by the trajectory and the initial value $P|_{t=0}$.

2.2. **Action and variations.** Recall that the Lagrangian density of System (1.10) was given in Equation (1.8) by

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho\mathrm{Tr}(P).$$

This Lagrangian density can also be written in terms of the variables $(\rho_1, \rho, v_1, v, \alpha_1, \rho\mathrm{Tr}P)$:

$$\mathcal{L}(\rho_1, \rho, v_1, v, \alpha_1, \rho\mathrm{Tr}P) := \frac{1}{2}\rho|v|^2 + \frac{\rho_1\rho}{2(\rho-\rho_1)}|v-v_1|^2 - U(\rho_1, \rho-\rho_1, \alpha_1) - \frac{1}{2}\rho\mathrm{Tr}P,$$

where

$$U(\rho_1, \rho_2, \alpha_1) := \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) + \rho_2\mathcal{E}_2\left(\frac{\rho_2}{1-\alpha_1}\right),$$

From now on, we will see $\mathcal{L}$ as a function of the variables $(\rho_1, \rho, v_1, v, \alpha_1, \rho\mathrm{Tr}P)$ only. One can now define the action associated to this Lagrangian:

$$\mathcal{A}\left[x_1, x, \alpha_1^l\right] := \int_0^T \int_\Omega \mathcal{L}(\rho_1, \rho, v_1, v, \alpha_1, \rho\mathrm{Tr}P)dxdt.$$

Let $\Phi = (\phi_1, \phi, \phi_{\alpha_1}) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ be a smooth vector field which is compactly supported in $(0,T)\times\Omega$. For every $\epsilon \in \mathbb{R}$, one can perturbate the trajectories and define $(x_{1,\epsilon}, x_\epsilon, \alpha_{1,\epsilon}^l) := (x_1, x, \alpha_1^l) + \epsilon\Phi$. To these modified trajectories, one can associate the corresponding densities, velocities and Reynolds stress tensor that we will denote $\rho_{1,\epsilon}, \rho_\epsilon, v_{1,\epsilon}, v_\epsilon, P_\epsilon$. We define

$$\mathcal{A}_\epsilon := \mathcal{A}[x_{1,\epsilon}, x_\epsilon, \alpha_{1,\epsilon}^l]$$

the perturbed action. Hamilton's stationary action principle states that the trajectory followed by the particles is a stationary point of the action, i.e. that

$$\left.\frac{\partial \mathcal{A}_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} = 0,$$

for every perturbation Φ. In order to simplify notations, we denote, for every quantity f_ϵ depending on ϵ, $\delta f := (\partial_\epsilon f_\epsilon)|_{\epsilon=0}$. δf is often called the variation of f. The stationary action principle is thus equivalent to require that the variation of the action is equal to zero: $\delta\mathcal{A} = 0$. We will also define $\zeta_1 := \phi_1 \circ x_1(t)^{-1}, \zeta := \phi \circ x(t)^{-1}$ and $\zeta_{\alpha_1} := \phi_{\alpha_1} \circ x_1(t)^{-1}$. From the stationary action principle, one obtains the following equality:

$$0 = \delta\mathcal{A} = \int_0^T \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\rho_1}\delta\rho_1 + \frac{\partial\mathcal{L}}{\partial\rho}\delta\rho + \frac{\partial\mathcal{L}}{\partial v_1}\delta v_1 + \frac{\partial\mathcal{L}}{\partial v}\delta v + \frac{\partial\mathcal{L}}{\partial\alpha_1}\delta\alpha_1 + \frac{\partial\mathcal{L}}{\partial(\rho\mathrm{Tr}P)}\delta(\rho\mathrm{Tr}P)\right)dxdt.$$

We thus need to compute the variations of all variables. First, it follows from the conservation of mass (2.6) and (2.7), as well as the formula (2.5) that

$$\delta\rho_1 = -\mathrm{div}(\rho_1\zeta_1) \quad \text{and} \quad \delta\rho = -\mathrm{div}(\rho\zeta).$$

As a consequence, we can integrate by parts to obtain that

$$\int_0^T \int_\Omega \frac{\partial\mathcal{L}}{\partial\rho_1}\delta\rho_1 = \int_0^T \int_\Omega \nabla\left(\frac{\partial\mathcal{L}}{\partial\rho_1}\right)\cdot\rho_1\zeta_1 \quad \text{and} \quad \int_0^T \int_\Omega \frac{\partial\mathcal{L}}{\partial\rho}\delta\rho = \int_0^T \int_\Omega \nabla\left(\frac{\partial\mathcal{L}}{\partial\rho}\right)\cdot\rho\zeta.$$

We now move on to the variation of the velocities. Since $v_1 \circ x_1(t) = v_1^l$ and $\delta v_1^l = \partial_t \phi_1$, we get that

$$\delta v_1 = \partial_t \zeta_1 + \frac{\partial \zeta_1}{\partial x} v_1 - \frac{\partial v_1}{\partial x} \zeta_1.$$

Let us define

$$K_1 := \frac{1}{\rho_1} \left(\frac{\partial \mathcal{L}}{\partial v_1} \right)^T, \quad \text{hence} \quad \rho_1 K_1^T = \frac{\partial \mathcal{L}}{\partial v_1}.$$

An integration by part coupled by the conservation of ρ_1 yields

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial \mathcal{L}}{\partial v_1} \delta v_1 &= \int_0^T \int_\Omega \rho_1 K_1 \cdot \left(\partial_t \zeta_1 + \frac{\partial \zeta_1}{\partial x} v_1 - \frac{\partial v_1}{\partial x} \zeta_1 \right) \\ &= - \int_0^T \int_\Omega \rho_1 \zeta_1 \cdot \left(\partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x} \right)^T K_1 \right). \end{aligned}$$

In a similar way, we define

$$K := \frac{1}{\rho} \left(\frac{\partial \mathcal{L}}{\partial \rho} \right)^T$$

and obtain that

$$\int_0^T \int_\Omega \frac{\partial \mathcal{L}}{\partial v} \delta v = - \int_0^T \int_\Omega \rho \zeta \cdot \left(\partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x} \right)^T K \right).$$

Concerning the variations of α_1, we compute that

$$\delta \alpha_1 = \zeta_{\alpha_1} - \frac{\partial \alpha_1}{\partial x} \zeta_1.$$

Finally, the equation satisfied by P implies that the variation of P is given by

$$\delta \mathrm{Tr} P = -2 \mathrm{Tr} \left(\frac{\partial \zeta}{\partial x} P \right) - \frac{\partial (\mathrm{Tr} P)}{\partial x} \zeta$$

(see [16]). In particular, the field ζ is needed to express the variations of P. This is why we use the trajectory $x(t, X)$ corresponding to the mean velocity v and responsible for the presence of lift terms (see [15] and below). It follows that

$$\delta(\rho \mathrm{Tr} P) = -2 \mathrm{Tr} \left(\frac{\partial \zeta}{\partial x} \rho P \right) - \mathrm{div}(\rho \mathrm{Tr} P \zeta).$$

Hence, by integration by parts,

$$\int_0^T \int_{\mathbb{R}} \frac{\partial \mathcal{L}}{\partial (\rho \mathrm{Tr} P)} \delta(\rho \mathrm{Tr} P) = \int_0^T \int_{\mathbb{R}} \mathrm{Tr} \left(\frac{\partial \zeta}{\partial x} \rho P \right) = - \int_0^T \int_{\mathbb{R}} \mathrm{div}(\rho P) \zeta.$$

The stationary action principle can thus be written

$$\begin{aligned} 0 = \delta \mathcal{A} = &- \int_0^T \int_\Omega \rho_1 \zeta_1 \cdot \left(\partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x} \right)^T K_1 - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho_1} \right) + \frac{1}{\rho_1} \frac{\partial \alpha_1}{\partial x} \frac{\partial \mathcal{L}}{\partial \alpha_1} \right) \\ &- \int_0^T \int_\Omega \rho \zeta \cdot \left(\partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x} \right)^T K - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho} \right) + \frac{1}{\rho} \mathrm{div}(\rho P)^T \right) \\ &+ \int_0^T \int_\Omega \frac{\partial \mathcal{L}}{\partial \alpha_1} \zeta_{\alpha_1}. \end{aligned}$$

Since ζ_1, ζ and ζ_{α_1} are arbitrary and independent of each other, each of the integrands must vanish and we obtain the following Euler-Lagrange equations:

$$0 = \partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x}\right)^T K_1 - \nabla\left(\frac{\partial \mathcal{L}}{\partial \rho_1}\right) + \frac{1}{\rho_1}\frac{\partial \alpha_1}{\partial x}\frac{\partial \mathcal{L}}{\partial \alpha_1}, \tag{2.8}$$

$$0 = \partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x}\right)^T K - \nabla\left(\frac{\partial \mathcal{L}}{\partial \rho}\right) + \frac{1}{\rho}\mathrm{div}(\rho P)^T, \tag{2.9}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_1}. \tag{2.10}$$

2.3. **Computation of the equation of the velocities.** We can now use the three Euler-Lagrange equations obtained to get the evolution equations for the velocities v_i, $i = 1, 2$. Note that Equation (2.8) can be simplified with the help of Equation (2.10). We also compute that

$$\begin{aligned} K_1 &= v_1 - v_2, \quad & \frac{\partial \mathcal{L}}{\partial \rho_1} &= \frac{1}{2}|v_1 - v_2|^2 - \frac{\partial U}{\partial \rho_1} + \frac{\partial U}{\partial \rho_2}, \\ K &= v_2, & \frac{\partial \mathcal{L}}{\partial \rho} &= \frac{1}{2}|v|^2 - \frac{1}{2}|v - v_2|^2 - \frac{\partial U}{\partial \rho_2}. \end{aligned}$$

Hence after adding (2.9) to (2.8), we obtain the equation for v_1:

$$\partial_t v_1 + (v_1 \cdot \nabla) v_1 + \mathrm{curl}(v_2) \wedge (v - v_1) + \nabla\left(\frac{\partial U}{\partial \rho_1}\right) + \frac{1}{\rho}\mathrm{div}(\rho P)^T = 0.$$

We also compute that (2.9) yields the equation for v_2:

$$\partial_t v_2 + (v_2 \cdot \nabla) v_2 + \mathrm{curl}(v_2) \wedge (v - v_2) + \nabla\left(\frac{\partial U}{\partial \rho_2}\right) + \frac{1}{\rho}\mathrm{div}(\rho P)^T = 0.$$

We still need to express these two equations without the dependence on α_1 coming from $(\partial U/\partial \rho_i)$. In order to do this, we define $g(\rho_1, \rho_2, \alpha_1) := \partial U/\partial \alpha_1$ and obtain with Equation (2.10) that

$$g(\rho_1, \rho_2, \alpha_1) = \frac{\partial U}{\partial \alpha_1} = -\frac{\partial \mathcal{L}}{\partial \alpha_1} = 0.$$

We compute that

$$g(\rho_1, \rho_2, \alpha_1) = \frac{\partial U}{\partial \alpha_1} = -\left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1'\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 \mathcal{E}_2'\left(\frac{\rho_2}{1-\alpha_1}\right) = p_2 - p_1,$$

hence (2.10) states that the two pressures are equal. Furthermore,

$$\begin{aligned} \frac{\partial g}{\partial \alpha_1} = \frac{\partial^2 U}{\partial \alpha_1^2} &= \frac{1}{\alpha_1}\left[2\left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1'\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_1}{\alpha_1}\right)^3 \mathcal{E}_1^{(2)}\left(\frac{\rho_1}{\alpha_1}\right)\right] \\ &\quad + \frac{1}{1-\alpha_1}\left[2\left(\frac{\rho_2}{1-\alpha_1}\right)^2 \mathcal{E}_2'\left(\frac{\rho_2}{1-\alpha_1}\right) + \left(\frac{\rho_2}{1-\alpha_1}\right)^3 \mathcal{E}_2^{(2)}\left(\frac{\rho_2}{1-\alpha_1}\right)\right] \\ &= \left(\frac{\rho_1}{\alpha_1}\right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 U_{22}, \end{aligned}$$

where

$$U_{ii} := \frac{1}{\alpha_i}\left[2\mathcal{E}_i'\left(\frac{\rho_i}{\alpha_i}\right) + \frac{\rho_i}{\alpha_i}\mathcal{E}_i^{(2)}\left(\frac{\rho_i}{\alpha_i}\right)\right] = \frac{\partial^2 U}{\partial \rho_i^2} = \frac{1}{\alpha_i}\left.\frac{d^2 \rho \mathcal{E}_i(\rho)}{d\rho^2}\right|_{\rho = \rho_i/\alpha_i} > 0,$$

since we assume that $\rho\mathcal{E}_i(\rho)$ is a convex function of ρ for $i = 1, 2$. We are thus allowed to use the implicit functions theorem to express α_1 as a function of ρ_1 and ρ_2: $\alpha_1 = \alpha_1(\rho_1, \rho_2)$. For $i = 1, 2$, we obtain that

$$\begin{aligned}\nabla\left(\frac{\partial U}{\partial \rho_i}\right) =& \frac{\partial^2 U}{\partial \rho_i \partial \rho_1}\nabla\rho_1 + \frac{\partial^2 U}{\partial \rho_i \partial \rho_2}\nabla\rho_2 + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1}\nabla\alpha_1 \\ =& \left(\frac{\partial^2 U}{\partial \rho_i \partial \rho_1} + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1}\frac{\partial \alpha_1}{\partial \rho_1}\right)\nabla\rho_1 + \left(\frac{\partial^2 U}{\partial \rho_i \partial \rho_2} + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1}\frac{\partial \alpha_1}{\partial \rho_2}\right)\nabla\rho_2 \\ =& \nabla(U_i),\end{aligned}$$

where

$$U_i(\rho_1, \rho_2) := \frac{\partial U}{\partial \rho_i}(\rho_1, \rho_2, \alpha_1(\rho_1, \rho_2))$$

as claimed in Equation (1.11). We also compute that

$$\frac{\partial^2 U}{\partial \rho_1 \partial \alpha_1} = -\frac{1}{\alpha_1}\left[2\frac{\rho_1}{\alpha_1}\mathcal{E}_1'\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1^{(2)}\left(\frac{\rho_1}{\alpha_1}\right)\right] = -\frac{\rho_1}{\alpha_1}U_{11},$$

and that

$$\frac{\partial^2 U}{\partial \rho_2 \partial \alpha_1} = \frac{\rho_2}{\alpha_2}U_{22}.$$

Finally, we can use the identity

$$\frac{\partial \alpha_1}{\partial \rho_i} = -\frac{\partial g/\partial \rho_i}{\partial g/\partial \alpha_1} = -\frac{(\partial^2 U/\partial \alpha_1 \partial \rho_i)}{(\partial^2 U/\partial \alpha_1^2)}$$

to obtain that

$$a_i := \frac{\partial U_i}{\partial \rho_i} = \frac{U_{11}U_{22}\left(\frac{\rho_{3-i}}{\alpha_{3-i}}\right)^2}{\left(\frac{\rho_1}{\alpha_1}\right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 U_{22}} > 0,$$

and

$$\frac{\partial U_1}{\partial \rho_2} = \frac{U_{11}U_{22}\frac{\rho_1}{\alpha_1}\frac{\rho_2}{\alpha_2}}{\left(\frac{\rho_1}{\alpha_1}\right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 U_{22}} = \frac{\partial U_2}{\partial \rho_1} =: b > 0.$$

It follows that $a_1 a_2 - b^2 = 0$ as claimed.

Remark. In the case where one phase is incompressible, there is no dependence on the variable α_1 and the derivation is easier. However, the system obtained still has the same structure, and the equality $a_1 a_2 = b^2$ is obtained by direct computation.

3. Hyperbolicity of the equations

We give here the proof that System (1.10) is hyperbolic. We first write (1.10) in matricial form:

$$\partial_t Y + \sum_{j=1}^{d} A_j(Y)\partial_j Y = 0.$$

The vector field Y takes values in $\mathbb{R}^{2+2d+d(d+1)/2} = \mathbb{R}^{(d+1)(d+4)/2}$, and the square matrices $A_j(Y)$ are smooth functions of Y. Let us also define, for every $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, the matrix

$$A(Y, \xi) := \sum_{j=1}^{d} A_j(Y)\xi_j.$$

Then saying that (1.10) is hyperbolic is equivalent to saying that, for every $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$, the matrix $A(Y,\xi)$ is diagonalizable with real eigenvalues.

Since the property of hyperbolicity is invariant under C^1 change of variables, we choose to set

$$Y := \begin{pmatrix} \rho_2 \\ v_2 \\ \rho \\ v \\ \tilde{P} \end{pmatrix}, \quad \text{with} \quad \tilde{P} := P > 0 \text{ if } d = 1, \quad \tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix} \text{ if } d = 2, \quad \tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix} \text{ if } d = 3$$

(recall that P is a symmetric second order tensor). With this variables, the matrix $A(Y,\xi)$ can be written as a block matrix:

$$A(Y,\xi) = \begin{pmatrix} v_2 \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha\xi & (v\cdot\xi) I_d + \xi(v_2 - v)^T & (\beta + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & v \cdot \xi & \rho\xi^T & 0 \\ \delta\xi & 0 & (\gamma + P/\rho)\xi & (v\cdot\xi) I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v\cdot\xi) I_d \end{pmatrix}.$$

The coefficients α, β, γ and δ are given by

$$\alpha := a_2 - b, \quad \beta := b, \quad \gamma := \frac{1}{\rho}(\rho_1 a_1 + \rho_2 b), \quad \delta := \frac{1}{\rho}[\rho_1(b - a_1) + \rho_2(a_2 - b)].$$

When $d = 1$, the matrices $C(\xi)$ and $D(\xi)$ are scalars given by $C(\xi) := \xi$, $D(\xi) := 2P\xi$. When $d = 2$, one has

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \text{ and } D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When $d = 3$,

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}, \quad D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where $(P\xi)_i$ denotes the i-th component of the vector $P\xi$.

In order to prove the hyperbolicity, the first step is to compute the characteristic polynomial of the matrix $A(Y,\xi)$. This polynomial is given by the following proposition.

Proposition 1. *The characteristic polynomial of the matrix $A(Y,\xi)$ is given by*

$$\begin{aligned} \chi_{A(Y,\xi)}(\lambda) =& (v\cdot\xi - \lambda)^{(d+2)(d-1)/2+1} \left[(v\cdot\xi - \lambda)^2 - \xi^T P \xi\right]^{d-2} \\ & \times \left\{2\rho_2\delta(v\cdot\xi - \lambda)(v_2 - v)^T \left[P - (\xi^T P\xi)\right]\xi + \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right] q(\lambda)\right\}, \end{aligned} \tag{3.1}$$

where

$$q(\lambda) := \left[(v_2\cdot\xi - \lambda)^2 - \alpha\rho_2\right]\left[(v\cdot\xi - \lambda)^2 - 3\xi^T P\xi - \rho\gamma\right] - \rho_2\delta\left(3\xi^T P\xi + \rho\beta\right). \tag{3.2}$$

Let us also denote $A'(Y,\xi)$ the matrix of the simpler system (1.16) *(without lift terms). The characteristic polynomial of this system is given by*

$$\chi_{A'(Y,\xi)} = (v_2\cdot\xi - \lambda)^{d-1}(v\cdot\xi - \lambda)^{d(d-1)/2+1}\left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]^{d-1} q(\lambda), \tag{3.3}$$

where $q(\lambda)$ is as in Equation (3.2).

The proof of this Proposition is technical and requires a lot of linear algebra computations. Hence we defer it to Appendix A.

Now that we have a formula for the characteristic polynomial of the matrices $A(Y,\xi)$ and $A'(Y,\xi)$, we can study the hyperbolicity of Systems (1.10) and (1.16). We will investigate the cases $d=1$, $d=2$ and $d=3$ separately.

3.1. **Case $d=1$.** When $d=1$, the hyperbolicity is given by the following proposition.

Proposition 2. *There exists a constant $C=C(\rho_1,\rho_2,P)$ such that, in dimension 1, System* (1.10) *and System* (1.16) *are hyperbolic when $|v_1-v_2|<C$.*

Proof. When $d=1$, the second order tensor P is a scalar, as well as the vector ξ. Since we have that $|\xi|=1$, we deduce that $\xi=\pm 1$ and $P-\xi^T P\xi=0$. Hence the second line of (3.1) vanishes and the characteristic polynomial is equal to

$$\chi_{A(Y,\xi)}=(v\cdot\xi-\lambda)\left(\left[(v_2\cdot\xi-\lambda)^2-\alpha\rho_2\right]\left[(v\cdot\xi-\lambda)^2-3\xi^T P\xi-\rho\gamma\right]-\rho_2\delta\left(3\xi^T P\xi+\rho\beta\right)\right).$$

Without loss of generality, we can suppose that $\xi=1$. We thus obtain that

$$q(\lambda)=\left[(v_2-\lambda)^2-\alpha\rho_2\right]\left[(v-\lambda)^2-3P-\rho\gamma\right]-\rho_2\delta\left(3P+\rho\beta\right). \tag{3.4}$$

We need to show that $q(\lambda)$ has four real roots. Since the general study of the roots of this polynomial is complicated, we first restrict ourselves to the case $v_1=v_2=v$. If we define $X:=(v-\lambda)^2$, then

$$q(\lambda)=(X-\alpha\rho_2)(X-3P-\rho\gamma)-\rho_2\delta(3P+\rho\beta)=:Q(X).$$

Hence $q(\lambda)=0$ if and only if $Q(X)=0$. Since Q is of degree 2, we have an explicit formula for its roots denoted $r_\pm$:

$$r_\pm=\frac{\alpha\rho_2+\rho\gamma+3P\pm\sqrt{\Delta}}{2},$$

where

$$\begin{aligned}\Delta&:=\left(\alpha\rho_2+\rho\gamma+3P\right)^2-4\left[(\alpha\rho_2(3P+\rho\gamma)-\rho_2\delta(3P+\rho\beta)\right]\\&=\left(\alpha\rho_2-\rho\gamma-3P\right)^2+4\rho_2\delta(3P+\rho\beta).\end{aligned} \tag{3.5}$$

We recall that $\alpha,\beta,\gamma,\delta$ are given by

$$\alpha:=a_2-b,\quad \beta:=b,\quad \gamma:=\frac{1}{\rho}(\rho_1a_1+\rho_2b),\quad \delta:=\frac{1}{\rho}[\rho_1(b-a_1)+\rho_2(a_2-b)].$$

It follows that

$$\alpha\rho_2+\rho\gamma=\rho_1a_1+\rho_2a_2>0,\quad \alpha\rho_2-\rho\gamma=\rho_2a_2-\rho_1a_1-2\rho_2b=\rho\delta-\rho\beta,$$

hence

$$\begin{aligned}\Delta=&(\rho\delta-\rho\beta-3P)^2+4\rho_2\delta(3P+\rho\beta)\\=&(\rho\delta+\rho\beta+3P)^2-4\rho_1\delta(3P+\rho\beta).\end{aligned}$$

We can combine the two previous lines to obtain

$$\Delta=\frac{\rho_1}{\rho}(\rho\delta-\rho\beta-3P)^2+\frac{\rho_2}{\rho}(\rho\delta+\rho\beta+3P)^2>0,$$

since $\rho\beta + 3P = \rho b + 3P > 0$. Hence the two roots $r_\pm$ are real and distinct. If we now return to (3.5), we see that

$$\begin{aligned}\Delta &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 4\left[3P\rho_2(\alpha - \delta) + \rho_2\rho(\alpha\gamma - \delta\beta)\right] \\ &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 4\left[3P\frac{\rho_1\rho_2}{\rho}(a_2 + a_1 - 2b) + \rho_2\rho_1(a_1 a_2 - b^2)\right] \\ &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 12P\frac{\rho_1\rho_2}{\rho}(a_2 + a_1 - 2b),\end{aligned}$$

where the last equality comes from the relation (1.13). From the expressions of a_1, a_2 and b obtained in Section 2, we see that

$$a_1 + a_2 - 2b = \gamma\left(\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2}\right)^2, \tag{3.6}$$

where γ is a positive constant. It follows that

$$r_\pm = \frac{\rho_1 a_1 + \rho_2 a_2 + 3P \pm \sqrt{(\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 12P\frac{\rho_1\rho_2}{\rho}\gamma\left(\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2}\right)^2}}{2}.$$

We see that $r_+ > 0$ and $r_- \geq 0$, with $r_- = 0$ if and only if $\rho_1/\alpha_1 = \rho_2/\alpha_2$, i.e. if and only if the two material densities are equal. Since by hypothesis the two material densities are different, we also have $r_- > 0$. Hence $\chi_{A(Y,\xi)}$ has five distinct roots, given by

$$\lambda_1 = v, \quad \lambda_{2\pm} = v \pm \sqrt{r_+}, \quad \lambda_{3\pm} = v \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P)$ such that $\chi_{A(Y,\xi)}$ has five disctinct roots for every v_1, v_2 such that $|v_1 - v_2| \leq C(\rho_1, \rho_2, P)$. Hence the matrix $A(Y, \xi)$ is diagonalizable with real eigenvalues and System (1.10) is strictly hyperbolic. The proof for the system without lift terms is the same since these terms vanish when $d = 1$.

□

3.2. **Case $d = 2$.** We now move on to the case $d = 2$. Recall that we aim to prove the following result:

Proposition 3. *Let θ be defined as in Equation* (1.17) *of Theorem* (1)*:*

$$\theta := \rho_1\rho_2\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 - \frac{1}{2}\left(\rho_1\sqrt{a_1} + \rho_2\sqrt{a_2}\right)^2.$$

If $\rho P < \theta I_d$ or $\rho P > \theta I_d$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that Systems (1.10) *and* (1.16) *are hyperbolic, whenever $|v_1 - v_2| < C$.*

Proof. In dimension 2, the characteristic polynomial of $A(Y, \xi)$ is given by

$$\chi_{A(Y,\xi)} = (v\cdot\xi - \lambda)^3\left(\left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]q(\lambda) + 2\rho_2\delta(v\cdot\xi - \lambda)(v_2 - v)^T\left[P - (\xi^T P\xi)\right]\xi\right),$$

where $q(\lambda)$ was defined in Equation (3.2). Let us first look at the polynomial

$$\tilde{q}(\lambda) := \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]q(\lambda) + 2\rho_2\delta(v\cdot\xi - \lambda)(v_2 - v)^T\left[P - (\xi^T P\xi)\right]\xi,$$

such that

$$\chi_{A(Y,\xi)} = (v\cdot\xi - \lambda)^3\tilde{q}(\lambda).$$

When $v_1 = v_2 = v$, $\tilde{q}$ reduces to

$$\tilde{q}(\lambda) = \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]q(\lambda).$$

By mimicking the case $d = 1$, we deduce that $\tilde{q}$ has six roots, given by

$$\lambda_{1\pm} = v \cdot \xi \pm \sqrt{\xi^T P \xi}, \quad \lambda_{2\pm} = v \cdot \xi \pm \sqrt{r_+}, \quad \lambda_{3\pm} = v \cdot \xi \pm \sqrt{r_-},$$

with

$$r_\pm = \frac{\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P \xi \pm \sqrt{\left(\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P \xi\right)^2 - 12 \xi^T P \xi \frac{\rho_1 \rho_2}{\rho} \gamma \left(\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2}\right)^2}}{2}. \tag{3.7}$$

Let us show that the six roots are all distinct. From the case $d = 1$, we see that $\lambda_{2\pm}$ and $\lambda_{3\pm}$ are always distinct. We also see with Equation (3.7) that $r_+ > \xi^T P \xi$, hence $\lambda_{1\pm}$ and $\lambda_{2\pm}$ are also always different. If finally we suppose that $\lambda_{1\pm}$ and $\lambda_{3\pm}$ are not distinct, we obtain from Equations (3.7) and (3.6) the following equality:

$$\rho \xi^T P \xi = \rho_1 \rho_2 (a_1 + a_2 - 2b) - \frac{1}{2}(\rho_1^2 a_1 + \rho_2^2 a_2 + 2\rho_1 \rho_2 b),$$

which yields after replacing b by $\sqrt{a_1 a_2}$,

$$\rho \xi^T P \xi = \rho_1 \rho_2 (\sqrt{a_1} - \sqrt{a_2})^2 - \frac{1}{2}(\rho_1 \sqrt{a_1} + \rho_2 \sqrt{a_2})^2 = \theta,$$

where θ was defined in (1.17). Since we supposed that $\rho P > \theta I_d$ or that $\rho P < \theta I_d$, and $\xi^T \xi = 1$, we deduce that $\rho \xi^T P \xi \neq \theta$ and the six roots are real and distinct.

By continuity, there exists $C = C(\rho_1, \rho_2, P)$ such that $\tilde{q}(\lambda)$ has six disctinct real roots for every v_1, v_2 such that $|v_1 - v_2| \leq C(\rho_1, \rho_2, P)$. Up to reducing C, we can suppose that all six eigenvalues are different from $v \cdot \xi$. Since the six eigenvalues are distinct, there exist six corresponding eigenvectors. We are thus left to find three independent eigenvectors for the eigenvalue $\lambda = v \cdot \xi$. The matrix $A(Y, \xi) - (v \cdot \xi) I_d$ is given by

$$A(Y,\xi) - (v \cdot \xi) I_d = \begin{pmatrix} v_2 \cdot \xi - v \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & \xi (v_2 - v)^T & (\beta + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & D(\xi) & 0 \end{pmatrix}$$

Let X be an eigenvector of $A(Y, \xi)$ for the eigenvalue $(v \cdot \xi)$. We write X as a block vector compatible with $A(Y, \xi)$ and solve the linear system $[A(Y, \xi) - (v \cdot \xi) I_d] X = 0$. After some computations, we obtain the following three independent eigenvectors:

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ \xi^\perp \\ \frac{(v_2 - v) \cdot \xi^\perp}{\gamma - \beta} \\ 0 \\ u_2 \end{pmatrix}, \quad X_3 = \begin{pmatrix} \rho_2 \\ (v \cdot \xi - v_2 \cdot \xi)\xi \\ \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \\ 0 \\ u_3 \end{pmatrix}, \tag{3.8}$$

where $\xi^\perp, u_1, u_2, u_3 \in \mathbb{R}^2$ are four vectors such that

$$\xi^\perp \neq 0, \quad \xi \cdot \xi^\perp = 0, \quad u_1 \neq 0, \quad C(\xi) u_1 = 0, \quad C(\xi) u_2 = -\frac{(v_2 - v) \cdot \xi^\perp}{\gamma - \beta} \left(\gamma + \frac{P}{\rho}\right) \xi,$$

and

$$C(\xi) u_3 = \left[-\rho_2 \delta - \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \left(\gamma + \frac{P}{\rho}\right) \right] \xi.$$

In order to see that the three vectors are well defined, we first note that $\gamma - \beta = \rho_1 (b - a_1)/\rho \neq 0$, since otherwise $b = a_1$ and from $a_1 a_2 = b^2$ one obtains $a_1 = a_2 = b$, which is incompatible with the

assumption that $a_1 + a_2 - 2b > 0$. Furthermore, we see that the matrix $C(\xi)$ has always a rank 2 submatrix when $\xi \neq 0$, which implies that $C(\xi)$ is onto and the dimension of its kernel is one. Thus the vectors u_1, u_2, u_3 are well defined. Hence the matrix $A(Y,\xi)$ is diagonalizable and System (1.10) is only weakly hyperbolic.

Case of the system without lift terms. Finally, let us investigate the case where the lift terms vanish. In this case, the characteristic polynomial is given by

$$\chi_{A'(Y,\xi)}(\lambda) = (v_2 \cdot \xi - \lambda)(v \cdot \xi - \lambda)^2[(v \cdot \xi - \lambda)^2 - \xi^T P \xi]q(\lambda).$$

When $v_1 = v_2 = v$, it follows from the case $d = 1$ that $q(\lambda)$ has four distinct real roots, given by

$$\lambda_{1\pm} = v \cdot \xi \pm \sqrt{r_+}, \qquad \lambda_{2\pm} = v \cdot \xi \pm \sqrt{r_-}.$$

Hence there exists a constant $C = C(\rho_1, \rho_2, P)$ such that q has four distinct real roots, for any $|v_1 - v_2| < C(\rho_1, \rho_2, P)$. This implies that $\chi_{A'(Y,\xi)}$ also only has real roots for these values of v_1, v_2. As in the previous case, up to reducing C, we can assume that the roots of q are distinct from $v \cdot \xi$, from $v_2 \cdot \xi$ and from $v \cdot \xi \pm \sqrt{\xi^T P \xi}$ by using the condition $\rho P < \theta I_d$ or $\rho P > \theta I_d$. It follows that most of the eigenvalues have a multiplicity one, except for $v \cdot \xi$, which has multiplicity two (or three if $v_2 \cdot \xi = v \cdot \xi$). Thus we need to find two eigenvectors for this eigenvalue $v \cdot \xi$, and three when $v_2 \cdot \xi = v \cdot \xi$. We compute that

$$A'(Y,\xi) - (v \cdot \xi) I_d = \begin{pmatrix} v_2 \cdot \xi - v \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v_2 \cdot \xi - v \cdot \xi) I_d & (\beta + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & D(\xi) & 0 \end{pmatrix}$$

We find two independent eigenvectors, given by

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} \rho_2 \\ (v_2 \cdot \xi - v \cdot \xi)\xi \\ \frac{\rho_2(\alpha - \delta) + (v_2 \cdot \xi - v \cdot \xi)^2}{\gamma - \beta} \\ 0 \\ u_3 \end{pmatrix}, \tag{3.9}$$

where $u_1 \neq 0$, $C(\xi)u_1 = 0$ and

$$C(\xi)u_3 = \left[-\rho_2 \delta - \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \left(\gamma + \frac{P}{\rho} \right) \right] \xi.$$

Note that, when $v_2 \cdot \xi = v \cdot \xi$, a third eigenvector for the eigenvalue $v \cdot \xi$ is

$$X_2 = \begin{pmatrix} 0 \\ \xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0. \tag{3.10}$$

Finally, we can show that the condition $\rho P < \theta I_d$ or $\rho P > \theta I_d$ is necessary for the hyperbolicity of System (1.16). Indeed, if this condition does not hold, one can find a vector ξ such that $|\xi| = 1$ and that $q(v \cdot \xi \pm \sqrt{\xi^T P \xi}) = 0$, whenever v_1 and v_2 are close enough. In this case, $v \cdot \xi \pm \sqrt{\xi^T P \xi}$ is a root of $\chi_{A'(Y,\xi)}$ of multiplicity two. However, computations show that the kernel of the matrix $A'(Y,\xi) - (v \cdot \xi \pm \sqrt{\xi^T P \xi}) I_d$ is always of dimension 1. Hence the matrix $A'(Y,\xi)$ is not diagonalisable in this case and System (1.16) is not hyperbolic.

□

3.3. **Case $d = 3$.** In dimension 3, we need to prove the following proposition:

Proposition 4. *Let θ be defined as in Equation* (1.17) *of Theorem* (1)*:*

$$\theta := \rho_1\rho_2 \left(\sqrt{a_1} - \sqrt{a_2}\right)^2 - \frac{1}{2}\left(\rho_1\sqrt{a_1} + \rho_2\sqrt{a_2}\right)^2.$$

If $\rho P < \theta I_d$ or $\rho P > \theta I_d$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that System (1.16) *is hyperbolic, whenever $|v_1 - v_2| < C$. Under these assumptions, System* (1.10) *is weakly hyperbolic.*

Proof. We first prove the result concerning System (1.10). When $d = 3$, the characteristic polynomial of the system is given by

$$\chi_{A(Y,\xi)} = (v\cdot\xi - \lambda)^6 \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right] \tilde{q}(\lambda),$$

with

$$\tilde{q}(\lambda) = \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right] q(\lambda) + 2\rho_2\delta(v\cdot\xi - \lambda)(v_2 - v)^T \left[P - (\xi^T P\xi)\right]\xi$$

and $q(\lambda)$ was defined in Equation (3.2). When ξ is such that

$$(v_2 - v)^T \left[P - (\xi^T P\xi)\right]\xi = 0, \tag{3.11}$$

the characteristic polynomial can be written

$$\chi_{A(Y,\xi)} = (v\cdot\xi - \lambda)^6 \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]^2 q(\lambda).$$

Hence the algebraic multiplicity of $v\cdot\xi \pm \sqrt{\xi^T P\xi}$ is equal to two. However, computations show that the geometric multiplicity of this eigenvalue is equal to two if and only if the vectors $v_2 - v$ and ξ are proportional, and is equal to one when it is not the case. Since Equation (3.11) can be satisfied even when ξ and $v_2 - v$ are not proportional (for instance when ξ is an eigenvector of P), the matrix $A(Y,\xi)$ is not diagonalizable for every ξ and System (1.10) is not hyperbolic in dimension 3.

Case of the system without lift terms. Let us now consider System (1.16). When $d = 3$,

$$\chi_{A'(Y,\xi)} = (v_2\cdot\xi - \lambda)^2 (v\cdot\xi - \lambda)^4 \left[(v\cdot\xi - \lambda)^2 - \xi^T P\xi\right]^2 q(\lambda).$$

From the case $d = 1$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that q has four distinct real roots whenever $|v_1 - v_2| < C$. Up to reducing C, we can suppose that the four roots of q are distinct from $v\cdot\xi$, from $v_2\cdot\xi$ and from $v\cdot\xi \pm \sqrt{\xi^T P\xi}$.

We are thus left to find two independent eigenvectors for the eigenvalue $v_2\cdot\xi$, four for the eigenvalue $v\cdot\xi$ and two for the eigenvalues $v\cdot\xi \pm \sqrt{\xi^T P\xi}$. First, we see that the matrix $C(\xi)$ is of rank 3 since it always has an invertible 3×3 submatrix. Thus the dimension of its kernel is also 3. Hence the formulas for the vectors X_1 and X_3, that were defined in Equation (3.9) for the case $d = 2$ already provide 4 independent eigenvectors. Similarly, the subspace which is orthogonal to ξ is of dimension 2 when $d = 3$, hence the Equation (3.10) defines two independent eigenvectors for the eigenvalue $v_2\cdot\xi$.

Finally, the matrix $A'(Y,\xi) - (v\cdot\xi \mp \sqrt{\xi^T P\xi})I_d$ is equal to

$$\begin{pmatrix}
v_2\cdot\xi - v\cdot\xi \pm \sqrt{\xi^T P\xi} & \rho_2\xi^T & 0 & 0 & 0 \\
\alpha\xi & (v_2\cdot\xi - v\cdot\xi \pm \sqrt{\xi^T P\xi})I_d & (\beta + P/\rho)\xi & 0 & C(\xi) \\
0 & 0 & \pm\sqrt{\xi^T P\xi} & \rho\xi^T & 0 \\
\delta\xi & 0 & (\gamma + P/\rho)\xi & \pm\sqrt{\xi^T P\xi}I_d & C(\xi) \\
0 & 0 & 0 & D(\xi) & \pm\sqrt{\xi^T P\xi}I_d
\end{pmatrix}.$$

We obtain two eigenvectors of the form

$$X_4 = \begin{pmatrix} 0 \\ \pm\sqrt{\xi^T P\xi}\xi^\perp \\ 0 \\ (v_2\cdot\xi - v\cdot\xi \pm \sqrt{\xi^T P\xi})\xi^\perp \\ \mp\frac{v_2\cdot\xi - v\cdot\xi\pm\sqrt{\xi^T P\xi}}{\sqrt{\xi^T P\xi}} D(\xi)\xi^\perp \end{pmatrix}, \quad \text{where} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T\xi^\perp = 0.$$

Hence $A'(Y,\xi)$ is diagonalizable and System (1.16) is hyperbolic.

□

4. Other models

This section is devoted to the proofs of Theorems 2, 3 and 4. The proofs of these theorems is very similar to the proof of Theorem 1. As a consequence, the proofs are given without detailed computations.

4.1. **Case where the turbulence is carried by one phase.** In this part, we give the proof of Theorem 2. We thus consider a two-phase flow for which one of the two phases, say the phase 2, is turbulent. In other words, instead of assuming that the turbulence is created by the mixture of the two phases, as it was the case in System (1.10), we supppose that it is intrinsic to the phase 2. We choose for this two-phase flow the Lagrangian defined in Equation (1.19):

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho_2\mathrm{Tr}(P_2). \tag{4.1}$$

We use the superscript 2 for the tensor P_2 to insist on the fact that it is carried by the phase two. as a consequence, we assume that the evolution of P_2 is given by Equation (1.20):

$$\partial_t P_2 + (v_2\cdot\nabla)P_2 + \frac{\partial v_2}{\partial x}P_2 + P_2\left(\frac{\partial v_2}{\partial x}\right)^T = 0. \tag{4.2}$$

We impose again the conservation of the two densities:

$$\partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1,2. \tag{4.3}$$

We also define, for $i = 1;2$, the trajectory followed by the particles of the phase i, denoted x_i, such that

$$\partial_t x_i(t,X) = v_i(t,x_i(t,X)), \qquad x_i(0,X) = X.$$

The action associated to the Lagrangien (4.1) is given by

$$\mathcal{A}\left[x_1,x_2,\alpha_1^l\right] := \int_0^T\int_\Omega \mathcal{L}.$$

One can apply the stationary action principle under the constraints given by (4.3) and (4.2) to obtain the following system:

$$\begin{cases} \partial_t\rho_i + \mathrm{div}(\rho_i v_i) = 0, & i = 1,2, \quad \text{(4.4a)} \\ \partial_t v_1 + (v_1\cdot\nabla v_1) + \nabla(U_1) = 0, & \quad \text{(4.4b)} \\ \partial_t v_2 + (v_2\cdot\nabla v_2) + \nabla(U_2) + \dfrac{1}{\rho_2}\mathrm{div}(\rho_2 P_2)^T = 0, & \quad \text{(4.4c)} \\ \partial_t P_2 + (v_2\cdot\nabla)P_2 + \dfrac{\partial v_2}{\partial x}P_2 + P_2\left(\dfrac{\partial v_2}{\partial x}\right)^T = 0. & \quad \text{(4.4d)} \end{cases}$$

The potentials U_1 and U_2 are defined as in Section 1. The conservation of total momentum still holds:

$$\partial_t(\rho_1 v_1 + \rho_2 v_2) + \operatorname{div}\left(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + pI_d + \rho_2 P_2\right)^T = 0,$$

as well as the conservation of energy:

$$\partial_t(e_1 + e_2) + \operatorname{div}\left(e_1 v_1 + e_2 v_2 + p(\alpha_1 v_1 + \alpha_2 v_2) + \rho_2 P_2 v_2\right) = 0,$$

where

$$e_1 := \frac{1}{2}\rho_1 |v_1|^2 + \rho_1 \mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right), \qquad e_2 := \frac{1}{2}\rho_2 |v_2|^2 + \rho_2 \mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) + \frac{1}{2}\rho_2 \mathrm{Tr}(P_2)$$

and

$$p := \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1'\left(\frac{\rho_1}{\alpha_1}\right) = \left(\frac{\rho_2}{\alpha_2}\right)^2 \mathcal{E}_2'\left(\frac{\rho_2}{\alpha_2}\right).$$

We now recall the main result about System (4.4):

Proposition 5. *The hyperbolicity of System* (4.4) *can be caracterized as follows:*

- *When $d = 1$, System* (4.4) *is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that* (4.4) *is hyperbolic when $|v_1 - v_2| \leq C$.*
- *When $d = 2, 3$ let*

  $$\mu := \rho_1 a_1 - \frac{1}{2}\rho_2 a_2$$

  and assume that either $P_2 > \mu I_d$ or $P_2 < \mu I_d$. Then (4.4) *is hyperbolic whenever $|v_1 - v_2| \leq C$.*

Proof. We use here the notations of Section 3. The matrix of System (4.4) is given by

$$B(Y,\xi) := \begin{pmatrix} v_1\cdot\xi & \rho_1\xi^T & 0 & 0 & 0 \\ a_1\xi & (v_1\cdot\xi)I_d & b\xi & 0 & 0 \\ 0 & 0 & v_2\cdot\xi & \rho_2\xi^T & 0 \\ b\xi & 0 & (a_2 + \frac{P_2}{\rho_2})\xi & (v_2\cdot\xi)I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v_2\cdot\xi)I_d \end{pmatrix}$$

We compute that the characteristic polynomial $\chi_{B(Y,\xi)}$ of this matrix is given by

$$\chi_{B(Y,\xi)}(\lambda) = (v_1\cdot\xi - \lambda)^{d-1}(v_2\cdot\xi - \lambda)^{d(d-1)/2+1}\left[(v_2\cdot\xi - \lambda)^2 - \xi^T P_2 \xi\right]^{d-2} q_0(\lambda),$$

where

$$q_0(\lambda) := \left[(v_1\cdot\xi - \lambda)^2 - \rho_1 a_1\right]\left[(v_2\cdot\xi - \lambda)^2 - \rho_2 a_2 - 3\xi^T P_2 \xi\right] - \rho_1\rho_2 b^2.$$

Let us show that q_0 has four distinct real roots when v_1 and v_2 are close. When $v_1 = v_2 = v$, let us denote $X := (v\cdot\xi - \lambda)^2$. Then

$$q_0(\lambda) = X^2 - \left(\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_2\xi\right) X + 3\xi^T P_2 \xi \rho_1 a_1 =: Q_0(X).$$

Hence $q_0(\lambda) = 0$ if and only if $Q_0(X) = 0$. The roots of Q_0 are given by

$$r_\pm = \frac{\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_2\xi \pm \sqrt{\Delta}}{2},$$

where

$$\begin{aligned}\Delta :=& (\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_2\xi)^2 - 12\xi^T P_2\xi\rho_1 a_1 \\ =& (\rho_1 a_1 - \rho_2 a_2 - 3\xi^T P_2\xi)^2 + 4\rho_1\rho_2 a_1 a_2 > 0.\end{aligned}$$

Hence $r_\pm > 0$ and q_0 has four distinct real roots, given by

$$\lambda_{1\pm} := v\cdot\xi \pm \sqrt{r_+}, \qquad \lambda_{2\pm} := v\cdot\xi \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that q_0 has four distinct real roots, for any $|v_1 - v_2| < C$. For these values of $|v_1 - v_2|$, $\chi_{B(Y,\xi)}$ has only real roots. Since $r_\pm > 0$, the roots of q_0 are distinct from $v \cdot \xi$. The assumption $P_2 < \mu I_d$ or $P_2 > \mu I_d$ ensures that the roots of q_0 are also distinct from $v \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$. Thus, up to reducing C, we can always assume that the roots of q_0 are roots of $\chi_{B(Y,\xi)}$ of multiplicity one. It follows that $v_1 \cdot \xi$ is a root of multiplicity $d-1$, $v_2 \cdot \xi$ is of multiplicity $d(d-1)/2 + 1$ and $v_2 \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$ is of multiplicity $d-1$.

Let us show that $B(Y, \xi)$ is diagonalizable. We see that we can find $d-1$ eigenvectors for the eigenvalue $v_1 \cdot \xi$ which are of the form

$$X_1 := \begin{pmatrix} 0 \\ \xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

For the eigenvalue $v_2 \cdot \xi$, we can find $d(d-1)/2 + 1$ eigenvectors of the form

$$X_2 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix} \quad \text{and} \quad X_3 := \begin{pmatrix} \rho_1 \\ (v_2 \cdot \xi - v_1 \cdot \xi)\xi \\ \frac{(v_1 \cdot \xi - v_2 \cdot \xi)^2 - \rho_1 a_1}{b} \\ 0 \\ u_2 \end{pmatrix},$$

where $u_1 \neq 0$, $C(\xi) u_1 = 0$, and

$$C(\xi) u_2 = \left[\frac{\rho_1 a_1 - (v_1 \cdot \xi - v_2 \cdot \xi)^2}{b} \left(a_2 + \frac{P_2}{\rho_2} \right) - \rho_1 b \right] \xi.$$

Note that $C(\xi)$ is a matrix of size $d(d+1)/2 \times d$ and of rank d, hence the dimension of its kernel is $d(d-1)/2$.

Finally, we can find $d-1$ eigenvectors corresponding to the eigenvalue $v_2 \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$, given by

$$X_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^\perp \\ \pm \frac{1}{\sqrt{\xi^T P_2 \xi}} D(\xi) \xi^\perp \end{pmatrix}, \quad \text{where} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

Hence $B(Y, \xi)$ is diagonalizable and System (4.4) is hyperbolic.

□

4.2. **Case where each phase is turbulent.** We now give the proof of Theorem 2. We thus consider a two-phase flow in which each phase is endowed with a Reynolds stress tensor denoted P_i, $i = 1, 2$. The Lagrangian density is the following:

$$\mathcal{L} := \frac{1}{2}\rho_1 |v_1|^2 + \frac{1}{2}\rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2}\rho_1 \mathrm{Tr}(P_1) - \frac{1}{2}\rho_2 \mathrm{Tr}(P_2). \tag{4.5}$$

The densities are still conserved:

$$\partial_t \rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1, 2.$$

The evolution equations for P_1, P_2 are given by

$$\partial_t P_i + (v_i \cdot \nabla) P_i + \frac{\partial v_i}{\partial x} P_i + P_i \left(\frac{\partial v_i}{\partial x} \right)^T = 0, \qquad i = 1, 2.$$

The action associated to the Lagrangian density (1.22) is

$$\mathcal{A}\left[x_1, x_2, \alpha_1^l\right] := \int_0^T \int_\Omega \mathcal{L},$$

where the trajectories x_i are defined by

$$\partial_t x_i(t,X) = v_i(t, x_i(t,X)), \qquad x_i(0,X) = X.$$

The stationary action principle then gives the following system of equations:

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, & i = 1,2, \quad (4.6a) \\ \partial_t v_i + (v_i \cdot \nabla v_i) + \nabla(U_i) + \dfrac{1}{\rho_i}\operatorname{div}(\rho_i P_i)^T = 0, & i = 1,2 \quad (4.6b) \\ \partial_t P_i + (v_i \cdot \nabla) P_i + \dfrac{\partial v_i}{\partial x} P_i + P_i \left(\dfrac{\partial v_i}{\partial x}\right)^T = 0 & i = 1,2. \quad (4.6c) \end{cases}$$

The potentials U_1 and U_2 are defined as in Section 1. The conservation of total momentum still holds:

$$\partial_t(\rho_1 v_1 + \rho_2 v_2) + \operatorname{div}\left(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + p I_d + \rho_1 P_1 + \rho_2 P_2\right)^T = 0,$$

as well as the conservation of energy:

$$\partial_t(e_1 + e_2) + \operatorname{div}\left(e_1 v_1 + e_2 v_2 + p(\alpha_1 v_1 + \alpha_2 v_2) + \rho_1 P_1 v_1 + \rho_2 P_2 v_2\right) = 0,$$

where

$$e_1 := \frac{1}{2}\rho_1 |v_1|^2 + \rho_1 \mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) + \frac{1}{2}\rho_1 \operatorname{Tr}(P_1), \qquad e_2 := \frac{1}{2}\rho_2 |v_2|^2 + \rho_2 \mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) + \frac{1}{2}\rho_2 \operatorname{Tr}(P_2)$$

and

$$p := \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1'\left(\frac{\rho_1}{\alpha_1}\right) = \left(\frac{\rho_2}{\alpha_2}\right)^2 \mathcal{E}_2'\left(\frac{\rho_2}{\alpha_2}\right).$$

We now state the main result about System (4.6):

Proposition 6. *The hyperbolicity of System* (4.6) *can be caracterized as follows:*

- *When $d = 1$, System* (4.6) *is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that* (4.4) *is hyperbolic when $|v_1 - v_2| \leq C$.*
- *When $d = 2, 3$ assume that*

$$\frac{1}{3} P_1 \leq P_2 \leq 3 P_1. \quad (4.7)$$

Then (4.4) *is hyperbolic whenever $|v_1 - v_2| \leq C$.*

Proof. We use here the notations of Section 3. The matrix of System (4.4) is given by

$$C(Y,\xi) := \begin{pmatrix} v_1 \cdot \xi & \rho_1 \xi^T & 0 & 0 & 0 & 0 \\ (a_1 + \frac{P_1}{\rho_1})\xi & (v_1 \cdot \xi) I_d & C(\xi) & b\xi & 0 & 0 \\ 0 & D_1(\xi) & (v_1 \cdot \xi) I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 \cdot \xi & \rho_2 \xi^T & 0 \\ b\xi & 0 & 0 & (a_2 + \frac{P_2}{\rho_2})\xi & (v_2 \cdot \xi) I_d & C(\xi) \\ 0 & 0 & 0 & 0 & D_2(\xi) & (v_2 \cdot \xi) I_d \end{pmatrix},$$

where

$$Y := \begin{pmatrix} \rho_1 \\ v_1 \\ \tilde{P}_1 \\ \rho_2 \\ v_2 \\ \tilde{P}_2 \end{pmatrix}$$

and $D_i(\xi)$ is defined as in Section 3, but with P_i instead of P ($i = 1, 2$). We compute that the characteristic polynomial $\chi_{C(Y,\xi)}$ of this matrix is given by

$$\begin{aligned}\chi_{C(Y,\xi)}(\lambda) =&(v_1 \cdot \xi - \lambda)^{d(d-1)/2+1} \left[(v_1 \cdot \xi - \lambda)^2 - \xi^T P_1 \xi\right]^{d-2} \\ &\times (v_2 \cdot \xi - \lambda)^{d(d-1)/2+1} \left[(v_2 \cdot \xi - \lambda)^2 - \xi^T P_2 \xi\right]^{d-2} q_1(\lambda),\end{aligned}$$

where

$$q_1(\lambda) := \left[(v_1 \cdot \xi - \lambda)^2 - \rho_1 a_1 - 3\xi^T P_1 \xi\right] \left[(v_2 \cdot \xi - \lambda)^2 - \rho_2 a_2 - 3\xi^T P_2 \xi\right] - \rho_1 \rho_2 b^2.$$

Let us show that q_1 has four distinct real roots when v_1 and v_2 are close. When $v_1 = v_2 = v$, let us denote $X := (v \cdot \xi - \lambda)^2$. Then

$$\begin{aligned}q_1(\lambda) &= X^2 - \left(\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi\right) X + (\rho_1 a_1 + 3\xi^T P_1 \xi)(\rho_2 a_2 + 3\xi^T P_2 \xi) - \rho_1 \rho_2 a_1 a_2 \\ &=: Q_1(X).\end{aligned}$$

Hence $q_1(\lambda) = 0$ if and only if $Q_1(X) = 0$. The roots of Q_1 are given by

$$r_\pm = \frac{\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi \pm \sqrt{\Delta}}{2},$$

where

$$\begin{aligned}\Delta :=&(\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi)^2 - 4\left[(\rho_1 a_1 + 3\xi^T P_1 \xi)(\rho_2 a_2 + 3\xi^T P_2 \xi) - \rho_1 \rho_2 a_1 a_2\right] \\ =&(\rho_1 a_1 + 3\xi^T P_1 \xi - \rho_2 a_2 - 3\xi^T P_2 \xi)^2 + 4\rho_1 \rho_2 a_1 a_2 > 0.\end{aligned}$$

Hence $r_\pm > 0$ and q_1 has four distinct real roots, given by

$$\lambda_{1\pm} := v \cdot \xi \pm \sqrt{r_+}, \qquad \lambda_{2\pm} := v \cdot \xi \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that q_1 has four distinct real roots, for any $|v_1 - v_2| < C$. For these values of $|v_1 - v_2|$, $\chi_{C(Y,\xi)}$ has only real roots. Since $r_\pm > 0$, the roots of q_1 are distinct from $v \cdot \xi$. The assumption (1.24) ensures that the roots of q_1 are also distinct from $v \cdot \xi \pm \sqrt{\xi^T P_i \xi}$ for $i = 1, 2$. Thus, up to reducing C, we can always assume that the roots of q_1 are roots of $\chi_{C(Y,\xi)}$ of multiplicity one. It follows that $v_i \cdot \xi$ is a root of multiplicity $d(d-1)/2 + 1$ and $v_i \cdot \xi \pm \sqrt{\xi^T P_i \xi}$ is of multiplicity $d - 1$, for $i = 1, 2$.

Let us show that $C(Y, \xi)$ is diagonalizable. We see that we can find $d(d-1)/2 + 1$ eigenvectors for the eigenvalue $v_1 \cdot \xi$ which are of the form

$$X_1 := \begin{pmatrix} 0 \\ 0 \\ u_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad X_2 := \begin{pmatrix} \rho_2 a_2 + 3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 \\ 0 \\ u_2 \\ -\rho_2 b \\ \frac{v_2 \cdot \xi - v_1 \cdot \xi}{\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2}[3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 - 2P_2] b\xi \\ \frac{1}{(v_2 \cdot \xi - v_1 \cdot \xi)^2 - \xi^T P_2 \xi} D_2(\xi)[3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 - 2P_2] b\xi \end{pmatrix},$$

where $u_1 \neq 0$, $C(\xi)u_1 = 0$, and

$$C(\xi)u_2 = \left[\rho_2 b^2 - \left[\rho_2 a_2 + 3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2\right]\left(a_1 + \frac{P_1}{\rho_1}\right)\right]\xi.$$

Note that $C(\xi)$ is a matrix of size $d(d+1)/2 \times d$ and of rank d, hence it is onto and the dimension of its kernel is $d(d-1)/2$.

We can also find $d-1$ eigenvectors corresponding to the eigenvalue $v_1 \cdot \xi \pm \sqrt{\xi^T P_1 \xi}$, given by

$$X_3 := \begin{pmatrix} 0 \\ \xi^\perp \\ \pm\frac{1}{\sqrt{\xi^T P \xi}} D(\xi)\xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

Since the system is invariant under permutation of indices $1 \leftrightarrow 2$, one can find similar eigenvectors for the eigenvalues $v_2 \cdot \xi$ and $v_2 \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$. Hence $C(Y, \xi)$ is diagonalizable and System (4.6) is hyperbolic.

□

4.3. **Other systems.** In the previous sections, we showed that the presence of turbulence in a two-phase flow, modeled by a Reynolds tensor, enables to obtain a hyperbolic system coming from a variational principle. We showed that the turbulence can be added in various ways and gave three different examples. Theorem 4 states that the three examples given previously are the only ones for which the system of equations obtained is hyperbolic. Recall the statement of Theorem 4:

Theorem. *Consider the following general Lagrangian density:*

$$\mathcal{L} := \frac{1}{2}\rho_1 |v_1|^2 + \frac{1}{2}\rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2 \mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2} r \mathrm{Tr}(P), \tag{4.8}$$

endowed with the constraints

$$\partial_t \rho_i + \mathrm{div}(\rho_i v_i) = 0, \qquad i = 1, 2 \tag{4.9}$$

and

$$\partial_t P + (u \cdot \nabla)P + \frac{\partial u}{\partial x} P + P\left(\frac{\partial u}{\partial x}\right)^T = 0, \tag{4.10}$$

where

$$(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}.$$

Let us consider the system formed by Equations (4.9), (4.10) *and the Euler-Lagrange equations associated to the Lagrangian density* (4.8). *We denote by M the matrix of this system. Then, in dimension 1, all the eigenvalues of M are real when $v_1 = v_2$, whatever the choice of convex internal energies $\rho\mathcal{E}_i(\rho)$, $i = 1, 2$, if and only if the density r and the velocity u are compatible, i.e. the following equation is satisfied:*

$$\partial_t r + \mathrm{div}(ru) = 0.$$

Proof. We treat each possible value of $(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}$ separately. The case $(r, u) = (\rho, v)$ was treated in the study of System (1.10). The case $(r, u) = (\rho_2, v_2)$ was treated in the study of System (4.4). By symmetry, we are left to study the three following cases:

- $(r, u) = (\rho_1, v_2)$
- $(r, u) = (\rho, v_2)$

- $(r,u)=(\rho_1,v)$

4.3.1. *Case* $(r,u)=(\rho_1,v_2)$. We obtain the following system:

$$\begin{aligned}
&\partial_t\rho_i+\operatorname{div}(\rho_i v_i)=0, \qquad i=1,2,\\
&\partial_t v_1+(v_1\cdot\nabla)v_1+\nabla(U_1+\frac{1}{2}\mathrm{Tr}P)=0,\\
&\partial_t v_2+(v_2\cdot\nabla)v_2+\nabla(U_2)+\frac{1}{\rho_2}\operatorname{div}(\rho_1 P)^T-\frac{\rho_1}{2\rho_2}\nabla(\mathrm{Tr}P)=0,\\
&\partial_t P+(v_2\cdot\nabla)P+\frac{\partial v_2}{\partial x}P+P\left(\frac{\partial v_2}{\partial x}\right)^T=0.
\end{aligned}$$

In $1d$, the characteristic polynomial is given by

$$\begin{aligned}
\chi(\lambda)=&(v_2-\lambda)\left[(v_1-\lambda)^2(v_2-\lambda)^2-\rho_1 a_1(v_2-\lambda)^2\right.\\
&\left.-\rho_2\left(a_2+\rho_1\frac{P}{\rho_2^2}\right)(v_1-\lambda)^2+\rho_1\rho_2\left[a_1\left(a_2+\rho_1\frac{P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]\right].
\end{aligned}$$

Let us look at the case $v_1=v_2=v$. If we denote $X:=(v-\lambda)^2$, we get that

$$\frac{\chi(\lambda)}{v-\lambda}=X^2-\left(\rho_1 a_1+\rho_2 a_2+\frac{\rho_1 P}{\rho_2}\right)X+\rho_1\rho_2\left[a_1\left(a_2+\frac{\rho_1 P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]=:Q(X).$$

The discriminant of $Q(X)$ is

$$\begin{aligned}
\Delta:=&\left(\rho_1 a_1+\rho_2 a_2+\frac{\rho_1 P}{\rho_2}\right)^2-4\rho_1\rho_2\left[a_1\left(a_2+\frac{\rho_1 P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]\\
=&\left(\rho_1 a_1-\rho_2 a_2-\frac{\rho_1 P}{\rho_2}\right)^2+4\rho_1\rho_2\left(b+\frac{P}{\rho_2}\right)^2>0.
\end{aligned}$$

Hence Q has two real roots. This gives four roots for χ. In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$0<a_1\left(a_2+\frac{\rho_1 P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2=\frac{P}{\rho_2^2}\left(\rho_1 a_1-2\rho_2 b-P\right).$$

Hence there are some admissible values of a_1,b,P such that χ has complex roots.

4.3.2. *Case* $(r,u)=(\rho,v_2)$. We obtain the following system:

$$\begin{aligned}
&\partial_t\rho_i+\operatorname{div}(\rho_i v_i)=0, \qquad i=1,2,\\
&\partial_t v_1+(v_1\cdot\nabla)v_1+\nabla(U_1+\frac{1}{2}\mathrm{Tr}P)=0,\\
&\partial_t v_2+(v_2\cdot\nabla)v_2+\nabla(U_2)+\frac{1}{\rho_2}\operatorname{div}(\rho P)^T-\frac{\rho_1}{2\rho_2}\nabla(\mathrm{Tr}P)=0,\\
&\partial_t P+(v_2\cdot\nabla)P+\frac{\partial v_2}{\partial x}P+P\left(\frac{\partial v_2}{\partial x}\right)^T=0.
\end{aligned}$$

In $1d$, the characteristic polynomial is given by

$$\begin{aligned}\chi(\lambda) =&(v_2-\lambda)\left[(v_1-\lambda)^2(v_2-\lambda)^2-\rho_1 a_1(v_2-\lambda)^2\right.\\ &\left.-\rho_2\left[a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right](v_1-\lambda)^2+\rho_1\rho_2\left[a_1\left(a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]\right].\end{aligned}$$

Let us look at the case $v_1=v_2=v$. If we denote $X:=(v-\lambda)^2$, we get that

$$\begin{aligned}\frac{\chi(\lambda)}{v-\lambda} =&X^2-\left(\rho_1 a_1+\rho_2 a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)X+\rho_1\rho_2\left[a_1\left(a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]\\ =&: Q(X).\end{aligned}$$

The discriminant of $Q(X)$ is

$$\begin{aligned}\Delta :=&\left(\rho_1 a_1+\rho_2 a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)^2-4\rho_1\rho_2\left[a_1\left(a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2\right]\\ =&\left(\rho_1 a_1-\rho_2 a_2-(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)^2+4\rho_1\rho_2\left(b+\frac{P}{\rho_2}\right)^2>0.\end{aligned}$$

Hence Q has two real roots. This gives four roots for χ. In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$0<a_1\left(a_2+(\rho_1+3\rho_2)\frac{P}{\rho_2^2}\right)-\left(b+\frac{P}{\rho_2}\right)^2=\frac{P}{\rho_2^2}\left[(\rho_1+3\rho_2)a_1-2\rho_2 b-P\right].$$

Hence there are some admissible values of a_1, b, P such that χ has complex roots.

4.3.3. *Case* $(r,u)=(\rho_1,v)$. We obtain the following system:

$$\begin{aligned}&\partial_t\rho_i+\operatorname{div}(\rho_i v_i)=0,\qquad i=1,2,\\ &\partial_t v_1+(v_1\cdot\nabla)v_1+\operatorname{curl}(v_2)\wedge(v-v_1)+\nabla(U_1)+\frac{\rho_2}{2\rho}\nabla(\operatorname{Tr}P)+\frac{1}{\rho}\operatorname{div}(\rho_1 P)^T=0,\\ &\partial_t v_2+(v_2\cdot\nabla)v_2+\operatorname{curl}(v_2)\wedge(v-v_2)+\nabla(U_2)-\frac{\rho_1}{2\rho}\nabla(\operatorname{Tr}P)+\frac{1}{\rho}\operatorname{div}(\rho_1 P)^T=0,\\ &\partial_t P+(v\cdot\nabla)P+\frac{\partial v}{\partial x}P+P\left(\frac{\partial v}{\partial x}\right)^T=0.\end{aligned}$$

In $1d$, the characteristic polynomial is given by

$$\begin{aligned}\chi(\lambda)=&\left[(v_1-\lambda)^2(v_2-\lambda)^2-\rho_1\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)(v_2-\lambda)^2-\rho_2\left(a_2+\rho_1\frac{P}{\rho^2}\right)(v_1-\lambda)^2\right.\\ &\left.+\rho_1\rho_2\left[\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)\left(a_2+\rho_1\frac{P}{\rho^2}\right)-\left(b+(2\rho_1+\rho_2)\frac{P}{\rho^2}\right)^2\right]\right](v-\lambda).\end{aligned}$$

Let us look at the case $v_1=v_2=v$. If we denote $X:=(v-\lambda)^2$, we get that

$$\begin{aligned}\frac{\chi(\lambda)}{v-\lambda} =&X^2-\left[\rho_1\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)+\rho_2\left(a_2+\rho_1\frac{P}{\rho^2}\right)\right]X\\ &+\rho_1\rho_2\left[\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)\left(a_2+\rho_1\frac{P}{\rho^2}\right)-\left(b+(2\rho_1+\rho_2)\frac{P}{\rho^2}\right)^2\right]=:Q(X).\end{aligned}$$

The discriminant of $Q(X)$ is

$$\begin{aligned}\Delta :=& \left[\rho_1\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)+\rho_2\left(a_2+\rho_1\frac{P}{\rho^2}\right)\right]^2\\ &-4\rho_1\rho_2\left[\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)\left(a_2+\rho_1\frac{P}{\rho^2}\right)-\left(b+(2\rho_1+\rho_2)\frac{P}{\rho^2}\right)^2\right]\\ =&\left[\rho_1\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)-\rho_2\left(a_2+\rho_1\frac{P}{\rho^2}\right)\right]^2+4\rho_1\rho_2\left(b+(2\rho_1+\rho_2)\frac{P}{\rho^2}\right)^2>0.\end{aligned}$$

Hence Q has two real roots. This gives four roots for χ. In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$\begin{aligned}0<&\left(a_1+3\frac{\rho_1}{\rho^2}P+2\frac{\rho_2}{\rho^2}P\right)\left(a_2+\rho_1\frac{P}{\rho^2}\right)-\left(b+(2\rho_1+\rho_2)\frac{P}{\rho^2}\right)^2\\ <&\frac{P}{\rho^2}\left(\rho_1(a_1+a_2-2b)+2\rho(a_2-b)-P\right).\end{aligned}$$

Hence there are some admissible values of a_1, a_2, b, P such that χ has complex roots.

□

Acknowledgments

The author would like to thank Charlotte Perrin and Sergey Gavrilyuk for their careful reading and suggestions.

Funding

The author was supported by iMPT and INRAE.

Appendix A. Computation of the characteristic polynomial

We compute here the characteristic polynomial of the matrix $A(Y,\xi)$ given at the beginning of Section 3, in order to get the eigenvalues of this matrix. We first state two lemmas which will be useful in the proof.

Lemma 1. *For any $\xi\in\mathbb{R}^d$, one has $C(\xi)D(\xi)=(\xi^TP\xi)I_d+P\xi\xi^T$, where $C(\xi)$ and $D(\xi)$ are the matrices defined in the beginning of Section 3.*

The proof of this lemma can be obtained by a direct computation. The second lemma is also elementary:

Lemma 2. *Let $x,y\in\mathbb{R}^d$. Consider the matrix $B:=I_d+xy^T$. Then the determinant of B is given by $\det(B)=1+x\cdot y$. Furthermore, when $\det(B)\neq 0$,*

$$B^{-1}=I_d-\frac{1}{1+x\cdot y}xy^T.$$

Proof. By density and continuity of the determinant, we can restrict ourselves to the case $x\cdot y\neq 0$. We see that the linear space $y^\perp$ is a subspace of the eigenspace of B associated to the eigenvalue 1, of dimension $d-1$. Furthermore, x is an eigenvector of B for the eigenvalue $1+x\cdot y$. Hence B is diagonalizable and $\det(B)=1+x\cdot y$. The formula for B^{-1} can be checked by computing the product BB^{-1}. □

We will proceed by using the Gaussian elimination method to obtain an upper block triangular matrix. Since the matrix $A(Y,\xi)$ is quite large, we will perform the operations on the blocks rather than the coefficients. When doing this, one has to be careful that blocks generally do not commute with each other. Hence it is important to keep in mind that operations on rows of blocks are made through left multiplication, while operations on block columns are performed through right multiplication. By definition, the characteristic polynomial of $A(Y,\xi)$ is given by

$$\chi_{A(Y,\xi)}(\lambda) := \begin{vmatrix} v_2\cdot\xi-\lambda & \rho_2\xi^T & 0 & 0 & 0 \\ \alpha\xi & (v\cdot\xi-\lambda)I_d+\xi(v_2-v)^T & (\beta+P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & v\cdot\xi-\lambda & \rho\xi^T & 0 \\ \delta\xi & 0 & (\gamma+P/\rho)\xi & (v\cdot\xi-\lambda)I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v\cdot\xi-\lambda)I_d \end{vmatrix}.$$

We multiply the last block column by $-(v\cdot\xi-\lambda)^{-1}D(\xi)$ and add it to the fourth block column. By Lemma 1, the characteristic polynomial is equal to:

$$\chi_{A(Y,\xi)}(\lambda) = \begin{vmatrix} v_2\cdot\xi-\lambda & \rho_2\xi^T & 0 & 0 & 0 \\ \alpha\xi & (v\cdot\xi-\lambda)I_d+\xi(v_2-v)^T & (\beta+P/\rho)\xi & N_0 & C(\xi) \\ 0 & 0 & v\cdot\xi-\lambda & \rho\xi^T & 0 \\ \delta\xi & 0 & (\gamma+P/\rho)\xi & M_0 & C(\xi) \\ 0 & 0 & 0 & 0 & (v\cdot\xi-\lambda)I_d \end{vmatrix}. \tag{A.1}$$

where we defined

$$M_0 := \frac{1}{v\cdot\xi-\lambda}\left([(v\cdot\xi-\lambda)^2-\xi^TP\xi]I_d - P\xi\xi^T\right),$$

and

$$N_0 := \frac{-1}{v\cdot\xi-\lambda}\left[(\xi^TP\xi)I_d + P\xi\xi^T\right].$$

We now multiply the third block column of (A.1) by $-\rho\xi^T/(v\cdot\xi-\lambda)$ and add it to the fourth block column. We obtain that

$$\chi_{A(Y,\xi)}(\lambda) = \begin{vmatrix} v_2\cdot\xi-\lambda & \rho_2\xi^T & 0 & 0 & 0 \\ \alpha\xi & (v\cdot\xi-\lambda)I_d+\xi(v_2-v)^T & (\beta+P/\rho)\xi & N_1 & C(\xi) \\ 0 & 0 & v\cdot\xi-\lambda & 0 & 0 \\ \delta\xi & 0 & (\gamma+P/\rho)\xi & M_1 & C(\xi) \\ 0 & 0 & 0 & 0 & (v\cdot\xi-\lambda)I_d \end{vmatrix}, \tag{A.2}$$

where

$$N_1 := \frac{-1}{v\cdot\xi-\lambda}\left[(\xi^TP\xi)I_d + 2P\xi\xi^T + \rho\beta\xi\xi^T\right], \tag{A.3}$$

and

$$\begin{aligned} M_1 :=& \frac{1}{v\cdot\xi-\lambda}\left(\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]I_d - 2P\xi\xi^T - \rho\gamma\xi\xi^T\right) \\ =& \frac{(v\cdot\xi-\lambda)^2-\xi^TP\xi}{v\cdot\xi-\lambda}\left[I_d - \frac{1}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\left(2P+\rho\gamma\right)\xi\xi^T\right]. \end{aligned}$$

By Lemma 2, it follows that

$$M_1^{-1} = \frac{v\cdot\xi-\lambda}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\left[I_d+\frac{1}{1-\frac{2\xi^TP\xi+\rho\gamma}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}}\frac{1}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}(2P+\rho\gamma)\xi\xi^T\right]$$

$$= \frac{v\cdot\xi-\lambda}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\left[I_d+\frac{1}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}(2P+\rho\gamma)\xi\xi^T\right] \tag{A.4}$$

(recall that $\xi^T\xi=1$). We now multiply the fourth block column of (A.2) by $-M_1^{-1}\delta\xi$ to the right and add it to the first block column. We obtain that the charactersitic polynomial is equal to the determinant of an upper block triangular matrix:

$$\chi_{A(Y,\xi)}(\lambda) = \begin{vmatrix} B_1 & B_2 \\ 0 & B_3 \end{vmatrix} = |B_1||B_3|, \tag{A.5}$$

where

$$B_1 := \begin{pmatrix} v_2\cdot\xi-\lambda & \rho_2\xi^T \\ \alpha\xi - N_1M_1^{-1}\delta\xi & (v\cdot\xi-\lambda)I_d+\xi(v_2-v)^T \end{pmatrix},$$

$$B_2 := \begin{pmatrix} 0 & 0 & 0 \\ (\beta+P/\rho)\xi & N_1 & C(\xi) \end{pmatrix},$$

and

$$B_3 := \begin{pmatrix} v\cdot\xi-\lambda & 0 & 0 \\ (\gamma+P/\rho)\xi & M_1 & C(\xi) \\ 0 & 0 & (v\cdot\xi-\lambda)I_d \end{pmatrix}.$$

In view of (A.5), we need to compute the determinants of the matrices B_1 and B_3. Since B_3 is block triangular, its determinant is equal to the product of the determinants of each diagonal block. By Lemma 2,

$$\begin{aligned} |B_3| =& (v\cdot\xi-\lambda)|M_1||(v\cdot\xi-\lambda)I_d| \\ =& (v\cdot\xi-\lambda)\left[\frac{(v\cdot\xi-\lambda)^2-\xi^TP\xi}{v\cdot\xi-\lambda}\right]^d\left(1-\frac{2\xi^TP\xi+\rho\gamma}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\right)(v\cdot\xi-\lambda)^{d(d+1)/2} \\ =& (v\cdot\xi-\lambda)^{d(d-1)/2+1}\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]^{d-1}\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]. \end{aligned}$$

We now compute the determinant of B_1. Let us define

$$E := (v\cdot\xi-\lambda)I_d+\xi(v_2-v)^T = (v\cdot\xi-\lambda)\left[I_d+\frac{1}{v\cdot\xi-\lambda}\xi(v_2-v)^T\right]. \tag{A.6}$$

By Lemma 2,

$$E^{-1} = \frac{1}{v\cdot\xi-\lambda}\left[I_d-\frac{1}{1+\frac{v_2\cdot\xi-v\cdot\xi}{v\cdot\xi-\lambda}}\frac{1}{v\cdot\xi-\lambda}\xi(v_2-v)^T\right] = \frac{1}{v\cdot\xi-\lambda}\left[I_d-\frac{1}{v_2\cdot\xi-\lambda}\xi(v_2-v)^T\right]. \tag{A.7}$$

We now multiply the second block row of B_1 by $-\rho_2\xi^TE^{-1}$ and add it to the first block row. We obtain that

$$\begin{aligned} |B_1| =& \begin{vmatrix} v_2\cdot\xi-\lambda-\rho_2\xi^TE^{-1}(\alpha\xi-N_1M_1^{-1}\delta\xi) & 0 \\ \alpha\xi-N_1M_1^{-1}\delta\xi & E \end{vmatrix} \\ =& \left[v_2\cdot\xi-\lambda-\rho_2\xi^TE^{-1}(\alpha\xi-N_1M_1^{-1}\delta\xi)\right]|E|. \end{aligned}$$

By Lemma 2,

$$|E| = (v\cdot\xi-\lambda)^d\left(1+\frac{v_2\cdot\xi-v\cdot\xi}{v\cdot\xi-\lambda}\right) = (v_2\cdot\xi-\lambda)(v\cdot\xi-\lambda)^{d-1}.$$

We are left to compute the scalar $\left[v_2\cdot\xi-\lambda-\rho_2\xi^TE^{-1}(\alpha\xi-N_1M_1^{-1}\delta\xi)\right]$. First, by Equation (A.7),

$$\xi^TE^{-1}=\frac{1}{v\cdot\xi-\lambda}\xi^T-\frac{1}{(v\cdot\xi-\lambda)(v_2\cdot\xi-\lambda)}(v_2-v)^T.$$

It follows that

$$\xi^TE^{-1}\xi=\frac{1}{v_2\cdot\xi-\lambda}.$$

On the other hand, Equation (A.4) gives that

$$M_1^{-1}\xi=\frac{v\cdot\xi-\lambda}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\left[1+\frac{1}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}(2P+\rho\gamma)\right]\xi.$$

Equation (A.3) then yields

$$\begin{aligned}N_1M_1^{-1}\xi=&\frac{-1}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\left[\xi^TP\xi+\frac{\xi^TP\xi}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}(2P+\rho\gamma)\right.\\&\left.+(2P+\rho\beta)\left(1+\frac{2\xi^TP\xi+\rho\gamma}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}\right)\right]\xi\\=&\frac{-1}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}\left[\xi^TP\xi\frac{(v\cdot\xi-\lambda)^2-3\xi^TP\xi}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}+\frac{2(v\cdot\xi-\lambda)^2}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}P+\rho\beta\right]\xi\\=&\frac{-1}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}\left[\xi^TP\xi+\rho\beta+2\frac{(v\cdot\xi-\lambda)^2P-(\xi^TP\xi)^2}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\right]\xi.\end{aligned}\tag{A.8}$$

Hence

$$\begin{aligned}&\xi^TE^{-1}N_1M_1^{-1}\xi\\=&\frac{-1}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}\left[\frac{\xi^TP\xi+\rho\beta}{v_2\cdot\xi-\lambda}+\frac{2\xi^TP\xi}{v\cdot\xi-\lambda}\right.\\&\left.-2\frac{(v\cdot\xi-\lambda)^2(v_2-v)^TP\xi-(\xi^TP\xi)^2(v_2-v)^T\xi}{(v\cdot\xi-\lambda)(v_2\cdot\xi-\lambda)[(v\cdot\xi-\lambda)^2-\xi^TP\xi]}\right]\\=&\frac{-1}{[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma](v_2\cdot\xi-\lambda)}\left[3\xi^TP\xi+\rho\beta+\frac{2(v\cdot\xi-\lambda)(v_2-v)^T\left[(\xi^TP\xi)-P\right]\xi}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\right],\end{aligned}$$

and

$$\begin{aligned}&v_2\cdot\xi-\lambda-\rho_2\xi^TE^{-1}(\alpha\xi-N_1M_1^{-1}\delta\xi)\\=&v_2\cdot\xi-\lambda-\frac{\alpha\rho_2}{v_2\cdot\xi-\lambda}\\&-\frac{\rho_2\delta}{[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma](v_2\cdot\xi-\lambda)}\left[3\xi^TP\xi+\rho\beta+\frac{2(v\cdot\xi-\lambda)(v_2-v)^T\left[(\xi^TP\xi)-P\right]\xi}{(v\cdot\xi-\lambda)^2-\xi^TP\xi}\right]\\=&\frac{\left[(v_2\cdot\xi-\lambda)^2-\alpha\rho_2\right]\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]-\rho_2\delta\left(3\xi^TP\xi+\rho\beta\right)}{(v_2\cdot\xi-\lambda)\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]}\\&+\frac{2\rho_2\delta(v\cdot\xi-\lambda)(v_2-v)^T\left[P-(\xi^TP\xi)\right]\xi}{(v_2\cdot\xi-\lambda)\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]}.\end{aligned}$$

Finally,

$$\begin{aligned}\chi_{A(Y,\xi)}(\lambda)=&(v\cdot\xi-\lambda)^{(d+2)(d-1)/2+1}\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]^{d-2}\\&\times\left[2\rho_2\delta(v\cdot\xi-\lambda)(v_2-v)^T\left[P-(\xi^TP\xi)\right]\xi+\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]q(\lambda)\right],\end{aligned}$$

where

$$q(\lambda) =: \left[(v_2\cdot\xi-\lambda)^2-\alpha\rho_2\right]\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]-\rho_2\delta\left(3\xi^TP\xi+\rho\beta\right),$$

as claimed in Proposition 1.

Case of the system without lift terms. When there are no lift terms the previous computations can be adapted to compute the characteristic polynomial of the modified system. Let us also denote $A'(Y,\xi)$ the matrix of System (1.16). It is given by

$$A'(Y,\xi)=\begin{pmatrix} v_2\cdot\xi & \rho_2\xi^T & 0 & 0 & 0\\ \alpha\xi & (v_2\cdot\xi)I_d & (\beta+P/\rho)\xi & 0 & C(\xi)\\ 0 & 0 & v\cdot\xi & \rho\xi^T & 0\\ \delta\xi & 0 & (\gamma+P/\rho)\xi & (v\cdot\xi)I_d & C(\xi)\\ 0 & 0 & 0 & D(\xi) & (v\cdot\xi)I_d \end{pmatrix}.$$

Thus the only difference is that one should replace the matrix E of (A.6) by

$$E' := (v_2\cdot\xi-\lambda)I_d,\quad \text{hence}\quad E'^{-1}=\frac{1}{v_2\cdot\xi-\lambda}I_d.$$

All computations of the proof stay unchanged until Equation (A.6). We deduce as in (A.5) that

$$\chi_{A'(Y,\xi)}=|B_1||B_3|,$$

with

$$|B_1|=\left[v_2\cdot\xi-\lambda-\rho_2\xi^TE'^{-1}(\alpha\xi-N_1M_1^{-1}\delta\xi)\right]|E'|$$

and

$$|B_3|=(v\cdot\xi-\lambda)^{d(d-1)/2+1}\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]^{d-1}\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]$$

as previously, substituting E' to E. We compute that

$$|E'|=(v_2\cdot\xi-\lambda)^d,\qquad \xi^TE'^{-1}\xi=\frac{1}{v_2\cdot\xi-\lambda}$$

and , with the help of (A.8),

$$\xi^TE'^{-1}N_1M_1\xi=-\frac{3\xi^TP\xi+\rho\beta}{\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right](v_2\cdot\xi-\lambda)}.$$

Hence

$$|B_1|=(v_2\cdot\xi-\lambda)^{d-1}\frac{\left[(v_2\cdot\xi-\lambda)^2-\rho_2\alpha\right]\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]-\rho_2\delta[3\xi^TP\xi+\rho\beta]}{(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma}.$$

Finally, the characteristic polynomial is equal to

$$\chi_{A'(Y,\xi)}=(v_2\cdot\xi-\lambda)^{d-1}(v\cdot\xi-\lambda)^{d(d-1)/2+1}\left[(v\cdot\xi-\lambda)^2-\xi^TP\xi\right]^{d-1}q(\lambda),$$

where

$$q(\lambda):=\left[(v_2\cdot\xi-\lambda)^2-\rho_2\alpha\right]\left[(v\cdot\xi-\lambda)^2-3\xi^TP\xi-\rho\gamma\right]-\rho_2\delta[3\xi^TP\xi+\rho\beta].$$

References

[1] P. G. Baines. *Topographic effects in stratified flows*. Cambridge university press, 2022.

[2] S. Benzoni-Gavage and D. Serre. *Multi-dimensional hyperbolic partial differential equations: First-order Systems and Applications*. OUP Oxford, 2006.

[3] C. Berthon, F. Coquel, J.-M. Hérard, and M. Uhlmann. An approximate solution of the riemann problem for a realisable second-moment turbulent closure. *Shock Waves*, 11:245–269, 2002.

[4] D. Bresch, C. Burtea, and F. Lagoutière. Mathematical justification of a compressible bifluid system with different pressure laws: a continuous approach. *Applicable Analysis*, 101(12):4235–4266, 2022.

[5] D. Bresch, G. Narbona-Reina, A. Burgisser, and M. Collombet. Mathematical topics in compressible flows from single-phase systems to two-phase averaged systems. *Studies in Applied Mathematics*, page e12739, 2024.

[6] A. Burgisser, M. Collombet, G. Narbona-Reina, and D. Bresch. Two-phase magma flow with phase exchange: Part ii. 1.5 d numerical simulations of a volcanic conduit. *Studies in Applied Mathematics*, page e12747, 2024.

[7] C. Burtea, S. Gavrilyuk, and C. Perrin. Hamilton's principle of stationary action in multiphase flow modeling. working paper or preprint, 2021.

[8] L. Chupin and T. Dubois. Non-isochoric stable granular models taking into account fluidisation by pore gas pressure. *Journal of Fluid Mechanics*, 979:A14, 2024.

[9] F. Coquel, J.-M. Hérard, K. Saleh, and N. Seguin. Two properties of two-velocity two-pressure models for two-phase flows. *Communications in Mathematical Sciences*, 12(3), 2014.

[10] É. Deléage. Well-posedness of reynolds averaged equations for compressible fluids with a vanishing pressure. *Mathematical Methods in the Applied Sciences*, 47(2):817–824, 2024.

[11] J. Dong, K. Inthavong, and J. Tu. Multiphase flows in biomedical applications. In *Handbook of multiphase flow science and technology*, pages 673–696. Springer, 2023.

[12] D. A. Drew and S. L. Passman. *Theory of multicomponent fluids*, volume 135. Springer Science & Business Media, 2006.

[13] F. Drui, A. Larat, S. Kokh, and M. Massot. Small-scale kinematics of two-phase flows: identifying relaxation processes in separated-and disperse-phase flow models. *Journal of Fluid Mechanics*, 876:326–355, 2019.

[14] S. Gavrilyuk. *Multiphase flow modeling via Hamilton's principle*. Springer, 2011.

[15] S. Gavrilyuk. 'Uncertainty' principle in two fluid-mechanics. *ESAIM: proceedings and surveys*, 69:47–55, 2020.

[16] S. Gavrilyuk and H. Gouin. Variational formulation for models of shear shallow water flows and ideal turbulence. *International Journal of Non-Linear Mechanics*, 2019.

[17] S. Gavrilyuk and R. Saurel. Mathematical and numerical modeling of two-phase compressible flows with micro-inertia. *Journal of Computational Physics*, 175(1):326–360, 2002.

[18] J. Giesselmann, C. Lattanzio, and A. E. Tzavaras. Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics. *Archive for Rational Mechanics and Analysis*, 223:1427–1484, 2017.

[19] J. Gray and C. Ancey. Multi-component particle-size segregation in shallow granular avalanches. *Journal of Fluid Mechanics*, 678:535–588, 2011.

[20] J. Jung, A. Hassanein, and R. W. Lyczkowski. Hemodynamic computation using multiphase flow dynamics in a right coronary artery. *Annals of biomedical engineering*, 34:393–407, 2006.

[21] A. Lefebvre and A. Lefebvre-Lepot. *Modélisation numérique d'écoulements fluide/particules*. PhD thesis, Université Paris Sud-Paris XI, 2007.

[22] D. Mitrović and M. Nedeljkov. Delta shock waves as a limit of shock waves. *Journal of Hyperbolic Differential Equations*, 4(04):629–653, 2007.

[23] B. Mohammadi and O. Pironneau. Analysis of the k-epsilon turbulence model. 1993.

[24] G. Narbona-Reina, D. Bresch, A. Burgisser, and M. Collombet. Two-phase magma flow with phase exchange: Part i. physical modeling of a volcanic conduit. *Studies in Applied Mathematics*, page e12741, 2024.

[25] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:235–257, 1918.

[26] S. B. Pope. Turbulent flows. *Measurement Science and Technology*, 12(11):2020–2021, 2001.

[27] K. Saleh and N. Seguin. Some mathematical properties of a barotropic multiphase flow model. *ESAIM: Proceedings and surveys*, 69:70–78, 2020.

[28] V. M. Teshukov. Gas-dynamic analogy for vortex free-boundary flows. *Journal of Applied Mechanics and Technical Physics*, 48:303–309, 2007.

[29] G. K. Vallis. *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 2017.

[30] D. C. Wilcox et al. *Turbulence modeling for CFD*, volume 2. DCW industries La Canada, CA, 1998.

[1]Aix Marseille Univ., CNRS, Centrale Marseille, I2M UMR CNRS 7373, 13000 Marseille, France

[2]Univ. Grenoble Alpes, CNRS, INRAE, IRD, Grenoble INP, IGE, 38000 Grenoble, France
Email address: emile.deleage@univ-amu.fr