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► **To cite this version:**

Camille Carvalho, Chrysoula Tsogka, Stéphanie Chaillat, Elsie A Cortes. Fast and accurate boundary integral equation methods for the multi-layer transmission problem. Proceedings of The 16th International Conference on Mathematical and Numerical Aspects of Wave Propagation., Jun 2024, Berlin, Germany. 10.17617/3.MBE4AA . hal-04785286

HAL Id: hal-04785286

<https://hal.science/hal-04785286v1>

Submitted on 15 Nov 2024

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Fast and accurate boundary integral equation methods for the multi-layer transmission problem

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Abstract

We consider a multi-layer transmission problem, which can be used for example to describe the light scattering in meta-materials (assemblings of various concentric penetrable materials). Our goal is to solve the multi-layer problem accurately with optimal discretization. Generally, the costs to solve this problem grow as more layers are introduced - solving this problem is thus particularly challenging for 3D models. For this reason, we use boundary integral equation (BIE) methods: they reduce the dimensionality of the problem and can provide high order accuracy. However, BIE methods suffer from the so-called close evaluation problem. We address it using modified representations. We further examine how to improve the speed of our method by optimizing the accuracy over number of discretization points ratio. In particular, we investigate whether the usual rule of thumb to mesh interfaces, based on the most constraining material, is necessary for the multi-layer transmission problem.

Keywords: boundary integral methods, close evaluation, multi-layered media, mesh adaptation, time-harmonic acoustic scattering

1 Problem Setting

We seek to describe how waves propagate through a medium of N concentric material layers L_i , with smooth boundary Γ_i , characterized by some wavenumber $k_i \in \mathbb{C}$, $i = 0, \dots, N-1$ (this wavenumber depends on the speed of light in the layer, therefore it contains the material properties of the layer). The light scattering by a plane wave $u^{in} = e^{ik_0 \vec{\alpha} \cdot \vec{x}}$, with wavenumber $k_0 > 0$ and angle $\alpha \in [0, 2\pi]$ in this medium leads to solving the transmission problem: $\Delta u_j + k_j^2 u_j = 0$ in $\dot{L}_j := L_j \setminus \{\Gamma_{j-1} \cup \Gamma_j\}$, $u_{j+1} = u_j$ on Γ_j , $\partial_{n_j} u_{j+1} = \beta_j \partial_{n_j} u_j$ on Γ_j , where u_j is the to-

tal field solution in the j th layer L_j , n_j is the outward unit normal on the boundary Γ_j , and $\beta_j := \frac{k_j^2}{k_{j+1}^2}$ denotes the *contrast* between adjacent layers L_j, L_{j+1} . Note that the Sommerfeld radiation condition needs to be satisfied at infinity (in L_0).

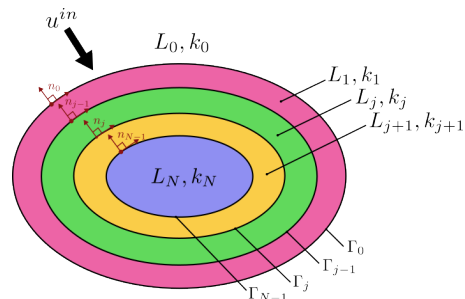


Figure 1: Schematic and notations of the problem.

2 Boundary Integral Equation System

We use boundary integral equation (BIE) methods to solve the transmission problem. This allows to reduce the problem by one dimension. Additionally, using representation formulae one can evaluate the field everywhere within the layers once the boundary data (trace u_i and normal trace $\partial_{n_i} u_i$ of the field on Γ_i , $i = 0, \dots, N-1$) is known. This leads to solving the following BIE system:

$$\begin{aligned}
 \left(\frac{1}{2} - D_{00}\right)[u_0] + S_{00}[\partial_{n_0} u_0] &= u^{in} \text{ on } \Gamma_0, \\
 D_{j-1,j}[u_{j-1}] + \beta_{j-1} S_{j-1,j}[\partial_{n_{j-1}} u_{j-1}] \\
 + \left(\frac{1}{2} - D_{jj}\right)[u_j] + S_{j,j}[\partial_{n_j} u_j] &= 0 \text{ on } \Gamma_j, \\
 D_{N-1,N}[u_{N-1}] - \beta_{N-1} S_{N-1,N}[\partial_{n_{N-1}} u_{N-1}] \\
 + u_N &= 0 \text{ on } \Gamma_{N-1}.
 \end{aligned} \tag{1}$$

Above, $D_{i,j}$, $S_{i,j}$ represent the double-layer potential and single-layer potential, respectively, defined on Γ_i for $i = j-1, j$, evaluated on Γ_j :

$$D_{i,j}[u_j](x) = \int_{\Gamma_i} \frac{\partial \Phi_j}{\partial n_i}(x, y) u_j(y) d\sigma_y, \quad x \in \Gamma_j$$

$$S_{i,j}[\partial_{n_i} u_j](x) = \int_{\Gamma_i} \Phi_j(x, y) \partial_{n_i} u_j(y) d\sigma_y, \quad x \in \Gamma_j$$

with the fundamental solution to the Helmholtz equation $\Phi_j(x, y) := \frac{i}{4} H_0^{(1)}(k_j |x - y|)$ with H_0^1 denoting the Hankel function of first kind.

Discretizing (1) requires to pay attention to the following: (i) layer potentials $D_{i,i}, S_{i,i}$ are weakly singular integrals (singularities appear when $x = y$ in the associated kernel), (ii) $D_{i-1,i}, S_{i-1,i}$ are nearly singular integrals (especially when the layers are thin), and (iii) the use of high contrasts β_i may require we solve larger linear systems (more quadrature points). We will use the Kress product rule with Nyström's method [1] to address (i), classically used to tackle weakly singular integrals with log singularities. We present below how we address (ii) and (iii).

3 Treating Close Evaluation Error

We treat the close evaluation problem where it occurs with the boundary regularized integral equation formulation (BRIEF) method [2]. This method relies on modified representations of layer potentials using auxiliary functions solutions of the equation. Given an evaluation point x_i^b on Γ_i , we define ψ_j , $j \in \llbracket 0, N - 1 \rrbracket$:

$$\psi_j(x) = u_j(x_i^b) g_j(x) + \partial_{n_i} u_j(x_i^b) f_j(x), \quad x \in \Gamma_j$$

with g_j and f_j satisfying the Helmholtz equation associated to k_j in \dot{L}_j , with the boundary conditions $g_j(x_i^b) = 1$, $\partial_{n_i} g_j(x_i^b) = 0$, $f_j(x_i^b) = 0$, $\partial_{n_i} f_j(x_i^b) = 1$.

We consider the case when two adjacent boundaries are sufficiently close together such that they impose this nearly-singular behavior on each other. Then for boundaries Γ_{j-1} and Γ_j , x_{j-1}^b is the corresponding closest point to x_j^b and vice versa. Then we modify (1) so that the boundary data satisfies on Γ_j ,

$$\begin{aligned} \frac{1}{2} u_j(x_j^b) &= \frac{1}{2} \psi_j(x_j^b) - D_{j-1,j} [u_{j-1} - \psi_{j-1}] \\ &\quad + \beta_{j-1} S_{j-1,j} [\partial_{n_{j-1}} u_{j-1} - \partial_{n_{j-1}} \psi_{j-1}] \\ &\quad + D_{j,j} [u_j - \psi_j] - S_{j,j} [\partial_{n_j} u_j - \partial_{n_j} \psi_j] (x_j^b). \end{aligned}$$

The method is also applicable in the layers to treat close evaluation error for $x \in \dot{L}_j$ close to some x_i^b . Figure 2 shows an example of how BRIEF reduces the near-singular error in a 3-layer problem defined by $ks = (k_0, k_1, k_2)$ and circular interfaces of radii $Rs = (R_0, R_1)$.

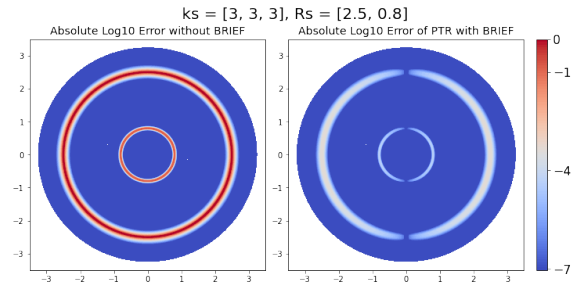


Figure 2: Error in the PTR solution without (left) and with BRIEF (right). Higher orders of error in the solution occur in the regions near the interfaces of radii R_s .

4 Mesh Adaptation

We are interested in developing a grid adaptivity method that optimally discretizes the boundary data such that we can accurately represent the solutions. A general assumption is that the highest wavenumber at each interface should determine how to refine our mesh. Consider for example the two-layer scattering problem (see Figure 3). For contrast $\beta_0 < 1$ (aka $k_1 > k_0$), the general rule typically sets the discretization based on k_1 , in order to capture the highly oscillatory scattered field in the interior. However, since waves are trapped by this configuration, the solution in the exterior layer does not need as much refinement to be accurate. We propose to determine a new, more optimal discretization particular to this problem that depends on the contrasts β_j in the system, characterizing the interactions between successive layers.

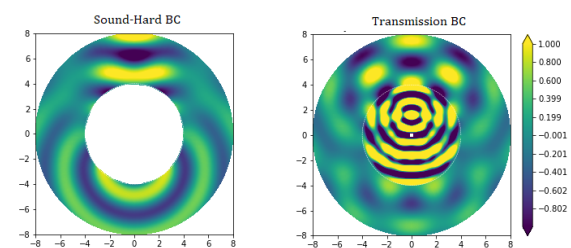


Figure 3: Real part of the scattered field for a two-layer problem with $\beta_0 = \frac{1}{9}$ for sound-hard (left) versus transmission (right) boundary conditions.

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