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Abstract

We consider the scattering of an incident plane wave by a penetrable obstacle characterized by some material parameter $\varepsilon \in \mathbb{C}$. We consider a Trefftz Discontinuous Galerkin (DG) approach, with a basis comprised of plane waves. This approach chooses solutions of the homogeneous Helmholtz equation as basis functions in each cell. This reduces the computational cost by eliminating the volumic portion of the weak form. Additionally, the basis functions are chosen to better mimic the expected behavior of the solutions (compared to standard polynomial approaches). Due to the penetrable scatterer, the behavior of the basis function will change with respect to the parameter ε, leading to challenges especially for the so-called sign-changing problem ($\varepsilon < 0$). We derive the approach and show well-posedness at the discrete level. Numerical examples (with application to surface plasmons for sign-changing problems) will be presented.

Keywords: Helmholtz, Discontinuous Galerkin, Trefftz Method, Scattering, Complex Media

1 Problem setting

For simplicity we consider time-harmonic scattering of a plane wave by a penetrable obstacle $\mathcal{D}$ in a homogeneous background medium in two dimensions. We consider the plane wave $u^{\text{in}} = e^{ikz}$ with wavenumber $k > 0$, $\mathcal{D}$ is a smooth, positively oriented convex scatterer characterized by $\varepsilon = \varepsilon_m \in \mathbb{C}$ (while the background is such that $\varepsilon = 1$). The goal is to accurately solve the transmission problem: Let $\varepsilon(x) = 1$, $x \in E := \mathbb{R}^2 \setminus \bar{\mathcal{D}}$ and $\varepsilon(x) = \varepsilon_m \in (-\infty, -1) \cup (0, +\infty)$, $x \in \mathcal{D}$ denote the piece-wise constant permittivity characterizing this problem. The total field u satisfies the following boundary value problem:

$$\begin{cases} \text{Find } u = u^{\text{in}} + u^{\text{sc}} \in H^1_{\text{loc}}(\mathbb{R}^2) \text{ such that:} \\ \nabla \cdot (\varepsilon^{-1} \nabla u) + k^2 u = 0, \quad \text{in } \mathbb{R}^2 \\ [u]_{\partial\mathcal{D}} = 0, \quad [\varepsilon^{-1} \partial_n u]_{\partial\mathcal{D}} = 0 \\ \lim_{r\to\infty} \sqrt{r}(\partial_r - ik) u^{\text{sc}} = 0 \end{cases} \tag{1}$$

It is well-known that (1) is well-posed for $\varepsilon_m \in \mathbb{C} \setminus \mathbb{R}_-$, and as well as for $\varepsilon_m \in \mathbb{R}_- \setminus \{-1\}$ using the T-coercivity theory [1]. While Finite Element Method (FEM) can be used to approximate (1), in some cases (especially when $\varepsilon_m < -1$) there exist *scattering resonances* close to the real axis impeding FEM to capture, in practice, the solution [3]. This is due to exponential behaviors at the boundary of the scatterer, due to the underlying resonance. Therefore we use the Plane Wave Discontinuous Galerkin Method (PWDG) framework to address this challenge.

2 PWDG Method

In the spirit of [4], we artificially truncate the domain to a disk $\Omega \supset \mathcal{D}$ and impose the radiation condition at $\partial\Omega$ (a Dirichlet-to-Neumann map could also be applied). We consider a mesh triangulation Ω_h, conforming at $\partial\mathcal{D}$ (and locally symmetric to guarantee well-posedness of the discrete problem when $\varepsilon_m \in \mathbb{R}_- \setminus \{-1\}$ [2]). We denote our mesh skeleton $\Gamma_h = \Gamma_h^I \bigcup \Gamma_h^B$ (the union of interior faces and boundary faces). The PWDG is a Trefftz method, it uses in each cell of Ω_h basis functions which are solutions of the given Helmholtz-like problem. As plane waves form a natural solution to the Helmholtz equation, we introduce the space: $PW^p_{k_\varepsilon}(\mathbb{R}^2) = \{v \in C^\infty(\mathbb{R}^2) | v = \sum_{j=1}^p \varphi_j e^{ik_\varepsilon \mathbf{d}_j \cdot \mathbf{x}}, \forall j, \varphi_j \in \mathbb{C}\}$, where $\mathbf{d}_j = (d_x, d_y) \in \mathbb{C}^2$ (with $\sqrt{d_x^2 + d_y^2} = 1$) denotes the incidence and amplitude (it contains both propagative and evanescent plane waves), k_ε is the material wavenumber ($k_\varepsilon = k$ in $\Omega \setminus \bar{\mathcal{D}}$ and $k_\varepsilon = \sqrt{\varepsilon_m} k$ in $\mathcal{D}$). We then introduce the discrete space $V(\Omega_h) = \{v \in L^2(\Omega) : v|_{\mathcal{K}} \in PW^p_{k_\varepsilon}(\mathcal{K}) \quad \forall \mathcal{K} \in \Omega_h\}$. From (1) we derive a discrete weak form on the skeleton Γ_h: for all $v \in V(\Omega_h)$,

$$\int_{\Gamma_h} \overline{v}(n \cdot ik\hat{\sigma}) d\Gamma_h - \int_{\Gamma_h} \varepsilon^{-1} \partial_n \overline{v} \hat{u} \, d\Gamma_h = 0 \tag{2}$$

where $\hat{u}$ and $\hat{\sigma}$ are auxiliary variables representing numerical fluxes in the solution and its gra-

dient. These variables are defined as follows:

$$ik\hat{\sigma} = \begin{cases} \{\{\varepsilon^{-1}\nabla u\}\} - \alpha ik[[u]], & \text{in } \Gamma_h^I \\ \nabla u - (1-\delta)(-\nabla u + iku n + gn), & \text{in } \Gamma_h^B \end{cases}$$

$$\hat{u} = \begin{cases} \{\{u\}\} - \frac{1}{ik}\beta[[\varepsilon^{-1}\nabla u]], & \text{in } \Gamma_h^I \\ u - \delta\left(\frac{1}{ik}\partial_n u - u - \frac{1}{ik}g\right), & \text{in } \Gamma_h^B \end{cases}$$

where the average $\{\{\cdot\}\}$ and jump $[[\cdot]]$ between adjacent cells are defined as

$$\{\{\varphi\}\} = \frac{1}{2}(\varphi^+ + \varphi^-)$$

$$[[\varphi]] = \begin{cases} \varphi^+ n^+ + \varphi^- n^- & \text{if } \varphi \text{ is scalar valued} \\ \varphi^+ \cdot n^+ + \varphi^- \cdot n^- & \text{if } \varphi \text{ is vector valued} \end{cases}$$

with $\alpha, \beta, \delta \in \mathbb{R}^+$ as chosen parameters. Contrary to standard PWDG, the numerical fluxes depend on ε (consistent with the problem's transmission conditions), which may lead to additional difficulties when $\varepsilon_m < 0$. In order to obtain these boundary fluxes, we assume a first order impedance boundary condition ($g = \partial_n u^{\text{in}} - iku^{\text{in}}$ in the numerical experiments). Then one can show that (2) boils down to solving the weak form: *Find* $u \in V(\Omega_h)$ *such that*

$$A(u,v) = f(v) \quad \forall v \in V(\Omega_h) \tag{3}$$

$$\begin{aligned} A(u,v) = &\int_{\Gamma_h^I} [[\overline{v}]] \cdot \{\{\varepsilon^{-1}\nabla u\}\} - \alpha ik[[\overline{v}]] \cdot [[u]] \\ &- [[\varepsilon^{-1}\nabla \overline{v}]]\{\{u\}\} - \frac{i}{k}\beta[[\varepsilon^{-1}\nabla \overline{v}]][[\varepsilon^{-1}\nabla u]]\, d\Gamma_h^I \\ &+ \int_{\Gamma_h^B} -(1-\delta)iku\overline{v} + (2-\delta)\partial_n u\overline{v} \\ &- (1+\delta)u\partial_n\overline{v} - \delta\frac{i}{k}\partial_n u\partial_n\overline{v}\, d\Gamma_h^B \end{aligned}$$

and $f(v) = \int_{\Gamma_h^B} g(\frac{\delta}{ik}\partial_n\overline{v} + (1-\delta)\overline{v})\, d\Gamma_h^B$. We can show that

Proposition 1 *Problem* (3) *is well-posed in* $V(\Omega_h)$.

The scheme of the proof is as follows: show that

$$\begin{aligned} |v|_{DG}^2 = &k^{-1}\left\|\beta^{1/2}[[\varepsilon^{-1}\nabla v]]\right\|_{\Gamma_h^I}^2 + k\left\|\alpha^{1/2}[[v]]\right\|_{\Gamma_h^I}^2 \\ &+ k\left\|(1-\delta)^{1/2}v\right\|_{\Gamma_h^B}^2 + k^{-1}\left\|\delta^{1/2}\partial_n v\right\|_{\Gamma_h^B}^2 \end{aligned}$$

defines a norm, then establish Fredholmness of the bilinear form $A(\cdot,\cdot)$ (or $A(\cdot, \mathsf{T}\cdot)$ for sign-changing ε [2]) over $L^2(\Omega_h) \times L^2(\Omega_h)$ via a Gårding inequality, and Holmgren uniqueness theorem. Error estimates should hold, and stability issues should be overcome as we use also evanescent plane waves [5].

3 Near-resonance Regime

We apply the method to parameters k, ε_m exciting a scattering resonant mode. In this regime, the solution is locally given by high amplitude, rapidly decaying Bessel and Hankel functions near the interface (see Figure 1). Due to this faster than polynomial decay, we propose to use a basis of evanescent plane waves in the Trefftz space chosen to well-approximate the expected decay of these functions [5]. Preliminary numerical experiments are underway and will be presented.

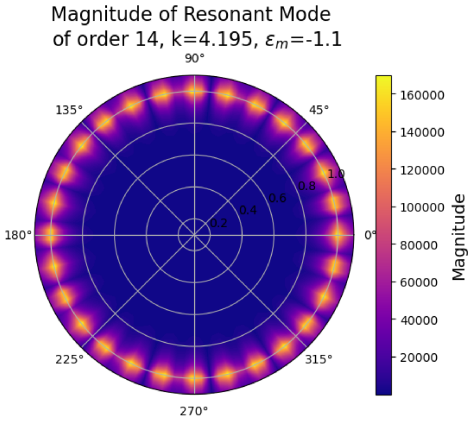

Figure 1: Sample scattering resonant mode.

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