



Stochastic closures for linear wave propagation in turbulence

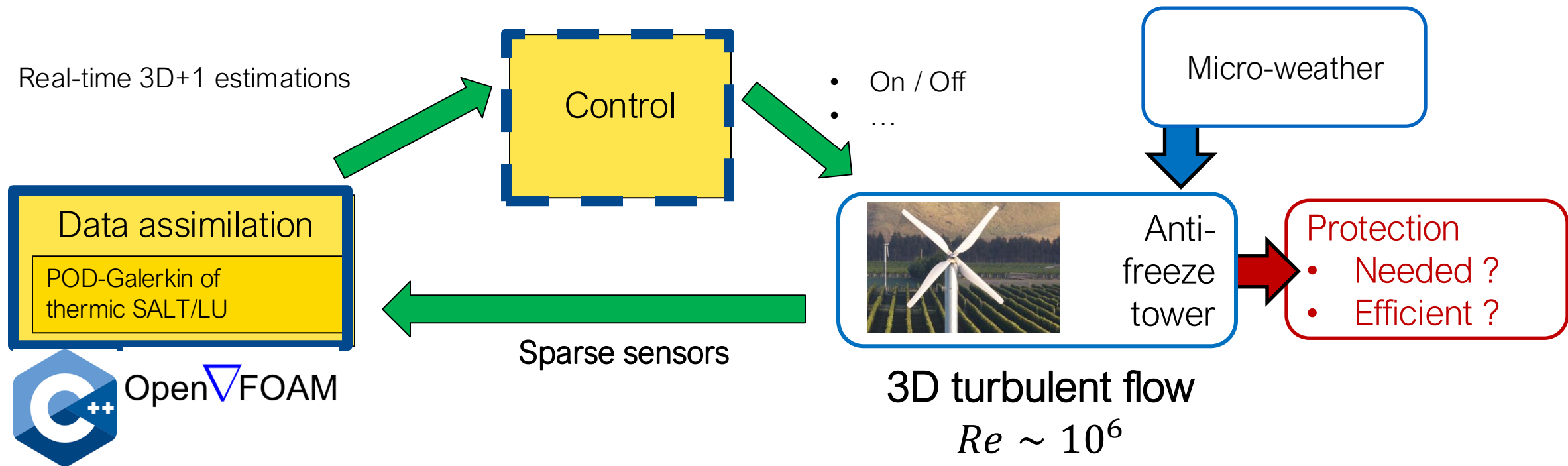
Valentin Resseguier, Erwan Hascoët,
Fabrice Collard, Bertrand Chapron



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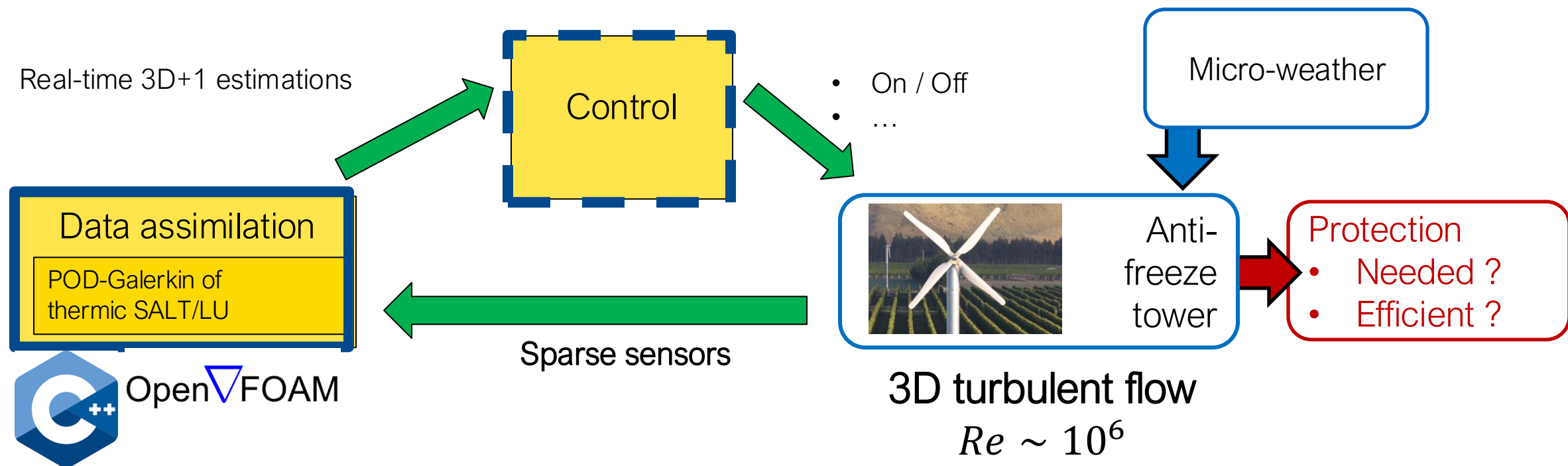
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SCALIAN

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Hiring:

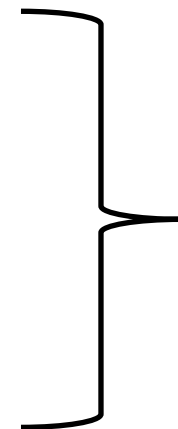
- **1-year postdoc position - INRAE**
- **Permanent research engineer position - SCALIAN**
(computer science)

Content

- I. Wave refraction due to turbulence
- II. Markovian closure
 - II.A. Generalized fast wave approximation
 - II.B. Calibration
 - II.C. Simplification for single ray dynamics
 - II.D. Waves trapped in a jet
- III. Beyond Markovian closure
 - II.A. Waves trapped in a SQG jet
 - II.B. Tests with satellite data
 - II.C. Calibration

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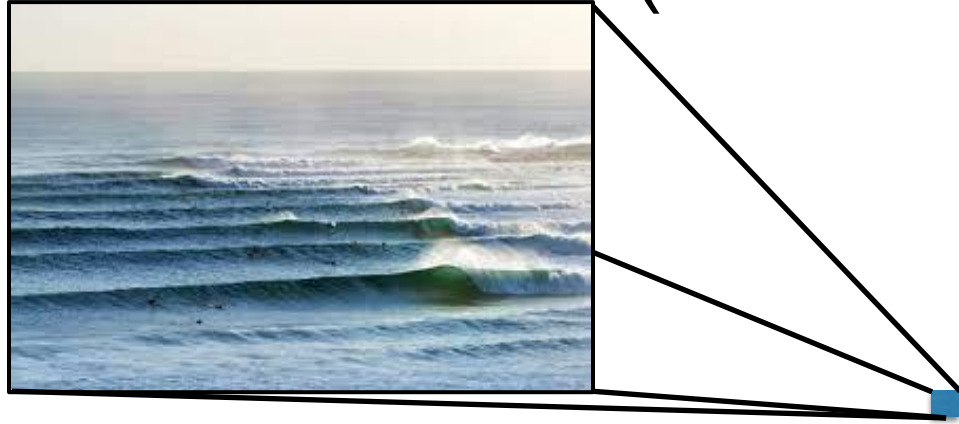


LU / SALT

Resseguier, Hascoet, Chapron (2024). JFM

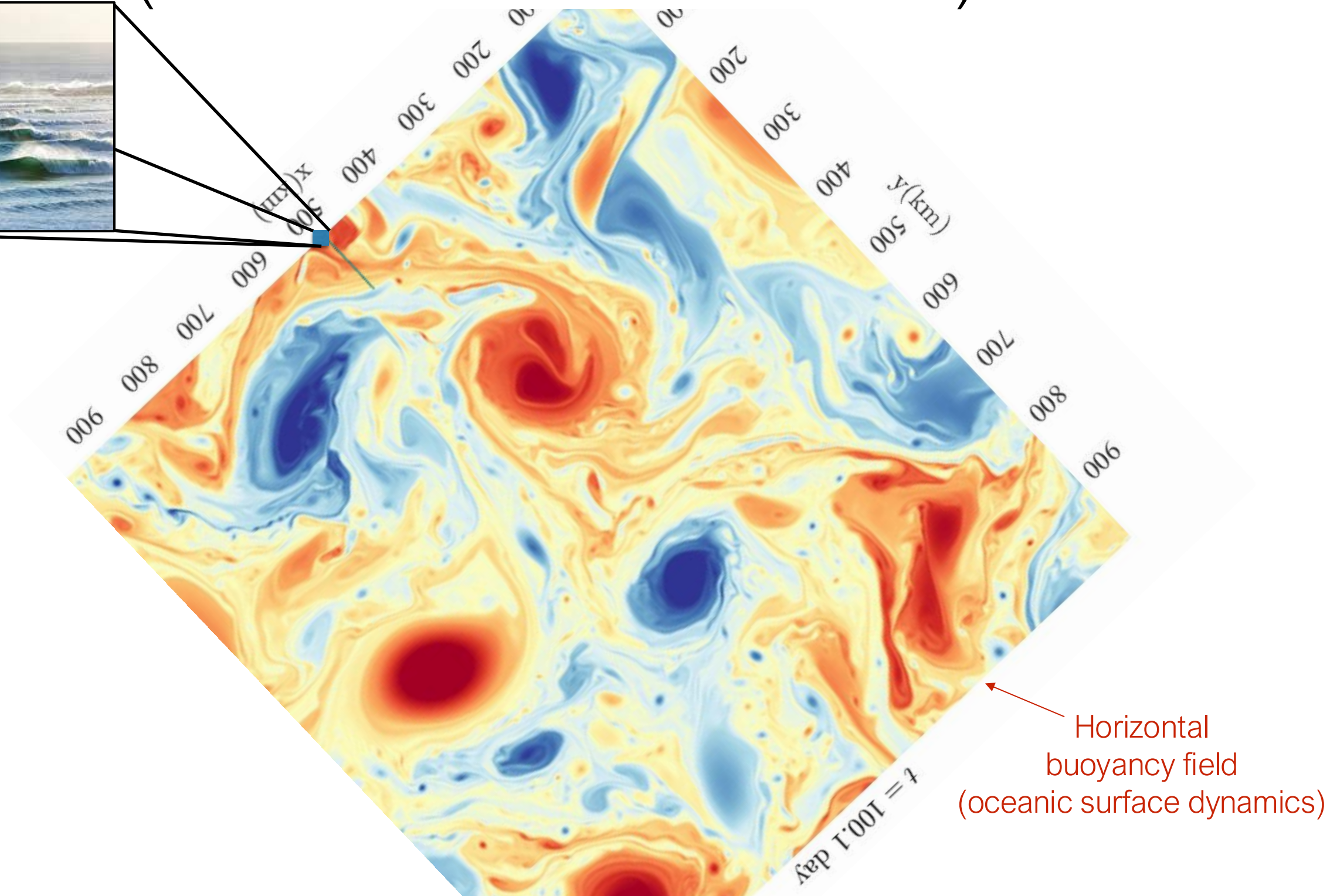
I. Wave refraction due to turbulence

Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)



Horizontal
buoyancy field
(oceanic surface dynamics)

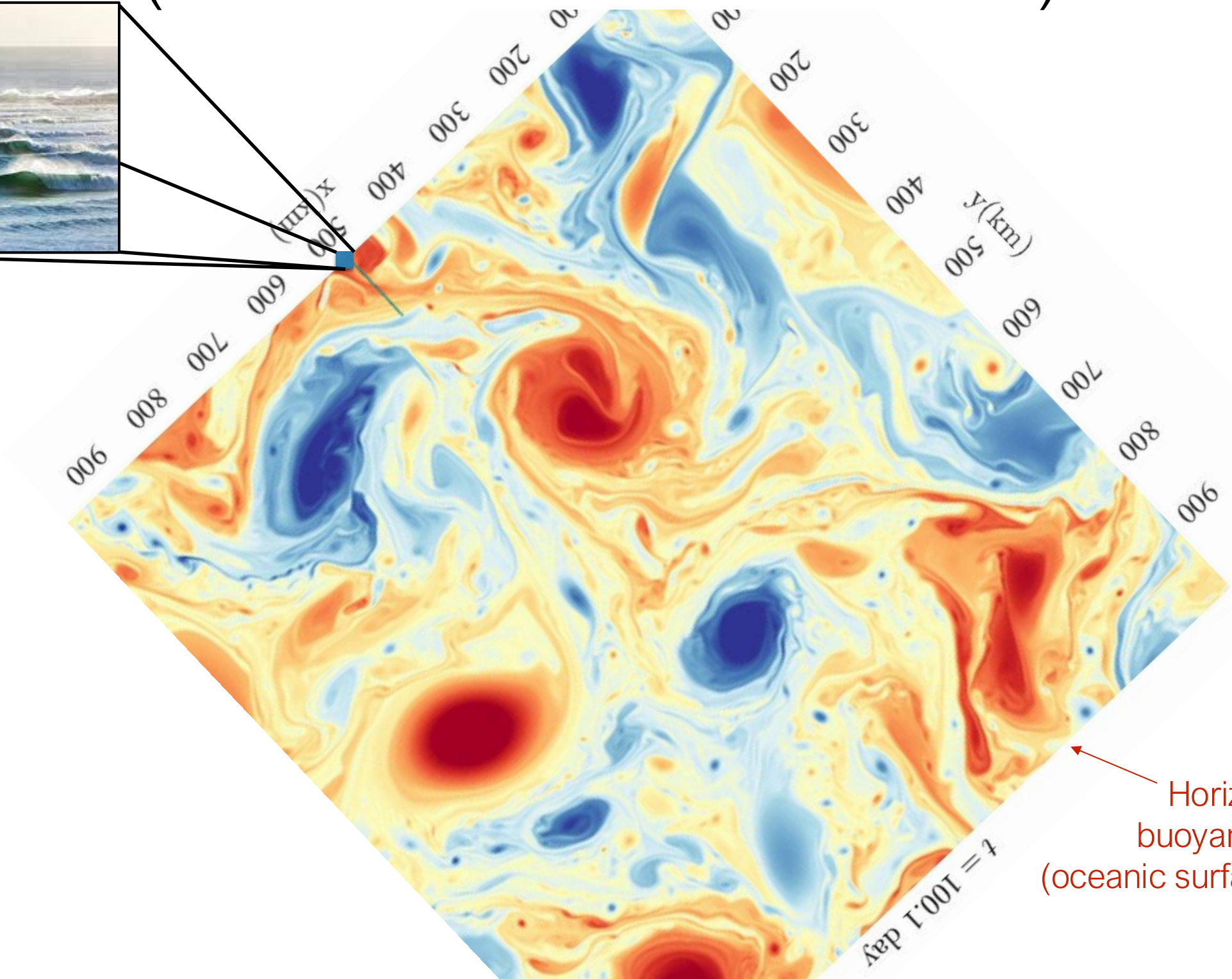
Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)



Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

$$he^{i\phi}$$



Horizontal buoyancy field (oceanic surface dynamics)

Current
velocity:
 v

Wave:
 $h e^{i\phi}$

$k = \nabla\phi$

Doppler
frequency:
 $\omega_0 \propto |k|^\alpha$

Dispersion ratio

At the first order in steepness ($=\|\nabla\phi\|a$)

$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k} = -\omega_0(k)$$

$$-\frac{\partial\phi}{\partial t} = \omega(k) = v \cdot k + \omega_0(k)$$

Current
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Wave phase transport
with a current velocity v

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Dispersion ratio

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Wave phase transport
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$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k} =$$

Frequency without currents

$$-\omega_0(k)$$

$$-\frac{\partial\phi}{\partial t} = \omega(k) = v \cdot k + \omega_0(k)$$

Total group velocity u_g :

$$v + C_g^0$$

Wave:

$$he^{i\phi}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler frequency:

$$\omega_0 \propto |k|^\alpha$$

Ray tracing / method of characteristic

$$\frac{dX_r}{dt} = \nabla_k \omega = C_g^0 + v \quad \Bigg| \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\frac{dk}{dt} = -\nabla_x \omega = -\nabla v^T k \quad \Bigg| \quad \text{Wave-vector}$$

Conservation of action spectral density N

$$\frac{dN}{dt} = 0 \quad \Bigg| \quad N = \frac{h^2}{\omega_0(k)} \quad \Bigg| \quad \text{Amplitude}$$

Total group velocity u_g :

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Ray tracing / method of characteristic

Function of k

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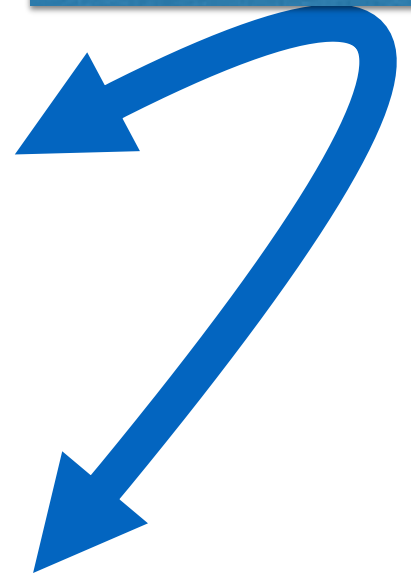
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Function of x

Conservation of action spectral density N

$$\frac{dN}{dt} = 0 \quad | \quad N = \frac{h^2}{\omega_0(k)} \quad | \quad \text{Amplitude}$$

Non-linear coupling



Total group velocity u_g :

$$v + C_g^0$$

Ray tracing / method of characteristic

Wave:

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Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler frequency:

$$\omega_0 \propto |k|^\alpha$$

Function of k

$$\frac{dX_r}{dt} = \nabla_k \omega = C_g^0 + v \quad \text{Ray}$$

Main contribution

Refraction & contraction/dilatation

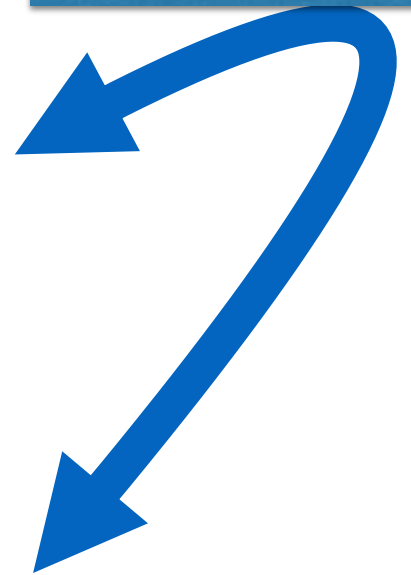
$$\frac{dk}{dt} = -\nabla_x \omega = -\nabla v^T k \quad \text{Wave-vector}$$

Function of x

Conservation of action spectral density N

$$\frac{dN}{dt} = 0 \quad \left| \quad N = \frac{h^2}{\omega_0(k)} \quad \right| \quad \text{Amplitude}$$

Non-linear coupling



Total group velocity u_g :

$$v + C_g^0$$

Ray tracing / method of characteristic

Wave: $he^{i\phi}$

Group velocity without current $C_g^0 = \nabla_k \omega_0$

$\nabla = \nabla_x$

Doppler frequency: $\omega_0 \propto |k|^\alpha$

$$\frac{dX_r}{dt} = \nabla_k \omega = C_g^0 + v \quad \text{Ray}$$

Fonction of k
Main contribution

Change orientation of C_g^0

$$\frac{dk}{dt} = -\nabla_x \omega = -\nabla v^T k \quad \text{Wave-vector}$$

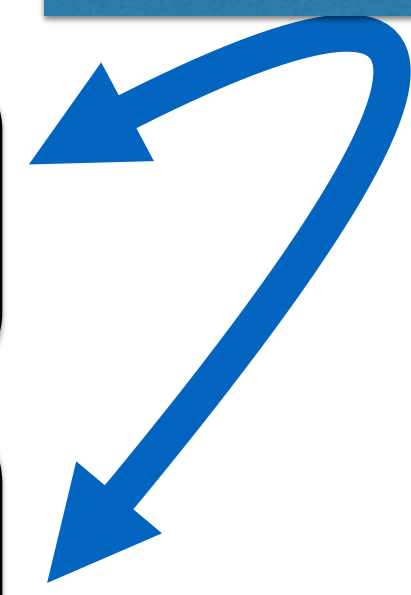
Fonction of x

Refraction & contraction/dilatation

Conservation of action spectral density N

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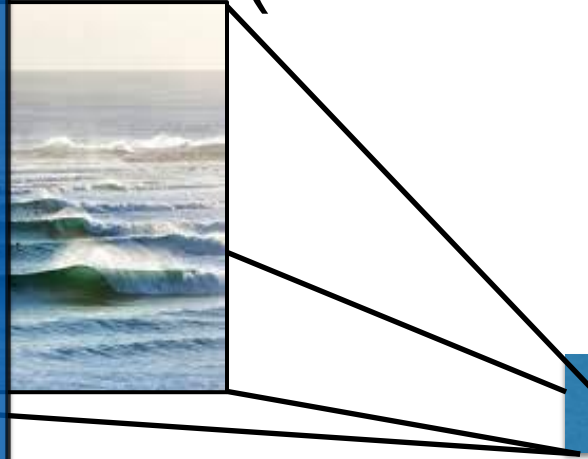
Non-linear coupling



Numerical example of wave (swell)
traveling inside **turbulence**
(oceanic surface currents)

Wave:

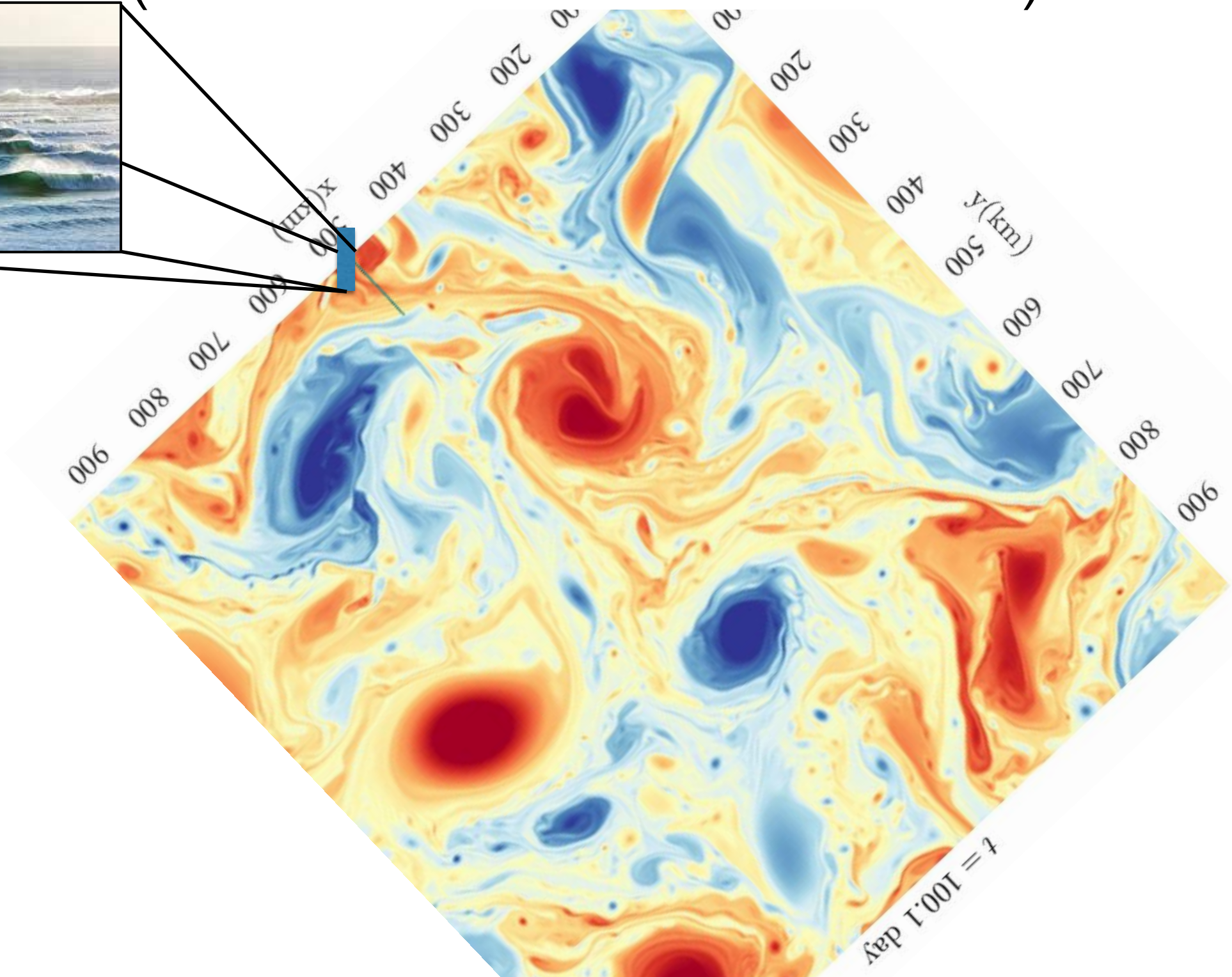
$$he^{i\phi}$$



Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

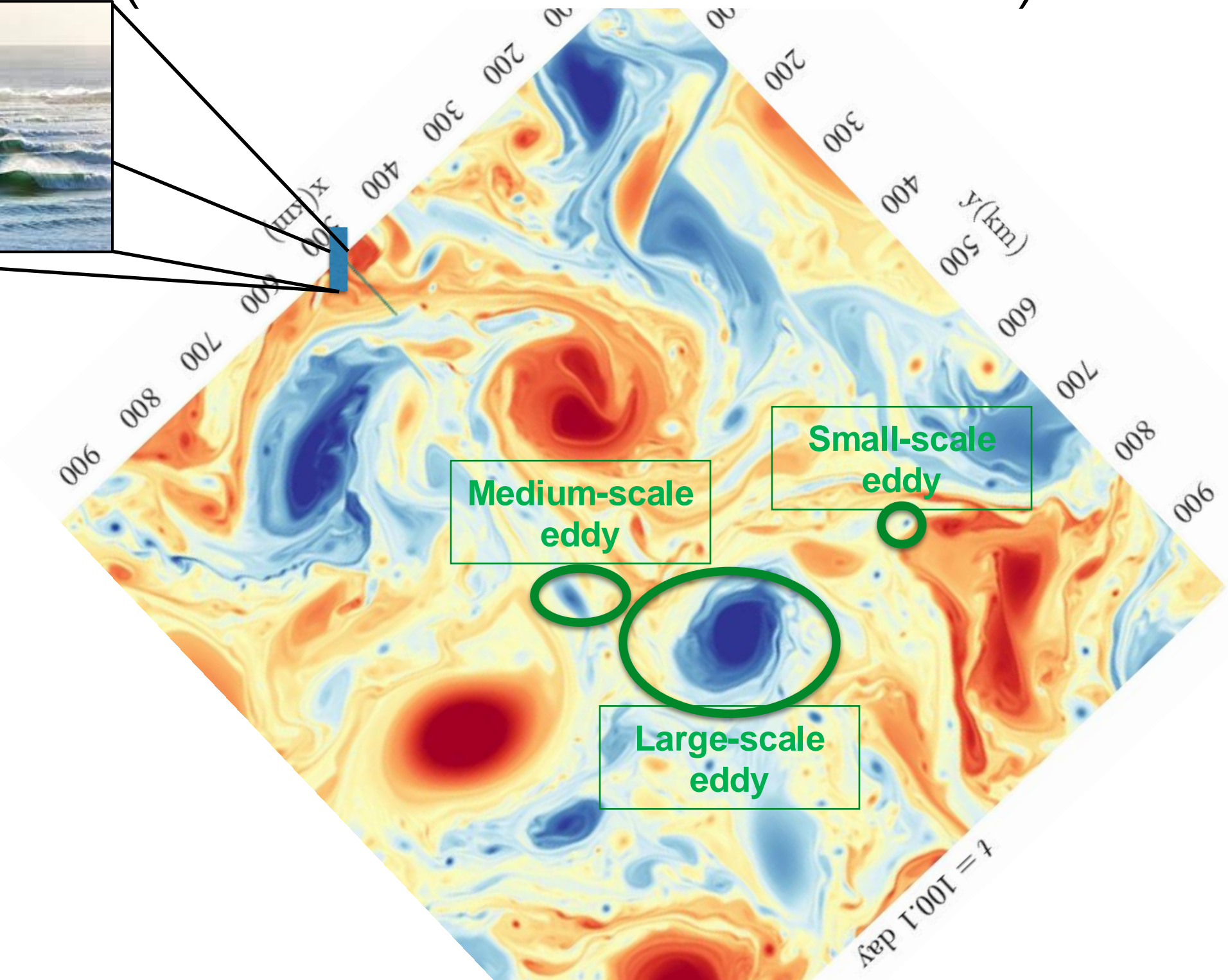
$$he^{i\phi}$$



Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

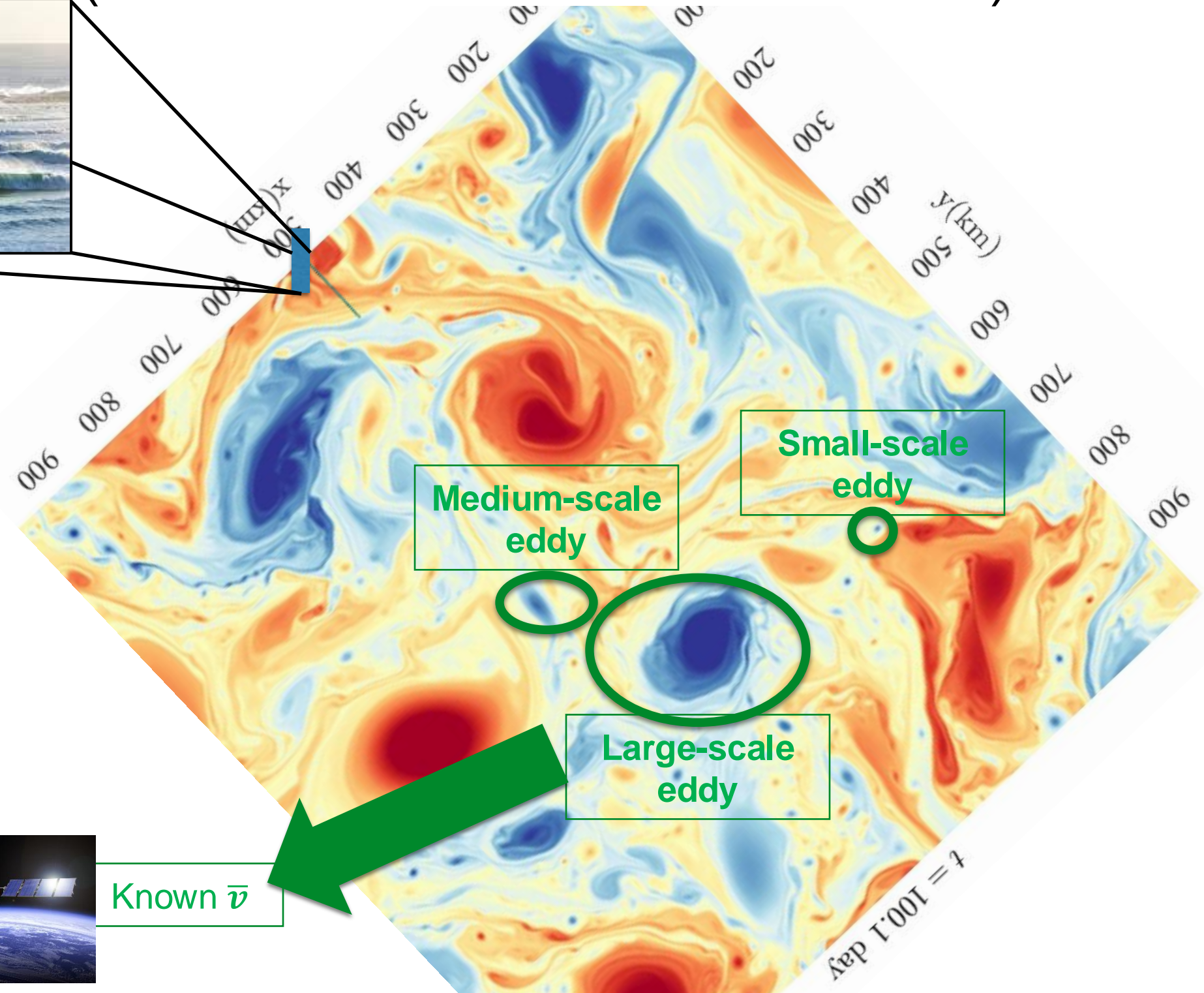
$$he^{i\phi}$$



Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

$$he^{i\phi}$$

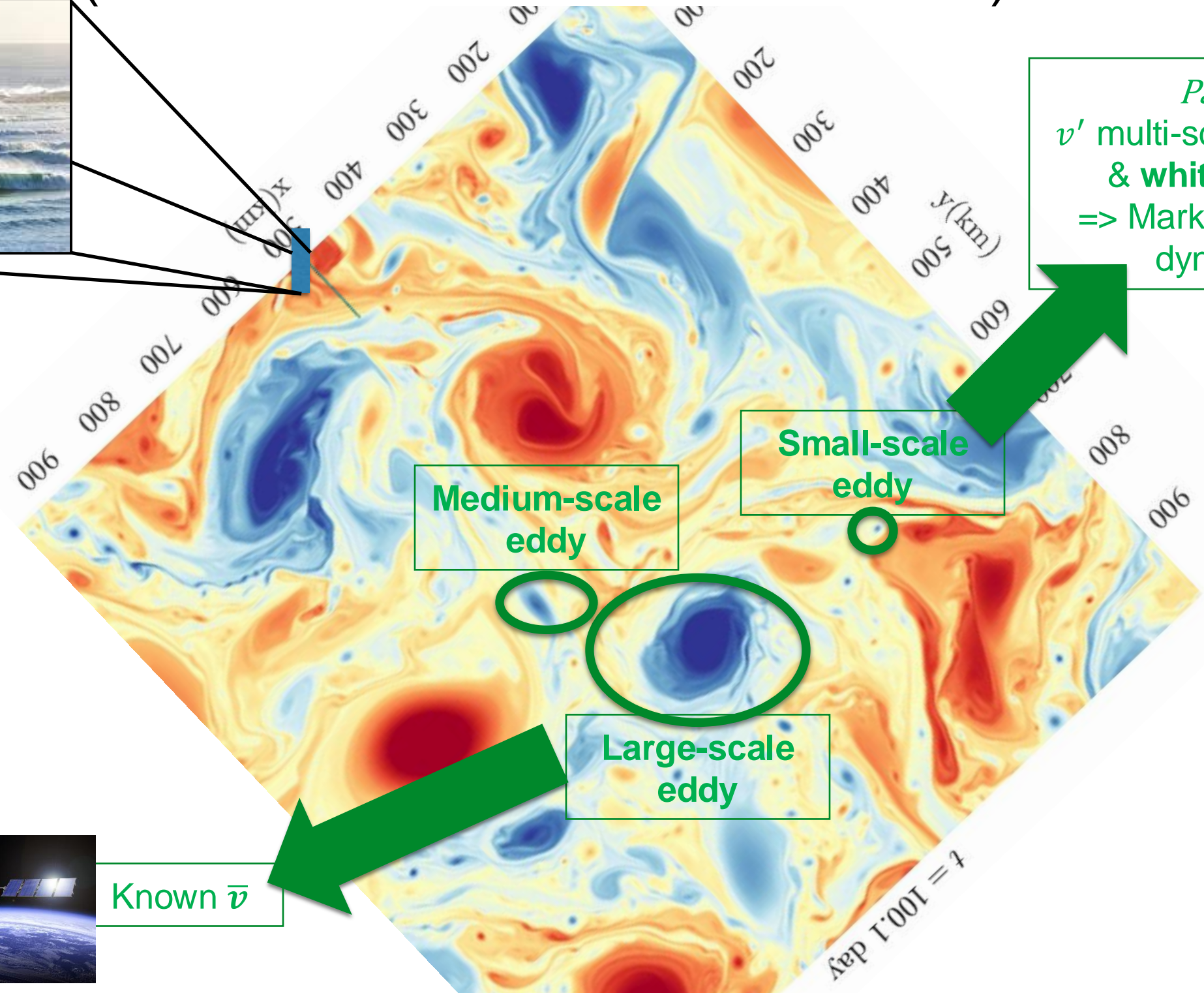


Known \bar{v}

Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

$$he^{i\phi}$$



Part II.
 v' multi-scale in space
& **white in time**
=> Markovian wave dynamics

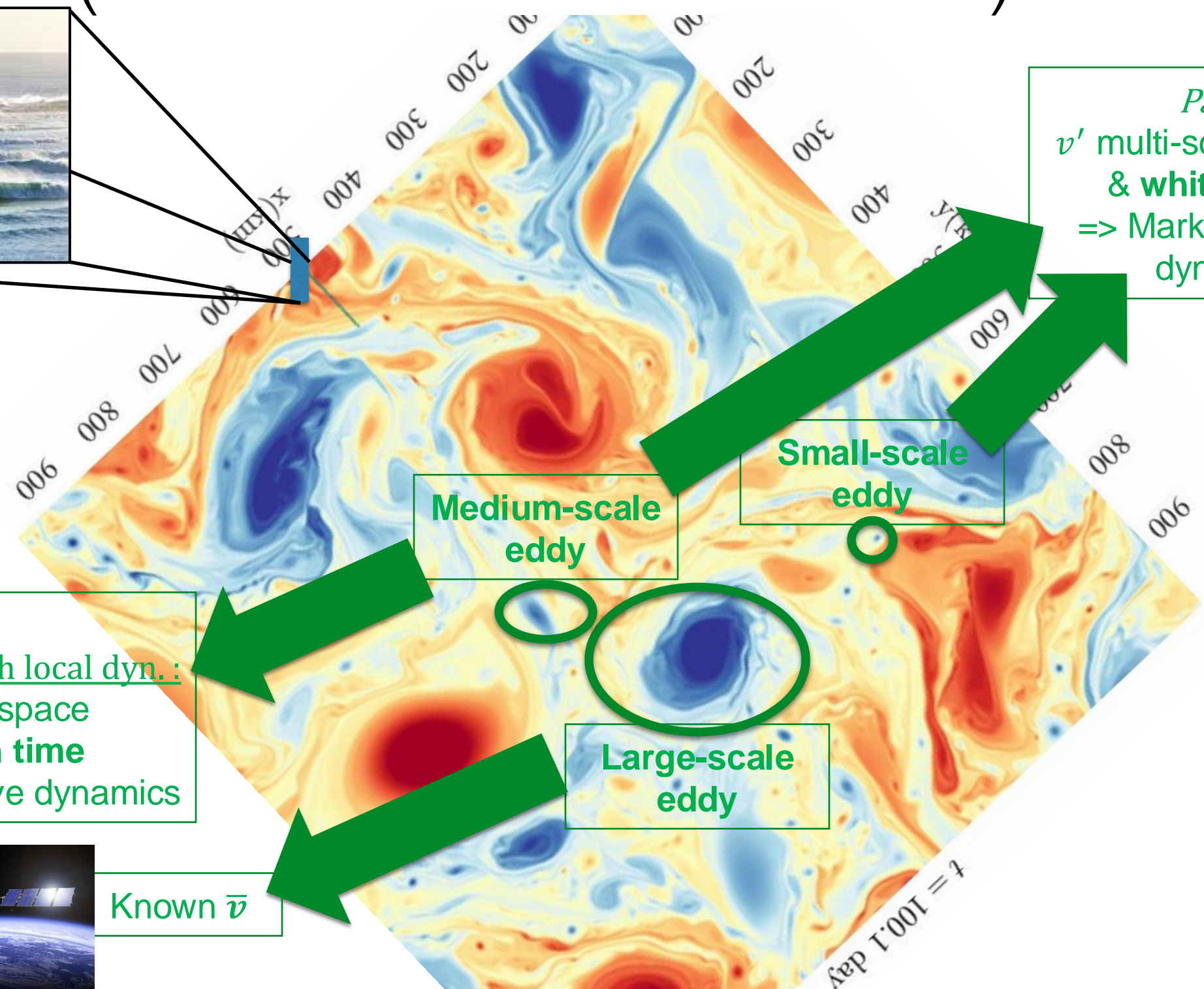


Known \bar{v}

Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

$$he^{i\phi}$$



Part II.
 v' multi-scale in space
 & **white in time**
 => Markovian wave dynamics

Part III.
 For strong currents with local dyn.:
 v' multi-scale in space
 & **multi-scale in time**
 => non-Markovian wave dynamics



Known \bar{v}

II. Markovian closure

II.A. Generalized fast wave approximation (GenFWA)

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma \circ \frac{dB_t}{dt}$$

$$= \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

Wave:

$$he^{i\phi}$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Generalized fast wave approx.

Smallest scale turbulence v' decorrelates along the wave propagation

- Validity :

$$\epsilon = \frac{\text{(Along-ray } v' \text{ correlation time)}}{\left(\text{characteristic time of wave group properties evolution} \right)} = \frac{\left(\frac{l_{v'}}{\|C_g^0\|} \right)}{\left(\frac{1}{\|\nabla v\|} \right)}$$
$$= \frac{l_{v'}}{l_v} \frac{\|v\|}{\|C_g^0\|} \ll 1$$

- Limitations :

- Fast waves $\left(\frac{\|C_g^0\|}{\|v\|} \gg 1 \right)$
- Small-scale currents $\left(\frac{l_{v'}}{l_v} \ll 1 \right)$
- Moderate current gradients $(\|\nabla v\| \ll 1)$
(\Rightarrow steep spectrum \Rightarrow non-local current dynamics)

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma \circ \frac{dB_t}{dt}$$

$$= \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t)

Wave:

$$he^{i\phi}$$

Group velocity
without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Stratonovich dispersion ratio

At the first order in steepness ($=\|\nabla\phi\|a$)

$$\frac{\partial \phi}{\partial t} + \left(\bar{v} + \sigma \circ \frac{dB_t}{dt} \right) \cdot \underbrace{\nabla \phi}_{=k} = -\omega_0(\mathbf{k})$$

Large scale group velocity:

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Small scale group velocity:

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Wave phase transport
with a current velocity v

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Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma \circ \frac{dB_t}{dt} \\ = \sum_i \xi_i \circ \frac{dW_t^i}{dt} \\ (\delta\text{-correlated in } t)$$

Wave:

$$he^{i\phi}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

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Frequency without currents

Large scale group velocity:

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Small scale group velocity:

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(δ -correlated in t)

Wave:

$$h e^{i\phi}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Stratonovich stochastic ray tracing

$$\frac{dX_r}{dt} = C_g^0 + \bar{v} + \sigma \circ \frac{dB_t}{dt} \quad | \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\frac{dk}{dt} = -\nabla \left(\bar{v} + \sigma \circ \frac{dB_t}{dt} \right)^T k \quad | \quad \text{Wave-vector}$$

Conservation of action spectral density N

$$\frac{dN}{dt} = 0 \quad | \quad N = \frac{h^2}{\omega_0(k)} \quad | \quad \text{Amplitude}$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma \frac{dB_t}{dt} = \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous & isotropic)

Wave:

$$h e^{i\phi}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Itô stochastic ray tracing

$$\frac{dX_r}{dt} = C_g^0 + \bar{v} + \sigma \frac{dB_t}{dt} \quad | \quad \text{Ray}$$

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Time-uncorrelated model for v' :

Method of characteristic

$$v = \bar{v} + v'$$

$$v = \bar{v}$$

$$v = \bar{v} + \sigma \dot{B}$$

wave group amplitude

buoyancy

Deterministic
reference:
wave groups in
512 x 512 flow

wave group amplitude

buoyancy

Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

wave group amplitude

buoyancy

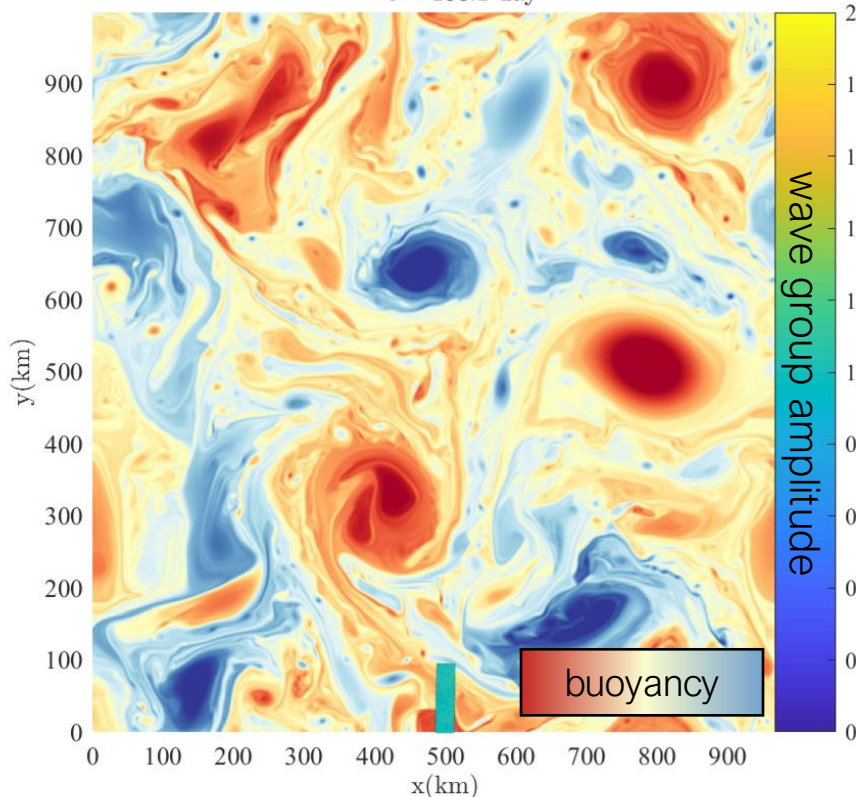
Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-uncorrelated v'

Time-uncorrelated model for v' :

Method of characteristic

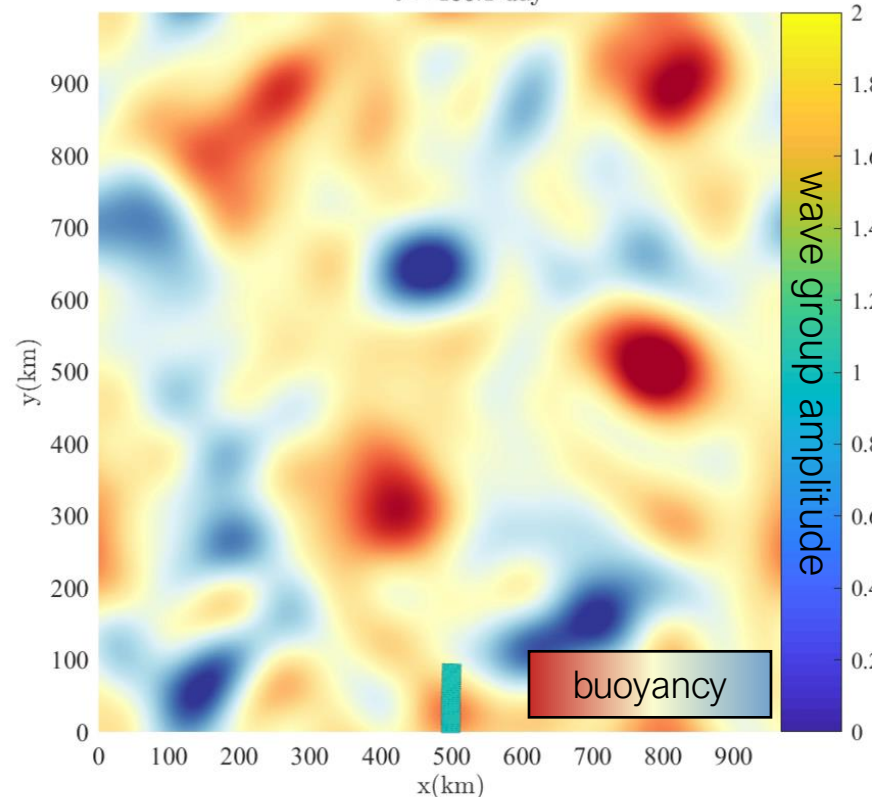
$$v = \bar{v} + v'$$

$t = 100.1$ day



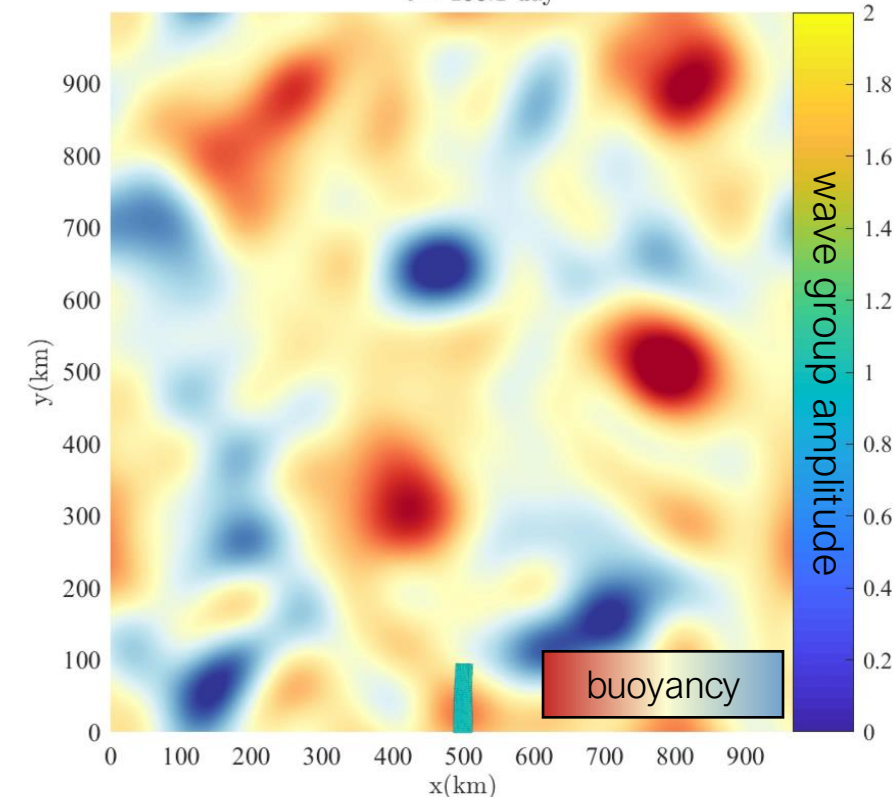
$$v = \bar{v}$$

$t = 100.1$ day



$$v = \bar{v} + \sigma \dot{B}$$

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Deterministic
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wave groups in
512 x 512 flow

Deterministic
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Our random
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wave groups in
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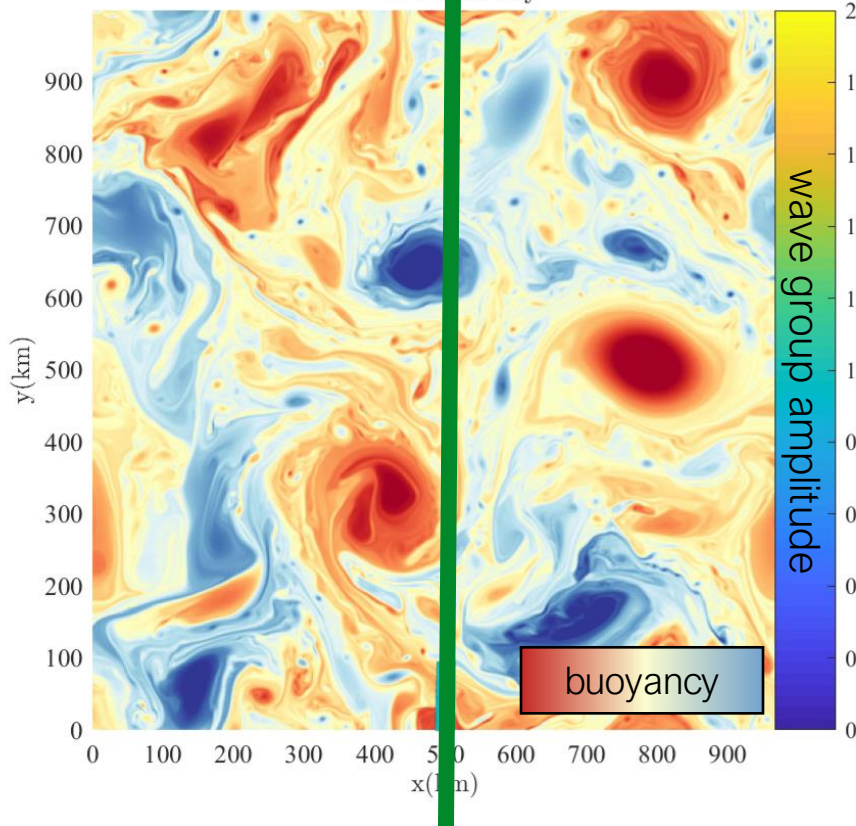
+ time-uncorrelated v'

Time-uncorrelated model for v' :

Method of characteristic

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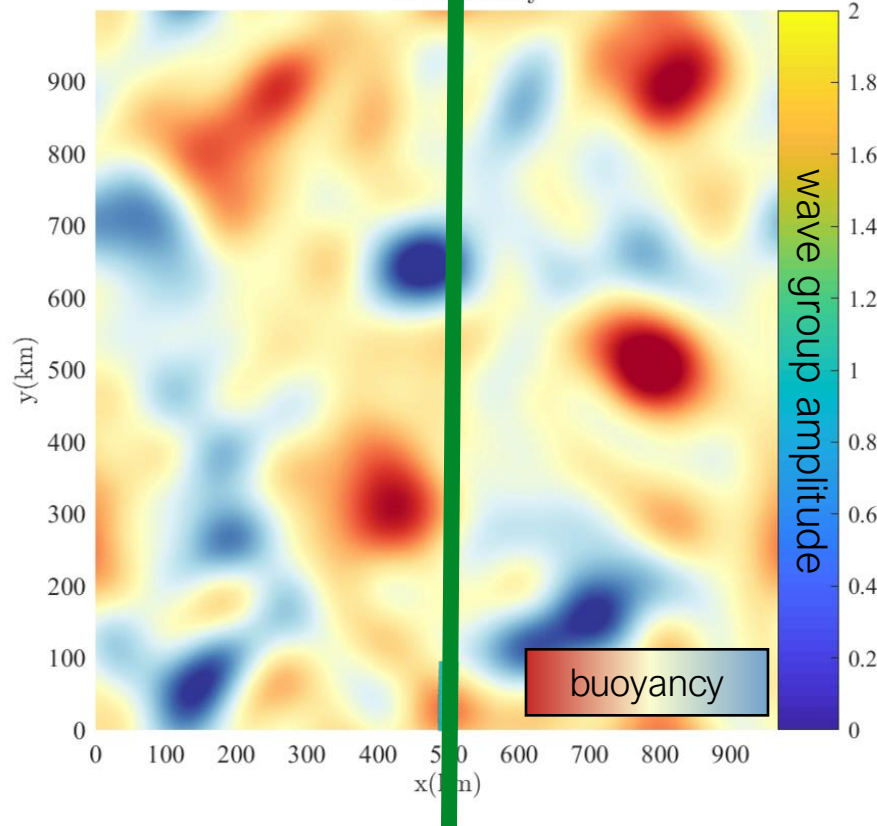
$t = 4.51$ day



Deterministic
reference:
wave groups in
512 x 512 flow

$$v = \bar{v}$$

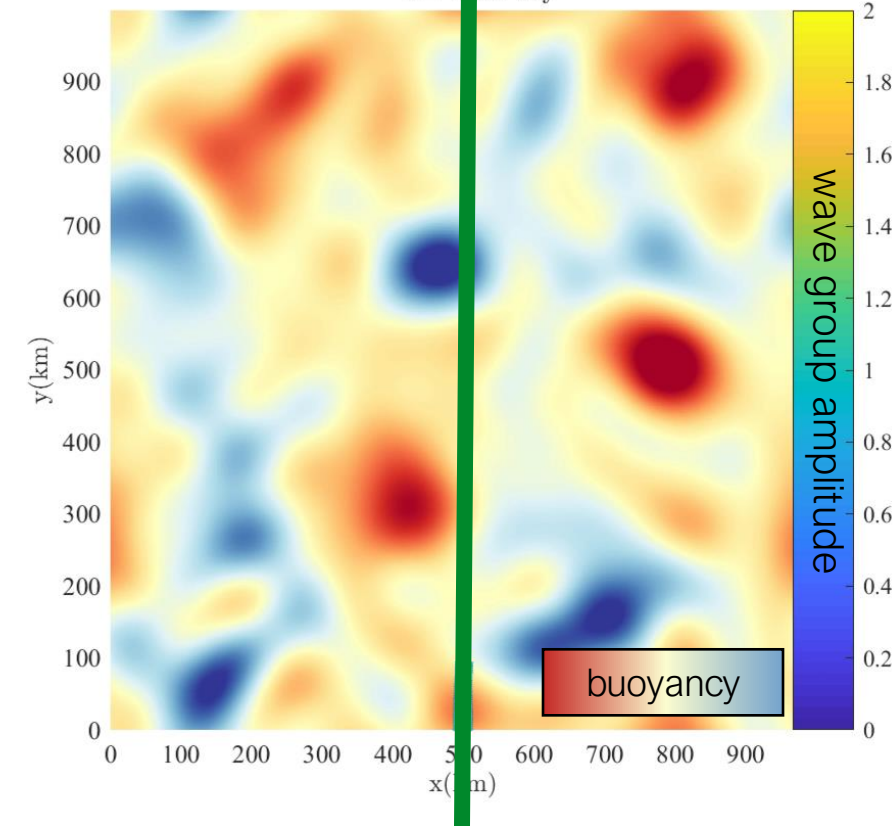
$t = 4.51$ day



Deterministic
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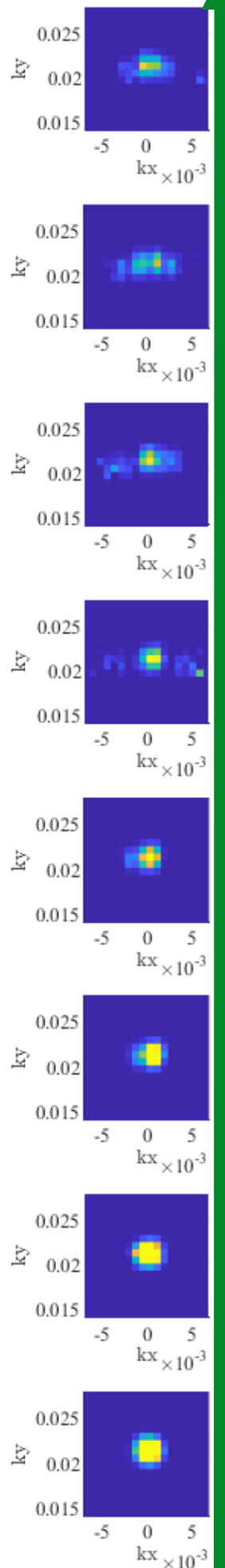
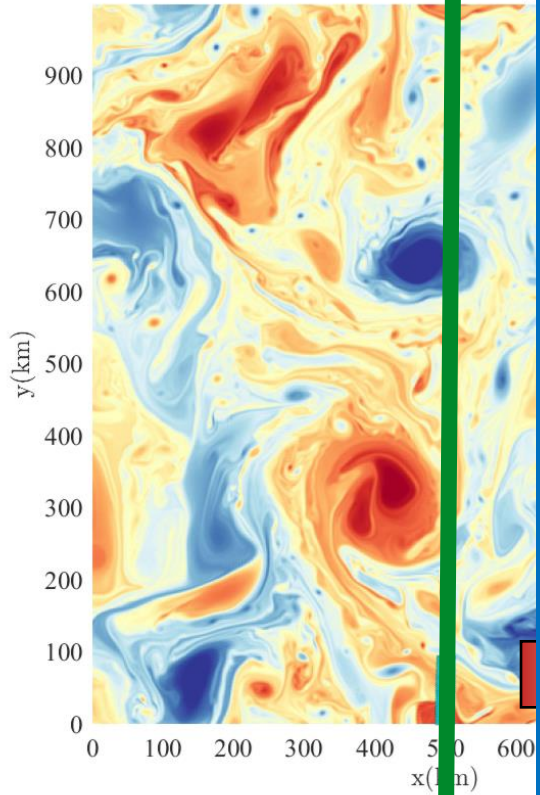
Our random
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Time-correlated model for v'

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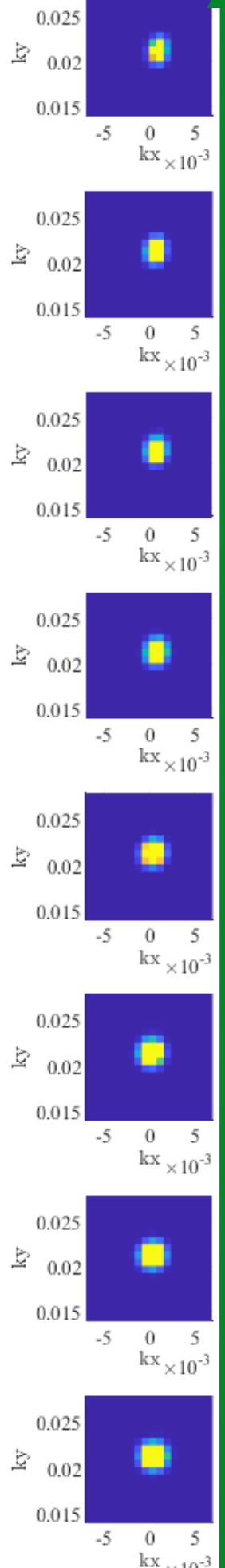
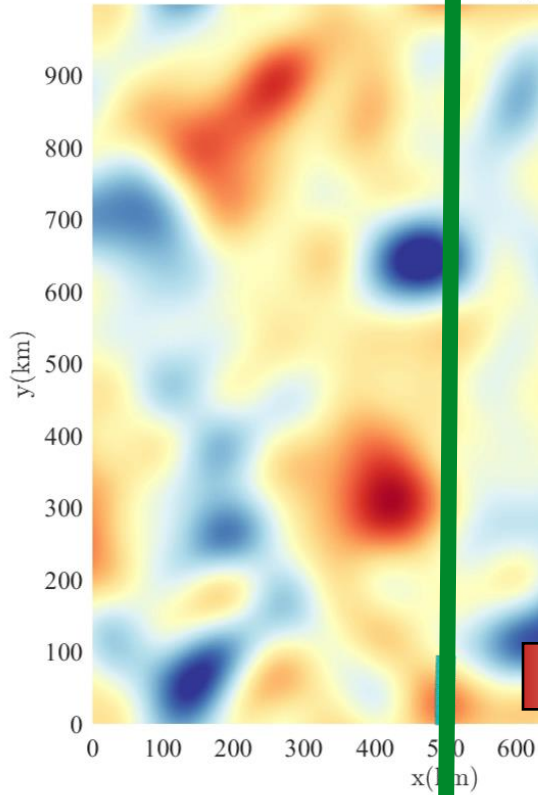
$t = 4.1$ day



Deterministic
reference
wave group
512 x 512 ft

$$v = \bar{v}$$

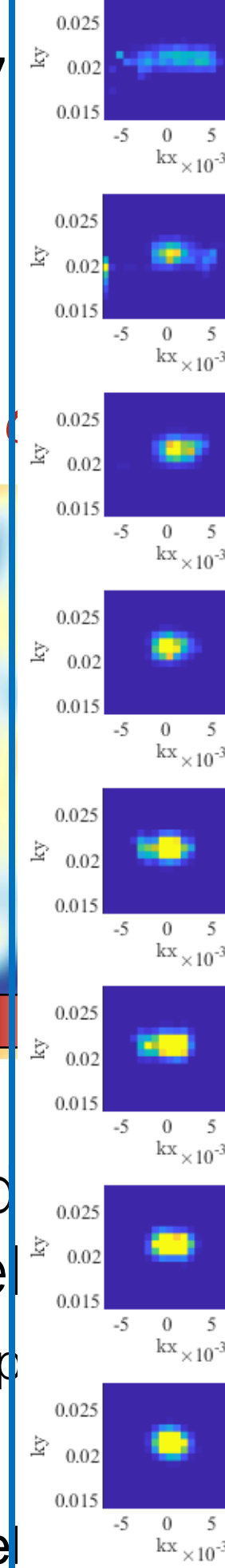
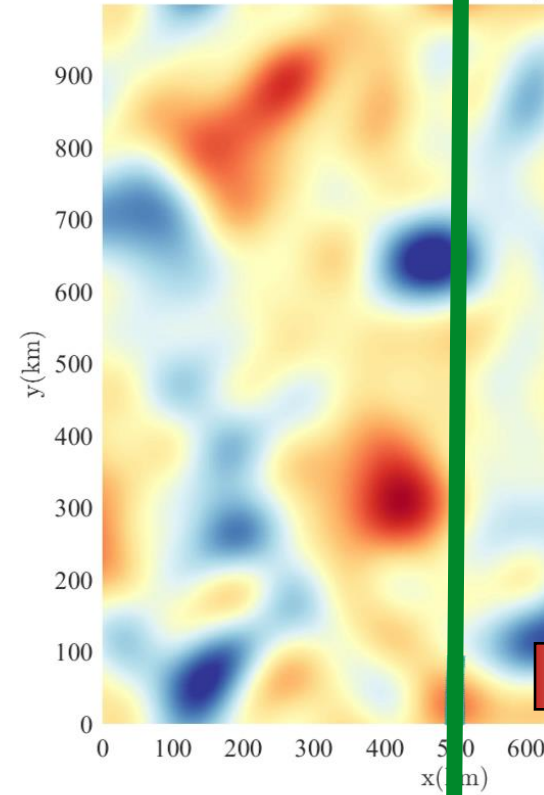
$t = 4.1$ day



Deterministic
benchmark
wave groups
smoothed flow \bar{v}

$$v = \bar{v} + v'$$

$t = 4.1$ day



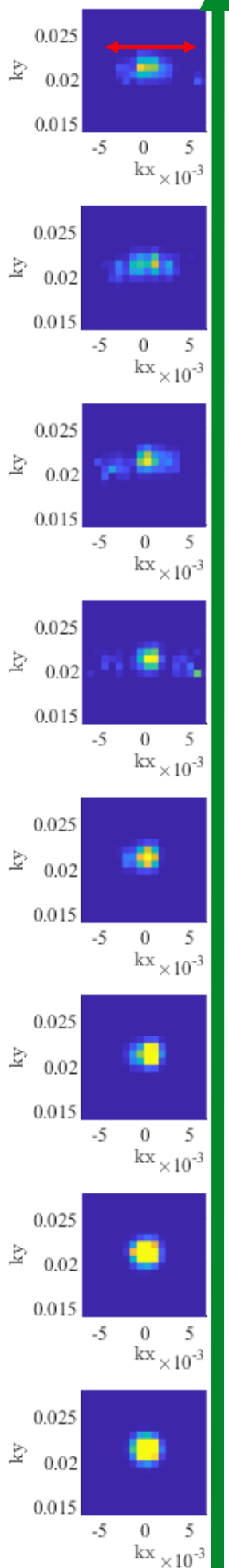
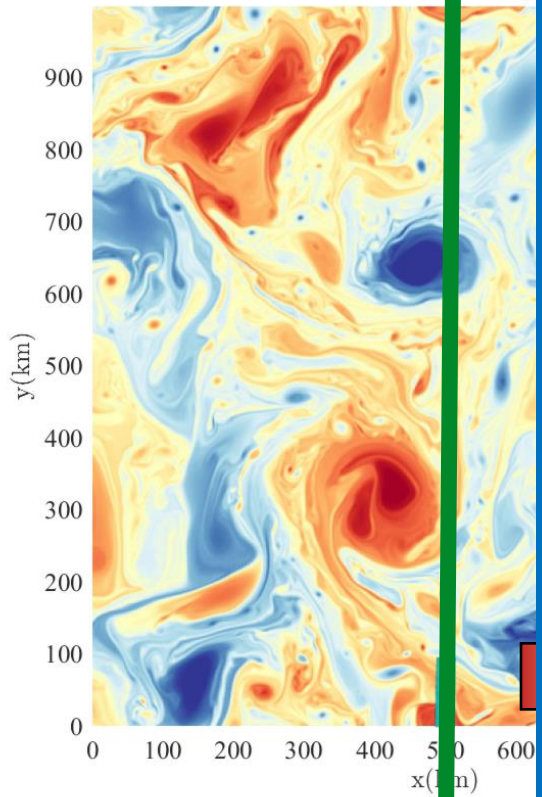
Our random
model
wave groups
smoothed flow \bar{v}
+ time-uncorrelated

Time-correlated model for v'

method of characteristics

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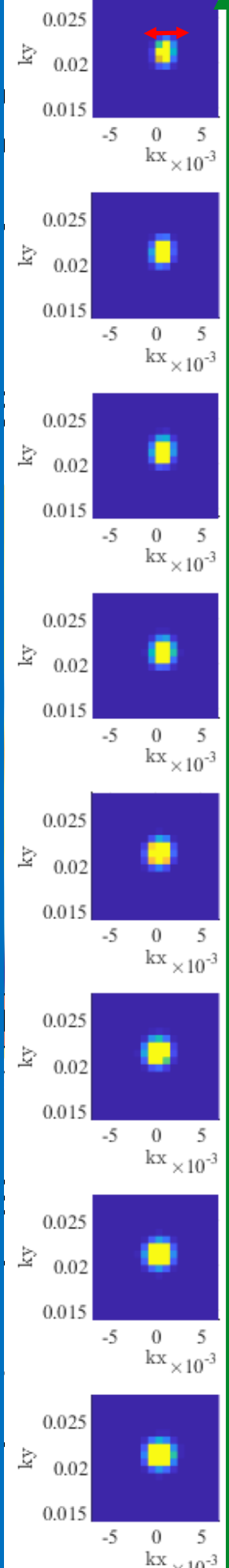
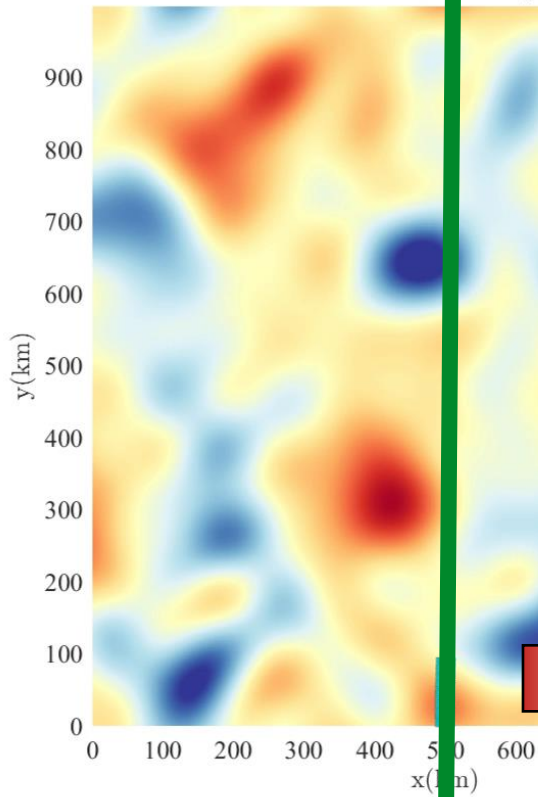
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Deterministic
reference
wave group
512 x 512 ft

$$v = \bar{v}$$

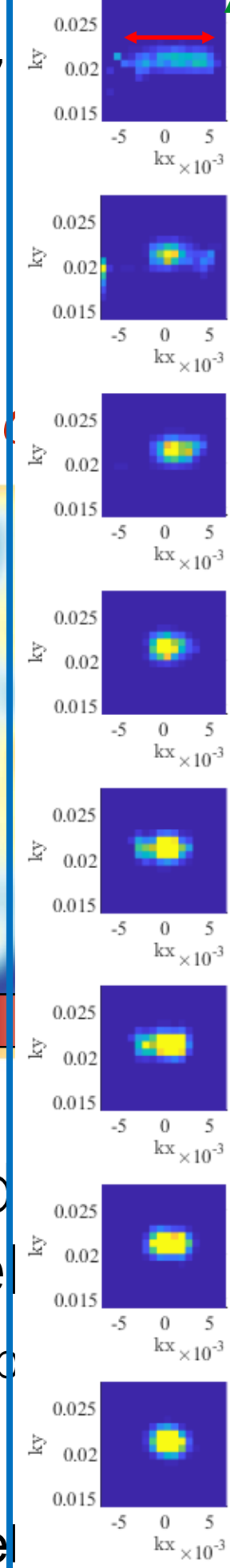
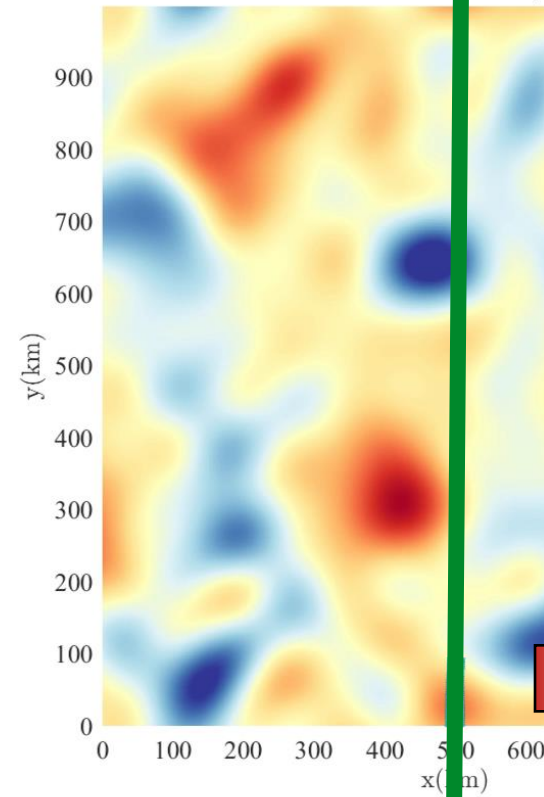
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Our random
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+ time-uncorrelated

II. Markovian closure

II.B. Calibration

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$
$$= \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

Time-uncorrelated model for v' : Calibration

$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Large scale
group velocity:

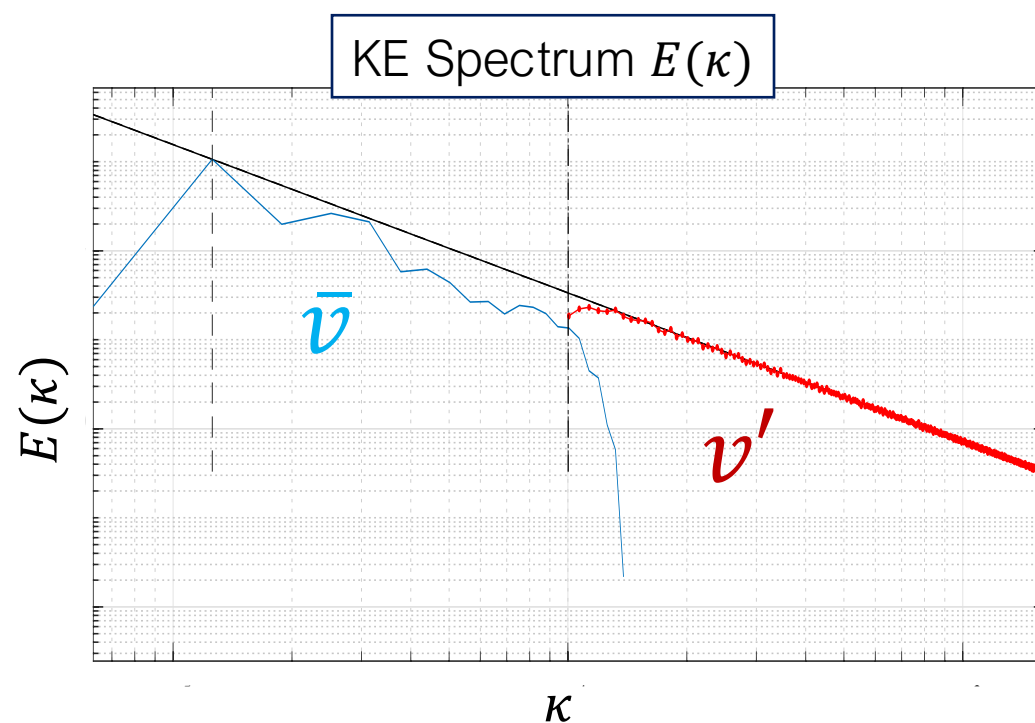
$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt} \\ = \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

Time-uncorrelated model for v' : Calibration



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

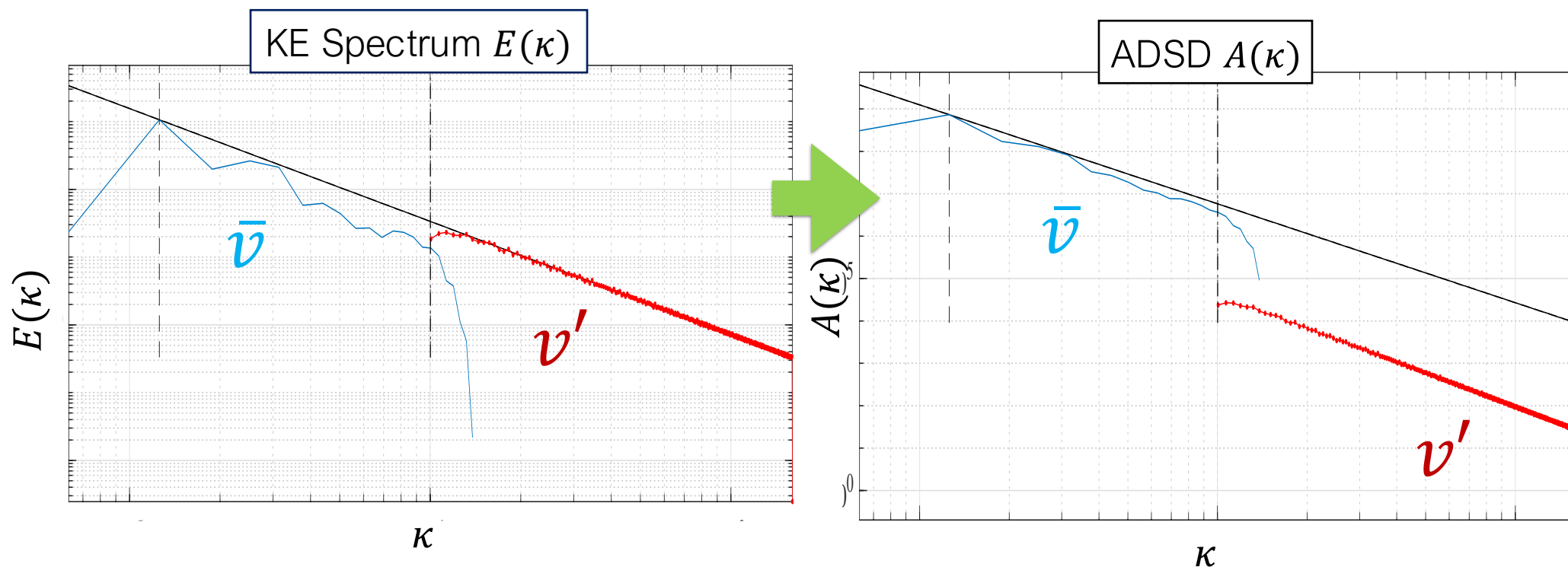
Small scale group velocity:

$$v' = \sigma \frac{dB_t}{dt} = \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

Time-uncorrelated model for v' : Calibration

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:
Resseguier, Pan & Fox-Kemper 2020

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

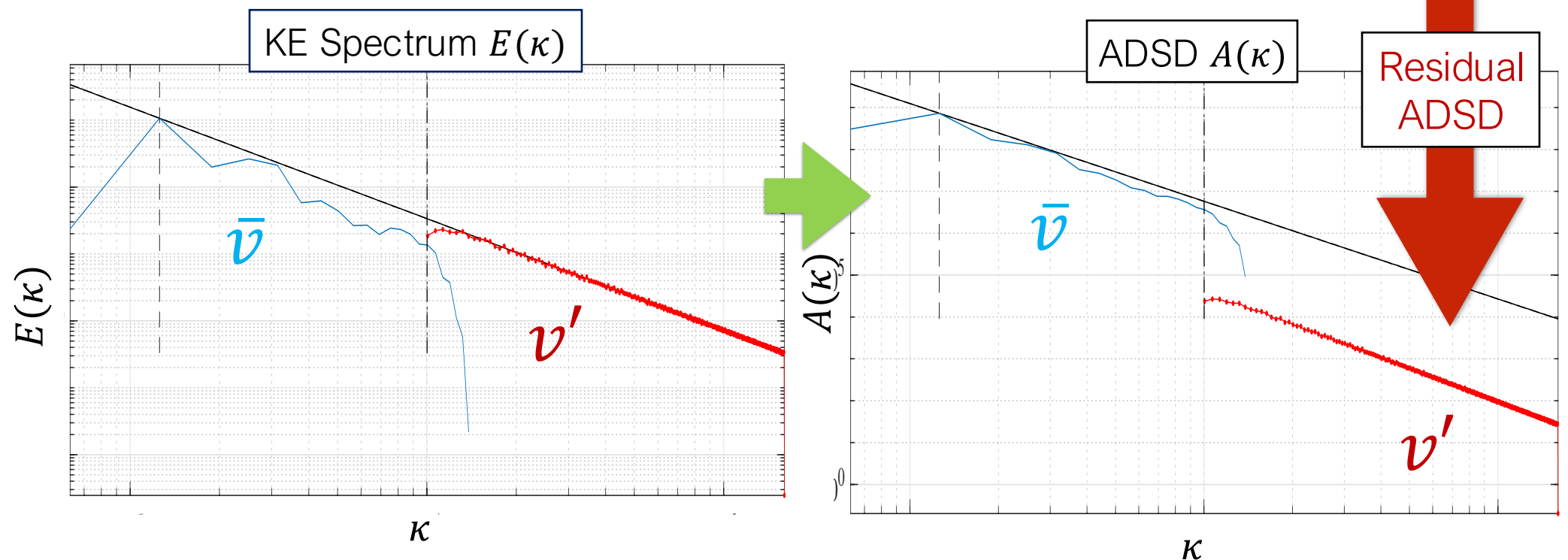
$$v' = \sigma \frac{dB_t}{dt} = \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

Time-uncorrelated model for v' : Calibration

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$

Correlation time along ray
 $\tau(\kappa) = \tau_{ray}(\kappa) = \frac{1/\kappa}{\|C_g^0\|}$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:
Resseguier, Pan & Fox-Kemper 2020

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

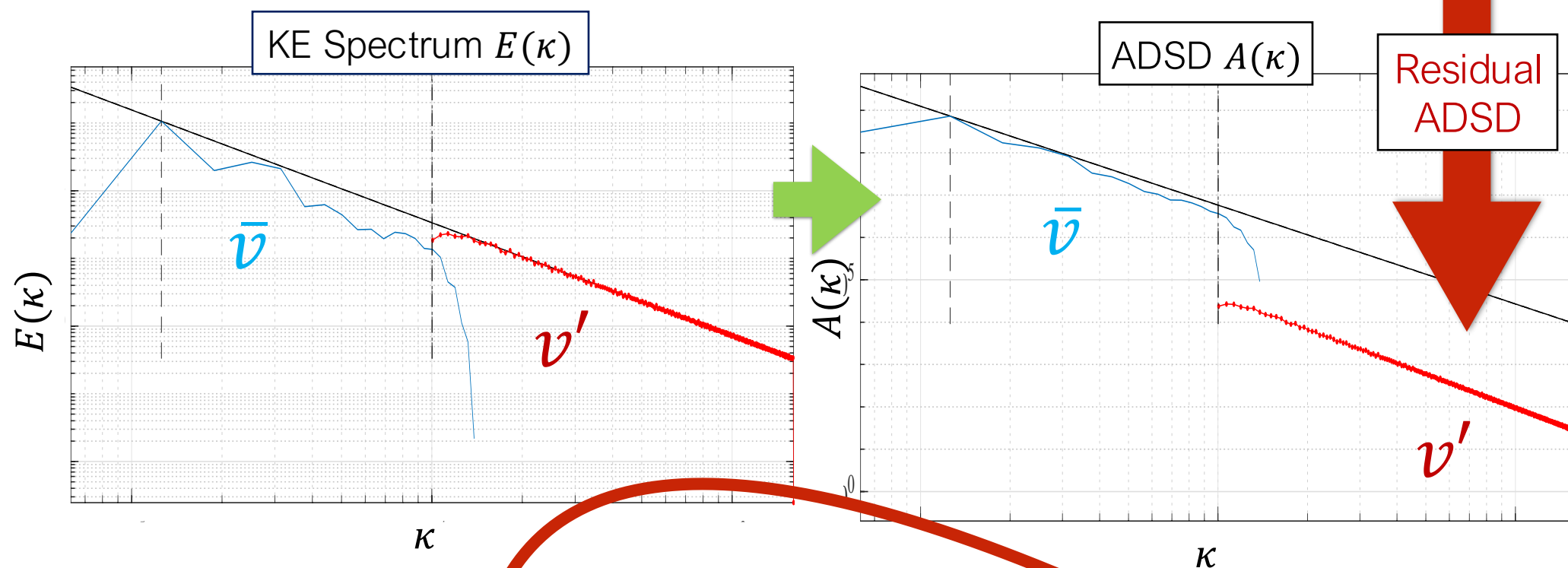
$$v' = \sigma \frac{dB_t}{dt} = \sum_i \xi_i \circ \frac{dW_t^i}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

Time-uncorrelated model for v' : Calibration

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$

Correlation time along ray
 $\tau(\kappa) = \tau_{ray}(\kappa) = \frac{1/\kappa}{\|C_g^0\|}$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:
Resseguier, Pan & Fox-Kemper 2020

II. Markovian closure

II.C. Simplification for single ray dynamics

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$

Small scale

group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Simplification for single ray dynamics

only 4 Brownian motions needed

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Simplification for single ray dynamics

only 4 Brownian motions needed

$$\frac{dX_r}{dt} = |C_g^0| \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} + \bar{v} + \sqrt{a_0} \begin{pmatrix} dB_t^{(1)} / dt \\ dB_t^{(2)} / dt \end{pmatrix} \quad \text{Ray}$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Simplification for single ray dynamics

only 4 Brownian motions needed

$$\frac{dX_r}{dt} = |C_g^0| \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} + \bar{v} + \sqrt{a_0} \begin{pmatrix} dB_t^{(1)} / dt \\ dB_t^{(2)} / dt \end{pmatrix} \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\frac{d \ln \kappa(t)}{dt} = -\bar{\sigma} \sin \zeta(t) + \gamma_0 + \sqrt{\gamma_0} \frac{dB_t^{(3)}}{dt} \quad \left| \begin{array}{l} \text{Wave-vector} \\ k = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} \end{array} \right.$$

$$d\theta_k(t) = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t)) dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt}$$

with $\zeta = 2(\theta_k + \bar{\phi})$ and potential $\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Simplification for single ray dynamics

only 4 Brownian motions needed

$$\frac{dX_r}{dt} = |C_g^0| \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} + \bar{v} + \sqrt{a_0} \begin{pmatrix} dB_t^{(1)}/dt \\ dB_t^{(2)}/dt \end{pmatrix} \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\frac{d \ln \kappa(t)}{dt} = -\bar{\sigma} \sin \zeta(t) + \gamma_0 + \sqrt{\gamma_0} \frac{dB_t^{(3)}}{dt} \quad \left| \quad \text{Wave-vector} \right.$$

$$d\theta_k(t) = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t)) dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt} \quad \left| \quad k = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} \right.$$

with $\zeta = 2(\theta_k + \bar{\phi})$ and potential $\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$

Conservation of action spectral density N

$$\mathbb{E} N(x, k, t) = \iint dx_r^0 dk^0 N^0(x_r^0, k^0) p(x, k | x_r^0, k^0, t)$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Simplification for single ray dynamics

only 4 Brownian motions needed

$$\frac{dX_r}{dt} = |C_g^0| \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} + \bar{v} + \sqrt{a_0} \begin{pmatrix} dB_t^{(1)}/dt \\ dB_t^{(2)}/dt \end{pmatrix} \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\frac{d \ln \kappa(t)}{dt} = -\bar{\sigma} \sin \zeta(t) + \gamma_0 + \sqrt{\gamma_0} \frac{dB_t^{(3)}}{dt} \quad \left| \quad \text{Wave-vector} \right.$$

$$d\theta_k(t) = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t)) dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt} \quad \left| \quad k = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix} \right.$$

with $\zeta = 2(\theta_k + \bar{\phi})$ and potential $\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$

Conservation of action spectral density N

$$\mathbb{E} N(x, k, t) = \iint dx_r^0 dk^0 N^0(x_r^0, k^0) p(x, k | x_r^0, k^0, t)$$

We can get analytic solutions !

Large scale
group velocity:

$$\bar{v} + C_g^0$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$
assumed to be cst.

Simplification for single ray dynamics

- Deterministic case -

Refraction & contraction/dilatation

$$\frac{d \ln \kappa(t)}{dt} = -\bar{\sigma} \sin \zeta(t)$$

$$d\zeta = -\frac{\partial \bar{v}}{\partial \zeta}(\zeta(t)) dt$$

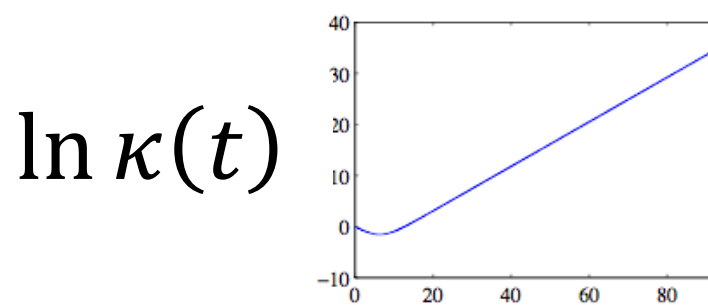
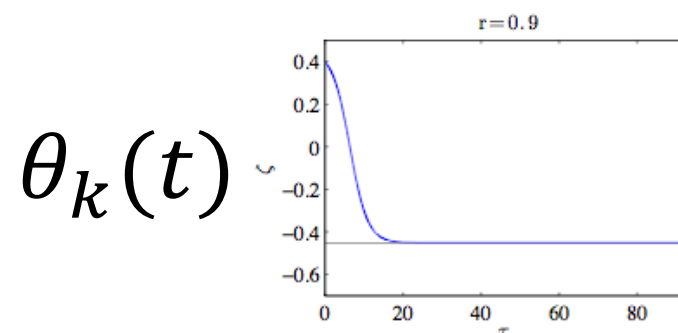
with $\zeta = 2(\theta_k + \bar{\phi})$ and potential $\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$

Wave-vector

$$k = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$$

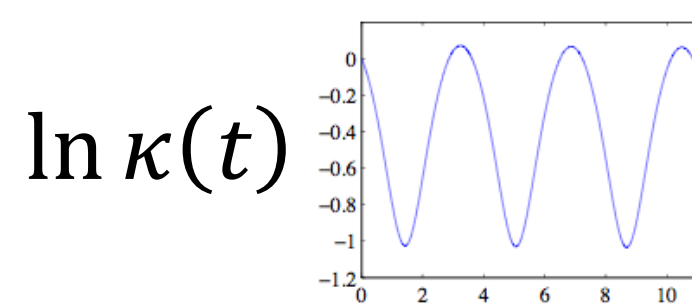
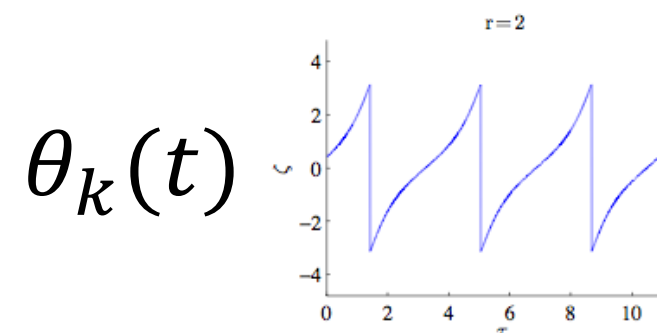
$$\bar{\omega} < \bar{\sigma}$$

Wave shortening



$$\bar{\omega} > \bar{\sigma}$$

Wave rotation



Large scale
group velocity:

$$\bar{v}$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$
assumed to be cst.

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Wave:

$$h e^{i\phi}$$

$$k = \nabla \phi = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$$

$$\zeta = 2(\theta_k + \bar{\phi})$$

$$\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$$

Simplification for single ray dynamics - Stochastic case -

We can solve the stationary Fokker-Planck for ζ ,

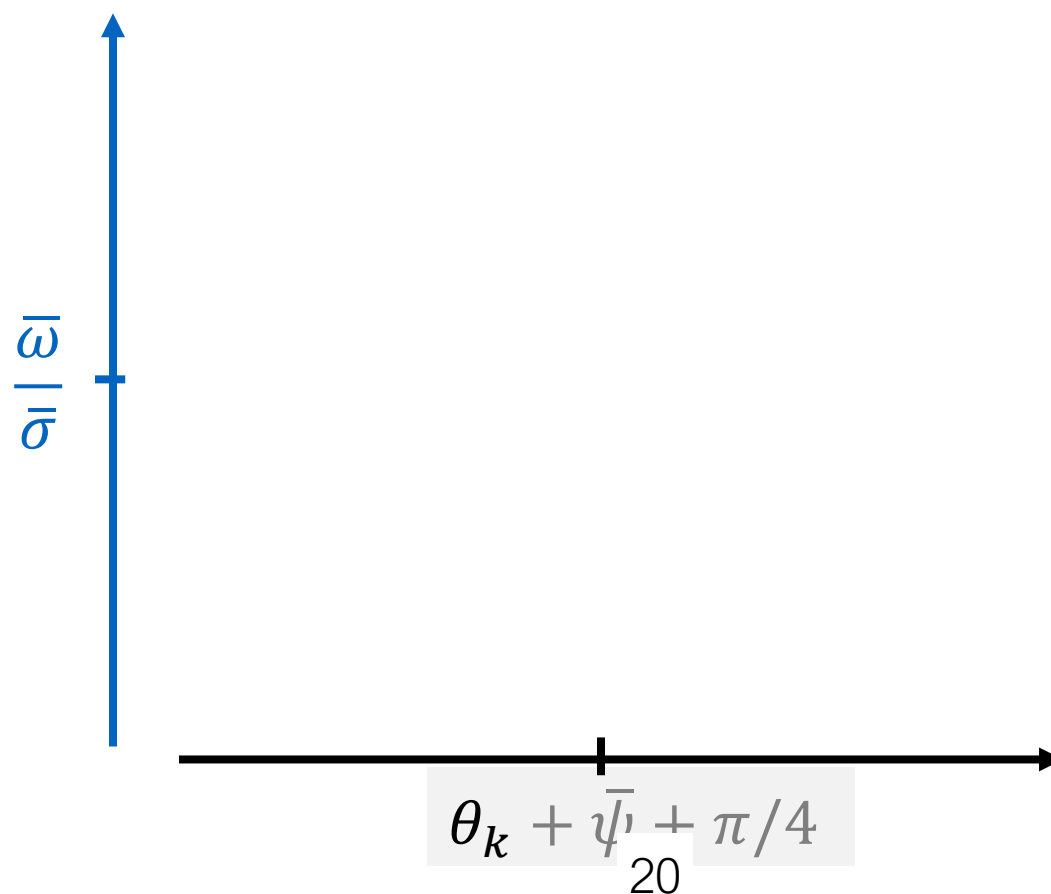
$$d\zeta = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t)) dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt}$$

\Rightarrow Up to periodicity constraints,

$$p(\zeta) \propto \exp\left(-\frac{2}{3} \frac{\bar{V}(\zeta)}{\gamma_0}\right)$$

and

$$\kappa(t) = \kappa(0) \exp\left(-\int_0^t \bar{\sigma} \sin \zeta(t') dt'\right) \exp\left(\gamma_0 t + \sqrt{\gamma_0} B_t^{(3)}\right)$$



Large scale
group velocity:

$$\bar{v}$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$
assumed to be cst.

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Wave:

$$he^{i\phi}$$

$$k = \nabla\phi = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$$

$$\zeta = 2(\theta_k + \bar{\phi})$$

$$\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$$

Simplification for single ray dynamics - Stochastic case -

We can solve the stationary Fokker-Planck for ζ ,

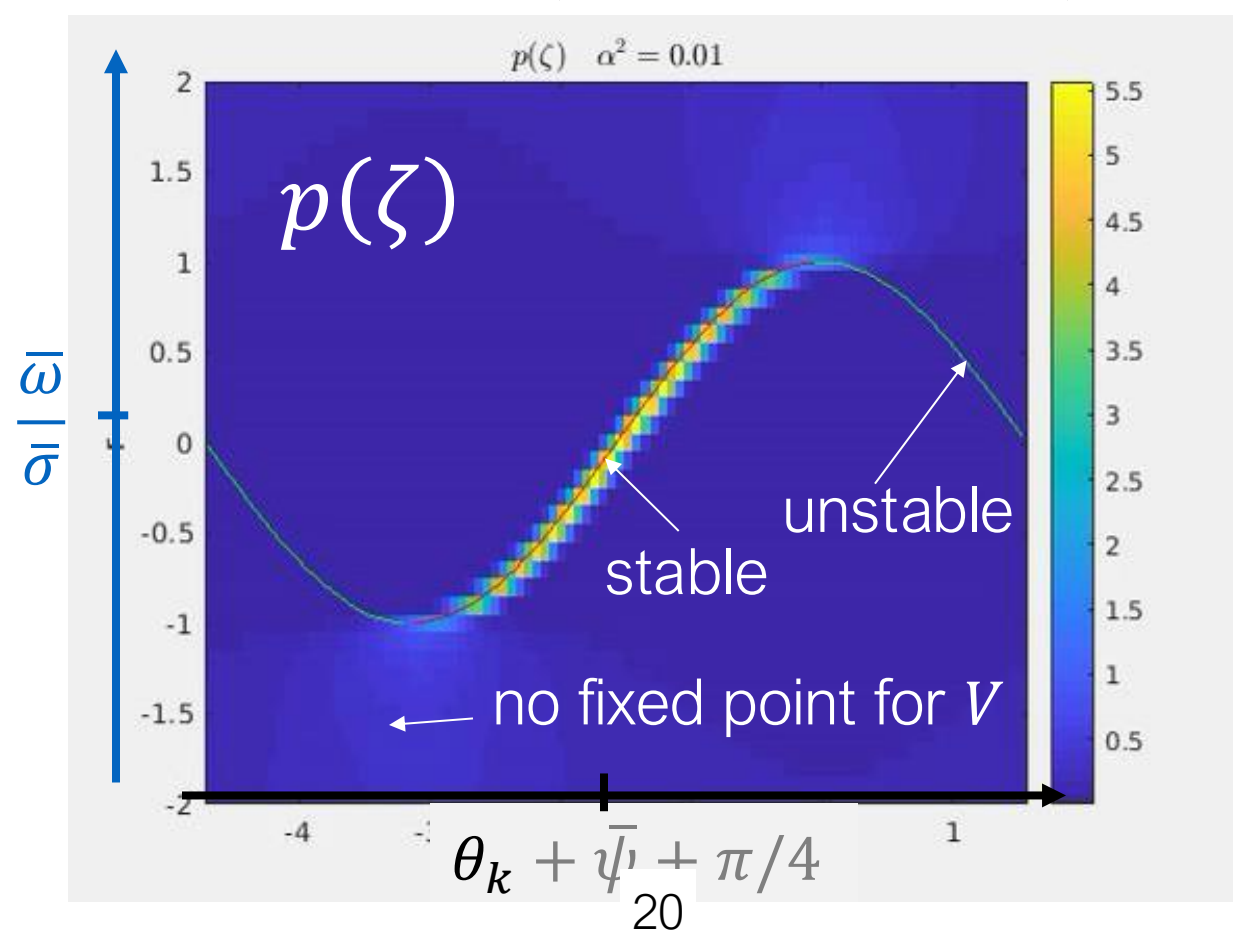
$$d\zeta = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t))dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt}$$

\Rightarrow Up to periodicity constraints,

$$p(\zeta) \propto \exp\left(-\frac{2\bar{V}(\zeta)}{3\gamma_0}\right)$$

and

$$\kappa(t) = \kappa(0) \exp\left(-\int_0^t \bar{\sigma} \sin \zeta(t') dt'\right) \exp\left(\gamma_0 t + \sqrt{\gamma_0} B_t^{(3)}\right)$$



Large scale
group velocity:

$$\bar{v}$$

$\bar{\sigma}, \bar{\omega}, \bar{\phi}$ related to $\nabla \bar{v}^T$
assumed to be cst.

Small scale
group velocity:

$$v' = \sigma \frac{dB_t}{dt}$$

(δ -correlated in t
homogenous
& isotropic)

$$a_0 = \frac{1}{2} \mathbb{E} \|\sigma dB_t\|^2 / dt$$

$$\gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Wave:

$$he^{i\phi}$$

$$k = \nabla\phi = \kappa \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$$

$$\zeta = 2(\theta_k + \bar{\phi})$$

$$\bar{V}(\zeta) = \bar{\omega} \zeta - \bar{\sigma} \sin \zeta$$

Simplification for single ray dynamics - Stochastic case -

We can solve the stationary Fokker-Planck for ζ ,

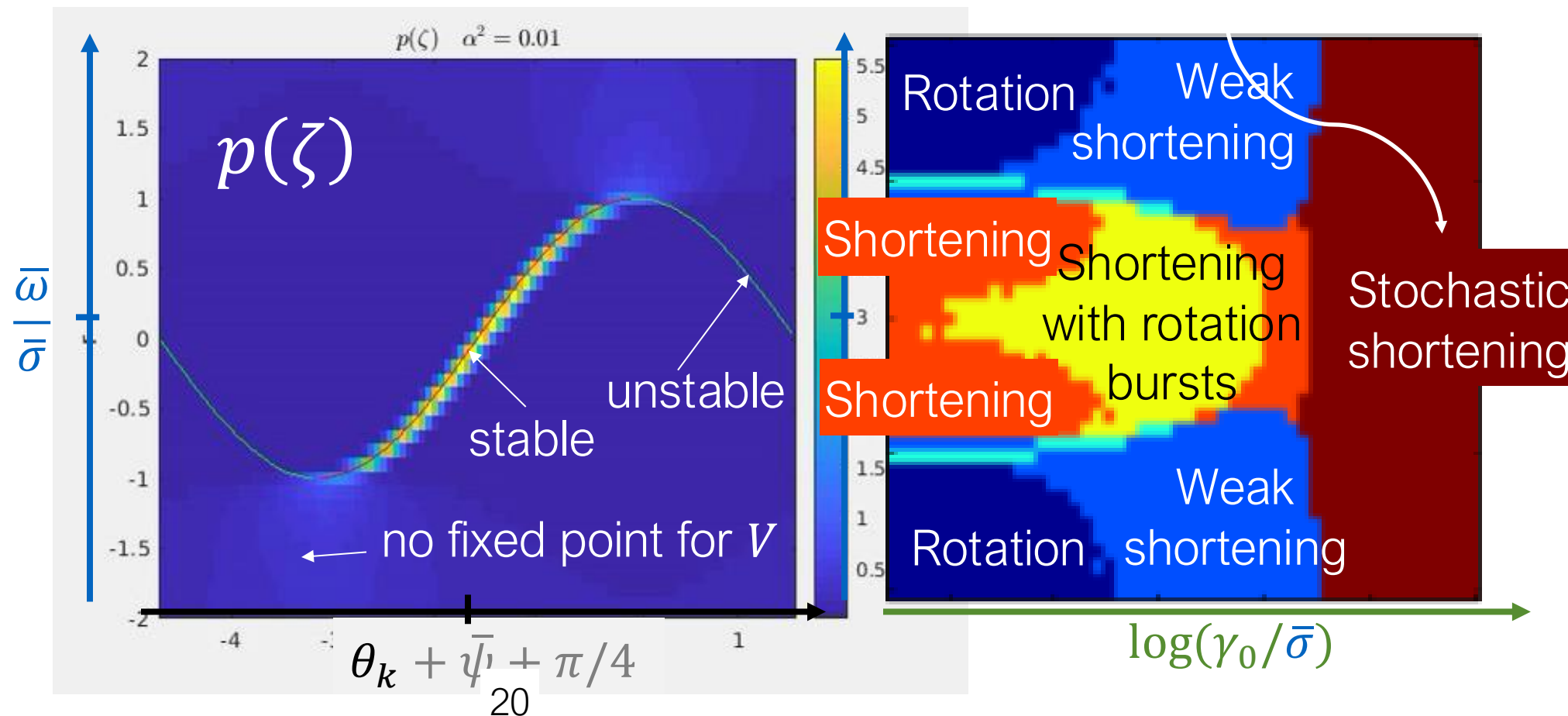
$$d\zeta = -\frac{\partial \bar{V}}{\partial \zeta}(\zeta(t))dt + \sqrt{3\gamma_0} \frac{dB_t^{(4)}}{dt}$$

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and

$$\kappa(t) = \kappa(0) \exp\left(-\int_0^t \bar{\sigma} \sin \zeta(t') dt'\right) \exp\left(\gamma_0 t + \sqrt{\gamma_0} B_t^{(3)}\right)$$



II. Markovian closure

II.D. Waves trapped in a jet

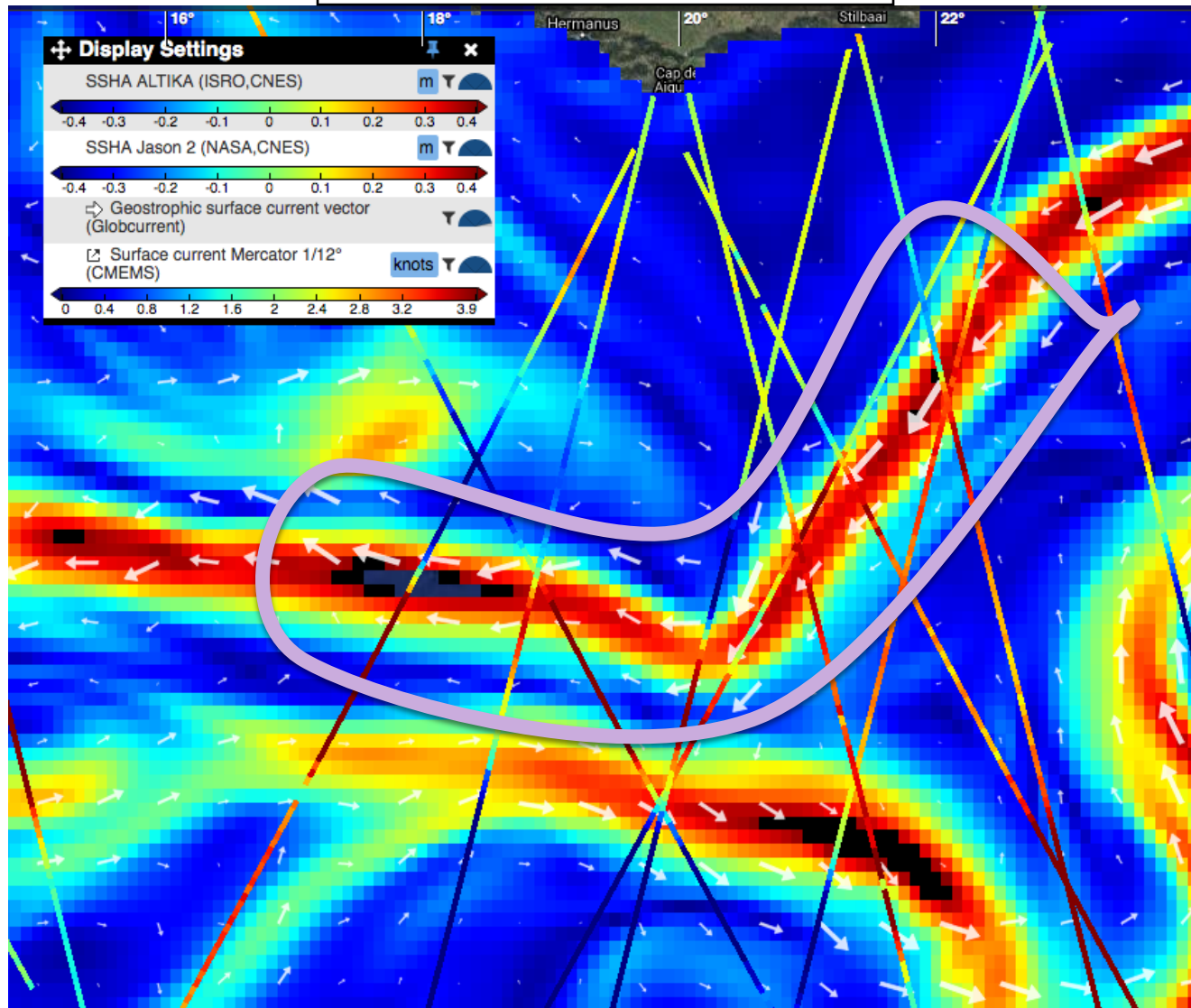
Sweel trapped in a jet



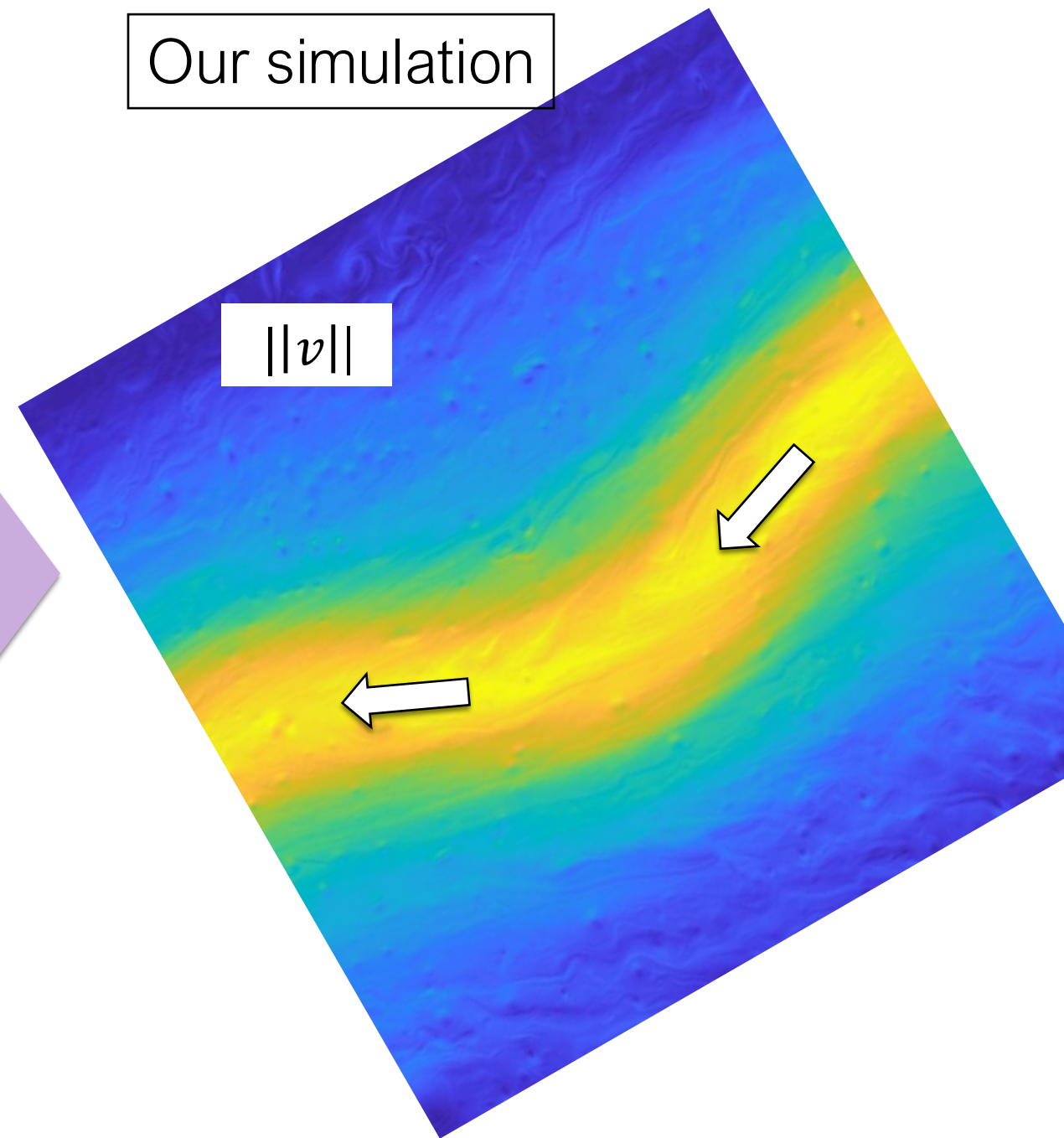
Sentinel 2

Idealized jet

Situation to mimic:



Our simulation



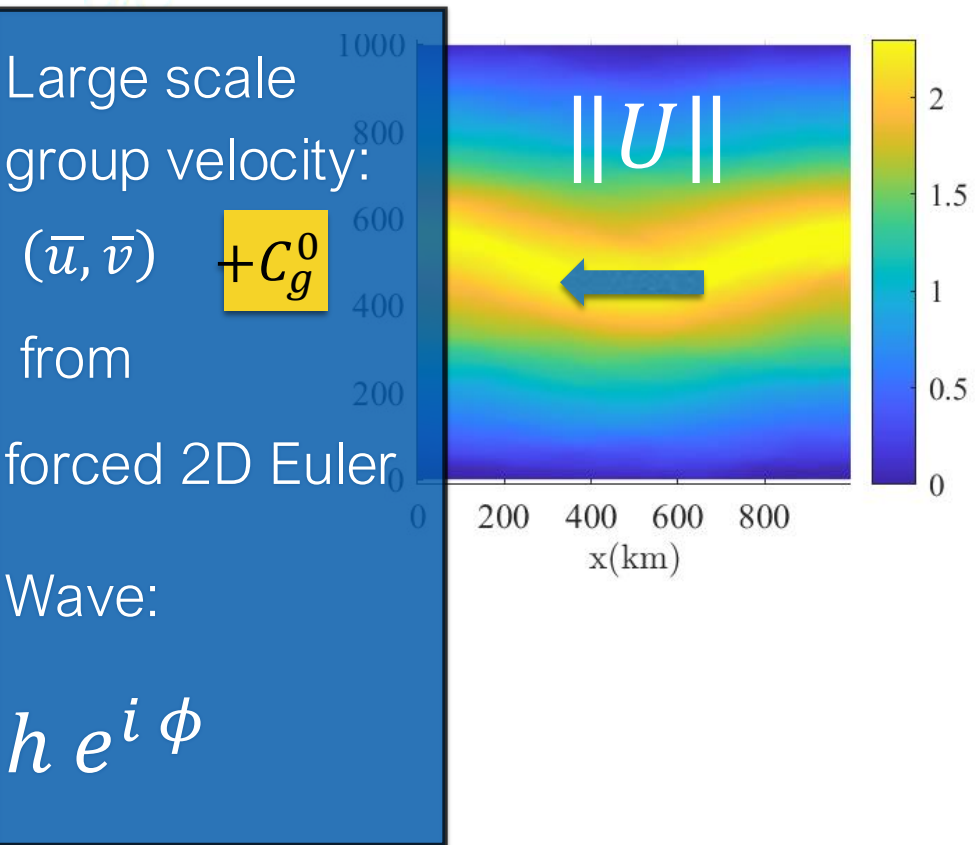
Agulhas current
Mercator current (01/01/2016)
+ SSHA weekly

Swells in a jet

forced 2D Euler
($U \sim 1\text{m/s}$)

ω

Ray tracing with
trapping + scattering

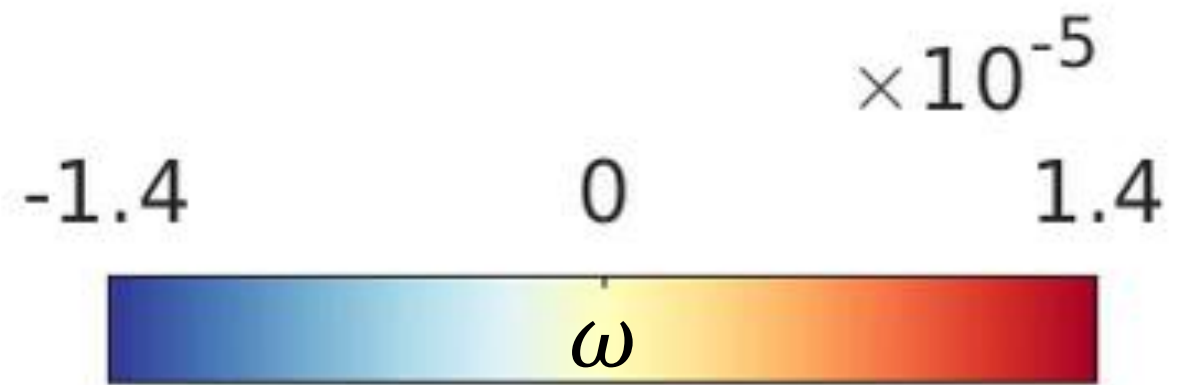
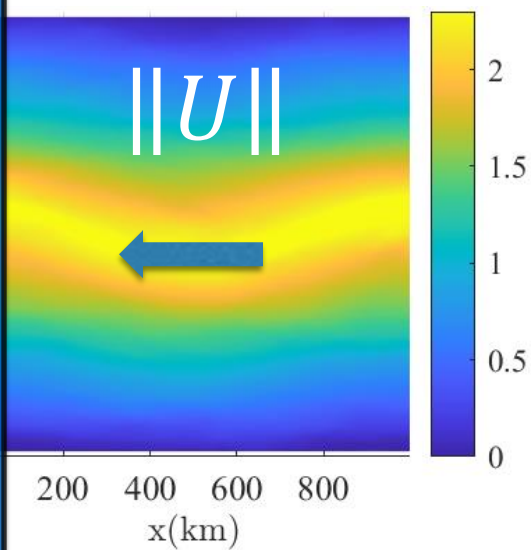


Swells in a jet

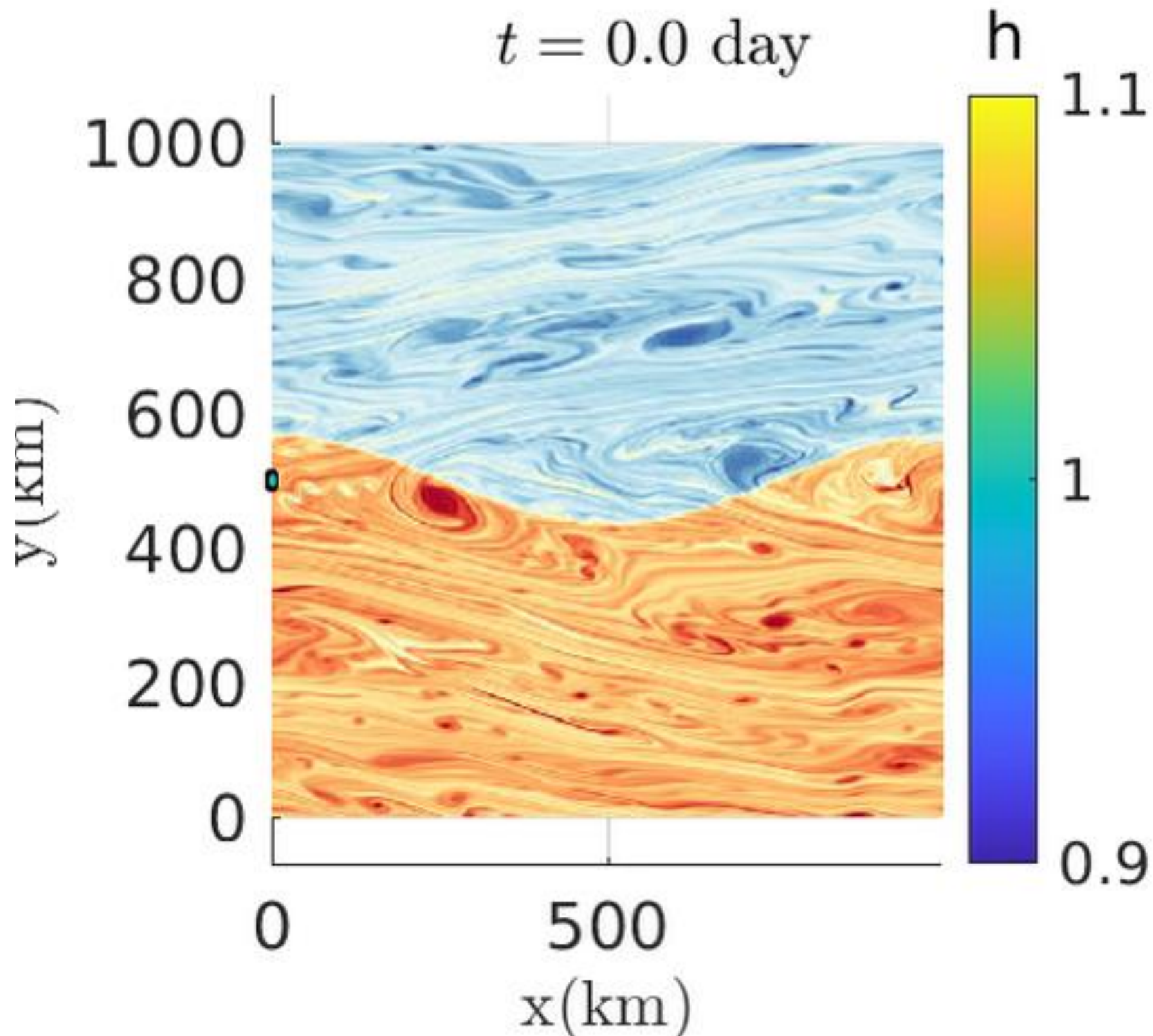
forced 2D Euler
($U \sim 1\text{m/s}$)

Ray tracing with
trapping + scattering

Large scale
group velocity:
 $(\bar{u}, \bar{v}) + C_g^0$
from
forced 2D Euler
Wave:
 $h e^{i\phi}$



$t = 0.0$ day



Swells in a jet

Resolution effect & stochastic closure

$$v = \bar{v} + v'$$

ω

$$v = \bar{v}$$

ω

$$v = \bar{v} + \sigma \dot{B}$$

ω

Deterministic
reference:

wave groups in
512 x 512 flow

Deterministic
benchmark:

wave groups in
smoothed flow \bar{v} (4 x 4)

Our random
model

wave groups in
smoothed flow \bar{v} (4 x 4)
+ time-uncorrelated v'

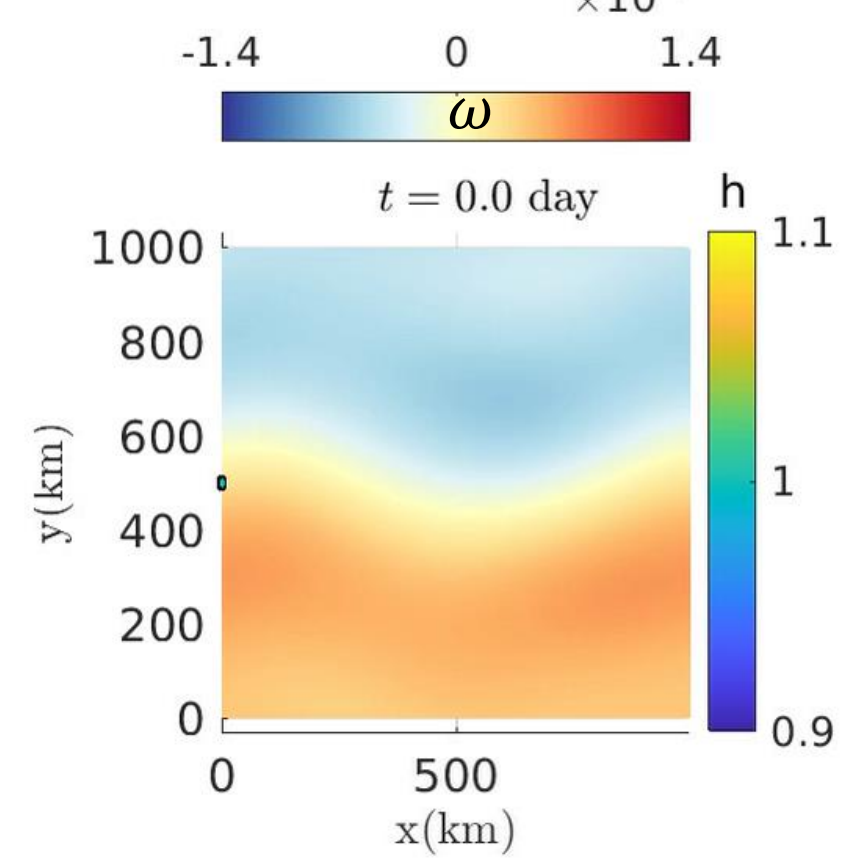
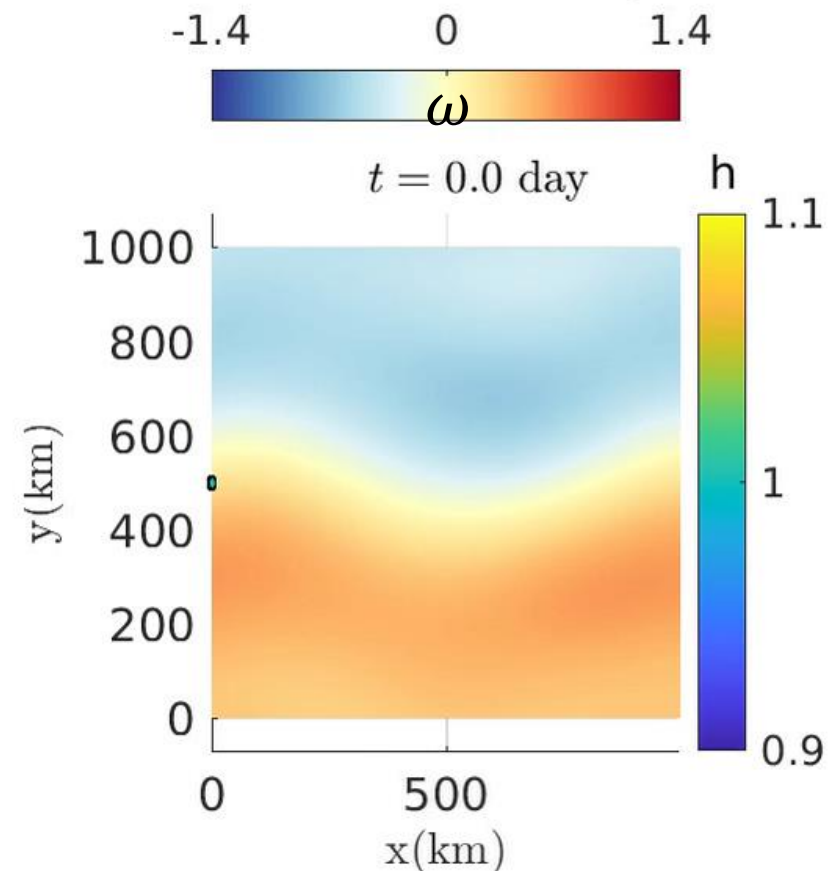
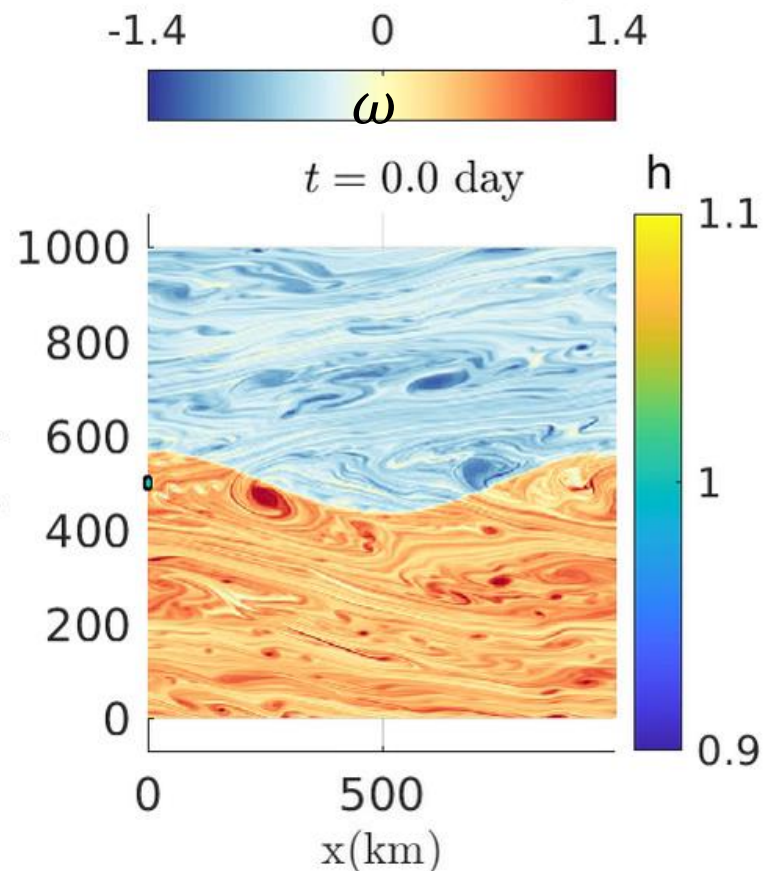
Swells in a jet

Resolution effect & stochastic closure

$$v = \bar{v} + v' \times 10^{-5}$$

$$v = \bar{v} \times 10^{-5}$$

$$v = \bar{v} + \sigma \dot{B} \times 10^{-5}$$



Deterministic reference:
wave groups in
512 x 512 flow

Deterministic benchmark:
wave groups in
smoothed flow \bar{v} (4 x 4)

Our random model
wave groups in
smoothed flow \bar{v} (4 x 4)
+ time-uncorrelated v'

Swells in a jet

Analytic solution

Large scale
group velocity:

$$\bar{u} \approx \bar{U}_0 - \frac{1}{2}\bar{\beta}y^2 + C_g^0$$

$$\bar{v} \approx 0$$

with $\bar{U}_0, \bar{\beta} < 0$

Small scale

group velocity:

$$v' = \sigma dB_t / dt$$

δ -correlated in t

Isotropic and

homogeneous in x

Wave:

$$h e^{i\phi}$$

Oscillating ray (x_r, y_r, k_x, k_y)

$$\frac{d^2 y_r}{dt^2} + \bar{\omega}_r^2 y_r = v_g^0 \sqrt{3\gamma_0} \dot{\beta}_t$$

$$\text{with } \bar{\omega}_r = \sqrt{|v_g^0 \bar{\beta}|} \text{ and } \gamma_0 = \frac{1}{8} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$$

Analytic stochastic solutions for

- Ray position & wave direction

$$y_r(t) = y_r(0) \cos(\bar{\omega}_r t) + \dots + Y_{\gamma_0} \sqrt{\bar{\omega}_r} \int_0^t \sin(\bar{\omega}_r(t-t')) d\beta_{t'}$$

- Wave number fluctuations
- Mean distribution of wave action and wave energy

II. Beyond Markovian closure

II.A. Waves trapped in a SQG jet

Large scale
group velocity:

$$\bar{v} + c_g^0$$

Small scale
group velocity:

$$v'$$

Wave:

$$he^{i\phi}$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Beyond validity of GFW

- Validity :

$$\epsilon = \frac{\text{(Along-ray } v' \text{ correlation time)}}{\left(\text{characteristic time of wave group properties evolution} \right)} = \frac{\left(\frac{l_{v'}}{\|c_g^0\|} \right)}{\left(\frac{1}{\|\nabla v\|} \right)}$$
$$= \frac{l_{v'}}{l_v} \frac{\|v\|}{\|c_g^0\|} \ll 1$$

- Limitations :

- Fast waves $\left(\frac{\|c_g^0\|}{\|v\|} \gg 1 \right)$
- Small-scale currents $(l_{v'} \ll 1)$
- Moderate current gradients $(\|\nabla v\| \ll 1)$
(\Rightarrow steep spectrum \Rightarrow non-local current dynamics)

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v'$$

Wave:

$$he^{i\phi}$$

Doppler

frequency:

$$\omega_0 \propto |k|^\alpha$$

Beyond validity of GFW

- Validity :

$$\epsilon = \frac{\text{(Along-ray } v' \text{ correlation time)}}{\left(\text{characteristic time of wave group properties evolution} \right)} = \frac{\left(\frac{l_{v'}}{\|C_g^0\|} \right)}{\left(\frac{1}{\|\nabla v\|} \right)}$$

$$= \frac{l_{v'}}{l_v} \frac{\|v\|}{\|C_g^0\|} \ll 1$$

- Limitations :

- Fast
 - Small
 - Mod
- GFW would fail for strong local current dynamics.
Example : Swells in a jet with flat spectrum,
typically forced SQG with $U \sim 1\text{m/s}$

(\Rightarrow steep spectrum \Rightarrow non-local current dynamics)

Time-correlated model for v'

Forward ray tracing

$$v = \bar{v} + v'$$

$$v = \bar{v}$$

$$v = \bar{v} + v'_{\text{cor}}$$

wave group amplitude



wave group amplitude



wave group amplitude



Vorticity

Vorticity

Vorticity

Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

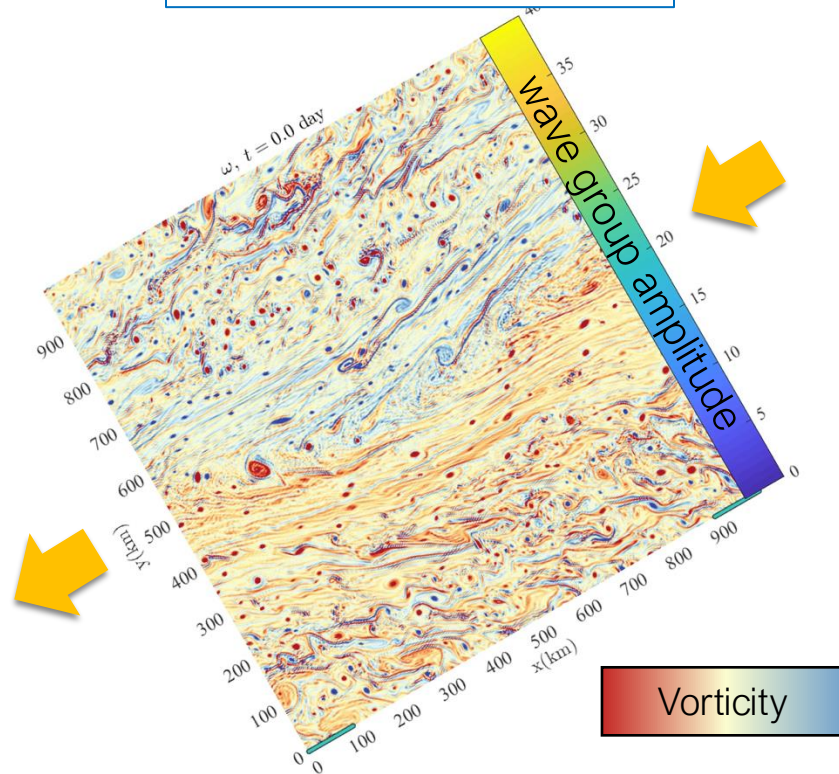
Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

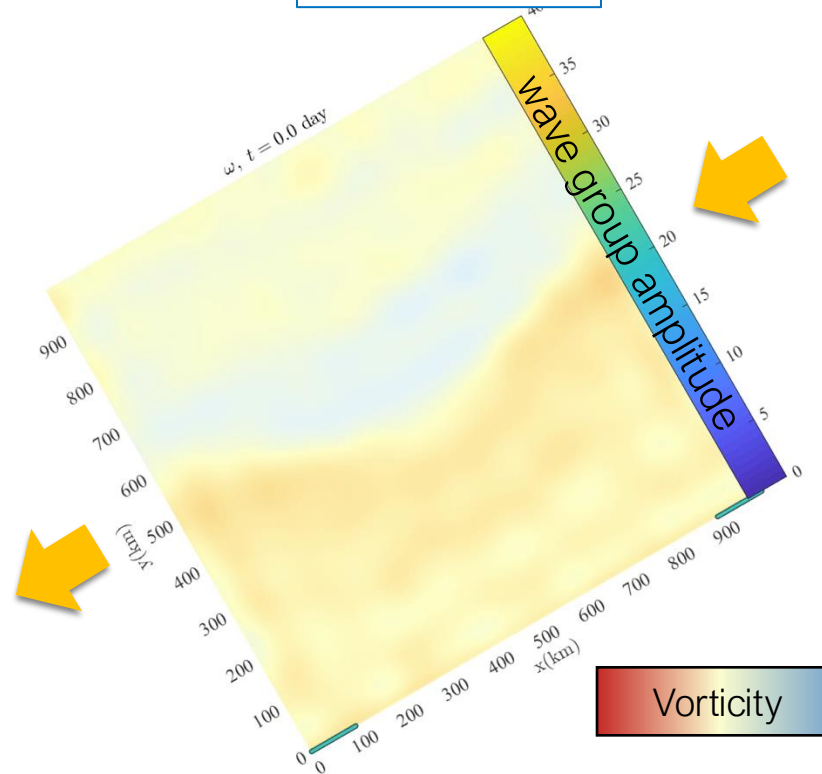
Forward ray tracing

$$v = \bar{v} + v'$$



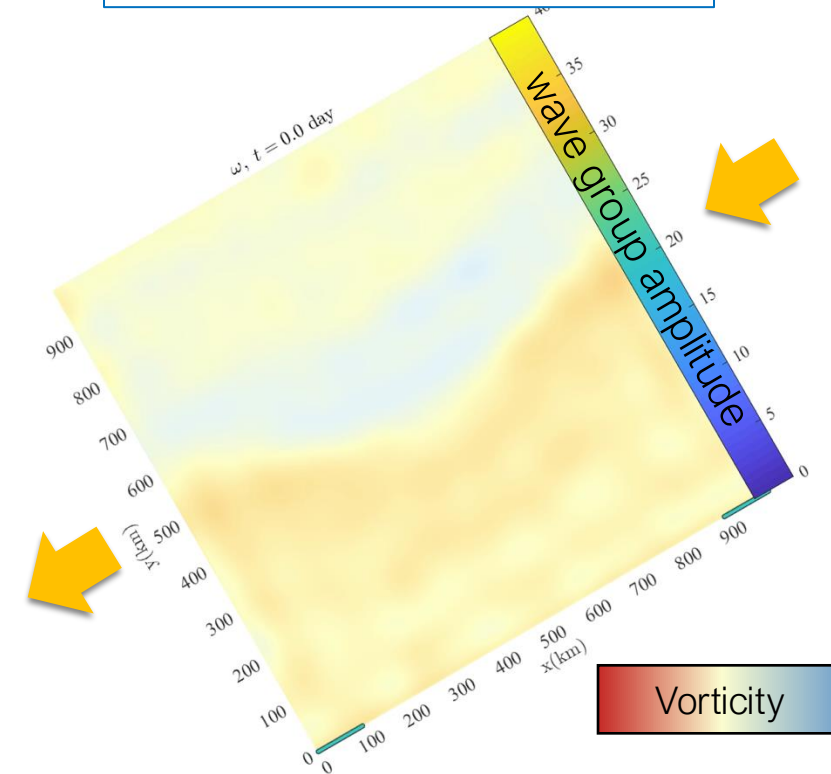
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

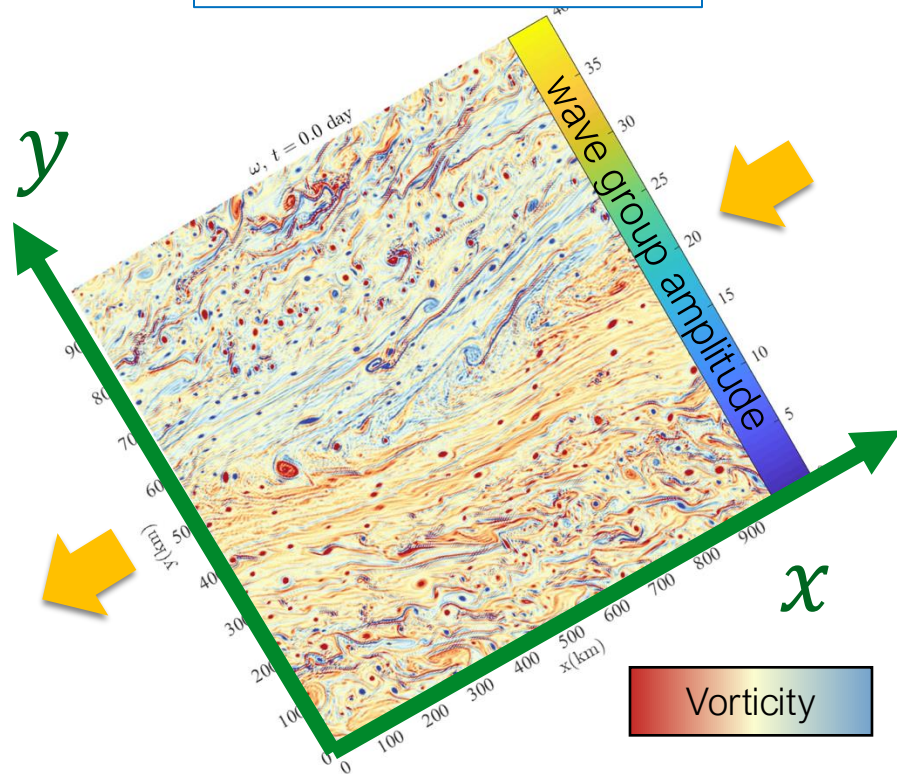


Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

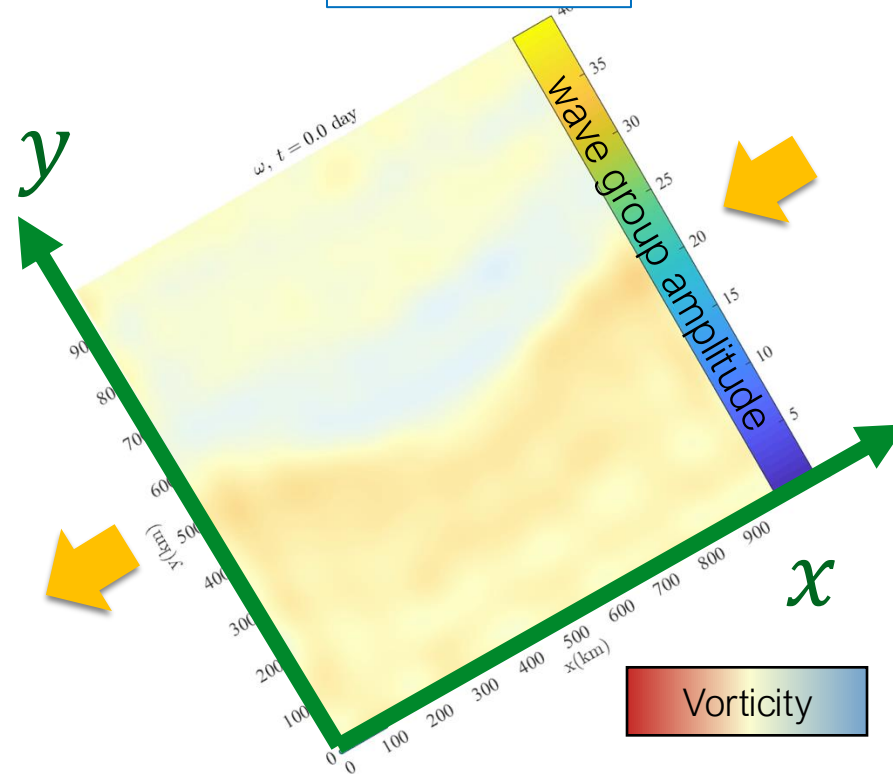
Forward ray tracing

$$v = \bar{v} + v'$$



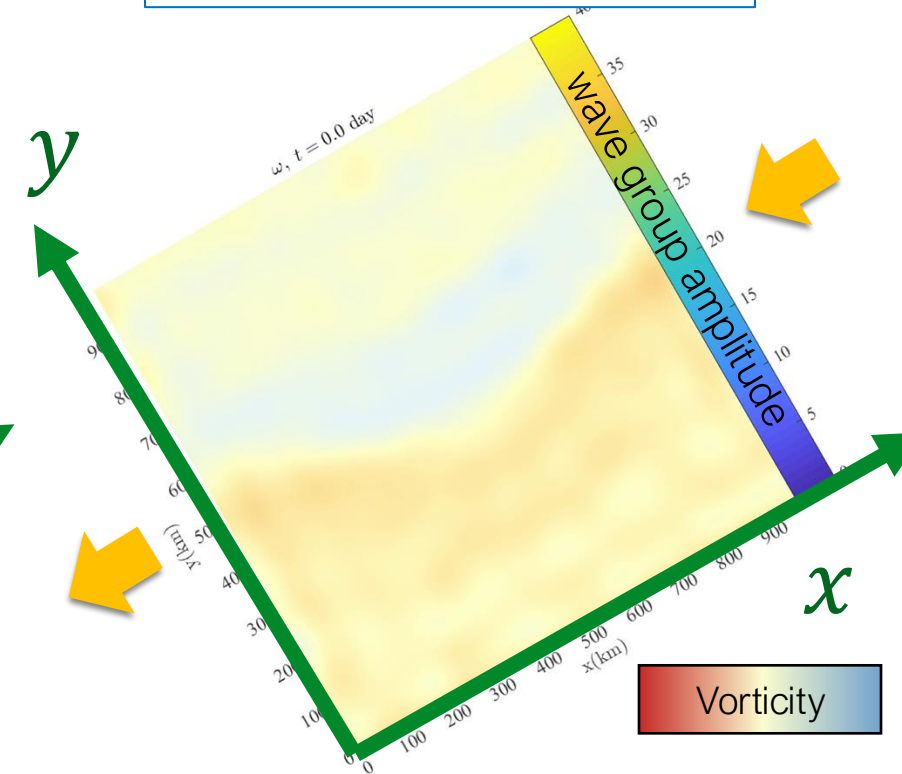
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

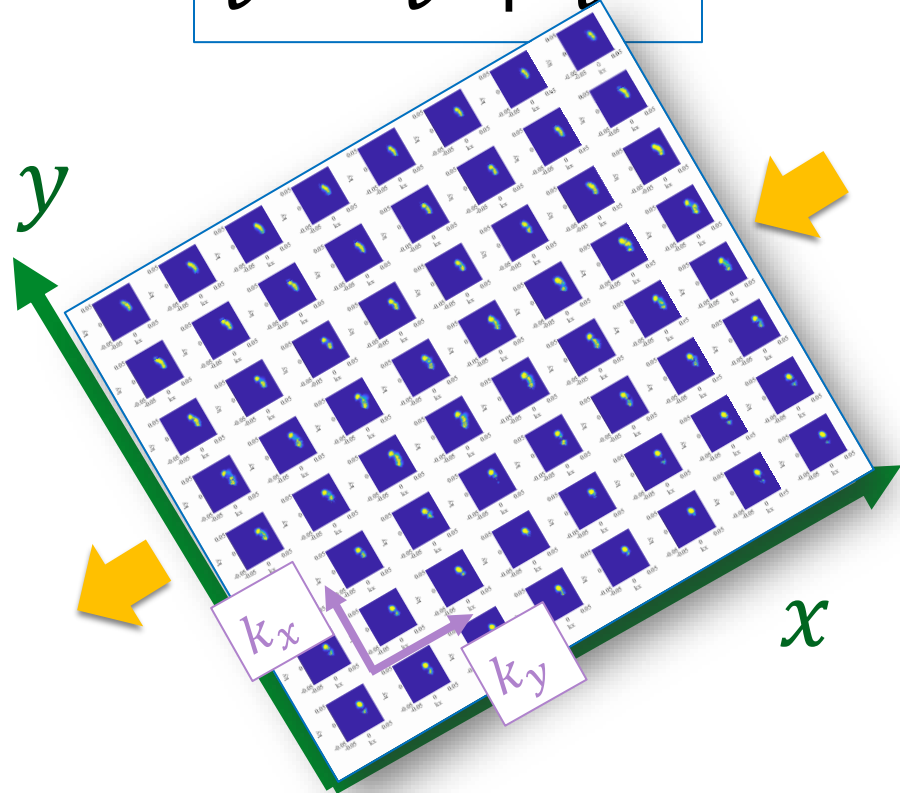


Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

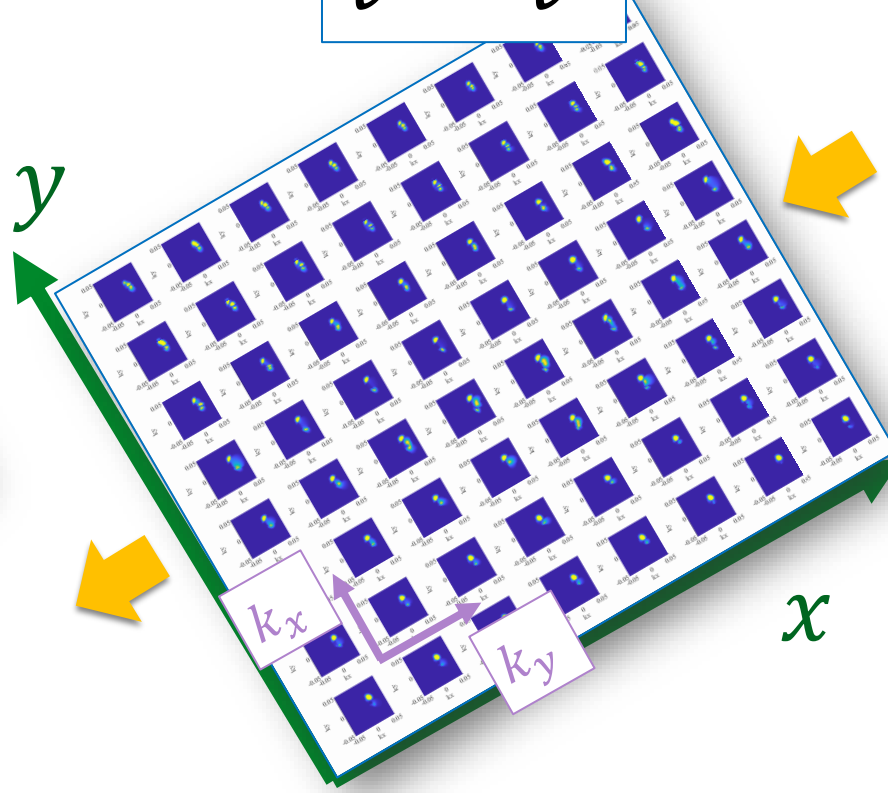
Wave spectra

$$v = \bar{v} + v'$$



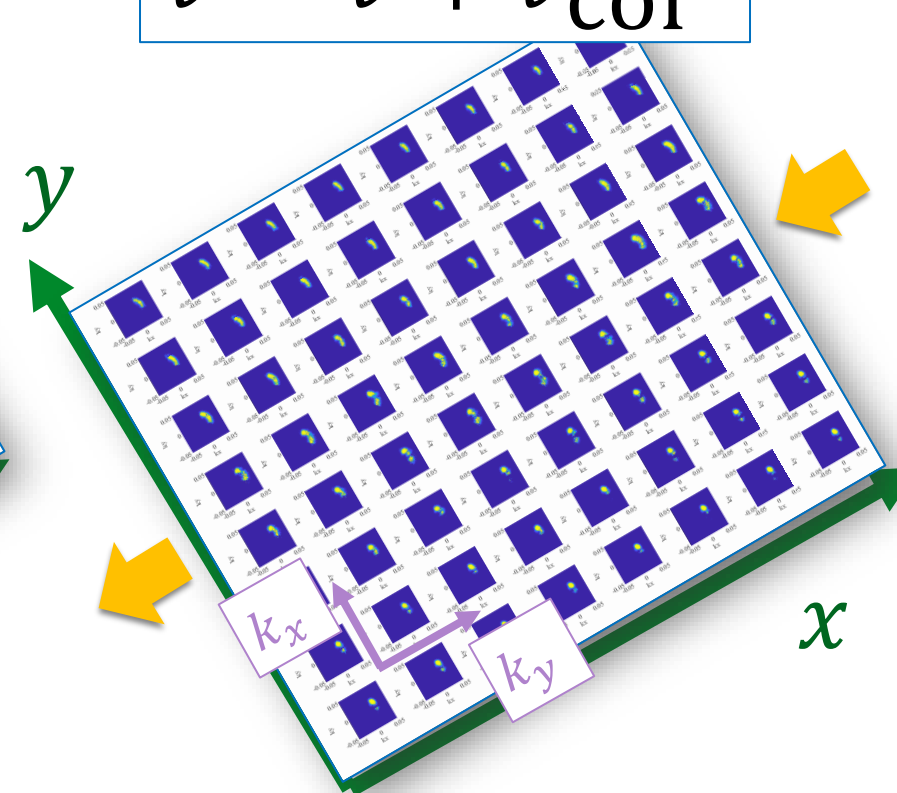
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

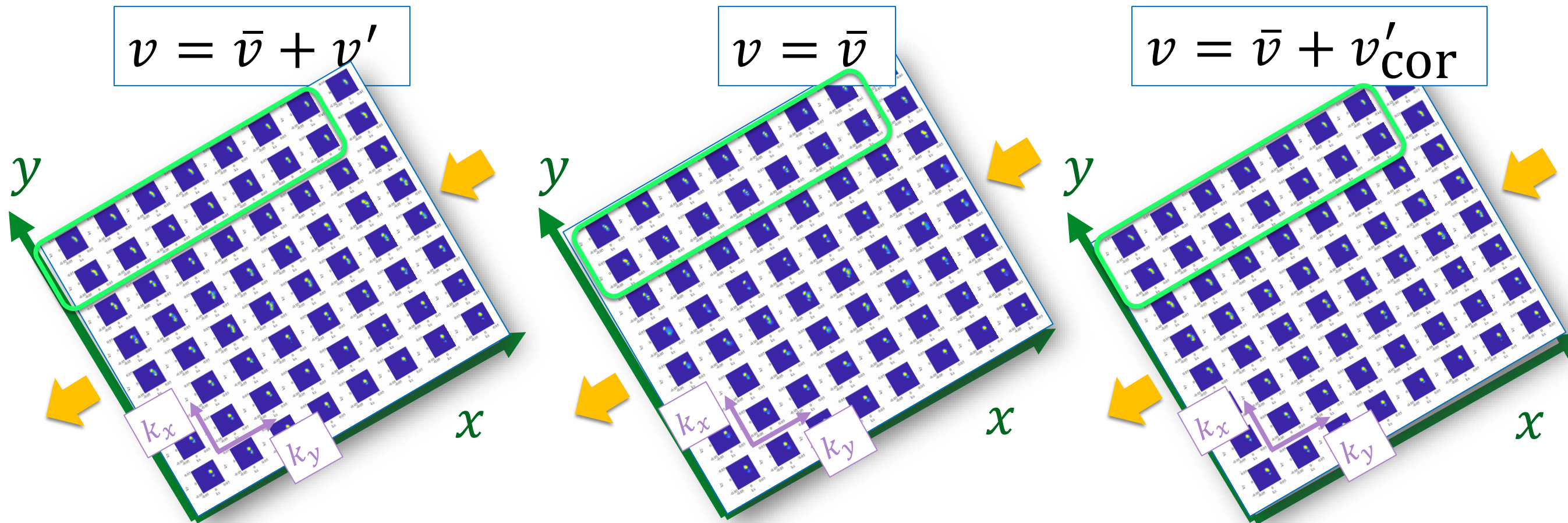
$$v = \bar{v} + v'_{\text{cor}}$$



Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

Wave spectra



Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

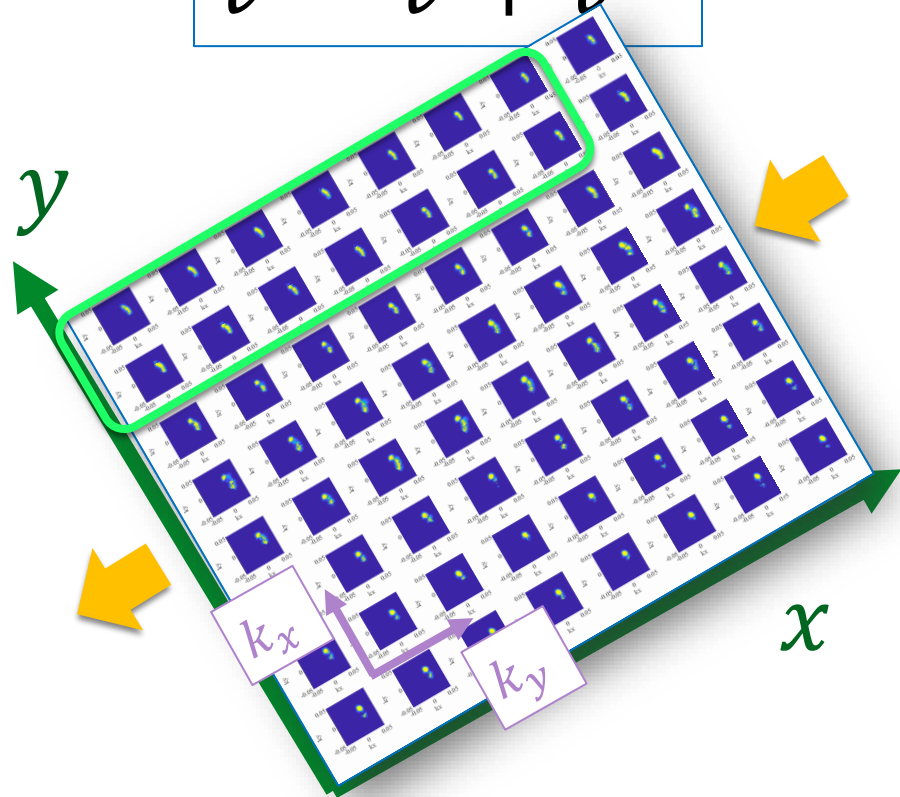
Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

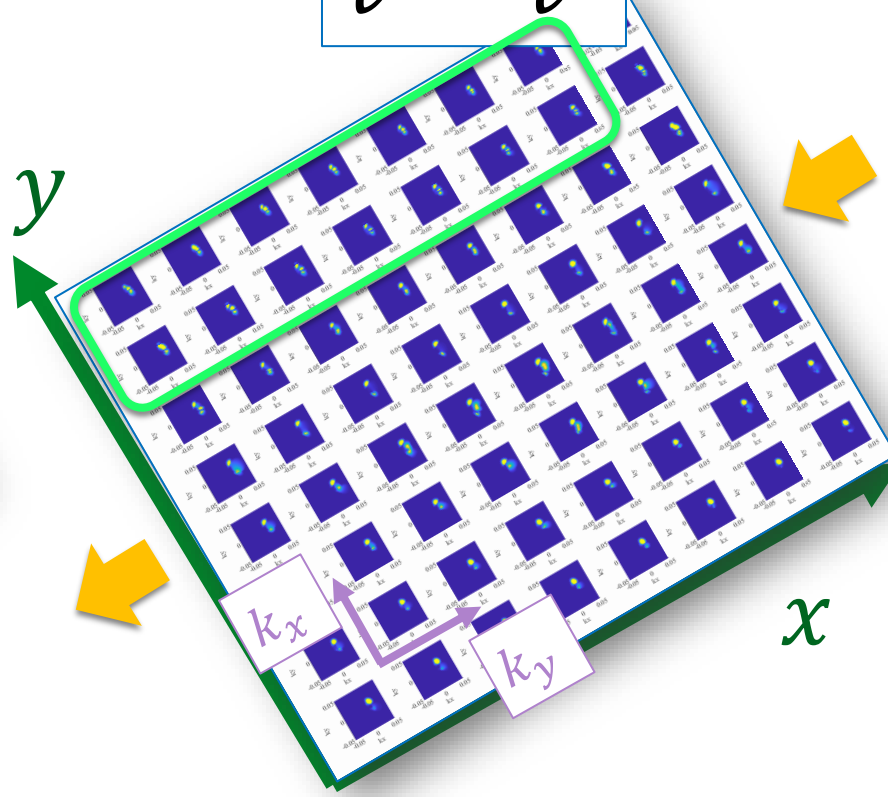
Time-correlated model for v'

Wave spectra

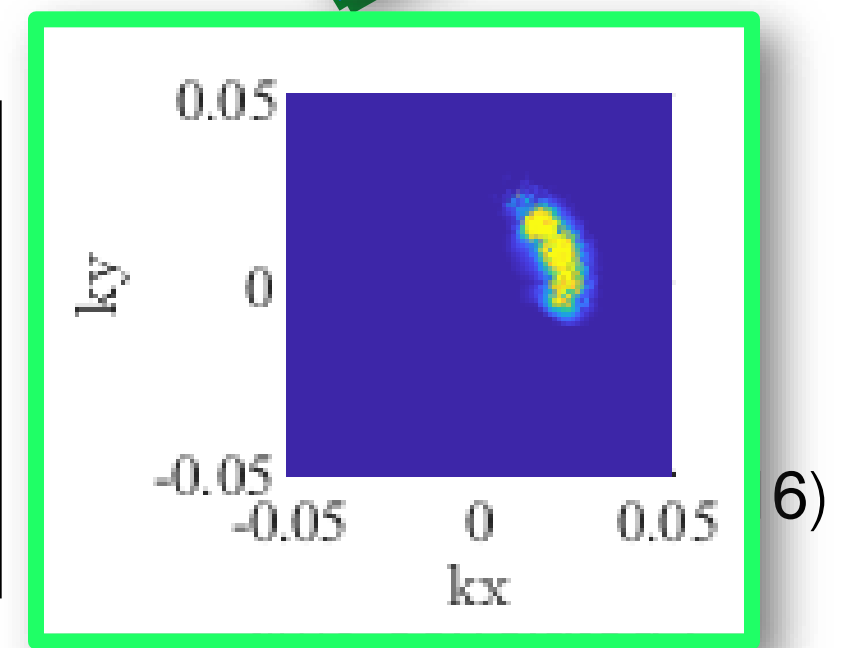
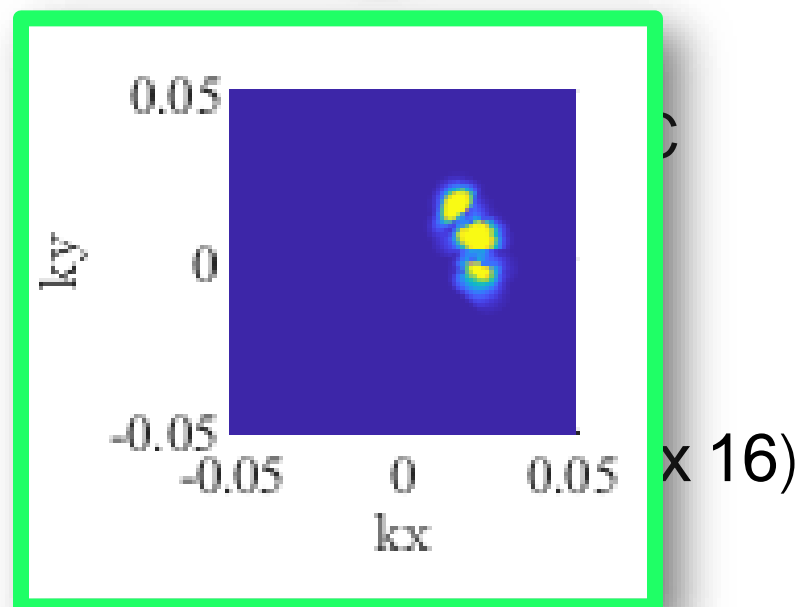
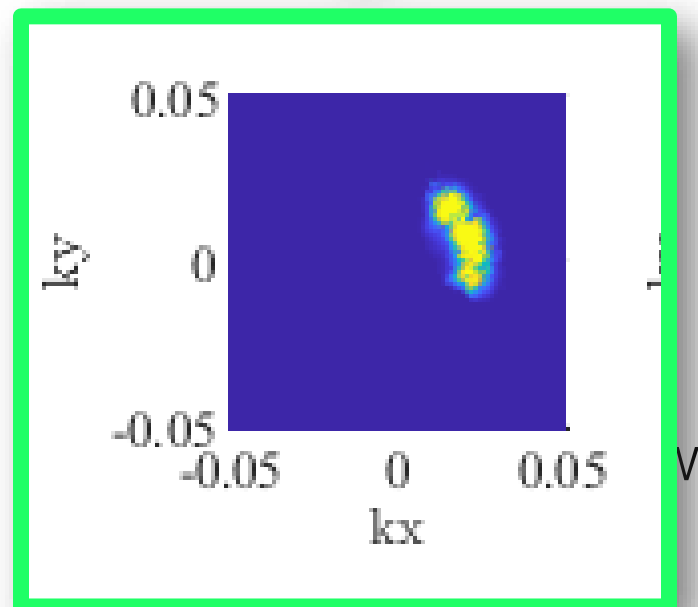
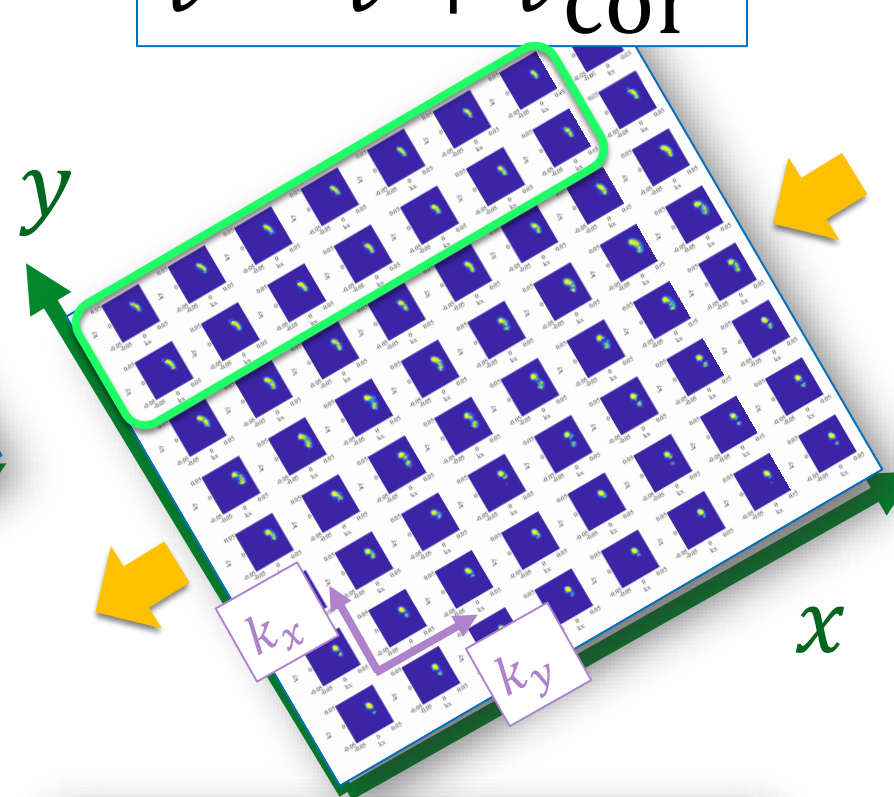
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



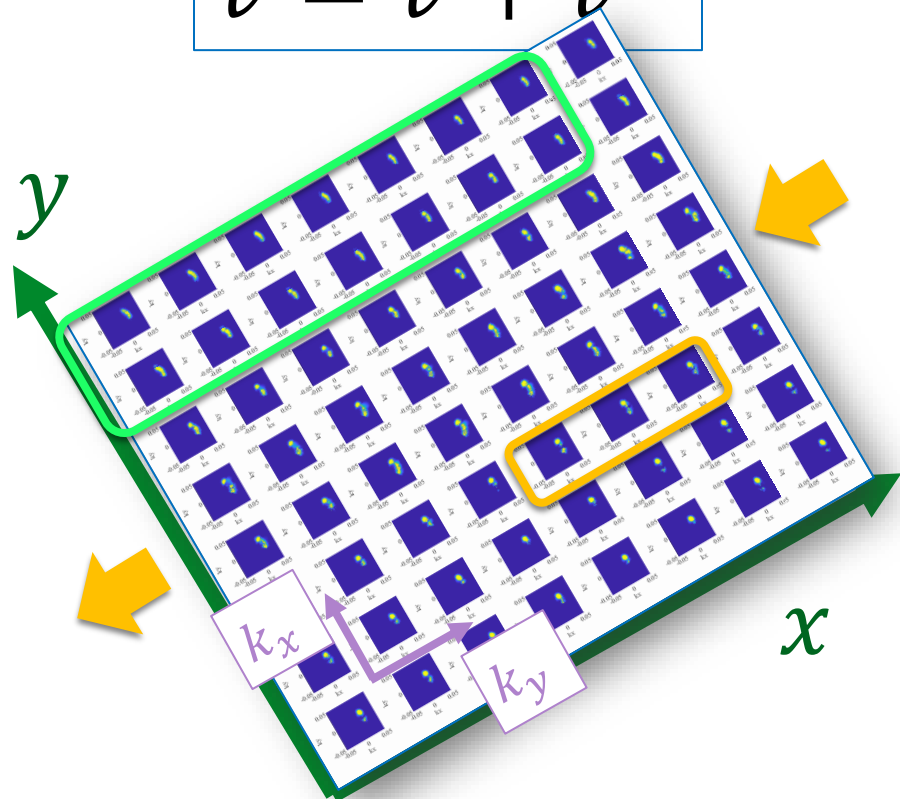
$$v = \bar{v} + v'_{\text{cor}}$$



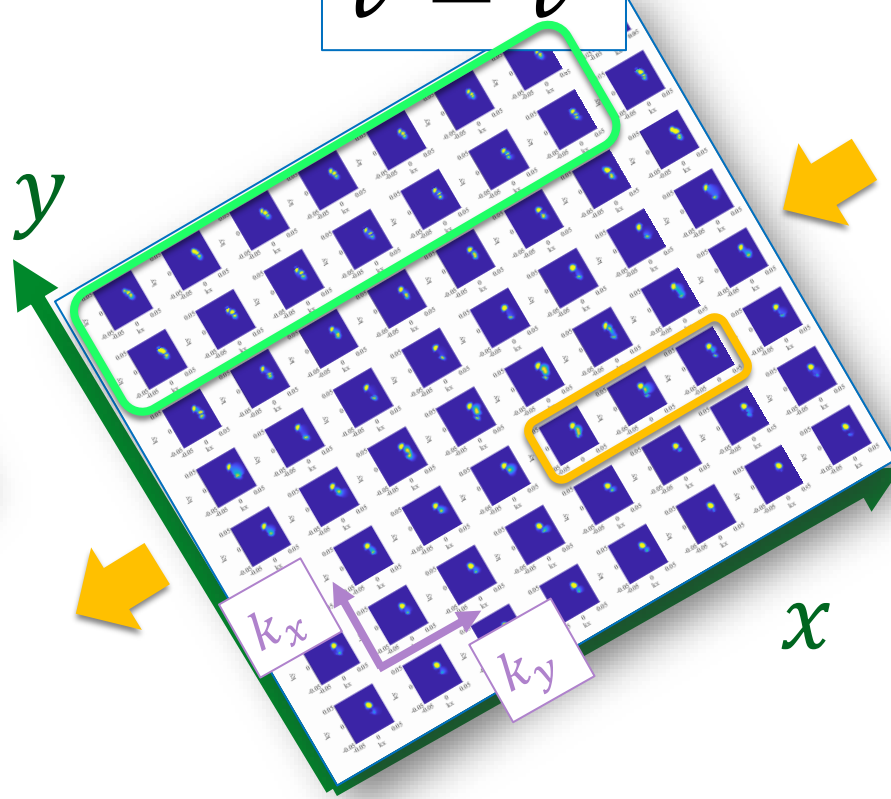
Time-correlated model for v'

Wave spectra

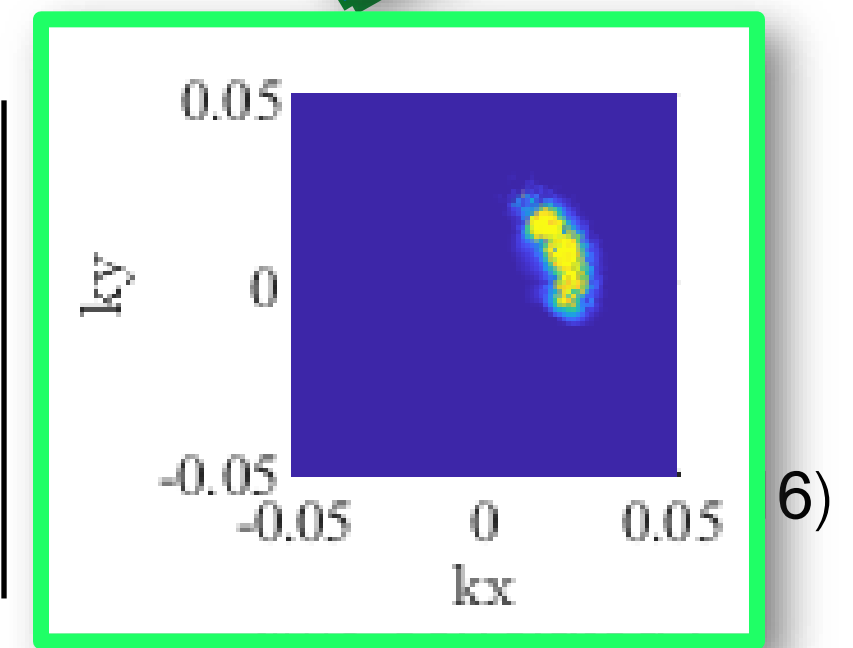
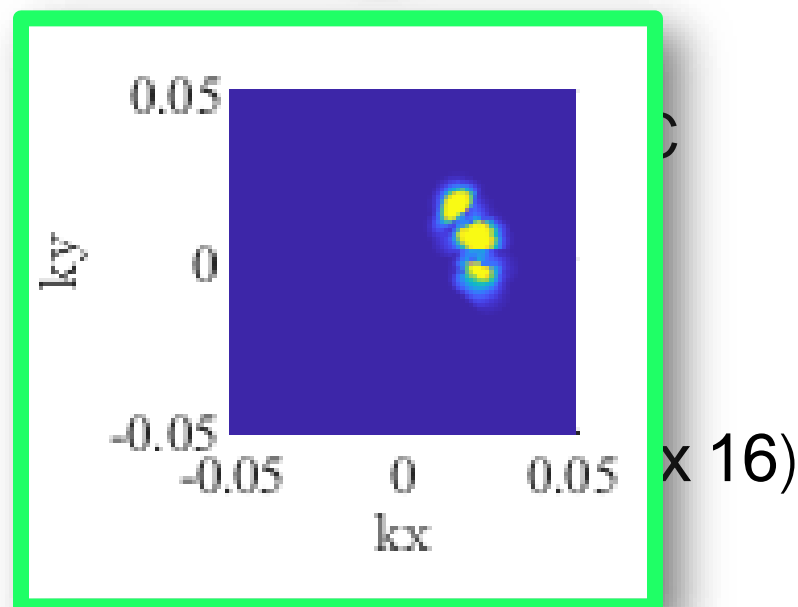
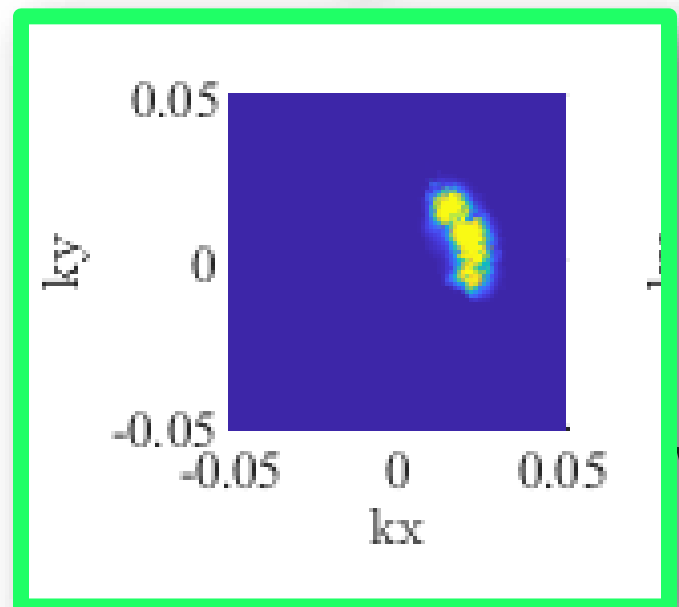
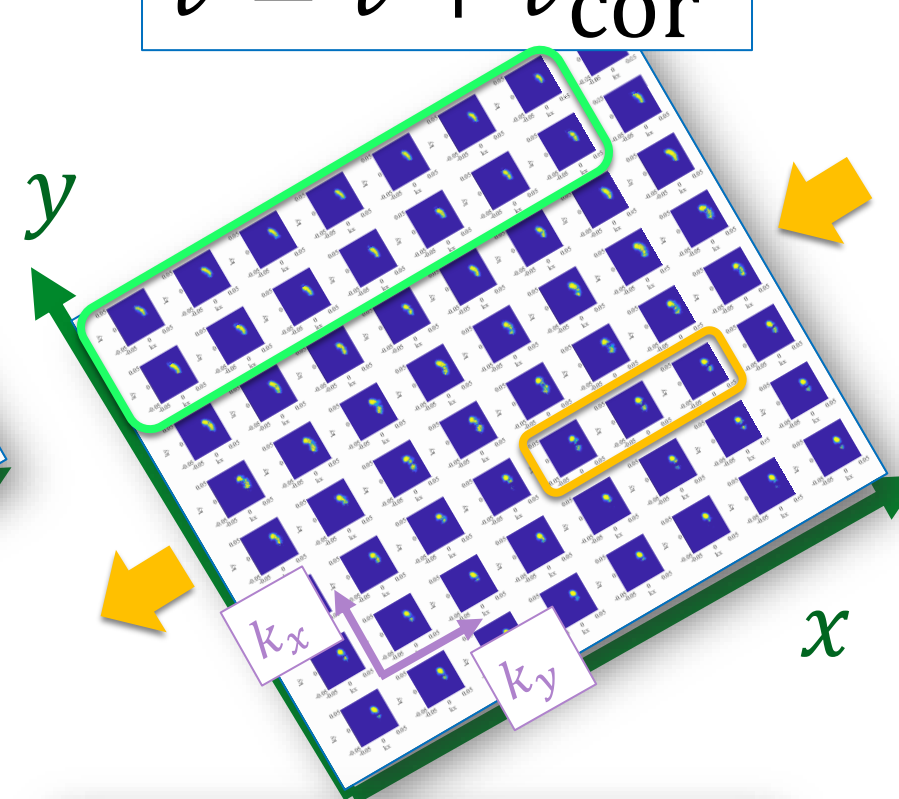
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



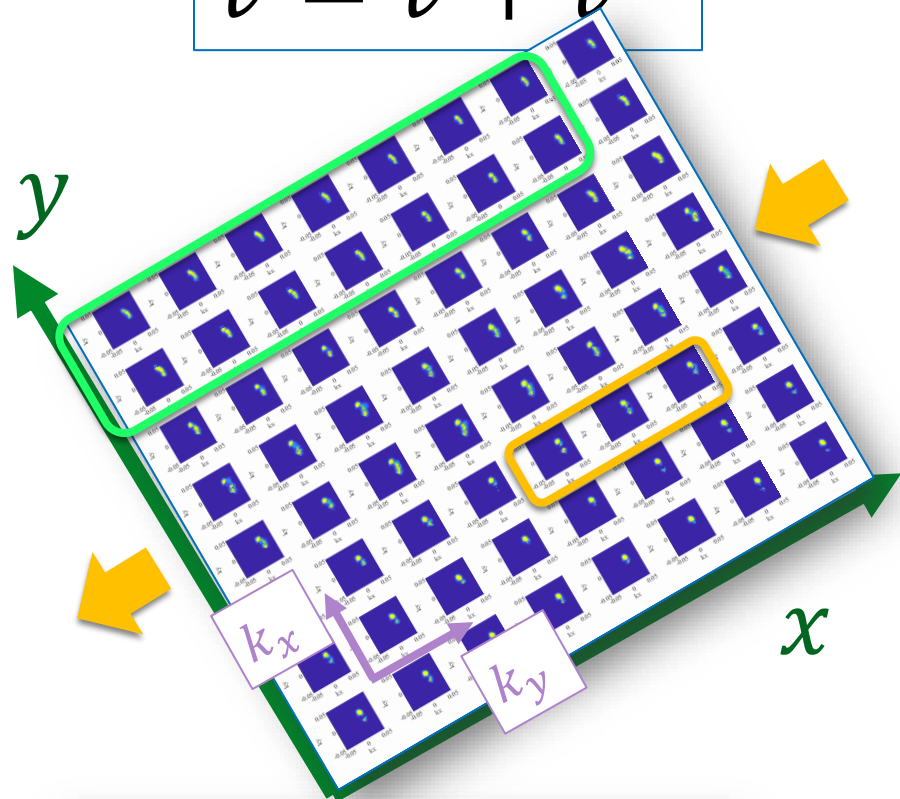
$$v = \bar{v} + v'_{\text{cor}}$$



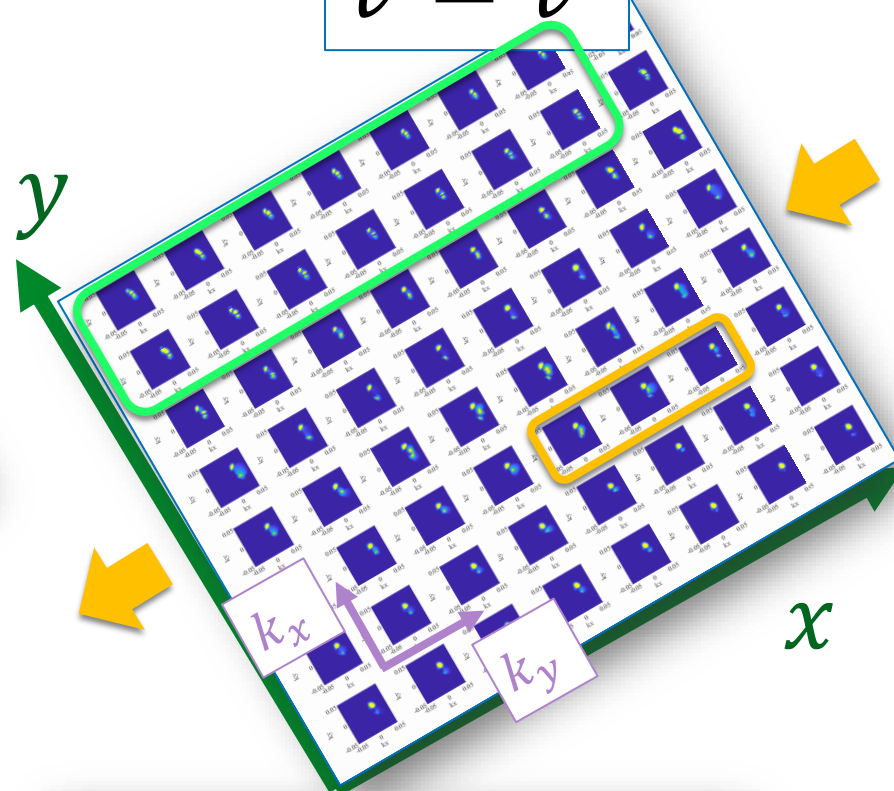
Time-correlated model for v'

Wave spectra

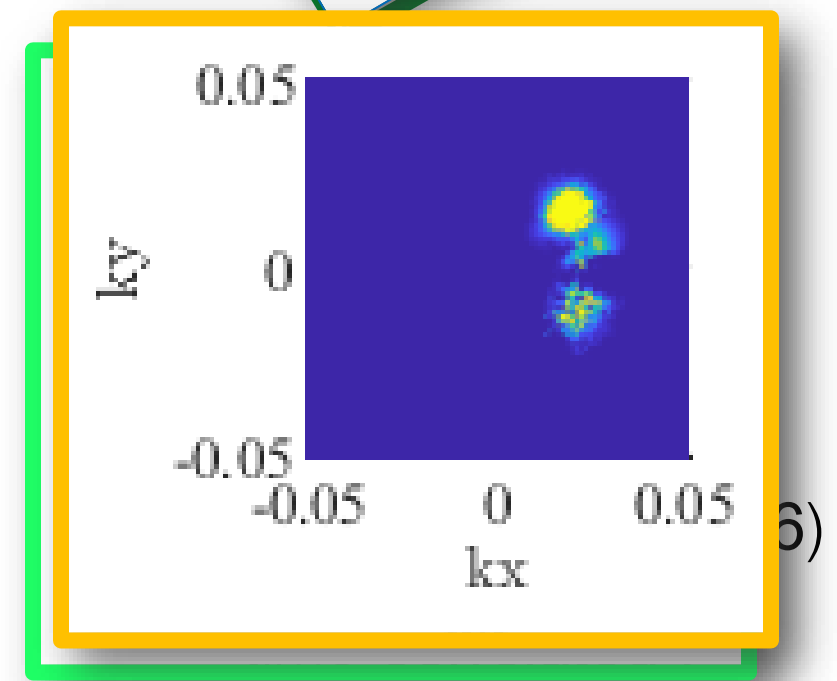
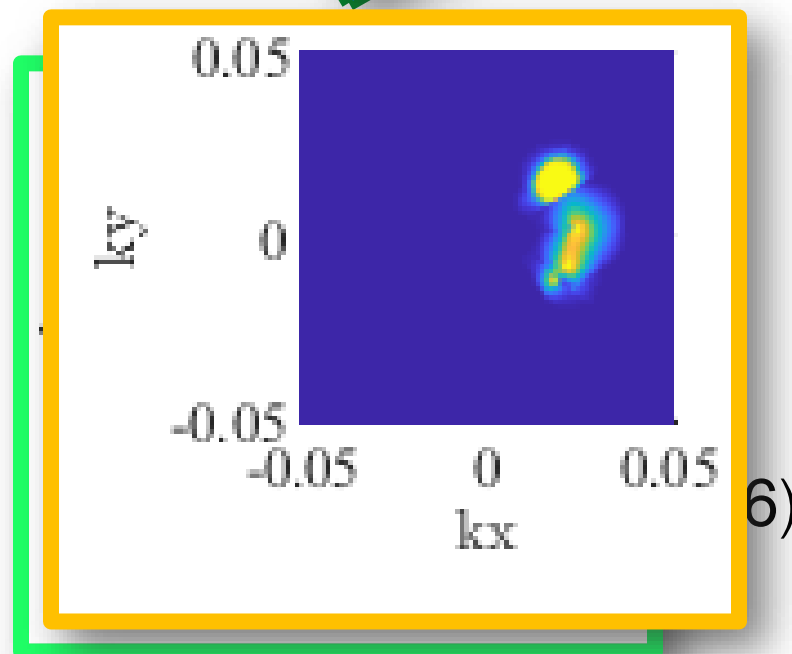
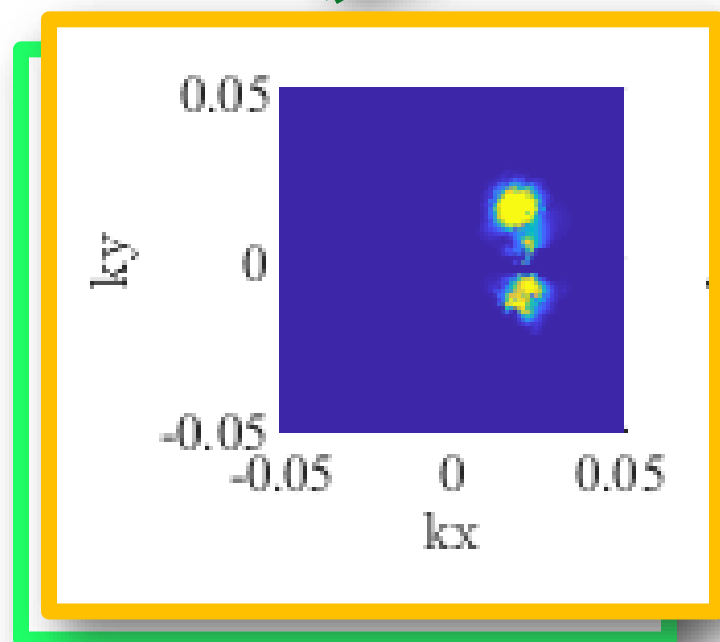
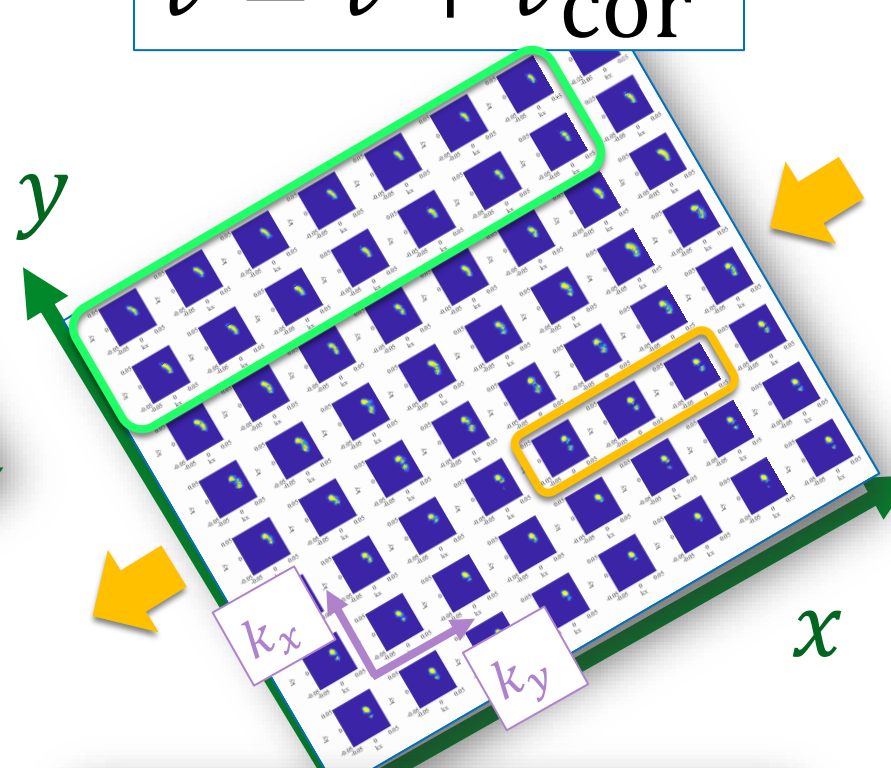
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



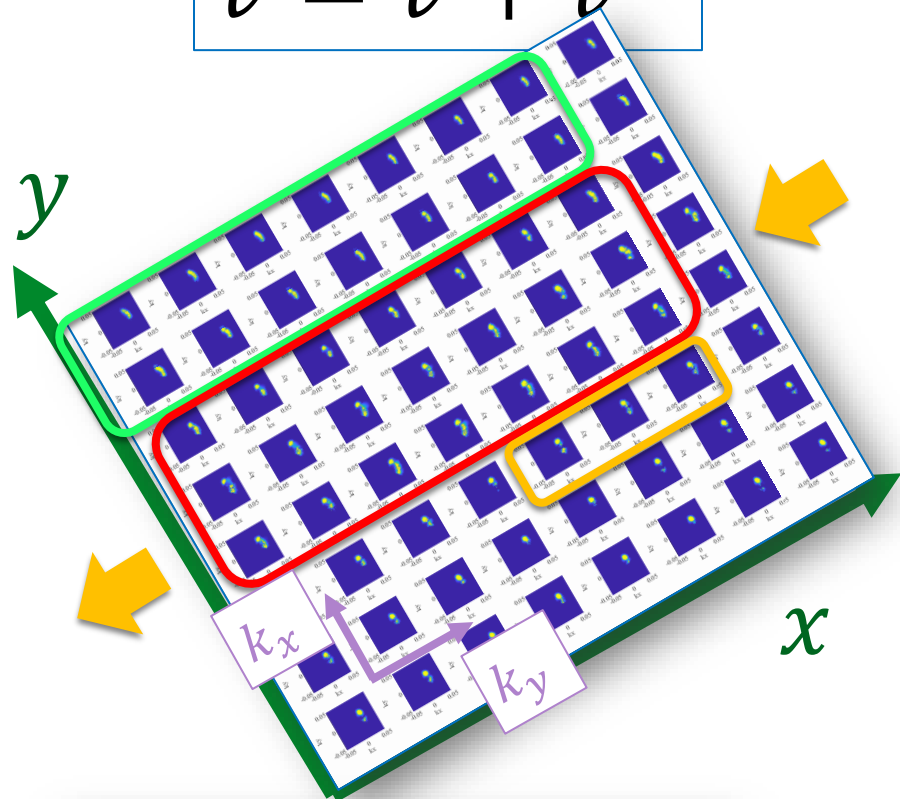
$$v = \bar{v} + v'_{\text{cor}}$$



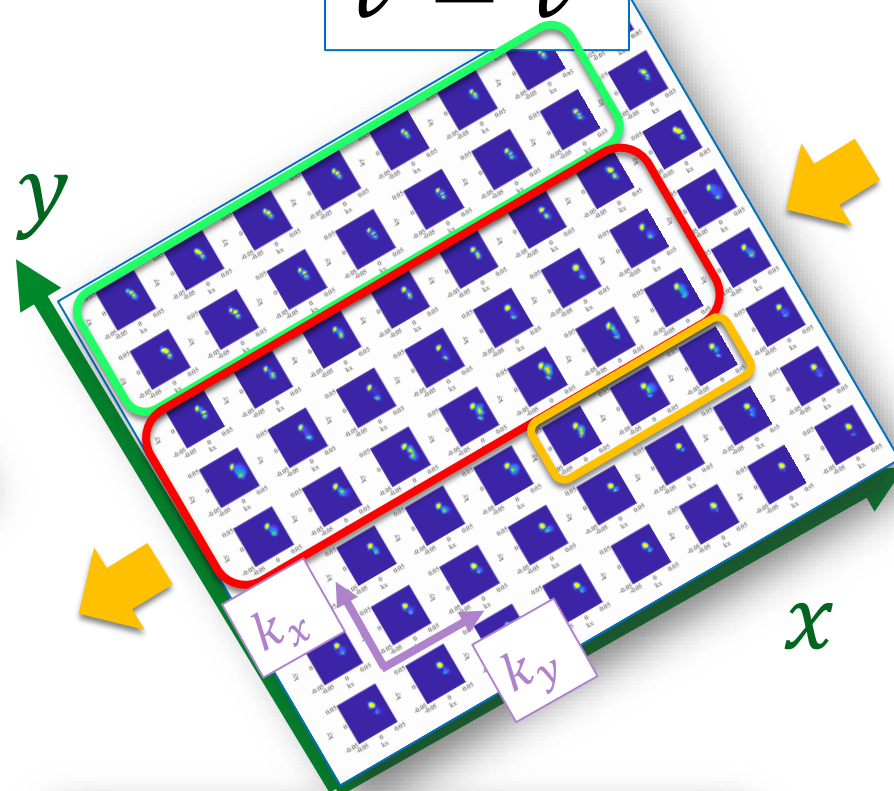
Time-correlated model for v'

Wave spectra

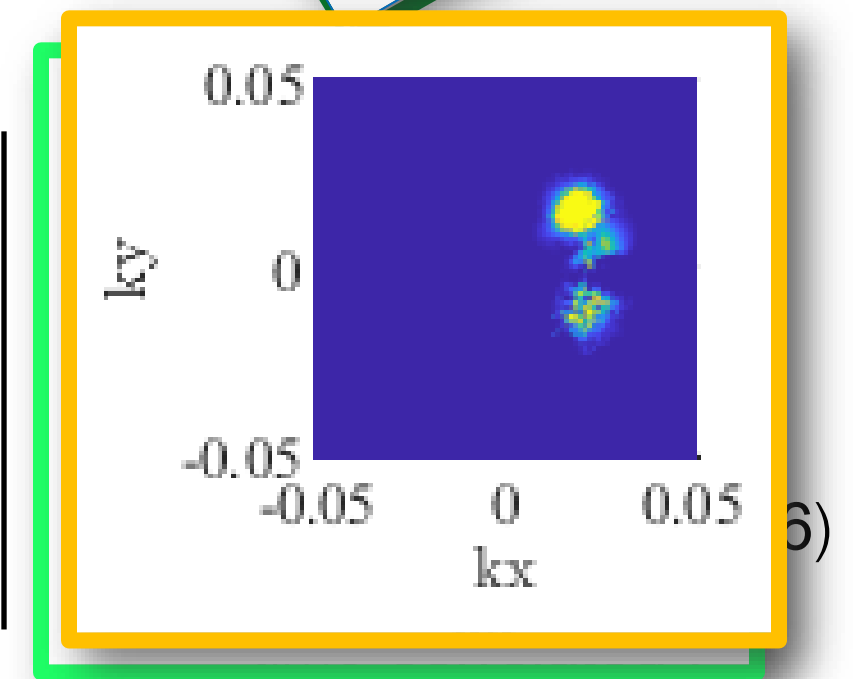
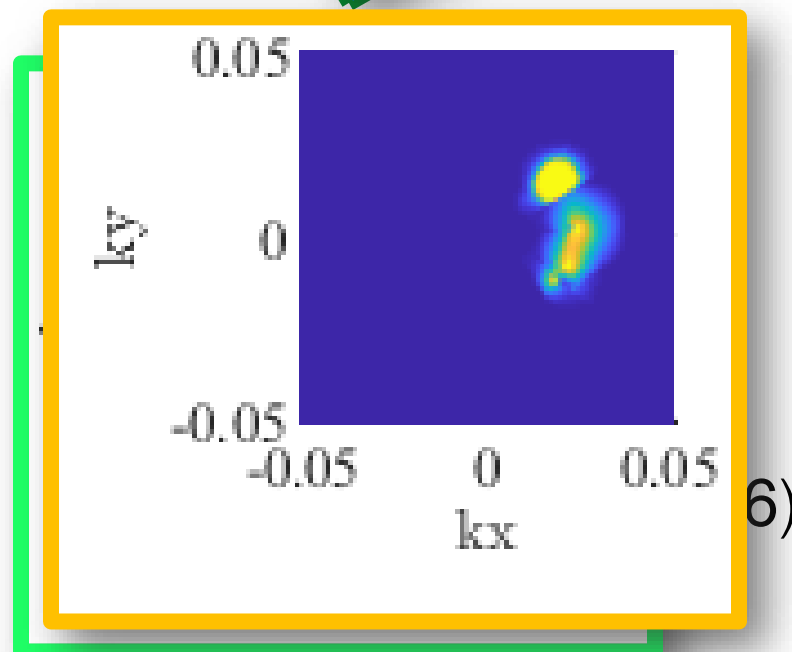
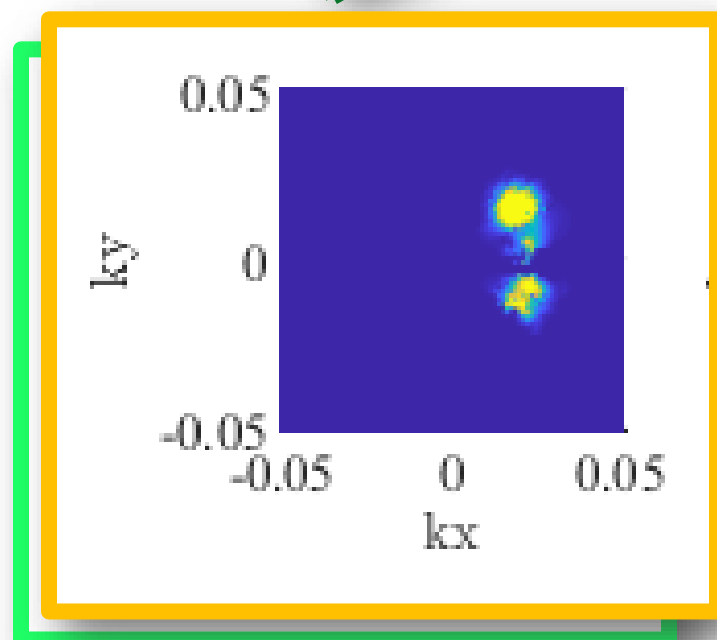
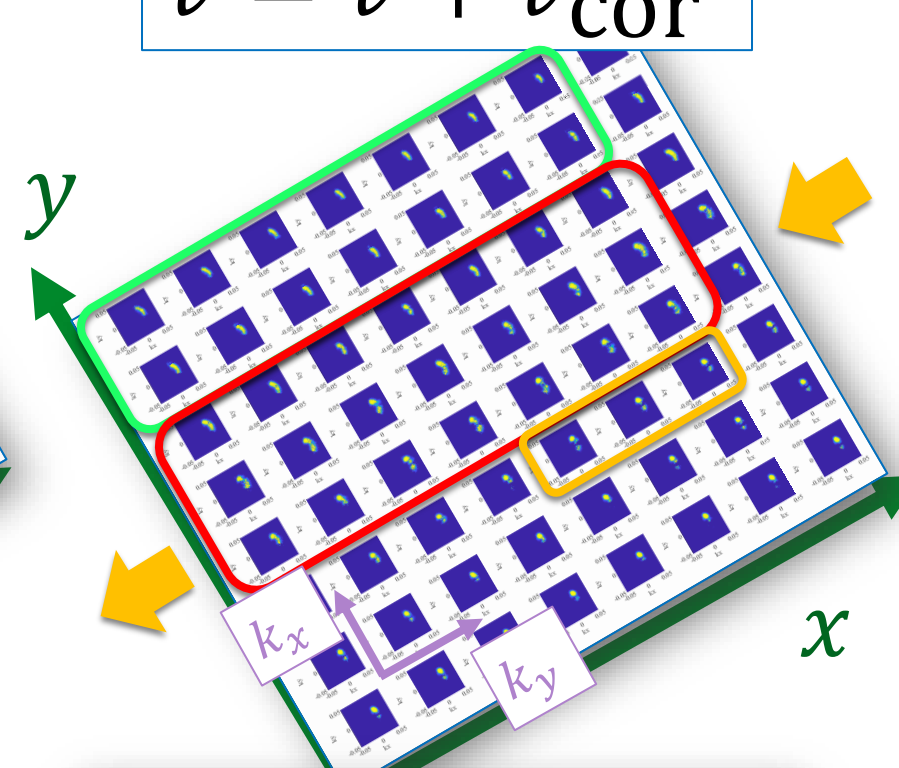
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



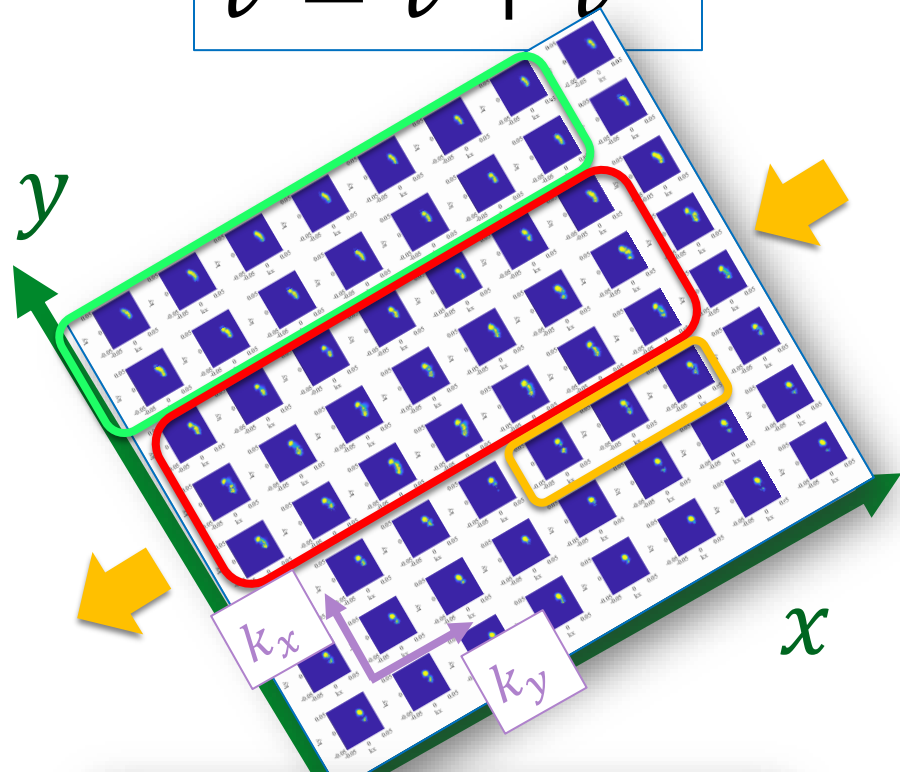
$$v = \bar{v} + v'_{\text{cor}}$$



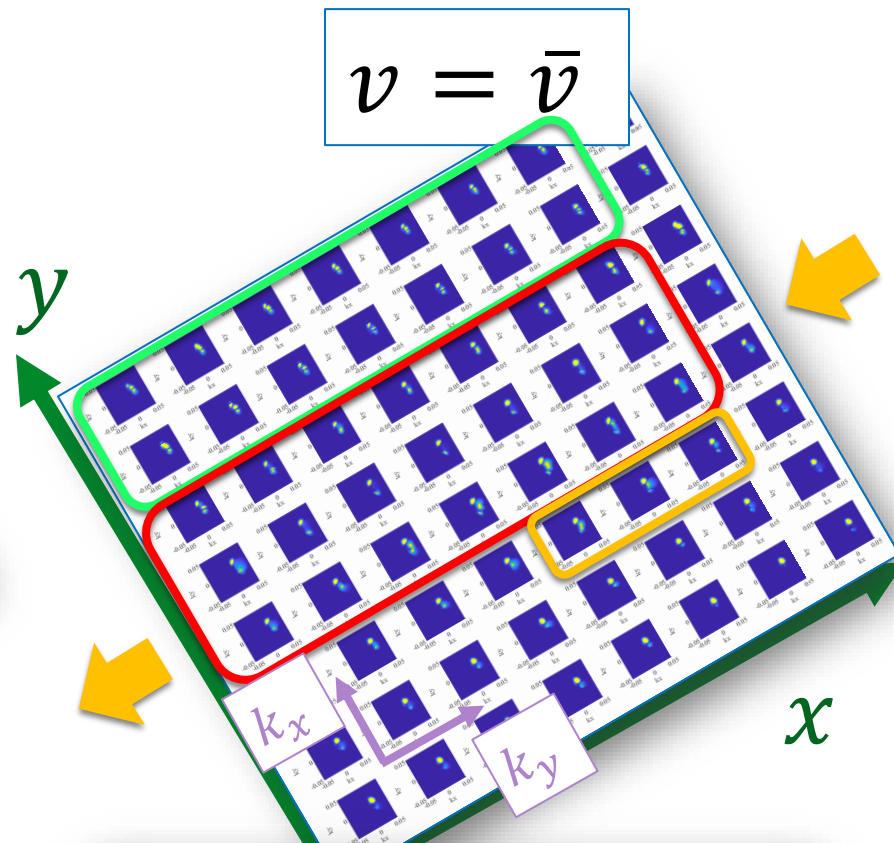
Time-correlated model for v'

Wave spectra

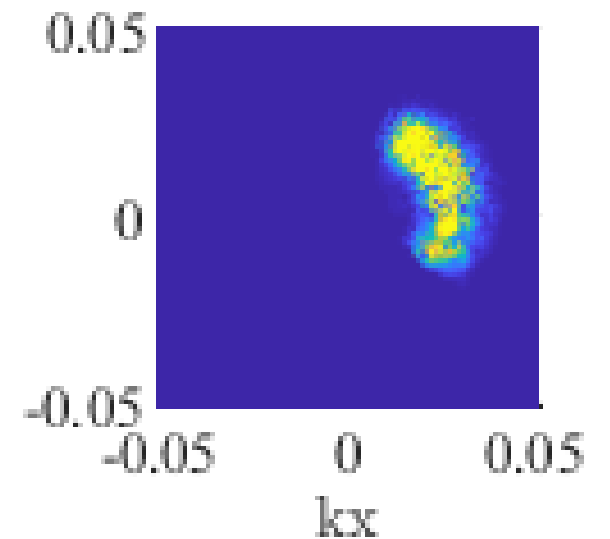
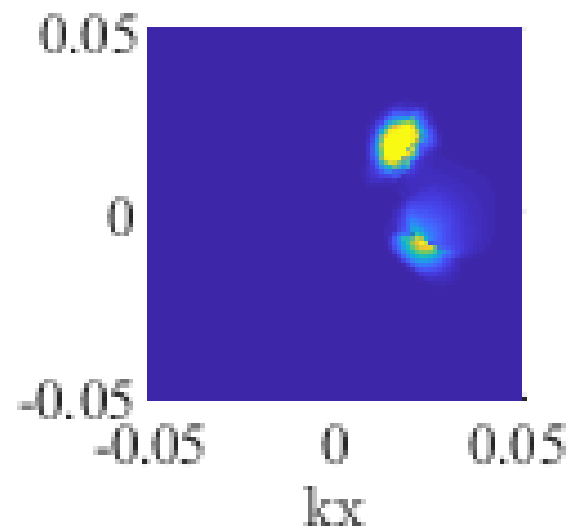
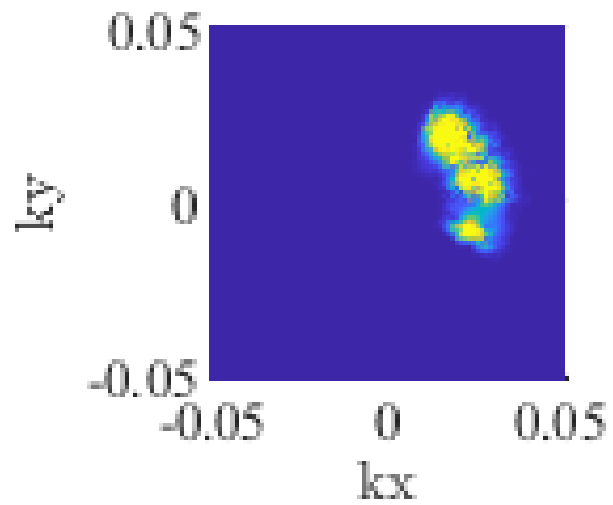
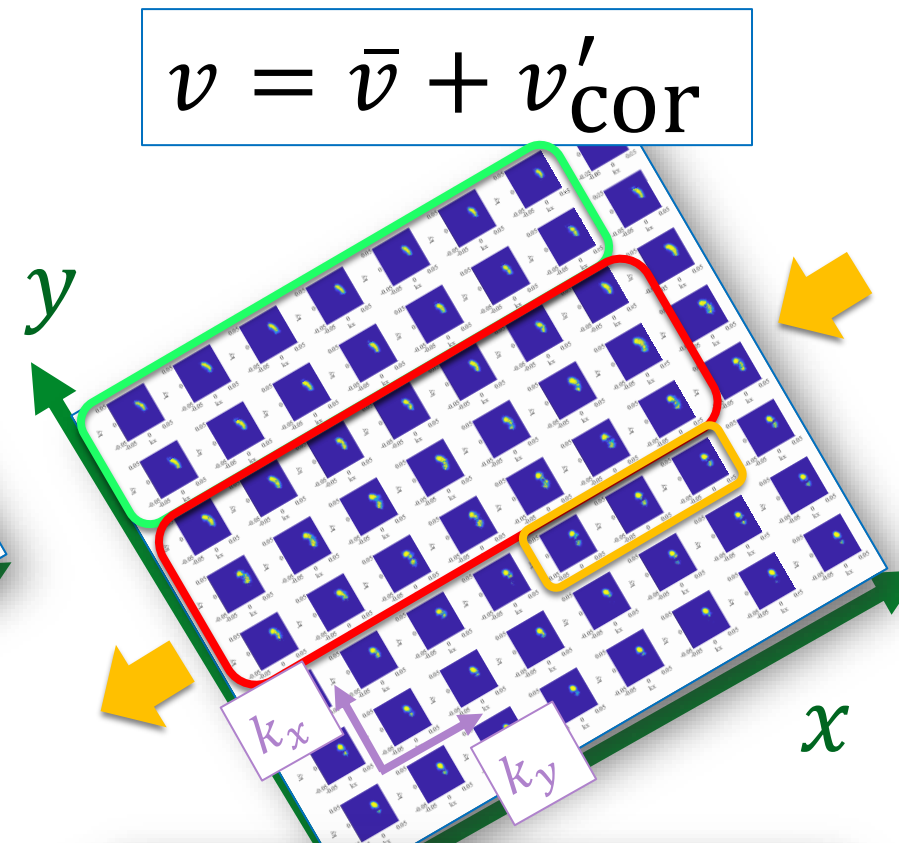
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



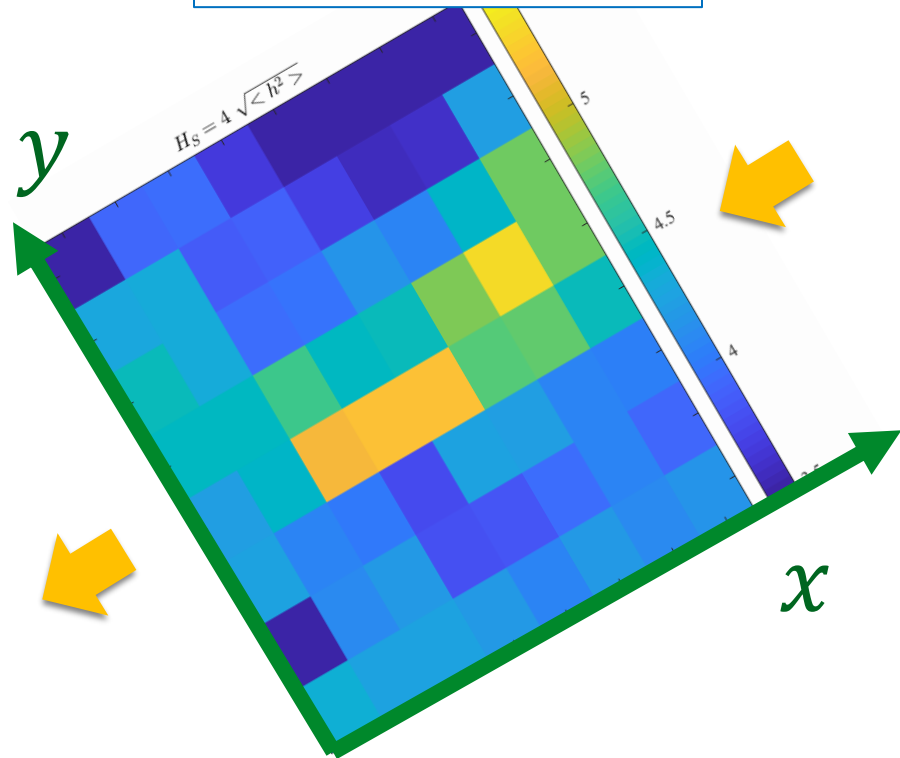
$$v = \bar{v} + v'_{\text{cor}}$$



Time-correlated model for v'

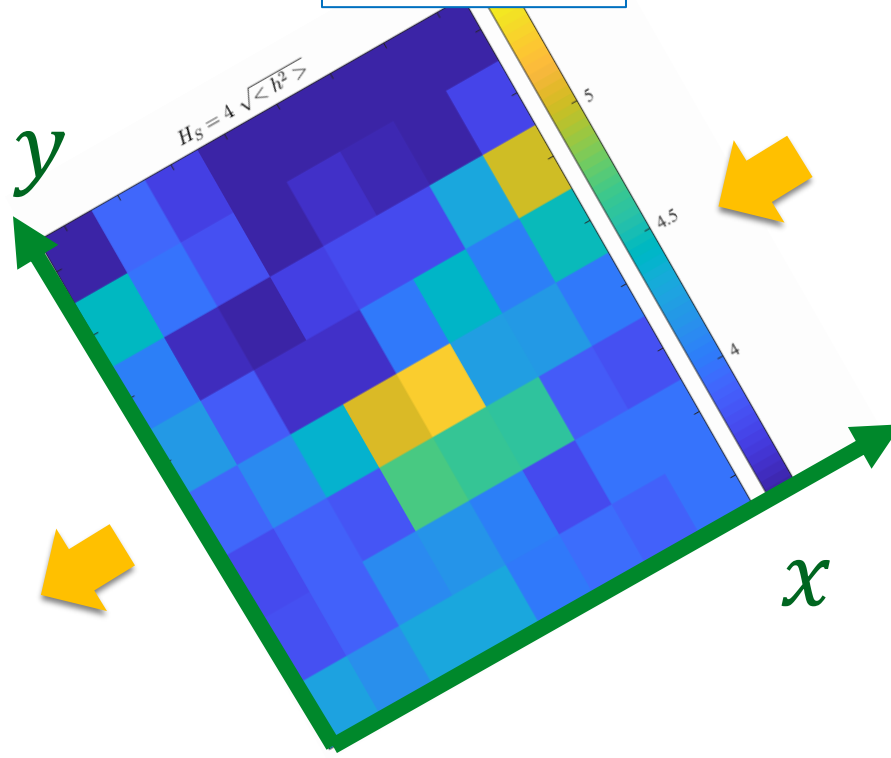
Large-scale significant wave height H_s

$$v = \bar{v} + v'$$



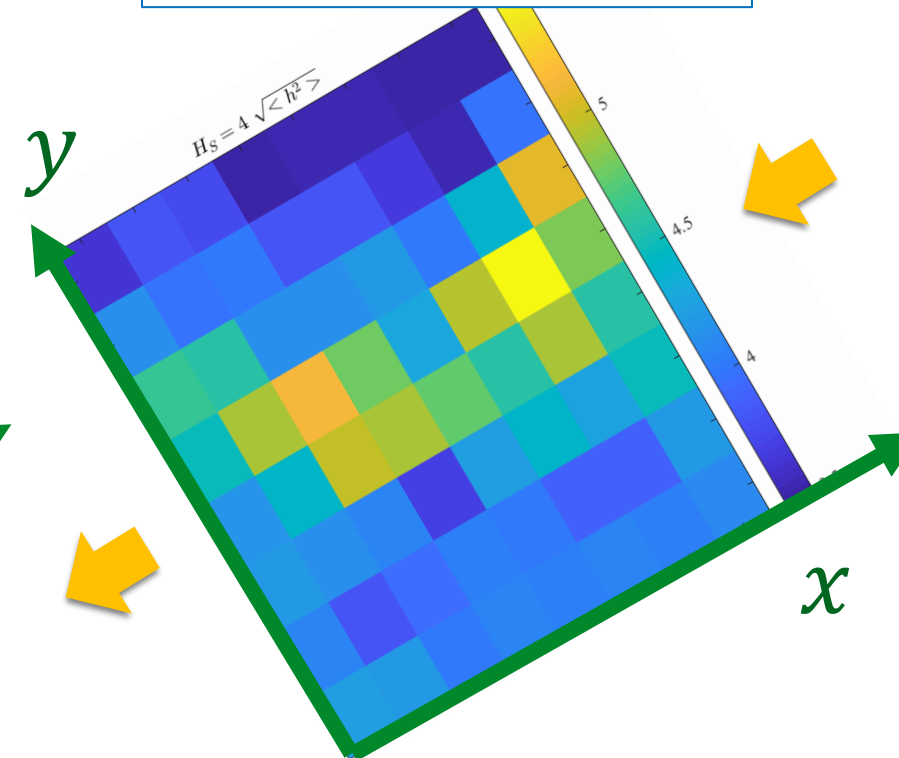
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

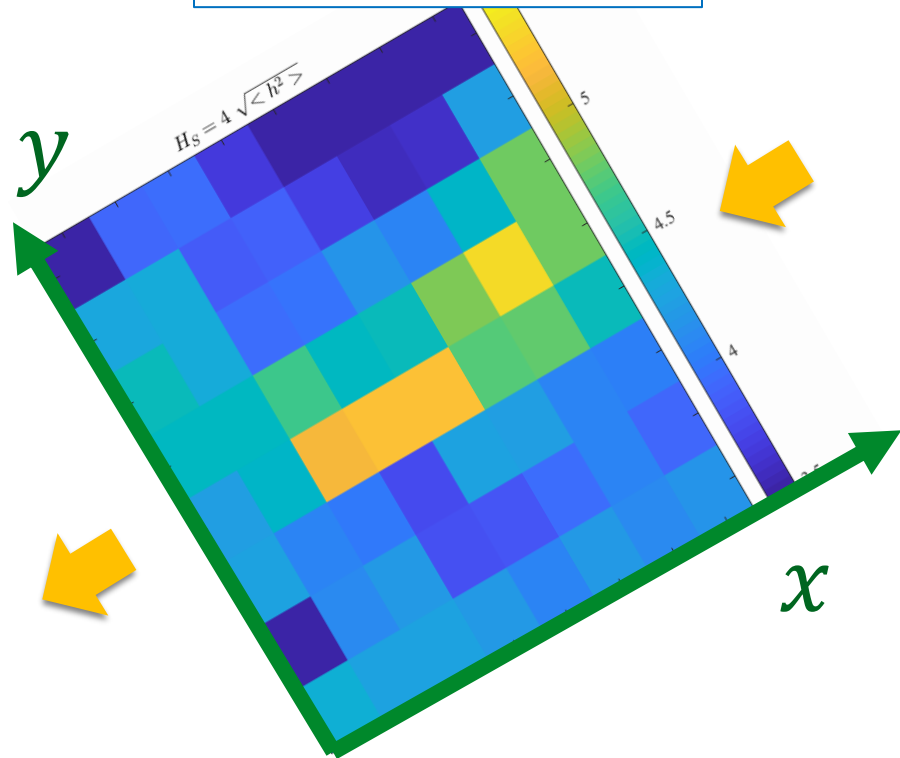


Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

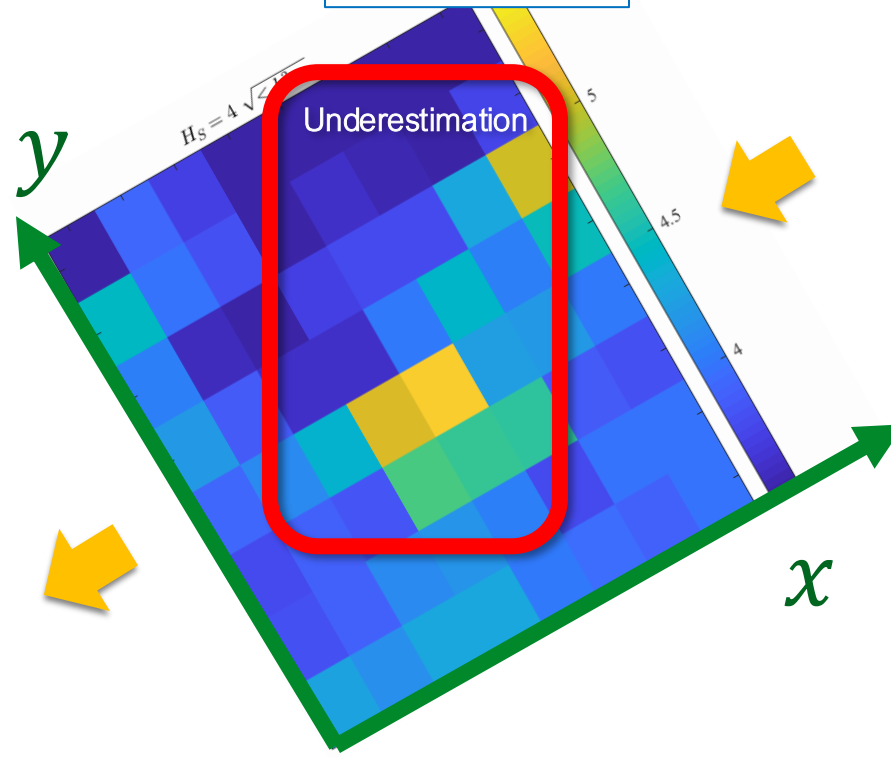
Large-scale significant wave height H_s

$$v = \bar{v} + v'$$



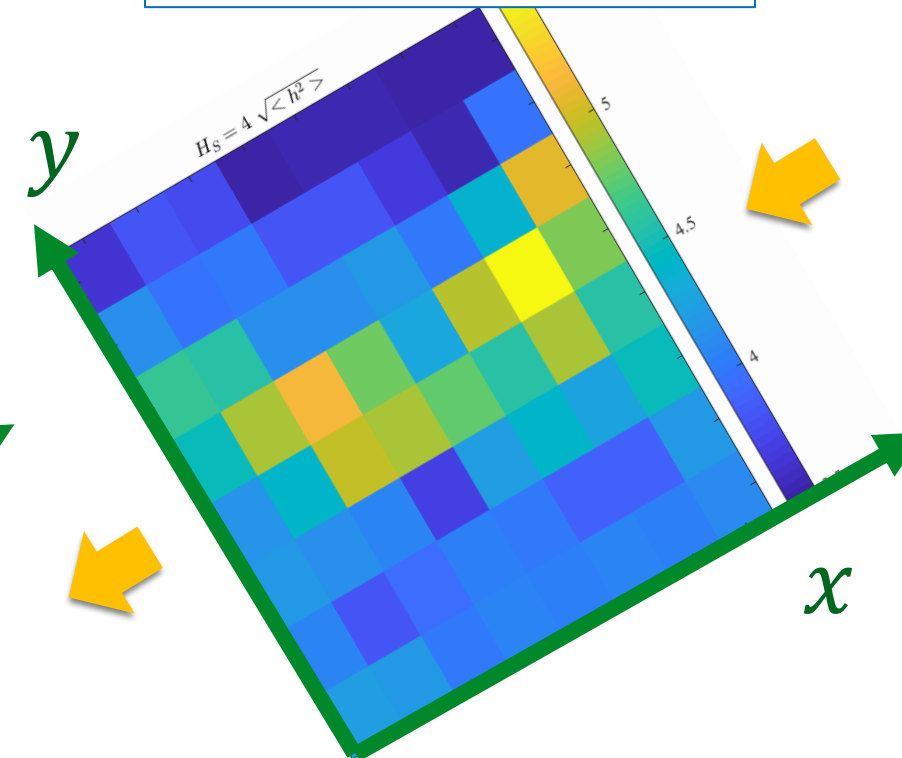
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

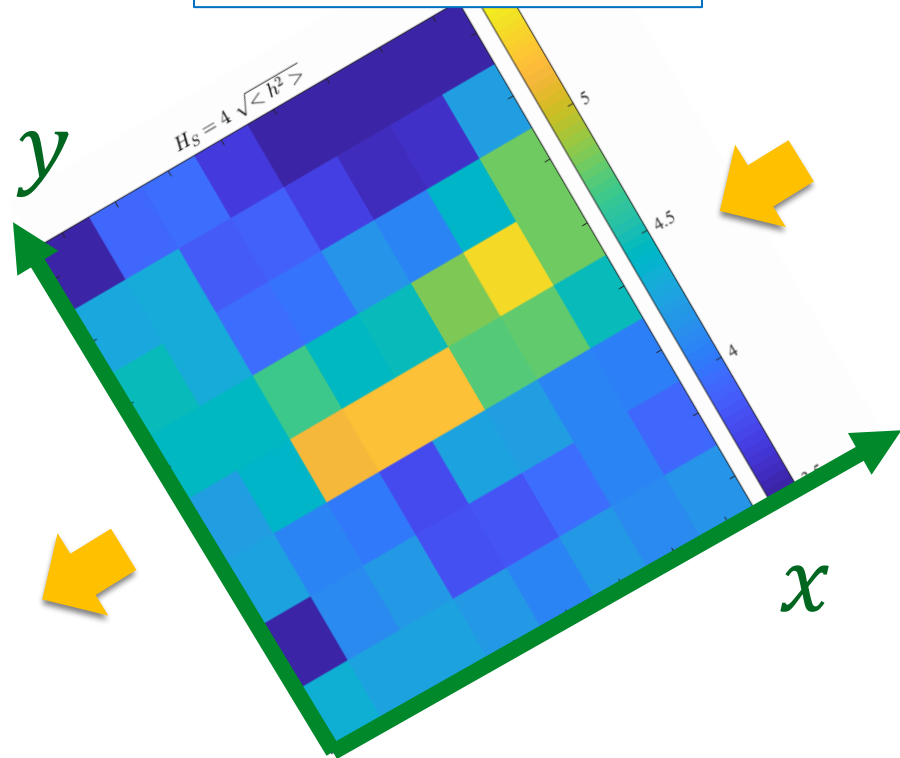


Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

Time-correlated model for v'

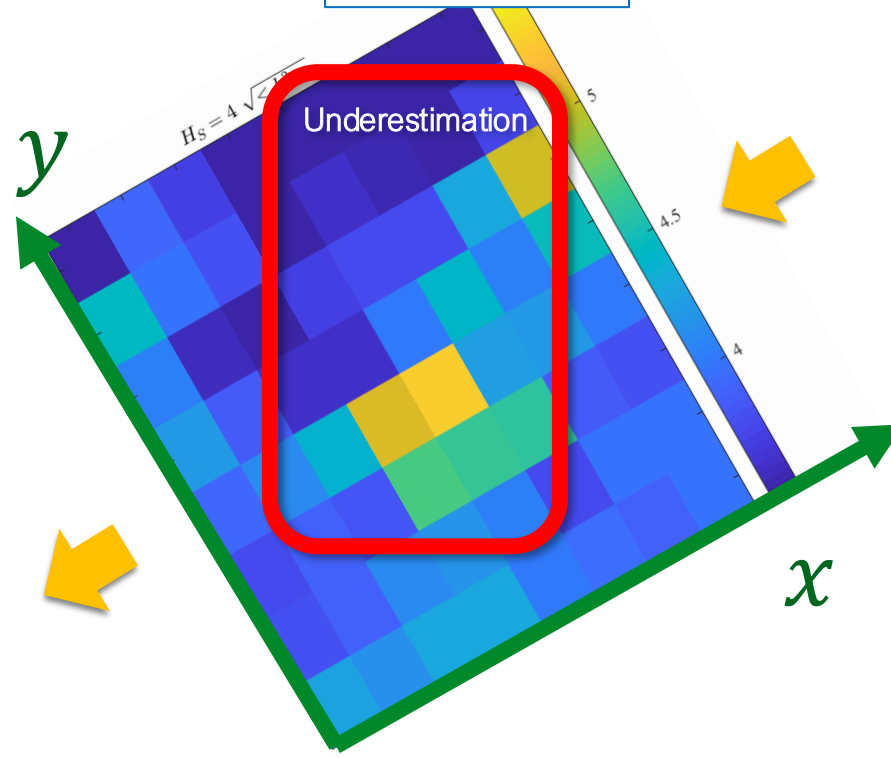
Large-scale significant wave height H_s

$$v = \bar{v} + v'$$



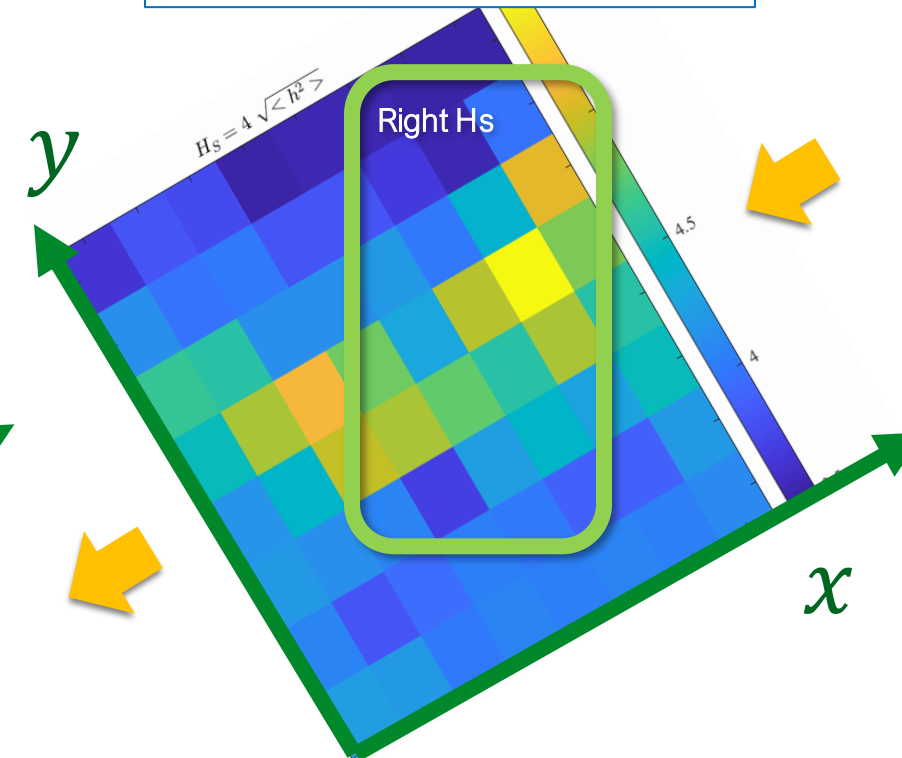
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

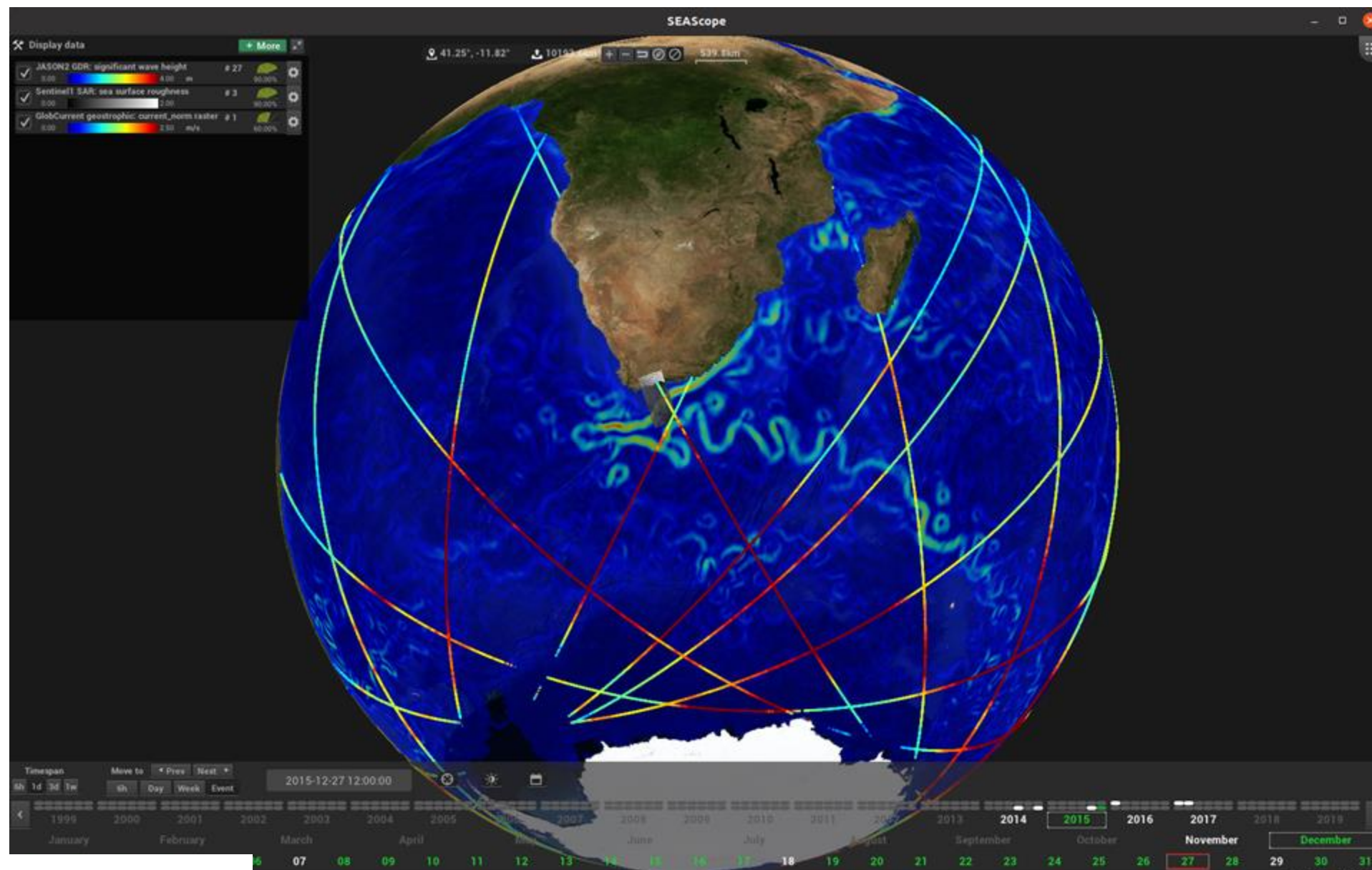


Our random
model
wave groups in
smoothed flow \bar{v} (16 x 16)
+ time-correlated v'

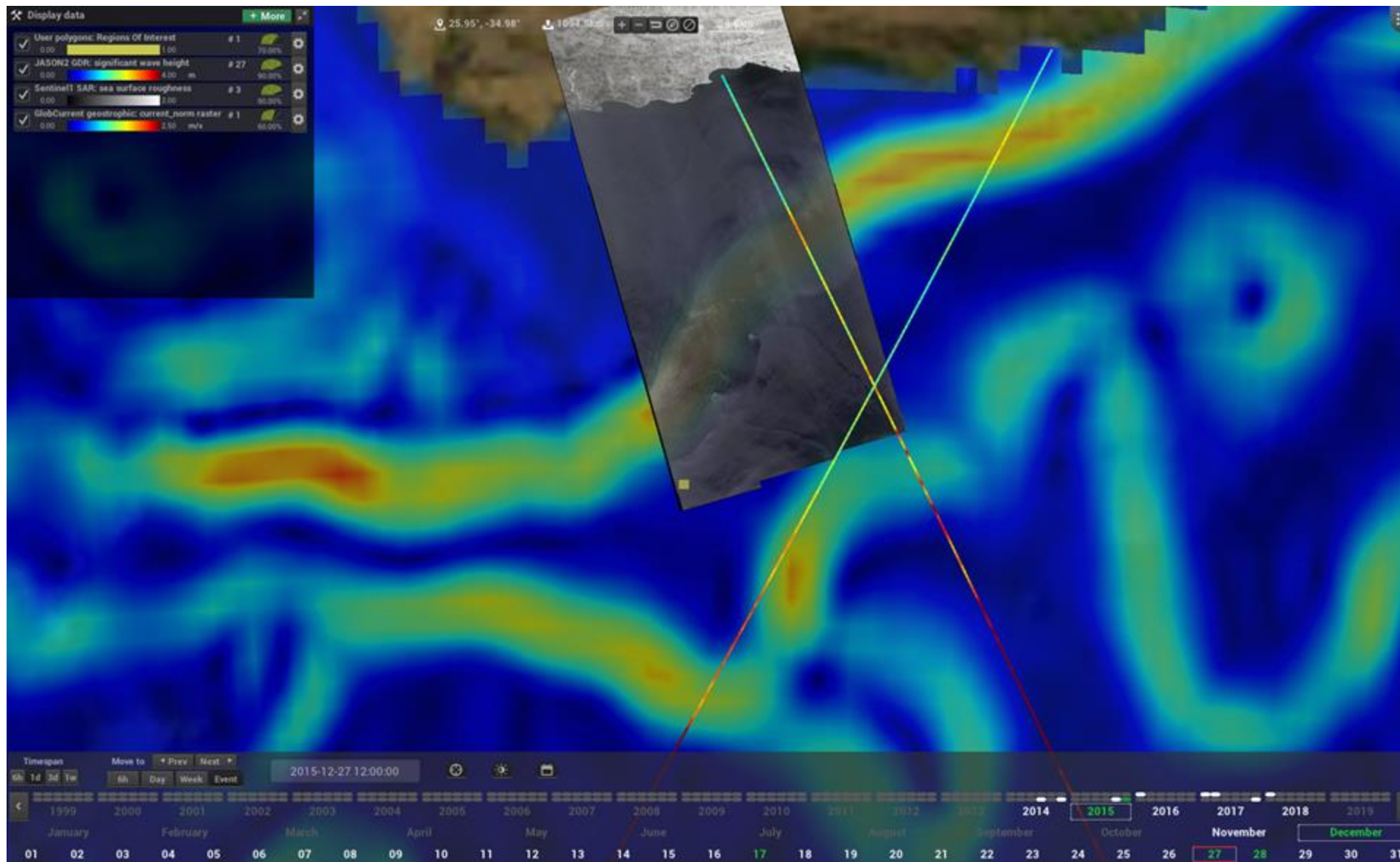
II. Beyond Markovian closure

II.B. Tests with satellite data

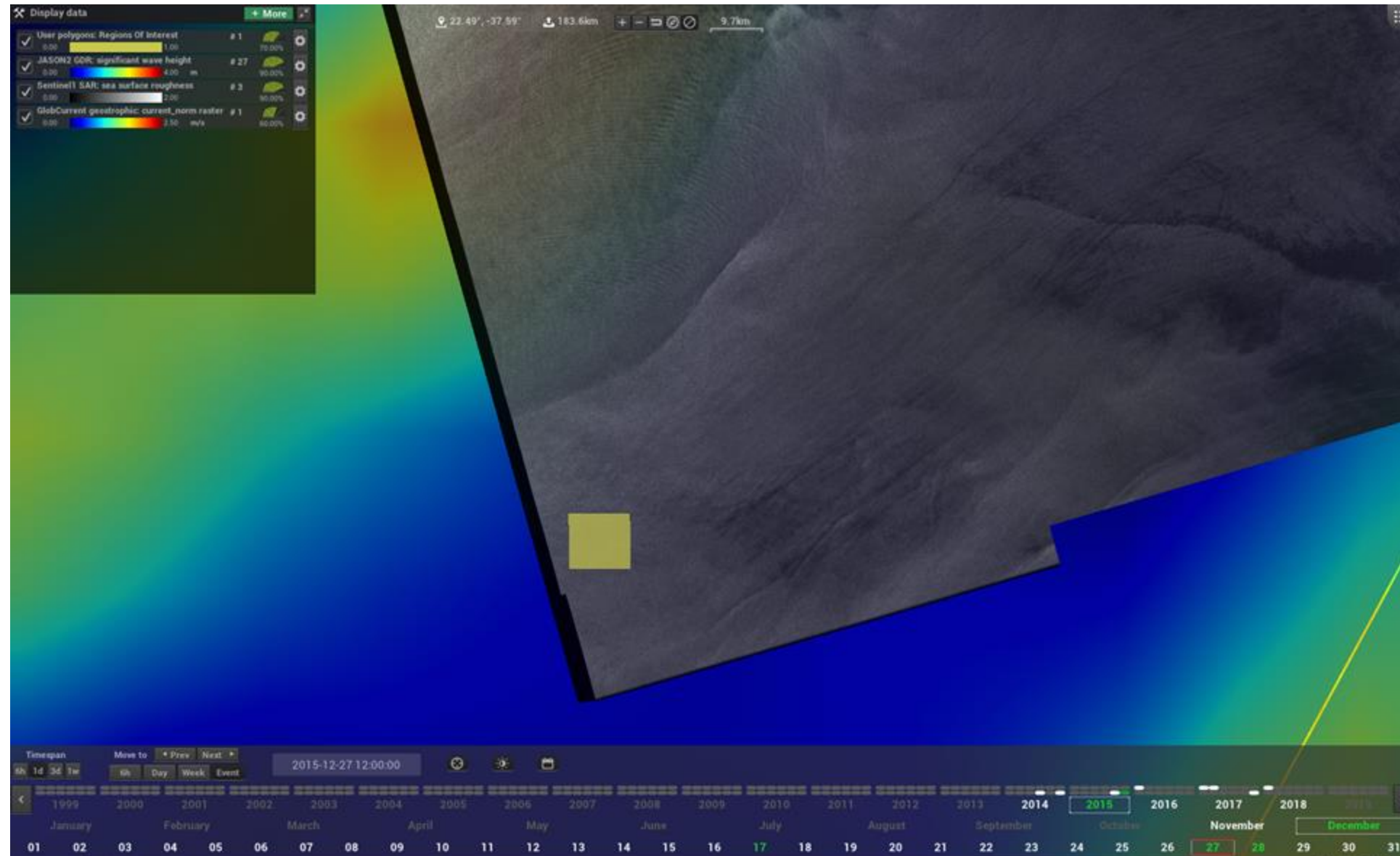
Tests with Sentinel-1 and Jason-2 satellite data



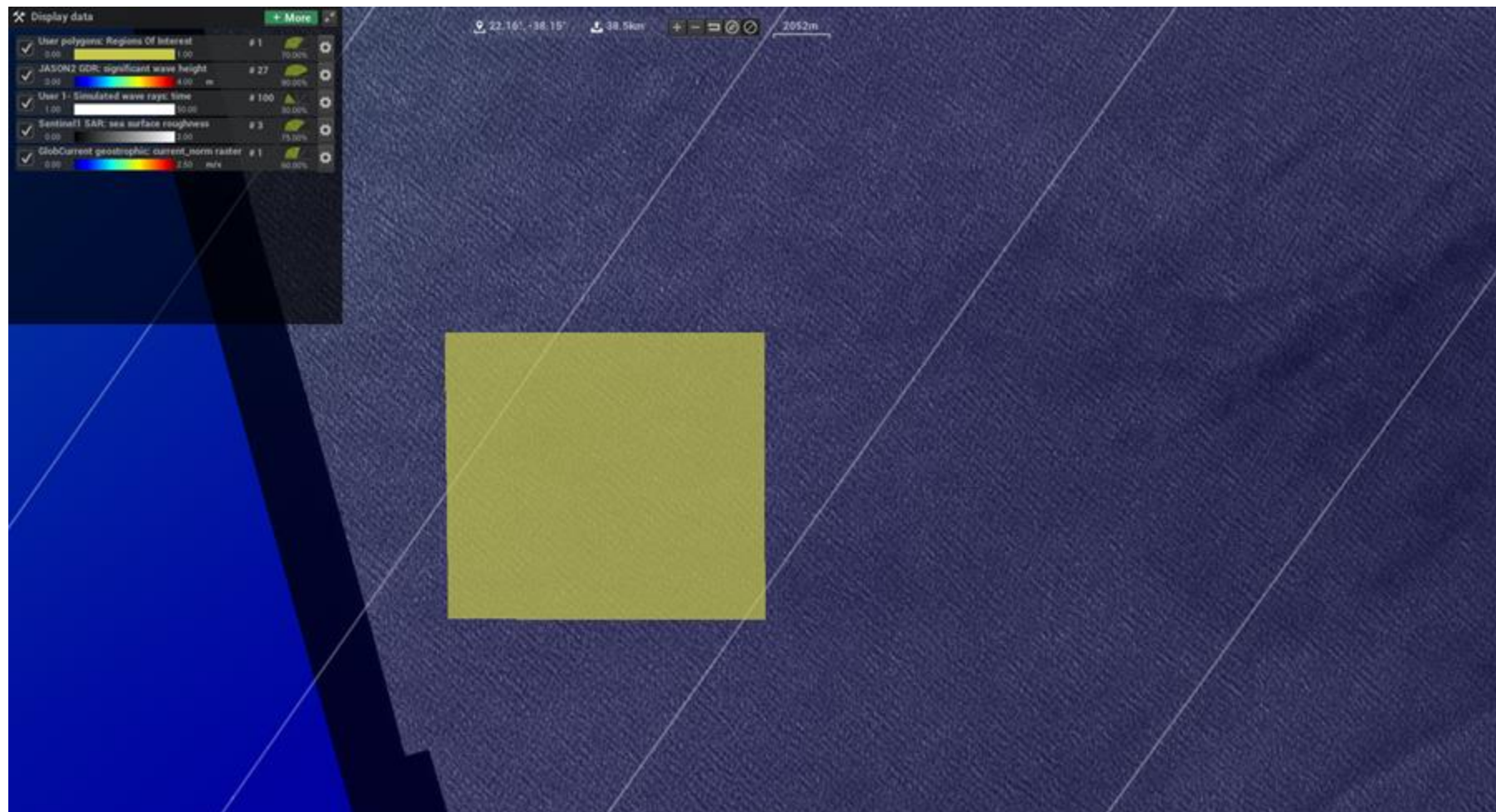
Tests with Sentinel-1 and Jason-2 satellite data



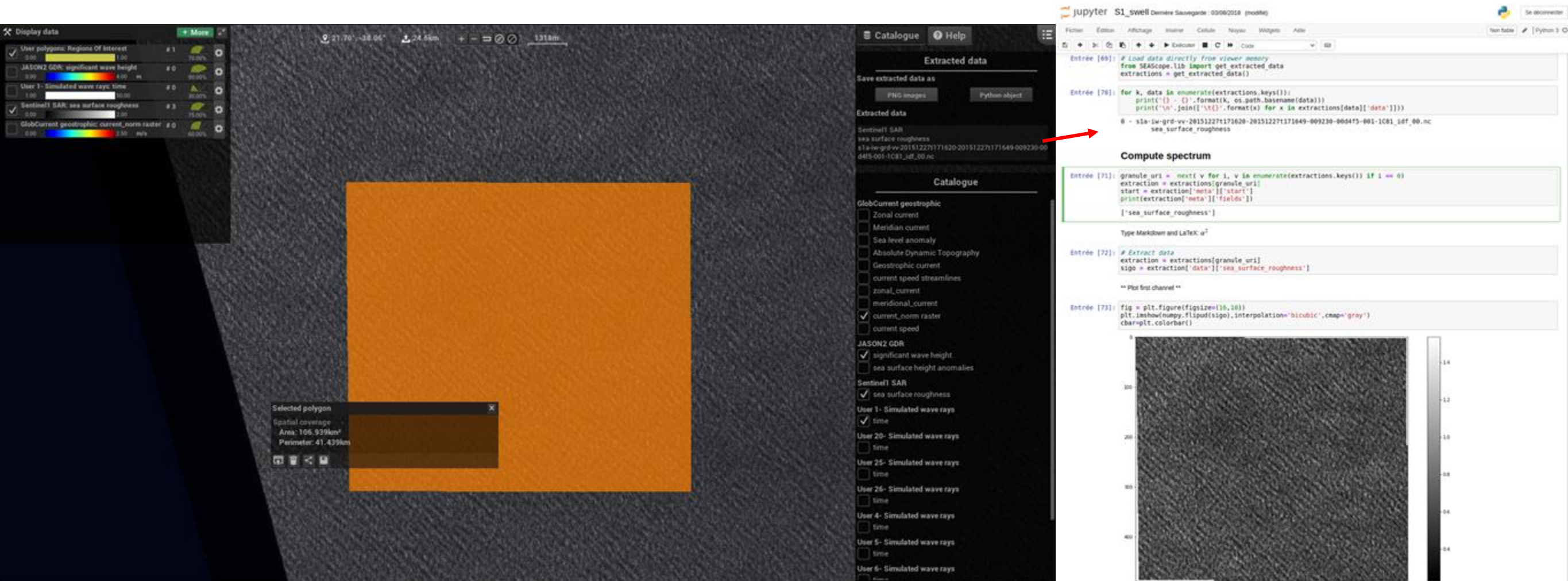
Tests with Sentinel-1 and Jason-2 satellite data



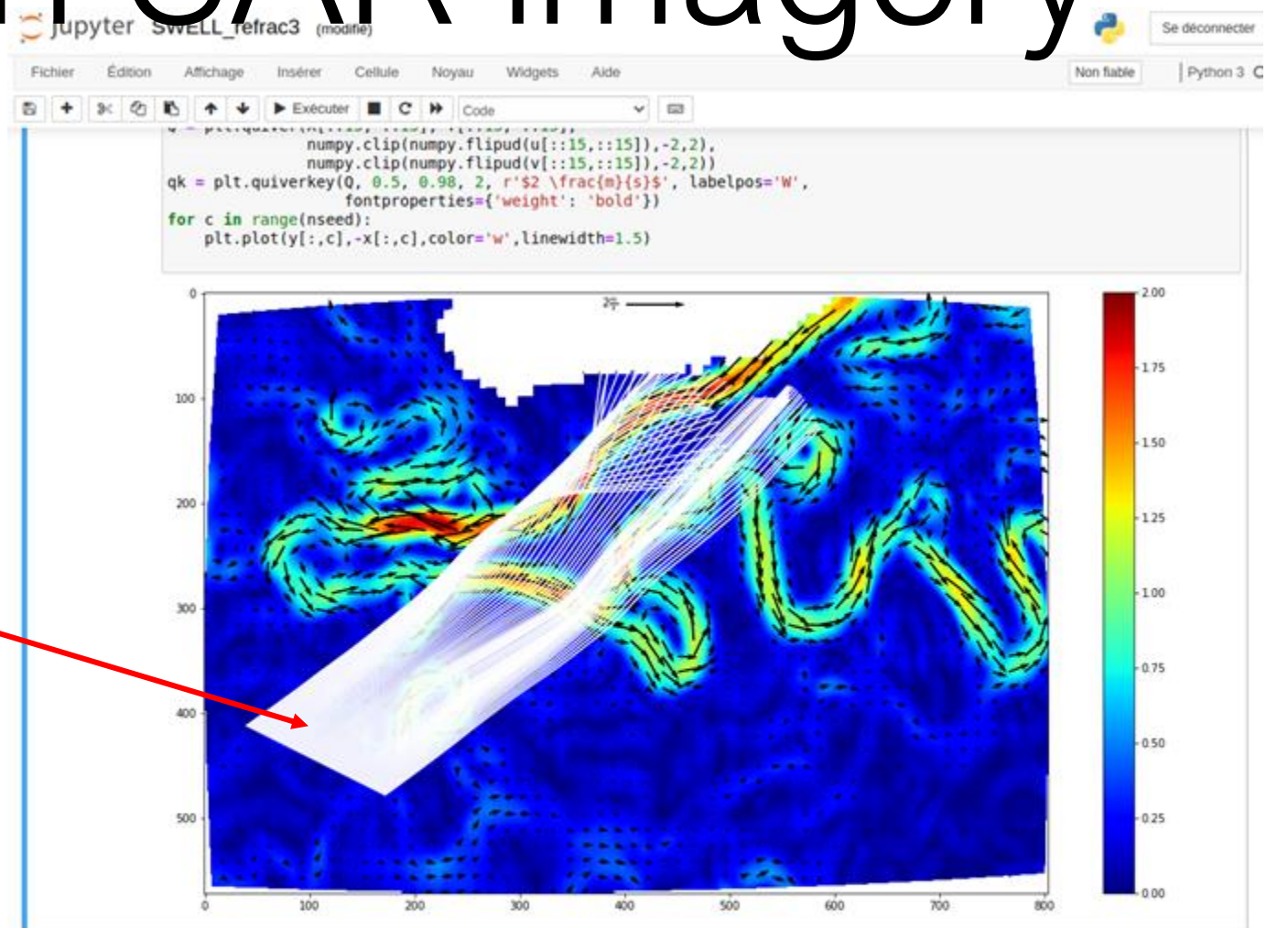
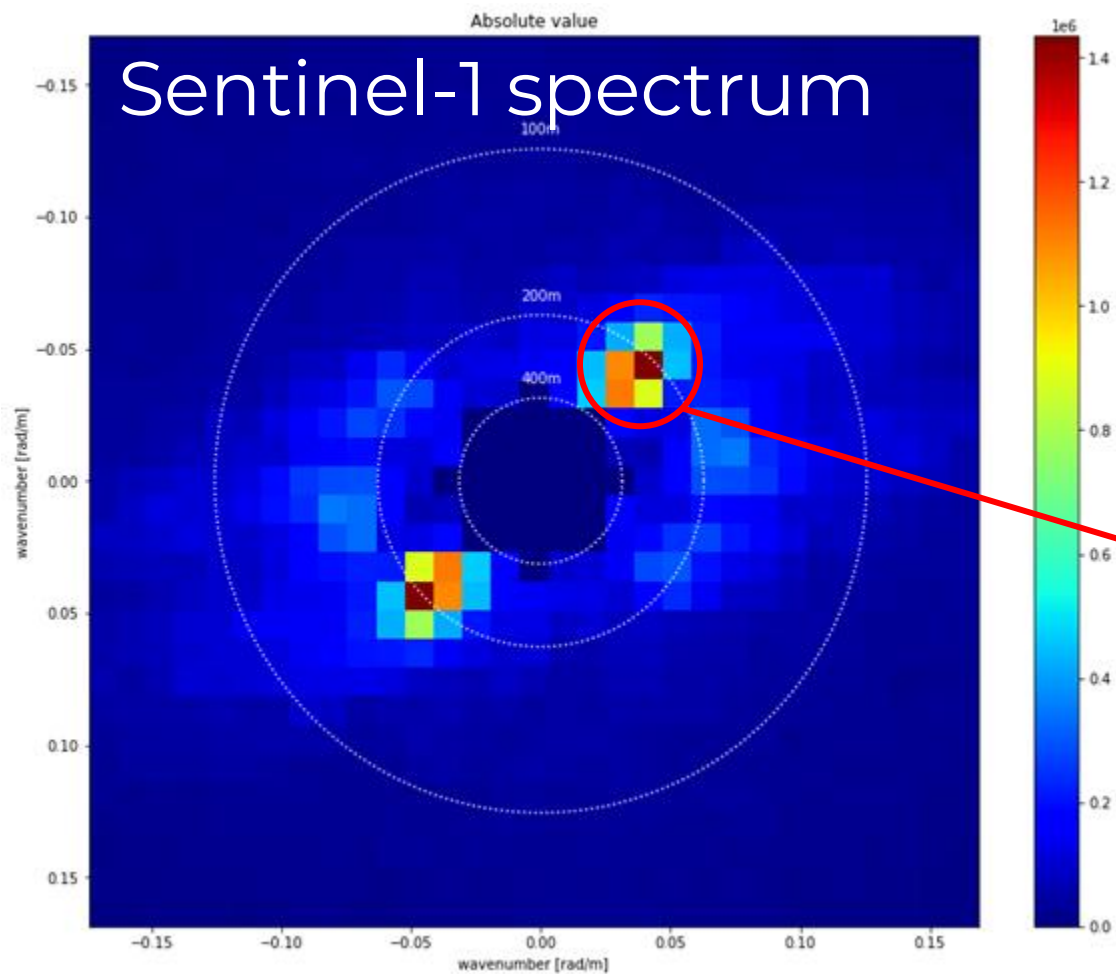
Tests with Sentinel-1 and Jason-2 satellite data



Estimation of the incident spectrum on SAR imagery



Estimation of the incident spectrum on SAR imagery

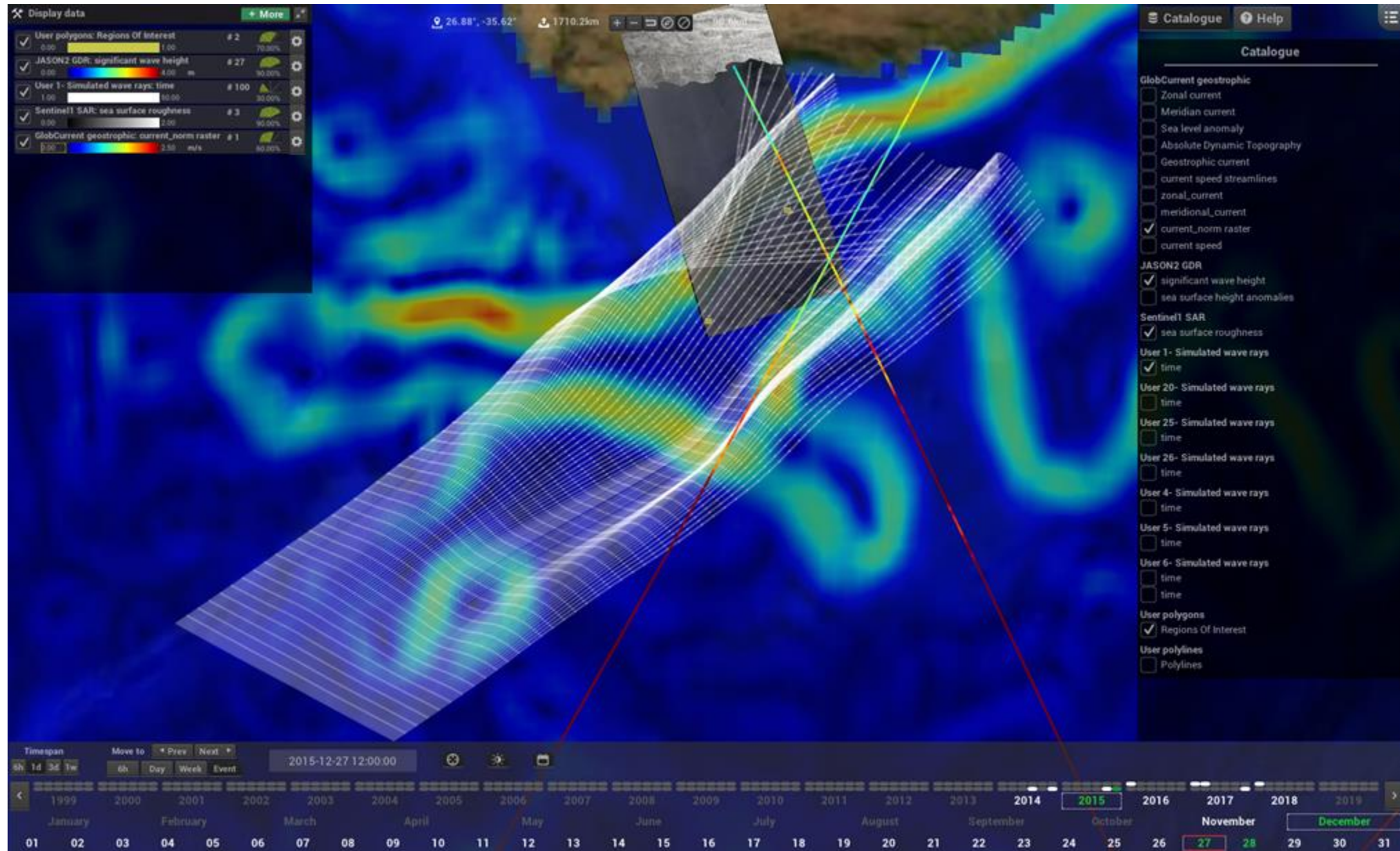


Export wave rays to SEAScope

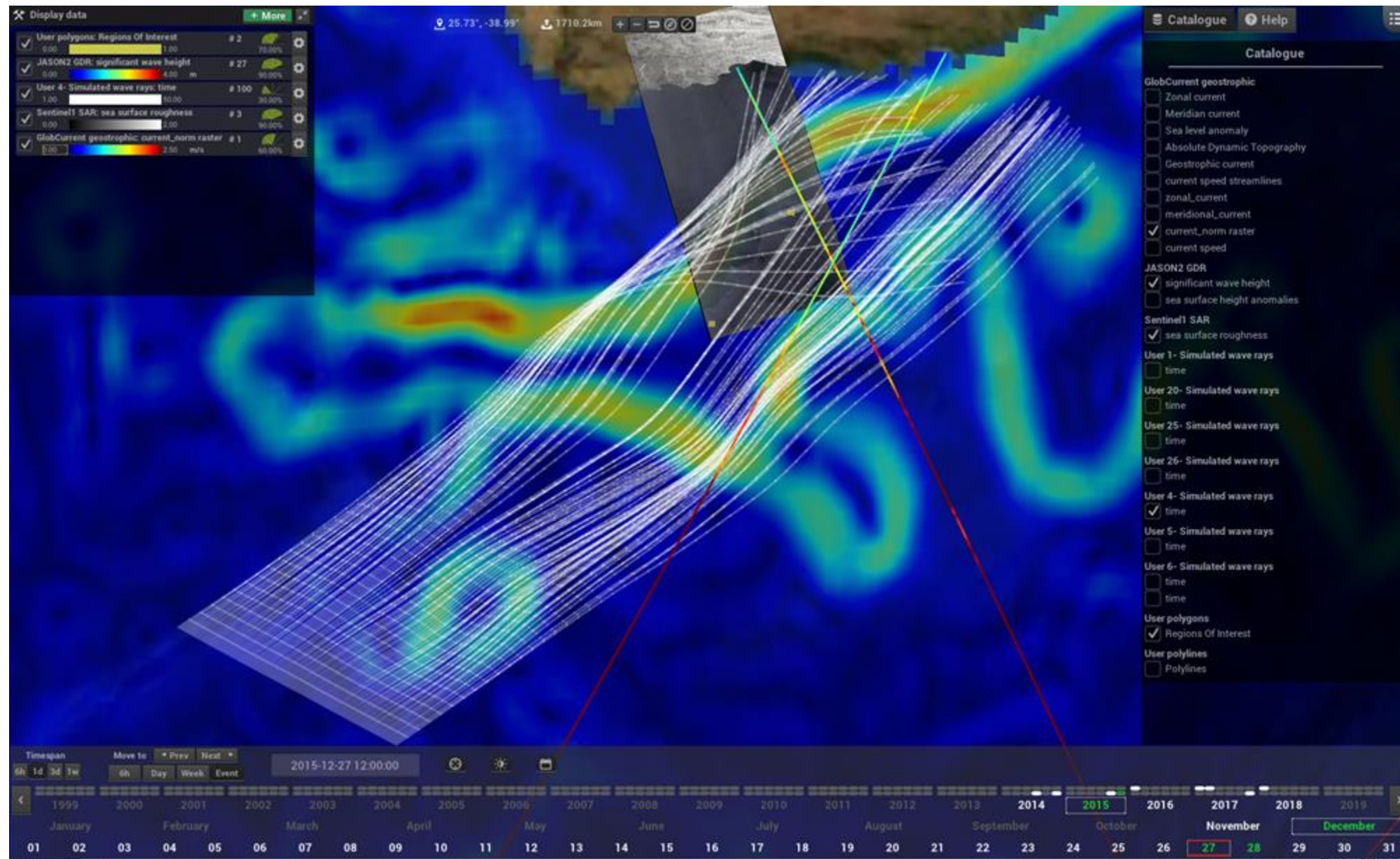
Define Collection:
** A granule has to belong to a collection **

```
Entrée [349]: from SEAScope.lib.utils import create_collection, init_ids
# Make sure to keep the collection identifier, we will need it later
collection_id, collection = create_collection('User 4- Simulated wave rays')
print(collection_id)
```

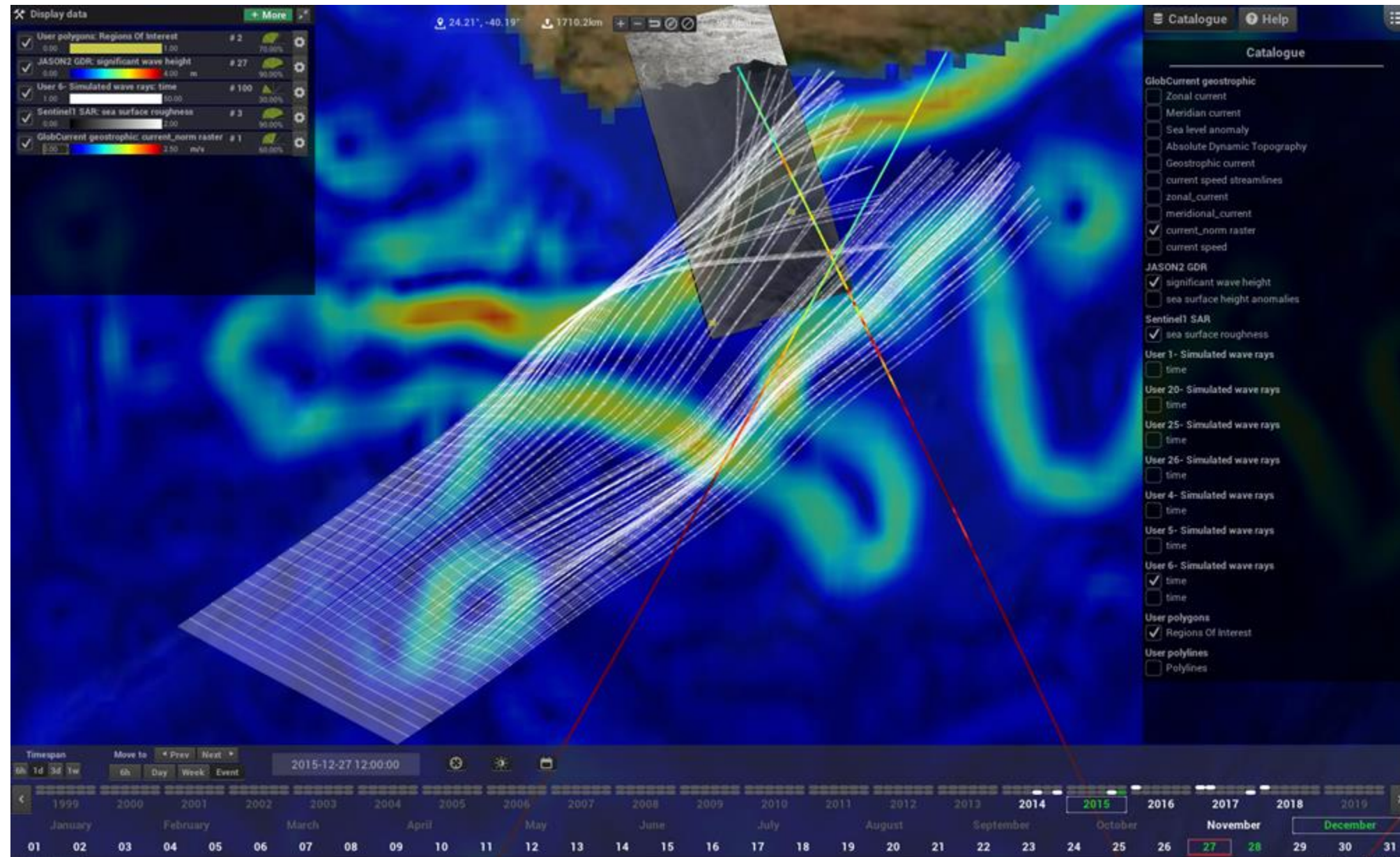

Determinist model



Stochastic model with homogeneous and time-correlated small-scale velocities

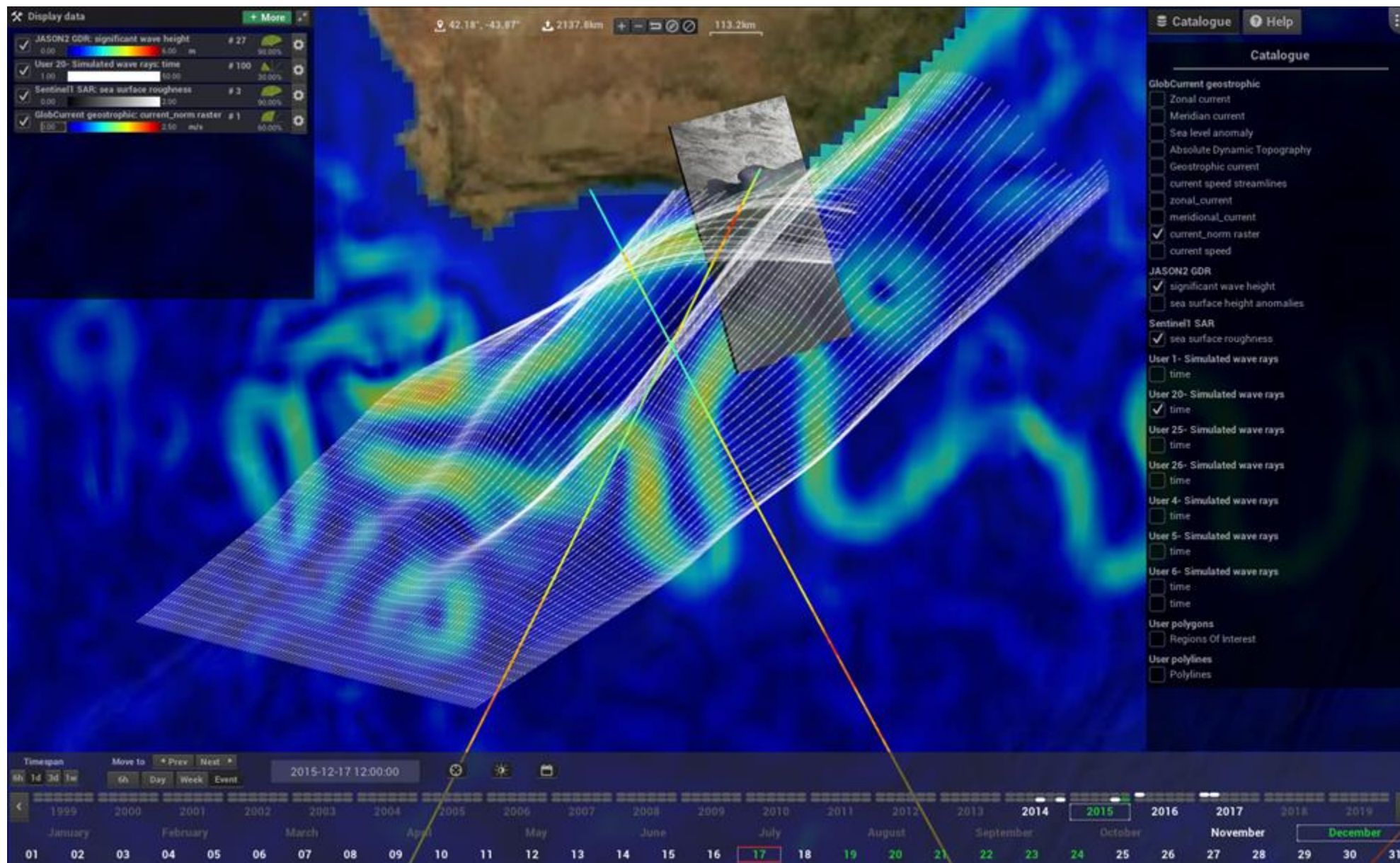


Stochastic model with heterogeneous and time-correlated small-scale velocities

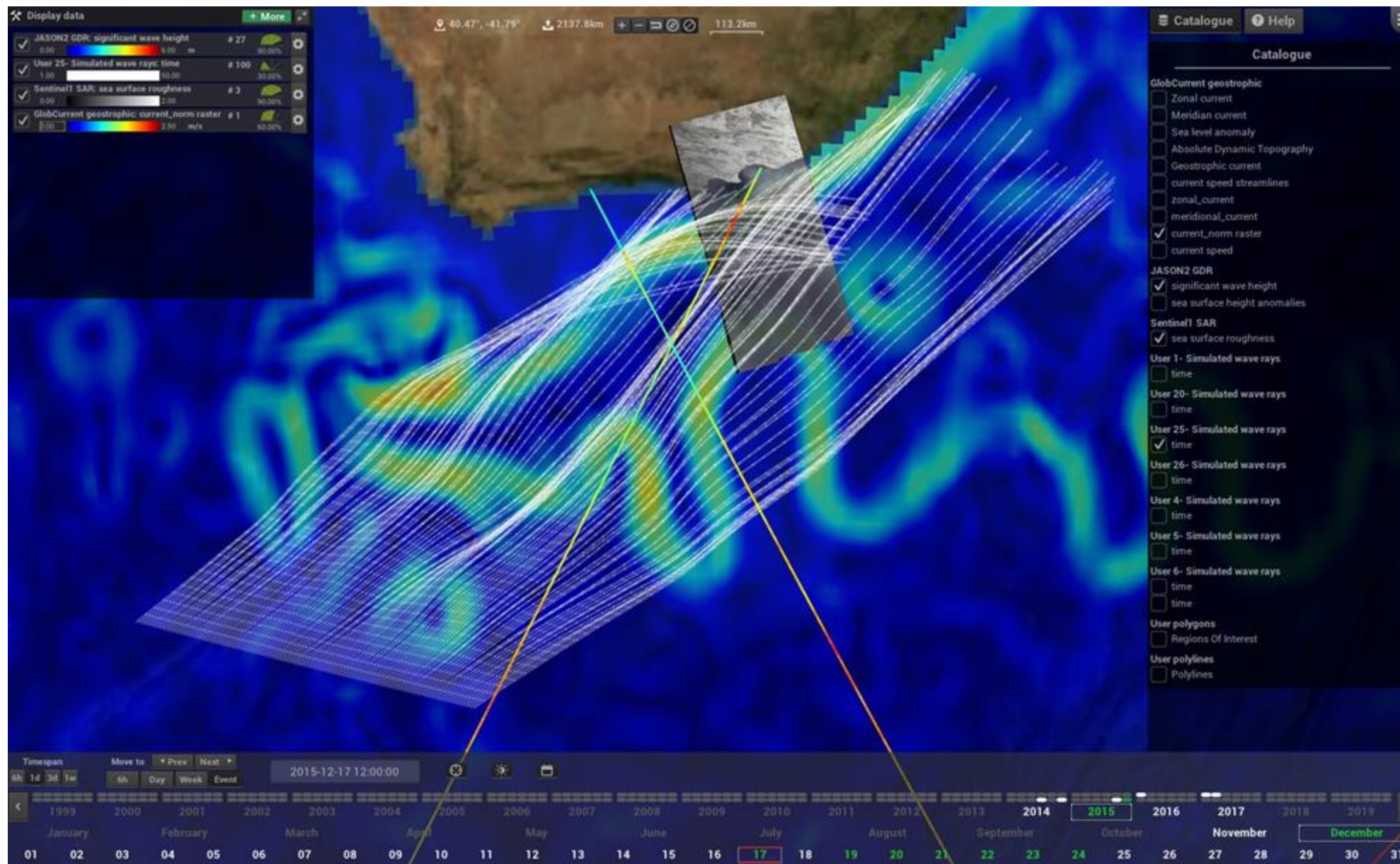


17 dec 2015

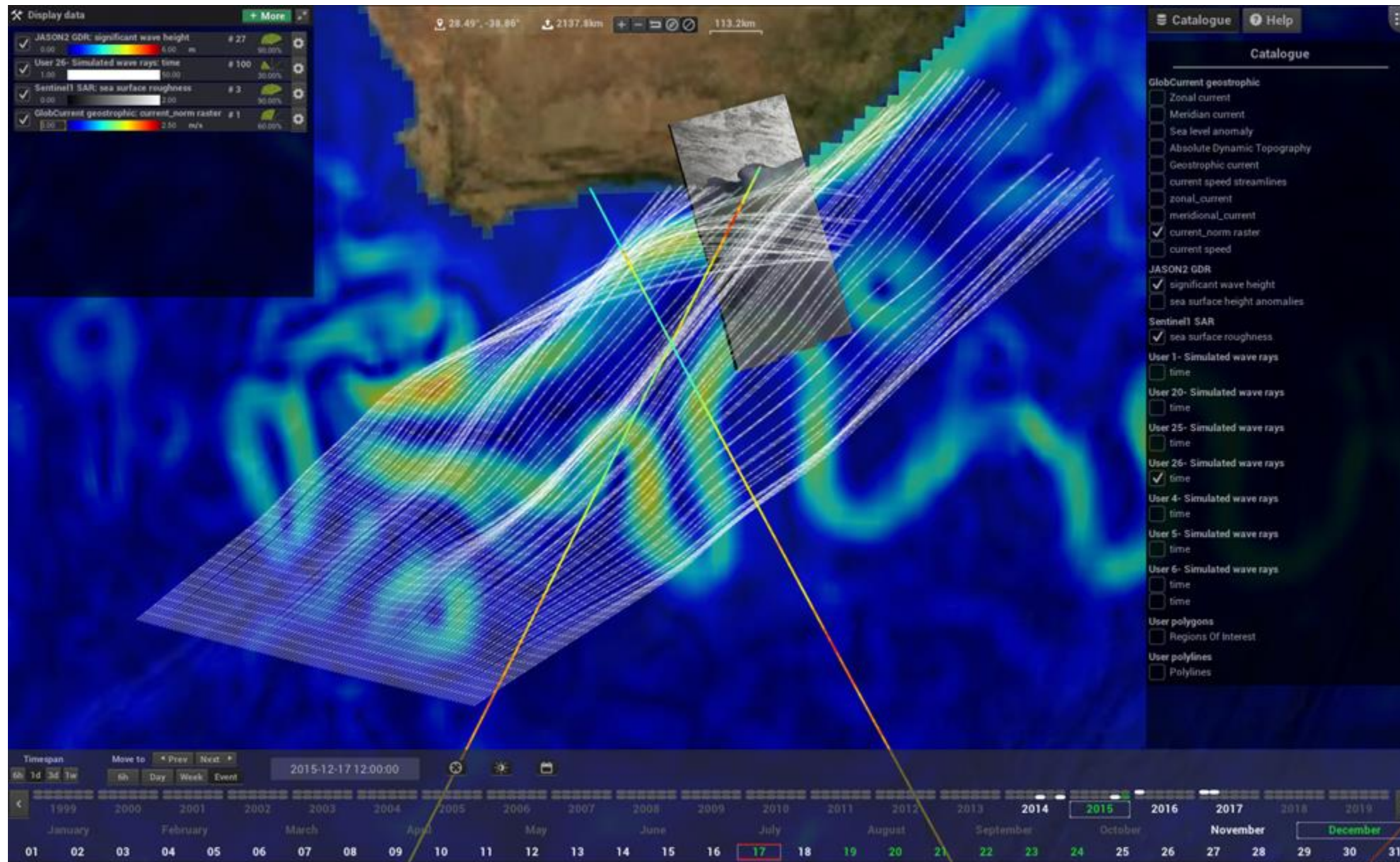
Determinist model



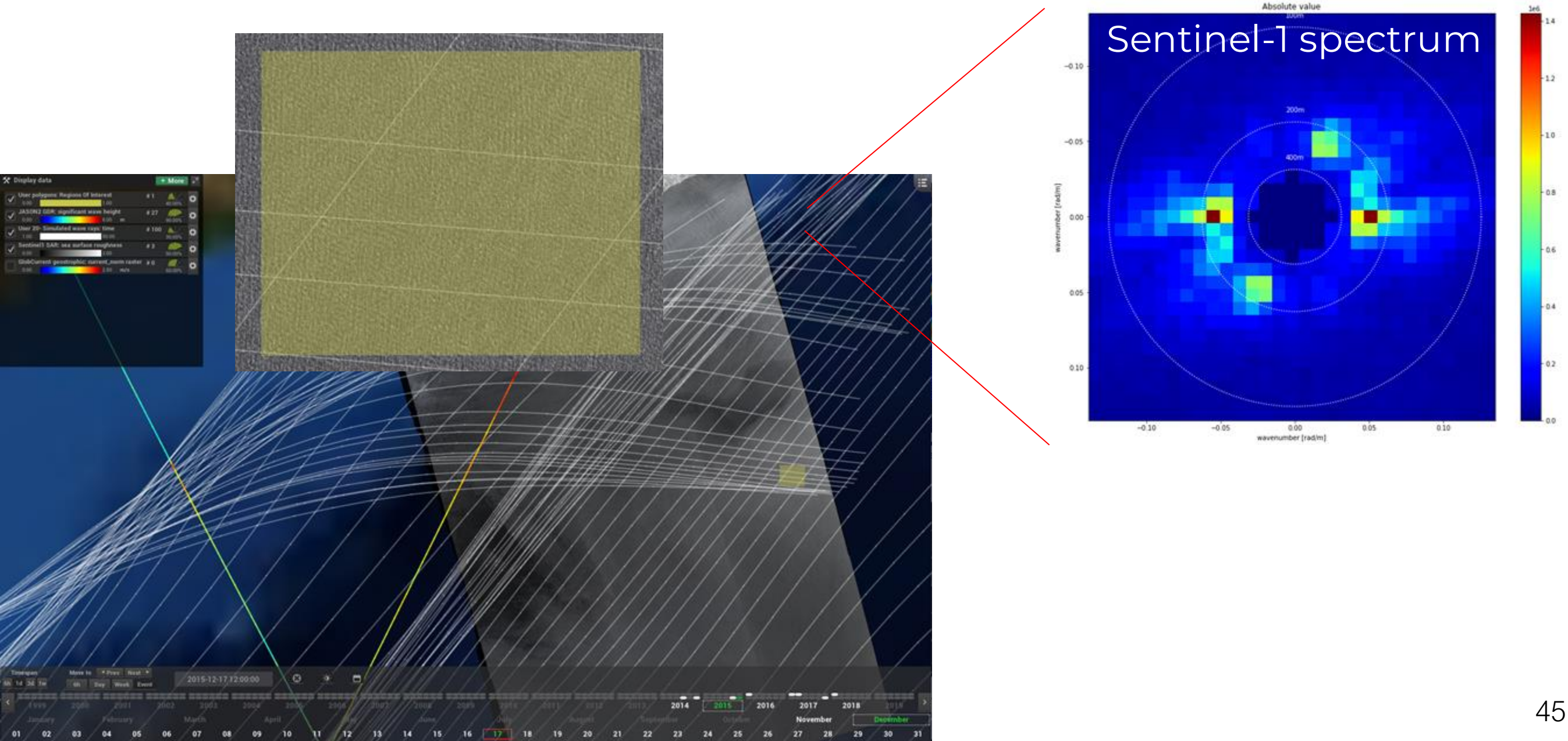
Stochastic model with homogeneous and time-correlated small-scale velocities



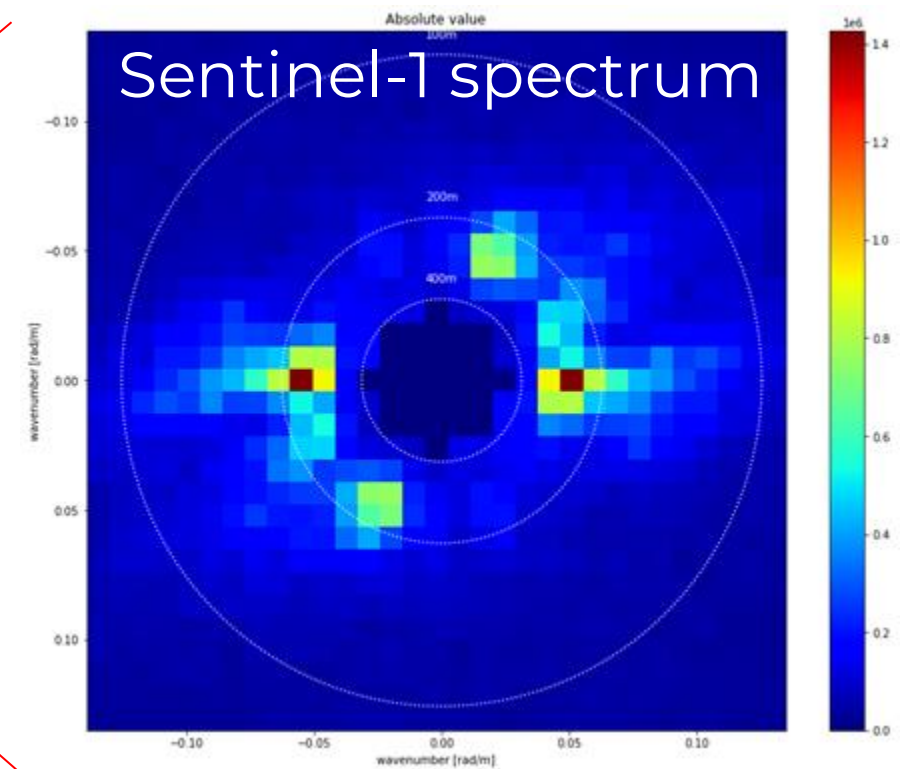
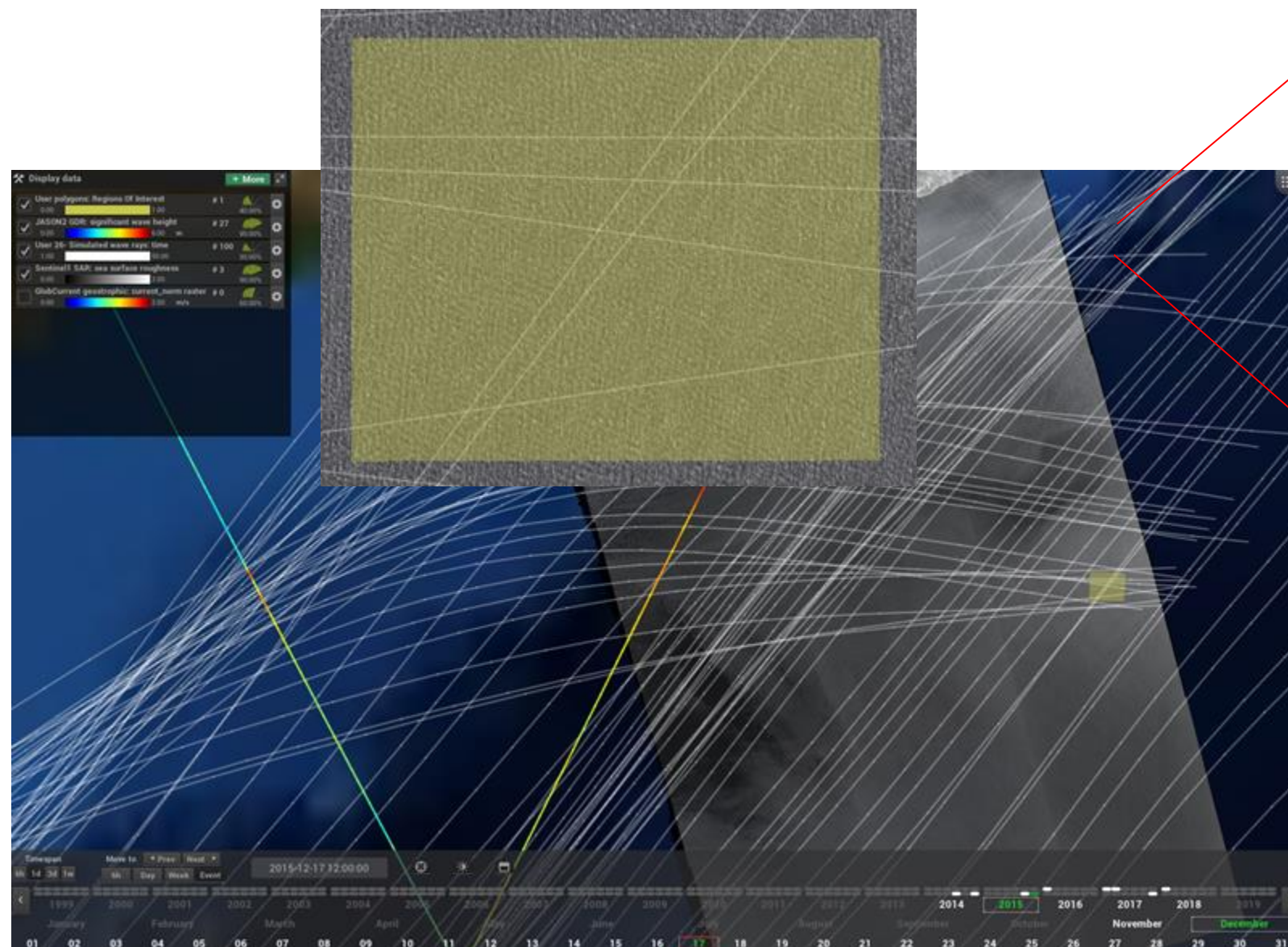
Stochastic model with heterogeneous and time-correlated small-scale velocities



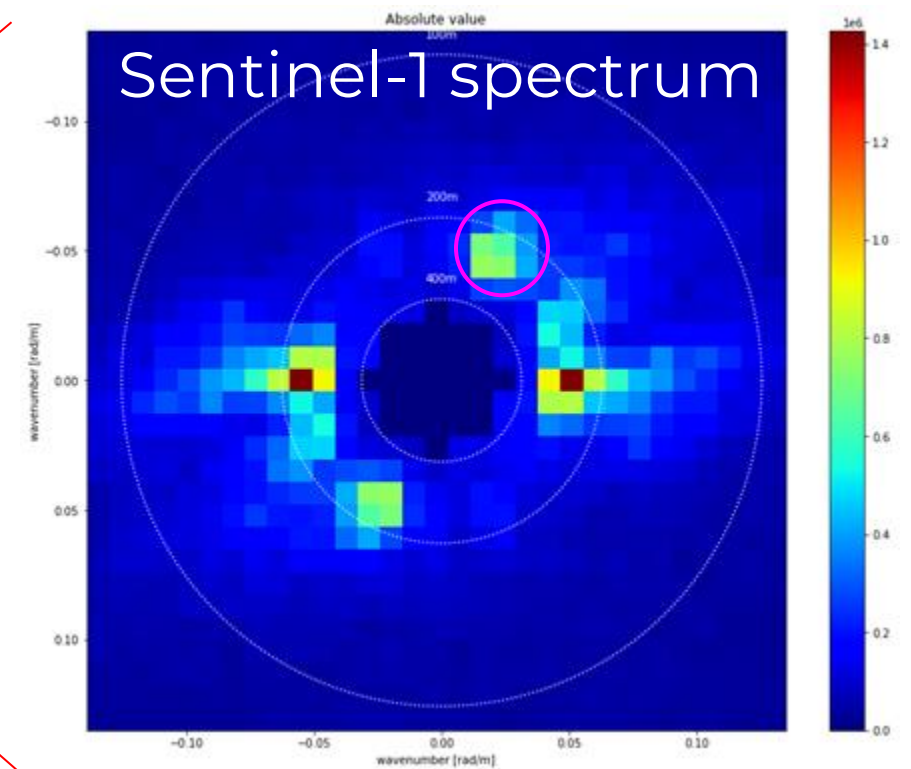
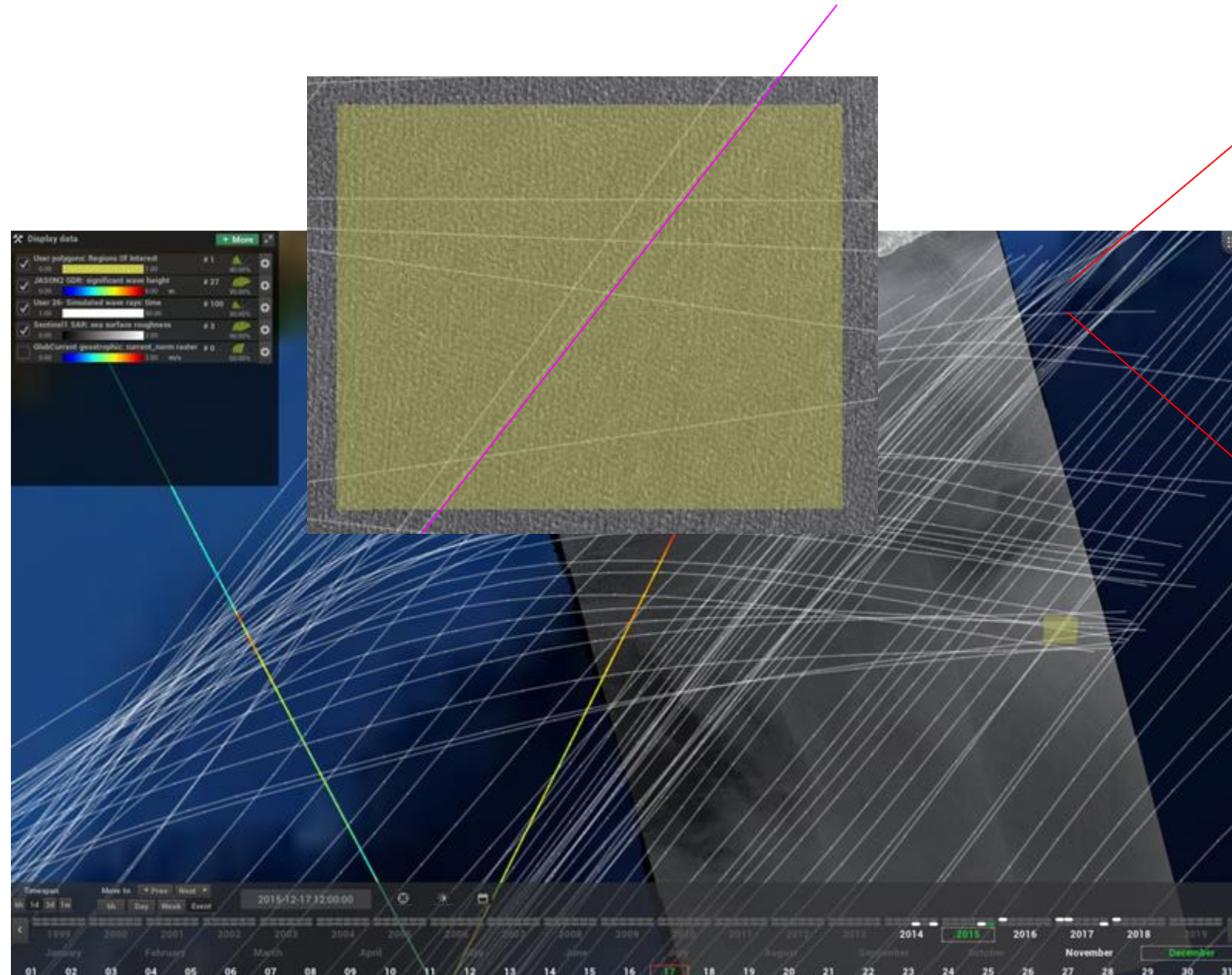
Determinist model



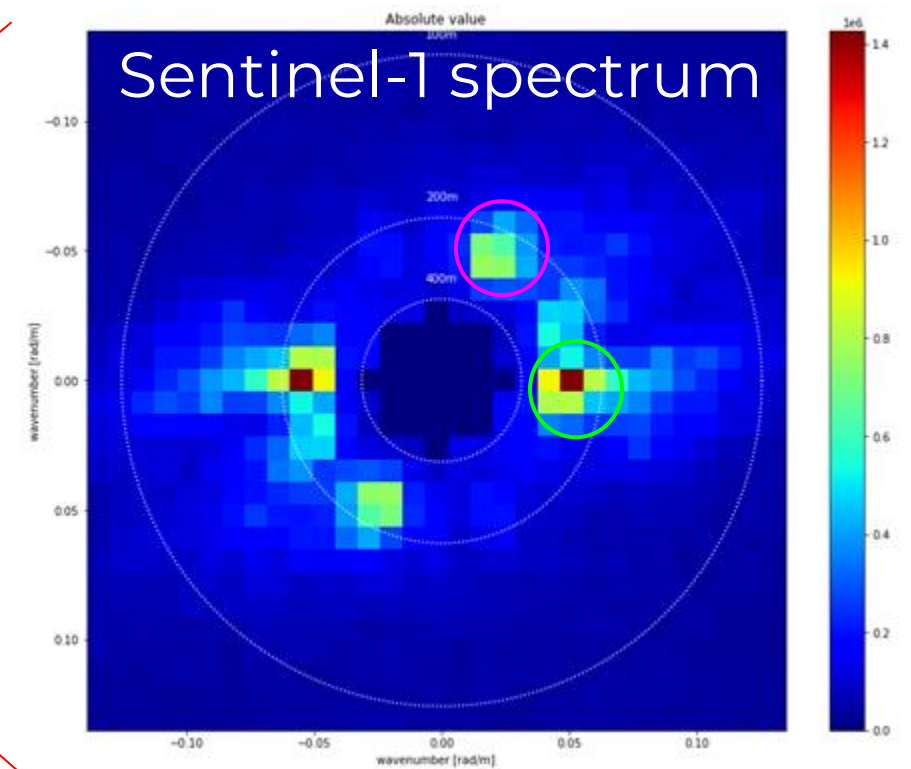
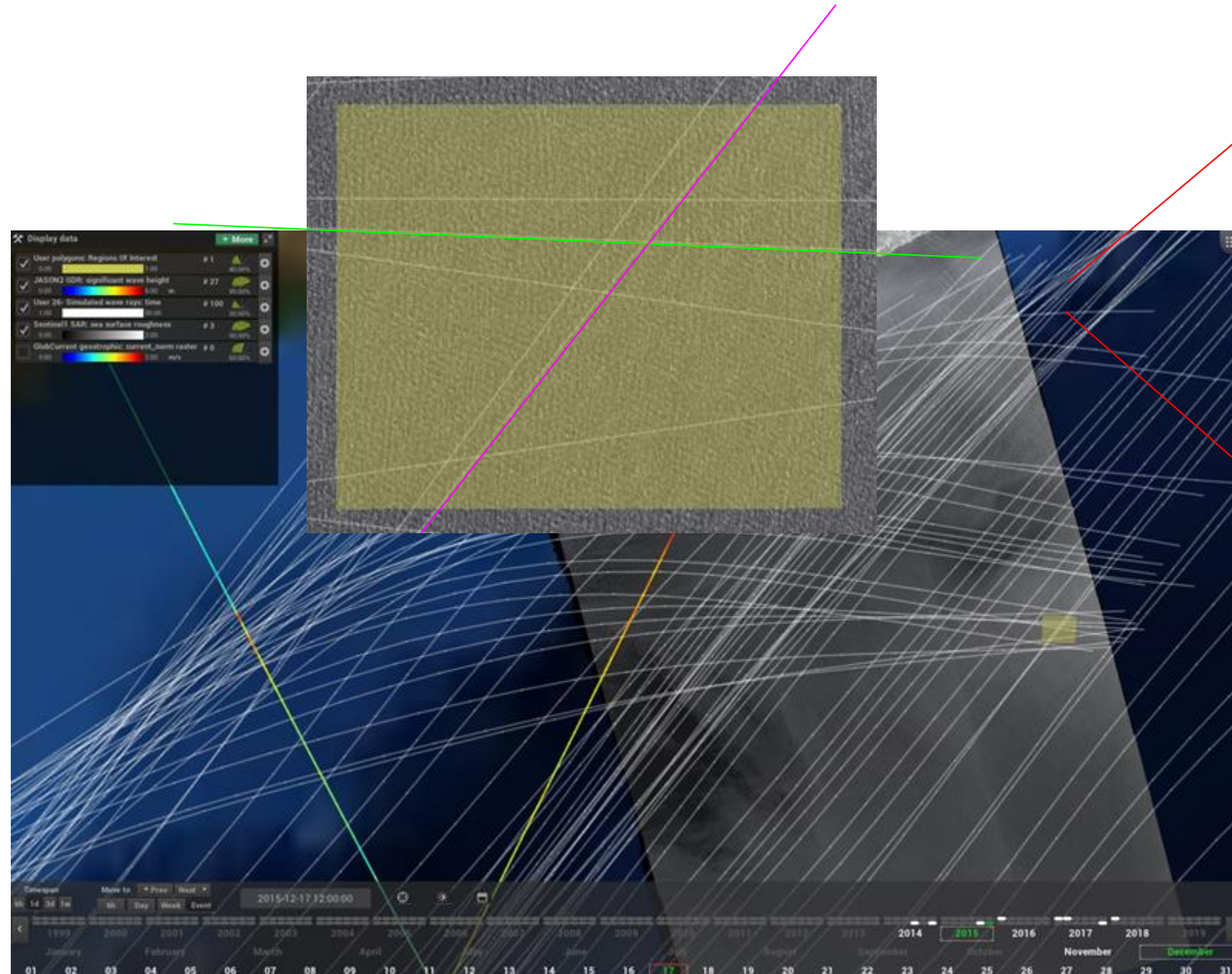
Stochastic model with heterogeneous and time-correlated small-scale velocities



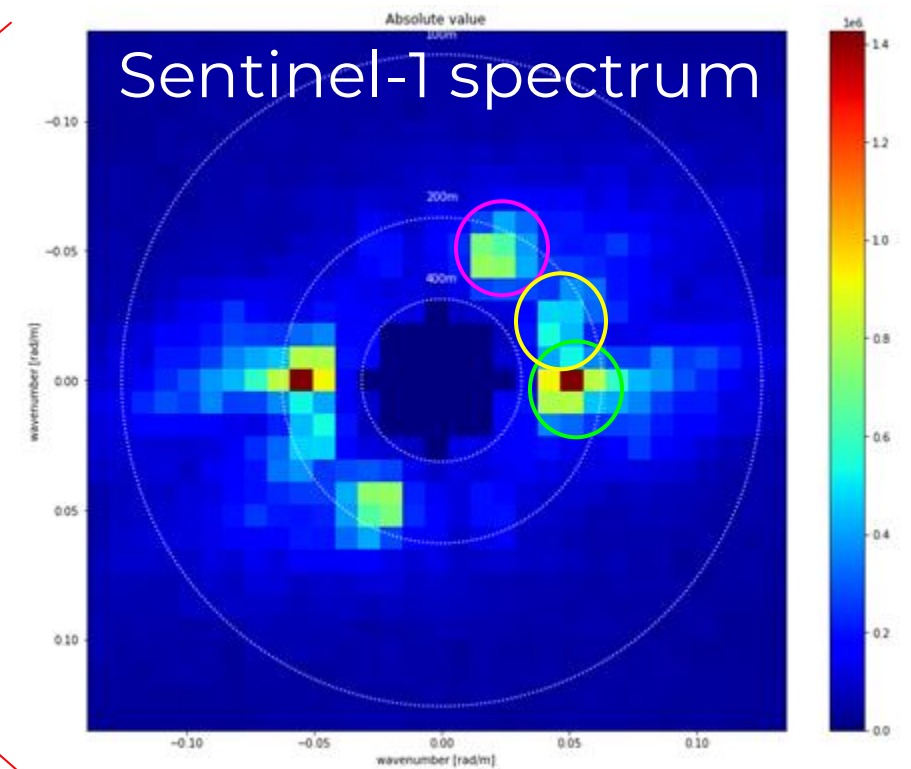
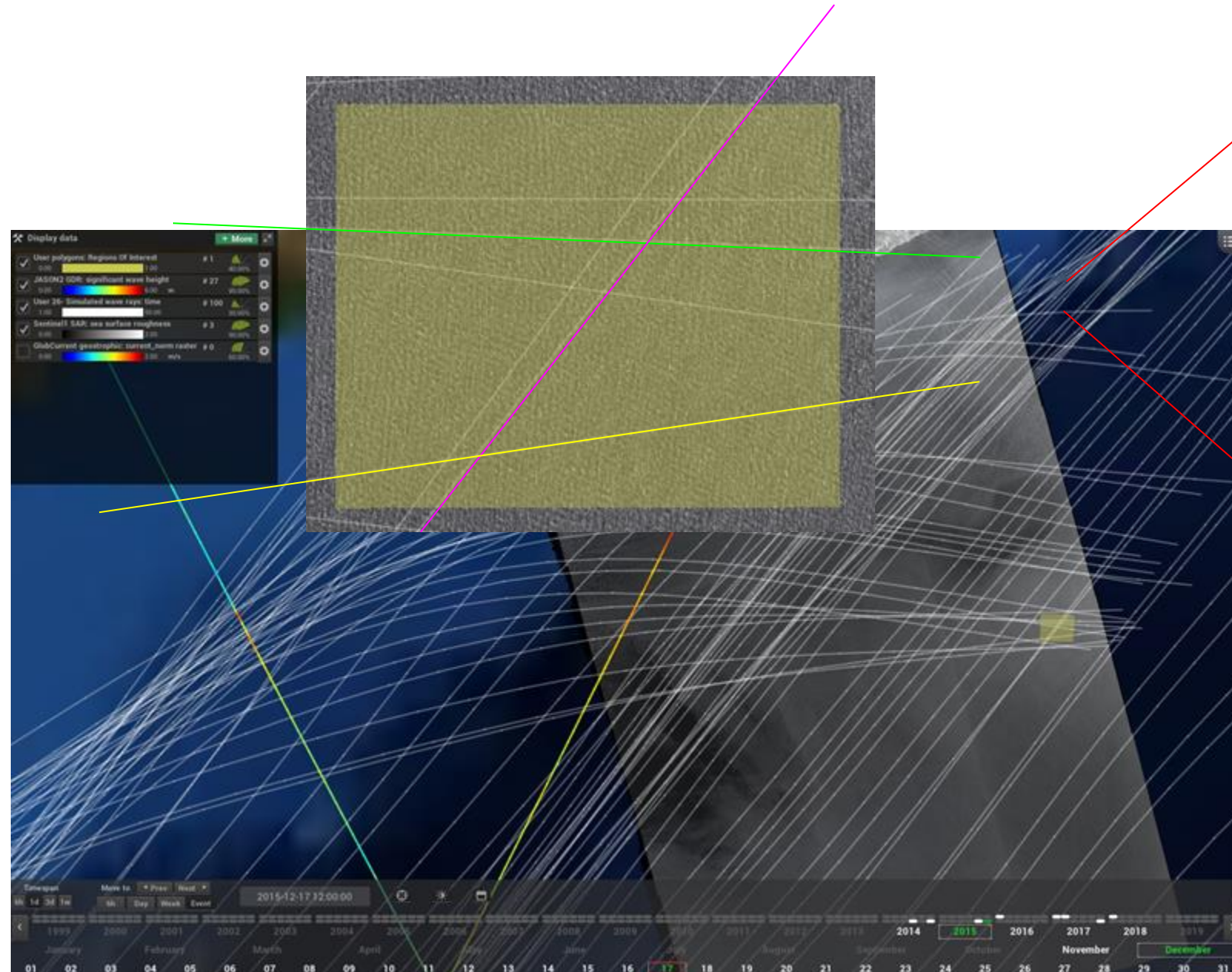
Stochastic model with heterogeneous and time-correlated small-scale velocities



Stochastic model with heterogeneous and time-correlated small-scale velocities



Stochastic model with heterogeneous and time-correlated small-scale velocities



II. Beyond Markovian closure

II.C. Callibration

Large scale
group velocity:

$$\bar{v} + c_g^0$$

Small scale
group velocity:

$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' :

Calibration

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

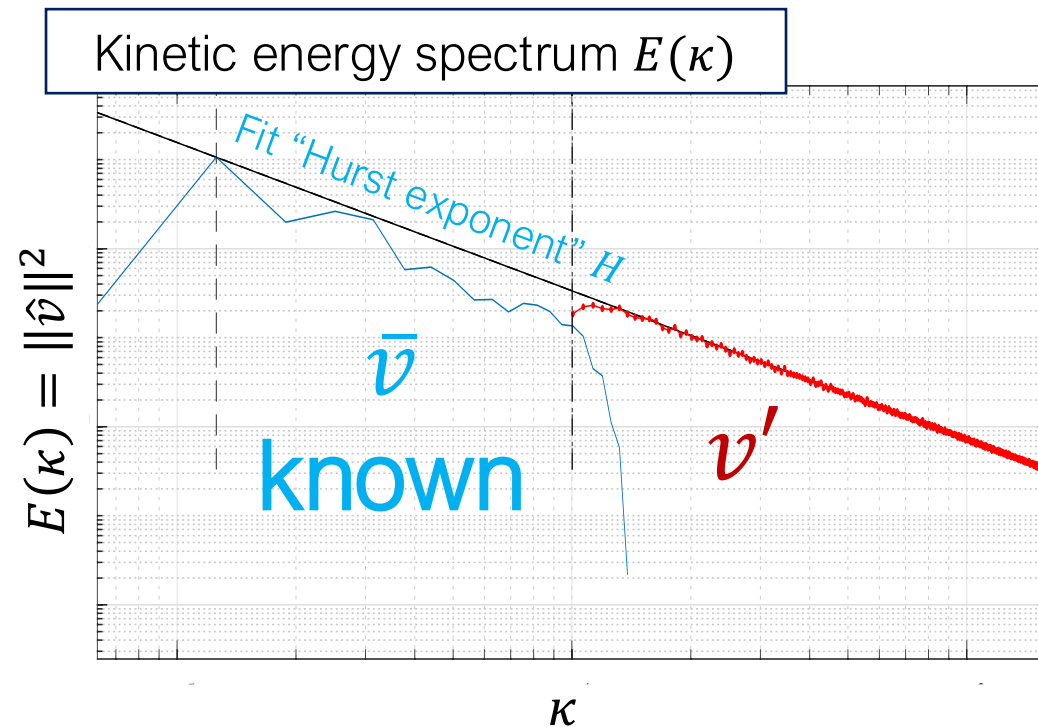
Small scale
group velocity:

$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' : Calibration



$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

v'

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' :

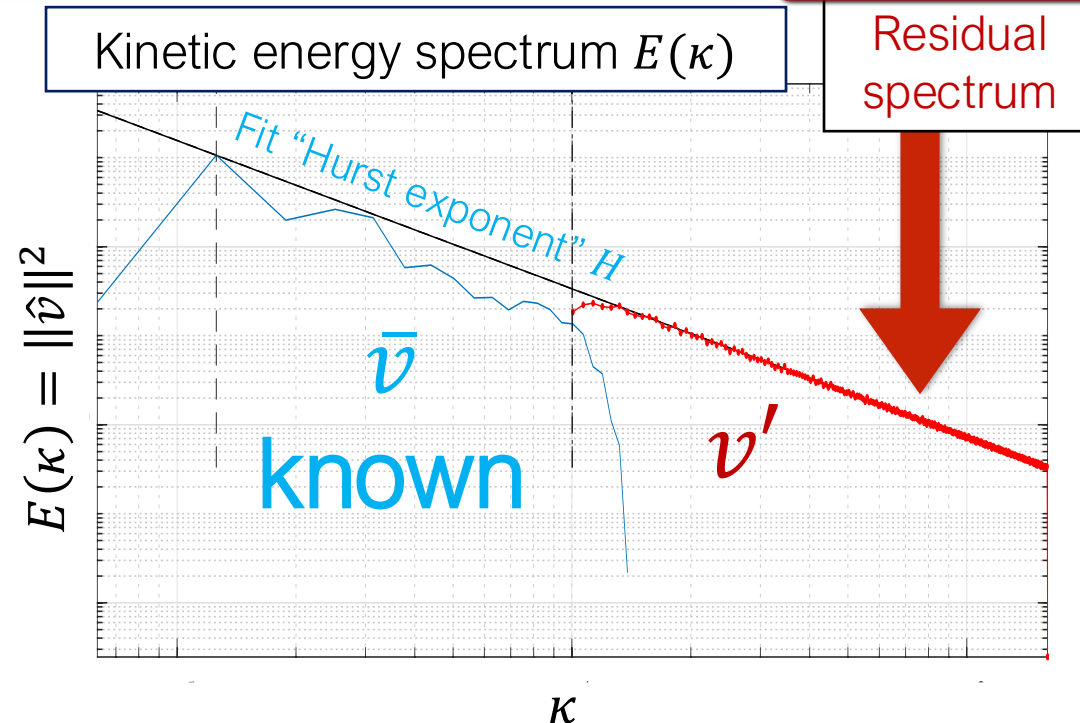
Calibration

Physical scale symmetry

$$\text{On-line fit : } A \|k\|^{-H}$$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$



$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:
 $\bar{v} + C_g^0$
 Small scale group velocity:
 v'
 Wave:
 $ae^{i\frac{z}{\epsilon}\phi}$

Time-correlated model for v' :

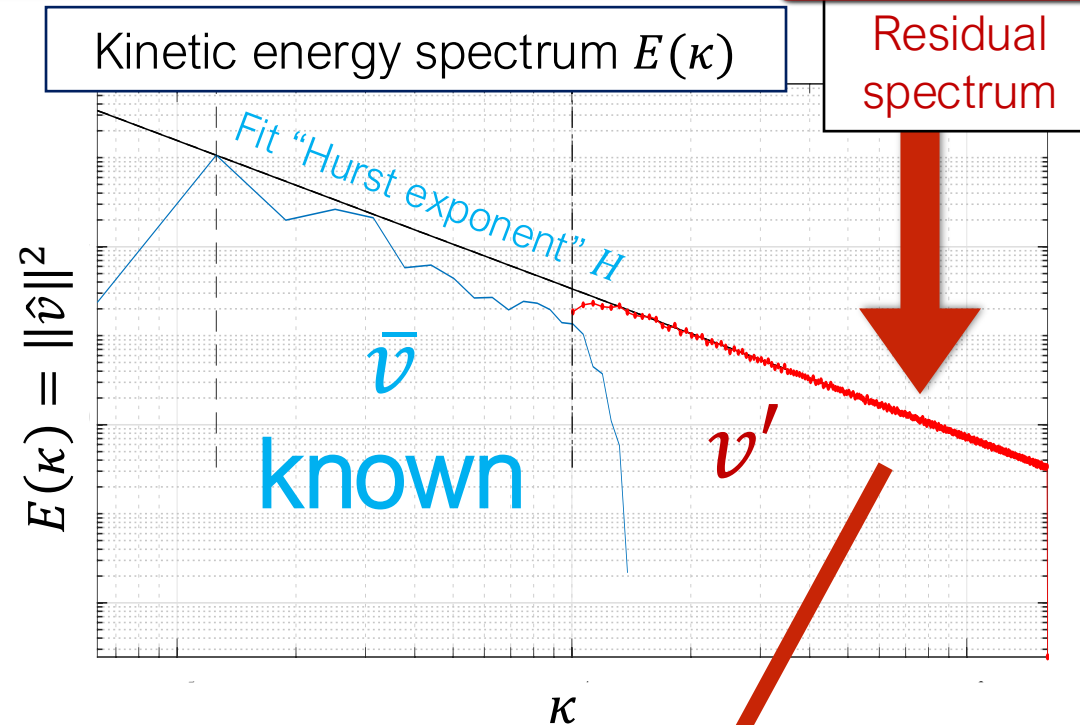
Calibration

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$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:
 $\bar{v} + C_g^0$
 Small scale group velocity:
 v'
 Wave:
 $ae^{i\frac{z}{\epsilon}\phi}$

Time-correlated model for v' :

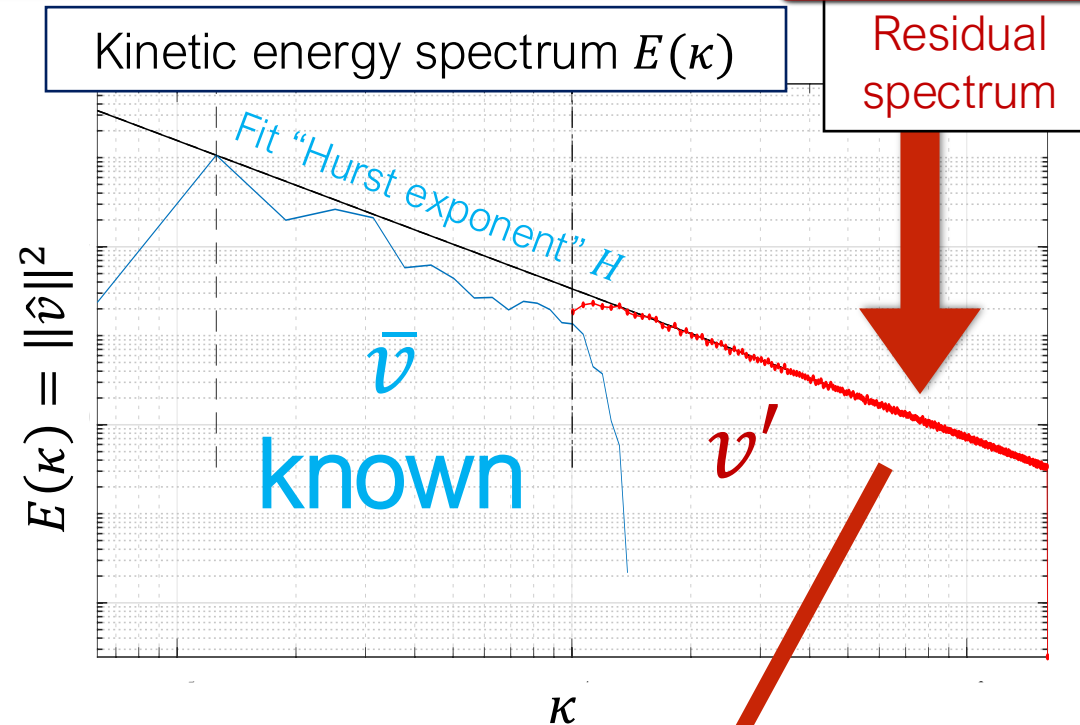
Calibration

Physical scale symmetry

On-line fit : $A \|k\|^{-H}$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$



$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:
 $\bar{v} + C_g^0$
 Small scale group velocity:
 v'
 Wave:
 $ae^{i\epsilon\phi}$

Time-correlated model for v' :

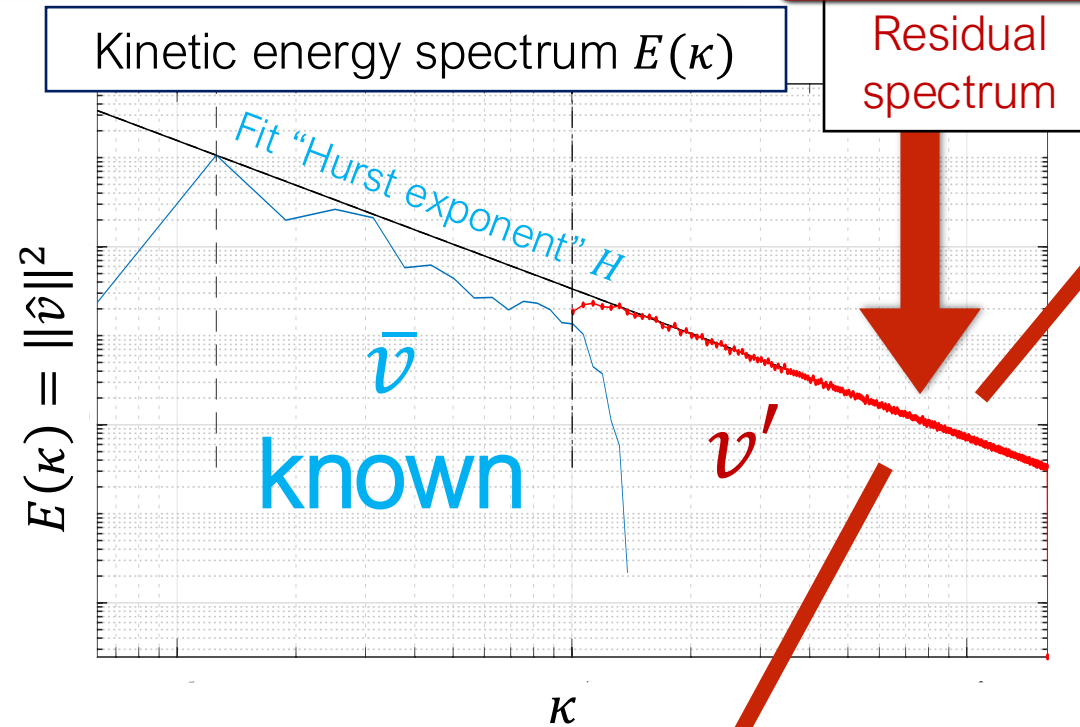
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Current correlation time

$$\tau(k) = \tau_{cur}(k) = \frac{1/k}{v_k} = \frac{1/k}{\sqrt{k E(k)}}$$

Filter h

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

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Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' :

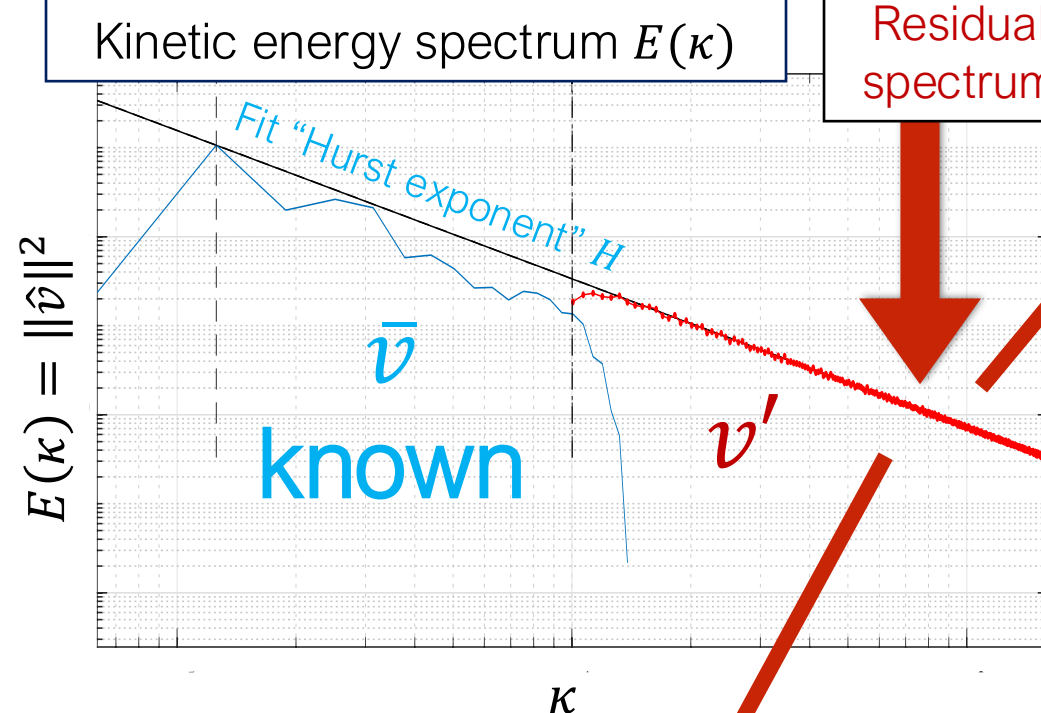
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Current correlation time

$$\tau(\kappa) = \tau_{cur}(\kappa) = \frac{1/\kappa}{v_k} = \frac{1/\kappa}{\sqrt{\kappa E(\kappa)}}$$

Multiscale noise B_t^c

From $B_t^c(x) = \iint \widehat{B}_t^c(k) e^{ik \cdot x} dx$,
Ornstein-Uhlenbeck process in time,
with correlation time $\tau(k)$ at the spatial scale k :

$$d\widehat{B}_t^c(k) = -\frac{1}{\tau(k)} \widehat{B}_t^c(k) dt + \sqrt{\frac{2}{\tau(k)}} (dB_t)(k)$$

and dB_t/dt "spatiotemporal white noise"
(cylindrical Wiener Process)

Filter h

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Conclusion

Conclusion

Generalized Fast Wave Approximation (GenFWA)

- Smallest scale turbulence decorrelates along the wave propagation
=> broader validity range compared to FWA
- Encodes both large-scale refraction and random scattering effects
- Takes into account wavenumber variation and handles strong heterogeneous flows, like localized jets with strong current gradients
- Provides both numerical and theoretical results
- Explains and quantifies ray trapping effects by jets, unlike FWA

- For strong currents with flat spectra,
we propose a new stochastic closure with stochastic currents multi-scale in space and time.
- Better explain satellite observations

In the future,

- we would like to use it as prior emulator of data assimilation
- Finite-size wave group with PiCIES (Tom Protin's PhD)