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1                   **FINITE-TIME OUTPUT REGULATION FOR LINEAR**  
2                   **TIME-VARYING HYPERBOLIC BALANCE LAWS\***

3                   YUBO BAI<sup>†</sup>, CHRISTOPHE PRIEUR<sup>‡</sup>, AND ZHIQIANG WANG<sup>§</sup>

4       **Abstract.** This work is concerned with the output regulation problem for a non-autonomous  
5 infinite dimensional system. Specially, a regulator for boundary controlled time-varying hyperbolic  
6 systems is designed. The disturbances can act within the space domain, and affect both boundaries  
7 and the to-be-controlled output. The to-be-controlled output comprises in-domain pointwise, distrib-  
8 uted and boundary outputs. The output regulation problem is solved in finite time. The regulator  
9 design is based on the solvability of the regulator equations. Due to the time-varying setting of the  
10 system and the generality of the to-be-controlled output, solving regulator equations becomes more  
11 challenging compared to the case of autonomous systems. A novel method is introduced to overcome  
12 this difficulty. By considering the regulator equations as a control system, we examine the dual  
13 system of the regulator equations and transform the solvability of the regulator equations into the  
14 validity of an observability-like inequality. Under the conditions regarding the boundary coupling  
15 term and the to-be-controlled output, we have proven this inequality. Additionally, time-varying  
16 setting also brings an advantage to the problem. Since the regulator equations is time-dependent, its  
17 solvability does not depend on the eigenmodes of the signal model. On the contrary, in the case of  
18 autonomous systems, its solvability depends on the relationship between the plant transfer behavior  
19 and the eigenmodes of the signal model.

20       **Key words.** hyperbolic systems, non-autonomous systems, output regulation

21       **MSC codes.** 93B52, 93C20, 35L40

22       **1. Introduction.** Control of partial differential equations (PDEs) has garnered  
23 significant attention due to its mathematical complexity and its applications in various  
24 other fields such as engineering and physics. One significant class of PDE systems is  
25 hyperbolic systems, which arise in many application scenarios such as open channels,  
26 gas flow pipelines, or road traffic flow models. The boundary stabilization of these  
27 hyperbolic systems has been considered in literature for decades, see, for instance, [5].  
28 Therein, the exponential stability of hyperbolic systems is studied. More recently,  
29 finite-time stabilization of hyperbolic systems has also received much attention. One  
30 can refer to [12, 13] for finite-time stabilization of homogeneous linear and quasilinear  
31 hyperbolic systems and to [10] for finite-time stabilization of linear time-varying hy-  
32 perbolic systems. In [10], a time-dependent backstepping method was used to design  
33 the state feedback control.

34       In this paper, we investigate the output regulation problem of the hyperbolic  
35 systems. For the output regulation problems, unlike the stabilization problems, the  
36 objective is to design feedback control such that the output of the system tracks a  
37 given reference and rejects the disturbances. There has been a very fruitful literature  
38 on the output regulation of the hyperbolic system. In [1, 2], boundary disturbance

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39 rejection for linear  $2 \times 2$  hyperbolic systems was considered by using the backstepping  
 40 approach. Concerning robust output regulation, [14] used backstepping method to de-  
 41 sign a robust state feedback regulator for boundary controlled linear  $2 \times 2$  hyperbolic  
 42 systems. Therein, the output to be controlled is assumed to be available for measure-  
 43 ment. Therefore, the regulator design is based on the internal model principle. Later  
 44 on, [17] generalized this work into general  $n \times n$  linear heterodirectional hyperbolic  
 45 systems, where the so-called  $p$ -copy internal model principle has to be fulfilled in order  
 46 to achieve the robust output regulation. In addition to the previous references, we  
 47 would also like to mention some works on the output regulation of other types of infi-  
 48 nite dimensional systems, including [26] for cascaded network of hyperbolic systems,  
 49 [21, 22] for heat equation, [3] for Korteweg-de Vries equation, [20] for beam equation,  
 50 [23] for thermoelastic system and [27] for infinite-dimensional nonlinear systems.

51 Concerning the finite-time output regulation of hyperbolic systems, which is the  
 52 focus of this article, the first result was obtained in [15]. In this paper, the back-  
 53 stepping method was used to design the feedback regulator for boundary controlled  
 54 linear  $2 \times 2$  time-invariant hyperbolic systems. Moreover, [16, 18] achieved finite-time  
 55 output regulation for general  $n \times n$  time-invariant hyperbolic systems with different  
 56 convergent time. These three works focus on autonomous hyperbolic systems.

57 This paper is concerned with the finite-time output regulation problem for linear  
 58 hyperbolic systems when the coupling coefficients of the system depend on both time  
 59 and space variables. Therein, the disturbances can act within the domain, affect-  
 60 ing both boundaries and the output to be controlled. The output to be controlled  
 61 comprises in-domain pointwise, distributed and boundary outputs. In this work, we  
 62 focus on the design of the feedback regulator, assuming that the system states, ref-  
 63 erence signal states, and disturbance states are known. Using the results from [10],  
 64 we transform the design of the feedback regulator into the solvability of regulator  
 65 equations.

66 Compared to the literature mentioned, in particular [15], this paper considers  
 67 non-autonomous hyperbolic systems, which introduces new challenges to the solvabil-  
 68 ity of regulator equations. As mentioned in [15], the regulator equations of time-  
 69 independent hyperbolic system can be expressed as ordinary differential equations  
 70 (ODEs) and can be explicitly solved. The solvability condition can be characterized  
 71 as the relationship between the signal model and the transfer behavior of the system.  
 72 However, under the time-varying setting, the regulator equations are PDEs rather  
 73 than ODEs. Due to the time-varying setting and the generality of the to-be-controlled  
 74 output, directly solving the regulator equations becomes difficult. We applied a novel  
 75 approach to address this challenge. We consider the regulator equations as a con-  
 76 trol system. Similarly to dealing with controllability problems, we examine the dual  
 77 system of the regulator equations. Then, the solvability of the regulator equations  
 78 is transformed into the validity of an observability-like inequality and Lyapunov-like  
 79 functions are used to prove this observability-like inequality. Through this method, we  
 80 can only obtain feedback gain function with  $L^2$  regularity over a finite time domain,  
 81 which restricts us to solving the output regulation problem only within a finite time  
 82 domain and considering only broad solution (with weak regularity) to the system.

83 In addition, due to our approach, we need assumptions on the dimensions of  
 84 the system and the to-be-controlled output, namely, the number of equations with  
 85 negative speeds (i.e., dimension of the input) is not less than the number of equations  
 86 with positive speeds, which is not less than the dimension of the to-be-controlled  
 87 output. In the meantime, time-varying setting also brings the following advantage: the  
 88 solvability of the regulator equations depends no longer on the relationship between

89 the plant transfer behavior and the eigenmodes of the signal model. In other words,  
 90 our approach relaxes the assumptions of [15, 16, 18]. Due to the discontinuity in  
 91 spatial variables of the dual system of the regulator equations, it is necessary to apply  
 92 specific techniques for the well-posedness. Inspired by the proof in [10], this is done  
 93 in Appendix A.

94 The remaining part of this paper is organized as follows. In Section 2, we introduce  
 95 the considered output regulation problem. Some preliminaries needed in the paper is  
 96 given in Section 3. Then, Section 4 presents the main results of this paper, namely the  
 97 design of the finite-time regulator. The well-posedness results for the broad solution  
 98 and the  $C^1$  solution are provided in Appendices A and B, respectively.

99 Throughout the paper, we use the following notation. For a domain  $\Omega$  in  $\mathbb{R}^n$ ,  
 100 a Banach space  $X$  and any nonnegative integer  $m$ , let  $C_U^m(\Omega; X)$  denote the vec-  
 101 tor space consisting of all functions  $f : \Omega \rightarrow X$  which, together with all their  
 102 partial derivatives  $D^\alpha f$  of orders  $|\alpha| \leq m$ , are bounded and uniformly continu-  
 103 ous on  $\Omega$ . For some constants  $T > 0$  and  $0 \leq t_0 < T$ , we define the domain  
 104  $\mathcal{D}(t_0) = \{(t, x) | t_0 < t < T, 0 < x < 1\}$ , and define the function space  $\mathcal{B}(t_0) =$   
 105  $C^0([t_0, T]; L^2(0, 1)) \cap C^0([0, 1]; L^2(t_0, T))$ . Let  $l$  belong to  $\mathbb{N}^+$  and let  $x_i, i = 0, 1, \dots, l$ ,  
 106 be some points in  $[0, 1]$  satisfying  $0 = x_0 < x_1 < \dots < x_l = 1$ . We define the do-  
 107 main  $\mathcal{D}_l(t_0) = \{(t, x) | t_0 < t < T, x \in \cup_{i=1}^l (x_{i-1}, x_i)\}$  and define the function space  
 108  $\mathcal{B}_l(t_0) = C^0([t_0, T]; L^2(0, 1)) \cap C_U^0(\cup_{i=1}^l (x_{i-1}, x_i); L^2(t_0, T))$ . For a vector  $\nu$  and a  
 109 matrix  $A$ , denote by  $\|\nu\|$  the Euclidean norm and by  $\|A\|$  the matrix norm of  $A$  as-  
 110 sociated to the Euclidean norm. For symmetric matrices  $P$  and  $Q$ ,  $P > 0$  ( $P \geq 0$ )  
 111 means that  $P$  is positive (nonnegative) definite, and  $P > Q$  ( $P \geq Q$ ) means  $P - Q > 0$   
 112 ( $P - Q \geq 0$ ). Denote by  $\text{Id}_n$  the  $n \times n$  identity matrix. Denote by  $\text{diag}(A_1, \dots, A_n)$  the  
 113 block diagonal matrix with matrices  $A_1, \dots, A_n$  on the diagonal, where  $A_i$  are square  
 114 matrices of potentially different sizes, and all off-diagonal blocks are zero matrices of  
 115 appropriate dimensions.

116 **2. Problem statement.** In this paper, combining the systems from [10, 18], we  
 117 consider the following linear time-varying  $n \times n$  hyperbolic system, for  $(t, x)$  in  $\mathcal{D}(t_0)$ ,

118 (2.1a)  $\quad \partial_t w(t, x) + \Lambda(t, x) \partial_x w(t, x) = A(t, x) w(t, x) + g_1(t, x) d(t),$

119 (2.1b)  $\quad w_+(t, 0) = Q(t) w_-(t, 0) + g_2(t) d(t),$

120 (2.1c)  $\quad w_-(t, 1) = u(t) + g_3(t) d(t),$

121 (2.1d)  $\quad w(t_0, x) = w^0(x),$

122 (2.1e)  $\quad y(t) = \mathcal{C}_t[w(t, \cdot)] + g_4(t) d(t).$

123 In (2.1),  $w : \mathcal{D}(t_0) \rightarrow \mathbb{R}^n$  is the state,  $w^0$  in  $L^2(0, 1)^n$  is the initial data at time  $t_0$ ,  
 124  $u(t)$  in  $\mathbb{R}^m$  is the control input,  $d(t)$  in  $\mathbb{R}^h$  is the disturbance and  $y(t)$  in  $\mathbb{R}^q$  is the  
 125 output to be controlled. The matrix  $A = (a_{ij})_{1 \leq i, j \leq n}$  couples the equations of the  
 126 system inside the domain, the matrix  $Q$  couples the equations of the system on the  
 127 boundary  $x = 0$  and the matrices  $g_i, i = 1, \dots, 4$  are disturbance input locations. Let  
 128 us make the following assumptions on all coefficients involved in (2.1).

129 *Assumption 2.1.* The matrix  $\Lambda$  is diagonal, namely

130 
$$\Lambda(t, x) = \text{diag}(\lambda_1(t, x), \dots, \lambda_n(t, x))$$

131 for every  $(t, x)$  in  $[0, \infty) \times [0, 1]$ .

132 *Assumption 2.2.* Assume that  $n \geq 2$ . Denote by  $m$  in  $\{1, \dots, n - 1\}$  the number  
 133 of equations with negative speeds and by  $p = n - m$  in  $\{1, \dots, n - 1\}$  the number of

134 equations with positive speeds. We assume that there exists some  $\varepsilon_0 > 0$  such that  
 135 for every  $(t, x)$  in  $[0, \infty) \times [0, 1]$ , we have

$$136 \quad (2.2) \quad \lambda_1(t, x) < \cdots < \lambda_m(t, x) < -\varepsilon_0 < 0 < \varepsilon_0 < \lambda_{m+1}(t, x) < \cdots < \lambda_n(t, x),$$

137 and, for every  $i$  in  $\{1, \dots, n-1\}$ ,

$$138 \quad (2.3) \quad \lambda_{i+1}(t, x) - \lambda_i(t, x) > \varepsilon_0.$$

139 Assumption 2.2 is identical to the assumption in [10], where finite-time stabilization  
 140 problem is considered. As stated in [10], (2.2) is expected for finite-time stabilization,  
 141 while (2.3) is mainly technical. All along this paper, for a vector (or vector-valued  
 142 function)  $\nu$  in  $\mathbb{R}^n$  and a matrix (or matrix-valued function)  $B$  in  $\mathbb{R}^{n \times n}$ , we use the  
 143 notation

$$144 \quad \nu = \begin{pmatrix} \nu_- \\ \nu_+ \end{pmatrix}, \quad B = \begin{pmatrix} B_{--} & B_{-+} \\ B_{+-} & B_{++} \end{pmatrix},$$

145 with  $\nu_-$  in  $\mathbb{R}^m$ ,  $\nu_+$  in  $\mathbb{R}^p$  and  $B_{--}$  in  $\mathbb{R}^{m \times m}$ ,  $B_{-+}$  in  $\mathbb{R}^{m \times p}$ ,  $B_{+-}$  in  $\mathbb{R}^{p \times m}$ ,  $B_{++}$  in  
 146  $\mathbb{R}^{p \times p}$ .

147 *Assumption 2.3.* The following regularities hold for  $\Lambda$ ,  $A$  and  $Q$ :

$$148 \quad \Lambda \in C^1([0, \infty) \times [0, 1])^{n \times n}, \quad A \in C^0([0, \infty) \times [0, 1])^{n \times n}, \quad Q \in C^0([0, \infty))^{p \times m},$$

$$149 \quad \Lambda, \partial_x \Lambda, A \in L^\infty((0, \infty) \times (0, 1))^{n \times n}, \quad Q \in L^\infty(0, \infty)^{p \times m}.$$

150 There exist constants  $M_0, M_1, M_Q > 0$  such that

$$151 \quad \|\Lambda\|_{L^\infty((0, \infty) \times (0, 1))^{n \times n}} \leq M_0, \quad \|\partial_x \Lambda\|_{L^\infty((0, \infty) \times (0, 1))^{n \times n}} \leq M_0,$$

$$152 \quad \|A\|_{L^\infty((0, \infty) \times (0, 1))^{n \times n}} \leq M_1, \quad \|Q\|_{L^\infty(0, \infty)^{p \times m}} \leq M_Q.$$

153 The output to be controlled  $y(t)$  in  $\mathbb{R}^q$  is modelled by the formal output operator  
 154  $\mathcal{C}_t$  which satisfies the following assumption.

155 *Assumption 2.4.* Given  $f_i = (f_{i-}, f_{i+})$ ,  $i = 0, 1, \dots, l$ , and  $c = (c_-, c_+)$  satisfying

$$156 \quad f_{i-} \in (C^0[0, +\infty))^{q \times m} \cap L^\infty(0, \infty)^{q \times m}, \quad f_{i+} \in (C^0[0, +\infty))^{q \times p} \cap L^\infty(0, \infty)^{q \times p},$$

$$157 \quad c_- \in C^0([0, +\infty); L^2(0, 1))^{q \times m} \cap L^\infty((0, +\infty); L^2(0, 1))^{q \times m},$$

$$158 \quad c_+ \in C^0([0, +\infty); L^2(0, 1))^{q \times p} \cap L^\infty((0, +\infty); L^2(0, 1))^{q \times p},$$

159 for any  $t \geq 0$  and  $\tilde{n}$  in  $\mathbb{N} \setminus \{0\}$ , the operator  $\mathcal{C}_t$  is defined by

$$160 \quad (2.4) \quad \begin{aligned} \mathcal{C}_t : (C^0[0, 1])^{n \times \tilde{n}} &\rightarrow \mathbb{R}^{q \times \tilde{n}} \\ \bar{\rho} &\mapsto \sum_{i=0}^l f_i(t) \bar{\rho}(x_i) + \int_0^1 c(t, x) \bar{\rho}(x) dx. \end{aligned}$$

161 There exist constants  $M_f, M_c > 0$  such that

$$162 \quad \max_{0 \leq i \leq l} \|f_i\|_{L^\infty(0, \infty)^{q \times n}} \leq M_f, \quad \|c\|_{L^\infty((0, +\infty); L^2(0, 1))^{q \times n}} \leq M_c.$$

163 Clearly, for any  $0 \leq t_0 < T$ ,  $\tilde{n}$  in  $\mathbb{N} \setminus \{0\}$  and  $\rho$  in  $\mathcal{B}(t_0)^{n \times \tilde{n}}$ ,  $(t \mapsto \mathcal{C}_t[\rho(t, \cdot)])$  is  
 164 in  $L^2(t_0, T)^{q \times \tilde{n}}$ . It comprises in-domain pointwise, distributed and boundary out-  
 165 puts. It encompasses the outputs used in [15, 16, 18, 26]. These output types are

166 widely applied in real-world problems that can be modeled by hyperbolic systems. For  
 167 boundary output, one can refer to [5, 6] for the boundary set-point control problem of  
 168 the Saint-Venant equations. In [4], heat exchanger with in-domain pointwise output  
 169 is considered.

170 The disturbance  $d(t)$  is in  $\mathbb{R}^h$ . The corresponding disturbance input locations  
 171 satisfy the following assumption.

172 *Assumption 2.5.* Matrix-valued functions  $g_i$ ,  $i = 1, 2, 3, 4$  are known and have the  
 173 following regularities

$$174 \quad g_1 \in C^0([0, \infty) \times [0, 1])^{n \times h}, \quad g_2 \in C^0([0, \infty))^{p \times h},$$

$$175 \quad g_3 \in C^0([0, \infty))^{m \times h}, \quad g_4 \in C^0([0, \infty))^{q \times h}.$$

176 The disturbance  $d(t)$  and the reference input  $r(t)$  in  $\mathbb{R}^q$  to be tracked by the output  
 177  $y(t)$  are the solutions to the following finite-dimensional signal model, for  $t > t_0$ ,

$$178 \quad (2.5) \quad \begin{aligned} \dot{v}(t) &= S(t)v(t), & v(t_0) &= v^0, \\ d(t) &= p_d(t)v(t), & r(t) &= p_r(t)v(t), \end{aligned}$$

179 where  $v^0$  is in  $\mathbb{R}^{n_v}$ . The coefficients of (2.5) satisfy the following assumption.

180 *Assumption 2.6.* Matrix-valued functions  $S$ ,  $p_d$  and  $p_r$  are known.  $S : [0, +\infty) \rightarrow$   
 181  $\mathbb{R}^{n_v \times n_v}$  is measurable and bounded on every finite subinterval of time,  $p_d$  is in  
 182  $C^0([0, \infty))^{h \times n_v}$  and  $p_r$  is in  $C^0([0, \infty))^{q \times n_v}$ .

183 By Assumption 2.6, there exists a unique continuous transition matrix  $\Psi : [0, +\infty)^2 \rightarrow$   
 184  $\mathbb{R}^{n_v \times n_v}$  of  $S$  such that the solution of (2.5) is given by  $v(t) = \Psi(t, t_0)v^0$ . One can  
 185 refer to [9, p. 5] for the properties of transition matrix  $\Psi$ . Denote by

$$186 \quad (2.6) \quad e_y(t) = y(t) - r(t)$$

187 the output tracking error. Inspired by [10, 18], let us give the notion of the uniform  
 188 finite-time output regulation that we are interested in.

189 **DEFINITION 2.7.** *The output  $y$  of the system (2.1) achieves the uniform finite-*  
 190 *time output regulation within settling time  $T_0$  if for any  $T > T_0$ , there exists a feedback*  
 191 *regulator  $u = \mathcal{K}_T[w, v]$ , such that for all  $0 \leq t_0 < T - T_0$ ,  $w^0$  in  $L^2(0, 1)^n$  and  $v^0$  in*  
 192  *$\mathbb{R}^{n_v}$ , the output tracking error  $e_y$  satisfies  $e_y = 0$  a.e. in  $(t_0 + T_0, T)$ .*

193 *Remark 2.8.* 1. Ensuring that the system output tracks a given reference  
 194 signal is a classic goal in control theory. The output regulation for linear  
 195 finite-dimensional system is well-understood and is well introduced in, for  
 196 example, [19, 25].  
 197 2. The uniformity means that the output regulation is achieved uniformly to the  
 198 initial time  $t_0$ .  
 199 3. The output regulation is considered in any finite interval  $(t_0, T)$  and the  
 200 regulator design  $\mathcal{K}_T$  is relative to  $T$ . This restriction is due to the machinery  
 201 of proof. See Subsection 4.2 for details.

202 **3. Preliminaries on characteristics.** In this section, let us introduce some  
 203 known facts on the characteristics associated with system (2.1) and the entry and  
 204 exit times for the interval  $[x_{i-1}, x_i]$ ,  $i = 1, \dots, l$ , see [10]. To this end, we use the  
 205 extension method introduced in [10] to extend  $\Lambda$  to a function of  $\mathbb{R}^2$  (still denoted  
 206 by  $\Lambda$ ) by keeping Assumptions 2.1 to 2.3. For every  $j = 1, \dots, n$ , let  $\chi_j$  be the flow

207 associated with  $\lambda_j$ , namely for every  $(t, x)$  in  $\mathbb{R}^2$ , the function  $s \mapsto \chi_j(s; t, x)$  is the  
 208 solution to the ODE, for  $s$  in  $\mathbb{R}$ ,

$$209 \quad (3.1) \quad \frac{\partial}{\partial s} \chi_j(s; t, x) = \lambda_j(s, \chi_j(s; t, x)), \quad \chi_j(t; t, x) = x.$$

210 The existence and uniqueness of the solutions to the ODE (3.1) follows the classical  
 211 theory. Moreover, since  $\lambda_j$  is bounded, the solution is global and has the regularity

$$212 \quad (3.2) \quad \chi_j \in C^1(\mathbb{R}^3),$$

213 and, for every  $(s, t, x)$  in  $\mathbb{R}^3$ , we have

$$214 \quad (3.3) \quad \partial_t \chi_j(s; t, x) = -\lambda_j(t, x) e^{\int_t^s \partial_x \lambda_j(\tau, \chi_j(\tau; t, x)) d\tau}, \quad \partial_x \chi_j(s; t, x) = e^{\int_t^s \partial_x \lambda_j(\tau, \chi_j(\tau; t, x)) d\tau}.$$

215 Next we introduce the entry and exit times for the interval  $[x_{i-1}, x_i]$ ,  $i = 1, \dots, l$ .  
 216 For  $j = 1, \dots, n$ ,  $t$  in  $\mathbb{R}$  and  $x$  in  $[0, 1]$ , let  $s_j^{\text{in},i}(t, x)$  and  $s_j^{\text{out},i}(t, x)$  be the entry and  
 217 exit times of the flow  $\chi_j(\cdot; t, x)$  inside the interval  $[x_{i-1}, x_i]$ , namely the respective  
 218 unique solutions to

$$219 \quad (3.4) \quad \begin{aligned} \chi_j(s_j^{\text{in},i}(t, x); t, x) &= x_i, & \chi_j(s_j^{\text{out},i}(t, x); t, x) &= x_{i-1}, & \text{if } j \in \{1, \dots, m\}, \\ \chi_j(s_j^{\text{in},i}(t, x); t, x) &= x_{i-1}, & \chi_j(s_j^{\text{out},i}(t, x); t, x) &= x_i, & \text{if } j \in \{m+1, \dots, n\}. \end{aligned}$$

220 The existence and uniqueness of  $s_j^{\text{in},i}(t, x)$  and  $s_j^{\text{out},i}(t, x)$  are guaranteed by (2.2) in  
 221 Assumption 2.2. From (3.2) and by the implicit function theorem, we have

$$222 \quad (3.5) \quad s_j^{\text{in},i}, s_j^{\text{out},i} \in C^1(\mathbb{R} \times [0, 1]), \quad i = 1, \dots, l, \quad j = 1, \dots, n.$$

223 Especially, we denote the entry and exit times for the interval  $[0, 1]$  as

$$224 \quad (3.6) \quad \begin{aligned} s_j^{\text{in}}(t, x) &= s_j^{\text{in},l}(t, x), & s_j^{\text{out}}(t, x) &= s_j^{\text{out},1}(t, x), & \text{if } j \in \{1, \dots, m\}, \\ s_j^{\text{in}}(t, x) &= s_j^{\text{in},1}(t, x), & s_j^{\text{out}}(t, x) &= s_j^{\text{out},l}(t, x), & \text{if } j \in \{m+1, \dots, n\}. \end{aligned}$$

225 Integrating the ODE (3.1) and using (2.2), we have the following bounds for every  $t$   
 226 in  $\mathbb{R}$  and  $x$  in  $[0, 1]$ ,

$$227 \quad (3.7) \quad t - s_j^{\text{in}}(t, x) < \frac{1}{\varepsilon_0}, \quad s_j^{\text{out}}(t, x) - t < \frac{1}{\varepsilon_0}, \quad j = 1, \dots, n.$$

228 Differentiating (3.4) and using (3.3), we see that for  $i = 1, \dots, l$ ,

$$229 \quad (3.8) \quad \begin{aligned} \partial_\mu s_j^{\text{in},i}(t, x) &= -\frac{\partial_\mu \chi_j(s_j^{\text{in},i}(t, x); t, x)}{\lambda_j(s_j^{\text{in},i}(t, x), x_i)}, & \text{if } j \in \{1, \dots, m\}, \\ \partial_\mu s_j^{\text{in},i}(t, x) &= -\frac{\partial_\mu \chi_j(s_j^{\text{in},i}(t, x); t, x)}{\lambda_j(s_j^{\text{in},i}(t, x), x_{i-1})}, & \text{if } j \in \{m+1, \dots, n\}, \end{aligned}$$

230 with  $\partial_\mu$  is  $\partial_t$  or  $\partial_x$ .

231 **4. Finite-time regulator.** In this section, we aim to find a finite-time regulator.  
 232 Let  $T > 0$  and  $0 \leq t_0 < T$ . We consider the following time-dependent regulator,

$$233 \quad (4.1) \quad u(t) = k_v(t)v(t) + \int_0^1 k_w(t, x)w(t, x)dx,$$

234 with feedback gain functions  $k_v : (0, T) \rightarrow \mathbb{R}^{m \times n_v}$  and  $k_w : \mathcal{D}(0) \rightarrow \mathbb{R}^{m \times n}$  to be  
 235 determined later. By applying (4.1) to the system (2.1) and taking the signal model  
 236 (2.5) into account, we have the closed-loop system for  $(t, x)$  in  $\mathcal{D}(t_0)$ ,

$$237 \quad (4.2a) \quad \dot{v}(t) = S(t)v(t), \quad v(t_0) = v^0,$$

$$238 \quad (4.2b) \quad \partial_t w(t, x) + \Lambda(t, x)\partial_x w(t, x) = A(t, x)w(t, x) + \tilde{g}_1(t, x)v(t),$$

$$239 \quad (4.2c) \quad w_+(t, 0) = Q(t)w_-(t, 0) + \tilde{g}_2(t)v(t),$$

$$240 \quad (4.2d) \quad w_-(t, 1) = k_v(t)v(t) + \int_0^1 k_w(t, x)w(t, x)dx + \tilde{g}_3(t)v(t),$$

$$241 \quad (4.2e) \quad w(t_0, x) = w^0(x),$$

$$242 \quad (4.2f) \quad e_y(t) = \mathcal{C}_t[w(t, \cdot)] - (p_r(t) - \tilde{g}_4(t))v(t),$$

243 where  $\tilde{g}_i = g_i p_d$ ,  $i = 1, 2, 3, 4$ , and  $e_y$  is defined in (2.6). Similar to [10], we consider  
 244 the broad solutions to (4.2b)–(4.2e). The definition of broad solution and the well-  
 245 posedness of (4.2b)–(4.2e) are given in Appendix A. We have the following well-  
 246 posedness result for (4.2b)–(4.2e).

247 **THEOREM 4.1.** *Let  $k_w$  be in  $L^\infty(\mathcal{D}(0))^{m \times n}$  and  $k_v$  be in  $L^2(0, T)^{m \times n_v}$ . Under  
 248 Assumptions 2.1 to 2.3 and 2.5, for every  $w^0$  in  $L^2(0, 1)^n$  and  $v$  in  $C^0[t_0, T]^{n_v}$ , there  
 249 exists a unique broad solution  $w$  in  $\mathcal{B}(t_0)^n$  to the system (4.2b)–(4.2e).*

250 Theorem 4.1 is a corollary of Theorem A.3 in Appendix A. Let us now state the main  
 251 result of this paper.

252 **THEOREM 4.2 (Finite-Time Regulator).** *Assume that Assumptions 2.1 to 2.6  
 253 hold and assume that there exist positive constants  $\varepsilon_Q$  and  $\varepsilon_f$  such that for all  $t \geq 0$ ,*

$$254 \quad (4.3) \quad Q(t)Q(t)^\top > \varepsilon_Q \text{Id}_p,$$

255 *and*

$$256 \quad (4.4) \quad f_{l_+}(t)f_{l_+}(t)^\top > \varepsilon_f \text{Id}_q.$$

257 *Let the settling time  $T_{\text{unif}}(\Lambda)$  be defined by*

$$258 \quad (4.5) \quad T_{\text{unif}}(\Lambda) = \sup_{t \geq 0} [s_{m+1}^{\text{out}}(s_m^{\text{out}}(t, 1), 0) - t].$$

259 *Then, the output  $y$  achieves the uniform finite-time output regulation within settling  
 260 time  $T_{\text{unif}}(\Lambda)$ . More precisely, for any  $T > T_{\text{unif}}(\Lambda)$ , there exist gain functions  $k_w$  in  
 261  $L^\infty(\mathcal{D}(0))^{m \times n}$  and  $k_v$  in  $L^2(0, T)^{m \times n_v}$  such that for all  $0 \leq t_0 \leq T - T_{\text{unif}}(\Lambda)$ ,  $w^0$   
 262 in  $L^2(0, 1)^n$  and  $v^0$  in  $\mathbb{R}^{n_v}$ , the output tracking error  $e_y$  of closed-loop system (4.2)  
 263 satisfies  $e_y = 0$  a.e. in  $(t_0 + T_{\text{unif}}(\Lambda), T)$ .*

264 **Remark 4.3.** 1. It follows from conditions (4.3) and (4.4) that the number  
 265 of equations with negative speeds  $m$  (the number of the control input), the  
 266 number of equations with positive speeds  $p$  and the number of outputs to be  
 267 controlled  $q$  should satisfy  $m \geq p \geq q$ .



- 268 2. When considering the case of a  $2 \times 2$  hyperbolic system with scalar output,  
 269 i.e.,  $m = p = q = 1$ , the conditions (4.3) and (4.4) are equivalent to  $|Q(t)|^2 \geq$   
 270  $\varepsilon_Q > 0$  and  $|f_{l+}(t)|^2 \geq \varepsilon_f > 0$  for all  $t \geq 0$ . If in addition the coefficients of  
 271 the system do not depend on time, the result of this paper does not recover  
 272 the result of [15], and vice versa. On the one hand, the example in [15,  
 273 Section 6] (see also the example in item 5 of this remark) shows that the  
 274 finite-time output regulation problem can be solved when condition (4.4) is  
 275 not satisfied. On the other hand, Lemma 1 from [15] provides the sufficient  
 276 and necessary conditions for the existence of a feedback regulator with time-  
 277 invariant feedback gains. The following example illustrates that, although the  
 278 conditions of Lemma 1 from [15] are not satisfied, we can still find a feedback  
 279 regulator with a time-dependent feedback gain. Consider the following  $2 \times 2$   
 280 system: for all  $(t, x)$  in  $(0, +\infty) \times (0, 1)$ ,

$$281 \quad (4.6a) \quad \partial_t w_1(t, x) - \partial_x w_1(t, x) = 0, \quad \partial_t w_2(t, x) + \partial_x w_2(t, x) = 0,$$

$$282 \quad (4.6b) \quad w_2(t, 0) = w_1(t, 0), \quad w_1(t, 1) = u(t), \quad w(0, x) = w^0(x),$$

$$283 \quad (4.6c) \quad y(t) = w_2(t, 1) - w_2(t, 1/2),$$

284 and consider the constant reference signal  $r(t) = v(t) \equiv v^*$  for some  $v^*$  in  
 285  $\mathbb{R} \setminus \{0\}$ . Direct calculation shows that the numerator  $N(s)$  of the transfer  
 286 function of (4.6) from  $u$  to  $y$  is  $N(s) = e^{-s} - e^{-s/2}$ . The conditions of  
 287 Lemma 1 from [15] are not satisfied since  $N(0) = 0$  and 0 is the eigenvalue  
 288 of signal model. However, conditions (4.3) and (4.4) are satisfied, which  
 289 implies that there exists time-dependent feedback regulator  $u$ . Indeed, by  
 290 the characteristic method, we have that for  $t \geq 2$ ,  $y(t) = u(t-2) - u(t-3/2)$ .  
 291 Then  $u(t) = -2tv^*$  solves the finite-time output regulation problem. Roughly  
 292 speaking, the advantage of Theorem 4.2 lies in the fact that the required  
 293 conditions (4.3) and (4.4) are independent of the signal model.

- 294 3. In (4.5),  $s_m^{\text{out}}$  and  $s_{m+1}^{\text{out}}$  defined in Section 3 are the exit time of the charac-  
 295 teristics for the interval  $[0, 1]$ . The settling time  $T_{\text{unif}}(\Lambda)$  has been introduced  
 296 in [10]. The main result of [10] is used in the proof of Theorem 4.2 (see The-  
 297 orem 4.4 below). Notice that the settling time  $T_{\text{unif}}(\Lambda)$  is only related to the  
 298 propagation speed  $\Lambda$  of the system. Here is an example of  $2 \times 2$  system to  
 299 compute the settling time: for all  $(t, x)$  in  $(0, +\infty) \times (0, 1)$ ,

$$300 \quad \partial_t w_1(t, x) - (1 + e^{-t})\partial_x w_1(t, x) = 0,$$

$$301 \quad \partial_t w_2(t, x) + (1 + 0.5 \sin(2\pi t))\partial_x w_2(t, x) = 0.$$

302 Direct calculation shows  $\chi_1(s; t, x) = -s + e^{-s} + t - e^{-t} + x$  and  $\chi_2(s; t, x) = s -$   
 303  $\cos(2\pi s)/(4\pi) - t + \cos(2\pi t)/(4\pi) + x$ . It is clear that  $s_2^{\text{out}}(t, 0) = t + 1$ . Denote  
 304  $h(t) = s_1^{\text{out}}(t, 1) - t$ . We have  $h(t)$  solves  $\Phi(h(t), t) = 0$ , where  $\Phi(h, t) =$   
 305  $1 - h + e^{-h-t} - e^{-t}$ . Taking the derivative of the relation  $\Phi(h(t), t) = 0$  and  
 306 using the fact that  $0.5 \leq h(t) \leq 1$ , we have that  $h'(t) \geq 0$ . Thus,  $h$  is a  
 307 bounded non-decreasing function and, consequently  $\lim_{t \rightarrow +\infty} h(t)$  exists and  
 308 is equal to  $\sup_{t \geq 0} h(t)$ . Letting  $t \rightarrow +\infty$  in the relation  $\Phi(h(t), t) = 0$ , we  
 309 obtain that  $\lim_{t \rightarrow +\infty} h(t) = 1$ . Therefore,  $T_{\text{unif}}(\Lambda) = 2$ .

- 310 4. Condition (4.3) is expected for the output regulation to be achieved. Here,  
 311 we provide an example to illustrate that when condition (4.3) is not satisfied,  
 312 the output regulation problem may have no solution. Let us consider the  
 313 following system: for all  $(t, x)$  in  $(0, +\infty) \times (0, 1)$ ,

$$314 \quad \partial_t w(t, x) + \Lambda \partial_x w(t, x) = 0,$$

$$\begin{aligned} w_+(t, 0) &= Qw_-(t, 0), \quad w_-(t, 1) = u(t), \\ w(0, x) &= w^0(x), \quad y(t) = w_+(t, 1), \end{aligned}$$

where  $\Lambda = \text{diag}(-2, -1, 1, 2)$ ,  $w_- = (w_1, w_2)^\top$ ,  $w_+ = (w_3, w_4)^\top$  and  $Q = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ . Clearly, condition (4.3) is not satisfied. By the characteristic method, for  $t > 2$ , the explicit representation of the output is

$$y(t) = \begin{pmatrix} w_3(t, 1) \\ w_4(t, 1) \end{pmatrix} = \begin{pmatrix} u_1(t - 3/2) + 2u_2(t - 2) \\ 2u_1(t - 1) + 4u_2(t - 3/2) \end{pmatrix}.$$

We observe that for  $t > 2$ ,  $w_4(t - 1/2, 1) = 2w_3(t, 1)$ . Then, for the constant signal  $r(t) = (1, 0)^\top$ , the finite-time output regulation can not be achieved. The null-controllability results in [11] do not require any assumptions about the structure of  $Q$ . This fact, to some extent, reflects the differences between the null-controllability and the output regulation.

5. Condition (4.4) is mainly technical. This assumption is needed because  $f_{l+}(t)f_{l+}(t)^\top > 0$  is necessary for the matrix  $\mathcal{P}(t)$  to be positive definite (see in particular (4.28) below). However, this condition is not necessary for the output of some systems to achieve output regulation. Indeed, consider system (4.6a) and (4.6b) again, but with a different output  $y(t) = w_2(t, 1/2)$ . By the characteristic method, we have that for  $t > 3/2$ ,  $y(t) = w_2(t, 1/2) = u(t - 3/2)$ . Then for any given reference signal  $r$ , the control  $u(t) = r(t + 3/2)$  enables the finite-time output regulation to be achieved.

Before providing the proof of Theorem 4.2, let us explain its difficulty and how we overcome these difficulties. In Theorem 4.1, we consider the broad solution to the system (4.2b)–(4.2e), which has only weak regularity  $\mathcal{B}(t_0)^n = C^0([t_0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(t_0, T)^n)$ , and we only consider the system (4.2b)–(4.2e) over finite time domain  $(t_0, T)$  rather than infinite time domain  $(t_0, \infty)$ . The reason for these restrictions lies in the time-varying nature of the system, which introduces new challenges in the output regulation problem. In detail, due to the time-varying setting, the regulator equations (see (4.9) below) are PDEs rather than ODEs as in [15]. In [15], the hyperbolic system to be considered is time independent, and thus, the solvability condition for the regulator equations can be characterized as the relationship between the signal model and the transfer behavior of the system. Furthermore, the solution to the regulator equations is independent of time, allowing for the design of regulator over infinite time domain  $(0, \infty)$  and the consideration of system over infinite time domain  $(0, \infty)$  as well. However, when considering time-varying system and accounting for boundary, pointwise as well as distributed outputs (2.4), directly finding solution to the regulator equations becomes challenging. To overcome new difficulties, we examine the dual system of the regulator equations, thereby transforming the issue of solvability of regulator equations over any finite time domain  $(0, T)$  into proving an observability-like inequality (as given in (4.22) below) regarding the solution to the dual system. Then we use Lyapunov-like functions (defined in (4.23) and (4.24) below) to prove observability-like inequality. Through this method, we can only find gain function  $k_v$  in  $L^2(0, T)^{m \times n_v}$ , rather than in more regular function spaces such as  $C^0[0, T]^{m \times n_v}$ . Furthermore, we cannot extend the gain function  $k_v$  to the infinite time domain  $(0, \infty)$ . Due to the regularity of gain function  $k_v$ , we can only consider system (4.2b)–(4.2e) over finite time domain  $(t_0, T)$  and consider the broad solution  $w$  in  $\mathcal{B}(t_0)^n$  to the system (4.2b)–(4.2e).

360 In Theorem 4.2, we provide sufficient conditions (4.3) and (4.4) for the solvability  
 361 of the regulator equations. In detail, (4.3) implies that the boundary coupling coeffi-  
 362 cient matrix  $Q$  is uniformly row full rank, and (4.4) implies that the to-be-controlled  
 363 output should include  $f_{l_+}(t)w_+(t, 1)$  and  $f_{l_+}$  is uniformly row full rank. These two  
 364 conditions ensure that any  $q$ -dimensional reference signal can be tracked. We use  
 365 these conditions in proving the observability-like inequality. These conditions are not  
 366 required for the output regulation problem in time-independent hyperbolic system, as  
 367 mentioned in [15, 16, 18]. They arise from time-varying settings and the machinery  
 368 of the proof.

369 Besides, time-varying setting also brings an advantage to the output regulation  
 370 problem. Since the regulator equations are time-dependent, its solvability no longer  
 371 relies on the relationship between the signal model and the transfer behavior of the  
 372 system. Output regulation problem can be achieved for any signal model (2.5).

373 We prove Theorem 4.2 in two subsections. In Subsection 4.1, we remove the  
 374 dependency of  $v$  in (4.2b)–(4.2d) and (4.2f) and provide the feedback gain  $k_w$  by  
 375 using the result in [10]. In Subsection 4.2, we prove that the regulator equation  
 376 admits a solution under the condition (4.4) and therefore, provide the feedback gain  
 377  $k_v$ .

378 **4.1. Removal of the dependency of  $v$ .** Let  $t_0$  be in  $[0, T - T_{\text{unif}}(\Lambda))$ . Inspired  
 379 by [15], we introduce a bounded invertible change of coordinates to eliminate the  
 380 dependency of  $v$  in (4.2b)–(4.2d) and (4.2f),

$$381 \quad (4.7) \quad z(t, x) = w(t, x) - \Pi(t, x)v(t),$$

382 with  $\Pi = [\Pi_{ij}] : \mathcal{D}(0) \rightarrow \mathbb{R}^{n \times n_v}$ . Then (4.2) takes the form: for  $(t, x)$  in  $\mathcal{D}(t_0)$ ,

$$383 \quad (4.8a) \quad \dot{v}(t) = S(t)v(t), \quad v(t_0) = v^0,$$

$$384 \quad (4.8b) \quad \partial_t z(t, x) + \Lambda(t, x)\partial_x z(t, x) = A(t, x)z(t, x),$$

$$385 \quad (4.8c) \quad z_+(t, 0) = Q(t)z_-(t, 0),$$

$$386 \quad (4.8d) \quad z_-(t, 1) = \int_0^1 k_w(t, x)z(t, x)dx,$$

$$387 \quad (4.8e) \quad z(t_0, x) = w^0(x) - \Pi(t_0, x)v^0,$$

$$388 \quad (4.8f) \quad e_y(t) = \mathcal{C}_t[z(t, \cdot)],$$

389 if  $\Pi$  is the solution to the regulator equations, for  $(t, x)$  in  $\mathcal{D}(0)$ ,

$$390 \quad (4.9a) \quad \partial_t \Pi(t, x) + \Lambda(t, x)\partial_x \Pi(t, x) = A(t, x)\Pi(t, x) - \Pi(t, x)S(t) + \tilde{g}_1(t, x),$$

$$391 \quad (4.9b) \quad \Pi_+(t, 0) = Q(t)\Pi_-(t, 0) + \tilde{g}_2(t),$$

$$392 \quad (4.9c) \quad \mathcal{C}_t[\Pi(t, \cdot)] = (p_r(t) - \tilde{g}_4(t)),$$

393 and

$$394 \quad (4.10) \quad k_v(t) = \Pi_-(t, 1) - \tilde{g}_3(t) - \int_0^1 k_w(t, x)\Pi(t, x)dx,$$

395 where  $\Pi(t, x) = (\Pi_-^\top, \Pi_+^\top)^\top(t, x)$  with  $\Pi_-(t, x)$  in  $\mathbb{R}^{m \times n_v}$  and  $\Pi_+(t, x)$  in  $\mathbb{R}^{p \times n_v}$ . The  
 396 finite-time stability of  $z$ -subsystem (4.8b)–(4.8e) follows from the following theorem,  
 397 which is the main result of [10].

398 THEOREM 4.4. *Under Assumptions 2.1 to 2.3, there exists a gain function  $k_w$  in*  
 399  *$L^\infty(\mathcal{D}(0))^{m \times n}$  such that for any  $w^0$  in  $L^2(0, 1)^n$ ,  $\Pi(t_0, \cdot)$  in  $L^2(0, 1)^{n \times n_v}$  and  $v^0$  in*  
 400  *$\mathbb{R}^{n_v}$ , the system (4.8b) and (4.8c) with feedback law (4.8d) is finite-time stable with*  
 401 *settling time  $T_{\text{unif}}(\Lambda)$  defined by (4.5).*

402 *Remark 4.5.* In [10],  $k_w$  can be defined for infinite time interval  $(0, \infty)$ . Thus,  $k_w$   
 403 does not depend on  $T$ .

404 If there exists a solution  $\Pi$  in  $\mathcal{B}(0)^{n \times n_v}$  to the regulator equations (4.9), we can  
 405 define the feedback gain function  $k_v$  in  $L^2(0, T)^{m \times n_v}$  by (4.10), and Theorem 4.2 is  
 406 deduced from Theorem 4.4. The remaining thing is to find a solution to the regulator  
 407 equations (4.9). This is the goal of the next subsection.

408 **4.2. Regulator equations.** In this section, we prove that under the assump-  
 409 tions of Theorem 4.2, the regulator equations (4.9) admit a solution. Postmultiply  
 410 (4.9) by  $\Psi(t, 0)$ , the transition matrix of  $S$ , and denote  $\hat{\Pi}(t, x) = \Pi(t, x)\Psi(t, 0)$ . This  
 411 yields the following equations, for  $(t, x)$  in  $\mathcal{D}(0)$ ,

$$412 \quad (4.11a) \quad \partial_t \hat{\Pi}(t, x) + \Lambda(t, x) \partial_x \hat{\Pi}(t, x) = A(t, x) \hat{\Pi}(t, x) + \hat{g}_1(t, x),$$

$$413 \quad (4.11b) \quad \hat{\Pi}_+(t, 0) = Q(t) \hat{\Pi}_-(t, 0) + \hat{g}_2(t),$$

$$414 \quad (4.11c) \quad \mathcal{C}_t[\hat{\Pi}(t, \cdot)] = \hat{g}_4(t),$$

415 where  $\hat{g}_1(t, x) = \tilde{g}_1(t, x)\Psi(t, 0)$ ,  $\hat{g}_2(t) = \tilde{g}_2(t)\Psi(t, 0)$  and  $\hat{g}_4(t) = (p_r(t) - \tilde{g}_4(t))\Psi(t, 0)$ .

416 *Remark 4.6.* The solvability of (4.11) does not depend neither on the signal ma-  
 417 trix  $S$  nor on the initial condition  $v^0$ . This property is essentially different from the  
 418 time independent case, where an ODE depending on  $S$  needs to be solved for  $\Pi$  (see  
 419 [15]).

420 The next lemma reduces the solvability of regulator equations (4.11) to the solv-  
 421 ability of a homogeneous equation.

422 LEMMA 4.7. *The regulator equations (4.11) have a solution  $\hat{\Pi}$  in  $\mathcal{B}(0)^{n \times n_v}$  if for*  
 423 *any  $F$  in  $L^2(0, T)^q$ , the homogeneous equations, for  $(t, x)$  in  $\mathcal{D}(0)$ ,*

$$424 \quad (4.12a) \quad \partial_t \phi(t, x) + \Lambda(t, x) \partial_x \phi(t, x) = A(t, x) \phi(t, x),$$

$$425 \quad (4.12b) \quad \phi_+(t, 0) = Q(t) \phi_-(t, 0),$$

$$426 \quad (4.12c) \quad \mathcal{C}_t[\phi(t, \cdot)] = F(t).$$

427 *admit a solution  $\phi$  in  $\mathcal{B}(0)^n$ .*

428 *Proof.* For  $i = 1, \dots, n_v$ , denote by  $\Pi^i(t, x)$  in  $\mathbb{R}^n$  the broad solution to the  
 429 following equations, for  $(t, x)$  in  $\mathcal{D}(0)$ ,

$$430 \quad (4.13a) \quad \partial_t \Pi^i(t, x) + \Lambda(t, x) \partial_x \Pi^i(t, x) = A(t, x) \Pi^i(t, x) + \hat{g}_1^i(t, x),$$

$$431 \quad (4.13b) \quad \Pi_+^i(t, 0) = Q(t) \Pi_-^i(t, 0) + \hat{g}_2^i(t),$$

$$432 \quad (4.13c) \quad \Pi_-^i(t, 1) = 0,$$

$$433 \quad (4.13d) \quad \Pi^i(0, x) = 0,$$

434 where  $\hat{g}_1^i$  and  $\hat{g}_2^i$  are the  $i$ th columns of  $\hat{g}_1$  and  $\hat{g}_2$  respectively. Due to the well-  
 435 posedness results (see Theorem A.3), there exists a unique broad solution  $\Pi^i$  in  $\mathcal{B}(0)^n$   
 436 to the system (4.13). For  $i = 1, \dots, n_v$ , denote by  $\phi^i$  the solution to (4.12) with  
 437  $F(t) = \hat{g}_4^i(t) - \mathcal{C}_t[\Pi^i(t, \cdot)]$ , where  $\hat{g}_4^i$  is the  $i$ th column of  $\hat{g}_4$ . Thus,  $\hat{\Pi} = (\Pi^1 + \phi^1, \Pi^2 +$   
 438  $\phi^2, \dots, \Pi^{n_v} + \phi^{n_v})$  is the solution to (4.11).  $\square$

439 Now we prove that under the assumptions of Theorem 4.2, the homogeneous  
 440 equations (4.12) admit a solution. By the well-posedness results (see Theorem A.3),  
 441 (4.12a) and (4.12b) together with the initial and boundary conditions

$$442 \quad (4.14) \quad \phi_-(t, 1) = u^0(t), \quad \phi(0, x) = \phi^0(x),$$

443 have a unique broad solution  $\phi$  in  $\mathcal{B}(0)^n$ , where  $u^0$  belongs to  $L^2(0, T)^m$  and  $\phi^0$   
 444 belongs to  $L^2(0, 1)^n$ . Then define the map  $\mathcal{F}_T$  as following

$$445 \quad (4.15) \quad \begin{aligned} \mathcal{F}_T : L^2(0, 1)^n \times L^2(0, T)^m &\rightarrow L^2(0, T)^q \\ (\phi^0, u^0) &\mapsto (t \mapsto \mathcal{C}_t[\phi(t, \cdot)]), \end{aligned}$$

446 where  $\phi$  in  $\mathcal{B}(0)^n$  is the broad solution to (4.14), (4.12a), and (4.12b). It follows that  
 447  $\mathcal{F}_T$  is a linear continuous map from  $L^2(0, 1)^n \times L^2(0, T)^m$  into  $L^2(0, T)^q$ .

448 We get that the homogeneous regulator equations (4.12) have a solution if the  
 449 map  $\mathcal{F}_T$  is onto. In order to decide whether  $\mathcal{F}_T$  is onto or not, we use the following  
 450 classical result of functional analysis (see Theorem 4.13 of [24, p. 100]).

451 **PROPOSITION 4.8.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces. Let  $\mathcal{F}$  be a linear con-*  
 452 *tinuous map from  $H_1$  into  $H_2$ . Then  $\mathcal{F}$  is onto if and only if there exists  $c > 0$  such*  
 453 *that*

$$454 \quad (4.16) \quad \|\mathcal{F}^*(\rho)\|_{H_1} \geq c\|\rho\|_{H_2}, \quad \forall \rho \in H_2,$$

455 where  $\mathcal{F}^*$  is the adjoint operator of  $\mathcal{F}$ .

456 In order to apply this proposition, we make explicit  $\mathcal{F}_T^*$  in the following lemma.

457 **LEMMA 4.9.** *Let  $\omega$  in  $L^2(0, T)^q$ . Let  $\theta$  in  $\mathcal{B}_l(0)^n$  be the unique broad solution to*  
 458 *the following equations (see Theorem A.3 for the well-posedness), for  $(t, x)$  in  $\mathcal{D}_l(0)$ ,*

$$(4.17a)$$

$$459 \quad \partial_t \theta(t, x) + \partial_x (\Lambda(t, x) \theta(t, x)) = -A(t, x)^\top \theta(t, x) - c(t, x)^\top \omega(t),$$

$$(4.17b)$$

$$460 \quad \theta_-(t, x_i^+) = \theta_-(t, x_i^-) - \Lambda_{--}(t, x_i)^{-1} f_{i-}(t)^\top \omega(t), \quad i = 1, \dots, l-1,$$

$$(4.17c)$$

$$461 \quad \theta_-(t, 0) = -\Lambda_{--}(t, 0)^{-1} [Q(t)^\top \Lambda_{++}(t, 0) \theta_+(t, 0) + (f_{0+}(t)Q(t) + f_{0-}(t))^\top \omega(t)],$$

$$(4.17d)$$

$$462 \quad \theta_+(t, x_i^-) = \theta_+(t, x_i^+) + \Lambda_{++}(t, x_i)^{-1} f_{i+}(t)^\top \omega(t), \quad i = 1, \dots, l-1,$$

$$(4.17e)$$

$$463 \quad \theta_+(t, 1) = \Lambda_{++}(t, 1)^{-1} f_{l+}(t)^\top \omega(t),$$

$$(4.17f)$$

$$464 \quad \theta(T, x) = 0.$$

465 Then

$$466 \quad (4.18) \quad \mathcal{F}_T^*(\omega) = (\theta(0, \cdot), f_{l-}^\top \omega - \Lambda_{--}(\cdot, 1) \theta_-(\cdot, 1)).$$

467 *Proof.* Let us first assume that  $(\Lambda, A, Q, c, \omega)$  is in  $C^2(\overline{\mathcal{D}(0)})^{n \times n} \times C^1(\overline{\mathcal{D}(0)})^{n \times n} \times$   
 468  $C^1[0, T]^{p \times m} \times C_U^1(\mathcal{D}_l(0))^{q \times n} \times C^1[0, T]^q$ ,  $f_i$  is in  $C^1[0, T]^{q \times n}$ ,  $i = 0, 1, \dots, l$ , and the  
 469 compatibility conditions

$$470 \quad (4.19) \quad \omega(T) = 0, \quad \omega'(T) = 0$$

471 hold. Let  $\phi^0$  in  $C^1[0, 1]^n$  and  $u^0$  in  $C^1[0, T]^m$  be such that

$$\begin{aligned}
472 \quad (4.20) \quad & \phi_+^0(0) = Q(0)\phi_-^0(0), \quad \phi_-^0(1) = u^0(0), \\
473 \quad & (u^0)'(0) = -\Lambda_{--}(0, 1)(\phi_-^0)'(1) + A_{-+}(0, 1)\phi_+^0(1) + A_{--}(0, 1)\phi_-^0(1), \\
474 \quad & \quad -\Lambda_{++}(0, 0)(\phi_+^0)'(0) + A_{++}(0, 0)\phi_+^0(0) + A_{+-}(0, 0)\phi_-^0(0) \\
475 \quad & = Q(t)[- \Lambda_{--}(0, 0)(\phi_-^0)'(0) + A_{-+}(0, 0)\phi_+^0(0) + A_{--}(0, 0)\phi_-^0(0)] + Q'(t)\phi_-^0(0).
\end{aligned}$$

476 Let  $\phi$  in  $C^1(\overline{\mathcal{D}(0)})^n$  be the  $C^1$  solution to (4.14), (4.12a), and (4.12b) (see The-  
477 orem B.2). Considering the boundary condition (4.12b) and the definition of the  
478 output in (2.4), we have that

$$\begin{aligned}
479 \quad (4.21) \quad & \mathcal{F}_T(\phi^0, u^0)(t) = \mathcal{C}_t[\phi(t, \cdot)] = \sum_{i=0}^l f_i(t)\phi(t, x_i) + \int_0^1 c(t, x)\phi(t, x)dx \\
480 \quad & = (f_{0+}(t)Q(t) + f_{0-}(t))\phi_-(t, 0) + \sum_{i=1}^l f_i(t)\phi(t, x_i) + \int_0^1 c(t, x)\phi(t, x)dx.
\end{aligned}$$

481 Let  $\theta$  in  $C_U^1(\mathcal{D}_l(0))^n$  be the  $C^1$  solution to (4.17) (see Theorem B.2). Then from  
482 (4.14), (4.17), (4.12a), and (4.12b), we obtain that, using integrations by parts,

$$\begin{aligned}
483 \quad 0 &= \sum_{i=1}^l \int_0^T \int_{x_{i-1}}^{x_i} \theta(t, x)^\top [\partial_t \phi(t, x) + \Lambda(t, x)\partial_x \phi(t, x) - A(t, x)\phi(t, x)] dx dt \\
484 \quad &= \sum_{i=1}^l \left\{ - \int_0^T \int_{x_{i-1}}^{x_i} [\partial_t \theta(t, x) + \partial_x (\Lambda(t, x)\theta(t, x)) + A(t, x)^\top \theta(t, x)]^\top \phi(t, x) dx dt \right. \\
485 \quad &\quad + \int_{x_{i-1}}^{x_i} [\theta(T, x)^\top \phi(T, x) - \theta(0, x)^\top \phi(0, x)] dx \\
486 \quad &\quad \left. + \int_0^T [\theta(t, x_i^-)^\top \Lambda(t, x_i)\phi(t, x_i) - \theta(t, x_{i-1}^+)^\top \Lambda(t, x_{i-1})\phi(t, x_{i-1})] dt \right\} \\
487 \quad &= \int_0^T \int_0^1 \omega(t)^\top c(t, x)\phi(t, x) dx dt - \int_0^1 \theta(0, x)^\top \phi^0(x) dx \\
488 \quad &\quad + \int_0^T \omega(t)^\top \left[ (f_{0+}(t)Q(t) + f_{0-}(t))\phi_-(t, 0) + \sum_{i=1}^l f_i(t)\phi(t, x_i) \right] dt \\
489 \quad &\quad - \int_0^T (f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1))^\top u^0(t) dt.
\end{aligned}$$

490 Consequently, it follows from (4.21) that

$$\begin{aligned}
491 \quad & \int_0^T \omega(t)^\top \mathcal{F}_T(\phi^0, u^0)(t) dt = \int_0^T \omega(t) \mathcal{C}_t[\phi(t, \cdot)] dt \\
492 \quad & = \int_0^1 \theta(0, x)^\top \phi^0(x) dx + \int_0^T (f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1))^\top u^0(t) dt,
\end{aligned}$$

493 which, together with Claim B.3 concludes the proof of Lemma 4.9.  $\square$

494 Recall  $\varepsilon_0$ ,  $M_0$ ,  $M_1$ ,  $M_Q$ ,  $M_f$ ,  $M_c$ ,  $\varepsilon_Q$  and  $\varepsilon_f$  defined in Assumptions 2.2 to 2.4  
495 and (4.3) and (4.4). In next lemma, we prove that under the assumptions of Theo-  
496 rem 4.2, the inequality (4.16) holds with respect to operator (4.18).

497 LEMMA 4.10. *Let the assumptions of Theorem 4.2 hold. Let  $\omega$  belong to  $L^2(0, T)^q$*   
 498 *and  $\theta$  in  $\mathcal{B}_l(0)^n$  be the broad solution to (4.17). Then there exists a constant  $c_T > 0$*   
 499 *such that*

$$(4.22) \quad \int_0^1 \|\theta(0, x)\|^2 dx + \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1)\|^2 dt \geq c_T \int_0^T \|\omega(t)\|^2 dt.$$

501 *Proof.* Let us first assume that  $(\Lambda, A, Q, c, \omega)$  is in  $C^2(\overline{\mathcal{D}(0)})^{n \times n} \times C^1(\overline{\mathcal{D}(0)})^{n \times n} \times$   
 502  $C^1[0, T]^{p \times m} \times C_U^1(\mathcal{D}_l(0))^{q \times n} \times C^1[0, T]^q$ ,  $f_i$  is in  $C^1[0, T]^{q \times n}$ ,  $i = 0, 1, \dots, l$ , and the  
 503 compatibility conditions (4.19) hold. Let  $\theta$  in  $C_U^1(\mathcal{D}_l(0))^n$  be the  $C^1$  solution to (4.17).  
 504 For  $i = 1, \dots, l$  and  $0 \leq t \leq T$ , let

$$(4.23) \quad V_{i+}(t) = e^{-Lt} \int_{x_{i-1}}^{x_i} e^{\alpha_i(x-x_{i-1})} \|\theta_+(t, x)\|^2 dx,$$

$$(4.24) \quad V_{i-}(t) = e^{-Lt} \int_{x_{i-1}}^{x_i} e^{\beta_i(x_i-x)} \|\theta_-(t, x)\|^2 dx,$$

507 with positive coefficients  $L$ ,  $\alpha_i$  and  $\beta_i$  to be chosen later. Denote  $V(t) = \sum_{i=1}^l (V_{i+}(t) +$   
 508  $V_{i-}(t))$ . The proof of (4.22) is based on identity  $V(0) = -\int_0^T \frac{dV}{dt}(t) dt$ , and the  
 509 main idea is as follows. First,  $V(0)$  is equivalent to  $\int_0^1 \|\theta(0, x)\|^2 dx$ . Next, we use  
 510 integration by parts to express term  $\beta_0 \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1)\|^2 dt -$   
 511  $\int_0^T \frac{dV}{dt}(t) dt$  as a quadratic form. Finally, by applying conditions (4.3) and (4.4) and  
 512 selecting appropriate constants  $L$ ,  $\alpha_i$  and  $\beta_i$ , we ensure that this quadratic form is  
 513 greater than or equal to  $c_T \int_0^T \|\omega(t)\|^2 dt$ . The weights of the Lyapunov-like functions  
 514 (4.23) and (4.24) are similar to those used in [10, 11, 13]. In [10, 11] the weights are  
 515 crucial to establish the well-posedness of the broad solutions.

516 Let us proceed with the proof. The time derivative of  $V_{i+}(t)$  along the  $C^1$  solution  
 517  $\theta$  to (4.17) is

$$\begin{aligned} 518 \quad \frac{dV_{i+}(t)}{dt} &= e^{-Lt} \int_{x_{i-1}}^{x_i} e^{\alpha_i(x-x_{i-1})} \theta_+(t, x)^\top [2\partial_t \theta_+(t, x) - L\theta_+(t, x)] dx \\ 519 &= e^{-Lt} \int_{x_{i-1}}^{x_i} e^{\alpha_i(x-x_{i-1})} \theta_+(t, x)^\top [-L\theta_+(t, x) - 2\partial_x(\Lambda_{++}(t, x)\theta_+(t, x)) \\ 520 &\quad - 2A_{++}(t, x)^\top \theta_+(t, x) - 2A_{+-}(t, x)^\top \theta_-(t, x) - 2c_+(t, x)^\top \omega(t)] dx \\ 521 &= e^{-Lt} \left\{ \int_{x_{i-1}}^{x_i} e^{\alpha_i(x-x_{i-1})} \theta_+(t, x)^\top [-(L\text{Id}_p - \alpha_i \Lambda_{++}(t, x) + \partial_x \Lambda_{++}(t, x) \right. \\ 522 &\quad \left. + 2A_{++}(t, x)^\top) \theta_+(t, x) - 2A_{+-}(t, x)^\top \theta_-(t, x) - 2c_+(t, x)^\top \omega(t)] dx \right. \\ 523 &\quad \left. - e^{\alpha_i(x_i-x_{i-1})} \theta_+(t, x_i^-)^\top \Lambda_{++}(t, x_i) \theta_+(t, x_i^-) \right. \\ 524 &\quad \left. + \theta_+(t, x_{i-1}^+)^\top \Lambda_{++}(t, x_{i-1}) \theta_+(t, x_{i-1}^+) \right\}. \end{aligned}$$

525 Similarly, the time derivative of  $V_{i2}(t)$  along the  $C^1$  solution  $\theta$  to (4.17) is

$$\begin{aligned} 526 \quad \frac{dV_{i-}(t)}{dt} &= e^{-Lt} \left\{ \int_{x_{i-1}}^{x_i} e^{\beta_i(x_i-x)} \theta_-(t, x)^\top [-(L\text{Id}_m + \beta_i \Lambda_{--}(t, x) + \partial_x \Lambda_{--}(t, x) \right. \\ 527 &\quad \left. + 2A_{--}(t, x)^\top) \theta_-(t, x) - 2A_{+-}(t, x)^\top \theta_+(t, x) - 2c_-(t, x)^\top \omega(t)] dx \right. \end{aligned}$$

$$\begin{aligned}
528 & -\theta_-(t, x_i^-)^\top \Lambda_{--}(t, x_i) \theta_-(t, x_i^-) \\
529 & + e^{\beta_i(x_i - x_{i-1})} \theta_-(t, x_{i-1}^+)^\top \Lambda_{--}(t, x_{i-1}) \theta_-(t, x_{i-1}^+) \Big\}.
\end{aligned}$$

530 Taking boundary and jump conditions (4.17b)–(4.17e) into account, we conclude that  
(4.25)

$$\begin{aligned}
531 & e^{Lt} \frac{dV(t)}{dt} = \sum_{i=1}^l \int_{x_{i-1}}^{x_i} [-2(e^{\alpha_i(x-x_{i-1})} c_+(t, x) \theta_+(t, x) + e^{\beta_i(x_i-x)} c_-(t, x) \theta_-(t, x))^\top \omega(t) \\
532 & - e^{\alpha_i(x-x_{i-1})} \theta_+(t, x)^\top (LId_p - \alpha_i \Lambda_{++}(t, x) + \partial_x \Lambda_{++}(t, x) + 2A_{++}(t, x)^\top) \theta_+(t, x) \\
533 & - e^{\beta_i(x_i-x)} \theta_-(t, x)^\top (LId_m + \beta_i \Lambda_{--}(t, x) + \partial_x \Lambda_{--}(t, x) + 2A_{--}(t, x)^\top) \theta_-(t, x) \\
534 & - 2\theta_+(t, x)^\top (e^{\alpha_i(x-x_{i-1})} A_{-+}(t, x)^\top + e^{\beta_i(x_i-x)} A_{+-}(t, x)) \theta_-(t, x)] dx \\
535 & - \sum_{i=1}^{l-1} \{ (e^{\alpha_i(x_i-x_{i-1})} - 1) \theta_+(t, x_i^+)^\top \Lambda_{++}(t, x_i) \theta_+(t, x_i^+) \\
536 & + e^{\alpha_i(x_i-x_{i-1})} [2\theta_+(t, x_i^+)^\top f_{i+}(t)^\top \omega(t) + \omega(t)^\top f_{i+}(t) \Lambda_{++}(t, x_i)^{-1} f_{i+}(t)^\top \omega(t)] \\
537 & - (e^{\beta_i(x_{i+1}-x_i)} - 1) \theta_-(t, x_i^-)^\top \Lambda_{--}(t, x_i) \theta_-(t, x_i^-) \\
538 & + e^{\beta_i(x_{i+1}-x_i)} [2\theta_-(t, x_i^-)^\top f_{i-}(t)^\top \omega(t) - \omega(t)^\top f_{i-}(t) \Lambda_{--}(t, x_i)^{-1} f_{i-}(t)^\top \omega(t)] \Big\} \\
539 & + \theta_+(t, 0)^\top \Lambda_{++}(t, 0) (e^{\beta_1 x_1} Q(t) \Lambda_{--}(t, 0)^{-1} Q(t)^\top + \Lambda_{++}(t, 0)^{-1}) \Lambda_{++}(t, 0) \theta_+(t, 0) \\
540 & + 2e^{\beta_1 x_1} \theta_+(t, 0)^\top \Lambda_{++}(t, 0) Q(t) \Lambda_{--}(t, 0)^{-1} (f_{0+}(t) Q(t) + f_{0-}(t))^\top \omega(t) \\
541 & + e^{\beta_1 x_1} \omega(t)^\top (f_{0+}(t) Q(t) + f_{0-}(t)) \Lambda_{--}(t, 0)^{-1} (f_{0+}(t) Q(t) + f_{0-}(t))^\top \omega(t) \\
542 & - e^{\alpha_l(1-x_{l-1})} \omega(t)^\top f_{l+}(t) \Lambda_{++}(t, 1)^{-1} f_{l+}(t)^\top \omega(t) - \theta_-(t, 1)^\top \Lambda_{--}(t, 1) \theta_-(t, 1).
\end{aligned}$$

543 Multiply (4.25) by  $-e^{-Lt}$  and integrate over  $(0, T)$ , and add

$$544 \quad \beta_0 \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1) \theta_-(t, 1)\|^2 dt$$

545 to both sides of (4.25) with positive coefficient  $\beta_0$  to be chosen later. Recall  $x_0 = 0$   
546 and  $x_l = 1$ . It follows from (4.17f) that

$$547 \quad (4.26) \quad V(0) + \beta_0 \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1) \theta_-(t, 1)\|^2 dt = R_1 + R_2,$$

548 where

$$549 \quad (4.27) \quad R_1 =$$

$$\begin{aligned}
550 & \int_0^T e^{-Lt} \sum_{i=1}^l \int_{x_{i-1}}^{x_i} [2(e^{\alpha_i(x-x_{i-1})} c_+(t, x) \theta_+(t, x) + e^{\beta_i(x_i-x)} c_-(t, x) \theta_-(t, x))^\top \omega(t) \\
551 & + e^{\alpha_i(x-x_{i-1})} \theta_+(t, x)^\top (LId_p - \alpha_i \Lambda_{++}(t, x) + \partial_x \Lambda_{++}(t, x) + 2A_{++}(t, x)^\top) \theta_+(t, x) \\
552 & + e^{\beta_i(x_i-x)} \theta_-(t, x)^\top (LId_m + \beta_i \Lambda_{--}(t, x) + \partial_x \Lambda_{--}(t, x) + 2A_{--}(t, x)^\top) \theta_-(t, x) \\
553 & + 2\theta_+(t, x)^\top (e^{\alpha_i(x-x_{i-1})} A_{-+}(t, x)^\top + e^{\beta_i(x_i-x)} A_{+-}(t, x)) \theta_-(t, x)] dx dt,
\end{aligned}$$

554 and

$$555 \quad (4.28) \quad R_2 = \int_0^T e^{-Lt} \Theta(t)^\top \mathcal{P}(t) \Theta(t) dt,$$



556 with

$$\begin{aligned}
557 \quad \Theta(t) &= \begin{pmatrix} \Theta_-(t) \\ \Theta_+(t) \\ \omega(t) \end{pmatrix}, \quad \mathcal{P}(t) = \begin{pmatrix} \mathcal{P}_-(t) & 0 & \mathcal{F}_-(t) \\ 0 & \mathcal{P}_+(t) & \mathcal{F}_+(t) \\ \mathcal{F}_-(t)^\top & \mathcal{F}_+(t)^\top & P_\omega(t) \end{pmatrix}, \\
558 \quad \Theta_-(t) &= (\theta_-(t, x_1^-)^\top, \dots, \theta_-(t, x_l^-)^\top)^\top, \quad \Theta_+(t) = (\theta_+(t, x_0^+)^\top, \dots, \theta_+(t, x_{l-1}^+)^\top)^\top, \\
559 \quad \mathcal{P}_-(t) &= \text{diag}(P_{1-}(t), \dots, P_{l-}(t)), \quad \mathcal{P}_+(t) = \text{diag}(P_{0+}(t), \dots, P_{(l-1)+}(t)), \\
560 \quad \mathcal{F}_-(t) &= (F_{1-}(t)^\top, \dots, F_{l-}(t)^\top)^\top, \quad \mathcal{F}_+(t) = (F_{0+}(t)^\top, \dots, F_{(l-1)+}(t)^\top)^\top, \\
561 \quad P_{i+}(t) &= (e^{\alpha_i(x_i - x_{i-1})} - 1)\Lambda_{++}(t, x_i), \quad P_{i-}(t) = -(e^{\beta_{i+1}(x_{i+1} - x_i)} - 1)\Lambda_{--}(t, x_i), \\
562 \quad F_{i+}(t) &= e^{\alpha_i(x_i - x_{i-1})} f_{i+}(t)^\top, \quad F_{i-}(t) = e^{\beta_{i+1}(x_{i+1} - x_i)} f_{i-}(t)^\top, \quad i = 1, \dots, l-1, \\
563 \quad P_{0+}(t) &= -\Lambda_{++}(t, 0)(e^{\beta_1 x_1} Q(t)\Lambda_{--}(t, 0)^{-1}Q(t)^\top + \Lambda_{++}(t, 0)^{-1})\Lambda_{++}(t, 0), \\
564 \quad F_{0+}(t) &= -e^{\beta_1 x_1} \Lambda_{++}(t, 0)Q(t)\Lambda_{--}(t, 0)^{-1}(f_{0+}(t)Q(t) + f_{0-}(t))^\top, \\
565 \quad P_{l-}(t) &= \beta_0 \Lambda_{--}(t, 1)^2 + \Lambda_{--}(t, 1), \quad F_{l-}(t) = -\beta_0 \Lambda_{--}(t, 1)f_{l-}(t)^\top, \\
566 \quad P_\omega(t) &= \beta_0 f_{l-}(t)f_{l-}(t)^\top + \sum_{i=1}^{l-1} \left[ e^{\alpha_i(x_i - x_{i-1})} f_{i+}(t)\Lambda_{++}(t, x_i)^{-1}f_{i+}(t)^\top \right. \\
567 \quad &\quad \left. - e^{\beta_{i+1}(x_{i+1} - x_i)} f_{i-}(t)\Lambda_{--}(t, x_i)^{-1}f_{i-}(t)^\top \right] + e^{\alpha_l(1 - x_{l-1})} f_{l+}(t)\Lambda_{++}(t, 1)^{-1}f_{l+}(t)^\top \\
568 \quad &\quad - e^{\beta_1 x_1} (f_{0+}(t)Q(t) + f_{0-}(t))\Lambda_{--}(t, 0)^{-1}(f_{0+}(t)Q(t) + f_{0-}(t))^\top.
\end{aligned}$$

569 Considering the left-hand side of (4.26), we have

$$\begin{aligned}
570 \quad (4.29) \quad V(0) + \beta_0 \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1)\|^2 dt \\
571 \quad \leq \tilde{M} \left( \int_0^1 \|\theta(0, x)\|^2 dx + \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t, 1)\theta_-(t, 1)\|^2 dt \right),
\end{aligned}$$

572 where

$$573 \quad (4.30) \quad \tilde{M} = \max\{\beta_0, \max_{i=1, \dots, l} \{e^{\alpha_i}, e^{\beta_i}\}\}.$$

574 Now we deal with the right-hand side of (4.26). Our aim is to choose suitable constants  
575  $L, \beta_0, \alpha_i, \beta_i, i = 1, \dots, l$ , such that  $R_1 + R_2 \geq \int_0^T \varepsilon e^{-Lt} \|\omega(t)\|^2 dt$  for some positive  
576 constant  $\varepsilon$ . Let us first deal with  $R_2$ . For any  $\varepsilon^* > 0$ , let  $\beta_0, \alpha_i, \beta_i, i = 1, \dots, l$ , large  
577 enough such that

$$\begin{aligned}
578 \quad \beta_0 \varepsilon_0 > 1, \quad e^{\beta_1 x_1} \varepsilon_0 \varepsilon_Q > M_0, \quad \alpha_i > 0, \quad \beta_{i+1} > 0, \quad i = 1, \dots, l-1, \\
579 \quad e^{\alpha_l(1 - x_{l-1})} \frac{\varepsilon_f}{M_0} \geq \varepsilon^* + \frac{M_f^2}{\varepsilon_0} \sum_{i=0}^{l-1} \left( \frac{e^{\alpha_i(x_i - x_{i-1})}}{e^{\alpha_i(x_i - x_{i-1})} - 1} + \frac{e^{\beta_{i+1}(x_{i+1} - x_i)}}{e^{\beta_{i+1}(x_{i+1} - x_i)} - 1} \right) \\
580 \quad + \frac{\beta_0 M_f^2}{\beta_0 \varepsilon_0 - 1} + 2e^{\beta_1 x_1} M_f^2 (1 + mpM_Q^2) \frac{e^{\beta_1 x_1} mpM_0 M_Q^2 \varepsilon_0^{-1} - e^{\beta_1 x_1} \varepsilon_Q + M_0 \varepsilon_0^{-1}}{e^{\beta_1 x_1} \varepsilon_0 \varepsilon_Q - M_0},
\end{aligned}$$

581 where  $\varepsilon_0$  is defined in Assumption 2.2,  $\varepsilon_Q$  is defined in (4.3),  $\varepsilon_f$  is defined in (4.4),  
582 and  $M_0$  and  $M_Q$  are defined in Assumption 2.3. Direct calculation shows that for all

583  $t$  in  $[0, T]$ ,

$$P_{i+}(t) > 0, \quad i = 0, \dots, l-1, \quad P_{i-}(t) > 0, \quad i = 1, \dots, l,$$

584 (4.31) 
$$P_\omega(t) \geq \varepsilon^* \text{Id}_q + \sum_{i=0}^{l-1} F_{i+}(t)^\top P_{i+}(t)^{-1} F_{i+}(t) + \sum_{i=1}^l F_{i-}(t)^\top P_{i-}(t)^{-1} F_{i-}(t).$$

585 Note that (4.3) and (4.4) are necessary for  $P_{0+}$  and  $P_\omega$  to be positive definite respec-  
586 tively. It follows from (4.31) and the Schur complement lemma (see [7, Appendix 5.5])  
587 that for all  $t$  in  $[0, T]$ ,  $\Theta(t)^\top \mathcal{P}(t) \Theta(t) \geq \varepsilon^* \|\omega(t)\|^2$ , and thus  $R_2 \geq \int_0^T \varepsilon^* e^{-Lt} \|\omega(t)\|^2 dt$ .

588 Now let us estimate  $R_1$ . For  $L$  large enough, we have

589 
$$R_1 \geq \int_0^T e^{-Lt} \sum_{i=1}^l \int_{x_{i-1}}^{x_i} [e^{\alpha_i(x-x_{i-1})} (L - (\alpha_i + 1)M_0 - 2pM_1) \|\theta_+(t, x)\|^2$$
  
590 
$$+ e^{\beta_i(x_i-x)} (L - (\beta_i + 1)M_0 - 2mM_1) \|\theta_-(t, x)\|^2$$
  
591 
$$- (e^{\alpha_i(x-x_{i-1})} + e^{\beta_i(x_i-x)}) M_1 (m \|\theta_+(t, x)\|^2 + p \|\theta_-(t, x)\|^2)$$
  
592 
$$+ 2(e^{\alpha_i(x-x_{i-1})} c_+(t, x) \theta_+(t, x) + e^{\beta_i(x_i-x)} c_-(t, x) \theta_-(t, x))^\top \omega(t)] dx dt$$
  
593 
$$\geq \int_0^T e^{-Lt} \sum_{i=1}^l \int_{x_{i-1}}^{x_i} [(L - (\tilde{M} + 1)M_0 \tilde{M} - 2nM_1 \tilde{M}) (\|\theta_+(t, x)\|^2 + \|\theta_-(t, x)\|^2)$$
  
594 
$$+ 2(e^{\alpha_i(x-x_{i-1})} c_+(t, x) \theta_+(t, x) + e^{\beta_i(x_i-x)} c_-(t, x) \theta_-(t, x))^\top \omega(t)] dx dt.$$

595 Notice that

596 
$$\left\| \int_{x_{i-1}}^{x_i} c_\pm(t, x) \theta_\pm(t, x) dx \right\|^2 \leq qnM_c^2 \int_{x_{i-1}}^{x_i} \|\theta_\pm(t, x)\|^2 dx.$$

597 It follows that

598 
$$R_1 \geq \int_0^T e^{-Lt} \sum_{i=1}^l \left[ \frac{L - (\tilde{M} + 1)M_0 \tilde{M} - 2nM_1 \tilde{M}}{qnM_c^2} \left\| \int_{x_{i-1}}^{x_i} c_+(t, x) \theta_+(t, x) dx \right\|^2 \right.$$
  
599 
$$+ \frac{L - (\tilde{M} + 1)M_0 \tilde{M} - 2nM_1 \tilde{M}}{qnM_c^2} \left\| \int_{x_{i-1}}^{x_i} c_-(t, x) \theta_-(t, x) dx \right\|^2$$
  
600 
$$\left. - 2\tilde{M} \|\omega(t)\| \left\| \int_{x_{i-1}}^{x_i} c_+(t, x) \theta_+(t, x) dx \right\| - 2\tilde{M} \|\omega(t)\| \left\| \int_{x_{i-1}}^{x_i} c_-(t, x) \theta_-(t, x) dx \right\| \right] dt$$

601 provided that  $L > (\tilde{M} + 1)M_0 \tilde{M} + 2nM_1 \tilde{M}$ . Therefore, we conclude that

602 
$$R_1 + R_2 \geq \int_0^T e^{-Lt} \left\{ \frac{\varepsilon^*}{2} \|\omega(t)\|^2 \right.$$
  
603 
$$+ \sum_{i=1}^l \left[ \frac{L - (\tilde{M} + 1)M_0 \tilde{M} - 2nM_1 \tilde{M}}{qnM_c^2} \left\| \int_{x_{i-1}}^{x_i} c_+(t, x) \theta_+(t, x) dx \right\|^2 \right.$$
  
604 
$$\left. - 2\tilde{M} \|\omega(t)\| \left\| \int_{x_{i-1}}^{x_i} c_+(t, x) \theta_+(t, x) dx \right\| + \frac{\varepsilon^*}{4l} \|\omega(t)\|^2 \right.$$

$$\begin{aligned}
& + \frac{L - (\tilde{M} + 1)M_0\tilde{M} - 2nM_1\tilde{M}}{qnM_c^2} \left\| \int_{x_{i-1}}^{x_i} c_-(t, x)\theta_-(t, x)dx \right\|^2 \\
& - 2\tilde{M}\|\omega(t)\| \left\| \int_{x_{i-1}}^{x_i} c_-(t, x)\theta_-(t, x)dx \right\| + \frac{\varepsilon^*}{4l}\|\omega(t)\|^2 \Big] \Big\} dt
\end{aligned}$$

provided that  $L > (\tilde{M} + 1)M_0\tilde{M} + 2nM_1\tilde{M}$ . Then we choose that

$$(4.32) \quad L \geq \frac{4lqnM_c^2\tilde{M}^2}{\varepsilon^*} + (\tilde{M} + 1)M_0\tilde{M} + 2nM_1\tilde{M}.$$

Consequently, we obtain that

$$(4.33) \quad R_1 + R_2 \geq \int_0^T \frac{\varepsilon^*}{2} e^{-Lt} \|\omega(t)\|^2 dt \geq \frac{\varepsilon^*}{2} e^{-LT} \int_0^T \|\omega(t)\|^2 dt.$$

Together with Claim B.3, this concludes the proof of Lemma 4.10 with  $c_T = \frac{\varepsilon^*}{2M} e^{-LT}$ .  $\square$

**Appendix A. Broad solutions.** We consider the following hyperbolic system, which includes all the systems of this paper. For  $(t, x)$  in  $\mathcal{D}_l(t_0)$ ,

$$(A.1a) \quad \partial_t w(t, x) + \Lambda(t, x)\partial_x w(t, x) = A(t, x)w(t, x) + J(t, x),$$

$$(A.1b) \quad w_+(t, x_i^+) = w_+(t, x_i^-) + \sigma^{i+}(t), \quad i = 1, \dots, l-1,$$

$$(A.1c) \quad w_+(t, 0) = Q(t)w_-(t, 0) + \sigma^{0+}(t),$$

$$(A.1d) \quad w_-(t, x_i^-) = w_-(t, x_i^+) + \sigma^{i-}(t), \quad i = 1, \dots, l-1,$$

$$(A.1e) \quad w_-(t, 1) = \int_0^1 L(t, \xi)w(t, \xi)d\xi + \sigma^{l-}(t),$$

$$(A.1f) \quad w(t_0, x) = w^0(x),$$

where  $w(t, x)$  in  $\mathbb{R}^n$  is the state, and  $w^0$  in  $L^2(0, 1)^n$  is the initial data. Functions  $J$  in  $L^2(\mathcal{D}(0))^n$ ,  $\sigma^- := (\sigma^{1-}, \dots, \sigma^{l-})$  in  $L^2(0, T)^{m \times l}$  and  $\sigma^+ := (\sigma^{0+}, \dots, \sigma^{(l-1)+})$  in  $L^2(0, T)^{p \times l}$  are the non-homogeneous terms. For the coefficients involved in system (A.1), let us make the following assumptions.

*Assumption A.1.* Assume that  $\Lambda$ ,  $A$  and  $Q$  satisfy Assumptions 2.1 to 2.3 and that  $L$  is in  $L^\infty(\mathcal{D}(0))^{m \times n}$  satisfying  $\|L\|_{L^\infty(\mathcal{D}(0))^{m \times n}} \leq M_1$  for  $M_1$  defined in Assumption 2.3.

Notice that  $L$  is defined over time interval  $(0, T)$ . The reason lies in the regularity of the feedback gain functions in this paper.

**A.1. Definition of broad solution.** Let us now introduce the definition of broad solution or so-called solution along the characteristics. This definition is similar to the definition of broad solution in [10]. Recalling the notations in Section 3, we introduce  $\bar{s}_j^{\text{in}}(t_0; t, x) = \max\{t_0, s_j^{\text{in}}(t, x)\}$  for  $j = 1, \dots, n$ , and

$$i(x) = i, \quad \text{if } x \in (x_{i-1}, x_i), \quad i = 1, \dots, l.$$

Similar to the methods used in [10], integrating the  $j$ th equation in (A.1a) along the characteristic  $\chi_j(s; t, x)$  and applying appropriate boundary, jump, or initial conditions, we obtain the following system of integral equation. For  $(t, x)$  in  $\mathcal{D}_l(t_0)$ ,

$$(A.2) \quad w_j(t, x) = I_j(w)(t, x) + \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t \sum_{k=1}^n a_{jk}(s, \chi_j(s; t, x)) w_k(s, \chi_j(s; t, x)) ds$$

$$+ \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t J_j(s, \chi_j(s; t, x)) ds,$$

where for  $j = 1, \dots, m$ ,

$$(A.3) \quad I_j(w)(t, x) = \begin{cases} \int_0^1 L_{j,:}(s_j^{\text{in}}(t, x), \xi) w(s_j^{\text{in}}(t, x), \xi) d\xi + \sum_{k=i(x)}^l \sigma_j^{k-}(s_j^{\text{in},k}(t, x)), & \text{if } s_j^{\text{in}}(t, x) > t_0, \\ w_j^0(\chi_j(t_0; t, x)) + \sum_{k=i(x)}^{i(\chi_j(t_0; t, x))-1} \sigma_{j-m}^{k-}(s_j^{\text{in},k}(t, x)), & \text{if } s_j^{\text{in}}(t, x) < t_0, \end{cases}$$

and for  $j = m+1, \dots, n$ ,

$$(A.4) \quad I_j(w)(t, x) = \begin{cases} Q_{j-m,:}(s_j^{\text{in}}(t, x)) w_-(s_j^{\text{in}}(t, x), 0) + \sum_{k=1}^{i(x)} \sigma_{j-m}^{(k-1)+}(s_j^{\text{in},k}(t, x)), & \text{if } s_j^{\text{in}}(t, x) > t_0, \\ w_j^0(\chi_j(t_0; t, x)) + \sum_{k=1+i(\chi_j(t_0; t, x))}^{i(x)} \sigma_{j-m}^{(k-1)+}(s_j^{\text{in},k}(t, x)), & \text{if } s_j^{\text{in}}(t, x) < t_0. \end{cases}$$

This leads to the following definition of the broad solution to the system (A.1) over  $(t, x)$  in  $\mathcal{D}_l(t_0)$ .

**DEFINITION A.2.** Let  $T > 0$ ,  $0 \leq t_0 < T$ ,  $w^0$  in  $L^2(0, 1)^n$ ,  $J$  in  $L^2(\mathcal{D}(0))^n$ ,  $\sigma^-$  in  $L^2(0, T)^{m \times l}$  and  $\sigma^+$  in  $L^2(0, T)^{p \times l}$  be fixed. We say that  $w$  is the broad solution to the system (A.1) over  $\mathcal{D}_l(t_0)$  if  $w$  is in  $\mathcal{B}_l(t_0)^n$  and if the integral equation (A.2) is satisfied for  $j = 1, \dots, n$ , for a.e.  $t_0 < t < T$  and a.e.  $x$  in  $(0, 1)$ .

**A.2. Well-posedness.** In this section, the well-posedness result is provided.

**THEOREM A.3.** Let  $T > 0$ . Under Assumption A.1, for every  $0 \leq t_0 < T$ ,  $w^0$  in  $L^2(0, 1)^n$ ,  $J$  in  $L^2(\mathcal{D}(0))^n$ ,  $\sigma^-$  in  $L^2(0, T)^{m \times l}$  and  $\sigma^+$  in  $L^2(0, T)^{p \times l}$ , there exists a unique broad solution  $w$  in  $\mathcal{B}_l(t_0)^n$  to (A.1) over  $\mathcal{D}_l(t_0)$ . Moreover, there exists  $C = C(T) > 0$  such that, for every  $0 \leq t_0 < T$ ,  $w^0$  in  $L^2(0, 1)^n$ ,  $J$  in  $L^2(\mathcal{D}(0))^n$ ,  $\sigma^-$  in  $L^2(0, T)^{m \times l}$  and  $\sigma^+$  in  $L^2(0, T)^{p \times l}$ , the broad solution  $w$  satisfies

$$(A.5) \quad \|w\|_{L^\infty((t_0, T); L^2(0, 1)^n)} + \|w\|_{L^\infty((0, 1); L^2(t_0, T)^n)} \leq C(\|w^0\|_{L^2(0, 1)^n} + \|J\|_{L^2(\mathcal{D}(0))^n} + \|\sigma^-\|_{L^2(0, T)^{m \times l}} + \|\sigma^+\|_{L^2(0, T)^{p \times l}}).$$

The proof is based on the proof of Theorem A.2 of [10]. We provide only a sketch of the proof here, highlighting the differences from the proof of Theorem A.2 presented in [10].

*Sketch of the proof of Theorem A.3.* The basic idea is the following fixed point method. A function  $w : \mathcal{D}(t_0) \rightarrow \mathbb{R}^n$  satisfies the integral equations (A.2) for a.e.  $t_0 < t < T$  and a.e.  $x$  in  $(0, 1)$  if and only if it is a fixed point of the map  $\mathcal{A} :$

665  $\mathcal{B}_l(t_0)^n \rightarrow \mathcal{B}_l(t_0)^n$  and  $(\mathcal{A}(w))_j(t, x)$  is given by the expression on the right-hand side  
 666 of (A.2). Let us now make  $\mathcal{B}_l(t_0)^n$  a Banach space by equipping it with the weighted  
 667 norm  $\|w\|_{\mathcal{B}_l(t_0)^n} = \|w\|_{\mathcal{B}_1} + \|w\|_{\mathcal{B}_2}$ , where

$$668 \quad \|w\|_{\mathcal{B}_1} = \max_{t \in [t_0, T]} e^{-\frac{L_1}{2}(t-t_0)} \sqrt{\int_0^1 \sum_{j=1}^n |w_j(t, x)|^2 e^{-L_2 x} dx},$$

$$669 \quad \|w\|_{\mathcal{B}_2} = \max_{x \in [0, 1]} e^{\frac{L_2}{2}(1-x)} \sqrt{\int_{t_0}^T \sum_{j=1}^n |w_j(t, x)|^2 e^{-L_1(t-t_0)} dt},$$

670 where  $L_1, L_2 > 0$  are constants independent of  $T, t_0, w_0, \sigma$  and  $J$  that will be fixed  
 671 below. The similar weight norms are also used in [11, 13]. Our goal is to show that,  
 672 for  $L_1, L_2 > 0$  large enough,

$$673 \quad (\text{A.6}) \quad \|\mathcal{A}(w^1) - \mathcal{A}(w^2)\|_{\mathcal{B}_l(t_0)^n} \leq \frac{1}{2} \|w^1 - w^2\|_{\mathcal{B}_l(t_0)^n}, \quad \forall w^1, w^2 \in \mathcal{B}_l(t_0)^n.$$

674 Actually, the proof of (A.6) is the same as in [10]. Indeed, we introduce  $w := w^1 - w^2$ ,  
 675 so that  $\mathcal{A}(w^1) - \mathcal{A}(w^2)$  is equal to the right-hand side of (A.2) with  $w^0 = 0, J = 0$ ,  
 676  $\sigma^+ = 0$  and  $\sigma^- = 0$ . This is a special case in [10]. Therefore, (A.6) is established by  
 677 following the proof in [10]. The remaining task is to verify that the estimate (A.5)  
 678 holds. Indeed, using (A.6) we obtain that the fixed point  $w$  of  $\mathcal{A}$  satisfies

$$679 \quad (\text{A.7}) \quad \frac{1}{2} \|w\|_{\mathcal{B}_l(t_0)^n} \leq \|\mathcal{A}(0)\|_{\mathcal{B}_l(t_0)^n},$$

680 and the straightforward computations show that

$$681 \quad (\text{A.8}) \quad \|w\|_{L^\infty((t_0, T); L^2(0, 1)^n)}^2 \leq e^{L_2} e^{L_1 h} \|w\|_{\mathcal{B}_1}^2, \quad \|w\|_{L^\infty((0, 1); L^2(t_0, T)^n)}^2 \leq e^{L_1 h} \|w\|_{\mathcal{B}_2}^2.$$

682 Then, the fixed point of  $\mathcal{A}$  satisfies the estimate (A.5) if the right-hand side of (A.5)  
 683 is the upper bound of  $\|\mathcal{A}(0)\|_{\mathcal{B}_l(t_0)^2}$ . By using changes of coordinate, (3.3) and (3.8),  
 684 we obtain the following estimates

$$685 \quad (\text{A.9}) \quad \int_0^1 \sum_{j=1}^n \left| \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t J_j(s, \chi_j(s; t, x)) ds \right|^2 e^{-L_2 x} dx \leq \frac{e^{M_0 T}}{\varepsilon_0} \|J\|_{L^2(\mathcal{D}(0))^n}^2,$$

$$686 \quad (\text{A.10}) \quad \int_{t_0}^T \sum_{j=1}^n \left| \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t J_j(s, \chi_j(s; t, x)) ds \right|^2 e^{-L_1(t-t_0)} dt \leq \frac{e^{M_0 T}}{\varepsilon_0^2} \|J\|_{L^2(\mathcal{D}(0))^n}^2,$$

$$687 \quad (\text{A.11}) \quad \int_0^1 \sum_{j=1}^n |I_j(0)(t, x)|^2 e^{-L_2 x} dx$$

$$688 \quad \leq 2e^{M_0 T} (M_0 \|\sigma^-\|_{L^2(0, T)^{m \times l}}^2 + M_0 \|\sigma^+\|_{L^2(0, T)^{p \times l}}^2 + \|w^0\|_{L^2(0, 1)^n}^2),$$

$$689 \quad (\text{A.12}) \quad \int_{t_0}^T \sum_{j=1}^n |I_j(0)(t, x)|^2 e^{-L_1(t-t_0)} dt$$

$$690 \quad \leq 2 \frac{e^{M_0 T}}{\varepsilon_0} (M_0 \|\sigma^-\|_{L^2(0, T)^{m \times l}}^2 + M_0 \|\sigma^+\|_{L^2(0, T)^{p \times l}}^2 + \|w^0\|_{L^2(0, 1)^n}^2),$$

691 where  $\varepsilon_0$  is defined in Assumption 2.2. It follows from (A.9) and (A.11) that

$$(A.13) \quad \|\mathcal{A}(0)\|_{\mathcal{B}_1}^2 \leq 2e^{M_0 T} \left( 2\|w^0\|_{L^2(0,1)^n}^2 + \frac{1}{\varepsilon_0} \|J\|_{L^2(\mathcal{D}(0))^n}^2 \right. \\ \left. + 2M_0 \|\sigma^-\|_{L^2(0,T)^{m \times l}}^2 + 2M_0 \|\sigma^+\|_{L^2(0,T)^{p \times l}}^2 \right).$$

Similarly, from (A.10) and (A.12) we obtain that

$$(A.14) \quad \|\mathcal{A}(0)\|_{\mathcal{B}_2}^2 \leq 2 \frac{e^{M_0 T + L_2}}{\varepsilon_0} \left( 2\|w^0\|_{L^2(0,1)^n}^2 + \frac{1}{\varepsilon_0} \|J\|_{L^2(\mathcal{D}(0))^n}^2 \right. \\ \left. + 2M_0 \|\sigma^-\|_{L^2(0,T)^{m \times l}}^2 + 2M_0 \|\sigma^+\|_{L^2(0,T)^{p \times l}}^2 \right).$$

Then, the estimate (A.5) for the fixed point of  $\mathcal{A}$  follows from (A.7), (A.8), (A.13), and (A.14).  $\square$

**Appendix B.  $C^1$  solutions.** In this section, we show that the broad solution is also  $C^1$  solution if the data of the system are smooth enough. Moreover, the continuous dependence of the broad solutions on the system data is given. In the proof of Lemmas 4.9 and 4.10,  $C^1$  solution is needed. Let us make the following assumptions for the coefficients involved in system (A.1).

*Assumption B.1.* Assume that  $\Lambda$ ,  $A$ ,  $Q$  and  $L$  satisfy Assumption A.1, and that  $\Lambda$ ,  $A$ ,  $Q$  and  $L$  are in  $C^2(\overline{\mathcal{D}(0)})^{n \times n}$ ,  $C^1(\overline{\mathcal{D}(0)})^{n \times n}$ ,  $C^1([0, T])^{p \times m}$  and  $C^1(\overline{\mathcal{D}(0)})^{m \times n}$ , respectively.

The  $C^1$  solution is given by the following theorem.

**THEOREM B.2.** *Let  $T > 0$ . Under Assumption B.1, for every  $0 \leq t_0 < T$ ,  $w^0$  in  $C_U^1(\cup_{i=1}^l (x_{i-1}, x_i))^n$ ,  $J$  in  $C_U^1(\mathcal{D}_l(0))^n$ ,  $\sigma^-$  in  $C^1([0, T])^{m \times l}$  and  $\sigma^+$  in  $C^1([0, T])^{p \times l}$  satisfying the compatibility conditions*

$$(B.1) \quad \sigma^{i\pm}(t_0) = w_{\pm}^0(x_i^{\pm}) - w_{\pm}^0(x_i^{\mp}), \quad i = 1, \dots, l-1, \\ \sigma^{0+}(t_0) = w_+^0(0) - Q(t_0)w_-^0(0), \quad \sigma^{l-}(t_0) = w_-^0(1) - \int_0^1 L(t_0, \xi)w^0(\xi)d\xi, \\ (\sigma^{i\pm})'(t_0) = -\Lambda_{\pm\pm}(t_0, x_i)((w_{\pm}^0)'(x_i^{\pm}) - (w_{\pm}^0)'(x_i^{\mp})) + A_{\pm\pm}(t_0, x_i)\sigma^{i\pm}(t_0) \\ - A_{\pm\mp}(t_0, x_i)\sigma^{i\mp}(t_0) + J_{\pm}(t_0, x_i^{\pm}) - J_{\pm}(t_0, x_i^{\mp}), \quad i = 1, \dots, l-1, \\ (\sigma^{0+})'(t_0) = J_+(t_0, 0) - Q(t_0)J_-(t_0, 0) - \Lambda_{++}(t_0, 0)(w_+^0)'(0) \\ + Q(t_0)\Lambda_{--}(t_0, 0)(w_-^0)'(0) + (A_{++}(t_0, 0) - Q(t_0)A_{-+}(t_0, 0))w_+^0(0) \\ + (A_{+-}(t_0, 0) - Q'(t_0) - Q(t_0)A_{--}(t_0, 0))w_-^0(0), \\ (\sigma^{l-})'(t_0) = -\Lambda_{--}(t_0, 1)(w_-^0)'(1) + A_{-+}(t_0, 1)w_+^0(1) + A_{--}(t_0, 1)w_-^0(1) + J_-(t_0, 1) \\ - \int_0^1 [L(t_0, \xi)(-\Lambda(t_0, \xi)(w^0)'(\xi) + A(t_0, \xi)w^0(\xi) + J(t_0, \xi)) + \partial_t L(t_0, \xi)w^0(\xi)]d\xi,$$

there exists a unique solution  $w$  in  $C_U^1(\mathcal{D}_l(t_0))^n$  to (A.1).

The proof follows the method in [11, Lemma 3.2] and [12, Lemma 2.1]. Here, we only provide a sketch of the proof, explaining how we apply the method from [11, Lemma 3.2] and [12, Lemma 2.1].

*Sketch of the proof of Theorem B.2.* Set, for  $u$  in  $C_U^0(\mathcal{D}_l(t_0))^n$ ,

$$\|u\|_0 := \max_{1 \leq i \leq n} \max_{(t,x) \in \mathcal{D}_i(t_0)} |e^{-L_1 t - L_2 x} u_i(t, x)|,$$

726 and for  $u$  in  $C_U^1(\mathcal{D}_l(t_0))^n$ ,

$$727 \quad \|u\|_1 := \max\{\|u\|_0, \|\partial_t u\|_0, \|\partial_x u\|_0\},$$

728 where  $L_1$  and  $L_2$  are two large, positive constants determined later. Set

$$729 \quad \mathcal{O} := \{v \in C_U^1(\mathcal{D}_l(t_0))^n \mid v(t_0, \cdot) = w^0, \partial_t v(t_0, \cdot) = -\Lambda(t_0, \cdot)(w^0)' + A(t_0, \cdot)w^0 + J(t_0, \cdot)\}.$$

730 For  $v$  in  $\mathcal{O}$ , let  $w = \mathcal{A}_1(v)$  be defined as follows: for  $j = 1, \dots, m$ ,

$$731 \quad (\text{B.2}) \quad w_j(t, x) = I_j(v)(t, x) + \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t \sum_{k=1}^n a_{jk}(s, \chi_j(s; t, x)) v_k(s, \chi_j(s; t, x)) ds$$

$$732 \quad \quad \quad + \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t J_j(s, \chi_j(s; t, x)) ds,$$

733 where  $I_j(v)(t, x)$  is defined in (A.3), and for  $j = m+1, \dots, n$ ,

$$734 \quad (\text{B.3}) \quad w_j(t, x) = I_j(w)(t, x) + \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t \sum_{k=1}^n a_{jk}(s, \chi_j(s; t, x)) v_k(s, \chi_j(s; t, x)) ds$$

$$735 \quad \quad \quad + \int_{\bar{s}_j^{\text{in}}(t_0; t, x)}^t J_j(s, \chi_j(s; t, x)) ds,$$

736 where  $I_j(w)(t, x)$  is defined in (A.4). Notice that for  $j = m+1, \dots, n$ ,  $I_j(w)(t, x)$  is  
 737 only involved with  $w_-$ , which is defined by (B.2). It follows from Assumption B.1 and  
 738 the compatibility conditions (B.1) that  $\mathcal{A}_1(\mathcal{O}) \subset \mathcal{O}$ . Direct calculation shows that  
 739 the fixed point of  $\mathcal{A}_1$  is the  $C^1$  solution to (A.1). Our aim is to show that, for  $L_1$  and  
 740  $L_2$  large enough,

$$741 \quad (\text{B.4}) \quad \|\mathcal{A}_1(v^1) - \mathcal{A}_1(v^2)\|_1 \leq \frac{1}{2} \|v^1 - v^2\|_1, \quad \forall v^1, v^2 \in \mathcal{O}.$$

742 We can directly use the method from [11, Lemma 3.2] and [12, Lemma 2.1] to prove  
 743 (B.4), since (A.1) is linear. Indeed, we introduce  $v := v^1 - v^2$ , so that  $w := \mathcal{A}_1(v^1) -$   
 744  $\mathcal{A}_1(v^2)$  is equal to the right-hand side of (B.2) and (B.3) with  $w^0 = 0$ ,  $J = 0$ ,  $\sigma^+ = 0$   
 745 and  $\sigma^- = 0$ . This is a special case in [12, Lemma 2.1]. Therefore, (B.4) is established  
 746 by following the proof in [12, Lemma 2.1].  $\square$

747 As for the continuous dependence of the broad solutions on the system data, one  
 748 can prove the following claim by using the same method as in [8, Theorem 3.5].

749 **CLAIM B.3.** *For  $\Lambda$ ,  $A$ ,  $Q$  and  $L$  satisfying Assumption A.1, and  $w^0$  in  $L^2(0, 1)^n$ ,  
 750  $J$  in  $L^2(\mathcal{D}(0))^n$ ,  $\sigma^-$  in  $L^2(0, T)^{m \times l}$  and  $\sigma^+$  in  $L^2(0, T)^{p \times l}$ , let  $w$  in  $\mathcal{B}_l(t_0)^n$  be the  
 751 broad solution to the system (A.1) over  $\mathcal{D}_l(t_0)$ . For  $k \geq 1$ ,  $\Lambda^k$ ,  $A^k$ ,  $Q^k$  and  $L^k$  satis-  
 752 fying Assumption B.1, and  $w^{0,k}$  in  $C_U^1(\cup_{i=1}^l (x_{i-1}, x_i))^n$ ,  $J^k$  in  $C_U^1(\mathcal{D}_l(0))^n$ ,  $\sigma^{-,k}$  in  
 753  $C^1([0, T])^{m \times l}$  and  $\sigma^{+,k}$  in  $C^1([0, T])^{p \times l}$  satisfying the compatibility conditions (B.1),  
 754 let  $w^k$  in  $C_U^1(\mathcal{D}_l(t_0))^n$  be the  $C^1$  solution to the system (A.1). Assume that*

$$755 \quad (\Lambda^k, A^k, Q^k, L^k, w^{0,k}, J^k, \sigma^{-,k}, \sigma^{+,k}) \rightarrow (\Lambda, A, Q, L, w^0, J, \sigma^-, \sigma^+) \quad \text{in}$$

$$756 \quad C^1(\overline{\mathcal{D}(0)})^{n \times n} \times C^0(\overline{\mathcal{D}(0)})^{n \times n} \times C^0([0, T])^{p \times m} \times L^\infty(\mathcal{D}(0))^{m \times n}$$

$$757 \quad \times L^2(0, 1)^n \times L^2(\mathcal{D}(0))^n \times L^2(0, T)^{m \times l} \times L^2(0, T)^{p \times l} \quad \text{as } k \rightarrow \infty.$$

758 *Then, we have  $w^k \rightarrow w$  in  $\mathcal{B}_l(t_0)^n$  as  $k \rightarrow \infty$ .*

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