# What are the physical and economic laws of ride-sharing under line shape? A techno-economic model featuring Waiting policies, Traffic equilibrium and Pricing strategies

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# What are the physical and economic laws of ride-sharing under line shape? A techno-economic model featuring Waiting policies, Traffic equilibrium and Pricing strategies

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## Abstract

Car traffic, while useful to its users owing to high service quality at moderate cost, exerts negative impacts on the environment local and global, henceforth on human health. Ride-sharing services enable their participants as Users (riders) or Agents (drivers) to make better use of seats in cars and to save money costs as well as environmental impacts, at the expense of some additional times of dwelling for rider boarding and alighting, of waiting and of transaction with the service platform. Line ride-sharing (LRS) is a specific kind of service fostering service participation as User (rider) or Agent (driver), by aligning and channeling both car runs and rides, thereby increasing run frequency (factor $\varphi$) and ride load per car (factor $\omega$). This article investigates the structural properties of an LRS service on a roadway link. From the stochastic properties of two flows of arrivals, Agents and Users, stem two physical laws about the number of users per agent run (a geometric distribution) and the wait time distributions according to the service Waiting Policy (WP) either Agent Waits (AW) or User Waits (UW). Then, service conditions $(\omega, \varphi)$ determine the expected values of agent rewards and of wait times by role U and A, contributing to their utility functions as modal options faced to service-neutral option N of non-participation. From the discrete choice of mode U, A or N by link users stem two economic laws: the inner-service split condition between U and A yields a relationship between frequency $\varphi$ and cab load $\omega$, while the outer-service split condition between N and service participation S={A,U} relates the service participation rate to $\omega$. These laws determine traffic equilibrium between the three modes at the link level under exogenous price schedule (user fare and agent reward). Further on, from service management aimed to just operator's profit or to system optimum (first or second best) also considering demand surplus and environmental impacts, the optimal service prices are endogenized, yielding specific forms of the two economic laws. By combining them in a "demanded volume" of cab load, the optimal states are straightforwardly obtained for the full range of link flows, according to waiting policy and regulation regime. A numerical experiment is performed to assess and compare the diverse service specifications with respect to the link flow.

### Keywords

Ride-sharing service; bi-sided platform; externalities; traffic equilibrium; optimal pricing

### Highlights

H1/ Service featured out in terms of run frequency and average car load

H2/ Physical laws of cab occupation and wait time by service role, as random variables

H3/ Economic laws linking service frequency and participation rate to cab occupancy

H4/ Waiting policies, regulation regimes and their combinations

### JEL codes

L11; L12; R41

# 1/ Introduction

## Background

In the connected era of mobility, ride-sharing between trip-makers is a platform kind of service relating one side of Agents (drivers supplying seats in their cars) and another side of Users (as riders in the said cars). The more agents there are, the lesser the wait times for Users, and conversely. Furthermore, the more User trips per Agent car run (factor $\omega$ of cab occupancy), the higher the agent reward on the basis of typical service price schedules. These positive interactions create value to the two sides: these are within-service "externalities" (from the individual perspective of service participants). The service also exerts influences out of its sphere: out-of-service externalities include the reduction of car traffic on roadways, yielding time gains to all car users, as well as lower environmental impacts.

In the "line" form of ride-sharing, in other words for a ride-sharing service under line shape, both the agents' car runs (also called cab runs) and the users' rides take place along a specific roadway path, thus called the line or more specifically the line link. Rider access to cars may be restricted to special stop points called "stations" (cf. Ecov's "Line" kind of service) or allowed at any point along the line path (cf. Ecov's "Line+"). The line shape, well known in public transit modes of passenger transportation, allows for focusing and channeling the trips of both Agents and Users. In a time period and by flow direction, agent trips give rise to service frequency (factor $\varphi$) that determines the wait times to incoming service participants. The service rules-of-engagement include a Waiting Policy (WP): "User Waits" (UW) as in bus or train lines, or "Agent Waits" (AW). The latter policy has been favored by Ecov since it reduces the burden of service access to the riders that are the "scarce matter" of ride-sharing services.

## Research questions

The characteristics of an LRS service on a given roadway link include the waiting policy, the transaction operations and related times on both sides and the price schedule of user fares and agent rewards. These rules-of-engagements set up the stage for the two factors of cab occupancy $\omega$ and cab frequency $\varphi$ (the number of agents in a time period).

Taking an abstract view of an LRS, the article addresses the following research questions: What are the structural properties of the service associated to the line shape? More specifically, distinguishing between physical properties and economic properties:

RQ1/ What are the "physical laws" of cab occupancy and wait times,

RQ2/ Are there "economic laws" about service frequentation on both sides and price conditions?

RQ3/ How do these laws shape out the operational state of the service as a system, including its effects on agents, users and other link trip-makers, on the operator and on the environment?

RQ4/ How does the system state depend on the service rules-of-engagement, as well as on regulation conditions that determine the operator strategy of service management aimed just to operator's profits or, more comprehensively, to social welfare also including the benefits to demand and the environment?

## Article objective and contributions

The article contributes to the physical and economic theory of ride-sharing services under line shape by providing a techno-economic model in five parts: as layers from bottom up, these are [i] traffic laws, [ii] demand behaviors, [iii] traffic equilibrium, [iv] service management, [v] service regulation.

[i] Traffic laws: beside the effects on link car flow and in turn on link run times, the specific traffic laws relate cab occupation and wait times on either side U or A to the period flows of type U and A respectively, depending on the waiting policy.

[ii] Demand behaviors: considering all link trip-makers as individual decision-makers, we model the utility function of each mode or role U, A and N (non-participant) on the basis of the related times and money terms. Postulating microeconomic rationality of all individuals, we devise a three-mode logit discrete choice model. The resulting modal shares combine with the link trip flow to yield the role flows as endogenous variables.

[iii] Traffic equilibrium: traffic laws and demand behaviors make a two-fold correspondence between role flows and the $(\omega, \varphi)$ pair of service factors, in converse directions: traffic laws from role flows to service conditions, demand behaviors from service conditions to role flows. The resulting equilibrium state is a specific kind of traffic equilibrium. We demonstrate the existence and quasi-uniqueness of an equilibrium state under exogenous price schedule (with agent reward related to cab occupancy). There is a cut-off value of cab occupancy that separates UW equilibrium states in a lower range from AW ones in an upper range. The cut-off value depends on the price parameters and on the service times by kind of trip leg – Run, Stop and Transaction, but not on the waiting policy.

[iv] Service management sets up the service prices (user fare, agent reward) as action levers to optimize an objective function. We devise a generic framework to accommodate diverse kinds of objective functions and derive the corresponding optimal prices. These prices combine service-created value to marginal costs of production and to the costs of externalities either within-service or out of it. Given the marginal costs of link congestion and environmental impacts, there is a ceiling value of cab occupancy on the optimal states. It is an upper bound on AW states as well as on UW ones.

[v] The Regulation layer of the model consists in setting up the service objective: either monopoly or system optimum first- or second- best. The generic formulation of service optimization enables us to address the different regimes, in combination to the waiting policies.

The upper layers [iii-v] of the model involve the lower layers [i-ii] through two specific relations that constitute two “economic laws” of ride-sharing services under line shape. On the one hand, service participation as User or Agent attracts a share of link trip-makers that depends on cab load together with exogenous set-ups: this relation between service participation rate and cab load can also be called the Outer Service Split (OSS) condition. On the other hand, the ratio between the U and A modal shares is equal to average cab occupancy: this determines the service frequency as a function of cab occupancy. The relation between frequency and cab load can also be called the Inner Service Split (ISS) condition.

Given the waiting policy and the regulation set-ups, the service economic laws of participation rate and frequency combine into a “demanded volume” function of cab load. Equilibrium states as well as optimal states can be determined by finding the value of cab load that makes the demanded volume equal to the link trip flow (either fixed or elastic).

## Research approach

The five-layer model stems from a constructive, theoretical approach to the research questions. The “traffic laws” sub-model involves elementary properties of stochastic processes. The “demand behaviors” sub-model is a combination of mode representation as in network traffic assignment (roadway and transit) with a discrete choice model of mode choice that is typical of Travel demand modeling. The “Traffic equilibrium” sub-model is inspired from the theory of path choice and traffic equilibrium in modal networks: the linkages between link flows and service factors $(\omega, \varphi)$ are specific and yield some non-conventional consequences on equilibrium states.

The "service management" and "service regulation" layers are typical of transport economics. The generic formulation of service optimization, owing to suitable parameterization, is cast into a primal-dual problem saddle point problem, of which the detailed outcomes allow for economic interpretation as well as for comparing waiting policies and regulation regimes.

## Article structure

The rest of the article is in eight parts. Section Two provides the system representation and the physical laws. Section Three deals with Traffic equilibrium: it gives a mathematical characterization of traffic equilibrium, formal properties of existence and uniqueness and a graphical solution scheme in the $(\varphi, \omega)$ plane of physical service conditions. Section Four addresses Service optimization and delivers the optimality conditions – notably the dual variables. Section Five develops the economic interpretation of an optimal state, including the notions of value creation by the service and its share between service participants and the operator, as well as the composition of optimal prices in relation to role utilities and social costs. Section Six emphasizes the two economic laws and traces out their consequences on service performance. A numerical experiment is carried out in Section Seven to demonstrate the model outcomes depending on specific conditions of waiting policy, regulation regimes and link flow size. Section Eight offers a Discussion and Section Nine a Conclusion.

**Tab.1: Notation.**

| *Service features* | *Territorial setting and link traffic* |
|---|---|
| $\nu$ number of riders per agent car run | $L$ link length |
| $\rho$ proportion of agent runs with riders | $H$ period duration |
| $\omega$ average rider load per agent car run | $\mathrm{T}_r^{\mathrm{FP}}$ run time function of role $r \in \{\mathrm{U}, \mathrm{N}\}$ under FP |
| $\varphi$ service frequency | $X_r^{\mathrm{FP}}$ car flow rate on link of role $r \in \{\mathrm{U}, \mathrm{N}\}$ |
| $r, s \in \mathfrak{R} \equiv \{\mathrm{A}, \mathrm{U}, \mathrm{N}\}$ service role | *Mobility and behaviors* |
| $\ell \in \mathcal{L} \equiv \{\mathrm{R}, \mathrm{S}, \mathrm{T}, \mathrm{W}\}$ trip leg | $q$ population size |
| $\delta_r^{\mathrm{W}}$ indicator variable of Wait assignment to $r$ | $y_r$ person flow of role $r$ |
| $\{\mathrm{IF}, \mathrm{JF}\}$ Isolated / Joint Flow Policy | $x_r$ car trip flow of role $r$ |
| *Time and money items* | $\beta_r^\ell$ Value-of-Time (VoT) by role $r$ and leg $\ell$ |
| $t_r$ trip time of role $r$, time $t_r^\ell$ by leg $\ell$ | $g_r$ generalized cost of $r$ to trip-maker |
| $\gamma_r$ modal constant of role $r$ | $V_r$ deterministic utility of $r$ |
| $\underline{m}$ base car cost per trip | $\varepsilon_r$ random perturbation of utility of role $r$ |
| $\dot{m}$ length factor of car cost | $U_r$ stochastic utility of $r$ |
| $\underline{\tau}_r$ base trip award ($r = \mathrm{A}$) or fare ($r = \mathrm{U}$) | $\theta$ concentration parameter of $\varepsilon_r$ |
| $\dot{\tau}_r$ length factor of award ($r = \mathrm{A}$) or fare ($r = \mathrm{U}$) | $p_r$ population proportion assigned to role $r$ |
| $\hat{\tau}_r$ average trip award ($r = \mathrm{A}$) or fare ($r = \mathrm{U}$) | $V_{\mathfrak{R}}$ expected maximum utility |
| $m_r$ money cost of role $r$ | D demand function |

# 2/ System representation and service physical laws

## 2.1/ Roadway path and traffic scene

A roadway link of network length denoted $L$, is considered in a given flow direction as a specific "traffic scene". It is studied on a daily basis, by within-day time period. A specific period of interest is considered, e.g. morning peak or evening peak or in-between, with time duration denoted $H$.

## 2.2/ Service roles as travel modes

With respect to a line ride-sharing service on the roadway link, three roles are identified for car trip-makers: service user as Rider (role U), service agent as driver (role A) or non-participant i.e. neutral (role N). The set of roles is denoted $\mathfrak{R} \equiv \{\mathrm{U}, \mathrm{A}, \mathrm{N}\}$.

On a per trip basis, the respective money expenses are denoted as $m_r$, and time expenses as $t_r$. These split into "leg times" $t_r^\ell$ according to trip phases or legs, $\ell \in \mathcal{L} \equiv \{\mathrm{R}, \mathrm{S}, \mathrm{T}, \mathrm{W}\}$:

- Leg R of car running on the link,
- Leg S of Stop or Dwelling for rider boarding and alighting, for Users and for those Agents that get "customers",
- Leg T of Transaction operations: getting information and possibly being assigned, Paying on the User side or Being paid on the Agent side, both using the service digital platform,
- Leg W of Waiting: either the User is required to Wait for the next Agent under the UW policy, or the Agent is asked to wait for an incoming user under the AW policy.

## 2.3/ Service protocol and physical laws

Ride-sharing is a bi-sided form of transport service as it involves people in two different ways: not only the service-to-demand form of Users, but also the cooperator-to-service form of Agents. These cooperate with the service coordination (platform) by supplying seats in their vehicles, the driving function including dwelling for rider boarding and alighting, possibly waiting for incoming riders to get to the car and other assistance (e.g. putting a foldable two-wheeler in the car trunk).

Service coordination between the two sides, Agents and Users, relies upon a digital interaction platform that achieves user and agent matching (in other words, the assignment of riders to cars and that of cars to riders) by suitable information collection and delivery, as well as fare collection from users and money compensation to agents.

*Transaction times*. These transactional operations are assumedly performed efficiently owing to high level of automation and suitable platform customer interface, with total time per trip of $t_{\mathrm{A}}^{\mathrm{T}}$ and $t_{\mathrm{U}}^{\mathrm{T}}$ on the agent and user side respectively.

*Stop times*. Also given is the stop time per user trip, $t_{\mathrm{U}}^{\mathrm{S}}$ on the user side, giving rise to a base stop time of $\underline{t}_{\mathrm{A}}^{\mathrm{S}}$ on the agent side if the agent car run is endowed with rides. Otherwise, the agent experiences no stop time.

The following Proposition is demonstrated in Appendix A on postulating that both flows of Users and Agents are independent Poisson processes with respective time rates $\psi_r \equiv y_r/H$.

***Proposition 1, Physical laws: Car occupation and time items according to waiting policy.***

1/ Car occupation: (i) The number of users per agent car run, a random variable denoted $\nu$, has a geometric distribution with parameter $\rho \equiv \frac{\psi_{\mathrm{U}}}{\psi_{\mathrm{U+A}}}$, and mean value $\frac{\rho}{1-\rho}$ denoted $\omega$, so that $\rho = \frac{\omega}{1+\omega}$.

(ii) The respective probabilities of getting users or not are $\Pr\{\nu > 0\} = \rho$ and $\Pr\{\nu = 0\} = 1 - \rho$.

2/ Under User Waits policy (UW): (i) the wait times of agents are negligible: $t_A^W = 0$. (ii) The user wait times are distributed $\mathrm{EXP}(\psi_\mathrm{A})$, i.e., exponentially distributed with parameter $\psi_\mathrm{A}$, so that $t_U^W = H/\varphi$.

3/ Under Agent Waits policy (AW): (i) the wait times of users are negligible: $t_U^W = 0$. (ii) The wait times of agents make a random variable (RV) that is distributed as the probability mixture of 0 and $\mathrm{EXP}(\psi_\mathrm{A})$, with respective coefficients $1-\rho$ and $\rho$, yielding average agent wait time of $t_A^W = \rho H/\varphi$.

We thus have that

$$\varphi \equiv y_\mathrm{A}, \tag{1a}$$

$$\omega \equiv \mathrm{E}[\nu] = y_\mathrm{U}/y_\mathrm{A}, \tag{1b}$$

$$\rho \equiv \Pr\{\nu > 0\} = \frac{\omega}{1+\omega}. \tag{1c}$$

Letting $\delta_\mathrm{A}^\mathrm{W}$ be the binary indicator of AW and $\delta_\mathrm{U}^\mathrm{W}$ that of UW, the option wait times are formulated generically as follows:

$$t_\mathrm{N}^\mathrm{W} = 0, \tag{2-N}$$

$$t_\mathrm{A}^\mathrm{W} = \delta_\mathrm{A}^\mathrm{W} \rho H/\varphi, \tag{2-A}$$

$$t_\mathrm{U}^\mathrm{W} = \delta_\mathrm{U}^\mathrm{W} H/\varphi. \tag{2-U}$$

The average agent stop time satisfies that $t_\mathrm{A}^\mathrm{S} = \Pr\{\nu > 0\}\,.\,\underline{t}_\mathrm{A}^\mathrm{S} + \Pr\{\nu = 0\}\,.\,0$, hence

$$t_\mathrm{A}^\mathrm{S} = \rho \underline{t}_\mathrm{A}^\mathrm{S}. \tag{3}$$

Link run times are denoted $t_\mathrm{U}^\mathrm{R}$ for Users or $t_\mathrm{N}^\mathrm{R}$ for role N to enable for possible distinction, using e.g. dedicated lanes for multi-occupied cars. An agent will get the same run time as a user if there is at least one user in the car, or the same run time as role N otherwise. Thus, on average,

$$t_\mathrm{A}^\mathrm{R} = \Pr\{\nu > 0\}\,.\,t_\mathrm{U}^\mathrm{R} + \Pr\{\nu = 0\}\,.\,t_\mathrm{N}^\mathrm{R} = \rho t_\mathrm{U}^\mathrm{R} + (1-\rho) t_\mathrm{N}^\mathrm{R}. \tag{4}$$

Table 2 summarizes the average leg times by trip leg for each mode.

**Tab. 2: Average trip leg time according to role.**

| Leg $\ell$ | Agent | User | Neutral |
|---|---|---|---|
| Run R | $\rho t_\mathrm{U}^\mathrm{R} + (1-\rho) t_\mathrm{N}^\mathrm{R}$ | $t_\mathrm{U}^\mathrm{R}$ | $t_\mathrm{N}^\mathrm{R}$ |
| Stop S | $\rho \underline{t}_\mathrm{A}^\mathrm{S}$ | $t_\mathrm{U}^\mathrm{S}$ | 0 |
| Transaction T | $t_\mathrm{A}^\mathrm{T}$ | $t_\mathrm{U}^\mathrm{T}$ | 0 |
| Wait W | $\delta_\mathrm{A}^\mathrm{W} \rho H/\varphi$ | $\delta_\mathrm{U}^\mathrm{W} H/\varphi$ | 0 |

Beside the waiting policy and the right-of-way assignment, service policy also includes (i) roadway link selection in relation to the local Mobility Organizing Authority, (ii) Access conditions of Rides to Car runs, assumed here at link endpoints only, (iii) the price schedule.

By assumption, for each car trip offered by an agent as a service run, a twofold reward (fee) is awarded to the agent: a base fee denoted $\underline{\tau}_\mathrm{A}$ applies whatever the number of riders (including modality Zero), plus a per ride contribution that depends on ride length, say $L\dot{\tau}_A$ with $\dot{\tau}_A$ the ride award rate per unit length. On average per service run, the money reward to the agent is thus

$$\hat{\tau}_\mathrm{A} = \underline{\tau}_\mathrm{A} + \omega L \dot{\tau}_\mathrm{A} \tag{5-A}$$

On the user side, we similarly distinguish between a fixed fare $\underline{\tau}_\mathrm{U}$ and a length-variable fare $L\dot{\tau}_U$: the ride fare is thus

$$\hat{\tau}_\mathrm{U} = \underline{\tau}_\mathrm{U} + L\dot{\tau}_\mathrm{U} \tag{5-U}$$

There certainly is a relationship between $\hat{\tau}_\mathrm{A}$ and $\hat{\tau}_\mathrm{U}$: some financial balance between $y_\mathrm{A}\hat{\tau}_\mathrm{A}$ and $y_\mathrm{U}\hat{\tau}_\mathrm{U}$ may be expected, up to external subsidizes e.g. from the local mobility organizing authority on a per run basis, to taxes and of course to platform remuneration. Such issues, of obvious interest to the service business model, are not addressed in this article.

## 2.4/ Modal money items, generalized costs and utility functions

The ride fare constitutes the basic money cost of the link trip to a service user:

$$M_\mathrm{U} \equiv \hat{\tau}_\mathrm{U} = \tau_\mathrm{U} + \omega L \dot{\tau}_\mathrm{U} \tag{6-U}$$

Service-neutral trip-makers are assumed to spend a fixed money cost of $\underline{m}$ plus a length-variable cost at unit length rate of $\dot{m}$, yielding link cost of

$$M_\mathrm{N} \equiv \underline{m} + L\dot{m} \tag{6-N}$$

To a service agent, the link cost amounts to car expense minus run income, yielding

$$M_\mathrm{A} \equiv M_\mathrm{N} - \hat{\tau}_\mathrm{A} = \underline{m} + L\dot{m} - \tau_\mathrm{A} - \omega L \dot{\tau}_\mathrm{A} \tag{6-A}$$

Some trip features remain unobserved in the model, notably the sub-paths up- and down-stream the roadway link. The unobserved features of the alternative modes $r$ are called "modal constants" measured as money costs and denoted $\gamma_r$. We shall denote $m_r \equiv M_r + \gamma_r$.

Thus, each mode as a choice option is characterized by a set of attributes: leg times, money costs, modal constant, customized to the individual attributes and circumstances. For each individual, the own preferences (e.g. Value of Time, VoT) and conditions of exposures (walk speed, money expense depending on car type…) are called behavioral parameters. By mode $r$ and trip leg $\ell$, the VoT denoted $\beta_r^\ell$ is the change rate of leg time to money. Among the different legs, only the agent run leg can vary depending on car occupation: from no riders to one or several, not only may the run time change (from $t_N^R$ to $t_U^R$) but also the agent VoT, say from $\beta_{A0}^R$ without riders to $\beta_{A1}^R$ with riders. On average, the agent value of run time is $\beta_A^R = (1-\rho)\beta_{A0}^R + \rho\beta_{A1}^R$. Yet, as the amount of run time may vary depending on the load state, we denote loosely

$$(\beta t)_A^R = (1-\rho)\beta_{A0}^R t_N^R + \rho\beta_{A1}^R t_U^R \tag{7}$$

At the trip level, the modal option $r$ induces a "travel generalized cost" of

$$g_r^\# \equiv m_r + \sum_{\ell\in\mathcal{L}} \beta_r^\ell . t_r^\ell \tag{8}$$

As cost is a disutility, the related deterministic utility function is a value, denoted

$$V_r \equiv -g_r^\# = -m_r - \sum_{\ell\in\mathcal{L}} \beta_r^\ell . t_r^\ell \tag{9}$$

Beyond the modal constants, there are other option attributes, individual attributes and choice conditions that are unobserved: they are modeled as random variables $[\varepsilon_r : r \in \Re]$ that turn the deterministic utility function $V_r$ into a "random utility function" denoted $U_r$ and defined as

$$U_r \equiv V_r + \varepsilon_r . \tag{10}$$

## 2.5/ Mode choice and option flows

Having modeled the individual preferences in the utility functions, the microeconomic principle of individual choice behavior is that, on every occurrence, the individual selects the option of maximum utility to him or her. Then, by aggregation over the random occurrences behind the random variables $\varepsilon_r$, the probability of choosing option $r$ comes out as

$$p_r \equiv \Pr\{U_r \geq U_s : \forall s \in \Re\}. \tag{11}$$

We shall resort to the multinomial logit model, in which the "perturbation" $\varepsilon_r$ is a random variable that follows the Gumbel distribution, and such that between the options the random family

$[\varepsilon_r: r \in \mathfrak{R}]$ are independent and identically distributed (this assumption reduces to "homoscedastic" for Gumbel variables and modal constants). Denoting as $\theta$ the concentration parameter of every $\varepsilon_r$, the usage frequency of option $r$ amounts to the following mathematical probability:

$$p_r = \frac{\exp(\theta.V_r)}{\exp(\theta.V_{\mathfrak{R}})} = \exp(\theta(V_r - V_{\mathfrak{R}})) \tag{12}$$

Wherein: $V_{\mathfrak{R}} \equiv \frac{1}{\theta}\ln\sum_{s\in\mathfrak{R}}\exp(\theta.V_s)$, called the satisfaction function, is the mean value of the maximum utility, $U_{\mathfrak{R}} \equiv \max\{U_s: \forall s \in \mathfrak{R}\}$. In the logit model, the maximum utility is also a Gumbel variable with concentration parameter $\theta$.

By assumption, each individual in the population of interest makes one and one trip only on the roadway link during the period under study. Thus the population size and the trip flow are equal and both are denoted as $q$. The choice model yields modal "demanded flows" $y_r^{\#}$ such that

$$y_r^{\#} = q.\,\mathrm{p}_r^{\#}. \tag{13}$$

By construct, it holds that $y_r^{\#} \geq 0$ and $\sum_{r\in\mathfrak{R}} y_r^{\#} = q$.

## 2.6/ Elastic demand

The trip demand $q$ may be related to the expected overall utility, $V_{\mathfrak{R}}$, through a demand function with respect to the expected overall generalized cost $g_{\mathfrak{R}} \equiv -V_{\mathfrak{R}}$:

$$q = \mathrm{D}(g_{\mathfrak{R}}). \tag{14}$$

"Fixed demand" corresponds to a constant function D.

## 2.7/ Car flows and traffic issues

The modal flow $y_r$ by travel mode $r \in \mathfrak{R}$ gives rise to a car flow $x_r$ that is equal to $y_r$ for $r \in \{\mathrm{A}, \mathrm{N}\}$ or null otherwise, i.e. $x_{\mathrm{U}} = 0$. Total car flow amounts to

$$x = x_{\mathrm{A}} + x_{\mathrm{N}} = y_{\mathrm{A}} + y_{\mathrm{N}} \tag{15}$$

The organization of an LRS may be combined to a specific arrangement of car lanes on the roadway link by dedicating one lane to multi-occupied vehicles, that is, "Isolated Flow" (IF), or not, that is, "Joint Flow" (JF). As agent cars with users are multi-occupied, the run time on the dedicated lane would be $t_U^R$, whereas that on common lanes is just $t_N^R$: hence the indices for travel time functions. The car flow rate eligible to lane dedication is

$$X_U^{\mathrm{IF}} \equiv y_{\mathrm{A}}.\Pr\{v > 0\}/H = \rho\varphi/H, \tag{15-U}$$

While that restricted to common lanes is

$$X_N^{\mathrm{IF}} \equiv (y_{\mathrm{N}} + y_{\mathrm{A}}.\Pr\{v = 0\})/H = (y_{\mathrm{N}} + (1-\rho)\varphi)/H. \tag{15-N}$$

We model the influence of a car trip flow by flow policy $\mathrm{FP} \in \{\mathrm{IF}, \mathrm{JF}\}$ as a travel time function $\mathrm{T}_r^{\mathrm{FP}}$ with respect to the related flow time rate $X_r^{\mathrm{FP}}$. Under Isolated Flow, then

$$t_U^R = \mathrm{T}_{\mathrm{U}}^{\mathrm{IF}}(X_U^{\mathrm{IF}}) \tag{16-I,U}$$

$$t_N^R = \mathrm{T}_{\mathrm{N}}^{\mathrm{IF}}(X_N^{\mathrm{IF}}) \tag{16-I,N}$$

While under Joint Flow, then $\mathrm{T}_{\mathrm{U}}^{\mathrm{JF}} = \mathrm{T}_{\mathrm{N}}^{\mathrm{JF}}$ and $X_U^{\mathrm{JF}} = X_N^{\mathrm{JF}} = x/H$, so that

$$t_U^R = t_N^R = \mathrm{T}_{\mathrm{U}}^{\mathrm{JF}}\left(X_U^{\mathrm{JF}}\right) = \mathrm{T}_{\mathrm{N}}^{\mathrm{JF}}(X_N^{\mathrm{JF}}) \tag{16-J}$$

Both flow volumes and run times are traffic conditions determining the local emissions of noise and of air pollutants, including GHG emissions of global outreach.

# 3/ Traffic equilibrium

## 3.1/ Supply sub-model

We select the vector $\boldsymbol{y}_{\mathfrak{R}} \equiv [y_r : r \in \mathfrak{R}]$ of modal flow volumes as "basic" subset of state variables from which all of the other exogenous variables are derived.

Vector $\boldsymbol{y}_{\mathfrak{R}}$ induces first the service conditions $(\varphi, \omega, \rho)$ in eqn. (1), then the leg time items $\boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}} \equiv [\, t^{\ell}_{r} : \ell \in \mathcal{L}, r \in \mathfrak{R}]$ in (2,3,4,15,16), next the service fee and fare $\hat{\boldsymbol{\tau}}_{\mathcal{S}} \equiv [\hat{\tau}_r : r \in \mathcal{S}]$ in (5) and the money items $\boldsymbol{m}_{\mathfrak{R}} \equiv [m_r : r \in \mathfrak{R}]$ in (6).

The supply sub-model is defined as the derivation of $\rho$, $\boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}$ and $\boldsymbol{m}_{\mathfrak{R}}$ from vector $\boldsymbol{y}_{\mathfrak{R}}$. It is denoted as

$$(\rho, \boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}, \boldsymbol{m}_{\mathfrak{R}}) = \mathbf{F_S}(\boldsymbol{y}_{\mathfrak{R}}) \tag{17}$$

Only non-negative option flows are considered, making the domain $\Delta_{\boldsymbol{y}} \equiv \{\boldsymbol{y}_{\mathfrak{R}} \geq 0\}$. We also consider the sub-domain $\Delta^{*}_{\boldsymbol{y}} \equiv \{\boldsymbol{y}_{\mathfrak{R}} \geq 0 : y_{\mathrm{A}} > 0\}$ where the supply function is continuous. On the contrary, if $y_{\mathrm{A}} = 0$ then $\omega$ is indeterminate and so is $\rho$; furthermore, $\varphi = 0$ leads to infinite wait times to users under UW. Under AW, if $\rho > 0$ then $\varphi = 0$ also entails infinite wait times to agents. We therefore extend the definition of the supply function on points such that $y_{\mathrm{A}} = 0$ in the following way: $t^{W}_{U} = t^{W}_{A} = +\infty$, under both waiting policies AW and UW.

## 3.2/ Demand sub-model

The "demand sub-model" is defined as the derivation of "demanded flows" $\boldsymbol{y}^{\#}_{\mathfrak{R}} \equiv [y^{\#}_{r} : r \in \mathfrak{R}]$ from the vector $(\rho, \boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}, \boldsymbol{m}_{\mathfrak{R}})$ through the formation of generalized costs in (7,8) and deterministic utilities in (9), expected maximum utility and demand volume in (14), option modal shares in (12), and finally demanded flows in (14). It is denoted as

$$\boldsymbol{y}^{\#}_{\mathfrak{R}} = \mathbf{F_D}(\rho, \boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}, \boldsymbol{m}_{\mathfrak{R}}) \tag{18}$$

The feasible domain of $\rho$ is $\Delta_{\rho} \equiv [0,1]$ and that of $(\boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}, \boldsymbol{m}_{\mathfrak{R}})$ is $\Delta_{\boldsymbol{t},\boldsymbol{m}} \equiv \mathbb{R}^{\mathcal{L}\times\mathfrak{R}+\mathfrak{R}}$. The composition of generalized cost is a linear combination, save for the multiplicative influence of $\rho$ in $g_A$: these are well-defined and continuous functions on the joint domain $\Delta_{\rho} \times \Delta_{\boldsymbol{t},\boldsymbol{m}}$. By the properties of the exponential function, $V_{\mathfrak{R}}$ and the $p^{\#}_{r}$ are well-defined and continuous functions of $(\rho, \boldsymbol{t}^{\mathcal{L}}_{\mathfrak{R}}, \boldsymbol{m}_{\mathfrak{R}})$. Postulating a continuous demand function D, so are $q$ and the demand flow functions $y^{\#}_{r}$.

Fig. 1 exhibits both the supply sub-model and the demand sub-model.

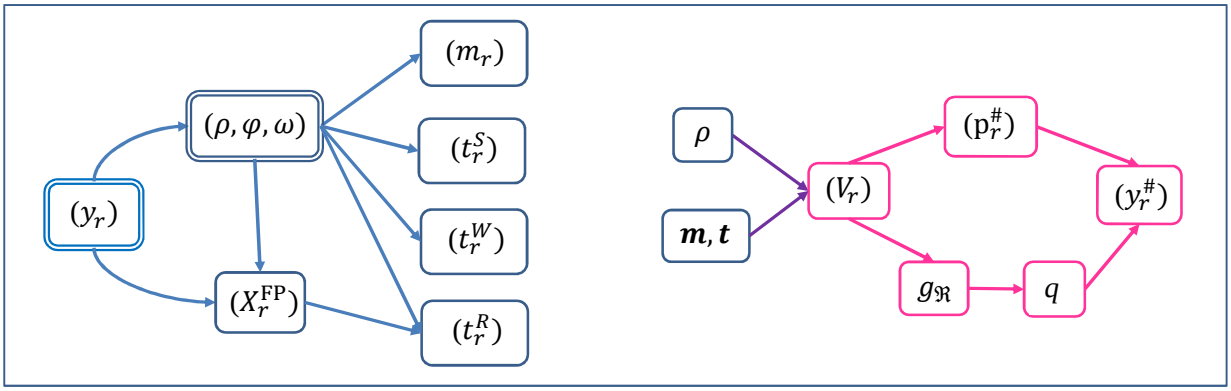


**Fig.1: (A) Supply sub-model, (B) Demand sub-model.**

## 3.3/ Supply-demand equilibrium

Definition. The system is in equilibrium if the system state $(\boldsymbol{y}_\Re, \rho, \boldsymbol{t}^{\mathcal{L}}_\Re, \boldsymbol{m}_\Re)$ satisfies jointly that

$$\left(\rho, \boldsymbol{t}^{\mathcal{L}}_\Re, \boldsymbol{m}_\Re\right) = \mathbf{F_S}(\boldsymbol{y}_\Re),$$
$$\boldsymbol{y}_\Re = \mathbf{F_D}\left(\rho, \boldsymbol{t}^{\mathcal{L}}_\Re, \boldsymbol{m}_\Re\right).$$

***Proposition 2: Equilibrium as a fixed-point problem***. Vector $\boldsymbol{y}_\Re$ is a supply-demand equilibrium iff it satisfies the fixed-point problem associated to the mapping $\mathbf{F_D} \circ \mathbf{F_S}$, i.e.,

$$\boldsymbol{y}_\Re = \mathbf{F_D} \circ \mathbf{F_S}(\boldsymbol{y}_\Re). \tag{19}$$

The conjunction of (17) and (18) at $\boldsymbol{y}_\Re$ implies (19) straightforwardly. Conversely, if (19) holds true then the system state $(\boldsymbol{y}_\Re, \mathbf{F_S}(\boldsymbol{y}_\Re))$ satisfies the equilibrium definition.

The FPP states that the "demanded flows" should match the "supplied ones".

By restricting the feasible set to $\Delta_{\boldsymbol{y}}^{\eta,Q} \equiv \{\boldsymbol{y}_\Re \geq 0 : y_\mathrm{A} \geq \eta, y_r \leq Q\}$ with some small $\eta > 0$ and some large $Q > 0$, and defining a modified demand function $\mathbf{F}_\mathbf{D}^{\eta,Q}$ such that $\mathbf{F}_{\mathbf{D},r}^{\eta,Q} \equiv \max\left\{\eta, \min\{Q, y_r^\#\}\right\}$, then the mapping $\mathbf{F}_\mathbf{D}^{\eta,Q} \circ \mathbf{F_S}$ is continuous on $\Delta_{\boldsymbol{y}}^{\eta,Q}$, and we can set parameters $\eta$ and $Q$ to values such that $\mathbf{F}_\mathbf{D}^{\eta,Q} \circ \mathbf{F_S}(\Delta_{\boldsymbol{y}}^{\eta,Q}) \subset \Delta_{\boldsymbol{y}}^{\eta,Q}$. Then, by the Brouwer theorem, there exists a solution to the fixed point problem.

Under UW and AW, null agent flow value $y_\mathrm{A} = 0$ yields infinite user wait time $t_U^W$: then $g_U^\# = +\infty$ if $\beta_U^W > 0$, $p_U^\# = 0$ and $y_U^\# = 0$. Similarly, $t_A^W = +\infty$, $g_A^\# = +\infty$ if $\beta_A^W > 0$, $p_A^\# = 0$ and $y_A^\# = 0$, which is consistent with $y_\mathrm{A} = 0$. The point $\boldsymbol{y}_\Re = (0, 0, \mathrm{D}(g_N^\#))$ is a trivial kind of equilibrium, along with the non-trivial kind that solves the truncated program with $\mathbf{F}_\mathbf{D}^{\eta,Q}$.

## 3.4/ Mathematical characterizations

As $p_r^\# = \exp(\theta V_{r-\Re})$, the equality between $\boldsymbol{y}_\Re$ and $\mathbf{F_D} \circ \mathbf{F_S}(\boldsymbol{y}_\Re)$ can also be put as

$$y_r = \exp(\theta V_{r-\Re}) \,.\, \mathrm{D}(g_\Re). \tag{20}$$

It is also equivalent to

$$\forall\, r \in \Re : \;\; g_r + \tfrac{1}{\theta} \ln y_r = g_\Re + \tfrac{1}{\theta} \ln \mathrm{D}(g_\Re). \tag{21}$$

This is in essence the Variational Inequality formulation of a logit traffic assignment model, as stated by Caroline Fisk (1980) and extended to elastic demand by Akamatsu & Matsumoto (1989). See Appendix B.

This system is equivalent to just

$$g_A + \tfrac{1}{\theta} \ln y_A = g_U + \tfrac{1}{\theta} \ln y_U, \tag{22a}$$
$$g_U + \tfrac{1}{\theta} \ln y_U = g_N + \tfrac{1}{\theta} \ln y_N, \tag{22b}$$
$$y_A + y_U + y_N = \mathrm{D}(g_\Re). \tag{22c}$$

As $y_U / y_A = \omega$, condition (22a) becomes

$$\tfrac{1}{\theta} \ln \omega = g_A - g_U. \tag{23a}$$

We call it the "inner service split" (ISS) condition since it compares the two options within the service. Given the option run times and $\omega$ (hence $\rho$), the ISS implies $\varphi$ as

$$\frac{1}{\theta} \ln \omega = m_A + (\beta t)_A^{RST} + \delta_\mathrm{A}^\mathrm{W} \beta_\mathrm{A}^\mathrm{W} \frac{\rho H}{\varphi} - m_U - (\beta t)_U^{RST} - \delta_\mathrm{U}^\mathrm{W} \beta_\mathrm{U}^\mathrm{W} \frac{H}{\varphi}$$

So that

$$\frac{H}{\varphi}(\delta_{\mathrm{U}}^{\mathrm{W}}\beta_{\mathrm{U}}^{\mathrm{W}} - \delta_{\mathrm{A}}^{\mathrm{W}}\beta_{\mathrm{A}}^{\mathrm{W}}\rho) = m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta}\ln\omega$$

The ISS gives $\varphi = y_A$ as a function $\hat{y}_A(\omega)$ with respect to $\omega$ (and the run times if variable):

$$\hat{y}_A(\omega) = \frac{H\left(\delta_{\mathrm{U}}^{\mathrm{W}}\beta_{\mathrm{U}}^{\mathrm{W}} - \delta_{\mathrm{A}}^{\mathrm{W}}\beta_{\mathrm{A}}^{\mathrm{W}}\rho\right)}{m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta}\ln\omega}.$$

The condition (22b) makes an "outer service split" (OSS) condition since it compares the outside option N to the inside option U. Knowing $\omega$ hence $\varphi = \hat{y}_A(\omega)$ and $\hat{y}_U(\omega) \equiv \omega.\hat{y}_A(\omega)$, then the OSS gives $\hat{y}_N(\omega)$ with respect to $\omega$:

$$\hat{y}_N(\omega) = \exp(\theta g_{U-N})\,\hat{y}_U(\omega). \tag{23b}$$

Substituting in (22c), we obtain a condition that

$$\hat{q}(\omega) \equiv \hat{y}_A(\omega)(1+\omega) + \hat{y}_N(\omega) = \mathrm{D}(g_{\Re}). \tag{23c}$$

It is a condition in $\omega$ only if the run times are fixed.

***Proposition 3***. Under fixed run times and fixed demand, then there exists a unique value $\omega^*$ solving the equilibrium characterization.

Proof: Function $\hat{y}_A(\omega)$ is monotonous with respect to $\omega$ and so is $\hat{q}(\omega)$.

## 3.5/ Geometric properties

The ISS condition determines the sign of a function of $\omega$, namely $\mathrm{F}_\omega \equiv m_{A-U} + (\beta t)_{A-U}^{RST} - \frac{1}{\theta}\ln\omega$. As $\varphi$ is non-negative, under UW it holds that $\mathrm{F}_\omega > 0$, while under AW it holds that $\mathrm{F}_\omega < 0$. Thus, UW equilibrium states belong to $\{\mathrm{F}_\omega > 0\}$ while their AW counterparts belong to $\{\mathrm{F}_\omega < 0\}$. The line $\{\mathrm{F}_\omega = 0\}$ delimits the respective sub-domains.

Under fixed run times, condition $\{\mathrm{F}_\omega = 0\}$ amounts to $\{\omega = \omega_0\}$, with $\mathrm{F}_\omega < 0$ for $\omega > \omega_0$ and $\mathrm{F}_\omega > 0$ for $\omega < \omega_0$. Thus, there exists a frontier load rate $\omega_0$ such that AW equilibrium states lie in $\{\omega > \omega_0\}$ and UW ones in $\{\omega < \omega_0\}$.

## 3.6/ Computational scheme

Under fixed demand volume $q$, the obvious way to solve the LRS traffic equilibrium is to search for $\omega$ satisfying (23c) at $q$. But in fact, the variations of the system state with respect to the demand volume are of primary relevance to understand the service performance.

Then an efficient utilization of (23c) is to make $q$ vary with $\omega$, that is, as function $\hat{q}(\omega)$, and to analyze any system performance $Z_q$ as a curve $(\hat{q}(\omega), Z_q)$ parameterized by $\omega$.

# 4/ Service management and its optimization

Table 3 provides supplementary notation to deal with the economic issues about the service.

**Tab.3: Supplementary notation.**

| *Costs and profit functions* | *Derived variables* |
|---|---|
| $\mathcal{C}_o(y_A, y_U)$ production costs of LRS service | $-S_r$ marginal link flow congestion cost of $y_r$ |
| $\dot{c}_r$ marginal production cost of $y_r$ | $\mathcal{X}_r \equiv \dot{c}_r + \frac{1}{\mu}(\chi_r - S_r)$ Social cost of $r$ |
| $\chi_U$ *value* to environment of U trip | $\hat{\psi} \equiv \psi - \eta \frac{\dot{\mathrm{D}}}{q}$ |
| $\chi_A, \chi_N$ *cost* to environment of A or N trip | $f \equiv 1 - \frac{\hat{\psi}}{\mu}$ operator's share of created value |
| Actor kind $k \in \{o, u, e, a\}$ | *Short-hand notations* |
| $\mathcal{P}_k$ profit function of actor type $k$ | $\frac{\partial x'}{\partial x} = \delta_{\{x=x'\}}$ for $x, x'$ in $\boldsymbol{x}$ |
| $\mathcal{J}$ generic service objective function | $\dot{\rho} \equiv \frac{\partial \rho}{\partial \omega} = (1+\omega)^{-2}$ |
| $\mu$ factor of $\mathcal{P}_o$ in $\mathcal{J}$ | $V_r^{\tau}, V_r^{\backslash\tau}$ Price & Out-of-price parts of utility |
| $\psi$ factor of $\mathcal{P}_u$ in $\mathcal{J}$ | $V_r^{W}, V_r^{\backslash W}$ Wait & Out-of-wait parts of utility |
| *Active variables* | $\dot{V}_{r,x} \equiv \frac{\partial V_r}{\partial x}$ ; $\dot{V}_{r,x}^{\backslash\tau}$ minus the price part |
| $\boldsymbol{x} \equiv (\tau_U, \tau_{A0}, \tau_{A1}, t_U^R, t_N^R, \varphi, \omega, q)$ primal vector | $\epsilon_{y_r:x}$ elasticity of $y_r$ to $x$ |
| $\xi$ dual variable of $\varphi = y_A^{\#}$ | $\tau_U^- \equiv \tau_U - \dot{c}_U$ |
| $\lambda$ dual variable of $y_U^{\#} = \omega y_A^{\#}$ | $\tau_A^+ \equiv \tau_A + \dot{c}_A$ |
| $\eta$ dual variable of $q = \mathrm{D}(g_{\Re}^{\#})$ | $\mathcal{U} \equiv \mu\tau_U^- + \chi_{U+N} - \lambda + S_{U-N}$ |
| $\sigma_s$ dual variable of $t_s^R = \mathrm{T}_s^R(\boldsymbol{y}_{\Re}^{\#})$ | $\mathcal{A} \equiv \mu\tau_A^+ + \chi_{A-N} - \lambda\omega - \xi - S_{A-N}$ |

## 4.1/ Service production and cost function

The service activity involves products of two kinds: user rides and agent runs, in respective quantities $y_U$ and $y_A$ at the period level. Putting aside the agent activities that the operator "buys" at specific cost $\tau_A.y_A$, the service involves some physical means on the link as well as its digital platform: the related production costs, denoted $\mathcal{C}_o$, are related to the activity levels:

$$\mathcal{C}_o = \mathcal{C}_o(y_A, y_U)$$

Around a production level $(\hat{y}_A, \hat{y}_U)$, the marginal cost of $y_r$ is denoted $\dot{c}_r \equiv \frac{\partial \mathcal{C}_o}{\partial y_r}$.

An affine linear specification may be considered to fix ideas:

$$\mathcal{C}_o = \underline{\mathcal{C}_o} + \dot{c}_A y_A + \dot{c}_U y_U.$$

## 4.2/ Actors' profit functions

The profit function of the operator is the difference between the commercial revenues $\tau_U.y_U$ and the two kinds of production costs:

$$\mathcal{P}_o = \tau_U.y_U - \tau_A.y_A - \mathcal{C}_o. \tag{24}$$

On the demand side, link users whatever their roles A, U or N with respect to the service, get on average an individual utility of $V_{\Re}$, yielding a net user surplus of $\mathcal{P}_u = qV_{\Re}$. In the frame of elastic demand, the net user surplus is

$$\mathcal{P}_u = \int_{g_{\Re}}^{\infty} \mathrm{D}(g)dg \tag{25}$$

Service externalities e.g. on the environment are related to the service activity levels. We expect service utilization to deliver environmental benefits about proportional to $y_U$ and likely also to $L$: this is modeled by a per-ride coefficient $\chi_U$. On the agent side, the negative externalities of their car traffic may be attributed to the service, at per-run cost of $\chi_A$. Likewise, the environmental impacts of the N-type may be targeted, at trip cost of $\chi_N$. Taken together, the net environmental benefits of the service are

$$\mathcal{P}_e = \chi_U y_U - \chi_A y_A - \chi_N y_N. \tag{26}$$

Furthermore, the Mobility Organizing Authority (code $a$) may be involved by subsidizing the service to make it financially viable. This applies if $\mathcal{P}_o < 0$ by subsidizing the service at level $(-\mathcal{P}_o)_+$. Assuming a marginal cost of public funds of $\mu_a$, the related cost to the authority is $\mu_a(-\mathcal{P}_o)_+$, yielding the specific agency profit function

$$\mathcal{P}_a = -\mu_a(-\mathcal{P}_o)_+. \tag{27}$$

## 4.3/ Objective function and regulation regimes

A service operator acting on its own may endeavor to maximize its profits. Such "selfish" operator behavior is denoted MO for Monopoly, as only one ridesharing service is considered on the link. It involves an objective function $\mathcal{J}_{\mathrm{MO}} \equiv \mathcal{P}_o$.

System optimization involves considering not only the operator's profits but also those to the other kinds of stakeholders. Omitting externalities and discarding subsidies, the objective function of the service as a system will then be $\mathcal{J}_{\mathrm{ou}} \equiv \mathcal{P}_o + \mathcal{P}_u$.

The consideration of environmental impacts leads to an extended system profit function of $\mathcal{J}_{\mathrm{oue}} \equiv \mathcal{P}_o + \mathcal{P}_u + \mathcal{P}_e$.

The inclusion of a service-viability scheme leads to an overall system profit of

$$\mathcal{J}_{\mathrm{ouea}} \equiv (\mathcal{P}_o)_+ + \mathcal{P}_u + \mathcal{P}_e + \mathcal{P}_a$$

The respective scopes o, o+u, o+u+e, o+u+e+a, correspond to different regimes of service regulation by the authority, leading to different behaviors of the service operator. A scope o+e may also be considered to model an "incentivized" service, rewarded by the authority for its environmental benefits.

To address each of the scopes, a generic objective function is formulated as follows:

$$\mathcal{J} = \mu\mathcal{P}_o + \psi\mathcal{P}_u + \mathcal{P}_e, \tag{28}$$

Wherein $\psi \in \{0,1\}$ is the binary indicator of Link Users inclusion, and $\mu$ takes value $1$ in the absence of public subsidizes or $\mu_a$ otherwise. This is because, under negative $\mathcal{P}_o$, only the authority's surplus applies, at level $-\mu_a(-\mathcal{P}_o)_+$, since the operator is compensated. On the $\{\mathcal{P}_o < 0\}$ domain, $-\mu_a(-\mathcal{P}_o)_+ = \mu_a\mathcal{P}_o$.

Together with parameters $\mu$ and $\psi$, the externality factors $\chi_U$, $\chi_A$ may be assigned relevant values, or just null values to discard the compensation of externalities, i.e., to let externalities out of the service as a system.

## 4.4/ Generic optimization program and relaxation scheme

The action levers available to the operator basically consist in service prices: $\tau_U$ on the User side and, on the Agent side, $\tau_{A0} \equiv \underline{\tau}_A$ and $\tau_{A1} \equiv L\dot{\tau}_A$. We denote them in vector form $\boldsymbol{\tau} \equiv (\tau_U, \tau_{A0}, \tau_{A1})$.

As active state variables, there are also:

- Service conditions $\varphi$ and $\omega$,

- Run times according to roles, $(t_U^R, t_N^R)$,
- Demand volume $q$ if it is elastic.

We denote as $\boldsymbol{x} \equiv (\tau_U, \tau_{A0}, \tau_{A1}, t_U^R, t_N^R, \varphi, \omega, q)$ the vector of active variables, and $x$ a generic component in $\boldsymbol{x}$. The other state variables are derived from $\boldsymbol{x}$:

- Role deterministic utility functions $V_r$,
- Expected maximum utility $V_{\mathfrak{R}}$,
- Role probabilities $p_r^\# = \exp(\theta(V_r - V_{\mathfrak{R}}))$,
- Role flows $y_r^\# = q.p_r^\#$.

All of them are considered now as functions of $\boldsymbol{x}$, yielding also $\mathcal{P}_o$, $\mathcal{P}_u$, $\mathcal{P}_e$, $\mathcal{P}_a$, as functions of $\boldsymbol{x}$, and in turn any objective function $\mathcal{J}$ that follows the generic pattern. The technical relations are modeled as specific constraints, to which we associate dual variables (as Lagrange multipliers):

- $\varphi = y_A^\#$ , with dual variable denoted $\xi$,
- $y_U^\# = \omega y_A^\#$, with dual variable denoted $\lambda$,
- $q = \mathrm{D}(g_{\mathfrak{R}}^\#)$, with dual variable $\eta$,
- $t_s^R = \mathrm{T}_s^R(\boldsymbol{y}_{\mathfrak{R}}^\#)$ for $s \in \{U, N\}$, with dual variables $\sigma_s$,

The problem of maximizing the objective function $\mathcal{J}$ with respect to vector of active variables $\boldsymbol{x}$ and subject to the technical constraints, is equivalent to determining a saddle-point of the following Lagrangian function with respect to the primal-dual pair of variables, $\boldsymbol{x}$ and $\boldsymbol{\alpha} \equiv (\xi, \lambda, \eta, \sigma_U, \sigma_N)$.

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\alpha}) \equiv \mathcal{J}(\boldsymbol{x}) - \xi\left(\varphi - y_A^\#\right) - \lambda\left(y_U^\# - \omega y_A^\#\right) - \eta(q - \mathrm{D}(g_{\mathfrak{R}}^\#)) - \sum_{s\in\{U,N\}} \sigma_s(t_s^R - \mathrm{T}_s^R(\boldsymbol{y}_{\mathfrak{R}}^\#)) \tag{29}$$

Wherein: $g_{\mathfrak{R}}^\# \equiv -V_{\mathfrak{R}}(\boldsymbol{x})$ and $\boldsymbol{y}_{\mathfrak{R}}^\#(\boldsymbol{x})$ depend on $\boldsymbol{x}$ only, as does $\mathcal{J}$ (either straightforwardly or through $\boldsymbol{y}_{\mathfrak{R}}^\#$ and $V_{\mathfrak{R}}$).

By construction of the Lagrangian function, a saddle-point is characterized by the first-order optimality conditions (FOCs) of the Lagrangian with respect to both the primal and dual variables:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \text{ for any } x \text{ in } \boldsymbol{x}, \tag{30a}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 0 \text{ for any } \alpha \text{ in } \boldsymbol{\alpha}. \tag{30b}$$

## 4.5/ Formal properties of an optimal state

The first-order optimality conditions lead to more explicit expressions that are established in Appendix D and exhibited in Table 4, using the following important auxiliary variables:

$$\tau_U^- \equiv \tau_U - \dot{c}_U, \tag{31a}$$

$$\tau_A^+ \equiv \tau_A + \dot{c}_A, \tag{31b}$$

$$\hat{\psi} \equiv \psi - \eta \frac{\dot{\mathrm{D}}}{q}, \tag{31c}$$

$$\mathcal{M} \equiv \frac{\mu - \hat{\psi}}{\theta}, \tag{31d}$$

$$\mathcal{U} \equiv \mu\tau_U^- + \chi_{U+N} - \lambda + S_{U-N}, \tag{31e}$$

$$\mathcal{A} \equiv \mu\tau_A^+ + \chi_{A-N} - \lambda\omega - \xi - S_{A-N}. \tag{31f}$$

In Appendix E, a straightforward computation scheme is provided for the uncongested demand case.

**Tab. 4. Optimality conditions with respect to primal and dual variables.**

| Variable | Related optimality condition | |
|---|---|---|
| $\tau_U$ | $\mathcal{U} = \mathcal{M}/p_N$ | (32a) |
| $\tau_{A0}, \tau_{A1}$ | $\mathcal{A} = -\mathcal{M}/p_N$ | (32b) |
| $t_U^R, t_N^R$ | $\sigma_s = \mu\varphi\left\{\frac{\partial V_A}{\partial t_s^R} + \omega\frac{\partial V_U}{\partial t_s^R}\right\} + (\hat{\psi}q - \mu\varphi)\frac{\partial V_N}{\partial t_s^R}$ | (32c) |
| $\varphi$ | $\lambda = -\mu\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau}$ | (32d) |
| $\omega$ | $\xi = \mu\varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right)$ | (32e) |
| $q$ | $\eta = \frac{\theta(S_N - \chi_N) + Z(\mu - \psi)}{\theta - Z\dot{\mathrm{D}}/q}$ | (32f) |
| $\xi$ | $\varphi = y_A^{\#}$ | (32g) |
| $\lambda$ | $y_U^{\#} = \omega y_A^{\#}$ | (32h) |
| $\eta$ | $q = \mathrm{D}(g_{\mathfrak{R}}^{\#})$ | (32i) |
| $\sigma_s$ | $t_s^R = \mathrm{T}_s^R(\boldsymbol{y}_{\mathfrak{R}}^{\#})$ | (32j) |

## 4.6/ On dual variables in equilibrium: economic interpretation

The dual variable $\lambda$ associated to constraint $y_U^{\#} = \omega y_A^{\#}$ satisfies that $\lambda/\mu = -\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} = -\dot{V}_{A\omega}^{\backslash\tau}$ : it is the marginal time cost of $\omega$ to each Agent in the service.

The dual variables $\sigma_N$ and $\sigma_U$ are associated to the formation of link run times. If types N and U share their roadway lanes, then the two variables collapse in one only ($\delta_{UN}^R = 1$). From the generic optimality condition (32c), $\sigma_s$ appears as a value of link decongestion owing to service involvement instead of non-commitment (N). Thus, by individual type, the derived variable $-S_r/\mu$ according to role $r$ is interpreted as the marginal traffic congestion cost of $y_r$ to link flowing.

The dual variable $\xi$ associated to constraint $\varphi = y_A^{\#}$ satisfies that $\xi/\mu = \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right)$ : it is *the marginal value of frequency at the car run level* to one agent ($\dot{V}_{A\varphi}$) as well as to the $\omega$ users ($\dot{V}_{U\varphi}$ each), times the number of car runs. It satisfies that

$$\frac{\xi}{\mu} = \frac{\mathrm{H}}{\varphi}\left(\delta_A^W\beta_A^W\rho + \omega\delta_U^W\beta_U^W\right) = (\beta t)_A^W + \omega(\beta t)_U^W. \tag{33}$$

Thus, $\xi/\mu$ is the cost of wait times to the car occupants at the level of a car run. It is equal to the marginal value of frequency to the link users, $\partial\mathcal{P}_u/\partial\varphi$ (cf. Appendix D).

The number of cab occupants, $1 + \omega$, times the wait time cost at the cab level, $(\beta t)_A^W + \omega(\beta t)_U^W$, is an amplified value of wait time still at the cab level. It satisfies that

$$(1 + \omega)\frac{\xi}{\mu} = (1 + \omega)\{(\beta t)_A^W + \omega(\beta t)_U^W\} = -(1 + \omega)\{V_A^W + \omega V_U^W\}. \tag{34}$$

# 5/ Value creation to individuals, operator and society

## 5.1/ Service utility and value creation *at the individual level*

Service participation by taking role U or A, denoted as the combined option $\mathcal{S} = \{U, A\}$, entails a combined utility $U_{\mathcal{S}} \equiv \max\{U_{\mathrm{U}}, U_{\mathrm{A}}\}$, with expected value denoted $V_{\mathcal{S}}$. In the multinomial logit model, $V_{\mathcal{S}}$ satisfies that

$$V_{\mathcal{S}} = \frac{1}{\theta}\mathcal{H}_{\rho} + \rho V_{\mathrm{U}} + (1-\rho)V_{\mathrm{A}}, \tag{5.1}$$

Wherein $\mathcal{H}_{\rho} \equiv -\rho\ln\rho - (1-\rho)\ln(1-\rho)$ denotes the entropy function of $[\rho, (1-\rho)]$ (hence it is positive), knowing that $\rho$ and $1-\rho$ are the respective probabilities of U and A conditionally to $\mathcal{S}$.

In turn, service availability entails a combined utility $U_{\Re} \equiv \max\{U_{\mathcal{S}}, U_{\mathrm{N}}\}$ to all individuals able to participate, whether they do or not. The resulting expected utility $V_{\Re}$, called the satisfaction function of options {U,A,N}, induces the demand surplus $\mathcal{P}_u$. In the logit model, $V_{\Re}$ satisfies that

$$V_{\Re} = \frac{1}{\theta}\mathcal{H}_{p_N} + p_{\mathcal{S}}V_{\mathcal{S}} + p_N V_N, \tag{5.2}$$

and, equivalently, the log-sum formula.

Furthermore, the utility difference is related to $p_N$: from the respective modal shares,

$$V_{\mathcal{S}-N} = \frac{1}{\theta}\ln\left(\frac{p_{\mathcal{S}}}{p_N}\right) = \frac{1}{\theta}\ln(p_N^{-1} - 1). \tag{5.3}$$

Then, the utility gain from $V_N$ to $V_{\Re}$ constitutes a value created by service availability to every potential participant. More generally, denoting $V_N^0$ the prior utility of link usage and $\Delta V_N \equiv V_N - V_N^0$, the utility variation from prior to current is

$$V_{\Re} - V_N^0 = V_{\Re} - V_N + \Delta V_N. \tag{5.4}$$

In the logit model, from the modal share of N it comes that

$$V_{\Re-N} \equiv V_{\Re} - V_N = -\frac{1}{\theta}\ln p_N = \frac{1}{\theta}\ln p_N^{-1}. \tag{5.5}$$

As function $\ln$ is increasing, this makes $p_N^{-1}$ some kind of utility function: furthermore, $\frac{1}{\theta}p_N^{-1}$ is a money-metric utility function, measuring the utility gain of service availability compared to option N only, to every individual as potential participant. If the current option N is equivalent to that in the situation prior to service creation, then $\frac{1}{\theta}p_N^{-1}$ is *the individual value of service creation* to potential customers. More generally, with respect to prior utility $V_N^0$, factor $e^{\theta\Delta V_N}$ modulates $\frac{1}{\theta}p_N^{-1}$ in the specific utility $\frac{1}{\theta}p_N^{-1}e^{\theta\Delta V_N}$. The outcome may be smaller than 1, meaning a utility reduction from prior situation without the service to the posterior situation that follows service creation.

Coming to factor $f \equiv 1 - \frac{\hat{\psi}}{\mu}$, under fixed demand (i.e. $\hat{\psi} = \psi$) it is equal to 1 if $\psi = 0$, i.e., selfish operator, or 0 if $\psi = 1$ and $\mu = 1$, i.e., service management oriented to first-best system optimum; second-best system optimization with $\mu > 1$ yields $f$ positive and smaller than 1. We may interpret it as the share of created-value recovered by the service operator. This makes $f\frac{1}{\theta}p_N^{-1}$ an amount of created value attributed to the operator.

## 5.2/ External costs and social values

Any individual of type $r$ exerts several effects external to the service: effects on the environment with money value $\chi_U$ for U or $-\chi_r$ for A and N, as well as effect on link flow with money value $-S_r$. Furthermore, types A and U induce marginal production costs to the service operator of $\dot{c}_r$. From the individual perspectives, these values and costs are external to them, so that the aggregate cost can be called the social cost associated to the role. It is denoted respectively

$$\mathcal{X}_r \equiv \dot{c}_r + \frac{1}{\mu}(\chi_r - S_r), \qquad \text{for } r \in \{A, N\} \tag{5.6a}$$

$$\mathcal{X}_U \equiv \dot{c}_U + \frac{1}{\mu}(-\chi_U - S_U). \tag{5.6b}$$

We denote $\mathcal{X} \equiv (\mathcal{X}_r)_r$ the vector of external costs according to the roles.

Coming to the individual utility $V_r$, it comprises three parts in relation to the service: the part related to service price is hereafter denoted as $V_r^\tau$, that to wait time as $V_r^W$, the rest $V_r^{\backslash W,\tau} \equiv V_r - V_r^\tau - V_r^W$.

As the price and wait time expense belong to the service sphere, we may consider $V_r^{\backslash W,\tau}$ as the residual utility out of that sphere, i.e., external to it.

Then, from the service standpoint, the social utility of each role is stated as

$$V_r^{\backslash W,\tau,\mathcal{X}} \equiv V_r^{\backslash W,\tau} - \mathcal{X}_r. \tag{5.7}$$

## 5.3/ Modal split and the composition of role utilities

The utility difference between service participation or not is

$$V_{\mathcal{S}-N} \equiv V_{\mathcal{S}} - V_N = \frac{1}{\theta}\mathcal{H}_\rho + \rho V_{\mathrm{U}} + (1-\rho)V_{\mathrm{A}} - V_{\mathrm{N}}.$$

The modal split between $\mathcal{S}$ and N in (5.3) makes an "outer service split" condition (OSS) relating $V_{\mathcal{S}-N}$ and $p_N^{-1}$. From the composition of role utilities,

$$V_{\mathcal{S}} - V_N = \frac{1}{\theta}\mathcal{H}_\rho + \rho(V_U^\tau + V_U^W + V_U^{\backslash W,\tau}) + (1-\rho)(V_A^\tau + V_A^W + V_A^{\backslash W,\tau}) - V_N$$

So that the OSS condition is restated as

$$-\rho(V_U^\tau + V_U^W) - (1-\rho)\left(V_A^\tau + V_A^W\right) = \frac{1}{\theta}\mathcal{H}_\rho + \rho V_U^{\backslash W,\tau} + (1-\rho)V_A^{\backslash W,\tau} - V_N - \frac{1}{\theta}\ln(p_N^{-1} - 1), \tag{5.8}$$

In which the left hand side corresponds to the service sphere.

Within the service, the modal split between U and A makes an "inner service split" (ISS) condition,

$$V_{U-A} \equiv V_U - V_A = \frac{1}{\theta}\ln\left(\frac{p_U}{p_A}\right) = \frac{1}{\theta}\ln(\omega)$$

From the composition of modal utilities, it can be restated as

$$V_{A-U}^\tau + V_{A-U}^W = -V_{A-U}^{\backslash W,\tau} - \frac{1}{\theta}\ln(\omega), \tag{5.9}$$

In which the left hand side corresponds to the service sphere.

## 5.4/ Optimal prices and the recovery of social costs

The auxiliary variables $\mathcal{M}$, $\mathcal{U}$ and $\mathcal{A}$ are instrumental to state the optimality conditions with respect to prices and establish explicit formulas for the optimized variables. $\mathcal{M}$ involves $\hat{\psi}$ and in turn $q$ and $S_N$. Under fixed demand, $\hat{\psi}$ reduces to only $\psi$, yielding a constant $\mathcal{M} = \frac{\mu - \psi}{\theta}$. At an optimal state, it holds that

$$\mathcal{U} = -\mathcal{A} = \frac{\mathcal{M}}{p_N} = f\frac{1}{\theta}p_N^{-1}.$$

It corresponds to the operator's amount of the service-created value at the individual level.

From the composition of $\mathcal{U}$ stems $\tau_U^- = \frac{\mathcal{U}+\lambda-\chi_{U+N}-S_{U-N}}{\mu}$: as $\tau_U = \dot{c}_U + \tau_U^-$, the optimum user price is $\tau_U = \dot{c}_U + f\frac{1}{\theta}p_N^{-1} - \dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} - \frac{S_{U-N}}{\mu} - \frac{\chi_{U+N}}{\mu}$, hence

$$\tau_U = f\frac{1}{\theta}p_N^{-1} - \dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} + \mathcal{X}_U - \mathcal{X}_N. \tag{5.10}$$

Thus, a user price $\tau_U$ optimized by specific service management comprises three parts: (i) the operator's amount of value created by the service at the individual level of link trip-makers, (ii) the marginal time cost of cab load to each agent, which is a congestion effect in the service sphere, (iii) $\mathcal{X}_U - \mathcal{X}_N$ the difference in social costs between U and N. In short, the user price pays the operator for its share of value creation, within-service congestion cost and the external effects beyond the service sphere.

Similarly, from the composition of $\mathcal{A}$ stems $\tau_A^+ = \frac{\mathcal{A}+\xi+\lambda\omega-\chi_{A-N}+S_{A-N}}{\mu}$: as $\tau_A = \tau_A^+ - \dot{c}_A$, the optimal agent reward is stated as $\tau_A = -\dot{c}_A - f\frac{1}{\theta}p_N^{-1} + \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - \omega\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} + \frac{S_{A-N}}{\mu} - \frac{\chi_{A-N}}{\mu}$, yielding

$$\tau_A = -f\frac{1}{\theta}p_N^{-1} + \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - \omega\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} - \mathcal{X}_{A-N}. \tag{5.11}$$

Thus, an agent reward $\tau_A$ optimized by specific service management comprises four parts:

(i) The opposite of the operator's amount of value created by the service at the individual level of link trip-makers, meaning that this amount is transferred from the agent to the operator,
(ii) The money value of frequency per cab run, meaning that the agent involvement converts wait times into economic value, through his or her involvement in the service,
(iii) The marginal time cost of $\omega$ at the agent level, i.e., a within-service congestion term: the related agent time is also converted into value by the agent commitment.
(iv) The difference in social costs between N and A, meaning that the differential social cost induced by service participation is attributed to the service operator not to the agent.

In short, the agent reward pays the agent for creating service value out of wait time and user's within-service congestion, while leaving to the operator its share of value creation, as well as the differential social cost with respect to N.

All in all, the optimized prices allow for (i) transferring value from users to agents, (ii) giving the operator its share of value creation, (iii) compensating social costs.

Factor $\frac{1}{\mu}$ is an attenuation coefficient lower than 1 if $\mu > 1$, meaning only partial attribution to the Agent, hence some diversion to the system level for the sake of service viability.

## 5.5/ On the operator income per cab run

On average per cab run, the operator recovers $\omega$ times the user fare $\tau_U$, minus the agent reward $\tau_A$. Under optimized prices, the gross operator income at the cab run level amounts to

$$\omega\tau_U - \tau_A = (1+\omega)f\frac{1}{\theta}p_N^{-1} - \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) + \omega\mathcal{X}_U + \mathcal{X}_A - (1+\omega)\mathcal{X}_N. \tag{5.12}$$

It breaks down into three parts:

(i) The full operator's amount of value created by the service at the cab run level, i.e., the number of occupants, $1+\omega$, times the amount per individual beneficiary,
(ii) The cost of wait time at the cab level, with a minus sign as the corresponding value is transferred from the operator to the agent,
(iii) The social costs relatively to non-participants.

As part (iii) is a compensation for externalities, the corresponding amount is supposed to be redirected to the community, presumably through the mobility authority. Yet, the authority may well consider that it is the existence of the service that generates such value to the environment and to the general link usage. A good way to attribute the compensation for externalities is to leave the created value to the service that generates it and reduces the externality, so as to enhance its incomes (hence its financial viability) and empower its development – strengthening the positive impacts.

## 5.6/ Modal split conditions: a reformulation

To an individual, the net price difference between "Paying as a User" and "Being paid as an Agent", is $\tau_{U+A} = \tau_U + \tau_A$. At an optimized state, it satisfies

$$\tau_{U+A} = \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - (1+\omega)\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} + \mathcal{X}_{U-A}. \tag{5.13}$$

Put in words, it is the sum of (i) the money equivalent of wait time at the cab level, (ii) the time cost of cab load $\omega$ amplified at the cab level by the number of all cab occupants (1 Agent and $\omega$ Users), (iii) $\mathcal{X}_{U-A} = \mathcal{X}_U - \mathcal{X}_A$ is the difference in social costs caused by U and A.

Going back to the inner service split condition, as $V_{A-U}^{\tau} = \tau_{U+A}$ and $\dot{V}_{A\rho}^{\backslash\tau} = \dot{V}_{A\rho}^{\backslash\tau,W} - \frac{\varphi}{\rho}\dot{V}_{A\varphi}$ we get

$$V_{A-U}^{W} + \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) + (1+\omega)\dot{\rho}\frac{\varphi}{\rho}\dot{V}_{A\varphi} = (1+\omega)\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau,W} + \mathcal{X}_{A-U} - V_{A-U}^{\backslash W,\tau} - \frac{1}{\theta}\ln(\omega)$$

In which the left hand side gathers the effects of frequency, while the RHS includes the difference in social costs. It is shown in Appendix (§E.1) that the LHS amounts to

$$V_{A-U}^{W} + \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) + (1+\omega)\dot{\rho}\frac{\varphi}{\rho}\dot{V}_{A\varphi} = (1+\omega)\frac{H}{\varphi}\left(\delta_U^W\beta_U^W + \dot{\rho}\delta_A^W\beta_A^W\right)$$

Thus, the ISS condition makes $\varphi$ a function of $\omega$ and of social costs $\mathcal{X}$: as $(1+\omega)\dot{\rho} = 1-\rho$,

$$(1+\omega)\frac{H}{\varphi}\left(\delta_U^W\beta_U^W + \dot{\rho}\delta_A^W\beta_A^W\right) = (1-\rho)\dot{V}_{A\rho}^{\backslash\tau,W} - V_{A-U}^{\backslash W,\tau,\mathcal{X}} - \frac{1}{\theta}\ln(\omega). \tag{5.14}$$

Furthermore, under optimized prices the service utility is stated as (cf. (E.5) in App. E.3)

$$\begin{aligned} V_S &= \frac{1}{\theta}\mathcal{H}_\rho + \rho\left\{V_U^W - \tau_U + V_U^{\backslash W,\tau}\right\} + (1-\rho)\left\{V_A^W + \tau_A + V_A^{\backslash W,\tau}\right\} \\ &= \frac{1}{\theta}\mathcal{H}_\rho - f\frac{1}{\theta}p_N^{-1} + \rho\left\{V_U^{\backslash W,\tau} - \mathcal{X}_{U-N}\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \mathcal{X}_{A-N}\right\} \end{aligned}$$

Because the two parts related to wait times and within-service congestion do vanish. Then, the utility difference between service participation and neutrality becomes

$$V_{S-N} = \frac{1}{\theta}\mathcal{H}_\rho - f\frac{1}{\theta}p_N^{-1} + \rho\left\{V_U^{\backslash W,\tau} - \mathcal{X}_U\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \mathcal{X}_A\right\} - \{V_N - \mathcal{X}_N\}$$

In turn, the OSS condition is restated as a relation linking $p_N^{-1}$ to $\omega$ and the social costs $\mathcal{X}$:

$$\ln(p_N^{-1} - 1) + f p_N^{-1} = \mathcal{H}_\rho + \theta\left\{\rho V_U^{\backslash W,\tau,\mathcal{X}} + (1-\rho)V_A^{\backslash W,\tau,\mathcal{X}} - V_N^{\backslash W,\tau,\mathcal{X}}\right\}. \tag{5.15}$$

# 6/ System performance according to demand volume and service strategy

## 6.1/ Service participation and value creation according to cab load

The OSS condition puts two functions in equality. The Left Hand Side (LHS) is a continuous, increasing function of $p_N^{-1}$ conditionally to factor $f \geq 0$ ([1]): the larger the factor, the higher the function. As $f_{\mathrm{SO}} = 0 < f_{\mathrm{S}\mu} < f_{\mathrm{MO}} = 1$, the LHS functions satisfy that $\mathrm{LHS}_{\mathrm{SO}} < \mathrm{LHS}_{\mathrm{S}\mu} < \mathrm{LHS}_{\mathrm{MO}}$.

The RHS is a function of only $\rho$ (hence of $\omega$) conditionally to external conditions $\mathcal{X}$: $\mathrm{RHS} = \frac{1}{\theta}\mathcal{H}_\rho + \rho V_U^{\mathcal{X}} + (1-\rho)V_A^{\mathcal{X}} - V_N^{\mathcal{X}}$, in which $V_A^{\mathcal{X}} = V_A^{\mathcal{X}}|_0 + \rho\dot{V}_A^{\backslash W,\tau}$ is an affine function of $\rho$ while $V_A^{\mathcal{X}}|_0$, $V_U^{\mathcal{X}}$ and $V_N^{\mathcal{X}}$ do not depend on it. Thus, the RHS function is a continuous function of $\rho \in [0,1]$, taking its values in a bounded interval (as $\mathcal{H}_\rho \in [0, \ln 2]$). It does not depend on waiting policies. Through $\mathcal{X}$ it involves service production, service regulation and externalities.

The solution $p_N^{-1}$ of an OSS condition must be larger than 1 since the LHS involves $\ln(p_N^{-1} - 1)$.

Given $\mathcal{X}$ and factor $f$, the OSS condition relates $p_N^{-1}$ to cab load $\omega$ in a unique and increasing way. At any $\omega$, decreasing $f$ yields a lower LHS hence a higher solution $p_N^{-1}$, hence higher value creation $\frac{1}{\theta}p_N^{-1}$ and higher service participation $p_S = 1 - p_N$. Given $f$, increasing $V_N^{\mathcal{X}}$ decreases $p_N^{-1}$, while increasing $V_U^{\mathcal{X}}$ or $V_A^{\mathcal{X}}$ increases $p_N^{-1}$.

## 6.2/ Service frequency and the location of optimal states

Short-hand notations $\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}} \equiv H(1+\omega)(\delta_U^W\beta_U^W + \dot{\rho}\delta_A^W\beta_A^W)$, $\mathrm{F}_{\mathrm{R}|\mathcal{X}} \equiv (1-\rho)\dot{V}_{A\rho}^{\backslash\tau,W} - V_{A-U}^{\mathcal{X}} - \frac{1}{\theta}\ln\omega$, enable us to restate the ISS condition as

$$\frac{\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}}{\varphi} = \mathrm{F}_{\mathrm{R}|\mathcal{X}} \tag{38}$$

It involves service production, traffic and regulation through $\mathcal{X}$, hence the RHS function $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$. The LHS function $\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}$ involves the service waiting policy WP exclusively of other respects (regulation etc.). Both $\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}$ and $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ are continuous functions of $\omega$. Function $\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}$ is non-negative and so is an optimal $\varphi$ (as it is equal to $y_A \geq 0$). Then, function $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ has to take a non-negative value to meet the ISS condition. In turn, any optimal cab load $\omega$ is required to satisfy that $\mathrm{F}_{\mathrm{R}|\mathcal{X}}(\omega) \geq 0$.

From its composition, $\mathrm{F}_{\mathrm{R}|\mathcal{X}} \to +\infty$ when $\omega \to 0^+$ and $\mathrm{F}_{\mathrm{R}|\mathcal{X}} \to -\infty$ when $\omega \to +\infty$: being continuous, it takes any real value positive or negative. The smallest root denoted $\bar{\omega}_{\mathcal{X}}$ such that $\mathrm{F}_{\mathrm{R}}(\omega) = 0$ makes a ceiling value on the feasible domain $[0, \bar{\omega}_{\mathcal{X}}[$.

[1] $f \equiv 1 - \frac{\hat{\psi}}{\mu}$ that measures the operator share of value created by the service: selfish strategy MO ($\mu = 1, \psi = 0$) corresponds to $f = 1$, 2nd best optimum to $f \in ]0,1[$ and first-best system optimum to $f = 0$. Under fixed demand, $\hat{\psi} = \psi \in \{0,1\}$ and $\mu \geq 1$ hence $f \geq 0$.

Then, as $\mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}$ is bounded, the quotient function $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}: \omega \mapsto \mathrm{F}_{\mathrm{L}}^{\mathrm{WP}}/\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ is continuous on $]0, \bar{\omega}_{\mathcal{X}}[$; it tends to zero when $\omega \to 0^{+}$ and to $+\infty$ when $\omega \to \bar{\omega}_{\mathcal{X}}$ from below. It gives the optimal frequency $\tilde{\varphi}$ with respect to cab load $\omega$, conditionally to $\mathcal{X}$.

At any feasible $\omega$, $\mathrm{F}_{\mathrm{L}}^{\mathrm{UW}}$ and $\mathrm{F}_{\mathrm{L}}^{\mathrm{AW}}$ determine the optimal frequencies $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{UW}}$ and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{AW}}$ respectively. If $\beta_A^W \leq (1+\omega)^2 \beta_U^W$ then $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{UW}} \geq \tilde{\varphi}_{\mathcal{X}}^{\mathrm{AW}}$, otherwise $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{UW}} < \tilde{\varphi}_{\mathcal{X}}^{\mathrm{AW}}$. In a general way, we have that

$$\tilde{\varphi}_{\mathcal{X}}^{\mathrm{UW}} = (1+\omega)^2 \frac{\beta_U^W}{\beta_A^W} \tilde{\varphi}_{\mathcal{X}}^{\mathrm{AW}},$$

so that the optimal frequency under UW increases with $\omega$ more strongly than that under AW.

The marginal production costs $\dot{c}_r$ and the regulation parameters $\mu, \psi, \chi_r$, as well as congestion costs $-S_r$, determine $\mathcal{X}$ hence function $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ and the related ceiling value $\bar{\omega}_{\mathcal{X}}$. In the absence of link flow congestion, $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ only involves fixed parameters. Null environmental compensations $\chi_r$ make $\mathrm{F}_{\mathrm{R}|\mathcal{X}}$ independent of the regulation regime, i.e., identical between selfish operator, 1st best and 2nd best system optimum, yielding in turn optimal frequencies that do not vary with the regulation regimes.

## 6.3/ Demand volume and the existence of optimal states

Both the ISS and OSS conditions are necessary conditions on optimal states. Given the waiting policy and the external costs $\mathcal{X}$, $p_N^{-1}$ and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ are specific functions of $\omega$ only. In turn, the following combination is a function of $\omega$ called the demanded volume function:

$$\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}} \equiv (1+\omega)\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}} \frac{p_N^{-1}}{p_N^{-1}-1} = \frac{(1+\omega)\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}}{1-p_N}$$

Both the numerator and denominator are continuous, non-negative functions; as $p_N^{-1}$ takes positive values strictly larger than 1 and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ varies from 0 to $+\infty$, the combined function $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ is continuous, non-negative and varies between 0 and $+\infty$ on the feasible domain $[0, \bar{\omega}_{\mathcal{X}}[$.

The existence of an optimum state is subjected to the necessary conditions established so far. Under fixed demand, if for an exogenous demand volume $q$ there is a feasible cab load $\omega \in [0, \bar{\omega}_{\mathcal{X}}[$ that solves

$$\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}} = q,$$

Then it is a candidate state for optimality (minimum or maximum).

If function $p_N^{-1}$ is increasing then so is $\omega \mapsto 1-p_N$ on the denominator, meaning a decreasing influence on $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$. But an increasing function $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ exerts an increasing influence on the demanded volume. If $p_N^{-1}$ is strictly increasing and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ strictly decreasing, then function $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ is strictly decreasing with respect to $\omega$.

As waiting policies do not influence $p_N^{-1}$, the demanded volume functions $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ depend on the waiting policy as do their parent functions $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$: then,

$$\tilde{Q}_{\mathcal{X}}^{\mathrm{UW}} = (1+\omega)^2 \frac{\beta_U^W}{\beta_A^W} \tilde{Q}_{\mathcal{X}}^{\mathrm{AW}}$$

Given $\mathcal{X}$, if $\beta_U^W \geq \beta_A^W$ then $\tilde{Q}_{\mathcal{X}}^{\mathrm{UW}}$ lies above $\tilde{Q}_{\mathcal{X}}^{\mathrm{AW}}$: then, for any $q$ the respective solutions $\tilde{\omega}_{\mathcal{X},q}^{\mathrm{WP}}$ satisfy that $\tilde{\omega}_{\mathcal{X},q}^{\mathrm{UW}} \leq \tilde{\omega}_{\mathcal{X},q}^{\mathrm{AW}}$. But if $\beta_U^W (1+\tilde{\omega}_{\mathcal{X},q}^{\mathrm{UW}})^2 < \beta_A^W$ then $\tilde{Q}_{\mathcal{X}}^{\mathrm{AW}}$ lies above $\tilde{Q}_{\mathcal{X}}^{\mathrm{UW}}$ at $\tilde{\omega}_{\mathcal{X},q}^{\mathrm{UW}}$, implying that $\tilde{\omega}_{\mathcal{X},q}^{\mathrm{AW}} < \tilde{\omega}_{\mathcal{X},q}^{\mathrm{UW}}$. The same property holds if $\beta_U^W (1+\tilde{\omega}_{\mathcal{X},q}^{\mathrm{AW}})^2 < \beta_A^W$.

Given the waiting policy, under fixed demand, fixed marginal production costs $\dot{c}_r$, no link flow congestion and null environmental compensations $\chi_r$, then the $(\mu, \psi)$ pair specifying the regulation regime influences the demanded volume function only through $p_N^{-1}$ on the basis of factor $f \equiv 1 - \frac{\psi}{\mu}$

that influences $p_N^{-1}$ in a decreasing way. Similarly, $1-p_N$ decreases with $f$ making function $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ increase with it. Thus, between two factors $f_1 < f_2$, $\tilde{Q}_{\mathcal{X},f_1}^{\mathrm{WP}} \geq \tilde{Q}_{\mathcal{X},f_2}^{\mathrm{WP}}$. For a given demand volume $q$, the respective solutions $\tilde{\omega}_1$ and $\tilde{\omega}_1$ satisfy that $\tilde{\omega}_1 < \tilde{\omega}_2$, meaning that the optimum cab load decreases with $f$:

$$\tilde{\omega}_{MO}^{\mathrm{WP}}(q) \leq \tilde{\omega}_{S\mu}^{\mathrm{WP}}(q) \leq \tilde{\omega}_{SO}^{\mathrm{WP}}(q)$$

Under these specifications, the $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ functions are identical between the regimes: if it is increasing, then

$$\tilde{\varphi}_{MO}^{\mathrm{WP}}(q) \leq \tilde{\varphi}_{S\mu}^{\mathrm{WP}}(q) \leq \tilde{\varphi}_{SO}^{\mathrm{WP}}(q)$$

## 6.4/ Cab load and frequency in relation to demand volume

Under fixed demand, fixed production and traffic conditions (constant $\dot{c}_r$ and $S_r$), and specific regulation levers $(\mu, \psi, \chi_r)$, the function $\omega \mapsto \tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ is well-defined on $[0, \bar{\omega}_{\mathcal{X}}[$ where it is non-negative and continuous. This demanded volume function is called "well-behaved" if it is monotonously increasing or decreasing. Then, it can be inverted into a monotonous function denoted $\tilde{\omega}_{\mathcal{X}}^{\mathrm{WP}}: q \mapsto \omega$ such that $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}(\omega) = q$.

By such univocal linkage between $q$ and $\omega$, the properties established with respect to $\omega$ transfer with respect to $q$ (with an adaptation if the dependency is decreasing). Specifically, OSS and ISS functions $p_N^{-1}$ and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ with respect to $\omega$ are adapted as functions $\hat{p}_N^{-1}$ and $\hat{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ with respect to $q$.

A well-behaved relation between $\omega$ and $q$ is an idealized property. In most instances we found several local intervals on which $q$ is fully increasing with respect to $\omega$, or fully decreasing. However, as $\omega$ approaches the ceiling value $\bar{\omega}_{\mathcal{X}}$ there is an increasing "last part" of $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ that is relatively large regarding both the range of $\omega$ and even more that of $q$ : we call it the "well-behaved range".

Well-behaved or not, in all cases the right way to link $p_N^{-1}$ and $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ to $q$ is to parameterize all of them by $\omega$ (Fig. 2). In the absence of congestion, the variable demand problem reduces to a single condition in $\omega$ only: deriving $V_{\mathfrak{R}}^{\#}$ hence $g_{\mathfrak{R}}^{\#}$ from $\omega$ through $\tilde{\varphi}$, the variable demand problem consists in finding $\omega$ such that

$$\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}(\omega) = \mathrm{D}\left(g_{\mathfrak{R}}^{\#}\right).$$

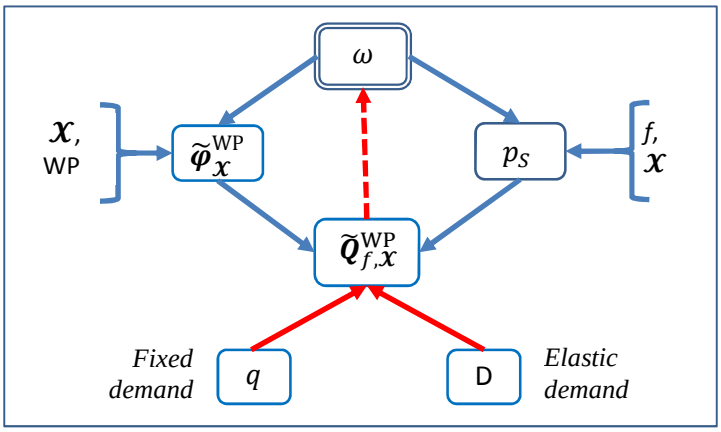


**Fig.2: The network of influences.**

## 6.5/ Optimal prices with respect to $\omega$ and $q$

Under given WP and fixed $\mathcal{X}$, it is straightforward to derive optimized prices $\tau_U$ and $\tau_A$ from $\omega$ through $p_N^{-1}$, $\tilde{\varphi}_{\mathcal{X}}^{\mathrm{WP}}$ and the related $V_{r,\rho}^{\backslash W}$, $\dot{V}_{r,\varphi}^{\backslash W}$ and $\dot{V}_{A\rho}^{\backslash \tau,W}$. The resulting $\tau_U$ and $\tau_A$ based on (5-10&11) correspond to demand level $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}(\omega)$.

Knowing $\tilde{\varphi}$, $\tau_U$ and $\tau_A$, the options' deterministic utilities $V_r$ are fully specified. This yields in turn the role modal shares $p_r$ and, combining to $\tilde{Q}$, the role flows $y_r$.

Knowing just $p_N^{-1}$ and $V_N$, a shorter path to the role deterministic utilities and flows is to derive $p_S = 1 - p_N$, $p_U = \rho . p_S$ and $p_A = (1-\rho)p_S$ ; then, $V_r = V_N + \frac{1}{\theta} \ln(p_r/p_N)$.

### 6.6/ Actors' profit functions

The respective actors' profits come from the role utilities (for $\mathcal{P}_u$) and role flows (for $\mathcal{P}_e$), complemented by the service prices for $\mathcal{P}_o$ and $\mathcal{P}_a$.

### 6.7/ On returns to scale

Returns to scale in service production can be assessed by relating total production costs $\mathcal{C}_o(y_A, y_U) + \tau_A y_A$ to the level of demand.

## 7/ Numerical experiment

The experiment is aimed to demonstrate the model outcomes depending on specific conditions, most notably the Waiting policy, the regulation regime and the size of the link trip flow.

### 7.1/ Case setting

An interurban motorway link with $L = 25$ km is considered during a time period of $H$ =1 hour.

The base costs of car holding and utilization are set up to $\underline{m} = 1$ € per trip and $\dot{m} = 0.20$ €/km.

As for time items, base stop times of $\underline{t}_{\mathrm{A}}^{S} = 3$ min and $t_{\mathrm{U}}^{S} = 2$ min are assumed, together with transaction times of $t_{\mathrm{A}}^{T} = 1$ min and $t_{\mathrm{U}}^{T} = 1.5$ min. A running speed of 70 km/h is assumed, yielding a link run time of 0.286 h.

Individual Values-of-Time are set to 15 €/h for every role and trip leg.

The modal constants are set to $\gamma_{\mathrm{N}} = 0$ € as N makes the reference option, $\gamma_{\mathrm{A}} = 1$ € conveying some reluctance, and $\gamma_{\mathrm{U}} = 3$ € to represent an increase of travel impedances on the trip legs up- and down-stream the roadway link. Logit parameter $\theta$ is set to 1 $€^{-1}$.

Marginal production costs are set to $(\dot{c}_A, \dot{c}_U) = (0.2, 0.3)$ € per trip. As the base production cost is dropped from the operator's profit function, the rest of it is called the operator's income.

Link flow congestion is neglected, and environmental costs are assigned null values, yielding social costs $\mathcal{X}_r = \dot{c}_r$. The MCPF ratio $\mu$ is set to 1.6, yielding specific factor $f_{S\mu} = 0.375$.

### 7.2/ Traffic equilibrium

The following price schedule is given: user fare of $\tau_{\mathrm{U}} = 2.8$ € per ride, agent reward with threshold $\tau_{\mathrm{A0}} = 0.5$ € per car run plus per-user bounty of $\tau_{\mathrm{A1}} = 2.5$ €. Then, the cut-off cab load separating UW and AW domains settles at $\omega^* = 0.446$. It is much sensitive to variations in inputs; on adjusting the data, it varied between $10^{-2}$ and $10^{4}$.

The variations of $y_{\mathrm{A}}$, $y_{\mathrm{U}}$, $y_{\mathrm{S}}$ and $\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}$ with respect to $\omega$ all exhibit sharp peaks around $\omega^*$ (Fig. 3): their values above 100 trips per hour are grouped in $\omega^* \pm 0.1$. The branches before $\omega^*$ correspond to UW and those after it to AW.

The parametric curves $(\tilde{Q}_{\mathcal{X}}^{\mathrm{WP}}, y_r)$ look like linear affine functions, be it only on $q \geq 20$ (Fig. 4). This is because the upper range corresponds to values of parameter $\omega$ close to $\omega^*$, so that $y_U \approx \omega^* y_A$, hence $y_S \approx (1+\omega^*) y_A$ and $y_N \approx (p_N^*/p_A^*) y_A$. Consequently, service frequency $y_A$ increases steadily with demand volume $q$; so does $y_U$, with lower slope owing to value of $\omega^*$ less than one half.

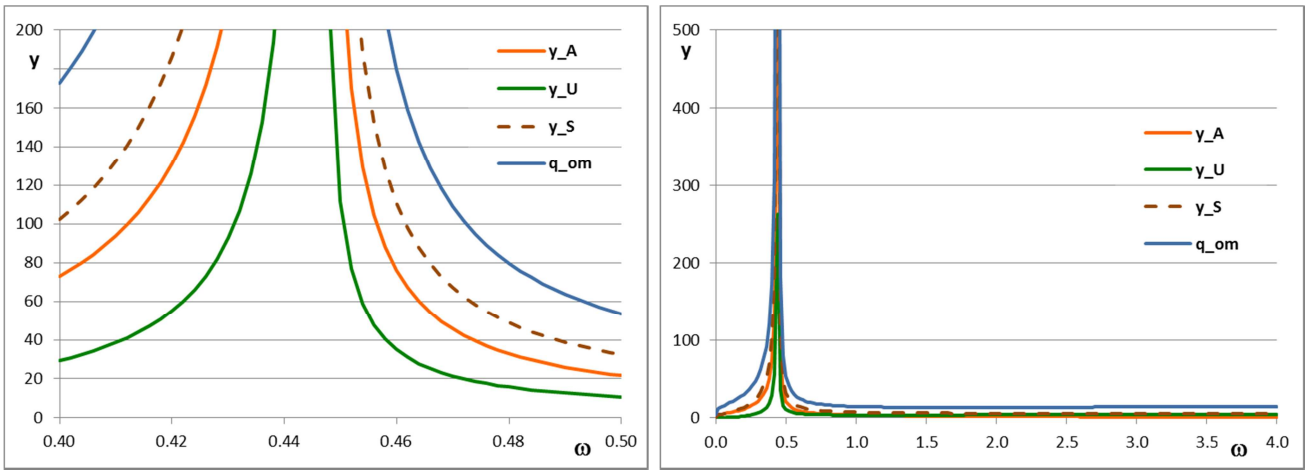


**Fig.3: Role flows and demanded volume with respect to cab load: (A) near to cut-off value, (B) wider perspective.**

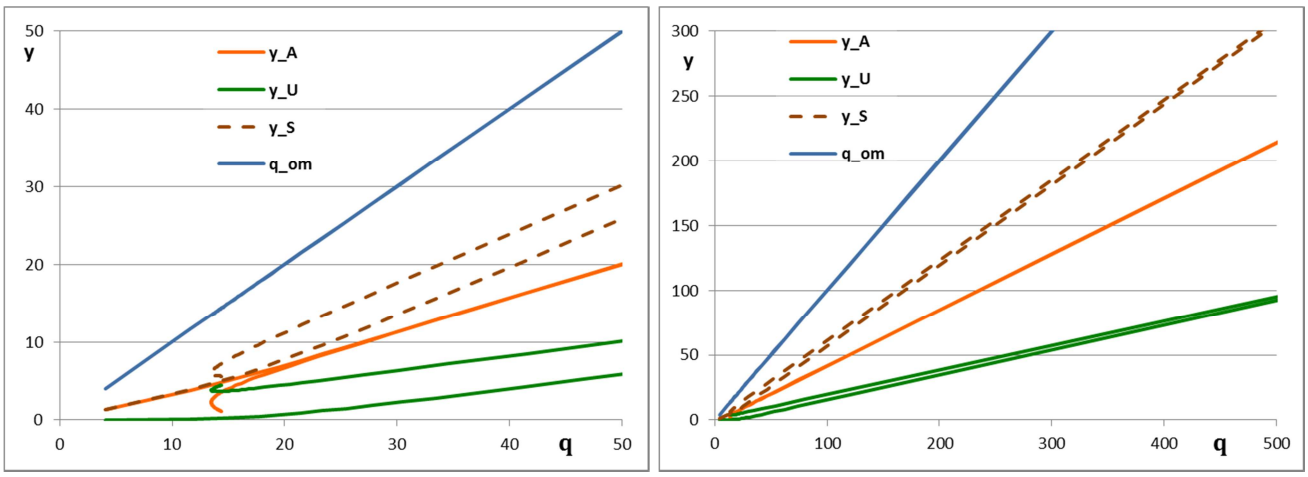


**Fig.4: Role flows as functions of link trip flows: (A) near to zero, (B) wider perspective.**

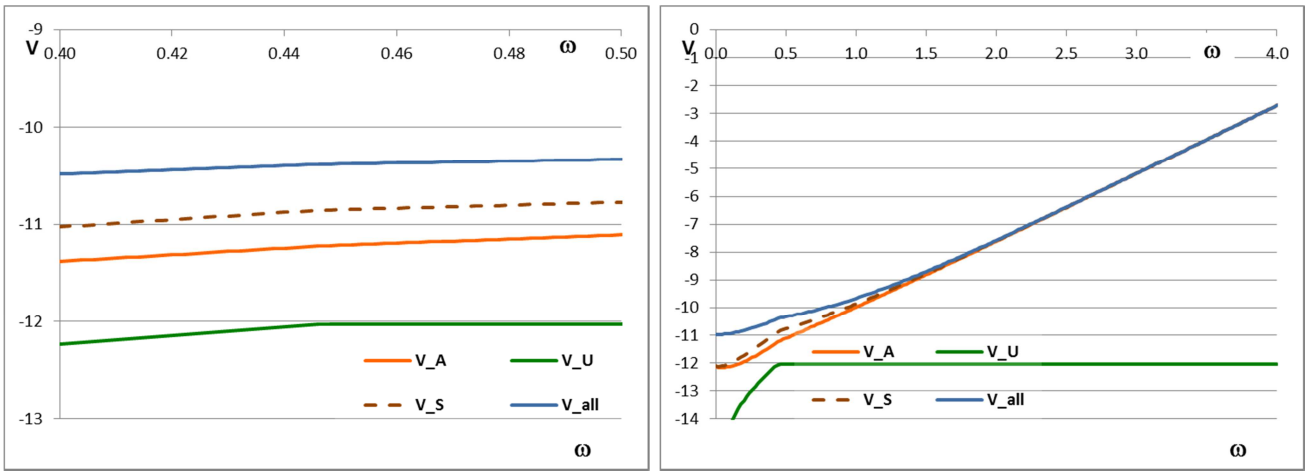


**Fig.5: Role utilities and service combined utility as functions of cab load: (A) near to cut-off value, (B) wider perspective.**

The options' deterministic utilities increase with $\omega$ (Fig. 5), mostly because of the affine relationship between it and $\tau_{\mathrm{A}}$. On the AW branch this effect dominates that of decreasing frequency yielding increasing wait time. On the UW branch, from 0 to $\omega^*$, the initial value of $V_R$ comes from that of $V_{\mathrm{N}}$ since the service share is low; as it increases, option A gets larger modal share, especially so on $\omega \geq 1$ where from $p_S \approx 1$ and $V_R \approx V_S \approx V_A$.

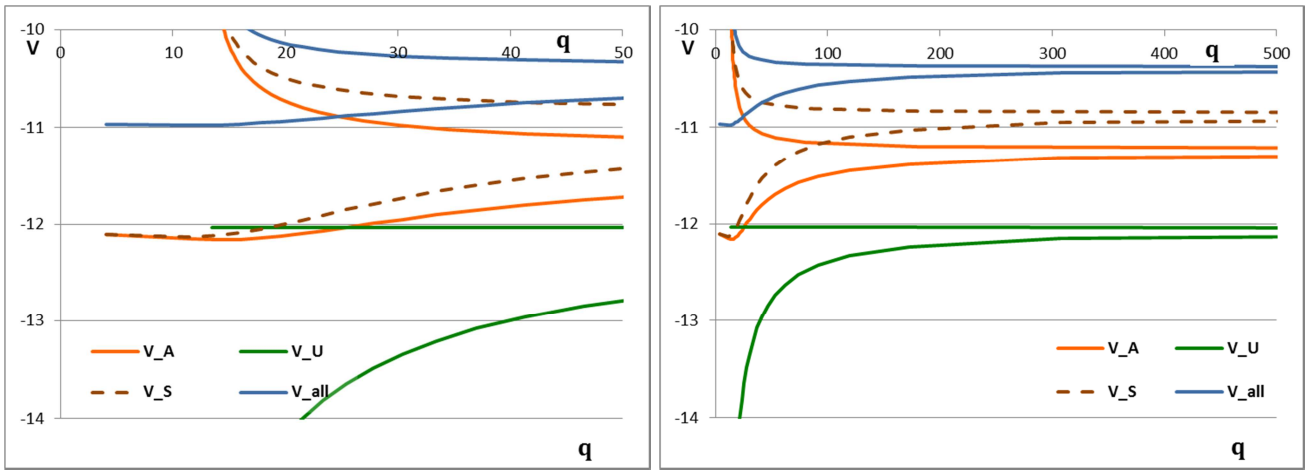


**Fig.6: Role utilities and service combined utility as functions of link trip flow: (A) small flows, (B) wider perspective.**

The utility curves with respect to $q$ are twofold (Fig. 6): the upper branches correspond to AW, with decreasing trends, and the lower ones to UW, with increasing trends. Each branch $q \mapsto V_r$ has a horizontal asymptote at its $V_r(\omega^*)$ level. For users, under UW utility increases with $q$ as higher frequency lowers wait times; under AW it keeps to a constant value. For agents, utility increases under UW yet keeps lower than the decreasing branch under AW.

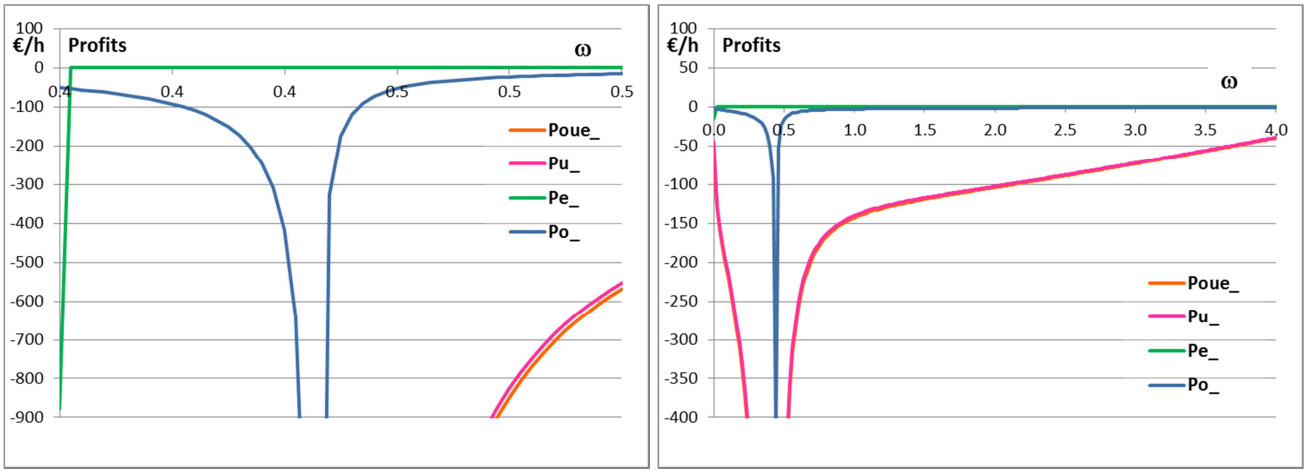


**Fig.7: Demand surplus and operator's income with respect to cab load: (A) near to cut-off value, (B) wider perspective.**

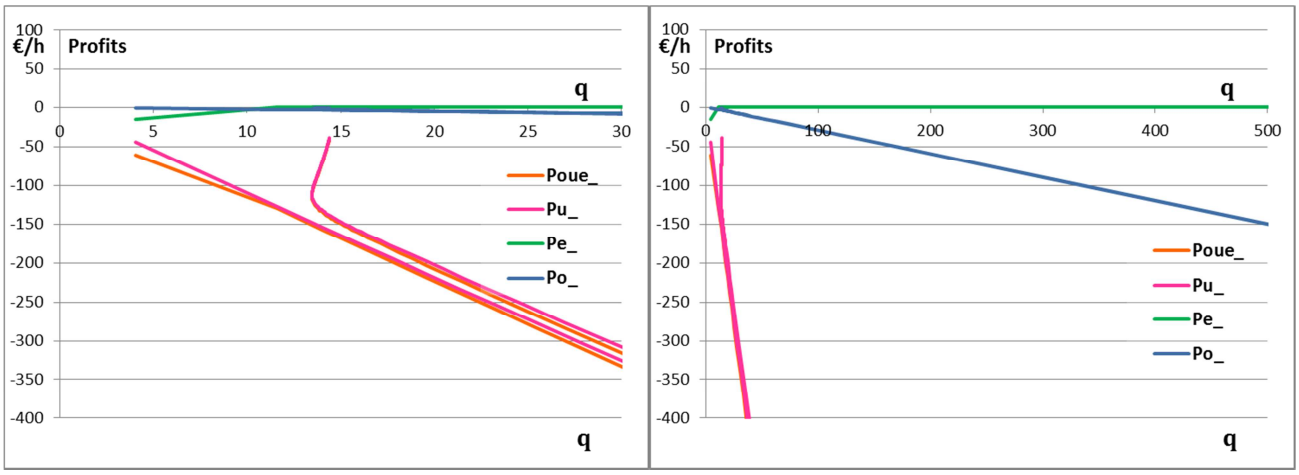


**Fig.8: Demand surplus and operator's income with respect to link flows: (A) small flows, (B) wider perspective.**

As both role utilities are higher under AW than under UW, so are the combined utilities $V_S$ and in turn $V_R$ (assuming constant $V_N$), resulting in higher user surplus function $\mathcal{P}_u$ under AW than under UW at all values of $\omega$ and $q$ (Fig. 7, 8).

With respect to $\omega$, $\mathcal{P}_u$ decreases down to $\omega^*$ then increases, mostly due to the variations of $\tilde{Q}_X^{\mathrm{WP}}$. The negative values of role utilities induce negative $V_R$: this is somewhat artificial since the trips are motivated by activities that yield positive utilities to their individuals. Adding some base utility $\underline{V}$ to all roles, making positive the role utilities, would make both $V_R$ and $\mathcal{P}_u$ positive, turning curves $(\omega, \mathcal{P}_u)$ and $(\tilde{Q}_X^{\mathrm{WP}}, \mathcal{P}_u)$ right up the abscissa axis and yielding an increasing trend according to $q$.

As for operator's profits $\mathcal{P}_o$, they decrease with $y_A$ (hence with $q$ on most of its range) because the net income per cab run, $\omega\tau_U^- - \tau_A^+ = -\tau_{\mathrm{A0}} - \dot{c}_A - \omega(\tau_{\mathrm{A1}} - \tau_{\mathrm{U}} + \dot{c}_U)$, is equal to $-2.7$ under the set-ups, which is negative at all levels of $\omega \geq 0$.

## 7.3/ Service optimization: cab load influences

From now on, the service price schedule is optimized depending on the regulation regime MO/Sμ/SO (hence factor $f$) and also on the waiting policy. Under the base set-ups, the ceiling value of optimal car loads settles at $\bar{\omega}_X = 54.5$. Like the equilibrium cut-off value, the ceiling value is highly sensitive to its inputs: over a range of instances we found values from 0.04 to 18,000. We call "realistic range" the interval $[0,4]$ of $\omega$, compared to the "full range" from 0 to 54.5, a value that corresponds to a full coach – about 15 fold the number of passengers in a full car. While unrealistic, the results for higher values of $\omega$ help us to grasp the numerical behavior of the mathematical model of service optimization, particularly so to sort out extremal points that can be either local minima or local maxima of the objective function. As a practical rule, for a given demand volume $q$, the last extremum closest to the ceiling value would be a maximum, that before it a minimum, and such alternation of maximum and minimum would go on downwards for lower values of $\omega$.

For each regulation regime, the service participation rate $p_{\mathrm{S}}$ (common to both waiting policies) increases with cab load $\omega$, up to a horizontal asymptote (Fig. 9). The SO limit share of about 86% is much higher than that of MO at about 48%. At $\omega = 4$ the participation rates reach about 90% of their limit values. Service frequency depends on waiting policy, not on regulation regime. Under UW it increases with $\omega$, reaching 10/h at $\omega$ =1 and 20/h at $\omega = 2.7$ (Fig. 10). Under AW it keeps to low values between 3 and 0.75/h, before taking off from $\omega \approx 20$, at first slowly then sharply on nearing the ceiling value. For realistic values of car load the AW frequency is very low. The declining phase from first peak of 2.85 at $\omega = 0.22$ to take-off beginning corresponds to local minima, whereas the subsequent rising phase corresponds to local maxima.

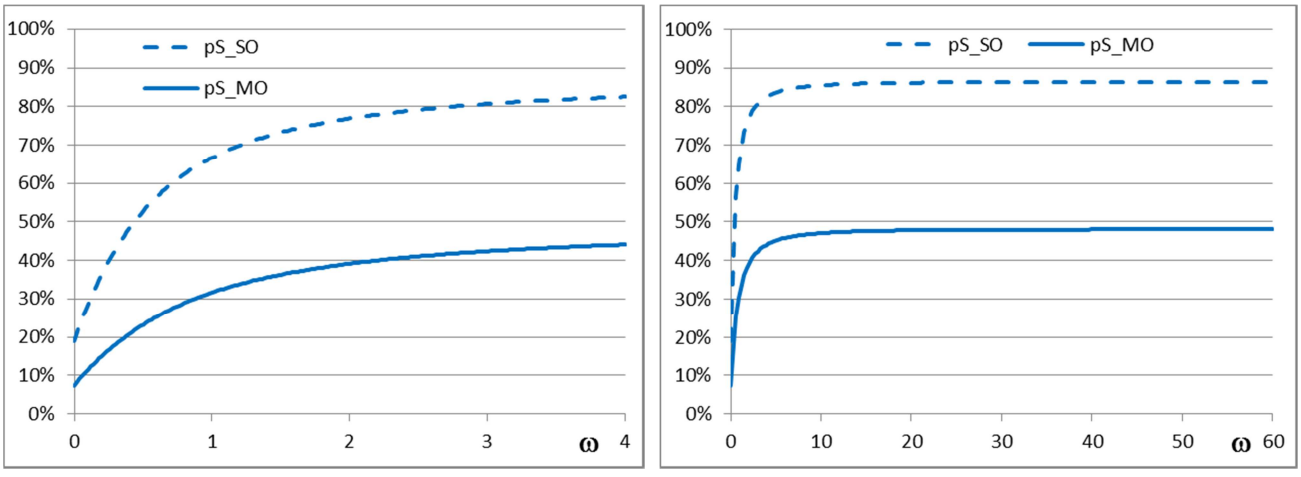


**Fig.9: Service participation rate under MO/SO with respect to cab load: (A) realistic range, (B) full range.**

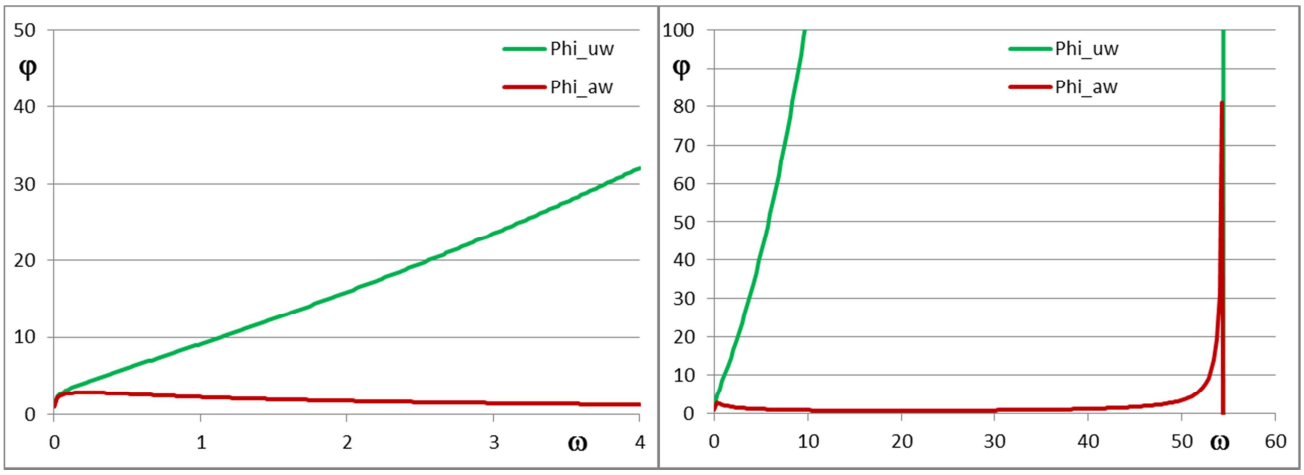


**Fig.10: Optimal frequency according to Waiting policy with respect to cab load: (A) realistic range, (B) full range.**

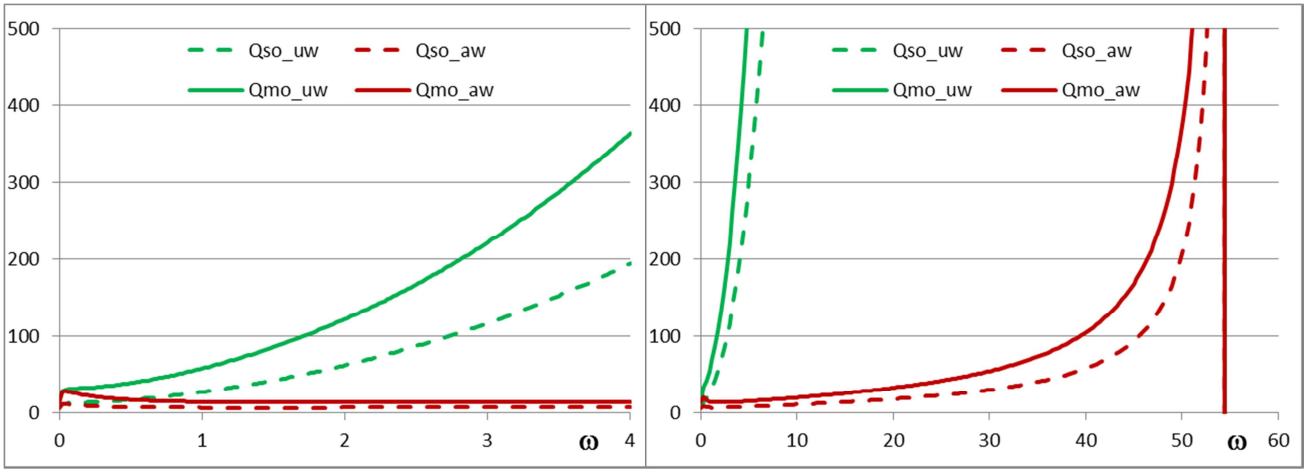


**Fig.11: Demanded volume with respect to cab load under UW / AW and MO / SO: (A) realistic range, (B) full range.**

Combining service frequency and participation rate yields four functions of demanded volume according to the pairs of waiting policy and regulation regime (Fig. 11). The UW volumes are increasing with $\omega$, whereas the AW ones mirror the variations of service frequency – yet with a much shorter declining phase, owing to the $(1+\omega)$ factor on the numerator. The increasing relations between $\omega$ and $q$ enable us to dismiss local minima in the analysis of service participation rate and frequency with respect to link flow.

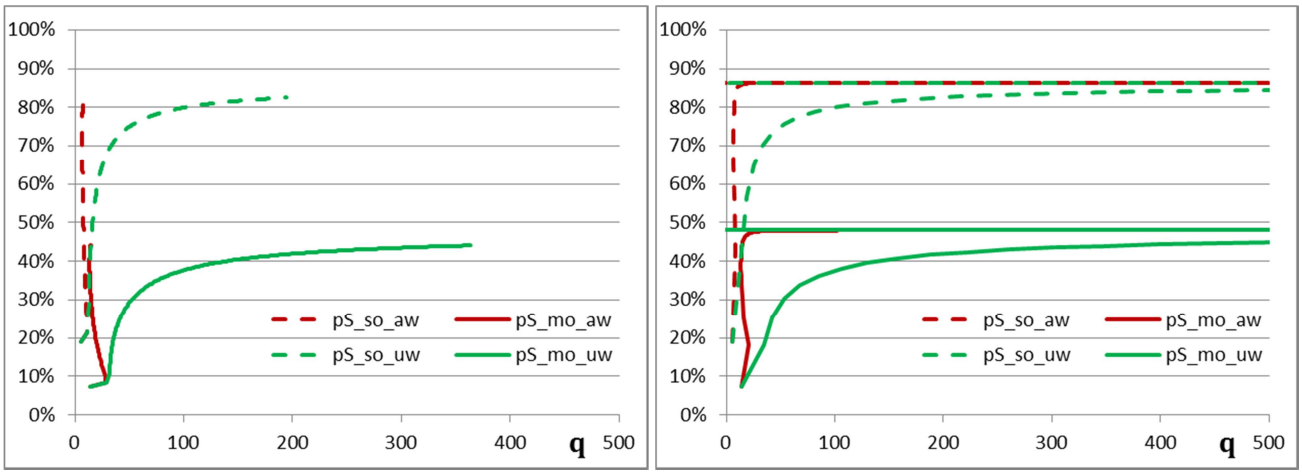


**Fig.12: Service participation rate under MO/SO with respect to demand volume: (A) realistic range, (B) full range.**

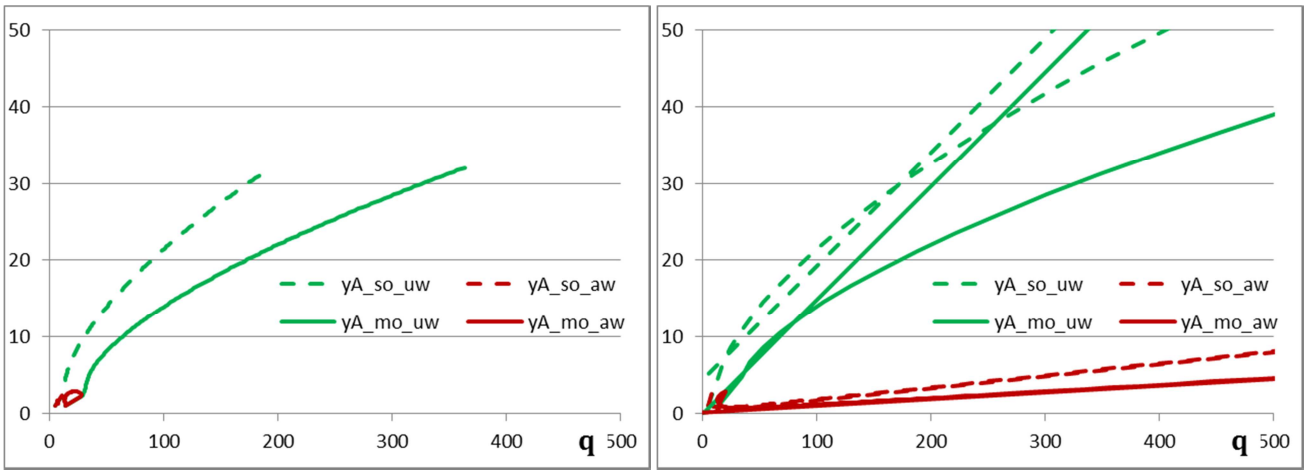


**Fig.13: Optimal frequency according to Waiting policy with respect to demand volume: (A) realistic range, (B) full range.**

With respect to $q$ the participation rate depends on waiting policy as well as on the regulation regime (Fig. 12). Under AW it quickly converges to the asymptote value. Under UW the variations are smoother: the participation rate reaches 90% of its asymptote value from link trip flow of about 200.

As for service frequency with respect to $q$ (Fig. 13), under AW the full range of $\omega$ is required to catch the steady increase; SO values are larger than their MO counterparts but remain at moderate level less than 10 runs/h at $q = 500$ trips/h. Under UW the smooth service frequency is much higher, surpassing 20 runs/h from about 200 trips/h under MO or 100 trips/h under SO (*the straight lines correspond to phantom last points in Excel*).

## 7.4/ Optimal prices

### 7.4.1 User fares

Optimized user fares $\tau_U$ with respect to cab load $\omega$ exhibit slow declines under AW, down to horizontal asymptotes at level higher for MO (at about 2.2 €) than for SO (at about 0.3 €) (Fig. 14). Once again the full range of $\omega$ is required to correctly interpret the variations of $\tau_U$ with respect to $q$, which are declining down to asymptotic levels (Fig. 15). Under UW the decrease is immediate under SO but not so quick under MO – because the initial levels at $\omega = 0$ and $q = 0$ are under the asymptote instead of above it.

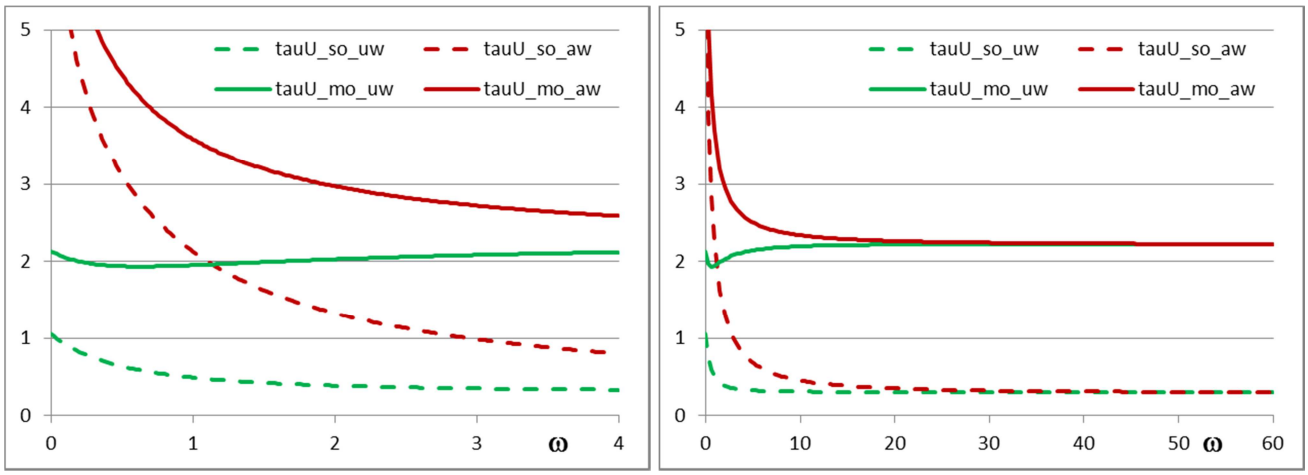


**Fig.14: User fare under UW / AW and MO / SO with respect to cab load: (A) realistic range, (B) full range.**

Given the waiting policy, the inequality between MO and SO values could be expected. Given the regulation regime, the asymptotic levels of the two waiting policies are identical, since both of them correspond to infinite frequency.

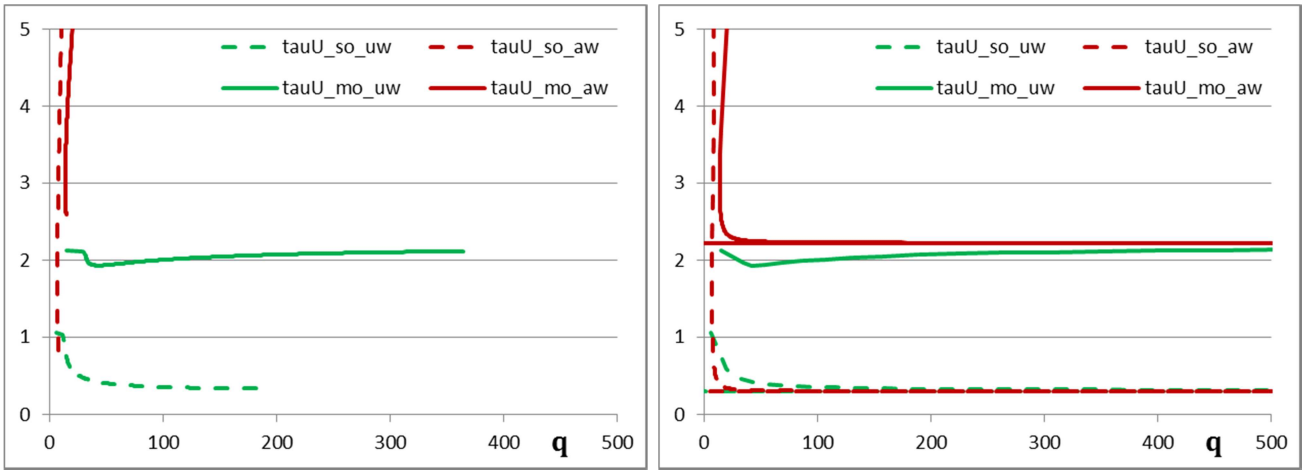


**Fig.15: User fare under UW / AW and MO / SO with respect to demand volume: (A) realistic range, (B) full range.**

### 7.4.2 Agent rewards

Optimized agent rewards $\tau_A$ exhibit more complex behaviors and take less expected values (Fig. 16 and 17 with respect to $\omega$ and $q$ respectively). Under UW the SO values are above their MO counterparts. Under AW the outcomes of the realistic range are dismissed by the full range: the optimal values of $\tau_A$ decline from very high down to fairly high for both MO and SO; SO is higher than MO for all cab loads but the curves cross for the high end of demanded volumes.

Negative values are reached under all conditions, e.g. from $\omega = 4$ under UW and MO. More significant is the persistency of low values under these conditions, as higher values could have been expected to motivate drivers. We interpret such behavior as an artifact of the logit model, induced by the linkage between the U and A options through $\omega$ in the ISS condition. Under AW the high levels of $\tau_A$ look more realistic: yet these outcomes would come along with per car passenger loads much higher than the seat capacity in typical cars.

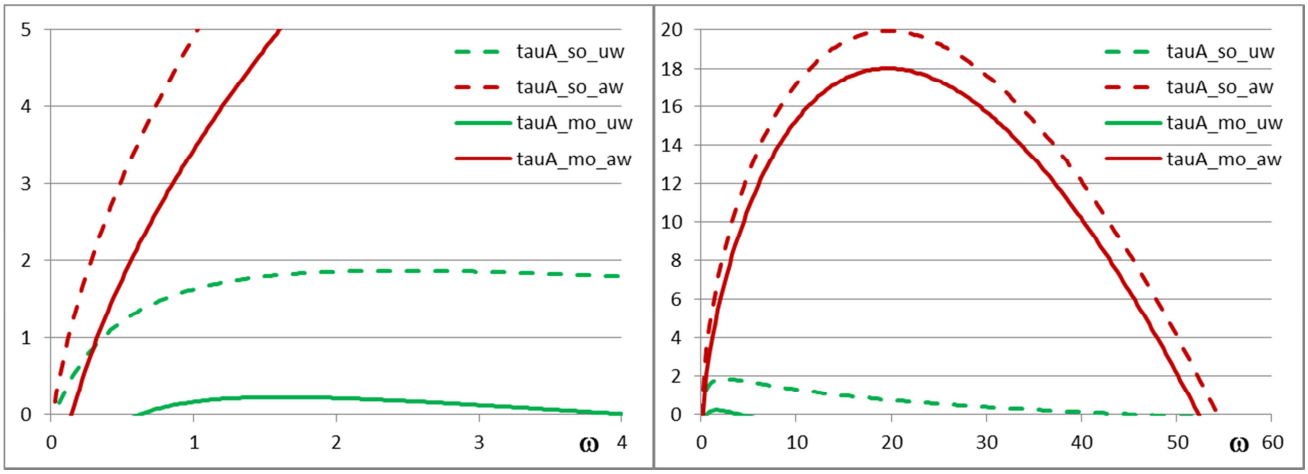


**Fig.16: Agent reward under UW / AW and MO / SO with respect to cab load: (A) realistic range, (B) full range.**

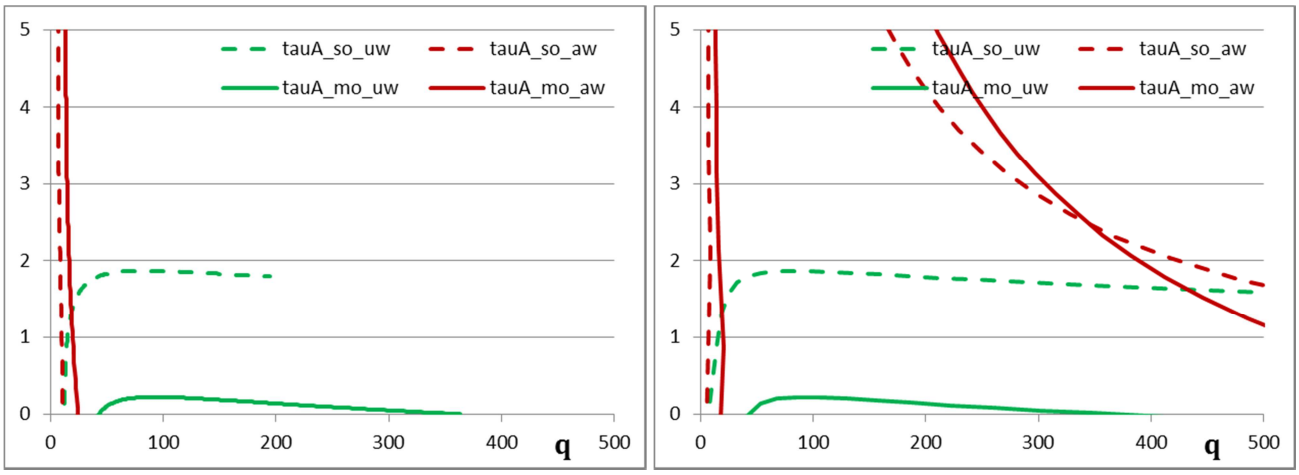


**Fig.17: Agent reward under UW / AW and MO / SO with respect to demand volume: (A) realistic range, (B) full range.**

## 7.5/ Role utilities

The case set-ups yield three reference values of modal utilities: the utility of alternative N, at level $V_N$ ≈-11.4, and also the role utilities out of wait time and prices, $V_A^{\backslash W,\tau}$ ≈-13.2 and $V_U^{\backslash W,\tau}$ ≈-9.2 (for large cab loads). In this case there is an ex-ante advantage to U over N, and an ex-ante disadvantage of A compared to N.

Taking into account the respective marginal production costs, we get a level $V_{U-A}^{\backslash W,\tau,\mathcal{X}} \approx 3$ or 4 of the base user's advantage over agents. Then, an AW policy will enlarge the utility gap, whereas an UW policy will narrow it. This explains for the large discrepancy between the agent rewards under AW versus UW: under AW the agent reward must compensate a larger utility gap than under UW.

The reference value of $V_{U-A}^{\backslash W,\tau,\mathcal{X}}$ can be identified between the $V_U$ and $V_A$ curves at the higher end of the range of cab loads under SO (Fig. 19) as well as under MO (Fig. 20), since the ceiling value is associated to infinite frequency meaning null wait times and small price gap of $\dot{c}_{U-A}$. By contrast, at $\omega = 1$ the agent and user utilities must be equal on the basis of the ISS condition (since $p_A = p_U$). In the same vein, when $\omega < 1$ it must hold that $V_A > V_U$. Such model behavior looks paradoxical compared to the naïve price schedule considered about Traffic equilibrium in §7.2: we would expect higher agent rewards under higher cab load, yielding higher agent utility.

Under all values of cab load, the utility levels of service participants are higher under SO than under MO (Fig. 20) where the service operator would capture all of the created value. Indeed, under MO the average utilities of A and U are below $V_N$: again a paradoxical outcome that stems from the basic random utility framework.

The relation between modal utilities and demand volume is exhibited in Fig. 21 and 22 for SO and MO respectively. The full range of cab load is required to identify $V_A(q)$ correctly (dismissing the local minima for small flows under AW). The user utility increases with $q$ in all cases; so does the joint utility $V_S$ at the service level and the joint utility $V_R$ at the link level. SO allows for rising $V_S$ and in turn $V_R$ well above $V_N$, while MO keeps $V_S$ below $V_N$ and achieves only slight progress of $V_R$ over $V_N$.

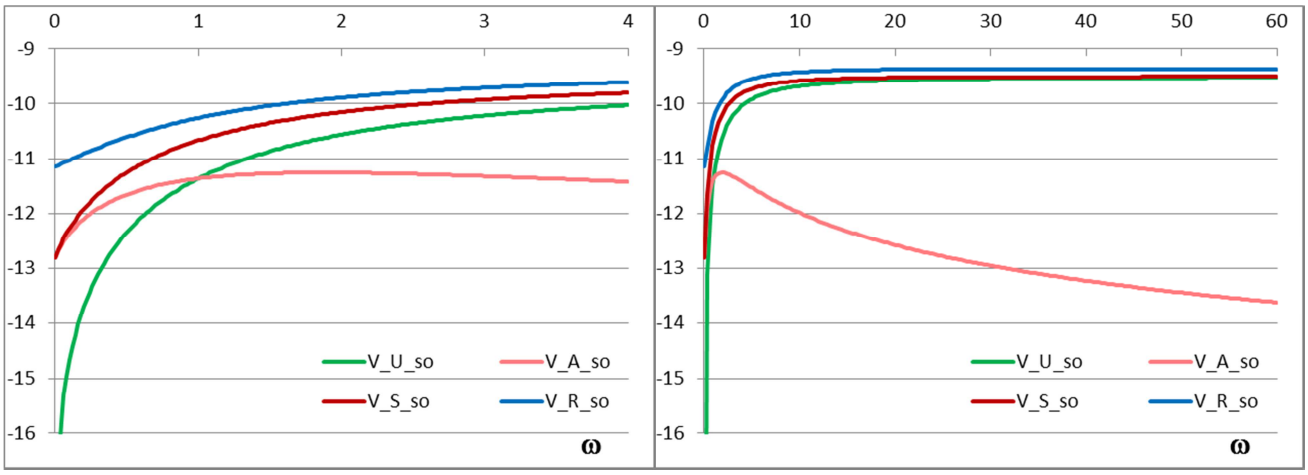


**Fig.19: Role utilities and service combined utility as functions of cab load under SO: (A) realistic range, (B) full range.**

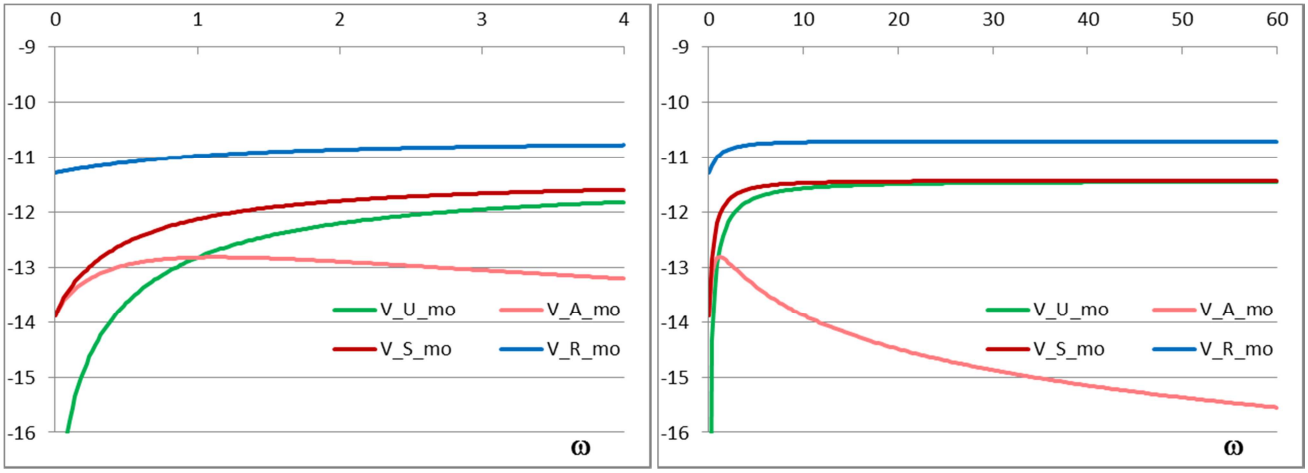


**Fig.20: Role utilities and service combined utility as functions of cab load under MO: (A) realistic range, (B) full range.**

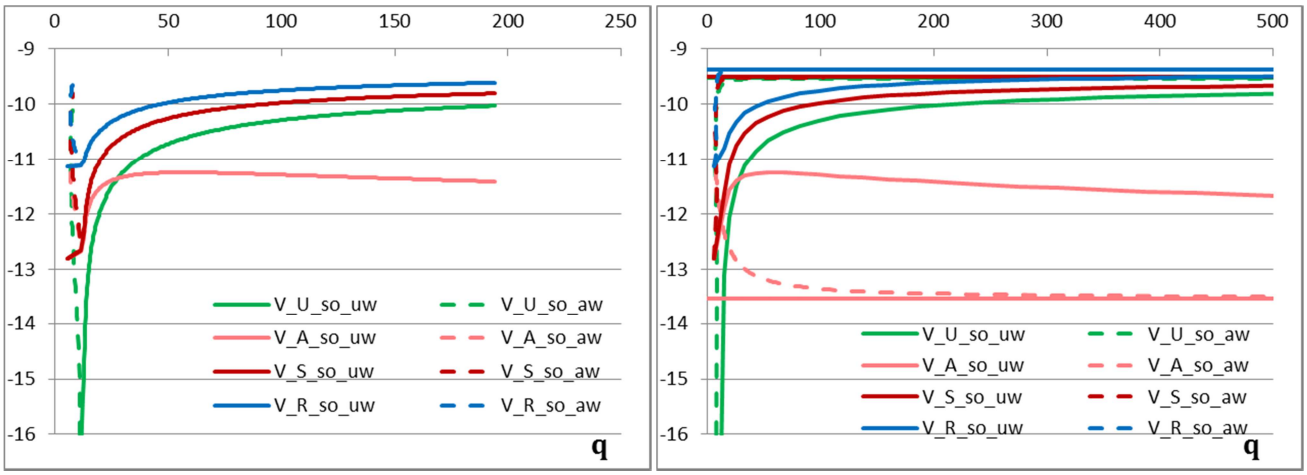


**Fig.21: Role utilities and service combined utility as functions of link flow under SO: (A) realistic range, (B) full range.**

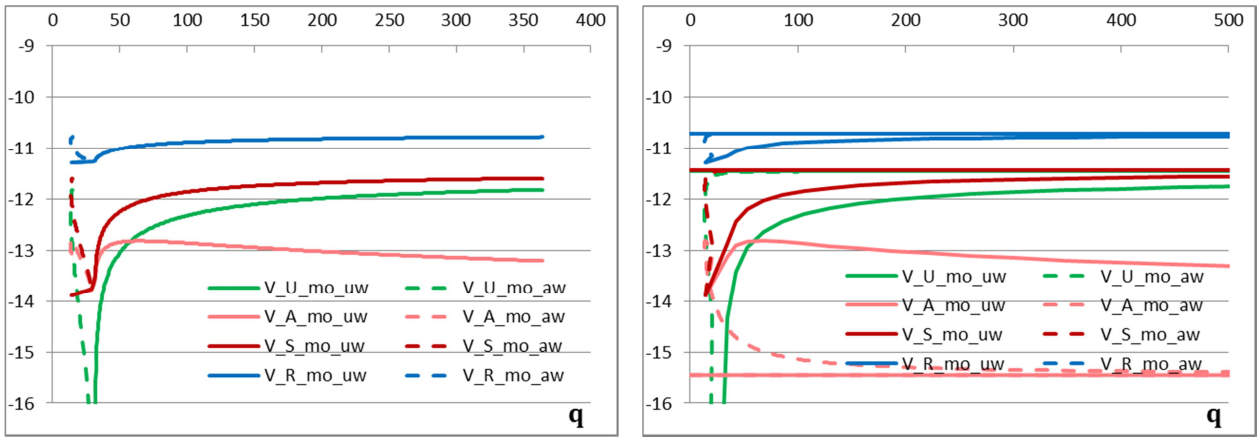


**Fig.22: Role utilities and service combined utility as functions of link flow under MO: (A) realistic range, (B) full range.**

## 7.6/ Actor's profits

Here, having neglected link traffic congestion as well as environmental effects, the social costs of service participation only consist in the marginal production costs. The consumer surplus $\mathcal{P}_u$ and the operator's profit function $\mathcal{P}_o$ (restricted to period net incomes) are exhibited with respect to cab load in Fig. 23 and 24 for SO and MO respectively.

The operator's profits are nonnegative but very small under SO and MO-AW in the realistic range of $\omega$, while that under MO-UW shows some increase starting from $\omega = 1$ and developing nicely, passing by a significant amount at $\omega = 4$. Demand surplus, as about Traffic equilibrium, should be read by reversing its sign: it increases with $\omega$ both under SO (with no difference between the waiting policies) and MO with significant advantage of UW over AW. The higher values of $\mathcal{P}_u$ with respect to $\omega$ under MO than under SO mean that the MO strategy will take better advantage of cab load.

The effects of regulation regimes and waiting policies are better understood on relating the actors' profits to the demand volume, see Fig. 25 and 26 for SO and MO respectively. At any $q$, demand surplus is higher under SO than under MO. Operator's profits are nonnegative and increasing with demand volume under both regimes, with MO levels higher than SO ones. Under SO they are higher under AW than under UW – meaning that the operator benefits more from agent exploitation.

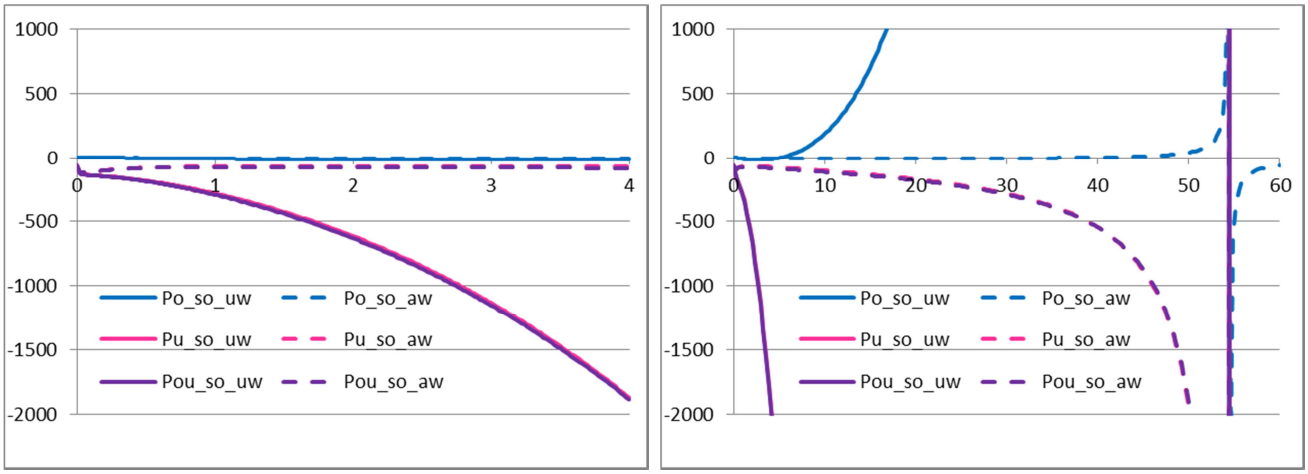


**Fig.23: Demand surplus and operator's income with respect to cab load under SO: (A) realistic range, (B) full range.**

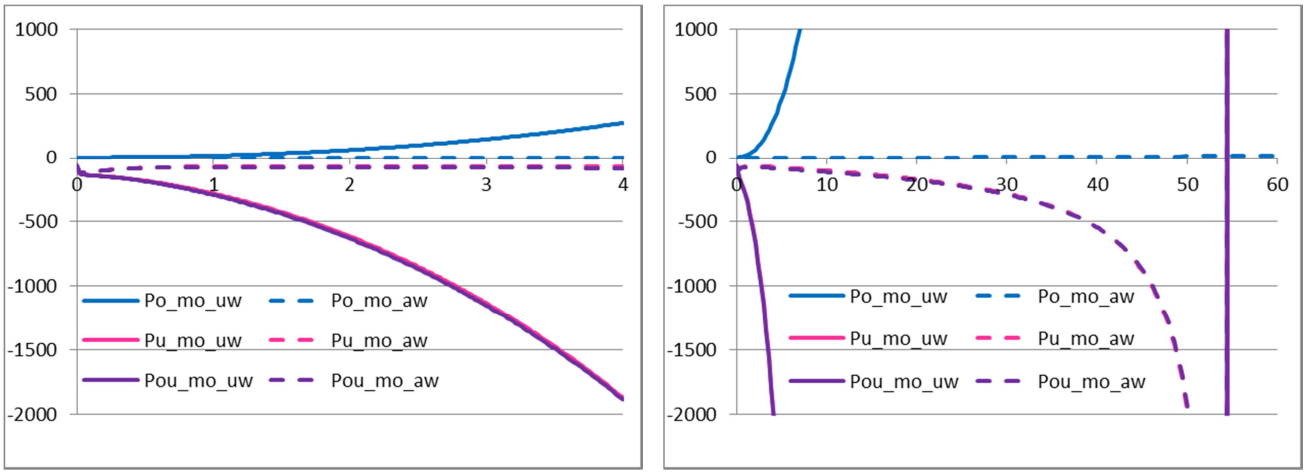


**Fig.24: Demand surplus and operator's income with respect to cab load under MO: (A) realistic range, (B) full range.**

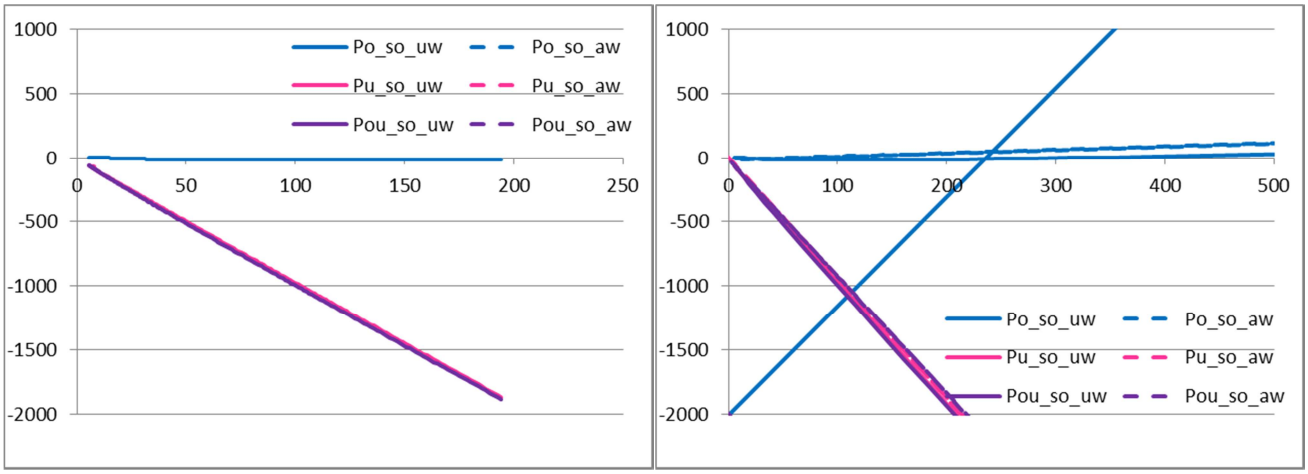


**Fig.25: Demand surplus and operator's income with respect to link trip flow under SO: (A) realistic range, (B) full range.**

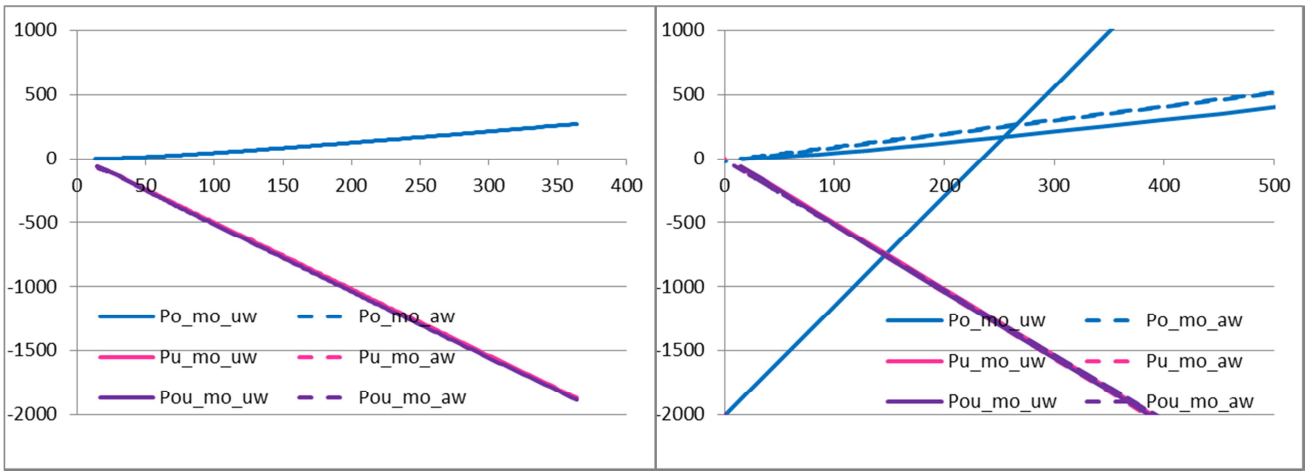


**Fig.26: Demand surplus and operator's income with respect to link trip flow under MO: (A) realistic range, (B) full range.**

# 8/ Discussion

## 8.1/ On model composition

The techno-economic model devised in this article is a five-layer representation of an LRS service as a system: from bottom up, (i) traffic laws, (ii) demand behaviors, (iii) traffic equilibrium, (iv) service management, (v) regulation.

The "traffic laws" model involves stochastic properties of large outreach (Poisson flows) to characterize cab occupation and wait times on the User and Agent sides. The consequences on the average cab load, agent stop time and agent run time are straightforward. We also set up the stage for congestible run times that may be identical or distinct between User and Neutral.

"Demand behaviors" are modeled in a classical manner – utility functions according to role, including prices and times of different kinds. We allowed for specific time valuation according to trip leg and individual role, and also for agent value of run time depending on car occupation. Only one demand segment is considered, with homogenous characteristics: demand heterogeneity deserves further research, especially about modal captivities (car-dependent people on the one hand, vs. transit-dependent people on the other hand) and also about the diversity of price-time tradeoffs (including income effects). At the individual level, a plain multinomial choice structure is represented: in the model treatment the joint utility on both sides of the service are considered, making a first step towards hierarchical choices. Yet, for transit-dependent people the choice set would be between option U and another option B (for Bus) of regular public transit.

"Traffic equilibrium" is modeled in a short-run perspective under exogenous prices. Demand involvement in roles A and U is a complex interplay, with mutual reinforcement between the role flows. There is a neat separation between the two waiting policies, on which the respective equilibrium states lie on either side of a cut-off cab load that depends on prices and times.

The "service management" layer adds up a model of service optimization with respect to service prices (user fare and agent reward). Along with the two waiting policies, different regimes of regulation are considered from selfish operator to 1$^{st}$ best system optimum passing by 2$^{nd}$ best. It is also possible to internalize side effects of the service on the environment and on link traffic. Generic formulas were provided to characterize optimal prices and interpret their composition: from social cost compensation, to specific within-service congestion and up to the value created by the service and its availability. Under optimized management, be it monopolistic or oriented to system optimum, the operator seeks to minimize its agent-related costs $\tau_A^+ . y_A$ : decreasing $\tau_A^+$ enables to keep $y_A$ to a moderately high level ensuring both satisfying frequency and productive cab occupation.

The "regulation" layer tops that of "service management": it mainly consists in exogenous specification of regulation regimes and conditions that determine the service objectives, and in the comparison of outcomes.

A sixth layer of "Territorial effects" remains to develop to address (i) accessibility benefits provided by the service to activities offered in places, especially so for transit-dependent people, (ii) population number and density that size up the accessibility benefits as well as some environmental effects, (iii) economic effects of changing money expenditure on the demand side and also the money flows through the inter-sectorial production, (iv) the service effects on the public purse – does the service constitute a high quality passenger transit at low cost to the public purse?

## 8.2/ Analytical properties

A series of analytical properties were established owing to simplifying assumptions: one roadway link, one time period under stationary regime, homogenous demand, multinomial logit discrete choice model.

On the "traffic laws" layer, the established properties are fairly general.

The "Demand behaviors" model makes a sound basis which is a first step towards more elaborate representation.

At the heart of the "Traffic equilibrium" model stands the relation between the two key service parameters (cab load, frequency) and the service flow: it is a twofold relation involving, on the demand side, the formation of role utilities from $(\omega, \varphi)$ and, on the service side, the formation of $(\omega, \varphi)$ from the role flows. We related the role choice probabilities to $(\omega, \varphi)$ and to the demand volume, in two characteristic conditions which we called ISS and OSS respectively. The IIA property of the logit model (IIA for Independence of Irrelevant Alternatives) was instrumental to cast these conditions into explicit analytical formulas.

The cut-off car load separating the respective domains of the two waiting policies corresponds to an idealized LRS service of infinite frequency – in which wait times would be null on both the U and A sides, thereby making equivalent the two waiting policies.

Concerning "Service management", we devised explicit composition formulas for the optimal prices, under generic form for a diversity of regulation types and conditions. In the respective compositions, the parts about social costs is classical; those on within-service effects and on service value creation are original contributions. Once again the IIA property of the logit model was instrumental to express the value created in simple form.

Furthermore, the analytical formulas were useful to devise a computation scheme that is efficient, be it only in the absence of link congestion. Cab load stands out as the key variable leading to optimal frequency, on the basis of the ISS condition, and to service participation rate, owing to the OSS condition. From these two outcomes we devised a "demanded volume" function relating the demand volume $q$ to cab load $\omega$: it gives a systematic method to analyze the service performances with respect to demand volume.

## 8.3/ Research directions

The model structure lends itself to sort out a bundle of research directions.

Regarding "Traffic laws", link congestion deserves to be modeled explicitly, especially so for situations where lane dedication is contemplated to prioritize high-occupancy vehicles. Such decision has high stakes, calling for specific assessment of benefits and costs. Detailed physical modeling is required to identify the respective contributions to link times of the different role flows: for instance, the user flow $y_U$ will cause specific insertions in the link flow and departures from it, while reducing the number of cars running on the link.

Concerning "Demand behaviors", we mentioned demand homogeneity and simple choice structure as first steps towards demand disaggregation and more complex choice structures. A multinomial probit makes an attractive model to represent the three roles, with more general stochastic dependencies, possibly also including distributed values-of-time to represent heterogeneous price-time trade-offs.

More elaborate Traffic and Demand sub-models obviously will lead to more complex Traffic equilibrium. The $(\omega, \varphi)$ pair of service characteristics, getting involved in broader and deeper nests of interactions, will be all the more important as key variables of the system state.

Demand disaggregation will also make service management more complex, by including several classes of demand surplus, possibly also class-related prices or waiting conditions.

Research directions also include multimodal competition with other passenger transit modes (notably regular transit lines or demand-responsive) or possibly multimodal cooperation e.g. by sharing costs, incomes and strategic objectives.

Lastly, along the economic network effects that typically arise in bi-sided platforms, comes the issue of LRS service development in network form, along multi-station lines, with transfer points and hubs.

Such spatial development is suitable to make the travel option U more widely available to people in territories, as well as to make role A more productive to individual trip-makers. An enlarged spatial form will also make multimodal competition and cooperation more complex and, potentially, more productive.

## 8.4/ Some hints about service design

While individual usages and lifestyles are often assumed to change in little time, compared to infrastructural investments, in practice it may take a number of years to change routines and habits. In France and other countries the development of ride-sharing services has been boosted by the Blablacar platform to plan interurban trips in advance (Astier et al., 2023), owing to large money gains to both the U and A sides on the basis of long distances and taking profit of the efficient motorway network. Suburban ride-sharing, despite many attempts and partial achievements in recent years, still keeps to a minor modal share. Will line ride-sharing over intermediate distance ranges have an intermediate fate? In fact it may capitalize on the combination of significant distances and trip frequency in the frame of daily mobility. This gives some ground to modeling average role performances (in wait times stop times, cab occupation, agent reward) in the role utility functions. Yet, the high service participation rates as predicted in the numerical experiment stand in sharp contrast to the low rates experienced in early implementations. Our model is a long-term one, leaving open the issue of phase transition from the prior situation dominated by private car usage. Based on the ECOV experiences, although agent enrolment requires specific action, the main challenge is to get riders on board. Special circumstances such as railway strikes, or Sustainable Mobility weeks, are likely to demonstrate LRS convenience at the places where such lines are implemented. However, public action from local authorities is more likely to drive changes – be it by facilitating service access or by subsidizing service operations and maybe also service participation. The optimal prices formulated here may constitute a basis for discussions between local authorities and service operators about subsidy levels.

# 9/ Conclusion

We devised a techno-economic model of a Line Ride Sharing service on a roadway link. The model has a five-layer structure, representing (i) traffic laws, (ii) demand behaviors, (iii) traffic equilibrium, (iv) service management, (v) system regulation. It involves cab frequency and above all cab load as key variables that determine the utilities of Agent and User roles at the trip level, hence the modal share between these roles and non-commitment (option N) and in turn the role trip flows. Beside the influences of role flows on link run times, the model captures specific effects: the mutual reinforcement between cab load and frequency, as well as within-service congestion. Regarding service management subjected to the classical set of regulation regimes, a generic formulation yields explicit formulas for the optimal action levers, notably User fare and Agent reward. The value created by the service to the individuals who can avail themselves of it is captured in a specific indicator: the operator's share of it depends on the regulation regime. Two traffic equilibrium conditions called "Inner service split" and "Outer service split" combine with optimality conditions to relate the service frequency and participation rate to the cab load, and in turn to the demand volume.

In the absence of link congestion, there exists a ceiling cab load that bounds optimal values from above; the "User waits" policy is more productive than "Agent waits" for medium and high link flows. Optimal states typically involve moderate Agent flows (some tens per hour) and large User flows.

Beside the research directions mentioned in the Discussion, the LRS orientation to high quality for the trip legs of in-vehicle running (car comfort) and wait for boarding (high frequency) may be extended to the Access legs of potential Users.

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# Appendix A: Car occupation & time items depending on waiting policy

It is postulated that the flows of Users and Agents are independent Poisson processes with respective time rates $\psi_r \equiv y_r/H$. On average over $H$, the expected number of individuals with role $r$ is then $\psi_r.H = y_r$. In each flow, the individuals arrive successively, with inter-arrival times between two successive individuals that are independent and distributed $\mathrm{EXP}(\psi_r)$: the average inter-arrival time is thus $1/\psi_r = H/y_r$.

## A1/ Number of users per agent car run

Let us consider any agent conditionally to time $\theta$ taken either [AW] from his instant of arrival to that of the next agent, [UW] from the arrival time of the previous agent to his own one. The users taking the car run are those arriving during time lapse $\theta$. Conditionally to $\theta$, the number $\nu_\theta$ of user arrivals is an RV distributed Poisson with parameter $\psi_\mathrm{U}\theta$, yielding

$$\Pr\{\nu_\theta = n\} = e^{-\psi_\mathrm{U}\theta}.\frac{(\psi_\mathrm{U}\theta)^n}{n!}, \forall n \in \mathbb{N} \qquad \text{(A.1)}$$

Deconditioning over $\theta$ which is distributed $\mathrm{EXP}(\psi_A)$, and denoting $\psi_{\mathrm{U+A}} \equiv \psi_\mathrm{U} + \psi_\mathrm{A}$,

$$\Pr\{\nu = n\} = \int_0^\infty \Pr\{\nu_\theta = n\}\, d\,\Pr\{\theta\}$$
$$= \int_0^\infty e^{-\psi_{\mathrm{U}}\theta}.\frac{(\psi_{\mathrm{U}}\theta)^n}{n!}e^{-\psi_{\mathrm{A}}\theta}\psi_{\mathrm{A}}\,d\theta$$
$$= \left(\frac{\psi_{\mathrm{U}}}{\psi_{\mathrm{U+A}}}\right)^n \frac{\psi_{\mathrm{A}}}{\psi_{\mathrm{U+A}}}\int_0^\infty \frac{(\psi_{\mathrm{U+A}}\theta)^n}{n!}e^{-\psi_{\mathrm{U+A}}\theta}\psi_{\mathrm{U+A}}d\theta$$

The latter integral is easily integrated by parts, yielding say $u_n$ such that

$$u_n \equiv \int_0^\infty \frac{(\psi\theta)^n}{n!}e^{-\psi\theta}\psi d\theta = \left[-\frac{(\psi\theta)^n}{n!}e^{-\psi\theta}\right]_0^\infty + \int_0^\infty \frac{(\psi\theta)^{n-1}}{(n-1)!}e^{-\psi\theta}\psi d\theta = u_{n-1}$$

So that $u_n = u_0 = 1$. Thus,

$$\Pr\{\nu = n\} = \left(\frac{\psi_{\mathrm{U}}}{\psi_{\mathrm{U+A}}}\right)^n \frac{\psi_{\mathrm{A}}}{\psi_{\mathrm{U+A}}} = (1-\rho).\rho^n,\ \forall n \in \mathbb{N} \tag{A.2}$$

We recognize a geometric distribution with parameter $\rho \equiv \frac{\psi_{\mathrm{U}}}{\psi_{\mathrm{U+A}}}$. Put in words, the number of users per agent is an RV that only depends on the ratio of their respective flows. The average value of such distribution is

$$\mathrm{E}[\nu] = \frac{\rho}{(1-\rho)} = \frac{\psi_{\mathrm{U}}}{\psi_{\mathrm{A}}}. \tag{A.3}$$

This implies the intuitive property that

$$\mathrm{E}[\nu] = \frac{y_{\mathrm{U}}}{y_{\mathrm{A}}}.$$

Denoting $\omega \equiv \frac{y_{\mathrm{U}}}{y_{\mathrm{A}}}$, then

$$\rho = \frac{y_{\mathrm{U}}}{y_{\mathrm{U+A}}} = \frac{\omega}{1+\omega}. \tag{A.4}$$

N.B. In the Poisson stochastic model, $\omega$ is just the average number of users per agent car. Its effect on the agent stop time is just to multiply the base stop time $\underline{t}_{\mathrm{A}}^{\mathrm{S}}$ by $\rho$ that is a probability. In case of several users boarding the car, we can expect them to board simultaneously under UW, or under AW maybe to board in turn – yet, if so then the stop time associated to all of them save the last one are embedded in the agent wait time.

Under either waiting policy, conditionally to $\theta$ the probability of No users riding in the agent car is $\Pr\{\nu_\theta = 0\}$, i.e. $\exp(-\psi_{\mathrm{U}}\theta)$. Deconditioning over $\theta$, we get

$$\Pr\{\nu = 0\} = \frac{\psi_{\mathrm{A}}}{\psi_{\mathrm{U+A}}} = 1 - \rho = \frac{1}{1+\omega}. \tag{A.5}$$

This outcome can be obtained more straightforwardly by considering the respective inter-arrival times $\theta_{\mathrm{U}}$ and $\theta_{\mathrm{A}}$ of users and agents: the probability of an agent not getting any user is the probability that there would be no user arriving during the inter-arrival time between himself and the next agent (AW) or between himself and the previous agent (UW), so that

$$\Pr\{\nu = 0\} = \Pr\{\theta_{\mathrm{A}} \le \theta_{\mathrm{U}}\} = \Pr\{\theta_{\mathrm{A}} = \min\{\theta_{\mathrm{A}}, \theta_{\mathrm{U}}\}\} = \frac{\psi_{\mathrm{A}}}{\psi_{\mathrm{A}}+\psi_{\mathrm{U}}} = \frac{1}{1+\omega} = 1 - \rho. \tag{A.6}$$

Thus, to the agent the probability of having to stop is

$$\Pr\{\nu > 0\} = 1 - \Pr\{\nu = 0\} = \rho. \tag{A.7}$$

It yields an average Agent stop time of

$$\mathrm{E}\left[t_{\mathrm{A}}^{\mathrm{S}}\right] = \rho\underline{t}_{\mathrm{A}}^{\mathrm{S}}. \tag{A.8}$$

## A2/ Under User Waits: User wait time

The seat capacity in cars is neglected, so that every user can board any car as soon as it arrives after the user own time of arrival at the origin station. Whatever the user time of arrival, from it the wait time up to an agent arrival is distributed $\mathrm{EXP}(\psi_\mathrm{A})$, from the memory-less property of the exponential distribution of agent inter-arrival times. Thus

$$\mathrm{E}[t_U^W] = \frac{1}{\psi_\mathrm{A}} = \frac{H}{y_\mathrm{A}} = \frac{H}{\varphi}. \tag{A.9}$$

As for Agents, there is no wait time on their side. In practice, an exception would be the arrival of additional users during the stop time of an agent dwelling for a first user to board in it.

## A3/ Under Agent Waits: Agent wait time

Conditionally to agent inter-arrival time $\theta$, letting $\{\nu_\theta = n\}$, from the properties of the Poisson process of user arrivals, the $n$ instants of user arrivals are independent RVs denoted $t_i$ with identical uniform distribution on $[0, \theta]$, hence with CDF as follows:

$$\mathrm{F}_\theta(z) \equiv \Pr\{t_i \le z \mid \theta\} = \min(1, \tfrac{z}{\theta}),\ \forall z \in \mathbb{R}^+.$$

The agent wait time $t_{W|\theta,n}$ conditionally to $\theta$ and $n$ is the maximum of the $n$ RVs $t_i$:

$$t_{\theta,n} \equiv \max\{t_i : i \in \{1,..n\}\}$$

Thus $\{t_{W|\theta,n} \le z \mid \theta\} = \cap_{i=1}^n \{t_i \le z \mid \theta\}$. From the independence between the $t_i$, the CDF of $t_{W|\theta,n}$ is

$\Pr\{t_{W|\theta,n} \le z \mid \theta\} = \prod_{i=1}^n \Pr\{t_i \le z \mid \theta\} = (F_{z\theta})^n$, wherein $F_{z\theta} \equiv \mathrm{F}_\theta(z)$.

Deconditioning over $n$,

$$\begin{aligned}
&\Pr\{t_{W|\theta} \le z \mid \theta\} = \sum_{n\ge 0} \Pr\{N_\theta = n\} \Pr\{t_{W|\theta,n} \le z \mid \theta, n\} \\
&= e^{-\psi_\mathrm{U}\theta} \sum_{n\ge 0} \frac{(\psi_\mathrm{U}\theta F_{z\theta})^n}{n!} \\
&= \exp(-\psi_\mathrm{U}\theta(1 - F_{z\theta})) = \exp(-\psi_\mathrm{U}(\theta - z)^+)
\end{aligned}$$

Deconditioning now over $\theta$,

$$\begin{aligned}
&\Pr\{t_W \le z\} = \int_0^\infty \Pr\{t_{W|\theta} \le z \mid \theta\}\, d\Pr\{\theta\} \\
&= \int_0^\infty \exp(-\psi_\mathrm{U}(\theta - z)^+)\, e^{-\psi_\mathrm{A}\theta} \psi_\mathrm{A} d\theta \\
&= \int_0^z e^{-\psi_\mathrm{A}\theta} \psi_\mathrm{A} d\theta + e^{\psi_\mathrm{U} z} \int_z^\infty e^{-\psi_{\mathrm{U+A}}\theta} \psi_\mathrm{A} d\theta \\
&= 1 - e^{-\psi_\mathrm{A} z} + e^{\psi_\mathrm{U} z} e^{-\psi_{\mathrm{U+A}} z} \frac{\psi_\mathrm{A}}{\psi_{\mathrm{U+A}}} \\
&= 1 - e^{-\psi_\mathrm{A} z} \rho \\
&= (1 - \rho) + \rho(1 - e^{-\psi_\mathrm{A} z})
\end{aligned}$$

This CDF is that of the probabilistic mixture of a Dirac variable at 0 and a variable $\mathrm{EXP}(\psi_\mathrm{A})$, with respective probabilities $1 - \rho$ and $\rho$, i.e., of the probabilities of having null or strictly positive number of users on board the Agent's car. Thus, under AW,

$$\mathrm{E}[t_A^W] = (1 - \rho).0 + \rho \frac{1}{\psi_\mathrm{A}} = \rho \frac{H}{y_\mathrm{A}} = \rho \frac{H}{\varphi}. \tag{A.10}$$

N.B. To satisfy the distributional assumptions, it requires that (i) user flow to be not so large compared to agent flow, so that the seat capacity in the car is mostly sufficient, (ii) the service coordination collects information about passage times from both the users and the agents, so that

the platform can know $\theta$ and $n$ "in advance" compared to the individuals: thus, the platform is able to tell to each agent whether to stop or not, and how many users will board the car.

# Appendix B: Traffic equilibrium

## B1/ System state and state vector

A line ride-sharing service as a system has a system state determined by the values of its endogenous variables. These include (cf. Figure 1):

(i) people trip flows $(y_r)$ according to service roles $r \in \mathfrak{R}$,

(ii) service quality variables $(\varphi, \omega, \rho)$,

(iii) the money and time items of the roles, denoted $t_r^\ell$ and $m_r^\ell$ by leg $\ell \in \{\mathrm{R}, \mathrm{S}, \mathrm{T}, \mathrm{W}\}$, depending on waiting policy $\# \in \{\mathrm{AW}, \mathrm{UW}\}$ and on the flow policy,

(iv) the deterministic utility functions $V_r^\#$ according to role $r$ and waiting policy #,

(v) the role probabilities $\mathrm{p}_r^\#$ among the $q$ individuals.

These probabilities sum up the preferences of the individuals, by assigning each of them to the optimal role according to deterministic features together with random circumstances.

## B2/ Structure of influences and basic state vector

The endogenous variables are related by the following set of conditions:

a/ $y_r \geq 0$ and $\sum_{r\in\mathfrak{R}} y_r = q$.

b/ quality formation: $\varphi = y_A$ and $\omega$ such that $y_U = \omega . y_A$, denoted as $(\varphi, \omega) = (\hat{\varphi}, \hat{\omega})(\boldsymbol{y}_\mathfrak{R})$.

c/ traffic laws yielding link run times according to roles and depending on flow policy.

d/ time and money item composition on the basis of $(\varphi, \omega, v)$ and according to waiting policy.

e/ modal utility function composition from money and time items.

f/ optimal choices: choice probabilities $\mathrm{p}_r^\#$ stem from random utility functions.

g/ flow assignment $y_r^\# = q . \mathrm{p}_r^\#, \forall r \in \mathfrak{R}$.

The flow vector $[y_r : r \in \mathfrak{R}]$ can make a basic state vector from which all of the other variables are derived. Such flow vector constitutes an equilibrium state if it satisfies that:

$$\forall r \in \mathfrak{R}: \; y_r = y_r^\#(\boldsymbol{y}_\mathfrak{R}) \tag{B.1}$$

It is a condition of quantitative balance (market clearing) between supply $(y_r)$ and demand $(y_r^\#)$.

Yet, as basic state vector it is more convenient to consider the $(\varphi, \omega)$ pair, from which stem $(\hat{y}_A, \hat{y}_U)$, service flow $\hat{y}_S = \hat{y}_A + \hat{y}_U$, neutral flow $\hat{y}_N = q - \hat{y}_S$, hence $v$ and the rest of endogenous variables. This basic state vector is feasible if $\varphi \geq 0, \omega \geq 0$ and $\varphi \leq q$.

A feasible vector $(\varphi, \omega)$ is an equilibrium state iff

$$\forall r \in \mathfrak{R}: \hat{y}_r(\varphi, \omega) = y_r^\#(\varphi, \omega) \tag{B.2}$$

## B3/ Fixed-point problem and iterative solution scheme

FPP in $(\varphi, \omega)$. A basic state vector $(\varphi, \omega)$ is in equilibrium if it is a fixed point for the mapping $(\varphi, \omega) \mapsto (\varphi', \omega')$ such that $\varphi' = \mathrm{y}_\mathrm{A}^\#(\varphi, \omega)$ and $\omega'$ satisfies that $\mathrm{y}_\mathrm{U}^\#(\varphi, \omega) = \omega' \mathrm{y}_\mathrm{A}^\#(\varphi, \omega)$.

FPP in $(y_A, y_U)$. A basic state vector $(y_A, y_U)$ is in equilibrium if it is a fixed point for the chained mapping $(y_A, y_U) \mapsto (\varphi, \omega)$ such that $\varphi = y_A$ and $\omega$ satisfies that $y_U = \omega y_A$, followed by $(\varphi, \omega) \mapsto (y_A^{\#}, y_U^{\#})$.

Iterative solution scheme. The basic strategy to solve an FPP is to progressively adapt a current state vector by combining it with its image through the mapping so as to obtain the next value. At step $k$, current state $(\varphi_k, \omega_k)$ induces image $(\hat{\varphi}_k, \hat{\omega}_k)$ : then the next step can be obtained as

$$\varphi_{k+1} \leftarrow \mathrm{Fn}[\varphi_k, \hat{\varphi}_k],$$
$$\omega_{k+1} \leftarrow \mathrm{Fn}[\omega_k, \hat{\omega}_k].$$

For instance, a convex combination scheme with step sizes $\zeta_k$ that decrease to zero:

$$\varphi_{k+1} \leftarrow \varphi_k + \zeta_k(\hat{\varphi}_k - \varphi_k) = (1 - \zeta_k)\varphi_k + \zeta_k \hat{\varphi}_k$$
$$\omega_{k+1} \leftarrow \omega_k + \zeta_k(\hat{\omega}_k - \omega_k) = (1 - \zeta_k)\omega_k + \zeta_k \hat{\omega}_k.$$

A similar strategy on the $(y_A, y_U)$ flow vector constitutes an equilibration algorithm well-known in network traffic assignment. The convex combination scheme is an instance of the "Method of Successive Averages". Yet the graphical scheme in Section 3 is a more straightforward strategy.

# C/ Equilibrium properties

## C.1/ Mathematical property of existence

On the demand side, model functions leading from $(\varphi, \omega)$ to $y_r^{\#}$ are continuous because they combine continuous elementary functions (e.g., deterministic utility $V_r$) in simple, continuous ways (e.g. $y_r^{\#}$ as a ratio of strictly positive functions $\exp(\theta . V_r)$ that are continuous owing to the continuity of the exponential function and that of $V_r$).

On the supply side, the derivation of $\hat{y}_r$ from $(\varphi, \omega)$ is continuous: $\hat{y}_A = \varphi$ is continuous everywhere, so are $\hat{y}_U = \varphi . \omega$ and $\hat{y}_N = q - \hat{y}_A - \hat{y}_U$.

Now, for any small parameter $\eta > 0$, function $(\hat{\varphi}, \hat{\omega}) = (y_A, \frac{y_U}{y_A})$ is continuous on the parameterized set $\Delta_{\boldsymbol{y}}^{\eta} \equiv \{\boldsymbol{y}_{\mathfrak{R}} \in \Delta_{\boldsymbol{y}} : y_A \geq \eta\}$. Defining $\hat{\varphi}_\eta \equiv \max\{\eta, \hat{\varphi}\}$ and $\hat{\omega}_\eta = y_U / \max\{\eta, y_A\}$, the function $\boldsymbol{y}_{\mathfrak{R}} \mapsto \boldsymbol{y}_{\mathfrak{R}}^{\#} \circ (\hat{\varphi}_\eta, \hat{\omega}_\eta)(\boldsymbol{y}_{\mathfrak{R}})$ is continuous on $\Delta_{\boldsymbol{y}}^{\eta}$ where it takes its values, hence, owing to Brouwer's theorem, it admits a fixed point there which we denote as $\boldsymbol{y}_\eta^*$.

Furthermore, any sequence of $(\eta_n > 0 : n \in \mathbb{N})$ such that $\eta_n \to 0$ for $n \to \infty$ yields an associated sequence of points $\boldsymbol{y}_{\eta(n)}^*$ in the overall feasible domain $\Delta_{\boldsymbol{y}}$. As it is a compact set, the sequence $(\boldsymbol{y}_{\eta(n)}^* : n \in \mathbb{N})$ admits at least one accumulation point, denoted $\boldsymbol{y}^*$. If $y_A^* > 0$ then $\boldsymbol{y}^*$ satisfies all of the model equations if, making it a solution to the unrestricted FPP, hence a traffic equilibrium state. But otherwise, if $y_A^* = 0$ for all such accumulation points, then the accumulation principle shows that there is no non-degenerate solution to the problem of traffic equilibrium.

## C.2/ Min-cost formulation of traffic equilibrium

In logit traffic assignment, each path or mode $r$ with cost function $g_r$ and flow $y_r$ gives rise to an impedance function as follows: $I_r \equiv g_r + \frac{1}{\theta} \ln y_r$.

This is because $y_r \exp(-\theta V_r) = y_r \exp(\theta g_r)$ is equal between options $r$ of positive flow $y_r$ and finite cost $g_r$ : taking logarithms, $\ln[y_r \exp(\theta g_r)] = \theta g_r + \ln y_r$, leading to the impedance function.

A logit traffic equilibrium under demand volume $q$ is a flow vector $[y_r : r \in \mathfrak{R}]$ such that, denoting $\hat{I}_{\mathfrak{R}} \equiv \min\{I_r : r \in \mathfrak{R}\}$,

$$y_r \geq 0$$

$$\sum\nolimits_{r\in\Re} y_r = q$$

$$y_r.\left(I_r - \hat{I}_\Re\right) = 0$$

The last condition states that only an option of minimum impedance can have positive flow.

## C.3/ VIP formulation of traffic equilibrium

Defining impedance functions $I_r \equiv g_r + \frac{1}{\theta}\ln y_r$, the VIP associated to mapping $\mathbf{I}_\Re \equiv [I_r : r \in \Re]$ is

"Find $\boldsymbol{y}^*_\Re \geq 0$ and satisfying $\sum_{r\in\Re} y^*_r = q$: denoting $\mathbf{I}^*_\Re \equiv \mathbf{I}_\Re(\boldsymbol{y}^*_\Re)$, such that
$\forall \boldsymbol{y}_\Re \geq 0$ with $\sum_{r\in\Re} y_r = q$, $\mathbf{I}^*_\Re.(\boldsymbol{y}_\Re - \boldsymbol{y}^*_\Re) \geq 0$."

It means that under option costs $\mathbf{I}^*_\Re$, no admissible assignment $\boldsymbol{y}_\Re$ can get total cost $\mathbf{I}^*_\Re.\boldsymbol{y}_\Re$ lower than at $\boldsymbol{y}^*_\Re$. Demand elasticity can be taken into account by considering domain $\boldsymbol{y}_\Re \geq 0$ and mapping $I_r \equiv g_r + \frac{1}{\theta}\ln y_r - \mathrm{D}^{(-1)}(\sum_{r\in\Re} y_r)$.

## C.4/ Nesting the service options in the choice universe

The three options {U,A,N} are put on the same level in the multinomial logit model. We may also think of a composite option "Service" $\mathcal{S} \equiv \{U, A\}$ and of a nested logit model with choice set {$\mathcal{S}$,N} on the first level and, on the second level, choice subset {U,A} at $\mathcal{S}$ as a nest (Fig.C.1). The combined utility function is $U_\mathcal{S} \equiv \max\{U_\mathrm{U}, U_\mathrm{A}\}$. In the logit framework, it is a Gumbel variable with concentration parameter $\theta$ and mean value

$$V_\mathcal{S} \equiv \frac{1}{\theta}\ln(e^{\theta V_\mathrm{U}} + e^{\theta V_\mathrm{A}})$$

Considering now the complementary probabilities $\rho = y_\mathrm{U}/y_\mathcal{S}$ and $1-\rho = y_\mathrm{A}/y_\mathcal{S}$, we get that

$$V_\mathcal{S} = \frac{1}{\theta}\ln e^{\theta V_\mathrm{U}}(1 + e^{\theta V_{\mathrm{A}-\mathrm{U}}}) = V_\mathrm{U} + \frac{1}{\theta}\ln(1+\frac{1}{\omega}) = V_\mathrm{U} + \frac{1}{\theta}\ln(\frac{\omega+1}{\omega}) = V_\mathrm{U} - \frac{1}{\theta}\ln\rho$$

$$V_\mathcal{S} = \frac{1}{\theta}\ln e^{\theta V_\mathrm{A}}(e^{\theta V_{\mathrm{U}-\mathrm{A}}} + 1) = V_\mathrm{A} + \frac{1}{\theta}\ln(\omega+1) = V_\mathrm{A} - \frac{1}{\theta}\ln(1-\rho)$$

Thus, decomposing $V_\mathcal{S} = \rho V_\mathcal{S} + (1-\rho)V_\mathcal{S} = \rho(V_\mathrm{U} - \frac{1}{\theta}\ln\rho) + (1-\rho)(V_\mathrm{A} - \frac{1}{\theta}\ln(1-\rho))$,

$$V_\mathcal{S} = \rho V_\mathrm{U} + (1-\rho)V_\mathrm{A} + \frac{1}{\theta}\mathcal{H}_\rho$$

Wherein $\mathcal{H}_\rho \equiv -\rho\ln\rho - (1-\rho)\ln(1-\rho)$ is the entropy of probability distribution $[\rho, 1-\rho]$.

The service deterministic utility is a convex combination of the U and A deterministic utilities, plus (up to factor $1/\theta$) the entropy function of their respective shares within the service (entropy is positive, making the service utility higher than the convex combination of its two constituents).

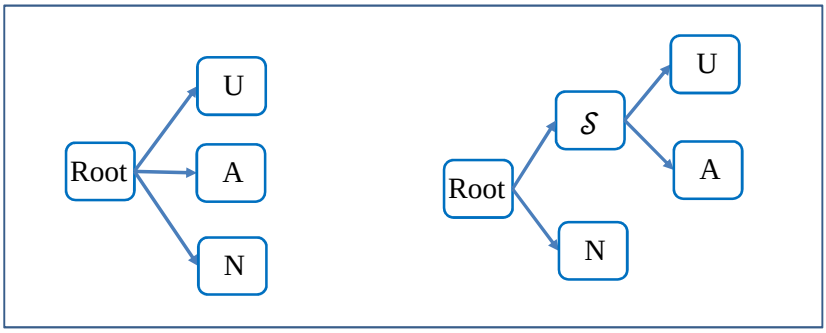


**Fig. C.1: (A) Multinomial choice model, (B) Nested choice model.**

# D/ Optimality conditions

## D.1/ Partial derivatives of utility functions

For role $N$, the deterministic utility function $V_N$ is a function of primal vector $\boldsymbol{x}$ with partial derivatives as follows with respect to any component $x$ in $\boldsymbol{x}$:

$$\frac{\partial V_N}{\partial x} = -\frac{\partial t_N^R}{\partial x}\beta_N^R$$

With $\frac{\partial t_N^R}{\partial x} = 1$ if $x = t_N^R$ or $\delta_{UN}^R$ if $x = t_U^R$ or 0 otherwise, with $\delta_{UN}^R$ the binary indicator of lane sharing between U and N (it is null under dedicated lanes).

Table D.1 provides the partial derivatives of $V_A$ and $V_U$ with respect to the different components $x$ in $\boldsymbol{x}$. We denote

$$\dot{\rho} \equiv \frac{\partial \rho}{\partial \omega} = (1+\omega)^{-2}. \tag{D.1}$$

Recall that $t_A^R = (1-\rho)t_N^R + \rho t_U^R$ and

$$(\beta t)_A^R = (1-\rho)\beta_{A0}^R t_N^R + \rho\beta_{A1}^R t_U^R.$$

The most intricate influence is

$\frac{\partial V_A}{\partial \omega} = \frac{\partial \hat{\tau}_A}{\partial \omega} - \frac{\partial (\beta t)_A^R}{\partial \omega} - \beta_A^S \underline{t}_A^S \frac{\partial \rho}{\partial \omega} - \delta_A^W \beta_A^W \frac{H}{\varphi}\frac{\partial \rho}{\partial \omega}$, hence

$$\frac{\partial V_A}{\partial \omega} = \tau_{A1} + \dot{\rho}(\beta_{A0}^R t_N^R - \beta_{A1}^R t_U^R - \beta_A^S \underline{t}_A^S - \delta_A^W \beta_A^W \frac{H}{\varphi}). \tag{D.2}$$

We also denote $\dot{V}_{A\rho}^{\backslash\tau,W} \equiv \frac{\partial V_A^{\backslash\tau,W}}{\partial \omega} = \beta_{A0}^R t_N^R - \beta_{A1}^R t_U^R - \beta_A^S \underline{t}_A^S$, $\dot{V}_{A\rho}^{\backslash\tau} \equiv \dot{V}_{A\rho}^{\backslash\tau,W} - \delta_A^W \beta_A^W \frac{H}{\varphi} = \dot{V}_{A\rho}^{\backslash\tau,W} - \frac{\varphi}{\rho}\dot{V}_{A\varphi}$, and $\dot{V}_{A\omega} \equiv \frac{\partial V_A}{\partial \omega} = \tau_{A1} + \dot{\rho}\dot{V}_{A\rho}^{\backslash\tau}$.

**Tab. D.1. Partial derivatives of $V_A$ and $V_U$.**

| $x$ | $\frac{\partial V_A}{\partial x}$ | $\frac{\partial V_U}{\partial x}$ |
|---|---|---|
| $\tau_U$ | $0$ | $-1$ |
| $\tau_{A0}$ | $+1$ | $0$ |
| $\tau_{A1}$ | $\omega$ | $0$ |
| $t_U^R$ | $-\rho\beta_{A1}^R - (1-\rho)\beta_{A0}^R\delta_{UN}^R$ | $-\beta_U^R$ |
| $t_N^R$ | $-\rho\beta_{A1}^R\delta_{UN}^R - (1-\rho)\beta_{A0}^R$ | $-\delta_{UN}^R\beta_U^R$ |
| $\varphi$ | $\dot{V}_{A\varphi} \equiv \delta_A^W\beta_A^W \mathrm{H}\rho\varphi^{-2}$ | $\dot{V}_{U\varphi} \equiv \delta_U^W\beta_U^W \mathrm{H}\varphi^{-2}$ |
| $\omega$ | $\dot{V}_{A\omega} \equiv \tau_{A1} + \dot{\rho}\dot{V}_{A\rho}^{\backslash\tau}$ | $0$ |
| $q$ | $0$ | $0$ |

## D.2/ Partial derivatives of demanded flows

Recalling the expected maximum utility $V_{\Re} \equiv \frac{1}{\theta}\ln\sum_{r\in\Re}\exp(\theta V_r)$, it holds that

$$\frac{\partial V_{\Re}}{\partial x} = \frac{1}{\theta}\frac{\sum_{r\in\Re}\theta\frac{\partial V_r}{\partial x}\exp(\theta V_r)}{\sum_{r\in\Re}\exp(\theta V_r)},$$

$$\frac{\partial}{\partial x} V_{\mathfrak{R}} = \sum_{r \in \mathfrak{R}} p_r^{\#} \frac{\partial V_r}{\partial x}. \tag{D.3}$$

In fact this formula is a more general property of discrete choice models: from $V_{\mathfrak{R}} \equiv \mathrm{E}[U_{\mathfrak{R}}]$ and $U_{\mathfrak{R}} \equiv \max U_r$, it follows that

$$\frac{\partial}{\partial x} V_{\mathfrak{R}} = \frac{\partial}{\partial x} \mathrm{E}[\max U_r] = \sum_{r \in \mathfrak{R}} p_r^{\#} \frac{\partial}{\partial x} \mathrm{E}[U_r | U_r = U_{\mathfrak{R}}] = \sum_{r \in \mathfrak{R}} p_r^{\#} \frac{\partial V_r}{\partial x}.$$

As $p_r^{\#} = \exp(\theta(V_r - V_{\mathfrak{R}}))$, it follows that

$$\frac{\partial}{\partial x} p_r^{\#} = \theta p_r^{\#} \left( \frac{\partial V_r}{\partial x} - \frac{\partial V_{\mathfrak{R}}}{\partial x} \right). \tag{D.4}$$

Hence $\frac{\partial}{\partial x} p_r^{\#} = \theta p_r^{\#} \left( \frac{\partial V_r}{\partial x} (1 - p_r^{\#}) - \sum_{s \neq r} p_s^{\#} \frac{\partial V_s}{\partial x} \right)$.

In turn, as $y_r^{\#} = q p_r^{\#}$ and $\frac{\partial}{\partial q} p_r^{\#} = 0$, we have that $\frac{\partial}{\partial x} y_r^{\#} = p_r^{\#} \frac{\partial q}{\partial x} + q \frac{\partial p_r^{\#}}{\partial x}$, hence

$$\frac{\partial}{\partial x} y_r^{\#} = p_r^{\#} \frac{\partial q}{\partial x} + \theta y_r^{\#} \left( \frac{\partial V_r}{\partial x} - \frac{\partial V_{\mathfrak{R}}}{\partial x} \right). \tag{D.5}$$

The service-related utilities have the following sensitivities to the user fare $\hat{\tau}_U$ and the overall agent fee $\hat{\tau}_A$:

| | | | |
|---|---|---|---|
| $\frac{\partial V_U}{\partial \tau_U} = -1$ | $\frac{\partial V_A}{\partial \tau_U} = 0$ | $\frac{\partial V_U}{\partial \tau_A} = 0$ | $\frac{\partial V_A}{\partial \tau_A} = 1$ |

Thus, with respect to price variables,

| | |
|---|---|
| $\frac{\partial V_{\mathfrak{R}}}{\partial \tau_U} = -p_U$ | $\frac{\partial V_{\mathfrak{R}}}{\partial \tau_A} = p_A$ |

The service-related flows have the following sensitivities to the price variables:

$$\frac{\partial y_U}{\partial \tau_U} = q \frac{\partial p_U}{\partial \tau_U} = q\theta p_U [(1 - p_U)(-1)] = -\theta y_U (1 - p_U)$$

$$\frac{\partial y_A}{\partial \tau_U} = q \frac{\partial p_A}{\partial \tau_U} = q\theta p_A p_U = \theta y_A p_U = \theta y_U p_A$$

$$\frac{\partial y_U}{\partial \tau_A} = q \frac{\partial p_U}{\partial \tau_A} = q\theta p_U [-p_A] = -\theta y_U p_A = -\theta y_A p_U$$

$$\frac{\partial y_A}{\partial \tau_A} = q \frac{\partial p_A}{\partial \tau_A} = q\theta p_A (1 - p_A) = \theta y_A (1 - p_A)$$

To summarize,

| | | | |
|---|---|---|---|
| $\frac{\partial y_U}{\partial \tau_U} = -\theta y_U (1 - p_U)$ | $\frac{\partial y_A}{\partial \tau_U} = \theta y_U p_A$ | $\frac{\partial y_U}{\partial \tau_A} = -\theta y_A p_U$ | $\frac{\partial y_A}{\partial \tau_A} = \theta y_A (1 - p_A)$ |

## D.3/ Partial derivatives of profit functions

Recalling that $\mathcal{P}_o = \hat{\tau}_U y_U^{\#} - \hat{\tau}_A y_A^{\#} - \mathcal{C}_o(y_A^{\#}, y_U^{\#})$, then

$$\frac{\partial}{\partial x} \mathcal{P}_o = y_U^{\#} \frac{\partial \hat{\tau}_U}{\partial x} + \hat{\tau}_U \frac{\partial y_U^{\#}}{\partial x} - y_A^{\#} \frac{\partial \hat{\tau}_A}{\partial x} - \hat{\tau}_A \frac{\partial y_A^{\#}}{\partial x} - \frac{\partial \mathcal{C}_o}{\partial y_A} \frac{\partial y_A^{\#}}{\partial x} - \frac{\partial \mathcal{C}_o}{\partial y_U} \frac{\partial y_U^{\#}}{\partial x}.$$

Defining $\tau_U^- \equiv \hat{\tau}_U - \dot{c}_U$ and $\tau_A^+ \equiv \hat{\tau}_A + \dot{c}_A$, both with $\dot{c}_r \equiv \frac{\partial \mathcal{C}_o}{\partial y_r}$, then

$$\frac{\partial}{\partial x} \mathcal{P}_o = y_U^{\#} \frac{\partial \hat{\tau}_U}{\partial x} + \tau_U^- \frac{\partial y_U^{\#}}{\partial x} - y_A^{\#} \frac{\partial \hat{\tau}_A}{\partial x} - \tau_A^+ \frac{\partial y_A^{\#}}{\partial x}. \tag{D.6}$$

As for demand surplus, under fixed demand $\mathcal{P}_u = q V_{\mathfrak{R}}$ yields

$$\frac{\partial}{\partial x}\mathcal{P}_u = V_\Re \frac{\partial q}{\partial x} + q\frac{\partial V_\Re}{\partial x} = q\frac{\partial V_\Re}{\partial x}$$

Since then $q$ is not included in the primal vector.

Under elastic demand, $\mathcal{P}_u = \int_{g_\Re}^{\infty} \mathrm{D}(g)dg$ so that $\frac{\partial}{\partial x}\mathcal{P}_u = -\mathrm{D}(g_\Re)\frac{\partial g_\Re}{\partial x}$, hence

$$\frac{\partial}{\partial x}\mathcal{P}_u = \mathrm{D}(g_\Re)\frac{\partial V_\Re}{\partial x}. \tag{D.7}$$

This formula encompasses the fixed demand case.

Assuming that $\mathrm{D}(g_\Re) = q$, then $\frac{\partial}{\partial x}\mathcal{P}_u = q\sum_{r\in\Re} p_r^{\#}\frac{\partial V_r}{\partial x}$, hence

$$\frac{\partial}{\partial x}\mathcal{P}_u = \sum\nolimits_{r\in\Re} y_r^{\#}\frac{\partial V_r}{\partial x}$$

Coming to environmental benefits,

$$\frac{\partial}{\partial x}\mathcal{P}_e = \chi_U\frac{\partial y_U^{\#}}{\partial x} - \chi_A\frac{\partial y_A^{\#}}{\partial x} - \chi_N\frac{\partial y_N^{\#}}{\partial x} \tag{D.8}$$

From these stem the partial derivatives of the objective function, $\mathcal{J} \equiv \mu\mathcal{P}_o + \psi\mathcal{P}_u + \mathcal{P}_e$ :

$$\frac{\partial}{\partial x}\mathcal{J} = \mu\frac{\partial\mathcal{P}_o}{\partial x} + \psi\frac{\partial\mathcal{P}_u}{\partial x} + \frac{\partial\mathcal{P}_e}{\partial x}$$

Replacing with previous intermediate outcomes,

$\frac{\partial}{\partial x}\mathcal{J} = \mu\left\{y_U^{\#}\frac{\partial\hat{\tau}_U}{\partial x} + \tau_U^{-}\frac{\partial y_U^{\#}}{\partial x} - y_A^{\#}\frac{\partial\hat{\tau}_A}{\partial x} - \tau_A^{+}\frac{\partial y_A^{\#}}{\partial x}\right\} + \psi\left\{\sum_{r\in\Re} y_r^{\#}\frac{\partial V_r}{\partial x}\right\} + \chi_U\frac{\partial y_U^{\#}}{\partial x} - \chi_A\frac{\partial y_A^{\#}}{\partial x} - \chi_N\frac{\partial y_N^{\#}}{\partial x}$, hence

$$\frac{\partial}{\partial x}\mathcal{J} = y_U^{\#}\left\{\mu\frac{\partial\hat{\tau}_U}{\partial x} + \psi\frac{\partial V_U}{\partial x}\right\} + \frac{\partial y_U^{\#}}{\partial x}\{\mu\tau_U^{-} + \chi_U\} - y_A^{\#}\left\{\mu\frac{\partial\hat{\tau}_A}{\partial x} - \psi\frac{\partial V_A}{\partial x}\right\} - \frac{\partial y_A^{\#}}{\partial x}\{\mu\tau_A^{+} + \chi_A\} - \chi_N\frac{\partial y_N^{\#}}{\partial x} + \psi y_N^{\#}\frac{\partial V_N}{\partial x} \tag{D.9}$$

**Tab. D.2. Partial derivatives of the objective function.**

| $x$ | $\frac{\partial\mathcal{J}}{\partial x}$ |
|---|---|
| $\tau_U$ | $(\mu-\psi)y_U^{\#} + (\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial\tau_U} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial\tau_U} - \chi_N\frac{\partial y_N^{\#}}{\partial\tau_U}$ |
| $\tau_{A0}$ | $(\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial\tau_A} - (\mu-\psi)y_A^{\#} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial\tau_A} - \chi_N\frac{\partial y_N^{\#}}{\partial\tau_A}$ |
| $\tau_{A1}$ | $\omega\left\{(\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial\tau_A} - (\mu-\psi)y_A^{\#} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial\tau_A} - \chi_N\frac{\partial y_N^{\#}}{\partial\tau_A}\right\}$ |
| $t_U^R,\ t_N^R$ | $\psi\frac{\partial V_U}{\partial x}y_U^{\#} + (\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial x} + \psi\frac{\partial V_A}{\partial x}y_A^{\#} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial x} - \chi_N\frac{\partial y_N^{\#}}{\partial x} + \psi y_N^{\#}\frac{\partial V_N}{\partial x}$ |
| $\varphi$ | $\psi\dot{V}_{U\varphi}y_U^{\#} + (\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial\varphi} + \psi\dot{V}_{A\varphi}y_A^{\#} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial\varphi} - \chi_N\frac{\partial y_N^{\#}}{\partial\omega}$ |
| $\omega$ | $(\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial\omega} - (\mu\tau_{A1} - \hat{\psi}\dot{V}_{A\omega})y_A^{\#} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial\omega} - \chi_N\frac{\partial y_N^{\#}}{\partial\omega}$ |
| $q$ | $(\mu\tau_U^{-} + \chi_U)\frac{\partial y_U^{\#}}{\partial q} - (\mu\tau_A^{+} + \chi_A)\frac{\partial y_A^{\#}}{\partial q} - \chi_N\frac{\partial y_N^{\#}}{\partial q}$ |

## D.4/ Partial derivatives of the Lagrangian function

From $\mathcal{L} \equiv \mathcal{J} - \xi\left(\varphi - y_A^{\#}\right) - \lambda\left(y_U^{\#} - \omega y_A^{\#}\right) - \eta(q - \mathrm{D}(g_{\mathfrak{R}}^{\#})) - \sum_{s\in\{U,N\}} \sigma_s(t_s^R - \mathrm{T}_s^R(\boldsymbol{y}_{\mathfrak{R}}^{\#}))$, it follows that for any primal variable $x$,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{J}}{\partial x} - \xi\left(\frac{\partial \varphi}{\partial x} - \frac{\partial y_A^{\#}}{\partial x}\right) - \lambda\left(\frac{\partial y_U^{\#}}{\partial x} - y_A^{\#}\frac{\partial \omega}{\partial x} - \omega\frac{\partial y_A^{\#}}{\partial x}\right) - \eta\left(\frac{\partial q}{\partial x} + \dot{\mathrm{D}}.\frac{\partial V_{\mathfrak{R}}}{\partial x}\right) - \sum_{s\in\{U,N\}} \sigma_s\left(\frac{\partial t_s^R}{\partial x} - \sum_{r\in\mathfrak{R}} \frac{\partial \mathrm{T}_s^R}{\partial y_r}\frac{\partial y_r^{\#}}{\partial x}\right) \tag{D.10}$$

Combining (D.10) with, from (D.3), $\dot{\mathrm{D}}.\frac{\partial V_{\mathfrak{R}}}{\partial x} = \frac{\dot{\mathrm{D}}}{q}\sum_{r\in\mathfrak{R}} y_r^{\#}\frac{\partial V_r}{\partial x}$, and letting $\hat{\psi} \equiv \psi - \eta\frac{\dot{\mathrm{D}}}{q}$, we get

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \left(\mu\frac{\partial \hat{\tau}_U}{\partial x} + \hat{\psi}\frac{\partial V_U}{\partial x}\right)y_U^{\#} + (\mu\tau_U^- + \chi_U - \lambda + S_U)\frac{\partial y_U^{\#}}{\partial x}\\ &- \left(\mu\frac{\partial \hat{\tau}_A}{\partial x} - \hat{\psi}\frac{\partial V_A}{\partial x} - \lambda\frac{\partial \omega}{\partial x}\right)y_A^{\#} - (\mu\tau_A^+ + \chi_A - \lambda\omega - \xi - S_A)\frac{\partial y_A^{\#}}{\partial x}\\ &- \chi_N\frac{\partial y_N^{\#}}{\partial x} + \hat{\psi}y_N^{\#}\frac{\partial V_N}{\partial x} - \xi\frac{\partial \varphi}{\partial x} - \eta\frac{\partial q}{\partial x} + S_N\frac{\partial y_N^{\#}}{\partial x} - \sigma_U\frac{\partial t_U^R}{\partial x} - \sigma_N\frac{\partial t_N^R}{\partial x}\end{aligned}$$

Wherein: $S_r \equiv \sum_{s\in\{U,N\}} \sigma_s\frac{\partial \mathrm{T}_s^R}{\partial y_r}$.

Replacing $y_N^{\#}$ and $\frac{\partial y_N^{\#}}{\partial x}$ by $q - y_A^{\#} - y_U^{\#}$ and $\frac{\partial q}{\partial x} - \frac{\partial y_A^{\#}}{\partial x} - \frac{\partial y_U^{\#}}{\partial x}$ respectively, it comes out that

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \left(\mu\frac{\partial \hat{\tau}_U}{\partial x} + \hat{\psi}\frac{\partial V_{U-N}}{\partial x}\right)y_U^{\#} + (\mu\tau_U^- + \chi_U + \chi_N - \lambda + S_U)\frac{\partial y_U^{\#}}{\partial x}\\ &- \left(\mu\frac{\partial \hat{\tau}_A}{\partial x} - \hat{\psi}\frac{\partial V_{A-N}}{\partial x} - \lambda\frac{\partial \omega}{\partial x}\right)y_A^{\#} - (\mu\tau_A^+ + \chi_A - \chi_N - \lambda\omega - \xi - S_A)\frac{\partial y_A^{\#}}{\partial x}\\ &+ \hat{\psi}q\frac{\partial V_N}{\partial x} - \xi\frac{\partial \varphi}{\partial x} - \eta\frac{\partial q}{\partial x} + (S_N - \chi_N)\frac{\partial q}{\partial x} - \sigma_U\frac{\partial t_U^R}{\partial x} - \sigma_N\frac{\partial t_N^R}{\partial x}\end{aligned}$$

Denoting

$$\mathcal{U} \equiv \mu\tau_U^- + \chi_{U+N} - \lambda + S_{U-N}$$
$$\mathcal{A} \equiv \mu\tau_A^+ + \chi_{A-N} - \lambda\omega - \xi - S_{A-N},$$

we get that

$$\frac{\partial \mathcal{L}}{\partial x} = \left(\mu\frac{\partial \hat{\tau}_U}{\partial x} + \hat{\psi}\frac{\partial V_{U-N}}{\partial x}\right)y_U^{\#} + \mathcal{U}\frac{\partial y_U^{\#}}{\partial x} - \left(\mu\frac{\partial \hat{\tau}_A}{\partial x} - \hat{\psi}\frac{\partial V_{A-N}}{\partial x} - \lambda\frac{\partial \omega}{\partial x}\right)y_A^{\#} - \mathcal{A}\frac{\partial y_A^{\#}}{\partial x} + \hat{\psi}q\frac{\partial V_N}{\partial x} - \xi\frac{\partial \varphi}{\partial x} + (S_N - \chi_N - \eta)\frac{\partial q}{\partial x} - \sigma_U\frac{\partial t_U^R}{\partial x} - \sigma_N\frac{\partial t_N^R}{\partial x} \tag{D.11}$$

This formula applies to every component $x$, yielding specific outcomes exhibited in Table D.3.

The partial derivatives of demanded flows can be replaced by equivalent statements in terms of elasticities: typically, $\frac{\partial y_r}{\partial x} = \frac{y_r}{x}\epsilon_{y_r:x}$. Then,

$$\frac{\partial \mathcal{L}}{\partial x} = \left(\mu\frac{\partial \hat{\tau}_U}{\partial x} + \hat{\psi}\frac{\partial V_{U-N}}{\partial x} + \mathcal{U}\frac{\epsilon_{y_U:x}}{x}\right)y_U^{\#} - \left(\mu\frac{\partial \hat{\tau}_A}{\partial x} - \hat{\psi}\frac{\partial V_{A-N}}{\partial x} - \lambda\frac{\partial \omega}{\partial x} + \mathcal{A}\frac{\epsilon_{y_A:x}}{x}\right)y_A^{\#} + \hat{\psi}q\frac{\partial V_N}{\partial x} - \xi\frac{\partial \varphi}{\partial x} + (S_N - \chi_N - \eta)\frac{\partial q}{\partial x} - \sigma_U\frac{\partial t_U^R}{\partial x} - \sigma_N\frac{\partial t_N^R}{\partial x} \tag{D.12}$$

**Tab. D.3. Partial derivatives of the Lagrangian function with respect to primal variables.**

| | |
|---|---|
| $x$ | $\frac{\partial \mathcal{L}}{\partial x}$ |
| $\tau_U$ | $(\mu - \hat{\psi}) y_U^{\#} + \mathcal{U} \frac{\partial y_U^{\#}}{\partial \tau_U} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial \tau_U}$ |
| $\tau_{A0}$ | $\mathcal{U} \frac{\partial y_U^{\#}}{\partial \tau_A} - (\mu - \hat{\psi}) y_A^{\#} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial \tau_A}$ |
| $\tau_{A1}$ | $\omega \left\{ \mathcal{U} \frac{\partial y_U^{\#}}{\partial \tau_A} - (\mu - \hat{\psi}) y_A^{\#} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial \tau_A} \right\}$ |
| $t_U^R, t_N^R$ | $\hat{\psi} \frac{\partial V_{U-N}}{\partial x} y_U^{\#} + \mathcal{U} \frac{\partial y_U^{\#}}{\partial x} + \hat{\psi} \frac{\partial V_{A-N}}{\partial x} y_A^{\#} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial x} + \hat{\psi} q \frac{\partial V_N}{\partial x} - \sigma_s$ |
| $\varphi$ | $\hat{\psi} \dot{V}_{U\varphi} y_U^{\#} + \mathcal{U} \frac{\partial y_U^{\#}}{\partial \varphi} + \hat{\psi} \dot{V}_{A\varphi} y_A^{\#} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial \varphi} - \xi$ |
| $\omega$ | $\mathcal{U} \frac{\partial y_U^{\#}}{\partial \omega} - (\mu \tau_{A1} - \hat{\psi} \dot{V}_{A\omega} - \lambda) y_A^{\#} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial \omega}$ |
| $q$ | $\mathcal{U} \frac{\partial y_U^{\#}}{\partial q} - \mathcal{A} \frac{\partial y_A^{\#}}{\partial q} + S_N - \chi_N - \eta$ |

## D.5/ Optimality conditions with respect to prices

Considering the price variables, the partial derivatives of the demanded flows are respectively:

$$\frac{\partial \mathcal{L}}{\partial \tau_U} = \left( \mu - \hat{\psi} + \mathcal{U} \frac{\epsilon_{y_U:\tau_U}}{\tau_U} \right) y_U^{\#} - \mathcal{A} \frac{\epsilon_{y_A:\tau_U}}{\tau_U} y_A^{\#}$$

$$\frac{\partial \mathcal{L}}{\partial \tau_A} = \mathcal{U} \frac{\epsilon_{y_U:\tau_A}}{\tau_A} y_U^{\#} - \left( \mu - \hat{\psi} + \mathcal{A} \frac{\epsilon_{y_A:\tau_A}}{\tau_A} \right) y_A^{\#}$$

The optimality condition with respect to $\tau_U$, $\frac{\partial \mathcal{L}}{\partial \tau_U} = 0$, states as

$$\left( \mu - \hat{\psi} + \mathcal{U} \frac{\epsilon_{y_U:\tau_U}}{\tau_U} \right) y_U^{\#} = \mathcal{A} \frac{\epsilon_{y_A:\tau_U}}{\tau_U} y_A^{\#}$$

At an optimal point where $y_U^{\#} = \omega y_A^{\#}$, if $y_U^{\#} \neq 0$ then

$$-\mathcal{U} \frac{\epsilon_{y_U:\tau_U}}{\tau_U} + \frac{1}{\omega} \mathcal{A} \frac{\epsilon_{y_A:\tau_U}}{\tau_U} = \mu - \hat{\psi}$$

The optimality condition with respect to $\tau_A$, $\frac{\partial \mathcal{L}}{\partial \tau_A} = 0$, gives that

$$\mathcal{U} \frac{\epsilon_{y_U:\tau_A}}{\tau_A} y_U^{\#} = \left( \mu - \hat{\psi} + \mathcal{A} \frac{\epsilon_{y_A:\tau_A}}{\tau_A} \right) y_A^{\#}$$

If $y_A^{\#} \neq 0$ and $y_U^{\#} = \omega y_A^{\#}$, then

$$\omega \mathcal{U} \frac{\epsilon_{y_U:\tau_A}}{\tau_A} - \mathcal{A} \frac{\epsilon_{y_A:\tau_A}}{\tau_A} = \mu - \hat{\psi}$$

If both optimality conditions are satisfied, we get that

$$-\mathcal{U} \frac{\epsilon_{y_U:\tau_U}}{\tau_U} + \frac{1}{\omega} \mathcal{A} \frac{\epsilon_{y_A:\tau_U}}{\tau_U} = \omega \mathcal{U} \frac{\epsilon_{y_U:\tau_A}}{\tau_A} - \mathcal{A} \frac{\epsilon_{y_A:\tau_A}}{\tau_A}$$

$$\mathcal{A} \left( \frac{1}{\omega} \frac{\epsilon_{y_A:\tau_U}}{\tau_U} + \frac{\epsilon_{y_A:\tau_A}}{\tau_A} \right) = \mathcal{U} \left( \omega \frac{\epsilon_{y_U:\tau_A}}{\tau_A} + \frac{\epsilon_{y_U:\tau_U}}{\tau_U} \right)$$

Under the logit model, as $y_U p_A = y_A p_U$, it holds that

| | | | |
|---|---|---|---|
| $\frac{\partial y_U}{\partial \tau_U} = -\theta y_U(1-p_U)$ | $\frac{\partial y_A}{\partial \tau_U} = \theta y_U p_A$ | $\frac{\partial y_U}{\partial \tau_A} = -\theta y_A p_U$ | $\frac{\partial y_A}{\partial \tau_A} = \theta y_A(1-p_A)$ |
| $\epsilon_{y_U:\tau_U} = -\theta\tau_U(1-p_U)$ | $\epsilon_{y_A:\tau_U} = \theta\tau_U p_U$ | $\epsilon_{y_U:\tau_A} = -\theta\tau_A p_A$ | $\epsilon_{y_A:\tau_A} = \theta\tau_A(1-p_A)$ |

Replacing and dividing both sides by $\theta$, we get the optimality conditions as

$$\mathcal{U}(1-p_U) + \tfrac{1}{\omega}\mathcal{A}p_U = \tfrac{\mu-\widehat{\psi}}{\theta}$$

$$\omega\mathcal{U}(1-p_A) - \mathcal{A}(1-p_A) = \tfrac{\mu-\widehat{\psi}}{\theta}$$

Hence that

$$\mathcal{A}\left(\tfrac{1}{\omega}p_U + 1 - p_A\right) = -\mathcal{U}(\omega p_A + 1 - p_U)$$

Using that $p_U = \omega p_A$, it comes out that

$$\mathcal{A} = -\mathcal{U}$$

Replacing in the first optimality condition and denoting $\mathcal{M} \equiv \frac{\mu-\widehat{\psi}}{\theta}$,

$$\mathcal{U}(1-p_U-p_A) = \mathcal{M}$$

Hence, as $1 - p_U - p_A = p_N$, that

$$\mathcal{U} = \mathcal{M}/p_N \tag{D.13a}$$

$$\mathcal{A} = -\mathcal{M}/p_N \tag{D.13b}$$

## D.6/ Optimality condition associated to frequency

With respect to frequency $\varphi$, the optimality condition $\frac{\partial \mathcal{L}}{\partial \varphi} = 0$ states as

$$\xi = (\widehat{\psi}\dot{V}_{U\varphi} + \mathcal{U}\frac{\epsilon_{y_U:\varphi}}{\varphi})y_U^{\#} + (\widehat{\psi}\dot{V}_{A\varphi} - \mathcal{A}\frac{\epsilon_{y_A:\varphi}}{\varphi})y_A^{\#}$$

In the logit model, $\epsilon_{y_r:\varphi} = \frac{\varphi}{y_r}\frac{\partial y_r^{\#}}{\partial \varphi} = \varphi\theta(\dot{V}_{r,\varphi} - \dot{V}_{\Re,\varphi})$ with $\dot{V}_{\Re,\varphi} = p_U\dot{V}_{U,\varphi} + p_A\dot{V}_{A,\varphi}$ so that

$$\epsilon_{y_U:\varphi} = \varphi\theta(\dot{V}_{U,\varphi}(1-p_U) - p_A\dot{V}_{A,\varphi})$$

$$\epsilon_{y_A:\varphi} = \varphi\theta(\dot{V}_{A,\varphi}(1-p_A) - p_U\dot{V}_{U,\varphi})$$

Thus, the optimality condition is restated as

$$\xi = (\dot{V}_{U\varphi}\{\widehat{\psi} + \mathcal{U}\theta(1-p_U)\} - \mathcal{U}\theta p_A\dot{V}_{A,\varphi})y_U^{\#} + (\dot{V}_{A\varphi}\{\widehat{\psi} - \mathcal{A}\theta(1-p_A)\} + \mathcal{A}\theta p_U\dot{V}_{U,\varphi})y_A^{\#}$$

Replacing $y_A^{\#}$ by $\varphi$ and $y_U^{\#}$ by $\varphi\omega$,

$$\xi = \varphi(\omega\dot{V}_{U\varphi}\{\widehat{\psi} + \mathcal{U}\theta(1-p_U) + \mathcal{A}\theta p_A\} + \dot{V}_{A\varphi}\{\widehat{\psi} - \mathcal{A}\theta(1-p_A) - \mathcal{U}\theta p_U\})$$

Under optimal prices yielding $\mathcal{A} = -\mathcal{U}$, then

$$\xi = \varphi(\omega\dot{V}_{U\varphi}\{\widehat{\psi} + \mathcal{U}\theta(1-p_U-p_A)\} + \dot{V}_{A\varphi}\{\widehat{\psi} + \mathcal{U}\theta(1-p_A-p_U)\})$$

$$\xi = \varphi(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi})\{\widehat{\psi} + \mathcal{U}\theta p_N\}$$

As $\widehat{\psi} + \mathcal{U}\theta p_N = \widehat{\psi} + \theta\mathcal{M} = \mu$,

$$\xi = \mu\varphi(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}) \tag{D.14}$$

## D.7/ Optimality condition associated to ride load per car run

With respect to average ride load per cab run, $\omega$, the optimality condition $\frac{\partial \mathcal{L}}{\partial \omega} = 0$ states as

$$\left(\mu\tau_{A1} - \hat{\psi}\dot{V}_{A\omega} - \lambda\right)y_A^{\#} = \mathcal{U}\frac{\partial y_U^{\#}}{\partial \omega} - \mathcal{A}\frac{\partial y_A^{\#}}{\partial \omega}$$

Which is equivalent to, as $\frac{1}{y_A}\frac{\partial y_U^{\#}}{\partial \omega} = \frac{y_U}{y_A}\frac{\epsilon_{y_U:\omega}}{\omega} = \epsilon_{y_U:\omega}$,

$$\lambda = \mu\tau_{A1} - \hat{\psi}\dot{V}_{A\omega} + \mathcal{A}\frac{\epsilon_{y_A:\omega}}{\omega} - \mathcal{U}\epsilon_{y_U:\omega}$$

In the logit model, $\epsilon_{y_r:\omega} = \omega\theta(\dot{V}_{r,\omega} - \dot{V}_{\Re,\omega})$ with $\dot{V}_{\Re,\omega} = p_U\dot{V}_{U,\omega} + p_A\dot{V}_{A,\omega} = p_A\dot{V}_{A,\omega}$ so that

$$\epsilon_{y_U:\omega} = -\omega\theta p_A\dot{V}_{A,\omega}$$

$$\epsilon_{y_A:\omega} = \omega\theta\dot{V}_{A,\omega}(1 - p_A)$$

Thus, the optimality condition is restated as

$$\lambda = \mu\tau_{A1} - \hat{\psi}\dot{V}_{A\omega} + \mathcal{A}\theta\dot{V}_{A,\omega}(1 - p_A) + \mathcal{U}\theta p_U\dot{V}_{A,\omega}$$

Under optimal prices yielding $\mathcal{A} = -\mathcal{U}$, then

$$\lambda = \mu\tau_{A1} + \dot{V}_{A\omega}\{-\hat{\psi} + \mathcal{U}\theta(p_U + p_A - 1)\}$$

$$\lambda = \mu\tau_{A1} - \mu\dot{V}_{A\omega}$$

As $\dot{V}_{A\omega} = \tau_{A1} + \dot{\rho}\dot{V}_{A\rho}^{\backslash\tau}$, finally

$$\lambda = -\mu\dot{\rho}\dot{V}_{A\rho}^{\backslash\tau} \tag{D.15}$$

## D.8/ Optimality condition associated to demand volume

With respect to demand volume as an active variable on its own, $q$, the optimality condition $\frac{\partial \mathcal{L}}{\partial q} = 0$ states as

$$\eta = S_N - \chi_N + \mathcal{U}\frac{\partial y_U^{\#}}{\partial q} - \mathcal{A}\frac{\partial y_A^{\#}}{\partial q}$$

$$\eta = S_N - \chi_N + \mathcal{U}p_U - \mathcal{A}p_A$$

$$\eta = S_N - \chi_N + \mathcal{U}(p_U + p_A)$$

$$\eta = S_N - \chi_N + \mathcal{U}(1 - p_N)$$

Hence, denoting $Z \equiv \frac{1-p_N}{p_N}$,

$$\eta = S_N - \chi_N + \mathcal{M}Z = S_N - \chi_N + Z\frac{\mu - \hat{\psi}}{\theta}$$

Recalling that $\hat{\psi} = \psi - \eta\frac{\dot{\mathrm{D}}}{q}$,

$$\theta(\eta - S_N + \chi_N) = Z(\mu - \psi + \eta\frac{\dot{\mathrm{D}}}{q})$$

$$\eta\left(\theta - Z\frac{\dot{\mathrm{D}}}{q}\right) = \theta(S_N - \chi_N) + Z(\mu - \psi)$$

$$\eta = \frac{\theta(S_N - \chi_N) + Z(\mu - \psi)}{\theta - Z\dot{\mathrm{D}}/q} \tag{D.16}$$

## D.9/ Optimality conditions with respect to link run times

With respect to any link run time $x = t_s^R$ with $s \in \{N, U\}$, the optimality condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ states as

$$\sigma_s = \hat{\psi}\dot{V}_{U-N}y_U^\# + \mathcal{U}\frac{\partial y_U^\#}{\partial x} + \hat{\psi}\dot{V}_{A-N}y_A^\# - \mathcal{A}\frac{\partial y_A^\#}{\partial x} + \hat{\psi}q\frac{\partial V_N}{\partial x}$$

Wherein $\dot{V}_{r-N} \equiv \frac{\partial V_{r-N}}{\partial x}$.

Recalling that $\frac{\partial y_r^\#}{\partial x} = \theta y_r^\#\left(\frac{\partial V_r}{\partial x} - \sum_s p_s^\#\frac{\partial V_s}{\partial x}\right)$, then, for $(r,s) = (U,A)$ or $(A,U)$, it holds that

$$\frac{\partial y_r^\#}{\partial x} = \theta y_r^\#\left(\frac{\partial V_{r-N}}{\partial x}(1 - p_s^\#) - p_s^\#\frac{\partial V_{s-N}}{\partial x}\right).$$

Replacing in $\mathcal{U}\frac{\partial y_U^\#}{\partial x} - \mathcal{A}\frac{\partial y_A^\#}{\partial x}$, then

$$\mathcal{U}\frac{\partial y_U^\#}{\partial x} - \mathcal{A}\frac{\partial y_A^\#}{\partial x} = \mathcal{U}\theta y_U^\#\left(\dot{V}_{U-N}(1 - p_U^\#) - p_A^\#\dot{V}_{A-N}\right) - \theta\mathcal{A}y_A^\#\left(\dot{V}_{A-N}(1 - p_A^\#) - p_U^\#\dot{V}_{U-N}\right)$$

At an optimal point, $y_A^\# = \varphi$, $y_U^\# = \varphi\omega$ and $\mathcal{A} = -\mathcal{U}$, so that

$$\begin{aligned}\mathcal{U}\frac{\partial y_U^\#}{\partial x} - \mathcal{A}\frac{\partial y_A^\#}{\partial x} &= \mathcal{U}\theta\varphi\{\dot{V}_{U-N}\omega(1 - p_U^\#) - \omega p_A^\#\dot{V}_{A-N} + \dot{V}_{A-N}(1 - p_A^\#) - p_U^\#\dot{V}_{U-N}\}\\ &= \mathcal{U}\theta\varphi\{\dot{V}_{U-N}\omega(1 - p_U^\# - p_A^\#) + \dot{V}_{A-N}(1 - p_A^\# - p_U^\#)\}\\ &= \mathcal{U}\theta\varphi p_N^\#\{\dot{V}_{A-N} + \omega\dot{V}_{U-N}\} = (\mu - \hat{\psi})\varphi\{\dot{V}_{A-N} + \omega\dot{V}_{U-N}\}\end{aligned}$$

So that the optimality condition becomes

$$\sigma_s = \mu\varphi\{\dot{V}_{A-N} + \omega\dot{V}_{U-N}\} + \hat{\psi}q\frac{\partial V_N}{\partial x}$$

$$\sigma_s = \mu\varphi\left\{\frac{\partial V_A}{\partial x} + \omega\frac{\partial V_U}{\partial x}\right\} + (\hat{\psi}q - \mu\varphi)\frac{\partial V_N}{\partial x} \tag{D.17}$$

As all $\frac{\partial V_r}{\partial x} \leq 0$ with respect to any link run time, $-\sigma_s$ appears as a congestion cost to the service participants. Under fixed demand and selfish operator behavior, i.e., $\psi = 0$, then $\hat{\psi} = 0$, yielding that $\sigma_s = \mu\varphi\{\dot{V}_{A-N} + \omega\dot{V}_{U-N}\}$. It shows that the congestion costs to link flowing from service participants are relative to those from non-participants (type N). Consequently, the $S_r$ variables are interpreted as the opposites of congestion costs.

# E/ Service optimality and traffic equilibrium

## E.1/ On the price difference between the U and A roles

The net price difference, to an individual, between "Paying as a User" and "Being paid as an Agent", is

$$\tau_{U+A} = \dot{c}_{U-A} + \frac{\xi + \lambda(1+\omega) - S_{U-A} - \chi_{U+A}}{\mu}$$

From their respective formulas,

$$\frac{\xi + \lambda(1+\omega)}{\mu} = \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - (1+\omega)\dot{\rho}\dot{V}_{A\rho}^{\setminus\tau}$$

Recalling that $\dot{\rho} = (1+\omega)^{-2}$ and $\dot{V}_{A\rho}^{\backslash\tau} = \dot{V}_{A\rho}^{\backslash\tau,W} - \frac{\varphi}{\rho}\dot{V}_{A\varphi}$, and using $1-\rho = (1+\omega)^{-1}$, it comes that

$$\begin{aligned}\frac{\xi + \lambda(1+\omega)}{\mu} &= \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - (1-\rho)\left(\dot{V}_{A\rho}^{\backslash\tau,W} - \frac{\varphi}{\rho}\dot{V}_{A\varphi}\right)\\ &= \varphi\left(\dot{V}_{A\varphi}\left(1 + \frac{1-\rho}{\rho}\right) + \omega\dot{V}_{U\varphi}\right) - (1-\rho)\dot{V}_{A\rho}^{\backslash\tau,W}\\ &= \frac{H}{\varphi}\left(\delta_A^W\beta_A^W + \omega\delta_U^W\beta_U^W\right) - \frac{1}{1+\omega}\dot{V}_{A\rho}^{\backslash\tau,W}\end{aligned}$$

Thus,

$$\tau_{U+A} = \dot{c}_{U-A} + \frac{H}{\varphi}\left(\delta_A^W\beta_A^W + \omega\delta_U^W\beta_U^W\right) - \frac{1}{1+\omega}\dot{V}_{A\rho}^{\backslash\tau,W} - \frac{S_{U-A} + \chi_{U+A}}{\mu} \tag{E.1}$$

## E.2/ The ISS condition of traffic equilibrium

At a traffic equilibrium, the ISS condition stems from

$$\frac{y_U}{y_A} = \frac{p_U}{p_A} = \exp(\theta V_{U-A})$$

As $y_U = \omega y_A$, it follows that $\frac{1}{\theta}\ln\omega = V_{U-A}$, or

$$V_{A-U} = -\frac{1}{\theta}\ln\omega \tag{E.2}$$

The utility difference $V_{A-U}$ splits into (i) the price difference $\tau_{A+U}$, (ii) the difference in wait time costs, denoted $V_{A-U}^W$, (iii) the rest of utility difference, denoted $V_{A-U}^{\backslash W,\tau}$:

$$V_{A-U}^W = \frac{\mathrm{H}}{\varphi}(\delta_U^W\beta_U^W - \delta_A^W\beta_A^W\rho) \tag{E.3a}$$

$$V_{A-U}^{\backslash W,\tau} \equiv V_{A-U} - \tau_{A+U} - V_{A-U}^W = (\beta t)_{U-A}^{RST} + \gamma_{U-A} - \tilde{m} \tag{E.3b}$$

The residual variable $V_{A-U}^{\backslash W,\tau}$ involves neither the prices nor the frequency. Thus, the ISS condition becomes

$$\tau_{A+U} + V_{A-U}^W = -V_{A-U}^{\backslash W,\tau} - \frac{1}{\theta}\ln\omega$$

Under service optimality, replacing $\tau_{A+U}$ using (E.1) yields the equivalent condition

$$\dot{c}_{U-A} + \frac{H}{\varphi}\left(\delta_A^W\beta_A^W + \omega\delta_U^W\beta_U^W\right) - \frac{1}{1+\omega}\dot{V}_{A\rho}^{\backslash\tau,W} - \frac{S_{U-A} + \chi_{U+A}}{\mu} + V_{A-U}^W = -V_{A-U}^{\backslash W,\tau} - \frac{1}{\theta}\ln\omega$$

Combining $\frac{H}{\varphi}\left(\delta_A^W\beta_A^W + \omega\delta_U^W\beta_U^W\right)$ to $V_{A-U}^W = \frac{H}{\varphi}\left(\delta_U^W\beta_U^W - \rho\delta_A^W\beta_A^W\right)$, the frequency-related part states as

$$\frac{H}{\varphi}\left(\delta_A^W\beta_A^W(1-\rho) + (1+\omega)\delta_U^W\beta_U^W\right) = \frac{H(1+\omega)}{\varphi}\left(\delta_U^W\beta_U^W + \rho\delta_A^W\beta_A^W\right)$$

The ISS condition thus becomes a characterization of $\varphi$ with respect to $\omega$:

$$\frac{H(1+\omega)}{\varphi}\left(\delta_U^W\beta_U^W + \rho\delta_A^W\beta_A^W\right) = \frac{1}{1+\omega}\dot{V}_{A\rho}^{\backslash\tau,W} - V_{A-U}^{\backslash W,\tau} + \dot{c}_{A-U} + \frac{\chi_{U+A} - S_{A-U}}{\mu} - \frac{1}{\theta}\ln\omega$$

Given $S_{A-U}$, it makes $\varphi$ a function of only $\omega$, the ISS function denoted

$$\tilde{\varphi} = \frac{H(1+\omega)\left(\delta_U^W \beta_U^W + \dot{\rho}\delta_A^W \beta_A^W\right)}{\frac{1}{1+\omega}\dot{V}_{A\rho}^{\backslash\tau,W} - V_{A-U}^{\backslash W,\tau} + \dot{c}_{A-U} + \frac{\chi_{U+A} - S_{A-U}}{\mu} - \frac{1}{\theta}\ln\omega} \tag{E.4}$$

## E.3/ On the service joint utility

Service participation on either side, $\mathcal{S} = \{U, A\}$, yields an expected utility $V_{\mathcal{S}}$ that satisfies

$$V_{\mathcal{S}} = \rho V_{\mathrm{U}} + (1-\rho)V_{\mathrm{A}} + \frac{1}{\theta}\mathcal{H}_\rho$$

Wherein $\mathcal{H}_\rho \equiv -\rho\ln\rho - (1-\rho)\ln(1-\rho)$ is the entropy function of the probability distribution $[\rho, 1-\rho]$ (cf. Appendix §C.4).

At optimal prices, on the U side,

$$V_{\mathrm{U}} = V_U^{\backslash W,\tau} - (\beta t)_U^W - \tau_U$$

$$V_{\mathrm{U}} = V_U^{\backslash W,\tau} - (\beta t)_U^W - \dot{c}_U - \frac{\mathcal{M}}{\mu}p_N^{-1} + \dot{\rho}\dot{V}_{A\rho} + \frac{S_{U-N}}{\mu} + \frac{\chi_{U+N}}{\mu}$$

On the A side,

$$V_{\mathrm{A}} = V_A^{\backslash W,\tau} - (\beta t)_A^W + \tau_A$$

$$V_{\mathrm{A}} = V_A^{\backslash W,\tau} - (\beta t)_A^W - \dot{c}_A - \frac{\mathcal{M}}{\mu}p_N^{-1} + \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right) - \omega\dot{\rho}\dot{V}_{A\rho} + \frac{S_{A-N}}{\mu} - \frac{\chi_{A-N}}{\mu}$$

The convex combination $\rho V_{\mathrm{U}} + (1-\rho)V_{\mathrm{A}}$ includes four terms as follows:

i) $\rho\left\{V_U^{\backslash W,\tau} - \dot{c}_U + \frac{S_{U-N}}{\mu} + \frac{\chi_{U+N}}{\mu}\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \dot{c}_A + \frac{S_{A-N}}{\mu} - \frac{\chi_{A-N}}{\mu}\right\}$,

ii) $-\rho\frac{\mathcal{M}}{\mu}p_N^{-1} - (1-\rho)\frac{\mathcal{M}}{\mu}p_N^{-1} = -\frac{\mathcal{M}}{\mu}p_N^{-1}$,

iii) $\dot{\rho}\dot{V}_{A\rho}\{\rho - (1-\rho)\omega\} = \dot{\rho}\dot{V}_{A\rho}\left\{\frac{\omega}{1+\omega} - \frac{1}{1+\omega}\omega\right\} = 0$,

iv) The time-related parts, namely

$$-\rho(\beta t)_U^W - (1-\rho)\left\{(\beta t)_A^W - \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right)\right\} = -\frac{1}{1+\omega}\left\{\omega(\beta t)_U^W + (\beta t)_A^W - \varphi\left(\dot{V}_{A\varphi} + \omega\dot{V}_{U\varphi}\right)\right\}$$

$$= -\frac{1}{1+\omega}\{\omega(\beta t)_U^W + (\beta t)_A^W - (\beta t)_A^W - \omega(\beta t)_U^W\} = 0$$

The two last terms vanish, so that

$$\rho V_{\mathrm{U}} + (1-\rho)V_{\mathrm{A}} = -\frac{\mathcal{M}}{\mu}p_N^{-1} + \rho\left\{V_U^{\backslash W,\tau} - \dot{c}_U + \frac{S_{U-N}+\chi_{U+N}}{\mu}\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \dot{c}_A + \frac{S_{A-N}-\chi_{A-N}}{\mu}\right\}$$

$$= -\frac{\mathcal{M}}{\mu}p_N^{-1} + \rho\left\{V_U^{\backslash W,\tau} - \dot{c}_U + \frac{S_U+\chi_U}{\mu}\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \dot{c}_A + \frac{S_A-\chi_A}{\mu}\right\} - \frac{S_N-\chi_N}{\mu}$$

All in all,

$$V_{\mathcal{S}} = \frac{1}{\theta}\mathcal{H}_\rho - \frac{\mathcal{M}}{\mu}p_N^{-1} + \rho\left\{V_U^{\backslash W,\tau} - \dot{c}_U + \frac{S_U+\chi_U}{\mu}\right\} + (1-\rho)\left\{V_A^{\backslash W,\tau} - \dot{c}_A + \frac{S_A-\chi_A}{\mu}\right\} - \frac{S_N-\chi_N}{\mu} \tag{E.5}$$

## E.4/ The OSS condition of traffic equilibrium

In traffic equilibrium, the OSS condition comes from

$$\frac{y_A + y_U}{y_N} = \frac{y_{\mathcal{S}}}{y_N} = \frac{p_{\mathcal{S}}}{p_N} = \exp(\theta V_{\mathcal{S}-N})$$

As $p_{\mathcal{S}} = 1 - p_N$, the OSS condition is restated as

$$\ln\frac{1-p_N}{p_N} = \ln(p_N^{-1}-1) = \theta V_{S-N} \tag{E.6}$$

Gathering with (E.5) and using that $\theta\frac{\mathcal{M}}{\mu} = 1-\frac{\hat{\psi}}{\mu}$, we get that

$$\begin{aligned}&\ln(p_N^{-1}-1) + \left(1-\frac{\hat{\psi}}{\mu}\right)p_N^{-1} = \mathcal{H}_\rho \\ &+\theta\left\{\rho\left[V_U^{\backslash W,\tau} - \dot{c}_U + \frac{S_U+\chi_U}{\mu}\right] + (1-\rho)\left[V_A^{\backslash W,\tau} - \dot{c}_A + \frac{S_A-\chi_A}{\mu}\right] - \left[V_{\mathrm{N}} + \frac{S_N-\chi_N}{\mu}\right]\right\}\end{aligned} \tag{E.7}$$

Given $\hat{\psi}$ and the $S_r$, it determines $p_N$ on the basis of $\omega$ only.

## E.4/ Solution in the absence of congestion

In the absence of link congestion, the role run times $t_{\mathrm{N}}^R$ and $t_{\mathrm{U}}^R$ are given. The agent run time is a convex combination of them with respective coefficients $1-\rho$ and $\rho$. The dual variables $\sigma_s$ are null, yielding null congestion costs $S_r$. This makes the relation between $\eta$ and $Z$ simpler:

$$\eta = \frac{Z(\mu-\psi)}{\theta - Z\dot{\mathrm{D}}/q}$$

Yielding that

$$\mu - \hat{\psi} = \mu - \psi + \frac{\eta\dot{\mathrm{D}}}{q} = \frac{\mu-\psi}{\theta-\frac{Z\dot{\mathrm{D}}}{q}}\left(\theta - \frac{Z\dot{\mathrm{D}}}{q} + \frac{Z\dot{\mathrm{D}}}{q}\right) = \frac{\theta}{\theta-\frac{Z\dot{\mathrm{D}}}{q}}(\mu-\psi)$$

Under fixed demand, $\dot{\mathrm{D}} = 0$, yielding $\mu - \hat{\psi} = \mu - \psi$.

As the LHS function in the OSS condition is increasing in $p_N^{-1}$, the OSS condition (knowing $\omega$) yields a unique solution $\tilde{p}_N$, inducing in turn $\mathcal{U}$ and $\mathcal{A}$. Taking a solution $\tilde{\varphi}$ of the ISS condition, the pair $(\omega, \tilde{\varphi})$ induces the roles' deterministic utilities $V_r^{\#}$ and in turn the $p_r^{\#}$. As both the ISS and OSS conditions are satisfied, it holds that $p_N^{\#} = \tilde{p}_N$.

Then, average load $\omega$ and demand volume $q$ are related by the definition of $\varphi$: $\omega$ must satisfy that

$$\tilde{\varphi} = qp_A^{\#} = q\frac{p_S^{\#}}{1+\omega} = q\frac{1-\tilde{p}_N}{1+\omega}.$$

The equation can be put in another way: on defining $\tilde{Q}_\omega \equiv \tilde{\varphi}\frac{1+\omega}{1-\tilde{p}_N}$, the optimum state $\omega$ at demand level $q$ is such that

$$\tilde{Q}_\omega = q$$

This relation can be interpreted reversely: that every $\omega$ induces an equilibrium state for a related demand volume $q$ with value $\tilde{Q}_\omega$. Then, using the parametric curve $\omega \mapsto \tilde{Q}_\omega$, we can relate the optimum frequency $\tilde{\varphi}$ and other outcomes to $q = \tilde{Q}_\omega$. This way, any parametric analysis with respect to $q$ is streamlined, very efficiently in fact.

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