



**HAL**  
open science

# On Kazhdan-Yom Din asymptotic Schur orthogonality for $K$ -finite matrix coefficients

Anne-Marie Aubert, Alfio Fabio La Rosa

► **To cite this version:**

Anne-Marie Aubert, Alfio Fabio La Rosa. On Kazhdan-Yom Din asymptotic Schur orthogonality for  $K$ -finite matrix coefficients. 2024. hal-04781699

**HAL Id: hal-04781699**

**<https://hal.science/hal-04781699v1>**

Preprint submitted on 15 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On Kazhdan-Yom Din's Asymptotic Orthogonality for $K$ -finite matrix coefficients of tempered representations

Anne-Marie Aubert, Alfio Fabio La Rosa

## Abstract

In a recent article, D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic form of Schur's orthogonality for tempered, irreducible, unitary representations of semisimple groups defined over local fields. In the non-Archimedean case, they established it for  $K$ -finite matrix coefficients. The purpose of this article is to prove the analogous result in the Archimedean case.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Recollections on Representation Theory</b>	<b>9</b>
<b>3</b>	<b>Asymptotic behaviour of representations</b>	<b>19</b>
<b>4</b>	<b>Asymptotic Orthogonality</b>	<b>36</b>

## 1 Introduction

Let  $G$  be a semisimple group over a local field, let  $K$  be a maximal compact subgroup of  $G$ . We fix a Haar measure on  $G$ , denoted  $dg$ . If  $H$  is the Hilbert space underlying a unitary representation of  $G$ , let  $H_K$  denote the space of  $K$ -finite vectors and  $H^\infty$  the space of smooth vectors.

In their recent work [10], D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic version of Schur's orthogonality relations. It should hold for matrix coefficients of tempered irreducible unitary representations of  $G$ , generalising Schur's well-known orthogonality relations for discrete series.

Following their article, we fix a norm on the Lie algebra  $\mathfrak{g}$  of  $G$ . By [10], Claim 5.2, we can choose it so that  $\text{Ad}K$  acts unitarily on  $\mathfrak{g}$ . We define the function

$$\mathbf{r}: G \longrightarrow \mathbb{R}_{\geq 0}, \quad \mathbf{r}(g) = \log(\max\{\|\text{Ad}(g)\|_{op}, \|\text{Ad}(g^{-1})\|_{op}\})$$

so that, given  $r \in \mathbb{R}_{>0}$ , we can introduce the corresponding ball

$$G_{<r} := \{g \in G \mid \mathbf{r}(g) < r\}.$$

Given this set-up, we are in position to state their conjecture.

**Conjecture 1.1** (Kazhdan-Yom Din, Asymptotic Schur's Orthogonality Relations). Let  $G$  be a semisimple group over a local field and let  $(\pi, H)$  be a tempered irreducible unitary representation of  $G$ . Then there exist  $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$  and  $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$  such that, for all  $v_1, v_2, v_3, v_4 \in H$ , the following holds:

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle} dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

Assuming that the matrix coefficients involved are  $K$ -finite, one has the following result:

**Theorem 1.2** ([10], Theorem 1.7). Let  $(\pi, H)$  be a tempered, irreducible, unitary representation of  $G$  and  $K$  be a maximal compact subgroup of  $G$ . Then there exists  $\mathbf{d}(\pi) \in \mathbb{Z}_{\geq 0}$  such that:

- (1) If  $G$  is non-Archimedean, there exists  $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$  such that, for all  $v_1, v_2, v_3, v_4 \in H_K$ , we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle} dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

- (2) If  $G$  is Archimedean, for any given non-zero  $v_1, v_2 \in H_K$ , there exists  $C(v_1, v_2) > 0$  such that

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} |\langle \pi(g)v_1, v_2 \rangle|^2 dg = C(v_1, v_2).$$

In the non-Archimedean case, the proof of (1) is achieved by first establishing the validity of the analogous version of (2). The polarisation identity allows the authors of [10] to define a form

$$D(\cdot, \cdot, \cdot, \cdot) : H_K \times H_K \times H_K \times H_K \longrightarrow \mathbb{C}$$

via the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle} dg.$$

In [10], Section 4.1, this form is shown to be  $G$ -invariant and one would like to invoke an appropriate form of Schur's lemma to argue as in the standard proof of Schur's orthogonality relations. That is, for fixed  $v_2, v_4 \in H_K$ , one defines the form

$$D(\cdot, v_2, \cdot, v_4) : H_K \times H_K \longrightarrow \mathbb{C}$$

and, for fixed  $v_1, v_3 \in H_K$ , the form

$$D(v_1, \cdot, v_3, \cdot) : H_K \times H_K \longrightarrow \mathbb{C}.$$

One applies Schur's lemma to these forms, which implies that each such form is a scalar multiple of the inner product on  $H$ . Upon comparing them, one obtains the desired orthogonality relations.

The appropriate version of Schur's lemma in the non-Archimedean case is provided by Dixmier's lemma, which can be applied since in the non-Archimedean setting the subspace of  $K$ -finite vectors  $H_K$  and the subspace of smooth vectors  $H^\infty$  coincide, the latter being equipped with the structure of a Fréchet representation of  $G$ , which is irreducible since  $H$  itself is irreducible.

The purpose of this article is to prove that the analogue of (1) in Theorem 1.2 holds in the Archimedean case. As explained in [10], Section 4.2, it suffices to prove the result for real semisimple groups (Theorem 4.4).

**Theorem 1.3.** Let  $(\pi, H)$  be a tempered, irreducible, Hilbert representation of a connected, semisimple Lie group  $G$  with finite centre. Let  $K$  be a maximal compact subgroup of  $G$ . Then there exists  $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$  such that, for all  $v_1, v_2, v_3, v_4 \in H_K$ , we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle} dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

We need to modify the strategy above to account for the fact that the space of  $K$ -finite vectors of a Hilbert space representation  $(\pi, H)$  of a real semisimple group does not afford a representation of  $G$ . It is, however, an admissible  $(\mathfrak{g}, K)$ -module.

Our approach relies crucially on the admissibility of irreducible, Hilbert representations of reductive Lie groups, a foundational theorem proved by Harish-Chandra. The theory of admissible  $(\mathfrak{g}, K)$ -modules then provides us with the appropriate version of Schur's lemma for  $(\mathfrak{g}, K)$ -invariant forms (Definition 2.11).

Hence, we are reduced to verify that  $D(\cdot, v_2, \cdot, v_4)$  and  $D(v_1, \cdot, v_3, \cdot)$  are, indeed,  $(\mathfrak{g}, K)$ -invariant. Having established this, to conclude the proof of Theorem 1.3, we can argue as in [10, Section 4].

From now on, to make the notation look more compact, given a Hilbert representation  $(\pi, H)$  of  $G$  and vectors  $v, w \in H$ , we set

$$\phi_{v,w}(g) := \langle \pi(g)v, w \rangle.$$

Let  $\dot{\pi}$  denote the following action of  $\mathfrak{g}$  on the smooth vectors  $H^\infty$  of  $H$ :

$$\dot{\pi}(X) := \frac{d}{dt} \pi(\exp(tX)v)|_{t=0}, \quad \text{for } v \in H^\infty \text{ and } X \in \mathfrak{g}.$$

For connected, semisimple Lie groups with finite centre,  $K$ -invariance is a consequence of  $\mathfrak{g}$ -invariance (Proposition 2.14). Therefore, the problem is establishing the  $\mathfrak{g}$ -invariance. Explicitly, we prove the following (Proposition 4.2).

**Proposition 1.4.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $(\pi, H)$  be a tempered, irreducible, Hilbert representation of  $G$ . Then, for all  $X \in \mathfrak{g}$ , and for all  $v_1, v_2, v_3, v_4 \in H_K$ , we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1, v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{\dot{\pi}(X)v_3, v_4}(g) dg$$

and

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{v_3, \dot{\pi}(X)v_4}(g) dg.$$

The key observation is that, by exploiting the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup  $P = MAN$  and with respect to the standard (for  $P$ ) parabolic subgroups of  $G$ , the expression

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1, v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg$$

reduces, roughly, to a sum of finitely many terms of the form

$$\int_K \langle \Gamma_{\lambda, l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu, m}(m_\lambda, \pi(k)v_3, w_4) \rangle_{L^2(M_\lambda)} dk.$$

Here,  $M_\lambda$  comes from a standard parabolic subgroup  $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$  of  $G$ . We denote  $\mathfrak{m}_\lambda$ ,  $\mathfrak{a}_{\lambda_0}$ ,  $\mathfrak{n}_{\lambda_0}$  the Lie algebras of  $M_\lambda$ ,  $A_{\lambda_0}$ ,  $N_{\lambda_0}$ , respectively. The functions  $\Gamma_{\lambda, l}$ ,  $\Gamma_{\mu, m}$ , as functions of  $m_\lambda$ , are analytic and square-integrable and arise from the asymptotic expansion of the matrix coefficients  $\phi_{\dot{\pi}(X)v_1, v_2}$  and  $\phi_{v_3, v_4}$ , respectively, relative to  $P_\lambda$ . The subscript in  $P_\lambda$  is meant to indicate that the parabolic subgroup is obtained, in an appropriate sense, from the datum of  $\lambda$ . Moreover,  $(\lambda, l)$  and  $(\mu, m)$  are related in a precise way.

We shall elaborate on these points later on. For the moment, let us point out that we reduced the initial problem to showing that, for every  $X \in \mathfrak{g}$ , and for all relevant pairs  $(\lambda, l)$  and  $(\mu, m)$ , the integral

$$\int_K \langle \Gamma_{\lambda, l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu, m}(m_\lambda, \pi(k)v_3, w_4) \rangle_{L^2(M_\lambda)} dk$$

equals

$$- \int_K \langle \Gamma_{\lambda, l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu, m}(m_\lambda, \pi(k)\dot{\pi}(X)v_3, w_4) \rangle_{L^2(M_\lambda)} dk.$$

We will prove that, if  $(\lambda, l)$  and  $(\mu, m)$  satisfy a certain condition (to be explained below), the functions  $\Gamma_{\lambda, l}(\cdot, v_1, w_2)$  and  $\Gamma_{\mu, m}(\cdot, v_3, w_4)$  are, in fact,  $Z(\mathfrak{g}_{\mathbb{C}})$ -finite, with  $Z(\mathfrak{g}_{\mathbb{C}})$  denoting the centre of the universal enveloping algebra of the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$ , and  $K \cap M_\lambda$ -finite. It will then follow from a theorem of Harish-Chandra (Theorem 2.17) that they are smooth vectors in the right-regular representation  $(R, L^2(M_\lambda))$

of  $M_\lambda$  .

The idea is to combine this observation with an appropriate form of Frobenius' reciprocity (Theorem 2.24), due to Casselman, to construct  $(\mathfrak{g}, K)$ -invariant maps

$$T_{w_2} : H_K \longrightarrow \text{Ind}_{\overline{P}_\lambda, K_\lambda} (H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_2}(v)(k)(m_\lambda) := \Gamma_{\lambda, l}(m_\lambda, \pi(k)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \text{Ind}_{\overline{P}_\lambda, K_\lambda} (H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_4}(v')(k)(m_\lambda) := \Gamma_{\lambda, l}(m_\lambda, \pi(k)v', w_4).$$

Here, the subgroup  $\overline{P}_\lambda$  is the parabolic subgroup opposite to  $P_\lambda$ . The notation  $\text{Ind}_{\overline{P}_\lambda, K}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$  stands for the space of  $K$ -finite vectors in the representation induced from the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K \cap M_\lambda)$ -module

$$H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

for an appropriately chosen admissible, unitary, sub-representation  $(\sigma, H_\sigma)$  of  $(R, L^2(M_\lambda))$ .

To apply the required form of Frobenius' reciprocity, we need to show that the maps

$$S_{w_2} : H_K \longrightarrow H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_2}(v)(m_\lambda) := \Gamma_{\lambda, l}(m_\lambda, v, w_2)$$

and

$$S_{w_4} : H_K \longrightarrow H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_4}(v')(m_\lambda) := \Gamma_{\lambda, l}(m_\lambda, v', w_4)$$

descend to  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant maps on  $H_K/\mathfrak{n}_{\lambda_0}H_K$ . Establishing this result is the technical heart of the article.

Assuming it, the integral

$$\int_K \langle \Gamma_{\lambda, l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu, m}(m_\lambda, \pi(k)v_3, w_4) \rangle_{L^2(M_\lambda)} dk$$

is nothing but

$$\langle \text{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X) \Gamma_{\lambda, l}(m_\lambda, v_1, w_2), \Gamma_{\mu, m}(m_\lambda, v_3, w_4) \rangle_{\text{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})},$$

where  $\langle \cdot, \cdot \rangle_{\text{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}$  is the inner product on  $\text{Ind}_{\overline{P}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ . We will see that this makes sense since the inducing data ensure unitarity. The sought equality will then follow from the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra.

To explain how the functions  $\Gamma_{\lambda, l}(\cdot, v_1, v_2)$  and  $\Gamma_{\mu, m}(\cdot, v_3, v_4)$  arise, we need to recall the main features of the asymptotic expansions of  $K$ -finite matrix coefficients of tempered representations. If  $\phi_{v, w}$  is such a matrix coefficient, then its asymptotic expansion relative to the minimal parabolic subgroup  $P$  of  $G$  can be thought of as a sum indexed by a countable collection

$$\Lambda := \{(\lambda, l)\}_{\lambda \in \mathcal{E}, l \in \mathbb{Z}_{\geq 0}^n : |l| \leq l_0}.$$

The set  $\mathcal{E}$  is a collection of complex-valued real-linear functionals on  $\text{Lie}(A)$  depending on  $(\pi, H)$  and not on the particular choice of  $v, w \in H_K$ . It is the set of **exponents** of  $(\pi, H)$ . The number  $n$  is the rank of  $G$  and  $l_0$ , too, depends on  $(\pi, H)$  only.

The term indexed by  $(\lambda, l)$  is multiplied by a complex coefficient  $c_{\lambda, l}(v, w)$ . The choice of  $v, w \in H_K$  determines the pairs in  $\mathcal{C}$  for which  $c_{\lambda, l}(v, w) \neq 0$ . If  $\lambda \in \mathcal{E}$ , there exists at least a pair of  $v, w \in H_K$  such that, for some  $l \in \mathbb{Z}_{\geq 0}^n$  with  $|l| \leq l_0$ , we have  $c_{\lambda, l}(v, w) \neq 0$ .

For any standard (for  $P$ ) parabolic subgroup  $P' = M'A'N'$  of  $G$ , the matrix coefficient  $\phi_{v, w}$  admits a similar asymptotic expansion. It can be thought of as a sum indexed by a countable collection

$$\Lambda' := \{(\nu, q)\}_{\nu \in \mathcal{E}', q \in \mathbb{Z}_{\geq 0}^r: |q| \leq q_0}.$$

Here,  $r \leq n$  is the dimension of  $A'$ , the set  $\mathcal{E}'$  consists of complex-valued real-linear functionals on  $\text{Lie}(A')$ . On regions on which both the expansion relative to  $P$  and the expansion relative to  $P'$  are meaningful, by comparing the two it turns out that the elements in  $\mathcal{E}'$  are precisely the restrictions to  $\text{Lie}(A')$  of the elements in  $\mathcal{E}$  and, making the appropriate identifications following from  $A' \subset A$ , each  $q$  is the projection to  $\mathbb{Z}_{\geq 0}^r$  of an  $l$  appearing in the expansion relative to  $P$ .

While in the expansion relative to  $P$  the term indexed by  $(\lambda, l)$  is multiplied by the complex coefficient  $c_{\lambda, l}(v, w)$ , the term indexed by  $(\nu, q)$  in the expansion relative to  $P'$  is multiplied by a real-analytic function

$$c_{\nu, q}(\cdot, v, w) : M' \rightarrow \mathbb{C}.$$

We need one more piece of information to explain how the functions  $\Gamma_{\lambda, l}(\cdot, v_1, v_2)$  and  $\Gamma_{\mu, m}(\cdot, v_3, v_4)$  arise: the construction of  $\mathbf{d}(\pi)$  in [10]. The idea is as follows. We can think of  $\lambda \in \mathcal{E}$  as an  $n$ -tuple of complex numbers  $(\lambda_1, \dots, \lambda_n)$ . It can be shown that there exist a finite sub-collection  $\mathcal{E}_0 \subset \mathcal{E}$  such that, for every  $\lambda \in \mathcal{E}$ , there exists  $\hat{\lambda} \in \mathcal{E}_0$  such that

$$\hat{\lambda} - \lambda \in \mathbb{Z}_{\geq 0}^n.$$

Moreover, any two distinct elements in  $\mathcal{E}_0$  are integrally inequivalent: their difference does not belong to  $\mathbb{Z}^n$ . By a result of Casselman (Theorem 3.2), for every  $\hat{\lambda} \in \mathcal{E}_0$  and for every  $i \in \{1, \dots, n\}$ , we have

$$\text{Re} \hat{\lambda}_i \leq 0$$

and it is clear that this holds for every  $\lambda \in \mathcal{E}$ .

For  $(\lambda, l) \in \Lambda$ , we introduce the set  $I_\lambda := \{i \in \{1, \dots, n\} | \text{Re} \lambda_i < 0\}$ , we define

$$\mathbf{d}_P(\lambda, l) := |I_\lambda^c| + \sum_{i \in I_\lambda^c} 2l_i \tag{1}$$

and we take the maximum,  $\mathbf{d}_P$ , as  $(\lambda, l)$  ranges over all the pairs with  $\lambda \in \mathcal{E}_0$ .

We can proceed analogously for every standard parabolic  $P'$  and obtain a non-negative integer  $\mathbf{d}_{P'}$ . The maximum over all  $P'$  is  $\mathbf{d}(\pi)$ .

Now, given  $\lambda \in \mathcal{E}_0$ , identifying  $I_\lambda$  with a subset of the simple roots determined by an order on the root system  $(\mathfrak{g}, \mathfrak{a})$ , we can construct a standard (for  $P$ ) parabolic subgroup  $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$  associated to  $I_\lambda$ . We will show that if  $(\lambda, l) \in \Lambda$  satisfies  $\lambda \in \mathcal{E}_0$  and  $\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi)$ , then  $\Gamma_{\lambda, l}(\cdot, v_1, v_2)$  is precisely the function  $c_{\nu, q}^{P_\lambda}(\cdot, v_1, v_2)$  with  $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$ , where  $\mathfrak{a}_{\lambda_0} := \text{Lie}(A_{\lambda_0})$ , and  $q$  equal to the projection of  $l$  to  $\mathbb{Z}_{\geq 0}^{\mathfrak{I}_\lambda}$ .

Finally, we mentioned that in the integral

$$\int_K \langle \Gamma_{\lambda, l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu, m}(m_\lambda, \pi(k)v_3, w_4) \rangle_{L^2(M_\lambda)} dk$$

the pairs  $(\lambda, l)$  and  $(\mu, m)$  must be related in a precise way. First of all,  $(\mu, m) \in \Lambda$  satisfies  $\mu \in \mathcal{E}_0$  and  $\mathbf{d}_P(\mu, m) = \mathbf{d}(\pi)$ . In addition, we must have  $I_\lambda = I_\mu$  (so that  $P_\lambda = P_\mu$ ) and  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ . The last condition, together with the unitarity of the representation  $(\sigma, H_\sigma)$  introduced above, is precisely what insures that  $\text{Ind}_{P_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$  is unitary.

Implementing the strategy sketched above requires gathering a number of intermediate results. Several are inspired from the chapter in [11] on Langland's classification of tempered representations. Here is a more detailed outline of the article.

**Section 2:** The first part includes a discussion of the  $(\mathfrak{g}, K)$ -module version of Schur's lemma (Theorem 2.13). In the second part, we recall the result of Harish-Chandra establishing that smooth,  $Z(\mathfrak{g}_\mathbb{C})$ -finite,  $K$ -finite, square-integrable functions on reductive groups are smooth vectors in the right-regular representation (Theorem 2.16). As a consequence, we prove that, on such a function, the action of  $\mathfrak{g}$  through differentiation is the same as the action of the Lie algebra through the right-regular representation (Proposition 2.18). After stating the basic facts on parabolically induced representations that we need, we discuss Casselman's version of Frobenius's reciprocity (Theorem 2.24).

**Section 3:** In the first part, we recall the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup and with respect to standard parabolic subgroups. We then explain in detail how the functions  $\Gamma_{\lambda, l}(\cdot, v_1, v_2)$ ,  $\Gamma_{\mu, m}(\cdot, v_3, v_4)$  arise. We begin by introducing an equivalence relation on the data indexing the asymptotic expansion relative to  $P$  of the  $K$ -finite matrix coefficients of a tempered, irreducible, Hilbert representation  $(\pi, H)$ . This equivalence relation is motivated by construction of  $\mathbf{d}(\pi)$  in [10] and it is meant to exploit the criteria for the computation of asymptotic integrals in Appendix A in loc. cit. Imposing the conditions on  $(\lambda, l)$  and  $(\mu, m)$  that we discussed above, we identify the functions  $\Gamma_{\lambda, l}(\cdot, v_1, v_2)$  and  $\Gamma_{\mu, m}(\cdot, v_3, v_4)$  with the coefficient functions in the asymptotic expansion relative to  $P_\lambda$  of  $\phi_{v_1, v_2}$  and  $\phi_{v_3, v_4}$  (Proposition 3.4). We then prove that they are smooth vectors in  $(R, L^2(M_\lambda))$  (Proposition 3.6). Combining Proposition 3.6 with the technical Lemma 3.7 and Lemma 3.8, we are in position to



construct unitary, admissible, finitely generated representations  $(\sigma_1, H_{\sigma_1})$  and  $(\sigma_2, H_{\sigma_2})$  whose direct sum is the unitary, admissible, finitely generated representation  $(\sigma, H_{\sigma})$  introduced above (Proposition 3.10).

**Section 4:** Having gathered the results we need, we prove Proposition 1.4 (Proposition 4.2). We begin with a computational Lemma which shows that second identity in Proposition 1.4 follows from the first (Lemma 4.1). The first part of the proof of Proposition 4.2 consists of an application of the considerations in Appendix A of [10] and a series of integral manipulations (justified in Lemma 4.3) aimed at showing that the integral

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \bar{\phi}_{v_3, v_4}(g) dg$$

can be computed in terms of a sum of integrals of the form

$$\int_K \langle \Gamma_{\lambda, l}(m_{\lambda}, \pi(k)v_1, w_2), \bar{\Gamma}_{\mu, m}(m_{\lambda}, \pi(k)v_3, w_4) \rangle dk$$

with the pairs  $(\lambda, l)$  and  $(\mu, m)$  both belonging to  $\Lambda$  with  $\lambda, \mu \in \mathcal{E}_0$ ,  $I_{\lambda} = I_{\mu}$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}_P(\mu, m) = \mathbf{d}(\pi).$$

We explained how to construct the unitary, finitely generated, admissible representation  $(\sigma, H_{\sigma})$  needed to apply Casselman's version of Frobenius' reciprocity. The discussion following Proposition 3.10 therefore gives  $(\mathfrak{g}, K)$ -equivariant maps

$$T_{w_2} : H_K \longrightarrow \text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_2}(v)(k)(m_{\lambda}) := \Gamma_{\lambda, l}(m_{\lambda}, \pi(k)v, w_2)$$

and

$$T_{w_4} : H_K \longrightarrow \text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_4}(v)(k)(m_{\lambda}) := \Gamma_{\mu, m}(m_{\lambda}, \pi(k)v, w_4).$$

The condition  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and the fact that, by the definition of  $I_{\lambda}$ , the functional  $\lambda|_{\mathfrak{a}_{\lambda_0}}$  is totally imaginary, shows that

$$\int_K \langle \Gamma_{\lambda, l}(m_{\lambda}, \pi(k)v_1, w_2), \bar{\Gamma}_{\mu, m}(m_{\lambda}, \pi(k)v_3, w_4) \rangle dk$$

is equal to

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

The  $(\mathfrak{g}, K)$ -equivariance of  $T_{w_2}$  gives

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle \text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})T_{w_2}(v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})},$$

by Corollary 2.20 we have

$$\begin{aligned} & \langle \text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_2}(v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} \\ &= -\langle T_{w_2}(v_1), \text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_4}(v_3) \rangle_{\text{Ind}_{\bar{\mathcal{P}}_{\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} \end{aligned}$$

and the  $(\mathfrak{g}, K)$ -equivariance of  $T_{w_4}$  gives

$$\langle T_{w_2}(v_1), \text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle T_{w_2}(v_1), T_{w_4}\dot{\pi}(X)v_3 \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})},$$

thus completing the proof of Proposition 4.2.

Finally, we proceed as explained in the first part of the Introduction to prove Theorem 1.3 (Theorem 4.4).

**Acknowledgements:** We would like to thank the anonymous referee for pointing out a serious gap in the previous version of the article.

## 2 Recollections on Representation Theory

Our presentation of the theory of  $(\mathfrak{g}, K)$ -modules follows [17]. To discuss its basic features, we need to gather some results on unitary representations of compact groups. We begin by recalling the basic notions in the study of representations of topological groups, which we always assume to be Hausdorff.

First, following [17], Section 1.1, let  $G$  denote a second-countable, locally compact group, equipped with a left Haar measure  $dg$ , and let  $V$  denote a complex topological vector space. We denote by  $\text{GL}(V)$  the group of invertible continuous endomorphisms of  $V$ . A **representation** of  $G$  on  $V$  is a strongly continuous homomorphism  $\pi : G \rightarrow \text{GL}(V)$ . Let  $(\pi, V)$  denote the datum of a representation of  $G$ . A subspace of  $V$  which is stable under the action of  $G$  through  $\pi$  is called an **invariant subspace**. A representation is said to be **irreducible** if the only closed invariant subspaces are the trivial subspace and  $V$  itself.

If  $(H, \langle \cdot, \cdot \rangle)$  is a separable Hilbert space, a representation  $\pi$  of  $G$  on  $H$  is termed a **Hilbert representation**. If, in addition,  $G$  acts by unitary operators through  $\pi$ , the representation is said to be **unitary**.

Next, following [14], Section 10, we introduce the basic features of the theory of vector-valued integration.

Let  $(X, dx)$  be a Radon measure space, let  $H$  be a Hilbert space and assume that

$$f : X \rightarrow H$$

is measurable. The function  $f$  is **integrable** if it satisfies the following two conditions:

- (1) For all  $v \in H$ ,

$$\int_X |\langle f(x), v \rangle| dx < \infty.$$

- (2) The map  $v \mapsto \int_X |\langle f(x), v \rangle| dx$  is a bounded conjugate-linear functional.

If  $f: X \rightarrow H$  is integrable, then, by the Riesz' representation theorem, there exists a unique element in  $H$ , denoted

$$\int_X f(x) dx,$$

such that, for all  $v \in H$ , we have

$$\left\langle \int_X f(x) dx, v \right\rangle = \int_X \langle f(x), v \rangle dx.$$

**Proposition 2.1.** Let  $(X, dx)$  be as above. Let  $H, E$  be Hilbert spaces,  $f: X \rightarrow H$  a measurable function and  $T: H \rightarrow E$  a bounded linear operator. Then the following holds:

(1) If

$$\int_X \|f(x)\| dx < \infty,$$

then  $f: X \rightarrow H$  is integrable.

(2) If  $f: X \rightarrow H$  is integrable, then so is  $Tf: X \rightarrow E$ . Moreover,

$$T \left( \int_X f(x) dx \right) = \int_X Tf(x) dx.$$

*Proof.* See [14], Proposition 10.8 and Proposition 10.9. □

Now, let  $(\pi, H)$  be a unitary representation of  $G$ . Let  $v \in H$  and  $f: G \rightarrow H$  be such that the map

$$g \mapsto f(g)\pi(g)v$$

is integrable. Let  $\pi(f)v$  denote the unique element in  $H$  such that, for all  $w \in H$ , we have

$$\langle \pi(f)v, w \rangle = \int_G f(g)\langle \pi(g)v, w \rangle dg.$$

**Proposition 2.2.** Let  $(\pi, H)$  be as above. If  $f \in L^1(G)$ , then, for all  $v \in H$ , the map  $g \mapsto f(g)\pi(g)v$  is integrable and the prescription

$$\pi(f): H \rightarrow H, v \mapsto \pi(f)v$$

defines a bounded linear operator.

*Proof.* See [14], Proposition 10.20. □

With the integral operators introduced in Proposition 2.2 at our disposal, we have all the tools needed to state the main results on the unitary representations of compact groups.

Let  $K$  be a compact group. Let  $\widehat{K}$  denote the set of equivalence classes of irreducible unitary representations of  $K$ . If  $(\pi, H)$  is a unitary representation, for each  $\gamma \in \widehat{K}$  let  $H(\gamma)$  denote the closure of the sum of all the closed invariant subspaces of  $H$  in the equivalence class of  $\gamma$ . We refer to  $H(\gamma)$  as the  $\gamma$ -isotypic component of  $H$ .

**Proposition 2.3.** Let  $K$  be a compact group. Let  $(\pi, H)$  be an irreducible unitary representation of  $K$ . Then  $H$  is finite-dimensional.

*Proof.* See [17], Proposition 1.4.2. □

Given Proposition 2.3, we can associate, to each  $\gamma \in \widehat{K}$ , the function

$$\chi_\gamma : K \longrightarrow \mathbb{C}, \quad \chi_\gamma(g) := \text{tr}\gamma(g),$$

the **character** of  $\gamma$ .

Recall that if  $\{(\pi_i, H_i) | i \in I\}$  is a countable family of unitary representations of a topological group  $G$ , we can construct a new unitary representation of  $G$ , the **direct sum**, on the Hilbert space completion of the algebraic direct sum of the  $H_i$ 's. We refer the reader to [17], Section 1.4.1, for the details of this construction. We let

$$\bigoplus_{i \in I} H_i$$

denote the direct sum of the family  $\{(\pi_i, H_i) | i \in I\}$ , dropping explicit reference to the  $\pi_i$ 's.

**Proposition 2.4.** Let  $K$  be a compact group. Let  $(\pi, H)$  be a unitary representation of  $K$ . Then  $(\pi, H)$  is the direct sum representation of its  $K$ -isotypic components; that is,

$$H = \bigoplus_{\gamma \in \widehat{K}} H(\gamma).$$

Moreover, let  $\alpha_\gamma$  denote the function

$$\alpha_\gamma(k) := \dim(\gamma) \overline{\chi_\gamma(k)}.$$

Then the following holds:

$$H(\gamma) = \pi(\alpha_\gamma)H.$$

*Proof.* See [17], Lemma 1.4.7. □

**Proposition 2.5.** Let  $K$  be a compact group. If  $(\pi, H)$  is a Hilbert space representation of  $K$ , then there exists an inner product on  $H$  that induces the original topology on  $H$  and for which  $K$  acts unitarily through  $\pi$ .

*Proof.* See [17], Lemma 1.4.8. □

We are finally ready to introduce  $(\mathfrak{g}, K)$ -modules.

**Definition 2.6.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $\mathfrak{g}$  denote its Lie algebra. Let  $K$  be a maximal compact subgroup of  $G$ , which we fix from now on, with Lie algebra  $\mathfrak{k}$ . A vector space  $V$ , equipped with the structure of  $\mathfrak{g}$ -module and  $K$ -module, is called a  **$(\mathfrak{g}, K)$ -module** if the following conditions hold:

(1) For all  $v \in V$ , for all  $X \in \mathfrak{g}$ , for all  $k \in K$ ,

$$kXv = \text{Ad}(k)Xkv$$

(2) For all  $v \in V$ , the span of the set

$$Kv := \{kv | k \in K\}$$

is a finite-dimensional subspace of  $V$ , on which the action of  $K$  is continuous.

(3) For all  $v \in V$ , for all  $Y \in \mathfrak{k}$ ,

$$\frac{d}{dt} \exp(tY)v|_{t=0} = Yv.$$

We remark that (3) implicitly uses the smoothness of the action of  $K$  on the span of  $Kv$ . This follows from the fact that a continuous group homomorphism between Lie groups is automatically smooth.

Let  $V$  and  $W$  be  $(\mathfrak{g}, K)$ -modules and let  $\text{Hom}_{\mathfrak{g}, K}(V, W)$  denote the space of  $\mathfrak{g}$ -morphisms that are also  $K$ -equivariant. Then  $V$  and  $W$  are said to be **equivalent** if  $\text{Hom}_{\mathfrak{g}, K}(V, W)$  contains an invertible element.

A  $(\mathfrak{g}, K)$ -module  $V$  is called **irreducible** if the only subspaces that are invariant under the actions of  $\mathfrak{g}$  and  $K$  are the trivial subspace and  $V$  itself. In this case, we have the following theorem:

**Theorem 2.7.** Let  $V$  be an irreducible  $(\mathfrak{g}, K)$ -module. Then the space  $\text{Hom}_{\mathfrak{g}, K}(V, V)$  is 1-dimensional.

*Proof.* This is the result actually proved in [17] Lemma 3.3.2, although the statement there says  $\text{Hom}_{\mathfrak{g}, K}(V, W)$ , for an unspecified  $W$ . We believe it is a typo.  $\square$

Let  $V$  be a  $(\mathfrak{g}, K)$ -module. Since, given each  $v \in V$ , the span of  $Kv$ , say  $W_v$ , is a finite-dimensional continuous representation of  $K$ , we can use Proposition 2.5 and then apply Proposition 2.4, thus decomposing  $W_v$  into a finite sum of finite-dimensional  $K$ -invariant subspaces of  $V$ . For  $\gamma \in \widehat{K}$ , we let  $V(\gamma)$  denote the sum of all the  $K$ -invariant finite dimensional subspaces in the equivalence class of  $\gamma$ . Then the discussion above implies that

$$V = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$$

as a  $K$ -module, with the direct sum indicating the algebraic direct sum. A  $(\mathfrak{g}, K)$ -module  $V$  is called **admissible** if, for all  $\gamma \in \widehat{K}$ ,  $V(\gamma)$  is finite-dimensional.

Given a unitary representation  $(\pi, H)$ , there exists a  $(\mathfrak{g}, K)$ -module naturally associated to it. To define it, recall that a vector  $v \in H$  is called **smooth** if the map

$$g \mapsto \pi(g)v$$

is smooth. Let  $H^\infty$  denote the subspace of smooth vectors of  $H$ . It is a standard fact that the prescription

$$\hat{\pi}(X) := \frac{d}{dt}\pi(\exp(tX))v|_{t=0},$$

for  $v \in H^\infty$  and  $X \in \mathfrak{g}$ , defines an action of  $\mathfrak{g}$  on  $H^\infty$ . Recall that a vector  $v \in H$  is  **$K$ -finite** if the span of the set

$$\pi(K)v := \{\pi(k)v | k \in K\}$$

is finite-dimensional. Let  $H_K$  denote the subspace of  $K$ -finite vectors of  $H$ . By [17], Lemma 3.3.5, with the action of  $\mathfrak{g}$  so defined and with the action of  $K$  through  $\pi$ , the space  $H_K \cap H^\infty$  is a  $(\mathfrak{g}, K)$ -module. The representation  $(\pi, H)$  is said to be **admissible** if  $H_K \cap H^\infty$  is admissible as a  $(\mathfrak{g}, K)$ -module and  $(\pi, H)$  is called **infinitesimally irreducible** if  $H_K \cap H^\infty$  is irreducible as a  $(\mathfrak{g}, K)$ -module. It is in general not true that a  $K$ -finite vector is smooth. However, if  $(\pi, H)$  is admissible, we have the following result:

**Theorem 2.8.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $(\pi, H)$  be an admissible representation of  $G$ . Then every  $K$ -finite vector is smooth.

*Proof.* See the proof [17], Theorem 3.4.10. □

In light of the following fundamental result of Harish-Chandra, Theorem 2.8 will play an important role in this article.

**Theorem 2.9.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $(\pi, H)$  be an irreducible, Hilbert representation of  $G$ . Then  $(\pi, H)$  is admissible.

*Proof.* See [13], Theorem 7.204. □

In the following, given a unitary representation  $(\pi, H)$ , we will write  $H_K$  for the  $(\mathfrak{g}, K)$ -module  $H_K \cap H^\infty$  even if  $(\pi, H)$  is not admissible. We believe it will not cause any confusion.

We are now in position to prove the version of Schur's lemma for sesquilinear forms that we will use in Section 3. It is given as Corollary 2.13 below. First, we need:

**Theorem 2.10.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $(\pi, H)$  be an admissible Hilbert representation of  $G$ . Then  $(\pi, H)$  is irreducible if and only if it is infinitesimally irreducible.

*Proof.* See [17], Theorem 3.4.11. □

**Definition 2.11.** Let  $V$  and  $W$  be  $(\mathfrak{g}, K)$ -modules. A sesquilinear form

$$B(\cdot, \cdot) : V \times W \longrightarrow \mathbb{C}$$

is  $(\mathfrak{g}, K)$ -invariant if it satisfies the following two conditions:

- (i) For all  $k_1, k_2 \in K$  and all  $v, w \in V$  we have

$$B(k_1v, k_2w) = B(v, w).$$

(ii) For all  $X \in \mathfrak{g}$  and all  $v, w \in V$  we have

$$B(Xv, w) = -B(v, Xw).$$

**Theorem 2.12.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $V$  be an admissible  $(\mathfrak{g}, K)$ -module. Suppose that there exist a  $(\mathfrak{g}, K)$ -module  $W$  and a non-degenerate  $(\mathfrak{g}, K)$ -invariant sesquilinear form

$$B(\cdot, \cdot) : V \times W \longrightarrow \mathbb{C}.$$

Then  $W$  is  $(\mathfrak{g}, K)$ -isomorphic to  $\overline{V}$ .

*Proof.* This is [17], Lemma 4.5.1, except for the fact that our form is sesquilinear. To account for it, we modify the definition of the map  $T$  in the reference by setting, for a given  $w \in W$ ,  $T(w)(v) = B(w, v)$  for all  $v \in V$ . This defines a map from  $W$  to  $\overline{V}$  obtained by sending  $w$  to  $T(w)$  which, by the argument in the reference, is a  $(\mathfrak{g}, K)$ -isomorphism.  $\square$

The next corollary is proved by adapting to our case the argument in [4], Proposition 8.5.12, and using the beginning of the proof of [11], Proposition 9.1.

**Corollary 2.13.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $(\pi, H)$  be an irreducible, admissible, Hilbert representation of  $G$ . Then, up to a constant, there exists at most one non-zero  $(\mathfrak{g}, K)$ -invariant sesquilinear form on  $H_K$ . In particular, if  $(\pi, H)$  is irreducible unitary, then every such form is a constant multiple of  $\langle \cdot, \cdot \rangle$ .

*Proof.* The irreducibility of  $H$  implies that of  $H_K$ , by Theorem 2.10 and by Theorem 2.8. Let  $B(\cdot, \cdot)$  be a  $(\mathfrak{g}, K)$ -invariant sesquilinear form. Consider the linear subspace  $V_0$  of  $H_K$  defined as

$$V_0 := \{v \in H_K \mid B(v, w) = 0 \text{ for all } w \in H_K\}.$$

Since  $B(\cdot, \cdot)$  is non-zero,  $V_0$  is a proper subspace of  $H_K$ . Since  $B(\cdot, \cdot)$  is moreover  $(\mathfrak{g}, K)$ -invariant, it follows that  $V_0$  is a  $(\mathfrak{g}, K)$ -invariant subspace of  $H_K$ , hence, by the irreducibility of  $H_K$ , it must be zero. Analogous considerations for the subspace

$$V^0 := \{w \in H_K \mid B(v, w) = 0 \text{ for all } v \in H_K\}$$

imply that  $B(\cdot, \cdot)$  is non-degenerate. By Theorem 2.12, the map  $v \mapsto T(v)$ ,  $T(v)(\cdot) := B(v, \cdot)$ , is a  $(\mathfrak{g}, K)$ -isomorphism. Since  $H_K$  is irreducible, the space  $\text{Hom}_{\mathfrak{g}, K}(H_K, H_K)$  is 1-dimensional by Theorem 2.7. Now, let  $B'(\cdot, \cdot)$  be another such form, with associated isomorphism  $T'$ . Then  $T(T')^{-1} = cI$ , for some  $c \in \mathbb{C}$ . For the last statement, the unitarity of  $(\pi, H)$  implies that  $\langle \cdot, \cdot \rangle$  is a  $(\mathfrak{g}, K)$ -invariant non-degenerate sesquilinear form and Theorem 2.9, with the discussion above, implies the result.  $\square$

Since we are assuming that  $G$  is connected, proving  $(\mathfrak{g}, K)$ -invariance reduces to proving  $\mathfrak{g}$ -invariance. Indeed, by Theorem 2.2, p. 256, in [9], any maximal compact subgroup  $K$  of  $G$  is connected. Therefore, by [12], Corollary 4.48, the exponential map

$$\exp : \mathfrak{k} \longrightarrow K$$

is surjective.

**Proposition 2.14.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $V$  be a  $(\mathfrak{g}, K)$ -module, let

$$B(\cdot, \cdot) : V \times V \longrightarrow \mathbb{C}$$

be a  $\mathfrak{g}$ -invariant sesquilinear form. Then  $B(\cdot, \cdot)$  is  $K$ -invariant.

*Proof.* Given any pair of vectors  $v, w \in V$ , we can find a finite-dimensional subspace of  $V$ , say  $W$ , which contains both and on which  $K$  acts continuously through a representation  $\pi$ . The restriction of the bilinear form  $B(\cdot, \cdot)$  to  $W$  is continuous. To prove that  $B(\pi(k)v, \pi(k)w) = B(v, w)$  for all  $k \in K$ , it suffices to prove that  $B(\pi(k)v, w) = B(v, \pi(k^{-1})w)$  for all  $k \in K$ . Given  $k \in K$ , let  $X \in \mathfrak{k}$  be such that  $k = \exp(X)$ . We begin by writing

$$B(\pi(k)v, w) = B(\pi(\exp X)v, w).$$

Since  $\pi(\exp X) = \exp \dot{\pi}(X)v$ , we obtain

$$B(\pi(\exp X)v, w) = B(\exp \dot{\pi}(X)v, w).$$

The continuity of  $B(\cdot, \cdot)$  on  $W$  gives

$$B(\exp \dot{\pi}(X)v, w) = \exp B(\dot{\pi}(X)v, w).$$

By the  $\mathfrak{g}$ -invariance of  $B(\cdot, \cdot)$ , we have

$$\exp B(\dot{\pi}(X)v, w) = \exp B(v, \dot{\pi}(-X)w)$$

and, finally,

$$\exp B(v, \dot{\pi}(-X)w) = B(v, \pi(\exp(-X))w).$$

□

Let us recall that any locally compact Hausdorff group  $G$  acts on the Hilbert space  $L^2(G)$  by the prescription

$$R(g)f(x) := f(xg).$$

The representation so obtained is unitary and if  $G$  is a Lie group the notion of smooth vectors in  $L^2(G)$  makes sense. In the next section, we will need a criterion to establish that certain functions are smooth vectors in  $L^2(G)$ . We will make use of the following notion:

**Definition 2.15.** Let  $G$  be a Lie group and let  $(\pi, H)$  be a Hilbert representation of  $G$ . The **Gårding subspace** of  $H$  is the vector subspace of  $H$  spanned by the set

$$\{\pi(f)v \mid v \in H, f \in C_c^\infty(G)\}.$$

**Proposition 2.16.** Let  $G$  be a Lie group with finitely many connected components, let  $(\pi, H)$  be a Hilbert representation of  $G$ . Then every vector in the Gårding subspace of  $H$  is a smooth vector in  $H$ .

*Proof.* See [17], Lemma 1.6.1. □



Recall that  $f \in C^\infty(G)$  is called  $Z(\mathfrak{g}_\mathbb{C})$ -**finite** if it is annihilated by an ideal of  $Z(\mathfrak{g}_\mathbb{C})$  of finite codimension. The criterion we need is the following result of Harish-Chandra:

**Theorem 2.17.** Let  $G$  be a group in the class  $\mathcal{H}$  as in [16], p. 192. Let  $f \in C^\infty(G)$  be  $K$ -finite and  $Z(\mathfrak{g}_\mathbb{C})$ -finite. Then there exists a function  $h \in C_c^\infty(G)$  which satisfies  $h(kgk^{-1}) = h(g)$  for all  $k \in K$  and for all  $g \in G$  and such that  $f * h = h$ . If  $f \in C^\infty(G)$ , in addition, is square-integrable, then  $f$  is a smooth vector in  $L^2(G)$ .

*Proof.* The first statement is [16], Proposition 14, p. 352. The second conclusion follows from the observation found at the beginning of the proof of Corollary 8.42 in [11] that  $f$  is in the Gårding subspace of  $L^2(G)$  and it is therefore smooth by Proposition 2.16. That  $f$  is indeed in the Gårding subspace of  $L^2(G)$  follows from the equality

$$R(\tilde{h})f = f * h,$$

where  $\tilde{h}(x) = h(x^{-1})$  and from the first statement. □

**Proposition 2.18.** Let  $G$  be a group in the class  $\mathcal{H}$ . Let  $f \in C^\infty(G)$  be  $K$ -finite,  $Z(\mathfrak{g}_\mathbb{C})$ -finite and square-integrable. Then, for every  $X \in \mathfrak{g}$ , we have

$$Xf = \dot{R}(X)f$$

where  $Xf: G \rightarrow \mathbb{C}$  is defined as

$$Xf(g) := \frac{d}{dt} [f(g \exp(tX))] |_{t=0} \quad (2)$$

*Proof.* By Theorem 2.17, there exists  $h \in C_c^\infty(G)$  such that

$$f = f * h.$$

From the equalities

$$Xf = X(f * h) = f * Xh$$

and

$$f * Xh = \dot{R}(\widetilde{Xh})f,$$

we obtain

$$Xf = \dot{R}(\widetilde{Xh})f.$$

Since

$$\dot{R}(\widetilde{Xh})f = \dot{R}(X)R(\tilde{h})f$$

and

$$R(\tilde{h})f = f * h = f,$$

we conclude

$$Xf = \dot{R}(X)f.$$

□

We will apply Proposition 2.18 to the group  $M$  in the Langlands decomposition of a parabolic subgroup  $P = MAN$  of a connected semisimple Lie group with finite centre. A group  $M$  of this form will not be connected, semisimple in general. However, it belongs to the class  $\mathcal{H}$  by [5], Lemma 9, p. 108.

We briefly recall the construction of parabolically induced representations. We refer the reader to [13], Chapter XI, for a more thorough account.

Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . The group  $K_M := K \cap M$  is a maximal compact subgroup of  $M$ . Let  $\lambda$  be a complex-valued real-linear functional on  $\mathfrak{a}$  and let  $(\sigma, H_\sigma)$  be a Hilbert representation of  $M$ . We define an action of  $G$  on the space of functions

$$\{f \in C(K, H_\sigma) \mid f(mk) = \sigma(m)f(k) \text{ for all } m \in K_M \text{ and all } k \in K\}$$

by declaring

$$\text{Ind}_P(\sigma, \lambda, g)f(k) := e^{(\lambda+\rho)(\mathfrak{h}(kg))} \sigma(\mathfrak{m}(kg))f(\mathfrak{k}(kg)),$$

where, if  $g = kman$  for some  $k \in K$ ,  $m \in M$ ,  $a \in A$ ,  $n \in N$ , we set  $\mathfrak{k}(g) := k$ ,  $\mathfrak{m}(g) := m$ ,  $\mathfrak{h}(g) := \log(a)$ ,  $\mathfrak{n}(g) := n$ . The symbol  $\rho$  denotes half of the sum of the positive restricted roots determined by  $\mathfrak{a}$  counted with multiplicities. On this space of functions, we introduce the norm

$$\|f\|_{\text{Ind}_P(\sigma, \lambda)} := \left( \int_K \|f(k)\|_\sigma^2 dk \right)^{\frac{1}{2}}$$

and, upon completing, we obtain a Hilbert representation of  $G$  which we denote  $\text{Ind}_P(\sigma, \lambda)$ . We will denote  $\text{Ind}_{P, K_M}(\sigma, \lambda)$  the space of  $K_M$ -finite vectors in  $\text{Ind}_P(\sigma, \lambda)$ .

**Proposition 2.19.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . Let  $\lambda$  be a complex-valued, real-linear, totally imaginary functional on  $\mathfrak{a}$  and let  $(\sigma, H_\sigma)$  be a unitary representation of  $M$ . Then  $\text{Ind}_P(\sigma, \lambda)$  is a unitary representation of  $G$ .

*Proof.* See [13], Corollary 11.39. □

**Corollary 2.20.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . Let  $\lambda$  be a complex-valued, real-linear, totally imaginary functional on  $\mathfrak{a}$  and let  $(\sigma, H_\sigma)$  be a unitary representation of  $M$ . Then, for every  $f_1, f_2 \in \text{Ind}_{P, K_M}(\sigma, \lambda)$  and for every  $X \in \mathfrak{g}$ , we have

$$\langle \text{Ind}_P(\sigma, \lambda, X)f_1, f_2 \rangle_{\text{Ind}_P(\sigma, \lambda)} = -\langle f_1, \text{Ind}_P(\sigma, \lambda, X)f_2 \rangle_{\text{Ind}_P(\sigma, \lambda)}.$$

*Proof.* This is a consequence of Proposition 2.19 and the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra on the space of smooth vectors ([18], p. 266). □

Next, we recall a form of Frobenius' reciprocity originally observed by Casselman. We first need some preparation.

First of all, we record the following.

**Lemma 2.21.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . If  $V$  is a  $(\mathfrak{g}, K)$ -module, then  $V/\mathfrak{n}V$  can be equipped with the structure of an  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -module in such a way that the quotient map

$$q: V \longrightarrow V/\mathfrak{n}V$$

is  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -equivariant.

*Proof.* It suffices to show that if  $v \in V$  is of the form  $v = Xw$  for some  $w \in V$  and  $X \in \mathfrak{n}$ , then, for all  $\xi \in K_M$ , we have

$$\xi v \in \mathfrak{n}V,$$

and, for all  $Y \in \mathfrak{m} \oplus \mathfrak{a}$ , we have

$$Yv \in \mathfrak{n}V.$$

Let  $\xi \in K_m$ . We have

$$\xi v = \xi Xw = \text{Ad}(\xi)X\xi w,$$

and, since  $K_M$ , being contained in  $M$ , normalises  $\mathfrak{n}$  by [12], Proposition 7.83, it follows that  $\text{Ad}(\xi)X \in \mathfrak{n}$ .

Let  $Y \in \mathfrak{m} \oplus \mathfrak{a}$ . We have

$$Yv = YXw = [Y, X]w + XYw.$$

The second term in the RHS belongs to  $\mathfrak{n}V$  because  $X \in \mathfrak{n}$  and the first belongs to  $\mathfrak{n}V$  because  $\mathfrak{n}$  is an ideal in  $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{an}$  by [12], Proposition 7.78.  $\square$

Let us recall that a  $(\mathfrak{g}, K)$ -module is **finitely generated** if it is a finitely generated  $U(\mathfrak{g}_{\mathbb{C}})$ -module. We say that a Hilbert representation  $(\pi, H)$  of  $G$  is finitely generated if  $H_K$  is finitely generated. We record the following result of Casselman.

**Theorem 2.22.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . Let  $V$  be an admissible, finitely generated  $(\mathfrak{g}, K)$ -module. Then  $V/\mathfrak{n}V$  is an admissible, finitely generated  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -module.

*Proof.* See [17], Lemma 4.3.1.  $\square$

**Corollary 2.23.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $P = MAN$  be a parabolic subgroup of  $G$ . If  $V$  is an irreducible  $(\mathfrak{g}, K)$ -module admitting an infinitesimal character, then  $V/\mathfrak{n}V$  is an admissible, finitely generated  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -module.

*Proof.* By [9], Theorem 2.2, p. 256,  $K$  is connected. By [13], Theorem 7.204,  $V$  is admissible. Combining [13], Example 1, p. 442 and [13], Corollary 7.207, it follows that  $V$  is finitely generated. The result now follows from Theorem 2.22.  $\square$

Let  $\mathfrak{p}$ ,  $\mathfrak{m}$ ,  $\mathfrak{a}$  and  $\mathfrak{n}$  denote the Lie algebras of  $P$ ,  $M$ ,  $A$  and  $N$ , respectively.

Let  $(\sigma, H_\sigma)$  be an admissible and finitely generated Hilbert representation of  $M$  which is unitary when restricted to  $K_M$ . Let  $\lambda$  be a complex-valued real-linear functional on  $\mathfrak{a}$ . Consider the  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -module  $H_{\sigma, K_M}^\lambda$  defined as

$$H_{\sigma, K_M}^\lambda := H_{\sigma, K_M} \otimes \mathbb{C}_{\lambda+\rho}$$

where the pair  $(\mathfrak{m}, K_M)$  acts on  $H_{\sigma, K_M}$  and  $\mathfrak{a}$  acts on  $\mathbb{C}_{\lambda+\rho}$  via the functional  $\lambda + \rho$ .

If  $V$  is a  $(\mathfrak{g}, K)$ -module and  $T \in \text{Hom}_{\mathfrak{g}, K}(V, \text{Ind}_{P, K_M}(\sigma, \lambda))$ , then we can define an element  $\hat{T} \in \text{Hom}_{\mathfrak{m} \oplus \mathfrak{a}, K_M}(V/\mathfrak{n}V, H_{\sigma, K_M}^\lambda)$  by setting

$$\hat{T}(v) := T(v)(1).$$

**Theorem 2.24.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $V$  be a  $(\mathfrak{g}, K)$ -module. Let  $(\sigma, H_\sigma)$  be an admissible and finitely generated Hilbert representation of  $M$  which is unitary when restricted to  $K_M$  and let  $\lambda$  be a complex-valued real-linear functional on  $\mathfrak{a}$ . Consider the  $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$ -module  $H_{\sigma, K_M}^\lambda$ . Then the map

$$\text{Hom}_{\mathfrak{g}, K}(V, \text{Ind}_{P, K_M}(\sigma, \lambda)) \longrightarrow \text{Hom}_{\mathfrak{m} \oplus \mathfrak{a}, K_M}(V/\mathfrak{n}V, H_{\sigma, K_M}^\lambda), \quad T \mapsto \hat{T}$$

is a bijection.

*Proof.* See [17], Lemma 5.2.3 and the discussion preceding it.  $\square$

For clarity, we point out that the formulation in [17] seems to contain some typos and so we modified it following [8], Theorem 4.9.

The inverse of the map  $T \mapsto \hat{T}$  is constructed as follows (see [17], Lemma 5.2.3 and Lemma 3.8.2. Alternatively, [8], Theorem 4.9).

Let  $S \in \text{Hom}_{\mathfrak{m} \oplus \mathfrak{a}, K_M}(V/\mathfrak{n}V, H_{\sigma, K_M}^\lambda)$ . Then we obtain an element  $\tilde{S} \in \text{Hom}_{\mathfrak{g}, K}(V, \text{Ind}_{P, K_M}(\sigma, \lambda))$  by setting

$$\tilde{S}(v)(k) := S(q(kv)),$$

where  $q: V \longrightarrow V/\mathfrak{n}V$  denotes the quotient map. Then the inverse of  $T \mapsto \hat{T}$  is given by the map

$$\text{Hom}_{\mathfrak{m} \oplus \mathfrak{a}, K_M}(V/\mathfrak{n}V, H_{\sigma, K_M}^\lambda) \longrightarrow \text{Hom}_{\mathfrak{g}, K}(V, \text{Ind}_{P, K_M}(\sigma, \lambda)), \quad S \mapsto \tilde{S}.$$

### 3 Asymptotic behaviour of representations

We begin by collecting the fundamental facts concerning asymptotic expansions of matrix coefficients of tempered representations. We refer the reader to [11], Chapter VIII, for a more thorough exposition of the topic.

Let  $G$  be a connected, semisimple Lie group with finite centre, let  $K$  be a fixed maximal compact subgroup of  $G$  corresponding to a Cartan decomposition of  $\mathfrak{g}$  as  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Let  $P = MAN$  denote the minimal parabolic subgroup of  $G$  with Lie algebra  $\mathfrak{p}$ . Given a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ , we call  $A$  the corresponding subgroup of  $P$  and  $M$

the centraliser of  $A$  in  $K$ . We fix a system  $\Delta$  of simple roots of the root system  $(\mathfrak{g}, \mathfrak{a})$ , we use  $\Delta^+$  to denote the corresponding set of positive roots.

Let  $\mathfrak{a}^+$  denote the set  $\{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \Delta\}$ . Then the subset of regular elements  $G^{\text{reg}}$  of  $G$  admits a decomposition as  $G^{\text{reg}} = K \exp(\mathfrak{a}^+) K$  and  $G$  itself admits a decomposition  $G = K \exp(\mathfrak{a}^+) K$ .

We write  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  and we identify it with the ordered set  $\{1, \dots, n\}$  in the obvious way. We adopt the following notation to simplify the appearance of the expansions we are going to work with.

For  $H \in \mathfrak{a}$  and  $l \in \mathbb{Z}_{\geq 0}^n$ , we set  $\alpha(H)^l := \prod_{i=1}^n \alpha_i(H)^{l_i}$ .

If  $\lambda$  is a real-linear complex-valued functional on  $\mathfrak{a}$ , since, for every  $H \in \mathfrak{a}$ , we have

$$\lambda(H) = \sum_{i=1}^n \lambda_i \alpha_i(H)$$

for some  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , we will often identify  $\lambda$  with the  $n$ -tuple  $(\lambda_1, \dots, \lambda_n)$ .

The next result is concerned with the expansion of  $K$ -finite matrix coefficients relative to  $P$ .

**Theorem 3.1.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $(\pi, H)$  be an irreducible, Hilbert representation of  $G$ . Then there exist a non-negative integer  $l_0$  and a finite set of real-linear complex-valued functionals on  $\mathfrak{a}$ , denoted  $\mathcal{E}_0$ , such that, for every  $v, w \in H_K$ , the restriction to  $\exp(\mathfrak{a}^+)$  of the matrix coefficient  $\phi_{v,w}$  admits a uniformly and absolutely convergent expansion as

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{l \in \mathbb{Z}_{\geq 0}^n : |l| \leq l_0} \sum_{k \in \mathbb{Z}_{\geq 0}^n} \alpha(H)^l e^{(\lambda-k)(H)} \langle c_{\lambda-k,l}(v), w \rangle,$$

where each  $c_{\lambda-k,l} : H_K \rightarrow H_K$  is a complex-linear map and  $\rho_{\mathfrak{p}}$  denotes half of the sum of the elements in  $\Delta^+$  counted with multiplicities.

*Proof.* By [13], Theorem 7.204, the representation  $(\pi, H)$  is admissible and therefore has an infinitesimal character. By [11], Theorem 8.32, we have the stated expansion for any  $\tau$ -spherical function (in the sense of [11], p.215)  $F$  on  $G$  of the form

$$F(g) = E_2 \pi(g) E_1,$$

where  $\tau_1$  and  $\tau_2$  are sub-representations of

$$\pi|_K \cong \bigoplus_{\gamma \in \tilde{K}} n_{\gamma} \gamma$$

of the form

$$\tau_1 := \bigoplus_{\gamma \in \Theta_1} n_{\gamma} \gamma \text{ and } \tau_2 := \bigoplus_{\gamma \in \Theta_2} n_{\gamma} \gamma$$

for finite collections  $\Theta_1, \Theta_2 \in \hat{K}$ , and  $E_1, E_2$  are the orthogonal projections to  $\tau_1, \tau_2$ , respectively. In this expansion, the set  $\mathcal{E}_0$ , the maps  $c_{\lambda-k, l}$  and the number  $l_0$  depend on  $\tau = (\tau_1, \tau_2)$  and we can expand  $\phi_{v, w}$  provided that  $v \in \tau_1$  and  $w \in \tau_2$ . To obtain an expansion valid for every  $v, w \in H_K$  and with  $F, l_0$  and the  $c_{\lambda-k, l}$  independent of  $\tau$ , we appeal to [2], Theorem in 8.8, which we can apply since  $(\pi, H)$  is finitely generated by [13], Corollary 7.207.  $\square$

We recall that if  $\nu, \nu'$  are real-linear complex-valued functionals on  $\mathfrak{a}$  such that  $\nu - \nu'$  is an integral linear combination of the simple roots, then we say that  $\nu$  and  $\nu'$  are **integrally equivalent**.

The set  $\mathcal{E}_0$  has the property that if  $\lambda, \lambda' \in \mathcal{E}_0$  with  $\lambda \neq \lambda'$ , then  $\lambda$  and  $\lambda'$  are not integrally equivalent.

If  $\nu$  and  $\nu'$  are integrally equivalent and  $\nu - \nu'$  is a non-negative integral combination of the simple roots, we write  $\nu \geq \nu'$ , thus introducing an order relation among integrally equivalent functionals on  $\mathfrak{a}$ .

If  $k \in \mathbb{Z}_{\geq 0}^n$  is such that the term

$$\alpha(H)^l e^{(\lambda-k)(H)} \langle c_{\lambda-k, l}(v), w \rangle$$

is non-zero for some  $\lambda \in \mathcal{E}_0$  and for some  $v, w \in H_K$ , then we say that  $\nu := (\lambda - k)$  is an **exponent**. The exponents which are maximal with respect to the order relation introduced above are called **leading exponents**:  $\mathcal{E}_0$  is precisely the set of leading exponents.

The following result is used crucially in [10] and in the following.

**Theorem 3.2.** Let  $(\pi, H)$  be an irreducible, tempered, Hilbert representation of  $G$ . Then every  $\lambda \in \mathcal{E}_0$  satisfies

$$\operatorname{Re} \lambda_i \leq 0$$

for every  $i \in \{1, \dots, n\}$ .

*Proof.* See [11], Theorem 8.53. Strictly speaking, in loc. cit. the theorem is formulated under some restrictions on  $G$ , but it is a convenient reference since we are adopting the same normalisation of the exponents. See [1], Proposition 3.7, p. 83, or [2], Corollary 8.12, for proofs for more general groups.  $\square$

We now turn to asymptotic expansions of matrix coefficients of  $(\pi, H)$  relative to standard (for  $P$ ) parabolic subgroups of  $G$ . We follow [11], Chapter VIII, Section 12.

Given a subset  $I \subset \{1, \dots, n\}$ , and recalling that we identified  $\Delta$  with  $\{1, \dots, n\}$ , we can associate to it a parabolic subgroup

$$P_I = M_I A_{I^c} N_{I^c}$$

of  $G$  containing  $P$  in such a way that  $\mathfrak{g}_{-\alpha} \subset \mathfrak{m}_I$  if and only if  $\alpha \in I$  (with  $\mathfrak{m}_I$  denoting the Lie algebra of  $M_I$ ). For the details, we refer the reader to [12], Chapter VII and

[11], Proposition 5.23.

First, we introduce the basis  $\{H_1, \dots, H_n\}$  of  $\mathfrak{a}$  dual to  $\Delta$ . We define the Lie algebra  $\mathfrak{a}_I$  as

$$\mathfrak{a}_I := \sum_{i \in I} \mathbb{R}H_i$$

and the group  $A_I$  as

$$A_I := \exp\left(\sum_{i \in I} \mathbb{R}\alpha_i\right).$$

We can then write  $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}_{I^c}$  and  $A = A_I A_{I^c}$ . The groups  $N_I$  and  $N_{I^c}$  correspond to the Lie algebras

$$\mathfrak{n}_I := \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} = 0} \mathfrak{g}_\beta \text{ and } \mathfrak{n}_{I^c} := \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} \neq 0} \mathfrak{g}_\beta.$$

We have

$$\rho = \rho_I + \rho_{I^c}$$

with

$$\rho_I := \frac{1}{2} \sum_{\beta \in \Delta^+ : \beta|_{\mathfrak{a}_{I^c}} = 0} (\dim \mathfrak{g}_\beta) \beta$$

and analogously for  $\rho_{I^c}$ . Denoting  $M_{0,I}$  the group corresponding to the Lie algebra

$$\mathfrak{m}_I = \mathfrak{m} \oplus \mathfrak{a}_I \oplus \mathfrak{n}_I \oplus \bar{\mathfrak{n}}_I,$$

the group  $M_I$  is then given by

$$M_I := Z_K(\mathfrak{a}_{I^c}) M_{0,I}.$$

Finally,  $K_I := K \cap M_I$  is a maximal compact subgroup of  $M_I$  and  $MA_I N_I$  is a minimal parabolic subgroup of  $M_I$ .

**Theorem 3.3.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $(\pi, H)$  be an irreducible, Hilbert representation of  $G$ . Let  $C$  be a compact subset of  $M_I$  satisfying  $K_I C K_I = C$ . Then there exists a positive real number  $R$  depending on  $C$  such that, for every  $m \in C$  and for every  $a = \exp H \in A_{I^c}$  which satisfies  $\alpha_i(H) > \log R$  for every  $i \in I^c$ , we have

$$\phi_{v,w}(m \exp H) = e^{-\rho_{I^c}(H)} \sum_{\nu \in \mathcal{E}_I} \sum_{q \in \mathbb{Z}_{\geq 0}^{I^c} : |q| \leq q_0} \alpha(H)^q e^{\nu(H)} c_{\nu,q}^{P_I}(m, v, w)$$

for every  $v, w \in H_K$ . Here,  $\mathcal{E}_I$  is a countable set of real-linear complex-valued functionals on  $\mathfrak{a}_{I^c}$ , each  $c_{\nu,q}^{P_I}$  extends to a real analytic function on  $M_I$  and satisfies

$$c_{\nu,q}^{P_I}(\xi_2 m \xi_1, v, w) = c_{\nu,q}^{P_I}(m, \pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for every  $\xi_1, \xi_2 \in K_I$ . Moreover, for every  $m \in M_I$  and  $w \in H_K$ , the map

$$H_K \longrightarrow \mathbb{C}, v \mapsto c_{\nu, q}^{P_I}(m, v, w)$$

is complex-linear and, for every  $m \in M_I$  and  $v \in H_K$ , the map

$$H_K \longrightarrow \mathbb{C}, w \mapsto c_{\nu, q}^{P_I}(m, v, w)$$

is conjugate-linear.

*Proof.* For a  $\tau$ -spherical function  $F$  as in the proof of Theorem 3.1, the result follows from [11], Theorem 8.45. To obtain an expansion independent of  $\tau$ , it suffices to prove that each  $F_{\nu - \rho_I^c}$  is independent of  $\tau$ .

Let  $m \in M_I$  and write  $m = \xi_2 a_I \xi_1$  for some  $a_I \in \overline{A_I^+}$ , where  $A_I^+$  is the positive Weyl chamber, and some  $\xi_1, \xi_2 \in K_I$ . Since

$$F_{\nu - \rho_I^c}(ma, v, w) = F_{\nu - \rho_I^c}(a_I a, \pi(\xi_1)v, \pi(\xi_2^{-1})w),$$

re-labeling things, it suffices to prove that  $F_{\nu - \rho_I^c}(\cdot, v, w)$  is independent of  $\tau$  as a function on  $\overline{A_I^+} A_{I^c}$ . By [11], Corollary 8.46, the functional  $\nu \in \mathcal{E}_{\mathcal{I}}$  is the restriction of an element in the set of exponents  $\mathcal{E}$  in the expansion relative to  $P$  and this set is independent of  $\tau$  by [2], Theorem 8.8. Therefore, it remains to prove that each  $c_{\nu, q}^{P_I}$  is independent of  $\tau$ . Since  $c_{\nu, q}^{P_I}$  is analytic on  $M_I$ , it suffices to prove that  $c_{\nu, q}^{P_I}(\cdot, v, w)$  as a function on  $A_I^+$  is independent of  $\tau$ . Given  $a_I \in A_I^+$ , we can find a compact subset  $C$  of  $M_I$  containing  $a_I$  such that  $K_I C K_I = C$ , and a positive  $R$  depending on  $C$ , such that for every  $H \in \mathfrak{a}_{I^c}$  satisfying  $\alpha_i(H) > \log R$  for every  $i \in I^c$ , the expansion of  $\phi_{v, w}(a_I a)$  relative to  $P$  and the expansion relative to  $P_I$  are both valid. Comparing them as in [11], p. 251, it follows that expansion relative to  $P_I$  is completely determined by the expansion relative to  $P$  and the latter is independent of  $\tau$  by Theorem 3.1.  $\square$

For every  $\nu \in \mathcal{E}_I$ , the term

$$\alpha(H)^q e^{(\nu - \rho_I^c)(H)} c_{\nu, q}^{P_I}(m, v, w)$$

is non-zero for some  $v, w \in H_K$  and some  $m \in M$ . The set  $\mathcal{E}_I$  is the set of **exponents relative to  $P_I$** .

We are ready to define the functions of the form  $\Gamma_{\lambda, I}$  discussed in the Introduction. The first step consists in associating a standard (for  $P$ ) parabolic subgroup of  $G$  to each  $\lambda \in \mathcal{E}_0$ .

Let  $(\pi, H)$  be an irreducible, tempered, Hilbert representation of  $G$  and let  $\lambda \in \mathcal{E}_0$ . We set  $I_\lambda := \{i \in \{1, \dots, n\} | \operatorname{Re} \lambda_i < 0\}$  which we identify with the subset  $\Delta_\lambda$  of  $\Delta$  defined as

$$\Delta_\lambda := \{\alpha_i \in \Delta | i \in I_\lambda\}.$$

The construction of standard parabolic subgroups from the datum of a subset of  $\Delta$  assigns to  $I_\lambda$  the standard parabolic subgroup  $P_\lambda$  defined by  $P_\lambda := P_{I_\lambda}$ . It admits a decomposition

$$P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0},$$



where

$$A_{\lambda_0} := A_{I_{\lambda_0}^c}.$$

The subgroup  $M$  admits a decomposition

$$M_{\lambda} = K_{\lambda} A_{\lambda} K_{\lambda},$$

where

$$A_{\lambda} := A_{I_{\lambda}}$$

and

$$K_{\lambda} := K \cap M_{\lambda}.$$

The group  $A$  decomposes as  $A = A_{\lambda} A_{\lambda_0}$ . We write  $\mathfrak{a}_{\lambda}$  and  $\mathfrak{a}_{\lambda_0}$  for  $\mathfrak{a}_{I_{\lambda}}$  and  $\mathfrak{a}_{I_{\lambda_0}^c}$ , respectively. Similarly, we write  $\rho_{\lambda}$  and  $\rho_{\lambda_0}$  for  $\rho_{I_{\lambda}}$  and  $\rho_{I_{\lambda_0}^c}$ , respectively.

We are going to introduce an equivalence relation on the data indexing the expansion of  $\phi_{v,w}$  relative to  $P$ . The definition is motivated by the construction of  $\mathbf{d}(\pi)$  in [10]. Let  $v, w \in H_K$ . We have

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{l \in \mathbb{Z}_{\geq 0}^n : |l| \leq l_0} \alpha(H)^l e^{\lambda(H)} \Phi_{\lambda,l}^{v,w}(H)$$

where

$$\Phi_{\lambda,l}^{v,w}(H) := \sum_{k \in \mathbb{Z}_{\geq 0}^n} e^{-k(H)} \langle c_{\lambda-k,l}(v_1), v_2 \rangle.$$

The terms in this expansion are indexed by the finite set

$$\mathcal{C} := \{(\lambda, l)\}_{\lambda \in \mathcal{E}_0, l \in \mathbb{Z}_{\geq 0}^n : |l| \leq l_0}. \quad (3)$$

We introduce a relation on  $\mathcal{C}$  by declaring that  $(\lambda, l) \sim (\mu, m)$  if  $I_{\lambda} = I_{\mu}$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and  $\text{res}_{I_{\lambda}^c} l = \text{res}_{I_{\mu}^c} m$ .

It is clear that  $\sim$  is an equivalence relation. We denote  $[\lambda, l]$  the equivalence class containing  $(\lambda, l)$ .

We can therefore re-group the expansion of  $\phi_{v,w}$  as follows:

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{[\lambda,l] \in \mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda', l') \in [\lambda, l]} \alpha(H_{\lambda})^{l'_{\lambda}} e^{\lambda'|_{\mathfrak{a}_{\lambda}}(H_{\lambda})} \Phi_{\lambda', l'}^{v,w}(H), \quad (4)$$

where

$$l_{\lambda_0} := \text{res}_{I_{\lambda_0}^c} l, \quad \alpha(H_{\lambda_0})^{l_{\lambda_0}} := \prod_{i \in I_{\lambda_0}^c} \alpha_i(H_{\lambda_0})^{l_i}, \quad l'_{\lambda} := \text{res}_{I_{\lambda}} l', \quad \alpha(H_{\lambda})^{l'_{\lambda}} := \prod_{i \in I_{\lambda}} \alpha(H_{\lambda})^{l'_i}$$

and  $H = H_{\lambda_0} + H_{\lambda}$  corresponds to the decomposition

$$\mathfrak{a}^+ = \mathfrak{a}_{\lambda_0}^+ \oplus \mathfrak{a}_{\lambda}^+.$$

We are also implicitly using the fact that  $\alpha(H)^l = \alpha(H_\lambda)^{l_\lambda} \alpha(H_{\lambda_0})^{l_{\lambda_0}}$  which follows from writing  $H$  with respect to the basis dual to  $\Delta$ .

Let us assume that  $[\lambda, l] \in \mathcal{C}/\sim$  satisfies

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

where  $\mathbf{d}_P(\lambda, l)$  is defined by (1).

For  $H_\lambda \in \mathfrak{a}_\lambda^+$ , we set

$$\Gamma_{\lambda, l}(\exp H_\lambda, v, w) := e^{-\rho(H)} \sum_{(\lambda', l') \in [\lambda, l]} \alpha(H_\lambda)^{l'_\lambda} e^{\lambda' |_{\mathfrak{a}_{\lambda_0}}(H_\lambda)} \Phi_{\lambda', l'}^{v, w}(H_\lambda). \quad (5)$$

Before establishing the properties of  $\Gamma_{\lambda, l}$ , let us pause to explain the motivation behind the condition on the equivalence class  $[\lambda, l]$ . The discussion that follows will be used only in Section 4. The reader who prefers to do so can skip to Proposition 3.4 without any loss of continuity.

Let  $v_1, v_2, v_3, v_4 \in H_K$ . We will be considering integrals of the form

$$\lim_{r \rightarrow \infty} \frac{1}{r \mathbf{d}(\pi)} \int_{\mathfrak{a}_{<r}^+} \phi_{v_1, v_2}(\exp H) \bar{\phi}_{v_3, v_4}(\exp H) \prod_{\beta \in \Delta^+} (e^{\beta(H)} - e^{-\beta(H)})^{\dim \mathfrak{g}_\beta} dH,$$

where

$$\mathfrak{a}_{<r}^+ := \mathfrak{a}^+ \cap \{H \in \mathfrak{a} \mid \beta(H) < r \text{ for all } \beta \in \Delta^+\}. \quad (6)$$

Treating these is the content of Appendix A in [10]. We remark that our region of integration is defined as to exclude the subset of  $\overline{\mathfrak{a}^+}$  where at least one of the simple roots vanishes. It is a set of measure zero.

We want to interpret Lemma A.5 in [10] in group-theoretic terms.

Let us consider the matrix coefficients  $\phi_{v_1, v_2}$  and  $\phi_{v_3, v_4}$ . By (4), on  $A^+ := \exp(\mathfrak{a}^+)$  they can be expanded as

$$\phi_{v_1, v_2}(\exp H) = e^{-\rho(H)} \sum_{[\lambda, l] \in \mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda |_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda', l') \in [\lambda, l]} \Psi_{\lambda', l'}^{v_1, v_2}(H) \quad (7)$$

and

$$\phi_{v_3, v_4}(\exp H) = e^{-\rho(H)} \sum_{[\mu, m] \in \mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu |_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu', m') \in [\mu, m]} \Psi_{\mu', m'}^{v_3, v_4}(H). \quad (8)$$

where, for  $(\lambda', l') \in [\lambda, l]$ , we set

$$\Psi_{\lambda', l'}^{v_1, v_2}(H) := \alpha(H_\lambda)^{l'_\lambda} e^{\lambda' |_{\mathfrak{a}_\lambda}(H_\lambda)} \Phi_{\lambda', l'}^{v_1, v_2}(H)$$

and similarly for  $(\mu', m') \in [\mu, m]$ .

Let  $[\lambda, l] \in \mathcal{C}/\sim$  and  $[\mu, m] \in \mathcal{C}/\sim$  be such that  $I_\lambda = I_\mu$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and

$$\mathbf{d}(\pi) = |I_\lambda| + \sum_{i \in I_\lambda} (l_i + m_i).$$

In view of the first condition, the third is equivalent to the requirement

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi) \text{ and } \mathbf{d}_P(\mu, m) = \mathbf{d}(\pi).$$

Consider the summand

$$e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'}) (H)} \Phi_{\lambda', l'}^{v_1, v_2} \overline{\Phi}_{\mu', m'}^{v_3, v_4}(H)$$

in the expansion of the product  $\phi_{v_1, v_2} \overline{\phi}_{v_3, v_4}$  on  $A^+$ .

Taking into account the factor  $e^{-2\rho(H)}$  and the fact that the term

$$\Omega(H) := \prod_{\beta \in \Delta^+} (e^{\beta(H)} - e^{-\beta(H)})^{\dim g_\beta} \quad (9)$$

is incorporated in the function  $\phi$  in [10], Lemma A.5 (compare section 4.7 in loc. cit.), this lemma shows that, as  $r \rightarrow \infty$ , the integral

$$\frac{1}{r^{\mathbf{d}(\pi)}} \int_{A_{<r}^+} e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'}) (H)} \Phi_{\lambda', l'}^{v_1, v_2} \overline{\Phi}_{\mu', m'}^{v_3, v_4}(H) \Omega(H) dH$$

tends to

$$C(\lambda, l, m) \int_{A_\lambda^+} e^{-2\rho_\lambda(H_\lambda)} \left[ \Psi_{\lambda', l'}^{v_1, v_2} \overline{\Psi}_{\mu', m'}^{v_3, v_4} \right] |_{\mathfrak{a}_\lambda} (H_\lambda) \Omega_\lambda(H_\lambda) dH_\lambda$$

where

$$\Omega_\lambda(H_\lambda) := \prod_{\beta \in \Delta_\lambda^+} (e^{\beta(H_\lambda)} - e^{-\beta(H_\lambda)})^{\dim g_\beta}, \quad (10)$$

with

$$\Delta_\lambda^+ := \{\beta \in \Delta^+ | \beta|_{\mathfrak{a}_{\lambda_0}} = 0\}$$

and the quantity  $C(\lambda, l, m)$  is given by

$$C(\lambda, l, m) := \int_{\{H \in A_{\lambda_0} | \text{ext}^{I_\lambda^c}(H) \in A_{<1}^+\}} \alpha(H_{\lambda_0})^{l_{\lambda_0} + m_{\mu_0}} dH_{\lambda_0} \quad (11)$$

Now, summing over all  $(\lambda', l') \in [\lambda, l]$  and over all  $(\mu', m') \in [\mu, m]$ , we obtain that the integral over  $A_{<r}^+$  of

$$e^{-2\rho(H)} \sum_{(\lambda', l') \in [\lambda, l]} \sum_{(\mu', m') \in [\mu, m]} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'}) (H)} \Phi_{\lambda', l'}^{v_1, v_2} \overline{\Phi}_{\mu', m'}^{v_3, v_4}(H) \Omega(H),$$

upon multiplying by  $\frac{1}{r^{\mathbf{d}(\pi)}}$  and letting  $r \rightarrow \infty$ , equals

$$C(\lambda, l, m) \int_{A_\lambda^+} e^{-2\rho_\lambda(H_\lambda)} \sum_{(\lambda', l') \in [\lambda, l]} \sum_{(\mu', m') \in [\mu, m]} \left[ \Psi_{\lambda', l'}^{v_1, v_2} \overline{\Psi}_{\mu', m'}^{v_3, v_4} \right] |_{\mathfrak{a}_\lambda} (H_\lambda) \Omega_\lambda(H_\lambda) dH_\lambda.$$

Finally, since

$$\Phi_{\lambda', l'}^{v_1, v_2}|_{\mathfrak{a}_\lambda}(H_\lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}^l} e^{-k(H_\lambda)} \langle c_{\lambda' - k, l'}(v_1), v_2 \rangle,$$

and similarly for  $\Phi_{\mu', m'}^{v_3, v_4}$ , the integral above equals

$$C(\lambda, l, m) \int_{A_\lambda^+} \Gamma_{\lambda, l}(\exp H_\lambda, v, w) \bar{\Gamma}_{\mu, m}(\exp H_\lambda, v, w) \Omega_\lambda(H_\lambda) dH_\lambda.$$

If  $[\lambda, l], [\mu, m] \in \mathcal{C}/\sim$  fail to satisfy any of the three conditions  $I_\lambda = I_\mu$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_\lambda}$  and

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi) = \mathbf{d}_P(\mu, m),$$

then, for every  $(\lambda', l') \in [\lambda, l]$  and for every  $(\mu', m') \in [\mu, m]$ , by the considerations in the proof of Claim A.6 and Lemma A.5 in [10], the integral

$$\frac{1}{r^{\mathbf{d}(\pi)}} \int_{A_{< r}^+} e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\mu')(H)} \Phi_{\lambda', l'}^{v_1, v_2} \bar{\Phi}_{\mu', m'}^{v_3, v_4}(H) \Omega(H) dH$$

vanishes as  $r \rightarrow \infty$ .

Therefore, the equivalence classes  $[\lambda, l] \in \mathcal{C}/\sim$  for which  $\Gamma_{\lambda, l}$  is defined are precisely the ones that may contribute a non-zero term to the expression

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{A_{< r}^+} \phi_{v_1, v_2}(\exp H) \bar{\phi}_{v_3, v_4}(\exp H) \Omega(H) dH.$$

Throughout the rest of this section, we fix an irreducible, tempered, Hilbert representation of a connected, semisimple Lie group  $G$  with finite centre.

To study the properties of  $\Gamma_{\lambda, l}$ , we begin by showing that it is equal to a function of the form  $c_{\nu, q}^{P_\lambda}$ . More precisely, we have:

**Proposition 3.4.** Let  $v, w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l),$$

Set  $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$  and  $q := l_{\lambda_0}$ . Then, for every  $H_\lambda \in \mathfrak{a}_\lambda^+$ , we have

$$\Gamma_{\lambda, l}(\exp H_\lambda, v, w) = c_{\nu, q}^{P_\lambda}(\exp H_\lambda, v, w).$$

*Proof.* For every  $H_\lambda \in \mathfrak{a}_\lambda^+$ , we can find a compact subset  $C$  of  $M_\lambda$  such that  $K_\lambda C K_\lambda = C$  and which contains  $H_\lambda$ , and a positive real  $R > 0$  such that if  $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$  satisfies  $\alpha_i(H_{\lambda_0}) > \log R$  for every  $i \in I_\lambda^c$ , then the expansion of  $\phi_{v, w}$  with respect to  $P$  and the expansion with respect to  $P_\lambda$  are both valid at  $H = H_\lambda + H_{\lambda_0}$ . Comparing them as in [11], p.251, we see that

$$c_{\nu, q}^{P_\lambda}(\exp H_\lambda, v, w) = \sum_{\lambda' \in \mathcal{E}_0: \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu} \sum_{l': |l'| \leq l_0 \text{ and } l'_{\lambda_0} = q} e^{-\rho_\lambda(H_\lambda)} \Psi_{\lambda', l'}^{v, w}(H_\lambda).$$

Since, by definition of  $\Gamma_{\lambda,l}(\cdot, v, w)$ , we have

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) = e^{-\rho(H_\lambda)} \sum_{(\lambda', l') \in [\lambda, l]} \Psi_{\lambda', l'}^{v, w}(H_\lambda),$$

recalling the definition of the equivalence relation that we imposed on  $\mathcal{C}$ , we only need to show that the set

$$\{\lambda' \in \mathcal{E}_0 \mid \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu\}$$

is equal to the set

$$\{\lambda \in \mathcal{E}_0 \mid I_{\lambda'} = I_\lambda \text{ and } \lambda'|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}\}.$$

Because of the assumption on  $[\lambda, l]$ , for every  $\lambda' \in \mathcal{E}_0$  such that  $\lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu$ , we have  $\operatorname{Re} \lambda'_j \neq 0$  for every  $j \in I_\lambda$ . Indeed, if there existed a  $j \in I_\lambda$  for which  $\operatorname{Re} \lambda'_j = 0$ , we would have

$$|I_{\lambda'}^c| \geq 1 + |I_\lambda^c|$$

and, since  $l'_{\lambda_0} = l_{\lambda_0}$ , this would imply

$$\mathbf{d}_P(\lambda', l') > |I_\lambda^c| + \sum_{i \in I_\lambda^c} 2l'_i \geq \mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

contradicting the maximality of  $\mathbf{d}(\pi)$ . Since, by Theorem 3.2, we have  $\operatorname{Re} \lambda'_i \leq 0$  for every  $i \in \{1, \dots, n\}$ , this concludes the proof.  $\square$

Theorem 8.45 in [11] and the discussion at the beginning of p. 251 in loc. cit. now show that  $\Gamma_{\lambda,l}(\cdot, v, w)$ , being equal to  $c_{\nu, q}^{P_\lambda}$ , extends to an analytic function on  $M_\lambda$ , which we denote again  $\Gamma_{\lambda,l}(\cdot, v, w)$ . If we decompose  $M_\lambda$  as

$$M_\lambda = K_\lambda M_\lambda K_\lambda,$$

and if we write  $m \in M_\lambda$  as  $m = \xi_2 \exp H_\lambda \xi_1$  for some  $\xi_1, \xi_2 \in K_\lambda$  and some  $H_\lambda \in \overline{\mathfrak{a}_\lambda^+}$ , then we have

$$\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w)$$

because  $c_{\nu, q}^{P_\lambda}(\cdot, v, w)$  exhibits the same behaviour.

We want to prove that  $\Gamma_{\lambda,l}(\cdot, v, w)$  belongs to  $L^2(M_\lambda)$  and it is  $Z(\mathfrak{m}_{\lambda\mathbb{C}})$ -finite. An application of Theorem 2.17 will imply that  $\Gamma_{\lambda,l}(\cdot, v, w)$  is a smooth vector in  $L^2(M_\lambda)$ . Similar ideas appear in [11], Chapter VIII, and in [15].

We recall that there exists an injective algebra homomorphism

$$\mu_{P_\lambda} : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow Z((\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0})_\mathbb{C}) \cong Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$$

which turns  $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$  into a free module of finite rank over  $\mu_{P_\lambda}(Z(\mathfrak{g}_\mathbb{C}))$  by ([6], Lemma 21).

**Proposition 3.5.** Let  $v, w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then  $\Gamma_{\lambda, l}(\cdot, v, w)$  belongs to  $L^2(M_\lambda)$ .

*Proof.* We argue as in the proof of [15], Lemma 4.10. By the proof of Proposition 3.4, we have  $\operatorname{Re}\lambda'_i < 0$  for every  $\lambda'$  appearing in the expansion of  $\Gamma_{\lambda, l}(\cdot, v, w)$  on  $A_\lambda^+$  and for every  $i \in I_\lambda$ . Since  $\Gamma_{\lambda, l}(\cdot, v, w)$  is analytic on  $\overline{A_\lambda^+}$ , we can apply [7], Theorem 4 and then argue as in [2], Theorem 7.5, to establish the desired square-integrability on  $\overline{A_\lambda^+}$ . The square-integrability on  $M_\lambda$  follows from combining the decomposition of  $M_\lambda$  as  $M_\lambda = K_\lambda \overline{A_\lambda^+} K_\lambda$ , the corresponding integral formula and the fact that if  $m = \xi_2 \exp H_\lambda \xi_1$ , for some  $H_\lambda \in \overline{\mathfrak{a}_\lambda^+}$  and some  $\xi_1, \xi_2 \in K_\lambda$ , then

$$\Gamma_{\lambda, l}(m, v, w) = \Gamma_{\lambda, l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w).$$

□

**Proposition 3.6.** Let  $v, w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every  $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$  and for every  $m \in M_\lambda$ , we have

$$X\Gamma_{\lambda, l}(m, v, w) = \Gamma_{\lambda, l}(m, \tilde{\pi}(X)v, w).$$

Moreover, the function  $\Gamma_{\lambda, l}(\cdot, v, w)$  is a smooth vector in the representation  $(R, L^2(M_\lambda))$  of  $M_\lambda$ .

*Proof.* For a given  $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$  and every  $g \in G$ , we have

$$X\phi_{v, w}(g) = \phi_{\tilde{\pi}(X)v, w}(g).$$

Therefore, the restriction of  $X\phi_{v, w}(\cdot)$  to  $M_\lambda A_{\lambda_0}$  satisfies

$$X\phi_{v, w}(ma) = \phi_{\tilde{\pi}(X)v, w}(ma).$$

Given  $m \in M_\lambda$  we can find a compact subset  $C$  of  $M_\lambda$  containing  $m$  such that  $K_\lambda C K_\lambda = C$  and a positive  $R$  depending on  $C$  such that if  $a = \exp H \in A_{\lambda_0}^+$  satisfies  $\alpha_i(H) > \log R$  for every  $i \in I_\lambda^c$ , then  $\phi_{\tilde{\pi}(X)v, w}(ma)$  may be expanded with respect to  $P_\lambda$ . Since  $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$ , the restriction of  $X\phi_{v, w}(\cdot)$  to  $M_\lambda A_{\lambda_0}$  can also be computed as the action of the differential operator  $X$  on the restriction of  $\phi_{v, w}(\cdot)$  to  $M_\lambda A_{\lambda_0}$ . For  $m \in M_\lambda$  and  $a \in A_{\lambda_0}^+$  as above, we expand the function so obtained with respect to  $P_\lambda$  and, as in the proof of (4.8) in [15], because of the convergence of the series, we can apply the differential operator term by term. By comparing the resulting expansion with the expansion of  $\phi_{\tilde{\pi}(X)v, w}(ma)$ , and invoking Corollary B.26 of [11], we obtain

$$Xc_{v, q}^{P_\lambda}(m, v, w) = c_{v, q}^{P_\lambda}(m, \tilde{\pi}(X)v, w)$$

for every  $\nu \in \mathcal{E}_I$  and every  $q \in \mathbb{Z}_{\geq 0}^{I_\lambda^c}$ . The first statement now follows from choosing  $\nu$  and  $q$  as in Proposition 3.4.

For the last statement, we need to show that  $\Gamma_{\lambda,l}(\cdot, v, w)$  is annihilated by an ideal of finite codimension in  $Z(\mathfrak{m}_{\lambda\mathbb{C}})$ ; the result will then follow from Theorem 2.17. Let  $J$  be the kernel of the infinitesimal character of  $(\pi, H)$ . Then  $J$  is an ideal of finite codimension in  $Z(\mathfrak{g}_{\mathbb{C}})$ . As observed in [5], p.182, the inverse image  $J_{\mathfrak{m}_\lambda}$  along the inclusion

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \longrightarrow Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}), \quad X \mapsto X \otimes 1$$

of the ideal generated by  $\mu_{P_\lambda}(J)$  in  $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$  is an ideal of finite codimension in  $Z(\mathfrak{m}_{\lambda\mathbb{C}})$ . This follows from the fact that the ideal generated by  $\mu_{P_\lambda}(J)$  is of finite codimension in  $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ , since  $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$  is a free module of finite type over  $\mu_{P_\lambda}(Z(\mathfrak{g}_{\mathbb{C}}))$  by [6], Lemma 21. Denoting  $\mu_{P_\lambda}(J)^e$  the ideal generated by  $\mu_{P_\lambda}(J)$ , we see that  $J_{\mathfrak{m}_\lambda}$  is precisely the kernel of the homomorphism

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \longrightarrow (Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})) / \mu_{P_\lambda}(J)^e, \quad X \mapsto (X \otimes 1) + \mu_{P_\lambda}(J)^e.$$

This exhibits  $J_{\mathfrak{m}_\lambda}$  as an ideal of finite codimension in  $Z(\mathfrak{m}_{\lambda\mathbb{C}})$ . Now, if  $X \in J_{\mathfrak{m}_\lambda}$ , then  $X \otimes 1$  belongs to  $\mu_{P_\lambda}(J)^e$ . Hence  $X \otimes 1$  can be written as

$$X \otimes 1 = \sum_{i=1}^r Y_i \mu_{P_\lambda}(Z_i)$$

with  $Y_i \in Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$  and  $Z_i \in J$ . For every  $i \in \{1, \dots, r\}$ , by (8.68) in [11], p. 251, the differential operator  $\mu_{P_\lambda}(Z_i)$  annihilates the function

$$F_{\nu-\rho_{\lambda_0}}(ma, v, w) := \sum_{q:|q| \leq q_0} c_{\nu,q}^{P_\lambda}(m, v, w) \alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Therefore,  $X \otimes 1$  annihilates it, as well. On the other hand, by the first of the proof, we have

$$(X \otimes 1) F_{\nu-\rho_{\lambda_0}}(ma, v, w) = \sum_{q:|q| \leq q_0} c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w) \alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Since the LHS vanishes identically on  $M_\lambda A_{\lambda_0}$ , it follows that

$$c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w) = 0$$

for every  $m \in M_\lambda$ . Choosing  $\nu$  and  $q$  as in Proposition 3.4, we find that  $\Gamma_{\lambda,l}(\cdot, v, w)$  is annihilated by  $J_{\mathfrak{m}_\lambda}$ .  $\square$

Let  $w \in H_K$ . The next two technical lemmata, together with Proposition 3.6, will be used to prove the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}, K_\lambda)$ -equivariance of the map

$$S_w : H_K \longrightarrow L^2(M_\lambda) \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w).$$

We are not claiming that for every  $w \in H_K$  this map is non-zero: the only thing we need to know is that, whenever  $w \in H_K$  is such that  $S_w$  is not identically zero, then

$S_w$  is  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}, K_\lambda)$ -equivariant. Having established this, we show that the existence of an admissible, finitely generated, unitary representation  $(\sigma, H_\sigma)$  of  $M_\lambda$  which will allow us to apply Theorem 2.24 in the way we explained in the Introduction. This is the content of the last two results of this section.

**Lemma 3.7.** Let  $v, w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C} / \sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every  $X \in \mathfrak{a}_{\lambda_0}$  and every  $m \in M_\lambda$ , we have

$$\Gamma_{\lambda, l}(m, \dot{\pi}(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda, l}(m, v, w).$$

*Proof.* We write  $m \in M_\lambda$  as  $m = \xi_2 a_\lambda \xi_2$  for some  $\xi_1, \xi_2 \in K_\lambda$  and some  $a_\lambda \in \overline{A_\lambda^+}$ . Then we have

$$\Gamma_{\lambda, l}(m, \dot{\pi}(X)v, w) = \Gamma_{\lambda, l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot{\pi}(X)v = \dot{\pi}(\text{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since  $M_\lambda$  centralises  $\mathfrak{a}_{\lambda_0}$  ([12], Proposition 7.82) and  $K_\lambda$  is contained in  $M_\lambda$ , we have

$$\Gamma_{\lambda, l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda, l}(a_\lambda, \dot{\pi}(X)\pi(\xi_1)v, \pi(\xi_2^{-1})w).$$

Therefore, re-labeling things, it suffices to prove that for every  $X \in \mathfrak{a}_{\lambda_0}$  and for every  $a_\lambda \in \overline{A_\lambda^+}$ , we have

$$\Gamma_{\lambda, l}(a_\lambda, \dot{\pi}(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda, l}(a_\lambda, v, w).$$

Moreover, since  $\Gamma_{\lambda, l}(\cdot, v, w)$  is analytic, it suffices to prove the identity for every  $a_\lambda \in A_\lambda^+$ .

Let  $a_\lambda = \exp H_\lambda \in A_\lambda^+$ . Then there exist a compact subset  $C$  of  $M_\lambda$  containing  $a_\lambda$  and such that  $K_\lambda C K_\lambda = C$ , and a positive  $R$  depending on  $C$  such that, for all  $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$  satisfying  $\alpha_i(H_{\lambda_0}) > \log R$  for every  $i \in I_\lambda^c$ , the expansion of  $\phi_{\dot{\pi}(X)v, w}(a_\lambda \exp H_{\lambda_0})$  relative to  $P$  and the expansion of  $\phi_{\dot{\pi}(X)v, w}(a_\lambda \exp H_{\lambda_0})$  relative to  $P_\lambda$  are both valid.

Setting  $H := H_\lambda + H_{\lambda_0}$  for  $H_{\lambda_0}$  as above, the first expansion gives

$$\phi_{\dot{\pi}(X)v, w}(H) = \sum_{\tilde{\lambda} \in \mathcal{E}} \sum_{\tilde{l} \in \mathbb{Z}_{\geq 0}^c: |\tilde{l}| \leq l_0} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda} - \rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(\dot{\pi}(X)v), w \rangle$$

By linearity we can assume that  $X = H_i$  for some  $i \in I_\lambda^c$ , where  $H_i$ , we recall, is the element in  $\mathfrak{a}_{\lambda_0}$  dual to the simple root  $\alpha_i$ .

Differentiating term by term and taking into account the computation

$$H_i \left[ \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda} - \rho)(H)} \right] = \tilde{l}_i \alpha(H)^{\tilde{l} - e_i} e^{(\tilde{\lambda} - \rho)(H)} + (\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho)(H_i) \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda} - \rho)(H)},$$



where  $e_i$  is the element in  $\mathbb{Z}_{\geq 0}^n$  having 1 as its  $i$ -th co-ordinate and 0 as every other co-ordinate, we observe that the only terms in the expansion

$$\phi_{v,w}(H) = \sum_{\tilde{\lambda} \in \mathcal{E}} \sum_{\tilde{l} \in \mathbb{Z}_{\geq 0}^n: |\tilde{l}| \leq l_0} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(v), w \rangle$$

that after differentiation by  $H_i \in \mathfrak{a}_{\lambda_0}$  can contribute a term of the form

$$c \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(v), w \rangle,$$

with  $c \in \mathbb{C}$ , to the expansion of  $\phi_{\tilde{\pi}(X)v,w}(H)$ , are precisely

$$\alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(v), w \rangle \text{ and } \alpha(H)^{\tilde{l}+e_i} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(v), w \rangle.$$

Assume  $\lambda \in \mathcal{E}_0$  and  $l \in \mathbb{Z}_{\geq 0}^n$  with  $|l| \leq l_0$  satisfy

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi).$$

Then, if the term

$$\alpha(H)^{l+e_i} e^{(\lambda-\rho)(H)} \langle c_{\lambda, l}(v), w \rangle$$

appeared in the expansion of  $\phi_{v,w}(H)$ , we would have

$$\mathbf{d}_P(\lambda, l + e_i) > |I_\lambda^c| + \sum_{i \in I_\lambda^c} 2l_i = \mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

contradicting the maximality of  $\mathbf{d}(\pi)$ . This reasoning shows that in the expansion

$$\phi_{\tilde{\pi}(H_i)v,w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu \in \mathcal{E}_I} \sum_{q \in \mathbb{Z}_{\geq 0}^n: |q| \leq q_0} \alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c_{\nu,q}^{P_\lambda}(a_\lambda, \tilde{\pi}(H_i)v, w)$$

relative to  $P_\lambda$ , the term indexed by  $(\nu, q)$  with  $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$  and  $q = l_{\lambda_0}$  satisfies

$$c_{\nu,q}^{P_\lambda}(a_\lambda, \tilde{\pi}(H_i)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i) c_{\nu,q}^{P_\lambda}(a_\lambda, v, w).$$

Indeed, the comparison in [11], p. 251, shows that

$$\alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c_{\nu,q}^{P_\lambda}(a_\lambda, \tilde{\pi}(H_i)v, w)$$

is the sum of all the terms in the expansion of  $\phi_{\tilde{\pi}(H_i)v,w}(H)$  relative to  $P$  which are indexed by couples  $(\tilde{\lambda}, \tilde{l})$  satisfying

$$\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \text{ and } \tilde{l}_{\lambda_0} = l_{\lambda_0}$$

and, as we saw, these are the terms of the form

$$(\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i) \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(v), w \rangle.$$

Finally, since

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = c_{\nu,q}^{P_\lambda}(a_\lambda, v, w)$$

by Proposition 3.4, we obtain

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i) \Gamma_{\lambda,l}(a_\lambda, \tilde{\pi}(H_i)v, w).$$

□

**Lemma 3.8.** Let  $v, w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then, for every  $X \in \mathfrak{n}_{\lambda_0}$  and every  $m \in M_\lambda$ , we have

$$\Gamma_{\lambda, l}(m, \dot{\pi}(X)v, w) = 0.$$

*Proof.* We write  $m \in M_\lambda$  as  $m = \xi_2 a_\lambda \xi_2$  for some  $\xi_1, \xi_2 \in K_\lambda$  and some  $a_\lambda \in \overline{A_\lambda^+}$ . Then we have

$$\Gamma_{\lambda, l}(m, \dot{\pi}(X)v, w) = \Gamma_{\lambda, l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot{\pi}(X)v = \dot{\pi}(\text{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since  $M_\lambda$  normalises  $\mathfrak{n}_{\lambda_0}$  ([12], Proposition 7.83) and  $K_\lambda$  is contained in  $M_\lambda$ , we have

$$\Gamma_{\lambda, l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda, l}(a_\lambda, \dot{\pi}(X')\pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for some  $X' \in \mathfrak{n}_{\lambda_0}$ . Therefore, re-labeling things, it suffices to prove that for every  $X \in \mathfrak{a}_{\lambda_0}$  and for every  $a_\lambda \in \overline{A_\lambda^+}$ , we have

$$\Gamma_{\lambda, l}(a_\lambda, \dot{\pi}(X)v, w) = 0.$$

Moreover, since  $\Gamma_{\lambda, l}(\cdot, v, w)$  is analytic, it suffices to prove the identity for every  $a_\lambda \in A_\lambda^+$ .

Let  $a_\lambda = \exp H_\lambda \in A_\lambda^+$ . Then there exist a compact subset  $C$  of  $M_\lambda$  containing  $H_\lambda$  and such that  $K_\lambda C K_\lambda = C$ , and a positive  $R$  depending on  $C$  such that, for all  $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$  satisfying  $\alpha_i(H_{\lambda_0}) > \log R$  for every  $i \in I_\lambda^c$ , the expansion of  $\phi_{\dot{\pi}(X)v, w}(a_\lambda \exp H_{\lambda_0})$  relative to  $P$  and the expansion of  $\phi_{\dot{\pi}(X)v, w}(a_\lambda \exp H_{\lambda_0})$  relative to  $P_\lambda$  are both valid.

Setting  $H := H_\lambda + H_{\lambda_0}$  for  $H_{\lambda_0}$  as above, the first expansion gives

$$\phi_{\dot{\pi}(X)v, w}(H) = \sum_{\tilde{\lambda} \in \mathcal{E}} \sum_{\tilde{l} \in \mathbb{Z}_{\geq 0}^n: |\tilde{l}| \leq l_0} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda} - \rho)(H)} \langle c_{\tilde{\lambda}, \tilde{l}}(\dot{\pi}(X)v), w \rangle$$

and the second gives

$$\phi_{\dot{\pi}(X)v, w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu \in \mathcal{E}_I} \sum_{q \in \mathbb{Z}_{\geq 0}^{I_\lambda^c}: |q| \leq q_0} \alpha(H_{\lambda_0})^q e^{(\nu - \rho_{\lambda_0})(H_{\lambda_0})} c_{\nu, q}^{P_\lambda}(a_\lambda, \dot{\pi}(X)v, w)$$

By [11], Corollary 8.46, each  $\nu - \rho_{\lambda_0}$  in the second expansion is of the form  $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}$  for some exponent  $\tilde{\lambda}$  in the first expansion. Therefore, it suffices to prove that if  $\lambda \in \mathcal{E}_0$  and  $l \in \mathbb{Z}_{\geq 0}^n$  with  $|l| \leq l_0$  satisfy

$$\mathbf{d}_P(\lambda, l) = \mathbf{d}(\pi),$$

then no term with exponent  $\tilde{\lambda} - \rho$  for which  $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}$  appears in the first expansion. Indeed, if we can show this, since by the comparison in [11], p. 251, the term

$$\alpha(H_{\lambda_0})^q e^{(\nu - \rho_{\lambda_0})(H_{\lambda_0})} c_{\nu, q}^{P_\lambda}(a_\lambda, \tilde{\pi}(X)v, w),$$

for  $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$  and  $q_{\lambda_0} = l_{\lambda_0}$  is the sum of all the terms in the expansion of  $\phi_{\tilde{\pi}(X)v, w}(H)$  relative to  $P$  which are indexed by couples  $(\tilde{\lambda}, \tilde{l})$  satisfying

$$\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \text{ and } \tilde{l}_{\lambda_0} = l_{\lambda_0},$$

it would follow that

$$c_{\nu, q}^{P_\lambda}(a_\lambda, \tilde{\pi}(X)v, w) = 0,$$

and therefore

$$\Gamma_{\lambda, l}(a_\lambda, \tilde{\pi}(X)v, w) = 0.$$

By linearity we can assume that  $X \in \mathfrak{g}_{-\alpha_i}$  for some  $i \in I_\lambda^c$  ([11], Proposition 5.23).

Computing as in [2], Lemma 8.16, we have

$$\phi_{\tilde{\pi}(X)v, w}(a) = \langle \tilde{\pi}(\text{Ad}(a)X)\pi(a)v, w \rangle = -e^{-\alpha_i(H)} \phi_{v, \tilde{\pi}(X)w}(a).$$

Hence every exponent in the expansion of  $\phi_{\tilde{\pi}(X)v, w}(a)$  relative to  $P$  is of the form  $\tilde{\lambda} = \lambda' - e_i$  for some  $\lambda' \in \mathcal{E}$ . Now, if there existed  $\lambda' \in \mathcal{E}$  with

$$(\lambda' - e_i)|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}},$$

we would have

$$\text{Re}(\lambda' - e_i)_i = \text{Re}\lambda_i = 0$$

since  $i \in I_\lambda^c$ . This means that  $\text{Re}\lambda'_i > 0$ , a contradiction. Indeed, since  $(\pi, H)$  is tempered, the real part of every co-ordinate of each leading exponent is at most zero by Theorem 3.2 and it follows that the same property holds for every element in  $\mathcal{E}$ . This concludes the proof.  $\square$

**Lemma 3.9.** Let  $w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then there exists a collection  $\{(\theta, H_\theta)\}_{\theta \in \Theta}$  of orthogonal irreducible sub-representations of  $L^2(M_\lambda)$ , the direct sum of which we denote  $(\sigma, H_\sigma)$ , such that the image of the  $(\mathfrak{m}_\lambda, K_\lambda)$ -equivariant map

$$S_w : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda}, \quad S_w(v)(m) := \Gamma_{\lambda, l}(m, v, w)$$

is the  $(\mathfrak{m}_\lambda, K_\lambda)$ -module  $H_{\sigma, K_\lambda}$ .

*Proof.* The map  $S_w$  is well-defined by Proposition 3.6. For every  $\xi \in K_\lambda$  and every  $m \in M_\lambda$ , we have

$$S_w(\pi(\xi)v)(m) = \Gamma_{\lambda, l}(m, \pi(\xi)v, w) = \Gamma_{\lambda, l}(m\xi, v, w) = R(\xi)S_w(v)(m).$$

By Proposition 3.6, for all  $X \in \mathfrak{m}_\lambda$  and for all  $m \in M_\lambda$ , we have

$$S_w(\dot{\pi}(X)v)(m) = X\Gamma_{\lambda,l}(m, v, w)$$

and, by Proposition 2.18, we have

$$X\Gamma_{\lambda,l}(m, v, w) = \dot{R}(X)\Gamma_{\lambda,l}(m, v, w).$$

Therefore

$$S_w(\dot{\pi}(X)v)(m) = \dot{R}(X)S_w(v)(m)$$

and this concludes the proof that  $S_w$  is  $(\mathfrak{m}_\lambda, K_\lambda)$ -equivariant.

In the proof of Proposition 3.6, we showed that, for each  $v \in H_K$ , the function  $\Gamma_{\lambda,l}(\cdot, v, w)$  is a  $Z(\mathfrak{m}_\lambda)$ -finite function in  $L^2(M_\lambda)$ . By [11], Corollary 8.42, there exist finitely many orthogonal irreducible sub-representations of  $(R, L^2(M_\lambda))$  such that  $\Gamma_{\lambda,l}(\cdot, v, w)$  is contained in their direct sum. It follows that there exists a (not necessarily finite) collection  $\{(\theta, H_\theta)\}_{\theta \in \Theta}$  of orthogonal irreducible sub-representations of  $(R, L^2(M_\lambda))$  such that  $S_w(H_K)$  is contained in their direct sum. Let  $(\sigma, H_\sigma)$  denote the direct sum of the sub-representations in this collection.

We need to show that the image of  $S_w$  is precisely  $H_{\sigma, K_\lambda}$ . We begin by observing that, for any given  $v \in H_K$ , we have  $\Gamma_{\lambda,l}(\cdot, v, w) \in H_{\sigma, K_\lambda}$ . Indeed, the  $K_\lambda$ -finiteness of  $v$  implies the existence of finitely many  $v_1, \dots, v_r \in H_K$  such that

$$R(K_\lambda)\Gamma_{\lambda,l}(\cdot, v, w) \in \text{span}\{\Gamma_{\lambda,l}(\cdot, v_i, w) \mid i \in \{1, \dots, r\}\}.$$

Hence,  $\Gamma_{\lambda,l}(\cdot, v, w)$  is  $K_\lambda$ -finite and, since it is a smooth vector in  $(R, L^2(M_\lambda))$  by Proposition 3.6, it belongs to  $H_\sigma \cap L^2(M_\lambda)_{K_\lambda} = H_{\sigma, K_\lambda}$  and it follows that  $S_w(H_K) \subset H_{\sigma, K_\lambda}$ . For the reverse inclusion, the irreducibility of each  $(\theta, H_\theta)$  implies that  $S_w(H_K) \cap H_{\theta, K_\lambda} = H_{\theta, K_\lambda}$ . Therefore  $H_{\sigma, K_\lambda}$  is contained in the image of  $S_w$ , completing the proof.  $\square$

**Proposition 3.10.** Let  $w \in H_K$ . Let  $[\lambda, l] \in \mathcal{C}/\sim$  be such that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l).$$

Then there exists an admissible, finitely generated, unitary representation  $(\sigma, H_\sigma)$  of  $M_\lambda$  such that the map

$$S_w: H_K \longrightarrow H_{\sigma, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w)$$

is  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant.

*Proof.* By Lemma 3.9, there exists a unitary representation  $(\sigma, H_\sigma)$  of  $M_\lambda$  such that  $S_w(H_K) = H_{\sigma, K_\lambda}$ . By Lemma 3.7, for all  $X \in \mathfrak{a}_{\lambda_0}$  and for all  $m \in M_\lambda$ , we have

$$S_w(\dot{\pi}(X)v)(m) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(m, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)S_w(v).$$

By Lemma 3.8, for all  $X \in \mathfrak{n}_{\lambda_0}$  and for all  $m \in M_\lambda$ , we have

$$S_w(\dot{\pi}(X)v)(m) = \Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = 0.$$

We thus obtained an  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant map

$$S_w: H_K \longrightarrow H_{\sigma, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w)$$

which factors through the quotient map

$$q: H_K \longrightarrow H_K/\mathfrak{n}_{\lambda_0}H_K$$

which is  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant by Lemma 2.21.

Since  $H_K$ , being irreducible (and hence admissible by [13], Theorem 7.204), has an infinitesimal character, by Corollary 2.23 the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -module  $H_K/\mathfrak{n}_{\lambda_0}H_K$  is admissible and finitely generated. It follows that

$$S_w(H_K) = H_{\sigma, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

is an admissible and finitely generated  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -module. The fact that  $\mathfrak{a}_{\lambda_0}$  acts by scalars, implies that  $H_{\sigma, K_\lambda}$  itself is finitely generated (as  $U(\mathfrak{m}_{\lambda\mathbb{C}})$ -module) and admissible.  $\square$

Proposition 3.10, in combination with Theorem 2.24 and the discussion following it, implies that the map

$$T_w: H_K \longrightarrow \text{Ind}_{\overline{P}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_w(v)(k)(m) := \Gamma_{\lambda,l}(m, \pi(k)v, w)$$

is  $(\mathfrak{g}, K)$ -equivariant. Here,  $\overline{P}_\lambda$  denotes the parabolic subgroup opposite to  $P_\lambda$  and we recall that the half-sum of positive roots determined by  $\overline{P}_\lambda$  is precisely  $-\rho_{\lambda_0}$ . The observation above follows from the fact that  $T_w = \check{S}_w$  in the notation of the discussion following Theorem 2.24.

## 4 Asymptotic Orthogonality

For a tempered, irreducible, Hilbert representation  $(\pi, H)$  of  $G$ , for  $v, w \in H$ , let

$$\phi_{v,w}(g) := \langle \pi(g)v, w \rangle$$

denote the associated matrix coefficient. By (2) of Theorem 1.2, there exists  $d(\pi) \in \mathbb{Z}_{\geq 0}$  such that

$$\lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} |\phi_{v,w}(g)|^2 dg < \infty$$

for all  $v, w \in H_K$ .

As in [10], Section 4.1, by the polarisation identity and by (2) of Theorem 1.2, the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi_{v_3, v_4}}(g) dg$$

is a well-defined form on  $H_K$  that is linear in the first and fourth variable, conjugate-linear in the second and the third.

We explained in the Introduction that the crucial point is the proof of Proposition 1.4. We begin with the following reduction.

**Lemma 4.1.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $(\pi, H)$  be a tempered, irreducible, Hilbert representation of  $G$ . If for all  $X \in \mathfrak{g}$  and for all  $v_1, v_2, v_3, v_4 \in H_K$  we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1, v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{\dot{\pi}(X)v_3, v_4}(g) dg$$

then the equality

$$\lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{v_3, \dot{\pi}v_4}(g) dg$$

holds for every  $X \in \mathfrak{g}$  and for every  $v_1, v_2, v_3, v_4 \in H_K$ .

*Proof.* We write

$$\phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) = \langle v_1, \pi(g^{-1})\dot{\pi}(X)v_2 \rangle \overline{\langle v_3, \pi(g^{-1})v_4 \rangle}$$

and since  $\langle \cdot, \cdot \rangle$  is Hermitian we have

$$\langle v_1, \pi(g^{-1})\dot{\pi}(X)v_2 \rangle \overline{\langle v_3, \pi(g^{-1})v_4 \rangle} = \phi_{v_4, v_3}(g^{-1}) \overline{\phi}_{\dot{\pi}(X)v_2, v_1}(g^{-1}).$$

Now, since  $G_{<r}$  is invariant under  $\iota(g) = g^{-1}$  and  $G$  is unimodular, we have

$$\int_{G_{<r}} \phi_{v_4, v_3}(g^{-1}) \overline{\phi}_{\dot{\pi}(X)v_2, v_1}(g^{-1}) dg = \int_{G_{<r}} \phi_{v_4, v_3}(g) \overline{\phi}_{\dot{\pi}(X)v_2, v_1}(g) dg$$

and therefore

$$\int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = \int_{G_{<r}} \phi_{v_4, v_3}(g) \overline{\phi}_{\dot{\pi}(X)v_2, v_1}(g) dg.$$

Applying complex conjugation, we obtain

$$\overline{\int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg} = \int_{G_{<r}} \phi_{\dot{\pi}(X)v_2, v_1}(g) \overline{\phi}_{v_4, v_3}(g) dg.$$

Assuming the validity of the first identity in the statement, we can write

$$\lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_2, v_1}(g) \overline{\phi}_{v_4, v_3}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{d(\pi)}} \int_{G_{<r}} \phi_{v_2, v_1}(g) \overline{\phi}_{\dot{\pi}(X)v_4, v_3}(g) dg.$$

Now, since

$$\int_{G_{<r}} \phi_{\dot{\pi}(X)v_2, v_1}(g) \overline{\phi}_{v_4, v_3}(g) dg = \overline{\int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg},$$

it follows that

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \overline{\lim_{r \rightarrow \infty} \int_{G_{<r}} \phi_{v_2, v_1}(g) \overline{\phi}_{\dot{\pi}(X)v_4, v_3}(g) dg}.$$

Observing that

$$\overline{\int_{G_{<r}} \phi_{v_2, v_1}(g) \overline{\phi}_{\dot{\pi}(X)v_4, v_3}(g) dg} = \int_{G_{<r}} \phi_{\dot{\pi}(X)v_4, v_3}(g) \overline{\phi}_{v_2, v_1}(g) dg$$

and that, using the invariance of  $G_{<r}$  under  $\iota(g) = g^{-1}$  and the unimodularity of  $G$ , we have

$$\int_{G_{<r}} \phi_{\dot{\pi}(X)v_4, v_3}(g) \overline{\phi}_{v_2, v_1}(g) dg = \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{v_3, \dot{\pi}(X)v_4}(g) dg,$$

we finally obtain

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, \dot{\pi}(X)v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{v_3, \dot{\pi}(X)v_4}(g) dg.$$

□

**Proposition 4.2.** Let  $G$  be a connected, semisimple Lie group with finite centre and let  $(\pi, H)$  be a tempered, irreducible, Hilbert representation of  $G$ . Then, for all  $X \in \mathfrak{g}$  and for all  $v_1, v_2, v_3, v_4 \in H_K$ , we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1, v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg = - \lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \phi_{v_1, v_2}(g) \overline{\phi}_{\dot{\pi}(X)v_3, v_4}(g) dg.$$

*Proof.* The integral formula for the Cartan decomposition, taking into account the fact that, except for a set of measure zero, every  $g \in G_{<r}$  can be written as  $g = k_2 \exp H k_1$ , for some  $k_1, k_2 \in K$  and some  $H \in \mathfrak{a}_{<r}^+$ , with  $\mathfrak{a}_{<r}^+$  as in (6), shows that  $\int_{G_{<r}} \phi_{\dot{\pi}(X)v_1, v_2}(g) \overline{\phi}_{v_3, v_4}(g) dg$  is equal to

$$\int_K \int_{\mathfrak{a}_{<r}^+} \int_K \phi_{\dot{\pi}(X)v_1, v_2}(k_2 \exp H k_1) \overline{\phi}_{v_3, v_4}(k_2 \exp H k_1) \Omega(H) dk_1 dH dk_2$$

with  $\Omega(H)$  defined in (9).

Arguing as in [10], p. 258, we can interchange the two innermost integrals in the RHS and, upon multiplying both sides by  $\frac{1}{r^{\mathbf{d}(\pi)}}$  and taking the limit as  $r \rightarrow \infty$ , the RHS can be computed as the integral over  $K \times K$  of

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}_{<r}^+} \phi_{\dot{\pi}(X)v_1, v_2}(k_2 \exp H k_1) \overline{\phi}_{v_3, v_4}(k_2 \exp H k_1) \Omega(H) dH.$$

By (7) and (8), we obtain

$$\phi_{v_1, v_2}(k_2 \exp H k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\lambda, l] \in \mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda_{\lambda_0}(H_{\lambda_0})} \sum_{(\lambda', l') \in [\lambda, l]} \Psi_{\lambda', l'}^{\pi(k_1)v_1, \pi(k_2^{-1})v_2}(H)$$

and

$$\phi_{v_3, v_4}(k_2 \exp H k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\mu, m] \in \mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu|_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu', m') \in [\mu, m]} \Psi_{\mu', m'}^{\pi(k_1)v_1, \pi(k_2^{-1})v_2}(H).$$

By [10] Lemma A.5 and Claim A.6, the only non-zero contributions to

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}_{<r}^+} \phi_{\dot{\pi}(X)v_1, v_2}(k_2 \exp H k_1) \bar{\phi}_{v_3, v_4}(k_2 \exp H k_1) \Omega(H) dH$$

may come from those  $[\lambda, l] \in \mathcal{C}/\sim$  and those  $[\mu, m] \in \mathcal{C}/\sim$  for which  $I_{\lambda} = I_{\mu}$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and

$$\mathbf{d}(\pi) = |I_{\lambda}| + \sum_{i \in I_{\lambda}} (l_i + m_i).$$

In view of the first condition, the third is equivalent to requiring that

$$\mathbf{d}(\pi) = \mathbf{d}_P(\lambda, l) = \mathbf{d}_P(\mu, m),$$

where  $\mathbf{d}_P(\lambda, l)$  and  $\mathbf{d}_P(\mu, m)$  are defined by (1).

By the discussion in Section 3 and by Proposition 3.4, the expression

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{\mathfrak{a}_{<r}^+} \phi_{\dot{\pi}(X)v_1, v_2}(k_2 \exp H k_1) \bar{\phi}_{v_3, v_4}(k_2 \exp H k_1) \Omega(H) dH$$

is equal to a finite sum terms of the form

$$C(\lambda, l, m) \int_{\mathfrak{a}_{\lambda}^+} \Gamma_{\lambda, l}(\exp H_{\lambda}, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu, m}(\exp H_{\lambda}, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \Omega_{\lambda}(H_{\lambda}) dH_{\lambda},$$

with  $C(\lambda, l, m)$  as in (11), the functions  $\Gamma_{\lambda, l}$  and  $\Gamma_{\mu, m}$  defined as in (5) and  $\Omega_{\lambda}(H_{\lambda})$  defined as in (10).

Taking into account the integration over  $K \times K$ , we proved that

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_K \int_{\mathfrak{a}_{<r}^+} \int_K \phi_{\dot{\pi}(X)v_1, v_2}(k_2 \exp H k_1) \bar{\phi}_{v_3, v_4}(k_2 \exp H k_1) \Omega(H) dk_1 dH dk_2$$

is equal to a finite sum of terms of the form

$$C(\lambda, l, m) \int_K \int_K \int_{\mathfrak{a}_{\lambda}^+} \Gamma_{\lambda, l}(\exp H_{\lambda}, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu, m}(\exp H_{\lambda}, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \Omega_{\lambda}(H_{\lambda}) dH_{\lambda} dk_1 dk_2.$$

By (1) of Lemma 4.3 and applying the Fubini-Tonelli theorem, we can interchange the two innermost integral and we therefore need to prove that

$$\int_K \int_{\mathfrak{a}_{\lambda}^+} \int_K \Gamma_{\lambda, l}(\exp H_{\lambda}, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu, m}(\exp H_{\lambda}, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \Omega_{\lambda}(H_{\lambda}) dk_1 dH_{\lambda} dk_2$$



is equal to

$$- \int_K \int_{\mathfrak{a}_\lambda^+} \int_K \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_3, \pi(k_2^{-1})v_4) \Omega_\lambda(H_\lambda) dk_1 dH_\lambda dk_2$$

We set

$$\mathcal{I}(\exp H_\lambda, k_1, k_2^{-1}) := \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4).$$

We apply the quotient integral formula ([3], Theorem 2.51) to write the integral

$$\int_K \int_{\mathfrak{a}_\lambda^+} \int_K \mathcal{I}(\exp H_\lambda, k_1, k_2^{-1}) \Omega(H_\lambda) dk_1 dH_\lambda dk_2$$

as

$$\int_K \int_{\mathfrak{a}_\lambda^+} \int_{K_\lambda \backslash K} \int_{K_\lambda} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, k_2^{-1}) \Omega_\lambda(H_\lambda) d\xi_1 d\dot{k}_1 dH_\lambda dk_2$$

and again to write it as

$$\int_{K/K_\lambda} \int_{K_\lambda} \int_{\mathfrak{a}_\lambda^+} \int_{K_\lambda \backslash K} \int_{K_\lambda} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, \xi_2^{-1} k_2^{-1}) \Omega_\lambda(H_\lambda) d\xi_1 d\dot{k}_1 dH_\lambda d\xi_2 d\dot{k}_2.$$

By (3) of Lemma 4.3, we can appeal to the Fubini-Tonelli theorem to interchange the two innermost integrals and to obtain

$$\int_{K/K_\lambda} \int_{K_\lambda} \int_{\mathfrak{a}_\lambda^+} \int_{K_\lambda} \int_{K_\lambda \backslash K} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, \xi_2^{-1} k_2^{-1}) \Omega_\lambda(H_\lambda) d\dot{k}_1 d\xi_1 dH_\lambda d\xi_2 d\dot{k}_2.$$

Now, combining the fact that  $M_\lambda^{\text{reg}} = K_\lambda A_\lambda^+ K_\lambda$ , the relevant integral formula and the fact that the complement of  $M^{\text{reg}}$  has measure zero in  $M$ , it follows that the integral

$$\int_{K_\lambda} \int_{\mathfrak{a}_\lambda^+} \int_{K_\lambda} \int_{K_\lambda \backslash K} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, \xi_2^{-1} k_2^{-1}) \Omega_\lambda(H_\lambda) d\dot{k}_1 d\xi_1 dH_\lambda d\xi_2$$

is equal to

$$\int_{M_\lambda} \int_{K_\lambda \backslash K} \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu,m}(m_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4) d\dot{k}_1 dm_\lambda.$$

For  $k_1 \in K$ , we define

$$f(k_1) := \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \rangle_{L^2(M_\lambda)}.$$

The function  $f$  is invariant under left-multiplication by  $K_\lambda$ . Indeed, if  $\xi \in K_\lambda$ , then

$$\Gamma_{\lambda,l}(m_\lambda, \pi(\xi k_1)\dot{\pi}(X)v_1, \pi(k_2)v_2) = \Gamma_{\lambda,l}(m_\lambda \xi, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2)v_2)$$

and similarly for the  $\Gamma_{\mu,m}$ -term. Since the right-regular representation of  $M_\lambda$  is unitary, we have

$$\langle \Gamma_{\lambda,l}(m_\lambda \xi, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2), \Gamma_{\mu,m}(m_\lambda \xi, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \rangle_{L^2(M_\lambda)} = f(k).$$

An application of the quotient integral formula ([3], Theorem 2.51) gives

$$\int_K f(k_1) dk_1 = \int_{K_\lambda \backslash K} \int_K f(\xi k_1) d\xi dk_1 = \text{vol}(K_\lambda) \int_{K_\lambda \backslash K} f(k_1) d\dot{k}_1.$$

By (2) in Lemma 4.3 and appealing again to the Fubini-Tonelli theorem, we interchange the integrals over  $M_\lambda$  and  $K_\lambda \backslash K$  to obtain that

$$\int_{K/K_\lambda} \int_{M_\lambda} \int_{K_\lambda \backslash K} \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \bar{\Gamma}_{\mu,m}(m_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4) d\xi_1 dm_\lambda d\xi_2$$

equals

$$\frac{1}{\text{vol}(K_\lambda)} \int_{K/K_\lambda} \int_K f(k_1) dk_1 d\dot{k}_2$$

which, in turn, equals

$$\frac{1}{\text{vol}(K_\lambda)} \int_{K/K_\lambda} \int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4) \rangle_{L^2(M_\lambda)} dk_1 d\dot{k}_2.$$

For fixed  $k_2 \in K$ , set  $w_2 := \pi(k_2^{-1})v_2$  and  $w_4 := \pi(k_2^{-1})v_4$ . We reduced the problem to proving that

$$\int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v_3, w_4) \rangle_{L^2(M_\lambda)} dk_1$$

equals

$$- \int_K \langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_3, w_4) \rangle_{L^2(M_\lambda)} dk_1.$$

By Proposition 3.10, there exists an admissible, finitely generated, unitary representation  $(\sigma_1, H_{\sigma_1})$  of  $M_\lambda$  such that the image of the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant map

$$S_{w_2} : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_2}(v)(m) := \Gamma_{\lambda,l}(m, v, w_2)$$

is precisely  $H_{\sigma_1, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$  and an admissible, finitely generated, unitary representation  $(\sigma_2, H_{\sigma_2})$  such that the image of the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant map

$$S_{w_4} : H_K \longrightarrow L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_4}(v)(m) := \Gamma_{\mu,m}(m, v, w_4)$$

is precisely  $H_{\sigma_2, K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$ . Let  $(\sigma, H_\sigma)$  denote the direct sum of  $(\sigma_1, H_{\sigma_1})$  and  $(\sigma_2, H_{\sigma_2})$ . It is an admissible, finitely generated, unitary representation which restricts to a unitary representation of  $K_\lambda$ . Since  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ , by the same computations as in Lemma 3.9 and Proposition 3.10 we obtain

$(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant maps

$$S_{w_2} : H_K \longrightarrow H_{\sigma, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_2}(v)(m) := \Gamma_{\lambda,l}(m, v, w_2)$$

and

$$S_{w_4} : H_K \longrightarrow H_{\sigma, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_4}(v)(m) := \Gamma_{\mu,m}(m, v, w_4)$$

factoring through the  $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$ -equivariant quotient map

$$q: H_K \longrightarrow H_K/\mathfrak{n}_{\lambda_0}H_K.$$

From the discussion following Proposition 3.10, we obtain  $(\mathfrak{g}, K)$ -equivariant maps

$$T_{w_2}: H_K \longrightarrow \text{Ind}_{\overline{\mathcal{P}}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_2}(v)(k_1)(m) := \Gamma_{\lambda, l}(m, \pi(k_1)v, w_2)$$

and

$$T_{w_4}: H_K \longrightarrow \text{Ind}_{\overline{\mathcal{P}}_\lambda, K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_4}(v)(k_1)(m) := \Gamma_{\mu, m}(m, \pi(k_1)v, w_4).$$

By definition of the inner product on  $\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ , we see that proving the sought identity is equivalent to proving that

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} = -\langle T_{w_2}(v_1), T_{w_4}(\dot{\pi}(X)v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

By the  $(\mathfrak{g}, K)$ -equivariance of  $T_{w_2}$ , we have

$$\langle T_{w_2}(\dot{\pi}(X)v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} = \langle \text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_2}(v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}$$

and, since  $\lambda|_{\mathfrak{a}_{\lambda_0}}$  is totally imaginary, from Corollary 2.20 we deduce

$$\begin{aligned} & \langle \text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_2}(v_1), T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})} \\ &= -\langle T_{w_2}(v_1), \text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X)T_{w_4}(v_3) \rangle_{\text{Ind}_{\overline{\mathcal{P}}_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})}. \end{aligned}$$

The result follows from the  $(\mathfrak{g}, K)$ -equivariance of  $T_{w_4}$ .  $\square$

**Lemma 4.3.** Let  $v_1, w_2, v_3, w_4 \in H_K$ . Let  $[\lambda, l], [\mu, m] \in \mathcal{C}/\sim$  be such that  $I_\lambda = I_\mu$ ,  $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$  and  $\mathbf{d}(\pi) = |I_\lambda| + \sum_{i \in I_\lambda} (l_i + m_i)$ . Then the following holds:

(1)

$$\int_K \int_{\mathfrak{a}_\lambda^+} |\Gamma_{\lambda, l}(\exp H_\lambda, \pi(k_1)v_1, w_2) \overline{\Gamma}_{\mu, m}(\exp H_\lambda, \pi(k_1)v_3, w_4)| dH_\lambda dk_1 < \infty$$

(2)

$$\int_{K_\lambda \backslash K} \int_{M_\lambda} |\Gamma_{\lambda, l}(m_\lambda, \pi(k)v_1, w_2) \overline{\Gamma}_{\mu, m}(m_\lambda, \pi(k)v_3, w_4)| dm_\lambda dk < \infty$$

(3) For any fixed  $H_\lambda \in \mathfrak{a}_\lambda^+$ , we have

$$\int_{K_\lambda \backslash K} \int_{K_\lambda} |\Gamma_{\lambda, l}(\exp H_\lambda, \pi(\xi k)v_1, w_2) \overline{\Gamma}_{\mu, m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)| d\xi dk < \infty.$$

*Proof.* For (1), we begin by observing that for fixed  $k \in K$ , the functions  $\Gamma_{\lambda, l}(\exp H_\lambda, \pi(k)v_1, w_2)$  and  $\overline{\Gamma}_{\mu, m}(\exp H_\lambda, \pi(k)v_3, w_4)$  are square-integrable on  $\mathfrak{a}_\lambda^+$  by Proposition 3.5. Therefore, we have

$$\int_{\mathfrak{a}_\lambda^+} |\Gamma_{\lambda, l}(\exp H_\lambda, \pi(k)v_1, w_2) \overline{\Gamma}_{\mu, m}(\exp H_\lambda, \pi(k)v_3, w_4)| dm_\lambda < \infty.$$

Hence, we can define the function

$$h: K \longrightarrow \mathbb{R}_{\geq 0}, \quad h(k) = \int_{\mathfrak{a}_\lambda^+} |\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4)| dH_\lambda$$

and the result will follow if we establish the continuity of  $h$ . The  $K$ -finiteness of  $v_1$  and  $v_3$  implies the existence of finitely many  $K$ -finite vectors  $v_1^{(1)}, \dots, v_1^{(p)}$  and finitely many  $K$ -finite vectors  $v_3^{(1)}, \dots, v_3^{(q)}$  such that

$$\pi(k)v_1 = \sum_{i=1}^p a_i(k)v_1^{(i)}, \quad \text{and} \quad \pi(k)v_3 = \sum_{j=1}^q b_j(k)v_3^{(j)}$$

for continuous complex-valued functions  $a_i$  and  $b_j$ . Let  $k_0 \in K$ . Then

$$|h(k) - h(k_0)|$$

is majorised by the integral over  $\mathfrak{a}_\lambda^+$  of

$$|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4) - \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_0)v_1, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_0)v_3, w_4)|.$$

By reverse triangle inequality, the integrand is majorised by

$$|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4) - \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_0)v_1, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, \pi(k_0)v_3, w_4)|$$

which, in turn, is less than or equal to

$$\sum_{i=1}^p \sum_{j=1}^q |a_i(k)\bar{b}_j(k) - a_i(k_0)\bar{b}_j(k_0)| |\Gamma_{\lambda,l}(\exp H_\lambda, v_1^{(i)}, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, v_3^{(j)}, w_4)|.$$

We obtained

$$|h(k) - h(k_0)| \leq \sum_{i=1}^p \sum_{j=1}^q |a_i(k)\bar{b}_j(k) - a_i(k_0)\bar{b}_j(k_0)| \int_{\mathfrak{a}_\lambda^+} |\Gamma_{\lambda,l}(\exp H_\lambda, v_1^{(i)}, w_2) \bar{\Gamma}_{\mu,m}(\exp H_\lambda, v_3^{(j)}, w_4)| dH_\lambda$$

and the continuity follows from the continuity of the  $a_i$ 's and  $b_j$ 's.

For (2), we begin by observing that for fixed  $k \in K$ , the functions  $\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2)$  and  $\bar{\Gamma}_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)$  are square-integrable on  $M_\lambda$  by Proposition 3.5. Therefore, we have

$$\int_{M_\lambda} |\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2) \bar{\Gamma}_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)| dm_\lambda < \infty.$$

Hence, we can define the function

$$h: K \longrightarrow \mathbb{R}_{\geq 0}, \quad h(k) = \int_{M_\lambda} |\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2) \bar{\Gamma}_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)| dm_\lambda.$$

Arguing as for (1), we obtain that  $h$  is continuous.

By the right-invariance of the Haar measure on  $M_\lambda$  and since

$$\Gamma_{\lambda,l}(m_\lambda, \pi(\xi k)v_1, w_2) = \Gamma_{\lambda,l}(m_\lambda \xi, \pi(k)v_1, w_2)$$

for every  $\xi \in K_\lambda$  (and similarly for the  $\Gamma_{\mu,m}$ -term), the function  $h$  is invariant under multiplication on the left by elements in  $K_\lambda$  and it therefore descends to a continuous function on  $K_\lambda \backslash K$ , concluding the proof of the first statement.

For (3), given a fixed  $H_\lambda \in \mathfrak{a}_\lambda^+$  the function

$$K_\lambda \longrightarrow \mathbb{C}, \quad \xi \mapsto \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)$$

is continuous. Indeed, let  $\xi_0 \in K_\lambda$ . Since  $\pi(k)v$  is  $K$ -finite, it is in particular  $K_\lambda$ -finite. Hence, there exist finitely many  $K_\lambda$ -finite vectors  $v_1, \dots, v_r$  such that

$$\pi(\xi)\pi(k)v = \sum_{i=1}^r c_i(\xi)v_i,$$

where each  $c_i$  is a complex-valued continuous function on  $K_\lambda$ . Therefore, the quantity

$$|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2) - \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_0 k)v_1, w_2)|$$

is bounded by

$$\sum_{i=1}^r |c_i(\xi) - c_i(\xi_0)| |\Gamma_{\lambda,l}(\exp H_\lambda, v_i, w_2)|$$

and the claim follows from the continuity of the  $c_i$ 's.

The same argument shows that, for fixed  $H_\lambda \in \mathfrak{a}_\lambda^+$ , the function

$$K_\lambda \longrightarrow \mathbb{C}, \quad \xi \mapsto \Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi)v_3, w_4)$$

is continuous and it follows that

$$\int_{K_\lambda} |\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2) \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)}| d\xi < \infty.$$

Hence, we can define the function

$$f : K \rightarrow \mathbb{R}_{\geq 0}, \quad f(k) = \int_{K_\lambda} |\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2) \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)}| d\xi$$

and argue as in the proof of (2).  $\square$

We can now complete the strategy outlined in the Introduction. For fixed  $v_2, v_4 \in H_K$ , we define

$$A_{v_2, v_4} := D(\cdot, v_2, \cdot, v_4),$$

which is linear in the first variable and conjugate linear in the second. For fixed  $v_1, v_3 \in H_K$ , we define

$$B_{v_1, v_3} := D(v_1, \cdot, v_3, \cdot),$$

which is conjugate-linear in the first variable and linear in the second.

**Theorem 4.4.** Let  $G$  be a connected, semisimple Lie group with finite centre. Let  $(\pi, H)$  be a tempered, irreducible, Hilbert representation of  $G$ . Then there exists  $\mathbf{f}(\pi) \in \mathbb{R}_{>0}$  such that, for all  $v_1, v_2, v_3, v_4 \in H_K$ , we have

$$\lim_{r \rightarrow \infty} \frac{1}{r^{\mathbf{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)v_3, v_4 \rangle} dg = \frac{1}{\mathbf{f}(\pi)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.$$

*Proof.* Fix  $v_2, v_4 \in H_K$ . By Proposition 4.2, we can apply Corollary 2.13 to the form  $A_{v_2, v_4}$ . Hence there exists  $c_{v_2, v_4} \in \mathbb{C}$  such that for all  $v_1, v_3 \in H_K$  we have

$$A_{v_2, v_4}(v_1, v_3) = c_{v_2, v_4} \langle v_1, v_3 \rangle.$$

Similarly, fixing  $v_1, v_3 \in H_K$ , by Proposition 4.2 and Lemma 4.1 there exists a  $d_{v_1, v_3} \in \mathbb{C}$  such that

$$\overline{B}_{v_3, v_1}(v_4, v_2) = d_{v_1, v_3} \langle v_4, v_2 \rangle,$$

since the left-hand side is conjugate-linear in the first variable. Hence, since

$$\overline{B}_{v_3, v_1}(v_4, v_2) = B_{v_1, v_3}(v_2, v_4),$$

we obtain

$$B_{v_1, v_3}(v_2, v_4) = d_{v_1, v_3} \overline{\langle v_2, v_4 \rangle}$$

By definition, we have

$$D(v_1, v_2, v_3, v_4) = A_{v_2, v_4}(v_1, v_3) = B_{v_1, v_3}(v_2, v_4),$$

so, for a vector  $v_0 \in H_K$  of norm 1, using (2) of Theorem 1.2, we obtain a real number  $C(v_0, v_0) > 0$  such that

$$D(v_0, v_0, v_0, v_0) = C(v_0, v_0) = c_{v_0, v_0} = d_{v_0, v_0}.$$

Computing  $D(v_1, v_0, v_3, v_0)$ , we have

$$d_{v_1, v_3} = c_{v_0, v_0} \langle v_1, v_3 \rangle.$$

Therefore, we obtained

$$D(v_1, v_2, v_3, v_4) = c_{v_0, v_0} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle},$$

showing that  $\mathbf{f}(\pi) := \frac{1}{C(v_0, v_0)}$  does not depend on the choice of  $v_0$ , as required.  $\square$

## References

- [1] A. Borel and N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Second Edition, Math. Surveys and Monographs, Vol. **67**, Amer. Math. Soci., 2000.
- [2] W. Casselman and D. Milićić, “Asymptotic Behavior of Matrix Coefficients of Admissible Representations”, *Duke Math. J.* **49**, No.4 (1982), 869–930.

- [3] G. B. Folland, *A Course in Abstract Harmonic Analysis*, Second Edition, CRC Press, 2015.
- [4] D. M. Goldfeld and J. Hundley, *Automorphic Representations and L-functions for the General Linear Group*, Vol. **1**, Cambridge University Press, 2011.
- [5] Harish-Chandra, *Harmonic analysis on real reductive groups I. The theory of the constant term*, Collected Papers Vol. IV, Springer-Verlag Berlin Heidelberg, Reprint 2014.
- [6] Harish-Chandra, *Invariant eigendistributions on a semisimple Lie group*, Collected Papers Vol. III, Springer-Verlag Berlin Heidelberg, Reprint 2014.
- [7] Harish-Chandra, *Supplement to “Some results on differential equations”*, Collected Papers Vol. III, Springer-Verlag Berlin Heidelberg, Reprint 2014.
- [8] H. Hecht and W. Schmid, “Characters, asymptotics and  $\mathfrak{n}$ -homology of Harish-Chandra modules”, *Acta Math.* **151** (1983), 49–151.
- [9] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics, Vol. **34**, Amer. Math. Soci., 2012.
- [10] D. Kazhdan and A. Yom Din, “On Tempered Representations”, *J. Reine Angew. Math.* **788** (2022), 239–280.
- [11] A. W. Knap, *Representation Theory of Semisimple Groups: an overview based on examples*, Princeton University Press, 1986.
- [12] A. W. Knap, *Lie Groups Beyond an Introduction*, Second Edition, Progress in Mathematics, Vol. **140**, Birkhäuser, 2002.
- [13] A. W. Knap and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University Press, 1995.
- [14] A. Knightly and C. Li, *Traces of Hecke Operators*, Math. Surveys and Monographs, Vol. **133**, Amer. Math. Soc., 2006.
- [15] R. P. Langlands, *On the Classification of Irreducible Representations of Real Algebraic Groups*, Math. Surveys and Monographs, Vol. **31**, 1988.
- [16] V. S. Varadarajan, *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics, Springer Berlin, 1977.
- [17] N. R. Wallach, *Real Reductive Groups I*, Academic Press, Inc., 1988.
- [18] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Grundlehren der mathematischen Wissenschaften **188**, Springer-Verlag, 1972.

(Anne-Marie Aubert) Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France.

*Email address:* anne-marie.aubert@imj-prg.fr

(Alfio Fabio La Rosa) Department of Mathematics, University of Luxembourg, Maison du Nombre, 6 Avenue de la Fonte, L-4364, Esch-sur-Alzette, Luxembourg.

*Email address:* fabio.larosa@uni.lu