Correspondences between affine Hecke algebras and applications

Anne-Marie Aubert

CORRESPONDENCES BETWEEN AFFINE HECKE ALGEBRAS AND APPLICATIONS

ANNE-MARIE AUBERT

Abstract. We review the construction of generalized affine Hecke algebras attached to Bernstein series of both smooth irreducible and enhanced L-parameters of p-adic reductive groups and apply it to the study of the Howe correspondence.

Contents

1. Introduction

Affine Hecke algebras are known to be very useful to describe the smooth complex representations of reductive p-adic groups, in terms of the supercuspidal representations of their Levi subgroups. Their use in the classification of smooth irreducible representations started with the seminal papers of Kazhdan and Lusztig [34] for Iwahori-spherical representations and of Lusztig [43], [44] for unipotent representations.

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For G a p-adic group, the theory of the Bernstein Center of [15] provides a decomposition of the category $\mathfrak{R}(G)$ of smooth representations of G as the direct product of a family of full subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, that are called Bernstein blocks. Here $\mathfrak{s}$ is the equivalence class of a pair (L,σ), where L is a Levi subgroup of G and σ is an irreducible supercuspidal representation of L (see §3 for more details). By [31], if G is symplectic group or an orthogonal group, each Bernstein block is Morita equivalent to the category of modules of an *extended affine Hecke algebra* $\mathcal{H}(G,\mathfrak{s})$, that is, the crossed product of an affine Hecke algebra by the group algebra of a finite group. By [59], for G an arbitrary p-adic group, and every $\mathfrak{s}=[L,\sigma]_G$ such that the restriction of σ to L^1 is multiplicity free, where L^1 is the subgroup of L generated by all its compact its subgroups, the Bernstein block $\mathfrak{R}^{\mathfrak{s}}(G)$ is Morita equivalent to the category of modules of an algebra $\mathcal{H}(G,\mathfrak{s})$, which is the crossed product of an affine Hecke algebra by a (2-cocycle) twisted group algebra of a finite group.

In recent works, we showed that affine Hecke algebras play a similar role on the Galois side of the local Langlands correspondence. In [8], we introduced a notion of *cuspidality* for enhanced Langlands parameters for G, which conjecturally puts irreducible supercuspidal representations of G in bijection with such enhanced L-parameters. We also define *cuspidal support maps* and *Bernstein series* for enhanced L-parameters, in analogy with Bernstein's theory of representations of p-adic groups. In [9], to every Bernstein series $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ of enhanced Langlands parameters for G we canonically associated an affine Hecke algebra $\mathcal{H}(G^\vee,\mathfrak{s}^\vee)$, again possibly extended by the twisted group algebra of a finite group. We proved that the simple modules of this algebra are naturally in bijection with the members of the Bernstein series, and that the set of central characters of the algebra is naturally in bijection with the collection of cuspidal supports of these enhanced Langlands parameters. These bijections send tempered or (essentially) square-integrable representations to the expected kind of Langlands parameters.

Furthermore, in [9] and [10], we proved that for many reductive p-adic groups (in particular for classical groups), if a Bernstein series $\mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}}(G))$ for G corresponds to a Bernstein series $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ of enhanced Langlands parameters for G via the local Langlands correspondence, then the algebras $\mathcal{H}(G,\mathfrak{s})$ and $\mathcal{H}(G^\vee,\mathfrak{s}^\vee)$ are Morita equivalent.

We review the construction of these algebras, and explain how these results can be apply to describe the theta correspondence for p-adic groups. The theory of this correspondence, also known as the Howe correspondence, was initiated by Roger Howe and builds upon the work of Weil in a representation theoretic treatment: a key role in the theory is played by the Weil representation (so-called, oscillator representation), a representation of a non-linear-algebraic group (the metaplectic group), described notably in [11]. It turned out that this correspondence has many important uses in local representation theory. Furthermore, its global counterpart (which agrees with local theta correspondence locally) provides one of a very few direct ways to explicitly construct automorphic forms.

We describe some of the recent results on the description of the Howe correspondence over both finite fields and non-archimedean local fields. For $(G_n,G'_{n'})$ an

irreducible reductive dual pair, formed of a p-adic symplectic group $\mathrm{Sp}_{2n}(F)$ and a p-adic orthogonal group $\mathrm{O}_{2n'}(F)$, we show in Theorem 5.1.4 that the compatibility the Howe correspondence with parabolic induction leads to a collection of correspondences between simple modules of algebras $\mathcal{H}(G_n, \mathfrak{s})$ and $\mathcal{H}(G'_{n'}, \mathfrak{s}')$, where $\mathfrak{s}'$ is explicitely described in term of $\mathfrak{s}$. We consider in particular the case of tempered representations and derive in Theorem 5.2.3 a correspondence at the level of C^*-algebras when $n' = n$ or $n' = n + 1$.

The study of the Howe correspondence has been also heavily influenced by the recent developments in the framework of the Langlands program, so that new results are formulated in this spirit (see notably [3], [25], and [24]). As a consequence of our results, we show that the Howe correspondence induces a collection of correspondences between simple modules of the algebras $\mathcal{H}(G_n^\vee, \mathfrak{s}^\vee)$ and $\mathcal{H}(G_{n'}'^\vee, \mathfrak{s}'^\vee)$ attached to Bernstein series of enhanced L-parameters of G_n and $G'_{n'}$, respectively. We studied in [4] the case where $\mathfrak{s} = [T, \mathrm{triv}]_{G_n}$ (with T a maximal torus in G and triv the trivial character of T) and $n' \in \{n, n+1\}$, via the Kazhdan-Lusztig parametrization of the simple modules of $\mathcal{H}(G_n, \mathfrak{s})$ and $\mathcal{H}(G'_{n'}, \mathfrak{s}')$ (in this case, we have $\mathfrak{s}' = [T', \mathrm{triv}]_{G'_{n'}}$ where T' is a maximal torus in G').

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2. Representations of finite reductive groups

Let k be a finite field and let $\overline{k}$ be a fixed algebraic closure of k. Let $\underline{\mathbb{G}}$ be a connected reductive algebraic group defined over k, and let $\mathbb{G} = \underline{\mathbb{G}}(k)$ denote the group of the k-rational points of $\underline{\mathbb{G}}$. The usual scalar product $(\ ,\ )_{\mathbb{G}}$ on the space of class functions on $\mathbb{G}$ is defined by

$$(f_1, f_2)_{\mathbb{G}} := |\mathbb{G}|^{-1} \sum_{g \in \mathbb{G}} f_1(g)\, \overline{f_2(g)}. \tag{2.0.1}$$

Let $\mathrm{Irr}(\mathbb{G})$ be the set of equivalence classes of irreducible representations of $\mathbb{G}$. For $\pi \in \mathrm{Irr}(\mathbb{G})$, we will also write π for its character. By Corollary 7.7 of [20], for any $\pi \in \mathrm{Irr}(\mathbb{G})$, there exists a k-rational maximal torus $\underline{\mathbb{T}}$ of $\underline{\mathbb{G}}$ and a character θ of $\mathbb{T}$ such that π occurs in the Deligne-Lusztig virtual character $R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta)$, *i.e.,* $(\pi, R_{\underline{\mathbb{T}}}^{\underline{\mathbb{G}}}(\theta))_{\mathbb{G}} \neq 0$.

Deligne and Lusztig introduced in [20] the following two regularity conditions: a character θ of $\mathbb{T}$ is said to be

- *in general position* if its stabilizer in $(\mathrm{N}_{\underline{\mathbb{G}}}(\underline{\mathbb{T}})/\underline{\mathbb{T}})(k)$ is trivial,
- *non-singular* if it is not orthogonal to any coroot.

If the center of $\underline{\mathbb{G}}$ is connected, then θ is non-singular if and only if it is in general position (see Proposition 5.16 in [20]).

We denote by $\underline{\mathbb{G}}^\vee$ the reductive connected group with root datum dual to that of $\underline{\mathbb{G}}$. The $\mathbb{G}$-conjugacy classes of pairs $(\underline{\mathbb{T}}, \theta)$ as above are in one-to-one correspondence with the $\mathbb{G}^\vee$-conjugacy classes of pairs $(\underline{\mathbb{T}}^\vee, s)$ where $\underline{\mathbb{T}}^\vee$ is a k-rational maximal torus of $\underline{\mathbb{G}}^\vee$ and s is a semisimple element of $\mathbb{G}^\vee$ belonging to $\underline{\mathbb{T}}^\vee$.

The *Lusztig series* $\mathcal{E}(\mathbb{G}, s)$ is defined to be the set

$$\left\{\pi \in \mathrm{Irr}(\mathbb{G})\colon (\pi, R_{\underline{\mathbb{T}}}^{\mathbb{G}}(\theta))_{\mathbb{G}} \neq 0, \text{ where } (\underline{\mathbb{T}}, \theta)_{\mathbb{G}} \text{ corresponds to } (\underline{\mathbb{T}}^\vee, s)_{\mathbb{G}^\vee}\right\}.$$

The set $\mathrm{Irr}(\mathbb{G})$ decomposes into a disjoint union

$$\mathrm{Irr}(\mathbb{G}) = \bigsqcup_{(s)} \mathcal{E}(\mathbb{G}, s), \tag{2.0.2}$$

where (s) runs over the $\mathbb{G}^\vee$-conjugacy classes of semisimple elements of $\mathbb{G}^\vee$. If θ is the trivial character of $\mathbb{T}$, then the representation π is called *unipotent*. By definition, $\mathcal{E}(\mathbb{G}, 1)$ is the subset of unipotent representations in $\mathrm{Irr}(\mathbb{G})$. We denote by $\underline{\mathbb{G}}^\vee(s)$ the centralizer of s in $\underline{\mathbb{G}}^\vee$, and by $\mathbb{G}^\vee(s)$ the group of k-rational points of $\underline{\mathbb{G}}^\vee(s)$.

Remark 2.0.3. If $\pi \in \mathcal{E}(\mathbb{G}, s)$ and $\underline{\mathbb{G}}^\vee(s)$ is a torus, then $\pi = \pm R_{\underline{\mathbb{T}}}^{\mathbb{G}}(\theta)$ with θ in general position (see Lemma 3.6.9 in [13]).

If the center of $\underline{\mathbb{G}}^\vee$ is connected then the group $\underline{\mathbb{G}}^\vee(s)$ is always connected. However, in general it may be disconnected, and we extend the notion of Deligne-Lusztig character as follows. We denote by $\underline{\mathbb{G}}^\vee(s)^\circ$ the identity component of $\underline{\mathbb{G}}^\vee(s)$, and by $\mathbb{G}^\vee(s)^\circ$ the group of k-rational points of $\underline{\mathbb{G}}^\vee(s)^\circ$. For $\underline{\mathbb{T}}^\vee$ a k-rational maximal torus of $\underline{\mathbb{G}}^\vee(s)^\circ$ and $\theta^\vee$ a character of $\mathbb{T}^\vee$, we set

$$R_{\underline{\mathbb{T}}^\vee}^{\mathbb{G}^\vee(s)}(\theta^\vee) := \mathrm{Ind}_{\mathbb{G}^\vee(s)^\circ}^{\mathbb{G}^\vee(s)}(R_{\underline{\mathbb{T}}^\vee}^{\underline{\mathbb{G}}^\vee(s)^\circ}(\theta^\vee)). \tag{2.0.4}$$

We define $\mathcal{E}(\mathbb{G}^\vee(s), 1)$ to be the set of irreducible constituents of $R_{\underline{\mathbb{T}}^\vee}^{\mathbb{G}^\vee(s)}(1)$.

By Theorem 4.23 of [39] when the center of $\underline{\mathbb{G}}^\vee$ is connected, and by §12 of [41] in general, there is a bijection

$$\mathcal{E}(\mathbb{G}, s) \xrightarrow{1-1} \mathcal{E}((\mathbb{G}^\vee(s), 1), \quad \pi \mapsto \pi^{\mathrm{u}}. \tag{2.0.5}$$

The bijection (2.0.5) satisfies the following properties:

(1) It sends a Deligne-Lusztig character $R_{\underline{\mathbb{T}}}^{\mathbb{G}}(\theta)$ in $\mathbb{G}$ (up to a sign) to a Deligne-Lusztig character $R_{\underline{\mathbb{T}}^\vee}^{\underline{\mathbb{G}}^\vee(s)}(1)$, where 1 denotes the trivial character of $\underline{\mathbb{T}}^\vee$ (see §12 of [41]).
(2) It preserves cuspidality in the following sense: if $s \in \mathbb{G}^\vee$, and $\mathcal{E}(\mathbb{G}, s)$ contains a cuspidal representation π_{c}, then
   (a) the largest k-split torus in the center of $\underline{\mathbb{G}}^\vee(s)$ coincides with the largest k-split torus in the center of $\underline{\mathbb{G}}^\vee$ (see (8.4.5) of [39]),
   (b) the unipotent representation $\pi_{\mathrm{c}}^{\mathrm{u}}$ is cuspidal.
(3) The dimension of every representation $\pi \in \mathcal{E}(\mathbb{G}, s)$ is given by

$$\dim(\pi) = \frac{|\mathbb{G}|_{p'}}{|\underline{\mathbb{G}}^\vee(s)(k)|_{p'}} \dim(\pi^{\mathrm{u}}), \tag{2.0.6}$$

where $|\mathbb{G}|_{p'}$ is the largest prime-to-p factor of the order of $\mathbb{G}$ (see Remark 13.24 of [21]).

2.1. **Centralizers of semisimple elements in classical groups.** Let $\underline{\mathbb{G}}$ be a classical group of rank n over $\overline{k}$, *i.e.,* $\underline{\mathbb{G}}$ is a symplectic group, a special orthogonal group or an orthogonal group. Let $\underline{\mathbb{T}}_n \simeq \overline{k}^\times \times \cdots \times \overline{k}^\times$ be a k-rational maximal torus of $\underline{\mathbb{G}}$. For $s = (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}_n$, we denote by $\nu_\lambda(s)$ the number of the λ_i's which are equal to λ, and by $\langle\lambda\rangle$ the set of all roots in $\overline{k}$ of the irreducible polynomial of λ over k. By §1.B of [7], the centralizer $\mathrm{C}_{\underline{\mathbb{G}}}(s)$ of s in $\underline{\mathbb{G}}$ decomposes as a product

$$\mathrm{C}_{\underline{\mathbb{G}}}(s) = \prod_{\langle\lambda\rangle \subset \{\lambda_1,\ldots,\lambda_n\}} \underline{\mathbb{G}}_{(\lambda)}(s), \tag{2.1.1}$$

where $\underline{\mathbb{G}}_{(\lambda)}(s)$ is a reductive quasi-simple group of rank equal to $\nu_\lambda(s) \cdot |\langle\lambda\rangle|$. The followings hold:

(i) if $\underline{\mathbb{G}} = \mathrm{Sp}_{2n}$, then $\underline{\mathbb{G}}_1(s) \simeq \mathrm{Sp}_{2\nu_1(s)}$ and $\underline{\mathbb{G}}_{-1}(s) \simeq \mathrm{Sp}_{2\nu_{-1}(s)}$,
(ii) if $\underline{\mathbb{G}} = \mathrm{SO}_{2n+1}$, then $\underline{\mathbb{G}}_1(s) \simeq \mathrm{SO}_{2\nu_1(s)+1}$ and $\underline{\mathbb{G}}_{-1}(s) \simeq \mathrm{O}^{\pm}_{2\nu_{-1}(s)}$,
(iii) If $\underline{\mathbb{G}} = \mathrm{O}^{\pm}_{2n}$, then $\underline{\mathbb{G}}_1(s) \simeq \mathrm{O}^{\pm}_{2\nu_1(s)}$ and $\underline{\mathbb{G}}_{-1}(s) \simeq \mathrm{O}^{\pm}_{2\nu_{-1}(s)}$,
(iv) If $\underline{\mathbb{G}} = \mathrm{U}_n$ or if $\lambda \neq \pm 1$, then $\underline{\mathbb{G}}_{(\lambda)}(s)$ is either a general linear group or a unitary group.

We set

$$\underline{\mathbb{G}}_{\neq}(s) := \prod_{\langle\lambda\rangle \subset \{\lambda_1,\ldots,\lambda_m\}, \lambda \neq \pm 1} \underline{\mathbb{G}}_{(\lambda)}(s). \tag{2.1.2}$$

By (2.1.1), we have

$$\mathrm{C}_{\underline{\mathbb{G}}^\vee}(s) = \underline{\mathbb{G}}_1(s)^\vee \times \underline{\mathbb{G}}_{-1}(s) \times \underline{\mathbb{G}}_{\neq}(s), \tag{2.1.3}$$

and hence a bijection

$$\begin{array}{ccccc} \mathcal{E}(\mathrm{C}_{\mathbb{G}^\vee}(s), 1) & \simeq & \mathcal{E}(\mathbb{G}_1(s)^\vee, 1) \times \mathcal{E}(\mathbb{G}_{-1}(s), 1) \times \mathcal{E}(\mathbb{G}_{\neq}(s), 1) \\ \pi^{\mathrm{u}} & \mapsto & \pi^{\mathrm{u}}_1 \otimes \pi^{\mathrm{u}}_{-1} \otimes \pi^{\mathrm{u}}_{\neq} \end{array}. \tag{2.1.4}$$

We obtain a one-to-one correspondence

$$\begin{array}{rccccc} \mathcal{L}_s\colon & \mathcal{E}(\mathbb{G}, s) & \to & \mathcal{E}(\mathbb{G}_1(s)^\vee, 1) \times \mathcal{E}(\mathbb{G}_{-1}(s), 1) \times \mathcal{E}(\mathbb{G}_{\neq}(s), 1) \\ & \pi & \mapsto & \pi^{\mathrm{u}}_1 \otimes \pi^{\mathrm{u}}_{-1} \otimes \pi^{\mathrm{u}}_{\neq} \end{array}. \tag{2.1.5}$$

2.2. **The Howe correspondence over finite fields.** We suppose that the characteristic of k is odd. Let N be a positive integer, and let W be a vector space over k of dimension $2N$, equipped with a nondegenerate alternated bilinear form $\langle\,,\,\rangle$. A pair of reductive subgroups of $\mathrm{Sp}(W) = \mathrm{Sp}_{2N}(k)$, where each one is the centralizer of the other, is called *reductive dual pair*. We study irreducible dual pairs, because these are the building blocks of all the others. Such pairs are of the following kinds:

- pairs of type I:
  - $(\mathrm{Sp}_{2n}(k), \mathrm{O}_{N'}(k))$ with $nN' = N$;
  - $(\mathrm{U}_n(k), \mathrm{U}_{n'}(k))$ with $nn' = 2N$;
- pairs of type II:
  - $(\mathrm{GL}_n(k), \mathrm{GL}_{n'}(k))$ with $nn' = 2N$.

A reductive dual pair $(\mathbb{G}, \mathbb{G}')$ is said to be in the *stable range* (with $\mathbb{G}'$ smaller) if the defining vector space for $\mathbb{G}$ has a totally isotropic subspace of dimension greater or equal than the dimension the defining vector space for $\mathbb{G}'$, e.g. the pairs $(\mathrm{Sp}_{2n}(k), \mathrm{O}_{2n'}(k))$ such that $n \geq 2n'$.

Howe introduced a correspondence $\Theta_{n,n'} \colon \mathfrak{R}(\mathbb{G}_n) \to \mathfrak{R}(\mathbb{G}_{n'})$ between the categories of complex representations of these subgroups. It is obtained from a particular representation of $\mathrm{Sp}_{2N}(k)$, called the *Weil* (or *oscillator*) *representation*. In order to define this representation, we must introduce the Heisenberg group. This is the group with underlying set $H(W) = W \times k$ and product

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle \right). \tag{2.2.1}$$

Let ϱ be an irreducible representation of $H(W)$. Its restriction to the center $\mathrm{Z}_{H(W)} \simeq k$ of $H(W)$ equals ψ_ϱ, for a certain character ψ_ϱ of k.

By the Stone-von-Neumann Theorem (see Theorem 2.I.2 in [49]), for any nontrivial character ψ of $\mathrm{Z}_{H(W)}$ there exists (up to equivalence) a unique irreducible representation ϱ of $H(W)$ such that $\psi_\varrho = \psi$. This representation is known as the Heisenberg representation. It depends on ψ, so we denote it by ϱ_ψ.

The natural action of $\mathrm{Sp}(W)$ on $H(W)$ fixes the elements of its center. Hence, for a fixed character ψ of k, the representations ϱ_ψ and $g \cdot \varrho_\psi$ agree on $\mathrm{Z}_{H(W)}$, for any $g \in \mathrm{Sp}(W)$. The Stone-von-Neumann Theorem implies that there is an operator $\omega_\psi(g)$ verifying

$$\varrho_\psi(g \cdot w, t) = \omega_\psi(g) \varrho_\psi(w, t) \omega_\psi(g)^{-1}. \tag{2.2.2}$$

This defines a projective representation ω_ψ of $\mathrm{Sp}(W)$, which can be lifted to an actual representation of $\mathrm{Sp}(W)$, known as the Weil representation.

The restriction of ω_ψ of $\mathrm{Sp}_{2N}(k)$ to $\mathbb{G}_n \times \mathbb{G}'_{n'}$ is

$$\omega_{\mathbb{G}_n, \mathbb{G}'_{n'}} = \sum_{\substack{\pi \in \mathrm{Irr}(\mathbb{G}) \\ \pi' \in \mathrm{Irr}(\mathbb{G}')}} \mathrm{m}_{\pi,\pi'} \, \pi \otimes \pi', \quad \text{where } \mathrm{m}_{\pi,\pi'} \in \mathbb{Z}_{\geq 0}. \tag{2.2.3}$$

Define $\Theta_{n'} \colon \mathbb{Z}\ \mathrm{Irr}(\mathbb{G}_n) \to \mathbb{Z}\ \mathrm{Irr}(\mathbb{G}'_{n'})$ by

$$\Theta_{n'}(\pi) := \left\{ \pi' \in \mathrm{Irr}(\mathbb{G}'_{n'}) \, : \, \mathrm{m}_{\pi,\pi'} \neq 0 \right\}, \quad \text{for } \pi \in \mathrm{Irr}(\mathbb{G}_n). \tag{2.2.4}$$

The occurrence of a irreducible representation π of $\mathbb{G}_n$ in the Howe correspondence for $(\mathbb{G}_n, \mathbb{G}'_{n'_\pi})$ with n'_π minimal (i.e., such that $\Theta_{\mathbb{G}'_{n'}}(\pi) = 0$ for any $n' < n'_\pi$) is referred to as the *first occurrence*.

2.2.1. *The Howe correspondence for unipotent representations.* Between members of a dual pair, the only ones having cuspidal unipotent representations are: $\mathrm{GL}_1(k)$, $\mathrm{Sp}_{2(a^2+a)}(k)$, $\mathrm{U}_{(a^2+a)/2}(k)$ (which have a unique such representation, say τ_a), and $\mathrm{O}_{2a^2}(k)$ (which has two: τ_a^{I} and $\tau_a^{\mathrm{II}} = \tau_a^{\mathrm{I}} \otimes \mathrm{sign}$).

From now on, we will only consider pairs formed by a symplectic group $\mathrm{Sp}_{2n}(k)$ and an orthogonal group $\mathrm{O}^\epsilon_{2n'}(k)$, where $\epsilon \in \{\pm\}$, with $\mathrm{O}^+_{2n'}(k)$ split and $\mathrm{O}^-_{2n'}(k)$ nonsplit. The following results were established in [1]:

(1) if π is a cuspidal irreducible representation of $\mathbb{G}_n = \mathrm{Sp}_{2n}(k)$, then $\Theta_{n'_\pi}(\pi)$ is a singleton $\{\pi'\}$ with π' cuspidal irreducible,
(2) if $\pi \in \mathrm{Irr}(\mathbb{G}_n)$ is unipotent then any $\pi' \in \Theta_{n'}(\pi)$ is unipotent,
(3) the representation τ_a of $\mathrm{Sp}_{2(a^2+a)}(k)$ corresponds to τ_a^{II} if ϵ is the sign of $(-1)^a$ and to τ_{a+1}^{I} otherwise.

Thus, the Howe correspondence between cuspidal unipotent representations is describe by the function $\theta\colon \mathbb{N} \to \mathbb{N}$, defined by

$$\theta(a) := \begin{cases} a & \text{if } \epsilon \text{ is the sign of } (-1)^a \\ a+1 & \text{otherwise.} \end{cases} \tag{2.2.5}$$

By [7], the Howe correspondence for unipotent representations of the dual pair $(\mathrm{Sp}_{2n}(k), \mathrm{O}^\epsilon_{2n'}(k))$ induces a correspondence between the parabolically induced representations $\mathrm{i}^{\mathrm{Sp}_{2n}(k)}_{\mathrm{Sp}_{2(a^2+a)}(k)\otimes\mathbb{T}}(\tau_a \otimes 1)$ and $\mathrm{i}^{\mathrm{O}^\epsilon_{2n'}(k)}_{\mathrm{O}^\epsilon_{2\theta(a)^2}(k)\otimes\mathbb{T}'}(\tau'_{\theta(a)} \otimes 1)$, where $\tau'_{\theta(a)} \in \{\tau^{\mathrm{I}}_{\theta(a)}, \tau^{\mathrm{II}}_{\theta(a)}\}$, and $\mathbb{T}$, $\mathbb{T}'$ are products of $\mathrm{GL}_1(k)$'s.

It induces a correspondence between the endomorphism algebras of these parabolically induced representations, and hence a correspondence Ω_{N_a,N'_a} between irreducible representations of Weyl groups of types B_{N_a} and $\mathrm{B}_{N'_a}$, where $N_a := n - a^2 - a$ and $N'_a := n - \theta(a)^2$.

The following conjectural explicit description of Ω_{N_a,N'_a} for pairs $(\mathrm{Sp}_{2n}(k), \mathrm{O}^\epsilon_{2n'}(k))$ was formulated in [7], and confirmed in [55], and in [45], independently.

Let us introduce first some definitions and notation: $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ where $\lambda_i \in \mathbb{Z}_{\geq 0}$ is called a partition of n if $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. We write $\lambda \dashv n$. We denote by $\mathcal{P}(n)$ set of partitions of n, and we write $\lambda \cup \mu$ for the partition of $n+m$ with parts $\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_l$. The usual order on partitions is defined by

$$\lambda \leq \lambda' \quad \text{if and only if} \quad \lambda_1 + \cdots + \lambda_i \leq \lambda'_1 + \cdots + \lambda'_i, \text{ for all } i \in \mathbb{N}.$$

We define another order on partitions as follows: for λ, λ' partitions of possibly different integers, we write

$$\lambda \preceq \lambda' \quad \text{if and only if} \quad \lambda'_{i+1} \leq \lambda_i \leq \lambda'_i, \text{ for all } i \in \mathbb{N}. \tag{2.2.6}$$

It says that $\lambda \preceq \lambda'$ if the Young diagram of λ is contained in the one of λ' and that we can go from the first to the second by adding at most one box per column. For instance, the partitions $\lambda = (4,1,1)$ of 6, and $\lambda' = (4,4,1,1)$ of 10 verify $\lambda \preceq \lambda'$. We observe that if two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ and $\lambda' = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l'})$ satisfy $\lambda \preceq \lambda'$ and $|\lambda| = |\lambda'|$, then we have $\lambda = \lambda'$. Indeed, by (2.2.6), we have $l \leq l'$, and

$$|\lambda| = \sum_{i=1}^{l} \lambda_i \leq \sum_{i=1}^{l} \lambda'_i = |\lambda| - (\lambda'_{l+1} + \cdots + \lambda'_{l'}),$$

thus, $l' = l$ and $\lambda_i = \lambda'_i$ for all $i \in \{1, \ldots, l\}$.

Irreducible characters of a Weyl group of type B_n or C_n are known to be parametrized by bipartitions of n (see for instance [19]). We denote by $\chi_{\xi,\eta}$ the irreducible character which corresponds to the bipartition (ξ, η) of n.

In the rest of this section we consider pairs $(\mathrm{Sp}_{2n}(k), \mathrm{O}^{\epsilon}_{2n'}(k))$ in the stable range (*i.e.,* such that that $n \geq 2n'$). We write $N_a(\zeta) := N_a - |\zeta|$ if ζ is a partition such that $|\zeta| \leq N_a$. Then we have:

(1) Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^{+}_{2n'})$ with a even and $(\mathrm{Sp}_{2n}, \mathrm{O}^{-}_{2n'})$ with a odd:

$$\Omega_{N_a,N'_a} = \sum_{r=0}^{\min(N_a,N'_a)} \sum_{(\xi,\zeta)\in\mathcal{P}_2(r)} \sum_{\eta,\eta'} \chi_{\xi,\eta} \otimes \chi_{\xi,\eta'},$$

where the third sum is over the partitions $\eta \dashv N_a(\xi)$ and $\eta' \dashv N'_a(\xi)$ such that $\zeta \preceq \eta$ and $\zeta \preceq \eta'$.

(2) Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^{+}_{2n'})$ with a odd and $(\mathrm{Sp}_{2n}, \mathrm{O}^{+}_{2n'})$ with a even:

$$\Omega_{N_a,N'_a} = \sum_{r=0}^{\min(N_a,N'_a)} \sum_{(\xi,\zeta)\in\mathcal{P}_2(r)} \sum_{\xi',\eta'} \chi_{\xi',\eta} \otimes \chi_{\xi,\eta'},$$

where the third sum is over the partitions $\xi' \dashv N_a(\eta)$ and $\eta' \dashv N'_a(\xi)$ such that $\xi \preceq \xi'$ and $\eta \preceq \eta'$.

In general, there exist representations $\pi \in \mathrm{Irr}(\mathbb{G})$ such that $\Theta_{n'}(\pi)$ contains more than one element. Hence, one may wonder if it would be possible to extract a *one-to-one correspondence*. Several approaches to this questions were considered:

- definition of the eta correspondence for dual pairs $(\mathrm{Sp}_{2n}(k), \mathrm{O}_{N'}(k))$ in the stable range in [28],
- construction of a one-to-one correspondence for unipotent representations of pairs of type II and of pairs in the stable range of the form $(\mathrm{O}^{\epsilon}_{2a^2}(k), \mathrm{Sp}_{2(a^2+a+N)}(k))$ or $(\mathrm{Sp}_{2(a^2+a+2)}(k), \mathrm{O}^{\epsilon}_{2a^2+N}(k))$ [6],
- construction of a one-to-one theta correspondence for unipotent representations of irreducible pairs of type I in the stable range in [22],
- extension of both eta and theta correspondences to all irreducible pairs of type I in [57].

Let $(\xi', \eta') \in \mathcal{P}_2(N'_a)$. We set

$$\Theta_{\xi',\eta'} := \left\{\chi_{\xi,\eta} \,:\, \chi_{\xi,\eta} \otimes \chi_{\xi',\eta'} \text{ occurs in } \Omega_{N_a,N'_a}\right\}. \tag{2.2.7}$$

Recall that a partition is called *symplectic* if each odd part appears with even multiplicity. There is a bijection between symplectic partitions of $2n$ and unipotent conjugacy classes of $\underline{\mathbb{G}}_n = \mathrm{Sp}_{2n}(\overline{k})$. We denote by $\mathcal{O}_\lambda$ the unipotent class associated to the symplectic partition λ. The Springer correspondence maps $\chi_{\xi,\eta}$ to a pair $(\mathcal{O}_\lambda, \rho)$ for some irreducible character ρ of $\mathrm{C}_{\underline{\mathbb{G}}_n}(u)/\mathrm{C}_{\underline{\mathbb{G}}_n}(u)^\circ$, where $u \in \mathcal{O}_\lambda$. In this situation we write $\lambda_{\xi,\eta} := \lambda$.

Definition 2.2.8. We say that $\chi_{\xi_{\max},\eta_{\max}} \in \Theta_{\xi',\eta'}$ is *maximal* if

$$\lambda_{\xi,\eta} \leq \lambda_{\xi_{\max},\eta_{\max}} \quad \text{for all } (\eta,\xi) \in \Theta_{\xi',\eta'}.$$

Remark 2.2.9. Since the order is not total, it is not clear a priori that a maximal representation exists, and if so, that it is unique.

Definition 2.2.10. We say that $\chi_{\xi_{\min},\eta_{\min}} \in \Theta_{\xi',\eta'}$ is *minimal* if

$$\lambda_{\xi_{\min},\eta_{\min}} \leq \lambda_{\xi,\eta} \quad \text{for all } (\eta,\xi) \in \Theta_{\xi',\eta'}.$$

A representation is said to be *extremal* if it is either maximal or minimal.

Let $(\xi',\eta') \in \mathcal{P}_2(N'_a)$. By [22],

(1) there exists a unique maximal representation $\chi_{\xi_{\max},\eta_{\max}} \in \Theta_{\xi',\eta'}$, it is given by

$$\xi_{\max} := \xi' \quad \text{and} \quad \eta_{\max} := (N_a - N'_a + \eta'_1 + \eta'_2, \eta'_3, \dots, \eta'_l).$$

(2) there exists a unique minimal representation $\chi_{\xi_{\min},\eta_{\min}} \in \Theta_{\xi',\eta'}$, it is given by

$$\xi_{\min} := \xi' \quad \text{and} \quad \eta_{\min} := (N_a - N'_a) \cup \eta'.$$

3. Smooth representations of p-adic reductive groups

Let F be a non Archimedean local field and W_F its abolute Weil group. We denote by $I_F, P_F \subset W_F$ the inertia group and the wild inertia group of F, respectively, and by k_F the residual field of F (a finite field with q elements, where q is a power of a prime number p). We denote by $W_F^{\mathrm{t}} := W_F \rtimes I_F$ the tame Weil group of F.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over F. We denote by G the group of the F-rational points of $\mathbf{G}$. Let L be a Levi subgroup of a parabolic subgroup P of G and let $\mathfrak{X}_{\mathrm{nr}}(L)$ denote the group of unramified characters of L. Let σ be an irreducible supercuspidal smooth representation of L and $\mathcal{O}$ the set of equivalence classes of representations L of the form $\sigma \otimes \chi$, with $\chi \in X_{\mathrm{nr}}(L)$. We write $\mathfrak{s} := (L,\mathcal{O})_G = [L,\sigma]_G$ for the G-conjugacy class of the pair $(L,\mathcal{O})$ and $\mathfrak{B}(G)$ for the set of such classes $\mathfrak{s}$. We set $\mathfrak{s}_L := (L,\mathcal{O})_L$.

We denote by $\mathfrak{R}^{\mathfrak{s}}(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π,V) such that every irreducible G-subquotient of π has its supercuspidal support in $\mathfrak{s}$. The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ in a direct product:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G). \tag{3.0.1}$$

Let $\mathrm{Irr}^{\mathfrak{s}}(G)$ denote the set of irreducible objects of the category $\mathfrak{R}^{\mathfrak{s}}(G)$. As a direct consequence of (3.0.1), we have

$$\mathrm{Irr}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G). \tag{3.0.2}$$

3.1. **Depth-zero supercuspidal representations of p-adic groups.** We denote by $\mathcal{B}(\mathbf{G},F)$ the Bruhat-Tits building of G and by $\mathcal{B}_{\mathrm{r}}(\mathbf{G},F) := \mathcal{B}(\mathbf{G}/\mathrm{Z}_{\mathbf{G}},F)$ the reduced Bruhat-Tits building of G. We have $\mathcal{B}(\mathbf{G},F) = \mathcal{B}_{\mathrm{r}}(\mathbf{G},F) \times (X_*(\mathrm{Z}_G) \otimes_{\mathbb{Z}} \mathbb{R})$, where $X_*(\mathrm{Z}_G)$ is the set of F-algebraic cocharacters of Z_G For each point x in $\mathcal{B}(\mathbf{G},F)$, we denote by $G_{x,0}$ the parahoric subgroup of G associated to x, by $G_{x,0^+}$ the pro-p unipotent radical of $G_{x,0}$, and $\mathbb{G}_{x,0}$ the quotient $G_{x,0}/G_{x,0+}$, (the points of) a connected reductive group over the residue field of F. Let $[x]$ denote the image of x in $\mathcal{B}_{\mathrm{r}}(\mathbf{G},F)$ and $G_{[x]}$ the stabilizer of $[x]$ under the action G on $\mathcal{B}_{\mathrm{r}}(\mathbf{G},F)$.

A *depth-zero* representation of G is a representation which admits non-zero invariant vectors under the action of the pro-p unipotent radical $G_{x,0+}$ of a parahoric subgroup $G_{x,0}$ of G.

Let π be any irreducible depth-zero supercuspidal representation of G. There exists a vertex $x \in \mathcal{B}_{\mathrm{r}}(\mathbf{G}, F)$ and an irreducible cuspidal representation π_x of $\mathbb{G}_x$, such that the restriction of π to $G_{x,0}$ contains the inflation $\mathrm{infl}(\pi_x)$ of π_x (see §1-2 of [50] or Proposition 6.6 of [53]). The normalizer $\mathrm{N}_G(G_{x,0})$ of $G_{x,0}$ in G is a totally disconnected group that is compact mod center, which, by the proof of (5.2.8) in [17], coincides with the fixator $G_{[x]}$ of $[x]$ under the action of G on $\mathcal{B}_{\mathrm{r}}(\mathbf{G}, F)$. Let $\widetilde{\pi}_x$ denote an extension of $\mathrm{infl}(\pi_x)$ to $G_{[x]}$. Then π is compactly induced from a representation of $G_{[x]}$:

$$\pi = \text{c-Ind}_{G_{[x]}}^{G}(\widetilde{\pi}_x). \tag{3.1.1}$$

The representation π is called *unipotent* if π_x is unipotent

3.2. **Depth-zero supercuspidal representations of p-adic classical groups.**

Example 3.2.1. Let E be an unramified quadratic extension of F, and let V be a vector space over E of dimension n. There are two classes of hermitian forms on V. They are distinguished by the parity of the valuation of the determinant of the form: we denote the form by $\langle\ ,\ \rangle^+$ if the valuation is even, and by $\langle\ ,\ \rangle^-$ otherwise. We denote the corresponding unitary groups by $\mathrm{U}_n^{\pm}(F)$. The group $\mathrm{U}_n^+(F)$ is quasi-split. If n is odd, then the groups $\mathrm{U}_n^+(F)$ and $\mathrm{U}_n^-(F)$ are isomorphic, if n is even, then $\mathrm{U}_n^-(F)$ is an inner form of $\mathrm{U}_n^+(F)$.

The standard maximal parahoric subgroups of $G = \mathrm{U}_n^{\pm}(F)$ are the groups $G_{x,0}$ such that $\mathbb{G}_{x,0}$ is isomorphic to the product of two unitary groups $\mathrm{U}_{n_1}(k_F)$ and $\mathrm{U}_{n_2}(k_F)$ with $n_1 + n_2 = n$. By §2.2 and (3.1.1), the supercuspidal unipotent representations of $\mathrm{U}_n(F)$ are the representations $\pi = \text{c-Ind}_{G_{[x]}}^{L}(\widetilde{\pi}_x)$, where $\pi_x = \tau_{a_1} \otimes \tau_{a_2}$ with $a_1^2 + a_1 = 2n_1$ and $a_2^2 + a_2 = 2n_2$. For $i \in \{1, 2\}$, we denote by λ_i the partition of n_i defined by $\lambda_i := (k_i, k_i - 1, k_i - 2, \ldots, 1,)$.

Example 3.2.2. Let $G = \mathrm{SO}_{2n+1}(F)$. The standard maximal parahoric subgroups $G_{x,0}$ of G have type $\mathrm{D}_{n_1} \times \mathrm{B}_{n_2}$ with $n_1 + n_2 = n$. We have $\mathbb{G}_{x,0} \simeq \mathrm{SO}_{2n_1}(k_F) \times \mathrm{SO}_{2n_2+1}(k_F)$. The corresponding depth-zero supercuspidal representation of G in compactly induced from the the inflation to $G_{x,0}$ of an irreducible cuspidal representation $\pi_x^1 \otimes \pi_x^2$ of $\mathbb{G}_{x,0}$. We have $\pi_x^1 \in \mathcal{E}(\mathrm{SO}_{2n_1}, (s_1))$ and $\pi_x^2 \in \mathcal{E}(\mathrm{SO}_{2n_2+1}, (s_2))$. By 2.1, the group W_{s_1} is of one of the following types

(1) $\mathrm{D}_{a_1^2} \times \mathrm{D}_{b_1^2}$ with $a_1^2 + b_1^2 = n_1$ and $a_1, b_1 \in \mathbb{Z}_{\geq 0}$,
(2) $\mathrm{A}^2_{a_1-1} \times \mathrm{A}^2_{\ell_1-1}$ with $a_1 = \frac{c_1+c_1^2}{2}$, $\ell_1 = \frac{d_1+d_1^2}{2}$, $a_1 + \ell_1 = n_1$ and $c_1, d_1 \in \mathbb{Z}_{\geq 0}$,
(3) $\mathrm{D}_{a_1^2} \times \mathrm{A}_{a_1-1}$ with $a_1 = \frac{c_1+c_1^2}{2}$, $a_1^2 + a_1 = n_1$ and $a_1, c_1 \in \mathbb{Z}_{\geq 0}$,

and W_{s_2} is of one of the following types

(1) $\mathrm{D}_{a_2^2} \times \mathrm{C}_{b_2^2+b_2}$ with $a_2^2 + b_2^2 + b_2 = n_2$ and $a_2, b_2 \in \mathbb{Z}_{\geq 0}$,
(2) $\mathrm{A}^2_{a_2-1} \times \mathrm{A}^2_{\ell_3-1}$ with $a_2 = \frac{c_2+c_2^2}{2}$, $\ell_2 = \frac{d_2+d_2^2}{2}$, $a_2 + \ell_1 = n_2$ and $c_2, d_2 \in \mathbb{Z}_{\geq 0}$,

(3) $\mathrm{D}_{a_2^2} \times \mathrm{A}_{a_2-1}$ with $a_2 = \frac{c_2+c_2^2}{2}$, $a_2^2 + a_2 = n_2$ and $a_2, c_2 \in \mathbb{Z}_{\geq 0}$,

(4) $\mathrm{C}_{b_2^2+b_2} \times \mathrm{A}_{a_2-1}$ with $a_2 = \frac{c_2+c_2^2}{2}$, $b_2^2 = b_2 + a_2 = n_2$ and $b_2, c_2 \in \mathbb{Z}_{\geq 0}$.

4. Affine Hecke algebras

Let $\mathcal{R} := (X, R, Y, R^\vee)$ be a root datum. We denote by W the group generated by the s_α for $\alpha \in R$. It is a finite Weyl group. We write $S := \{s_\alpha : \alpha \in \Delta\}$. Then (W, S) is a finite Coxeter system.

We choose, for every $s \in S$, a complex number q_s, such that

$$q_s = q_{s'} \quad \text{if } s \text{ and } s' \text{ are conjugate in } W. \tag{4.0.1}$$

Let $\mathbf{q}\colon S \to \mathbb{C}$ be the function $s \mapsto q_s$. We define a new $\mathbb{C}$-algebra $\mathcal{H}(W, \mathbf{q})$ which has a vector space basis $\{T_w : w \in W\}$. Here T_1 is the unit element and the T_W satisfy the following quadratic relations and braid relations

$$(T_s + 1)(T_s - q_s) = 0 \quad \text{and} \quad \underbrace{T_s T_{s'} T_s \cdots}_{m(s,s') \text{ terms}} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m(s,s') \text{ terms}} \quad \text{for any } s, s' \in S. \tag{4.0.2}$$

Fix $q \in \mathbb{R}_{>1}$ and let $\lambda, \lambda^*\colon R \to \mathbb{C}$ be functions such that

- if s_α and s_β are conjugate in W, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,
- if $\alpha^\vee \notin 2Y$, then $\lambda^*(\alpha) = \lambda(\alpha)$.

We note that $\alpha^\vee \in 2Y$ is only possible for short roots α in a type B component of the root system R.

For $\alpha \in R$ we write

$$q_{s_\alpha} := q^{\lambda(\alpha)} \quad \text{and (if } \alpha^\vee \in m_{\max}^\vee) \;\; q_{s'_\alpha} := q^{\lambda^*(\alpha)}. \tag{4.0.3}$$

We denote by ℓ the usual length function on W. Let $\mathcal{H}(W, \mathbf{q})$ denote the Iwahori-Hecke algebra of W. It has a basis $\{T_w : w \in W\}$ such that

$$\begin{array}{rcll} T_w T_v & = & T_{wv} & \text{if } \ell(w) + \ell(v) = \ell(wv), \\ (T_{s_\alpha} + 1)(T_{s_\alpha} - q_{s_\alpha}) & = & 0 & \text{if } \alpha \in \Delta. \end{array}$$

Let $\{\theta_x : x \in X\}$ denote the standard basis of $\mathbb{C}[X]$. Then the affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$ is the vector space $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathcal{H}(W, \mathbf{q})$ such that $\mathbb{C}[X]$ and $\mathcal{H}(W, \mathbf{q})$ are embedded as subalgebras, and for $\alpha \in \Delta$ and $x \in X$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = \left((q^{\lambda(\alpha)} - 1) + \theta_{-\alpha} \left(q^{\frac{\lambda(\alpha)+\lambda^*(\alpha)}{2}} - q^{\frac{\lambda(\alpha)-\lambda^*(\alpha)}{2}} \right) \right) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}}.$$

When $\alpha^\vee \notin 2Y$, the cross relation simplifies to

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = (q^{\lambda(\alpha)} - 1) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-\alpha}}.$$

4.1. **Affine Hecke algebras and Bernstein blocks.** Let $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$. We write $\mathfrak{s}_L := [L, \sigma]_L$ and denote by $W^{\mathfrak{s}}$ the extended finite Weyl group $\mathrm{N}_G(\mathfrak{s}_L)/L$. The set of roots of G with respect to L contains a root system $\Sigma^{\mathfrak{s}}$, namely the set of roots for which the associated Harish-Chandra μ-function has a zero on the orbit $\mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma$. This induces a semi-direct decomposition

$$W^{\mathfrak{s}} = W(\Sigma^{\mathfrak{s}}) \rtimes \mathfrak{R}^{\mathfrak{s}}, \tag{4.1.1}$$

where $W(\Sigma^{\mathfrak{s}})$ is the Weyl group of $\Sigma^{\mathfrak{s}}$ and $\mathfrak{R}^{\mathfrak{s}}$ is the stabilizer in $W^{\mathfrak{s}}$ of the set of positive roots. We have

$$L^1 = \bigcap_{\chi \in \mathfrak{X}_{\mathrm{nr}}(L)} \ker(\chi), \tag{4.1.2}$$

and we suppose that the restriction of σ to L^1 is multiplicity free. By [59], there exists a 2-cocycle

$$\natural^{\mathfrak{s}} \colon \mathfrak{R}^{\mathfrak{s}} \times \mathfrak{R}^{\mathfrak{s}} \longrightarrow \mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma]^{\times}, \tag{4.1.3}$$

such that we have a bijection

$$\mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow \mathrm{Irr}(\mathcal{H}(G, \mathfrak{s})), \tag{4.1.4}$$

where $\mathcal{H}(G, \mathfrak{s})$ is the twisted extended affine Hecke algebra of the form

$$\mathcal{H}(G, \mathfrak{s}) := \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) \rtimes \mathbb{C}[\mathfrak{R}^{\mathfrak{s}}, \natural^{\mathfrak{s}}]. \tag{4.1.5}$$

4.2. **Affine Hecke algebras and enhanced L-parameters.** We denote by $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil-Deligne group of F and by $G^\vee$ the complex connected reductive group with root datum dual to that of G. The results in [8], [9] apply to an arbitrary group G, but, for the simplicity of the exposition, we suppose here that G is a pure inner twist of an F-split group G^*.

A *Langlands parameter* – or L-parameter – is then a morphism $\varphi \colon W_F' \to G^\vee$ such that

- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is morphism of algebraic groups,
- $\varphi(w)$ is a semisimple element of $G^\vee$, for any $w \in W_F$.

Let $\mathrm{Z}_{G^\vee}$ denote the center of $G^\vee$.

For φ a given L-parameter, we define

$$S_\varphi := \mathrm{C}_{G^\vee}(\varphi(W_F')). \tag{4.2.1}$$

An *enhanced L-parameter* is a pair (φ, ρ) where φ is an L-parameter for G and $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, with $\mathcal{S}_\varphi := S_\varphi / S_\varphi^\circ$. For φ a given L-parameter, the representation ρ is called an *enhancement* of φ. We define an action of $G^\vee$ on the set of enhanced L-parameters by:

$$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho), \ \text{ for } g \in G^\vee, \tag{4.2.2}$$

where ${}^g\rho \colon h \mapsto \rho(g^{-1}hg)$. We denote by Φ_{e} set of $G^\vee$-conjugacy classes of enhanced L-parameters, and by $\Phi_{\mathrm{e}}(G)$ the subset formed by the ones that are relevant for G.

By applying (4.2.1) to the restriction $\varphi|_{W_F}$ of φ to W_F, we define the (possibly disconnected) complex reductive group

$$\mathcal{G}_\varphi := \mathrm{C}_{G^\vee}(\varphi(W_F \times \{1\})). \tag{4.2.3}$$

We denote by $\mathcal{G}_\varphi^\circ$ the identity component of $\mathcal{G}_\varphi$, and we set

- $u = u_\varphi := \varphi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$ (a unipotent element);
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{C}_{\mathcal{G}_\varphi}(u))$. We have $\mathcal{S}_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi)$.

An enhanced L-parameter $(\varphi, \rho) \in \Phi_{\mathrm{e}}$ is called *cuspidal* if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$),
- (u_φ, ρ) is a *cuspidal pair* in $\mathcal{G}_\varphi$ (see [40], and Remark 4.6, Definition 4.7 and Definition 4.11 in [5]).

We denote by $\Phi_{\mathrm{e,cusp}}(G)$ the set of $G^\vee$-conjugacy of cuspidal enhanced L-parameters for G.

Example 4.2.4. Let $G = \mathrm{Sp}_{2n}(F)$ and let $\iota_{G^\vee} \colon G^\vee \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ be the standard embedding. For every L-parameter φ for G, we define

$$I_\varphi := \{\tau \in \mathrm{Irr}(W_F) \,:\, \tau \text{ occurs in } \iota_{G^\vee} \circ \varphi|_{W_F}\}\,. \tag{4.2.5}$$

We have $I_\varphi = I_\varphi^{\mathrm{O}} \sqcup I_\varphi^{\mathrm{S}} \sqcup I_\varphi^{\mathrm{GL}}$, where I_φ^{O} (resp. I_φ^{S}) is the subset of I_φ formed by the representations that are orthogonal (resp. symplectic), and I_φ^{GL} is the maximal subset of I_φ which is formed by representations τ that are not selfdual and satisfy $\tau^\vee \in I_\varphi^{\mathrm{GL}}$ for every $\tau \in I_\varphi^{\mathrm{GL}}$. Hence, we have the following decomposition

$$\iota_{G^\vee} \circ \varphi = \bigoplus_{\tau \in I_\varphi^{\mathrm{O}}} \tau \boxtimes M_\tau \oplus \bigoplus_{\tau \in I_\varphi^{\mathrm{S}}} \tau \boxtimes M_\tau \oplus \bigoplus_{\tau \in I_\varphi^{\mathrm{GL}}} (\tau \oplus \tau^\vee) \boxtimes M_\tau \tag{4.2.6}$$

where M_τ is a multiplicity space of τ. Let m_τ denote the dimension of M_τ. We have

$$\mathcal{G}_\varphi \simeq \prod_{\tau \in I_\varphi^{\mathrm{O}}} \mathrm{Sp}_{m_\tau}(\mathbb{C}) \times \prod_{\tau \in I_\varphi^{\mathrm{S}}} \mathrm{O}_{m_\tau}(\mathbb{C}) \times \prod_{\tau \in I_\varphi^{\mathrm{GL}}} \mathrm{GL}_{m_\tau}(\mathbb{C}). \tag{4.2.7}$$

By [52], if the enhanced L-parameter (φ, ρ) is cuspidal, then we have

$$m_\tau = \begin{cases} d_\tau^2 + d_\tau \quad (\text{with } d_\tau \in \mathbb{Z}_{\geq 0}) & \text{if } \tau \in I_\varphi^{\mathrm{O}} \\ d_\tau^2 \quad (\text{with } d_\tau \in \mathbb{Z}_{>0}) & \text{if } \tau \in I_\varphi^{\mathrm{S}} \\ 1 & \text{if } \tau \in I_\varphi^{\mathrm{GL}}. \end{cases} \tag{4.2.8}$$

Let $\mathcal{C} \in \mathfrak{U}(\mathcal{G}_\varphi^\circ)$ and let $\mathcal{E}$ irreduciblebe an $\mathcal{G}_\varphi^\circ$-equivariant local system on $\mathcal{C}$. By [40], the IC-sheaf $\mathcal{F}_\rho := \mathrm{IC}(\mathcal{C}, \mathcal{E}_\rho)$ occurs as a summand of $\mathrm{i}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}^\circ}(\mathrm{IC}(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, for some triple $(\mathcal{P}, \mathcal{L}, (\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}}))$, where $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}^\circ$ with Levi subgroup $\mathcal{L}$ and $(\mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}})$ is a cuspidal unipotent pair in $\mathcal{L}$. Moreover, the triple $(\mathcal{P}, \mathcal{L}, \mathcal{C}_{\mathrm{cusp}}, \mathcal{E}_{\mathrm{cusp}})$ is unique up to $\mathcal{G}^\circ$-conjugation.

Let $\rho^\circ \in \mathrm{Irr}(A_{\mathcal{G}^\circ}(u))$. The cuspidal support of (u, ρ°), denoted by $\mathrm{Sc}^{\mathcal{G}^\circ}(u, \rho^\circ)$, is defined to be

$$(\mathcal{L}, (v, \rho^\circ_{\mathrm{cusp}}))_{\mathcal{G}^\circ}, \quad \text{where } v \in \mathcal{C}_{\mathrm{cusp}} \text{ and } \rho^\circ_{\mathrm{cusp}} \leftrightarrow \mathcal{E}_{\mathrm{cusp}}. \tag{4.2.9}$$

We set $\mathcal{T} := \mathrm{Z}^\circ_{\mathcal{L}}$ and $\mathcal{M} := \mathrm{C}_{\mathcal{G}}(\mathcal{T})$. The cuspidal support of (u,ρ) is a (well-defined) triple $(\mathcal{M}, v, \rho_{\mathrm{cusp}})_{\mathcal{G}}$, where $\rho^\circ_{\mathrm{cusp}}$ occurs in the restriction of ρ_{cusp} to $A_{\mathcal{G}^\circ}(u)$. The cuspidal support of (φ,ρ) is defined to be

$$\mathrm{Sc}(\varphi,\rho) := (\mathrm{C}_{{}^L G}(\mathcal{T}), (\varphi_v, \rho_{\mathrm{cusp}})). \tag{4.2.10}$$

Recall from Section 3.3.1 of [29] that the group of unramified characters of L is naturally isomorphic to $(\mathrm{Z}_{L^\vee} \rtimes I_F)^\circ_{W_F}$. We consider this as an object on the Galois side of the local Langlands correspondence and we write

$$\mathfrak{X}_{\mathrm{nr}}({}^L L) := (\mathrm{Z}_{L^\vee \rtimes I_F})^\circ_{W_F}. \tag{4.2.11}$$

Given $(\varphi,\rho) \in \Phi_{\mathrm{e}}(L)$ and $z \in (\mathrm{Z}_{L^\vee \rtimes I_F})_{W_F}$, we define $(z\cdot\varphi,\rho) \in \Phi_{\mathrm{e}}(L)$ by

$$z\cdot\varphi := \varphi \text{ on } I_F \times \mathrm{SL}_2(\mathbb{C}) \text{ and } (z\cdot\varphi)(\mathrm{Fr}_F) := \widetilde{z}\varphi(\mathrm{Fr}_F), \tag{4.2.12}$$

where $\widetilde{z} \in (\mathrm{Z}_{L^\vee \rtimes I_F})_{W_F}$ represents z.

An inertial equivalence class for $\Phi_{\mathrm{e}}(G)$ is the $G^\vee$-conjugacy class $\mathfrak{s}^\vee$ of a pair $({}^L L, \mathfrak{s}^\vee_L)$, where L is a Levi subgroup of G, ${}^L L = L^\vee \rtimes W_F$ and $\mathfrak{s}^\vee_L$ is a $\mathfrak{X}_{\mathrm{nr}}({}^L L)$-orbit in $\Phi_{\mathrm{e,cusp}}(L)$.

The *Bernstein series* $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ associated to $\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)$ is defined to be the fiber of $\mathfrak{s}^\vee$ under the map Sc defined in (4.2.10). By [8], the set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of enhanced L-parameters for G is partitioned, analogously to (3.0.2), as

$$\Phi_{\mathrm{e}}(G) = \prod_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi^{\mathfrak{s}^\vee}(G). \tag{4.2.13}$$

In [9], we canonically associated an affine Hecke algebra (possibly extended with a finite R-group) $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ to every Bernstein series $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ of enhanced Langlands parameters for G. While we considered only the first case of (4.2.1), our construction applies equally well to both the other cases. We showed that the simple modules of this algebra are naturally in bijection with the elements of the Bernstein series $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ and that the set of central characters of the algebra is naturally in bijection with the collection of cuspidal supports of these enhanced Langlands parameters.

We summerise the construction of $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$. By applying (4.2.1) to the restriction $\varphi|_{I_F}$ of φ to the inertia group $I_F \subset W_F$, we define the (possibly disconnected) complex reductive group

$$\mathcal{J}_\varphi := \mathrm{C}_{G^\vee}(\varphi(I_F \times \{1\})). \tag{4.2.14}$$

Let $R(\mathcal{J}^\circ, \mathcal{T})$ be the set of $\alpha \in X^*(\mathcal{T}) \backslash \{0\}$ which appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of $\mathcal{J}^\circ_\varphi$. By Proposition 3.9 in [9], $R(\mathcal{J}^\circ, \mathcal{T})$ is a root system. We denote by $W^\circ_{\mathfrak{s}^\vee}$ its Weyl group. Let $W_{\mathfrak{s}^\vee} := \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee)/L^\vee$. We have $W_{\mathfrak{s}^\vee} = W^\circ_{\mathfrak{s}^\vee} \rtimes \mathrm{Fr}_{\mathfrak{s}^\vee}$, where

$$\mathrm{Fr}_{\mathfrak{s}^\vee} := \left\{ w \in W_{\mathfrak{s}^\vee} \,:\, w(R(\mathcal{J}^\circ, \mathcal{T})^+) \subset R(\mathcal{J}^\circ, \mathcal{T})^+ \right\}.$$

We define a root datum

$$\mathcal{R}_{\mathfrak{s}^\vee} := (R_{\mathfrak{s}^\vee}, X^*(T_{\mathfrak{s}^\vee}), R^\vee_{\mathfrak{s}^\vee}, X_*(T_{\mathfrak{s}^\vee})),$$

where $T_{\mathfrak{s}^\vee} \simeq \mathfrak{s}^\vee_L$ and

$$R_{\mathfrak{s}^\vee} = \{ m_\alpha \, \alpha \,:\, \alpha \in R(\mathcal{J}^\circ, \mathcal{T})_{\mathrm{red}} \subset X^*(T_{\mathfrak{s}^\vee}) \},$$

with $m_\alpha \in \mathbb{Z}_{>0}$. The group $W_{\mathfrak{s}^\vee}$ acts on $\mathcal{R}_{\mathfrak{s}^\vee}$. In [9] we defined $W_{\mathfrak{s}^\vee}$-invariant functions

$$\lambda\colon m_{\mathfrak{s}^\vee} \to \mathbb{Q}_{>0} \quad \text{and} \quad \lambda^*\colon \{m_\alpha\alpha \in m_{\mathfrak{s}^\vee} : m_\alpha\alpha \in 2X_*(T_{\mathfrak{s}^\vee})\} \to \mathbb{Q}. \tag{4.2.15}$$

We choose semisimple subgroups $\mathcal{J}_i \subset \mathcal{J}^\circ$, normalized by $\mathrm{N}_G(\mathfrak{s})$, such that the derived group $\mathcal{J}^\circ_{\mathrm{der}}$ is the almost direct product of the $\mathcal{J}_i$, and the multiplication map

$$m_{\mathcal{J}^\circ}\colon \mathrm{Z}^\circ_{\mathcal{J}^\circ} \times \mathcal{J}_1 \times \cdots \times \mathcal{J}_d \to \mathcal{J}^\circ \tag{4.2.16}$$

is a surjective group homomorphism with finite central kernel. It induces a decomposition

$$\mathrm{Lie}(\mathcal{J}) = \mathrm{Z}_{\mathrm{Lie}(\mathcal{J}^\circ)} \oplus \mathrm{Lie}(\mathcal{J}_1) \oplus \cdots \oplus \mathrm{Lie}(\mathcal{J}_d). \tag{4.2.17}$$

We obtain an orthogonal, $W_{\mathfrak{s}^\vee}$-stable decomposition

$$R(\mathcal{J}^\circ, \mathcal{T}) = m_1 \sqcup \cdots \sqcup m_d, \quad \text{where } m_i := R(\mathcal{J}_i, \mathcal{J}_i \cap \mathcal{T}). \tag{4.2.18}$$

We let $\vec{r} := (m_1, \ldots, m_d)$ be an array of variables, corresponding to (4.2.16) in the sense that m_i is relevant for $\mathcal{J}_i$ and m_i only. We have

$$\mathcal{H}(G^\vee, \mathfrak{s}^\vee) = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{r}) \rtimes \mathbb{C}[\mathfrak{R}^{\mathfrak{s}^\vee}, \natural^{\mathfrak{s}^\vee}], \tag{4.2.19}$$

where $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{r})$ is a generalized version (with d indeterminates) of the algebras considered in §4 and $\natural^{\mathfrak{s}^\vee}\colon \mathfrak{R}^{\mathfrak{s}^\vee} \times \mathfrak{R}^{\mathfrak{s}^\vee} \to \mathbb{C}^\times$ is 2-cocycle. The cocycle is trivial when G is an inner twist of $\mathrm{GL}_n(F)$, a pure inner twist. of a quasi-split classical group or of the group $\mathrm{GSpin}_n(F)$ (see [9], [10]), and when G is the exceptional group of type G_2 (see [12]).

5. Howe correspondence

5.1. **A correspondence between Hecke algebras of p-adic groups.** We will consider here only the case of dual pairs formed by a p-adic symplectic group and a p-adic orthogonal group of even dimension. The other cases can be treated in a similar way. Let n and n' be two fixed non-negative integers, and let W_n be a symplectic space of (even) dimension $2n$ over F. The corresponding group of isometries is the symplectic group $G_n := \mathrm{Sp}(W_n)$. Let $V_{n'}$ be a quadratic space of dimension $2n'$ over F (i.e., a space endowed with a non-degenerate symmetric F-bilinear form). We denote by $\eta_{n'}$ the character of $F^\times$ associated to $F(\sqrt{\Delta}_{n'})/F$, where $\Delta_{n'}$ is the discriminant of the form. The orthogonal group $G'_{n'} := \mathrm{O}(V_{n'})$ is the corresponding group of isometries, and $(G_n, G'_{n'})$ is a dual pair in $\mathrm{Sp}(W_{2nn'})$. Fixing a non-trivial additive character ψ of F, we obtain the so-called Weil representation ω of the metaplectic cover $\mathrm{Mp}(W_{2nn'})$ of $\mathrm{Sp}_{W_{2nn'}}$. We can define a splitting $G_n \times G'_{n'} \to \mathrm{Mp}(W_{2nn'})$ (see [36]). By means of this splitting we obtain a Weil representation of $G_n \times G'_{n'}$. which we denote $\omega_{n,n'}$.

For every $\pi \in \mathrm{Irr}(G_n)$, the maximal π-isotypic quotient of $\omega_{n,n'}$ is of the form $\pi \otimes \Theta_{n'}(\pi)$, where $\Theta_{n'}(\pi)$ is a smooth representation of $G'_{n'}$, called the *full theta lift* of π to $W_{n'}$. The representation $\Theta_{n'}(\pi)$, when non-zero, has a unique irreducible quotient, denoted by $\theta_{n'}(\pi)$, which is called the *small theta lift* of π. Similarly, if

$\pi' \in \mathrm{Irr}(G'_{n'})$, the maximal π'-isotypic quotient of $\omega_{n,n'}$ is of the form $\Theta_n(\pi') \otimes \pi'$, where $\Theta_n(\pi')$ is a smooth representation of G_n, called the full theta lift of π' to V_n. The representation $\Theta_n(\pi')$, when non-zero, has a unique irreducible quotient, denoted $\theta_n(\pi')$, which is called the small theta lift of π'. These statements were first formulated by Howe in [33], and were proved by Waldspurger in [60] when the residual characteristic of F is odd, and by Gan and Takeda [26] in general.

We have $V_{n'} = V^{\mathrm{an}} \oplus H^{n'}$, where V^{an} is an anisotropic F-vector subspace, and H is the hyperbolic plane. For $m' \geq 0$, let $V_{m'} := V^{\mathrm{an}} \oplus H^{m'}$. The collections $\mathfrak{T} := \{W_m : m \geq 0\}$ and $\mathfrak{T}' := \{V_{m'} : m' \geq 0\}$ are called Witt tower of vector spaces. One can then consider a tower of the theta correspondence associated to the tower of reductive dual pairs $(G_n, G'_{m'})_{m' \geq 0}$. For an irreducible smooth representation π of G_n, we thus have the representation $\theta_{m'}(\pi)$. The smallest non-negative integer $n'_{\min}$ such that $\theta_{n'_{\min}}(\pi) \neq 0$ is called the *first occurrence index* of π for the Witt tower $\mathfrak{T}'$, and the representation $\theta^0(\pi) := \theta_{n'_{\min}}(\pi)$ is called the first occurrence of π for this Witt tower. Similarly, one can then consider a tower of the theta correspondence associated to the tower of reductive dual pairs $(G_m, G'_{n'})_{r \geq 0}$. For an irreducible smooth representation π' of G'_n, we have the representation $\theta_{m'}(\pi)$. The smallest non-negative integer $n'_{\min}$ such that $\theta_{n_{\min}}(\pi') \neq 0$ is called the *first occurrence index* of π' for the Witt tower $\mathfrak{T}$, and the representation $\theta^0(\pi') := \theta_{n_{\min}}(\pi)$ is called the first occurrence of π' for this Witt tower. By Chapter 3 in [49], such $n'_{\min}$ and $n_{\min}$ exist and $n'_{\min} \leq n'$ and $n_{\min} \leq n$. Moreover, we have $\Theta_{m'}(\pi) \neq 0$ for any $m' \geq n'_{\min}$ and $\Theta_m(\pi') \neq 0$ for any $m \geq n_{\min}$. By Theorem 2.5 in [35] (see also Chapter 3 in [49]), if the representation π is supercuspidal then $\Theta_{n'_{\min}}(\pi)$ is irreducible (and thus is equal to $\theta_{n'_{\min}}(\pi)$) and supercuspidal. Similarly, if π' is supercuspidal then $\Theta_{n_{\min}}(\pi')$ is irreducible (and thus is equal to $\theta_{n_{\min}}(\pi')$) and supercuspidal.

A Levi subgroup L_n of a parabolic subgroup P_n of G_n is isomorphic to $G_{n_0} \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$, where $n_0 + n_1 + \cdots + n_r = n$, and any $\pi \in \mathrm{Irr}(G_n)$ is isomorphic to a subquotient of the parabolic induced representation $\mathrm{i}_{L_n,P_n}(\sigma)$ where $\sigma \in \mathrm{Irr}_{\mathrm{sc}}(L_n)$. We have $\sigma \simeq \pi_{n'_0} \otimes \tau_1 \otimes \cdots \otimes \tau_r$, where $\pi_{n'_0} \in \mathrm{Irr}_{\mathrm{sc}}(G'_{n'_0})$ and $\tau_i \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_{m_i}(F))$ for $1 \leq i \leq r$. A Levi subgroup $L'_{n'}$ of a parabolic subgroup $P'_{n'}$ of $G'_{n'}$ is isomorphic to $G'_{n'_0} \times \mathrm{GL}_{n'_1}(F) \times \cdots \times \mathrm{GL}_{n'_{r'}}(F)$, where $n'_0 + n'_1 + \cdots + n'_{r'} = n'$ and any $\pi' \in \mathrm{Irr}(G'_n)$ is isomorphic to a subquotient of the representation $\mathrm{i}_{L'_n,P'_n}(\sigma')$, where $\sigma' \simeq \pi'_{n'_0} \otimes \tau'_1 \otimes \cdots \otimes \tau'_{r'} \in \mathrm{Irr}_{\mathrm{sc}}(L'_{n'})$ such that $\pi'_{n'_0} \in \mathrm{Irr}_{\mathrm{sc}}(G'_{n'_0})$ and $\tau'_j \in \mathrm{Irr}_{\mathrm{sc}}(\mathrm{GL}_{n'_j}(F))$ for $1 \leq j \leq r'$. The G_n-conjugacy class $(L_n, \sigma)_{G_n}$ of the pair (L_n, σ) is uniquely determined by π, is called the supercuspidal support of π, and denoted by $\mathrm{Sc}(\pi)$. Similarly, the $G'_{n'}$-conjugacy class of the pair $(L'_{n'}, \sigma'_{n'})$ is uniquely determined by π', is called the supercuspidal support of π', and denoted by $\mathrm{Sc}(\pi')$. We set

$$\overline{n} := n - n_0 \quad \overline{n}' := n' - n'_0, \quad d := n - n', \quad \text{and} \quad d_0 := n_0 - n'_0. \tag{5.1.1}$$

By [35], if $\Theta_{n'}(\pi) \neq \{0\}$ and $\pi' := \theta_{n'}(\pi)$, then

(1) if $\overline{n}' \leq \overline{n}$, the supercuspidal support of π' is

$$\left(L'_{n'}, \pi'_{n'_0} \otimes \eta_{n'}^{-1}\tau_1 \otimes \cdots \otimes \eta_{n'}^{-1}\tau_\ell \otimes \nu_F^{d-1} \otimes \nu_F^{d-2} \otimes \cdots \otimes \nu_F^{d_0}\right)_{G'_{n'}},$$

where $\pi'_{n'_0} := \theta^0(\pi_{n_0})$.

(2) if $\overline{n}' \geq \overline{n}$, the supercuspidal support of π is

$$\left(L_n, \pi_{n_0} \otimes \eta_{n'}\tau'_1 \otimes \cdots \otimes \eta_{n'}\tau'_{r'} \otimes \eta_{n'}\nu_F^{-d-1} \otimes \eta_{n'}\nu_F^{-d-2} \otimes \cdots \otimes \eta_{n'}\nu_F^{-d_0}\right)_{G_n}.$$

where $\pi_{n_0} := \theta^0(\pi'_{n'_0})$.

Let $L \in \{L_n, L'_{n'}\}$. Recall that a character of L is said to be unramified if it is trivial on all the compact subgroups of L. We denote by $\mathfrak{X}_{\mathrm{nr}}(L)$ the group of unramified characters of L.

Proposition 5.1.2. *For any $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G_n)$, where $\mathfrak{s} = [L_n, \sigma]_{G_n}$, we have $\theta_{n'}(\pi) \in \mathrm{Irr}^{\theta(\mathfrak{s})}(G_n)$, where*

$$\theta(\mathfrak{s}) := [L'_{n'}, \sigma']_{G'_{n'}}. \tag{5.1.3}$$

Proof. Let $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G_n)$, where $\mathfrak{s} = [L_n, \sigma]_{G_n}$ and $\pi' := \theta_{n'}(\pi)$. Thus, we have $\mathrm{Sc}(\theta_{n'}(\pi)) = (L'_{n'}, \sigma')_{G'_{n'}}$. We set $\mathfrak{s}' := [L'_{n'}, \sigma']_{G'_{n'}}$. Let $\chi \in \mathfrak{X}_{\mathrm{nr}}(L_n)$, and let $\pi_\chi \in \mathrm{Irr}(G_n)$ such that $\mathrm{Sc}(\pi_\chi) = (L_n, \chi \otimes \sigma)_{G_n}$. The character χ has the following form: $\chi = \mathrm{triv} \otimes \chi_1 \otimes \cdots \otimes \chi_r$, where $\chi_i \in \mathfrak{X}_{\mathrm{nr}}(\mathrm{GL}_{n_i}(F))$ for $1 \leq i \leq r$. By applying the previous formula (case (1)) to the representation π_χ we see that, if $\overline{n}' \leq \overline{n}$, then the supercuspidal support of $\theta_{n'}(\pi_\chi)$ is

$$\begin{aligned}\Big(L'_{n'}, \pi'_{n'_0} \otimes \chi_1\eta_{n'}^{-1}\tau_1 \otimes \cdots \otimes \chi_r\eta_{n'}^{-1}\tau_r \otimes \nu_F^{n'-n} \otimes \cdots \otimes \nu_F^{\overline{n}'-\overline{n}-1}\Big)_{G'_{n'}} \\ = (L'_{n'}, (\chi \otimes \underbrace{\mathrm{triv} \otimes \cdots \otimes \mathrm{triv}}_{n_0 - n'_0 \text{ times}}) \otimes \sigma')_{G'_{n'}}.\end{aligned}$$

The character $\chi \otimes \mathrm{triv}^{\otimes(n_0-n'_0)}$ of $L'_{n'}$ being unramified, we have $\theta_{n'}(\pi_\chi) \in \mathrm{Irr}^{\mathfrak{s}'}(G'_{n'})$. Since every representation in $\mathrm{Irr}^{\mathfrak{s}}(G_n)$ is of the form π_χ for some $\chi \in \mathfrak{X}_{\mathrm{nr}}(L_n)$, we get

$$\theta_{n'}(\mathrm{Irr}^{\mathfrak{s}}(G_n)) \subset \mathrm{Irr}^{\mathfrak{s}'}(G'_{n'}).$$

On the other hand, if $\overline{n}' \geq \overline{n}$, by definition of the representation π_χ its supercuspidal support is $(L_n, \chi \otimes \sigma)_{G_n}$, hence, by the formula (case (2)) applied to π, it equals

$$\left(L_n, \pi_{n_0} \otimes \eta_{n'}\chi_1\tau'_1 \otimes \cdots \otimes \eta_{n'}\chi_{r'}\tau'_{r'} \otimes \eta_{n'}\nu_F^{n'-n} \otimes \cdots \otimes \eta_{n'}\nu_F^{\overline{m}-\overline{n}-1}\right)_{G_n}.$$

Thus, by applying the formula (case (2)) to π_χ, we get

$$\mathrm{Sc}(\theta_{n'}(\pi_\chi)) = (L'_{n'}, \pi'_{n'_0} \otimes \chi_1\tau'_1 \otimes \cdots \otimes \chi_{r'}\tau'_{r'})_{G'_{n'}} = (L'_{n'}, \chi' \otimes \sigma')_{G'_{n'}},$$

where $\chi' := \mathrm{triv} \otimes \chi_1 \otimes \cdots \otimes \chi_{r'} \in \mathfrak{X}_{\mathrm{nr}}(L'_{n'})$. Hence, $\theta_{n'}(\pi_\chi) \in \mathrm{Irr}^{\mathfrak{s}'}(G'_{n'})$. □

Theorem 5.1.4. *For every $\mathfrak{s} \in \mathfrak{B}(G)$, the Howe correspondence $\pi \mapsto \theta_{n'}(\pi)$ for the reductive dual pair $(G_n, G'_{n'})$ induces a correspondence $E \mapsto \theta(E)$ between subsets of simple modules of the extended affine Hecke algebras $\mathcal{H}(G_n, \mathfrak{s})$ and $\mathcal{H}(G'_{n'}, \theta(\mathfrak{s}))$.*

Proof. By [31], the algebras $\mathcal{H}(G_n, \mathfrak{s})$ for any $\mathfrak{s} \in \mathfrak{B}(G_n)$ and $\mathcal{H}(G'_{n'}, \mathfrak{s}')$ for any $\mathfrak{s}' \in \mathfrak{B}(G'_{n'})$ are extended affine Hecke algebras (that is, the 2-cocycles are trivial in the case of classical groups). Then the result follows from the combination of Proposition 5.1.2 with (3.0.2). □

5.2. **A correspondence between completions of Hecke algebras.** To study representations of G it is often useful to consider various group algebras of G. There is the Hecke algebra $\mathcal{H}(G)$, defined to be the convolution algebra of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$. The category $\mathfrak{R}(G)$ is equivalent to the category of of nondegenerate $\mathcal{H}(G)$-modules. By letting G act on $\mathcal{H}(G)$ by left translation, we obtain from (3.0.1) a decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}, \tag{5.2.1}$$

with $\mathcal{H}(G)^{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(G)$. The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}(G)$.

From the point of view of noncommutative geometry, for the study of tempered representations of G, it is interesting to use the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$, whose spectrum coincides with the tempered dual $\mathrm{Irr}^{\mathrm{t}}(G)$ of G. Analogously, we have the following decomposition of $C^*_{\mathrm{r}}(G)$:

$$C^*_{\mathrm{r}}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} C^*_{\mathrm{r}}(G)^{\mathfrak{s}}. \tag{5.2.2}$$

Theorem 5.2.3. *We suppose that $n' = n$ or $n' = n+1$. Then the theta correspondence $\pi \mapsto \theta_{n'}(\pi)$ for the reductive dual pair $(G_n, G'_{n'})$ induces a correspondence between subsets of simple modules of $C^*_{\mathrm{r}}(G)^{\mathfrak{s}}$ and $C^*_{\mathrm{r}}(G)^{\theta(\mathfrak{s})}$.*

Proof. By Theorem 1.2 in [23], if $\pi \in \mathrm{Irr}^{\mathrm{t}}(G_n)$, then $\theta_{n'}(\pi) \in \mathrm{Irr}^{\mathrm{t}}(G'_{n'})$. On the other hand, by Theorem 6.2 in [58], when G is G_n or $G'_{n'}$, then $C^*_{\mathrm{r}}(G)^{\mathfrak{s}}$ is Morita equivalent to the reduced C^*-completion of $\mathcal{H}(G,\mathfrak{s})$ for each $\mathfrak{s} \in \mathfrak{B}(G)$. Thus, the result follows from Theorem 5.1.4. □

Remark 5.2.4. Mesland and Şengün already observed in [46] that certain cases of the theta correspondence can be described by using objects of C^*-algebraic nature. More precisely, they constructed Hilbert C^*-bimodules over C^*-algebras of groups in equal rank dual pairs and showed that the Rieffel induction functors implemented by these modules coincide with the theta correspondence.

5.3. **The Howe correspondence for depth-zero representations.** In [54], Pan proved that the theta correspondence preserves depth zero representations for type I reductive dual pairs (G, G') over p-adic fields of odd residual characteristic and is compatible with the theta correspondence for finite reductive dual pairs in the following sense. Let $(\pi, \mathcal{V})$ be a depth-zero representation of G. There exists a point x of $\mathcal{B}(\mathbf{G}, F)$ such that π has non-zero invariant vectors under $G_{x,0+}$ and an irreducible cuspidal representation π_x of $\mathbb{G}_{x,0}$ which is contained in $\mathcal{V}^{G_{x,0+}}$. If $(\pi', \mathcal{V}') \in \mathrm{Irr}(G')$ corresponds to $(\pi, \mathcal{V})$ via the theta correspondence, then there exists a point x' of $\mathcal{B}(\mathbf{G}', F)$ such that π' has non-zero invariant vectors under $G'_{x',0+}$ and an irreducible cuspidal representation $\pi_{x'}$ of $\mathbb{G}'_{x',0}$ which is contained in $\mathcal{V}^{G'_{x',0+}}$, and the representation $\pi_{x'}$ corresponds to π_x via the theta correspondence for reductive dual pair $(\mathbb{G}_{x,0}, \mathbb{G}'_{x',0})$ over k_F.

5.4. **The Howe correspondence for supercuspidal representations.**

5.4.1. *The Howe correspondence for positive-depth supercuspidal representations.* Loke and Ma gave in [38] a description of the local theta correspondence between tame supercuspidal representations in terms of the supercuspidal data. Tame supercuspidal representations were constructed in [61]. Let $\mathcal{D}(G_{n'})$ (resp. $\mathcal{D}(G'_{n'})$) be a set of supercuspidal data for G_n (resp. $G'_{n'}$). For a given supercuspidal Yu datum Σ, let $[\pi_\Sigma]$ denote the isomorphism class of the supercuspidal representation attached to Σ. In [30] an equivalence relation $\sim$ on $\mathcal{D}(G_n)$ was defined so that the map from $\overline{\mathcal{D}}(G_n) := \mathcal{D}(G_n)/\sim$ to $\mathrm{Irr}_{\mathrm{sc}}(G_n)$ given by $[\Sigma] \mapsto [\pi_\Sigma]$ is a bijection. We set

$$\overline{\mathcal{D}}_{\mathfrak{T}} := \bigcup_n \overline{\mathcal{D}}(G_n).$$

Using the moment maps and theta correspondences over finite fields, a *theta lift for supercuspidal data* was defined in [38]. It is a map

$$\vartheta_{n',\mathfrak{T}'} \colon \overline{\mathcal{D}}(G_{n'}) \hookrightarrow \overline{\mathcal{D}}_{\mathfrak{T}'}. \tag{5.4.1}$$

Suppose that $\Sigma \in \mathcal{D}(G_n)$ and $[\Sigma'] := \vartheta_{n,\mathfrak{T}'}([\Sigma]) \in \overline{\mathcal{D}}_{n'}(G')$ for some n'. By [38], we have $\theta_n(\pi_\Sigma) = \pi'_{\Sigma'}$, and, conversely, if π and π' are irreducible supercuspidal representations of G_n and $G'_{n'}$, respectively, then there exists $\Sigma \in \mathcal{D}(G_n)$ such that $\pi = \pi_\Sigma$ and $\pi' = \pi'_{\Sigma'}$, where $[\Sigma'] := \vartheta_{n',\mathfrak{T}'}([\Sigma])$.

5.5. **A correspondence between Hecke algebras for enhanced L-parameters.** Let $\mathfrak{s} = [L_n, \sigma]_{G_n} \in \mathfrak{B}(G_n)$. When F has characteristic zero, a local Langlands correspondence $\sigma \mapsto (\varphi_\sigma, \rho_\sigma)$ is available via endoscopy, due to Arthur and Mœglin (see [2] and [48]). By [27], this result extends to the case where F has positive characteristic.

We set

$$\mathfrak{s}^\vee := [L_n^\vee, (\varphi_\sigma, \rho_\sigma)]_{G_n^\vee}. \tag{5.5.1}$$

By [9], for $G \in \{G_n, G'_{n'}\}$, the algebra $\mathcal{H}(G, \mathfrak{s})$ is canonically isomorphic to the algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$. Thus, by Theorem 5.1.4, the Howe correspondence $\pi \mapsto \theta_{n'}(\pi)$ for the reductive dual pair $(G_n, G'_{n'})$ induces a correspondence $E \mapsto \theta(E)$ between subsets of simple modules of the extended affine Hecke algebras $\mathcal{H}(G_n^\vee, \mathfrak{s}^\vee)$ and $\mathcal{H}(G_{n'}'^\vee, \theta(\mathfrak{s})^\vee)$, and hence a correspondence

$$(\varphi, \rho) \mapsto (\theta_{n'}(\varphi), \theta_{n'}(\rho)) \tag{5.5.2}$$

between subsets of the series $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G_n)$ and $\Phi_{\mathrm{e}}^{\theta(\mathfrak{s})^\vee}(G'_{n'})$.

References

[1] J. Adams and A. Moy, "Unipotent representations and reductive dual pairs over finite fields", Trans. Amer. Math. Soc. **340** (1) (1993) 309–321.

[2] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, Colloquium Publications **61**, Amer. Math. Soc., 2013.

[3] H. Atobe and W.T. Gan, "Local theta correspondence of tempered representations and Langlands parameters", Invent. Math. **210** (2017), no. 2, 341–415.

[4] A.-M. Aubert, "Description de la correspondance de Howe en terme de classification de Kazhdan-Lusztig", Invent. math. **103** (1991), 379–415.

[5] A.-M. Aubert, "Local Langlands and Springer correspondences", in *Representations of Reductive p-adic Groups*, A.-M. Aubert, M. Mishra, A. Roche, S. Spallone (Eds.), Progress in Mathematics **328**, Birkhäuser, 2019, pp. 1–37.
[6] A.-M. Aubert, W. Kraśkiewicz and T. Przebinda, "Howe correspondence and Springer correspondence for dual pairs over a finite field", Proc. Sympos. Pure Math. **92** (2016) 17–44.
[7] A.-M. Aubert, J. Michel and R. Rouquier, "Correspondance de Howe pour les groupes réductifs sur les corps finis", Duke Math. J. **83** No.2 (1996), 353—397.
[8] A.-M. Aubert, A. Moussaoui and M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", Manuscripta Math. **157** (2018), no. 1-2, 121–192.
[9] A.-M. Aubert, A. Moussaoui and M. Solleveld, "Affine Hecke algebras for Langlands parameters", arXiv:1701.03593, 2017.
[10] A.-M. Aubert, A. Moussaoui and M. Solleveld, "Affine Hecke algebras for classical p-adic groups", arXiv:2211.08196, 2022.
[11] A.-M. Aubert and T. Przebinda, "A reverse engineering approach to the Weil representation", Central European J. Math. **12** (2014), 1500–1585.
[12] A.-M. Aubert and Y. Xu, "Hecke algebras for p-adic reductive groups and local Langlands correspondences for Bernstein blocks", Adv. Math. **436** (2024), Paper No. 109384, 45 pp.
[13] A.-M. Aubert and Y. Xu, "The explicit local Langlands correspondence for G_2", arXiv:2208.12391, 2022.
[14] P. Bakić and M. Hanzer, "Theta correspondence for p-adic dual pairs of type I", J. Reine Angew. Math. **776** (2021), 63–117.
[15] J. Bernstein, "Le "centre" de Bernstein", in: *Representations of reductive groups over a local field*, Travaux en Cours, 1–32, Ed. P. Deligne, Hermann, Paris, 1984.
[16] N. Bourbaki, *Lie Groups and Lie Algebras*, Elements of Math., Chapters 4-6, Springer-Verlag, Berlin, 2002.
[17] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée", Inst. Hautes Études Sci. Publ. Math. (1984), **60**, 197–376.
[18] C. Bushnell and P. Kutzko, "Smooth representations of reductive p-adic groups: structure theory via types", Proc. London Math. Soc. **77** (1998), no.3, 582–634.
[19] R. Carter, *Finite Groups of Lie Type, Conjugacy classes and complex characters*, Wiley Classics Library, 1993.
[20] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", Ann. of Math. (2) **103** (1976), no. 1, 103–161.
[21] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts **95**, Cambridge University Press, Cambridge, 2020.
[22] J. Epequin, "Extremal unipotent representations for the finite Howe correspondence", Journal of Algebra **535** (2019) 480–502.
[23] W.T. Gan and S. Takeda, "On the Howe duality conjecture in classical theta correspondence", in *Advances in the theory of automorphic forms and their L-functions*, 105–117, Contemp. Math. **664**, Amer. Math. Soc., Providence, RI, 2016.
[24] W. T. Gan and A. Ichino, "Formal degrees and local theta correspondence", Invent. math. **195** (2014), 509–672.
[25] W. T. Gan and G. Savin, "Representations of metaplectic groups I: Epsilon dichotomy and local Langlands correspondence", Compos. Math. **148** (2012), no. 6, 1655–1694.
[26] W. T. Gan and S. Takeda, "A proof of the Howe duality conjecture", J. Amer. Math. Soc. **29** (2016), 473–493.
[27] R. Ganapathy and S. Varma, "On the local Langlands correspondence for split classical groups over local function fields", J. Inst. Math. Jussieu **16** (2017), 987–107.

[28] S. Gurevich and R. Howe, "Small representations of finite classical groups", in *Representation theory, number theory, and invariant theory*, Progr. Math., vol.**323**, 2017, pp. 209–234.
[29] T. J. Haines, "The stable Bernstein center and test functions for Shimura varieties" pp. 118–186 in: *Automorphic Forms and Galois Representations*, London Math. Soc. Lecture Note Series **415**, Cambrige University Press, 2014.
[30] J. Hakim and F. Murnaghan, "Distinguished tame supercuspidal representations", IMRP (2008).
[31] V. Heiermann, "Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique: le cas des groupes classiques", Selecta Math. (N.S.) **17** (2011), no. 3, 713–756.
[32] V. Heiermann, "Local Langlands correspondence for classical groups and affine Hecke algebras", Math. Z. (2017) **287**, 1029–1052.
[33] R. Howe, "θ-series and invariant theory", in: Automorphic forms, representations and L-functions, American Mathematical Society, 1979, 275–285.
[34] D. Kazhdan and G. Lusztig, "Proof of the Deligne-Langlands conjecture for Hecke algebras", Invent. Math. **87** (1987), no. 1, 153–215.
[35] S. Kudla, "On the local theta-correspondence", Invent. math. **83** (1986), 229–255.
[36] S. Kudla, "Splitting metaplectic covers of dual reductive pairs", Isr. J. Math. **87** (1994), 361–401.
[37] S. Kudla and S. Rallis, "On first occurrence in the local theta correspondence", in *Automorphic representations, L-functions and applications: progress and prospects*, Ohio State Univ. Math. Res. Inst. Publ., **11**, de Gruyter, Berlin, 2005, 273–308.
[38] H. Y. Loke and J.-J. Ma, "Local theta correspondences between supercuspidal representations", Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no.4, 927–991.
[39] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. Math. Stud., Princeton, 1984.
[40] G. Lusztig, "Intersection cohomology complexes on a reductive group", Invent. Math. **75.2** (1984), 205–272.
[41] G. Lusztig, "On the representations of reductive groups with disconnected centre", Astérisque, tome **168** (1988), 157–166.
[42] G. Lusztig, "Cuspidal local systems and graded Hecke algebras", Publ. Math. Inst. Hautes Études Sci. **67** (1988), 145–202.
[43] G. Lusztig, "Classification of unipotent representations of simple p-adic groups", Int. Math. Res. Not. (1995), 517-589.
[44] G. Lusztig, "Classification of unipotent representations of simple p-adic groups, II", Representation Theorey **6** (2002), 243–289.
[45] J.-J. Ma, C. Qiu, and J. Zou, "Generic Hecke algebra and theta correspondence over finite fields", Adv. Math. **438** (2024), Paper No. 109444, 44 pp.
[46] B. Mesland and M.H. Şengün, "Equal rank local theta correspondence as a strong Morita equivalence", arXiv:2207.13484, 2022.
[47] C. Mœglin, "Multiplicité 1 dans les paquets d'Arthur aux places p-adiques", pp. 333–374 in: *On certain L-functions*, Clay Math. Proc. **13**, Amer. Math. Soc., 2011.
[48] C. Mœglin, "Paquets stables des séries discrètes accessibles par endoscopie tordue: leur paramètre de Langlands", Contemp. Math. **614** (2014), 295-–336
[49] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. **1291**, Springer.
[50] L. Morris, "Tamely ramified supercuspidal representations", Ann. scien. Éc. Norm. Sup. 4$^{\text{e}}$ série, **29** (1996), 639–667.
[51] L. Morris, "Level zero G-types", Compositio Mathematica **118** (1999), 135–157.
[52] A. Moussaoui, "Centre de Bernstein enrichi pour les groupes classiques", Represent. Theory **21** (2017), 172–246.

[53] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types", Comment. Math. Helv. **71** (1996), no. 1, 98–121.
[54] S.-Y. Pan, "Local theta correspondence of depth zero representations and theta dichotomy", J. Math. Soc. Japan **54** (2002), 793–845.
[55] S.-Y. Pan, "Howe correspondence of unipotent characters for a finite symplectic/even-orthogonal dual pair", Amer. J. Math. **146**:3 (2024), 813–869.
[56] S.-Y. Pan, "Lusztig correspondence and Howe correspondence for finite reductive dual pairs", Math. Ann. (2024).
[57] S.-Y. Pan, "On theta and eta correspondences for finite symplectic/orthogonal dual pair", arXiv:2006.06241, 2020.
[58] M. Solleveld, "On completions of Hecke algebras", in: A.-M. Aubert, M. Mishra, A. Roche, S. Spallone (Eds.), *Representations of Reductive p-adic Groups*, Progress in Mathematics **328**, Birkhäuser, 2019, pp. 207–262.
[59] M. Solleveld, "Endomorphism algebras and Hecke algebras for reductive p-adic groups", J. Algebra **606** (2022), 371–470.
[60] J.-L. Waldspurger, " Démonstration d'une conjecture de duality de Howe dans le cas p-adique, $p \neq 2$", in: Festschrift in honor of I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I, Weizmann, Jerusalem, 1990, 267–324.
[61] J.-K. Yu, "Construction of tame supercuspidal representations", J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.

Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France, anne-marie.aubert@imj-prg.fr