Size of multipartitions through level-rank duality

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SIZE OF MULTIPARTITIONS THROUGH LEVEL-RANK DUALITY

NICOLAS JACON

Abstract. We give short purely combinatorial proofs for certain formulae relating the size of an l-partition with the one of its e-quotient in the sense of Uglov. We deduce a way to compute the size of the e-core of multipartitions and the e-weight of a multipartition. All these multipartitions naturally appear in the context of level-rank duality for Ariki-Koike algebras and Cherednik algebras.

1. Introduction

It is well-known that the combinatorics of partitions play an important role in the context of the ordinary and modular representation theory of the symmetric group $\mathfrak{S}_n$. The simple modules of $\mathfrak{S}_n$ in characteristic zero are naturally labeled by these objects whereas they are indexed by certain subset of partitions called p-regular partitions in characteristic p. One usual way to study the modular representation theory of the symmetric group is to consider the associated decomposition matrix. This matrix, whose rows are indexed by the partitions of n, controls the representation theory of $\mathfrak{S}_n$ over a field of characteristic $p > 0$ by a process of modular reduction.

The decomposition matrix of the symmetric group is in fact a block diagonal matrix and one may study the blocks of this matrix independently. It is thus an important problem to understand how these blocks are distributed. To do this, one can associate to any partition by a purely combinatorial way, a couple given by a certain partition, called the p-core, and a p-tuple of partitions called the p-quotient. Two partitions indexing two rows of the matrix are in the same block if and only if they have the same p-core partitions (this is known and the Nakayama's conjecture, which is in fact a Theorem). In addition, the size of the p-quotient is an invariant for the block, the p-weight, which gives a good approximation of how the block is complicated.

Recently, there have been progresses on understanding these problems in the more general context of Ariki-Koike algebras at an e-root of unity and associated with a l-tuple $\mathbf{s}^l$ in $\mathbb{Z}^l$ (called a multicharge), see [5, 3] There is also a notion of decomposition matrix for these algebras, whose rows are this time indexed by the l-tuples of partitions (called l-partitions). One can associate to each l-partition $\boldsymbol{\lambda}^l$ an e-partition $\boldsymbol{\lambda}_e$ associated with a multicharge $\mathbf{s}_e$ in $\mathbb{Z}^e$. This datum then allows to define an analogue of the e-core in this context.

The map sending $(\boldsymbol{\lambda}^l, \mathbf{s}_l)$ to $(\boldsymbol{\lambda}_e, \mathbf{s}_e)$ has been originally defined by Uglov in the context of quantum groups. This map not only permits the generalization of the notion of core: it plays an important role in the representation theory of rational Cherednik algebras and Fock spaces for quantum groups.

(1) It is a cornerstone for the definition of the canonical basis for Fock spaces in higher level [6] in affine type A.
(2) It gives an equivalence (the level-rank duality) between the category $\mathcal{O}$ of different Cherednik algebras [1, 7, 8].

The aim of this note is to show several properties of this map. We relate the size of $\boldsymbol{\lambda}^l$ with $\boldsymbol{\lambda}_e$ and we show by a purely combinatorial argument that the size of the e-partitions associated with l-partitions of the same size and with the same core, have the same size (see Theorem 5). We give an explicit formula for this size (see Proposition 2). Then, generalizing results of [4], we give a formula for the size of the e-core l-partitions. We deduce a way to compute the size of any e-partition, given the datum of its core. This in particular gives a way to compute the e-weight of an l-partition (see Corollary 2).

2. Multipartitions, cores and quotients

We first quickly recall the main combinatorial notions we need to state our results. For more details, we refer to [2]. For $s \in \mathbb{Z}$ and $l \in \mathbb{N}$, we denote by $\mathbb{Z}^l[s]$ the set of $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$ such that $\sum_{0 \leq i \leq l-1} s_i = s$.

2.1. **Partitions.** A *partition* $\lambda := (\lambda_1, \dots, \lambda_r)$ of *size* $n \in \mathbb{Z}_{>0}$ is a sequence of non increasing integers of total sum $|\lambda| := n$. If $l \in \mathbb{N}$, an *l-partition* (or multipartition) $\boldsymbol{\lambda}$ is a sequence of partitions $(\lambda^0, \dots, \lambda^{l-1})$ such that $|\boldsymbol{\lambda}| := \sum_{0 \leq i \leq l-1} |\lambda^i| = n$.. The integer n is called the *size* of the l-partition $\boldsymbol{\lambda}$. The set of l-partitions of size n is denoted by $\Pi^l(n)$ and the set of l-partitions by Π^l.

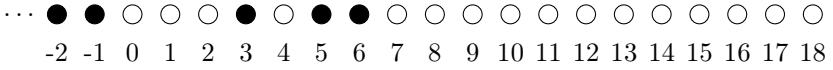
2.2. **Symbols.** By definition, a *symbol* of *charge* $s \in \mathbb{Z}$ is an infinite sequence of integers $X = (\beta_i)_{i<s}$ such that:

(1) For all $i < m$, we have $\beta_{i-1} < \beta_i$ (that is X is a strictly increasing sequence)
(2) There exists $N < m$ such that for all $j \leq N$, we have $\beta_j = j$.

We denote by $\mathrm{Symb}(s)$ the set of symbols of charge m. If $m \in \mathbb{Z}$ then the symbol $X = (\beta_i)_{i<s}$ such that $\beta_i = i$ for all $i < m$ is called the *trivial symbol* of charge m.

A symbol may be conveniently represented using its abacus configuration and we will use this configuration intensively in this paper. We associate to a symbol X an horizontal runner full of (an infinite number of) beads numbered in $\mathbb{Z}$. A bead numbered by $a \in \mathbb{Z}$ is colored in black if and only if $a \in X_j$ (we will say that the *position* of the bead is a). The others are written in white.

Example 1. The abacus associated to the symbol $X = (\dots, -3, -2, -1, 3, 5, 6)$ of charge 3 is



The charge of a symbol X may be conveniently read in the associated abacus as follows. For each black bead, if there exists a white bead at its left then replace the leftmost one with a black bead and the black bead itself with a white bead. We obtain the abacus of a trivial symbol. The charge of X is then the charge of the trivial symbol, that is the number associated with the leftmost white bead.

As for the notion of l-partition, one can now generalize this: an *l-symbol* is a collection of l symbols:

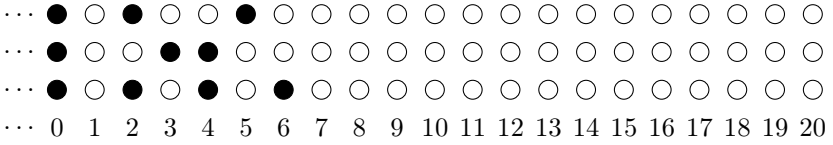
$$\mathbf{X} = (X_0, \dots, X_{l-1}).$$

The *multicharge* (or l-charge) of the symbol is the l-tuple $(s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$ where for all $j = 0, \dots, l-1$, the number s_j is the charge of $X_j = (\beta_i^j)_{i<s_j}$. An *$l$-symbol* $\mathbf{X} = (X_0, \dots, X_{l-1})$ can be represented using its abacus configuration. In this way, we associate to each X_j from $j = 0$ to $l-1$ an abacus as above and we write them from bottom to top so that the beads in the same column are numbered by the same integer. We call the associated object an *l-abacus*.

Example 2. Let $l = 3$ and let us consider the following 3-symbol:

$$X = ((\dots, -1, 0, 2, 4, 6), (\dots, -1, 0, 3, 4), (\dots, -1, 0, 2, 5)).$$

The associated 3-abacus is:



We can recover the multicharge by moving the black beads at the left in each runner. To each symbol $X = (\beta_i)_{i<s}$ (and thus to each abacus) of charge s we can canonically associate a partition $\lambda(X) = (\lambda_1, \dots, \lambda_r)$ such that for all $i \geq 1$, we have $\lambda_i = \beta_{s-i} + i - s$. Note that if $k >> 1$ then $\lambda_k = 0$. Regarding the abacus associated to the set of β-numbers, the parts of the partition are easily obtained by counting the numbers of white beads at the left of each black bead. Conversely, to any partition $(\lambda_1, \dots, \lambda_r)$, we can associate a set of β-numbers (and thus an abacus). Let $m \in \mathbb{Z}$. Then we define:

$$X^s(\lambda) = (\beta_i)_{i<s}$$

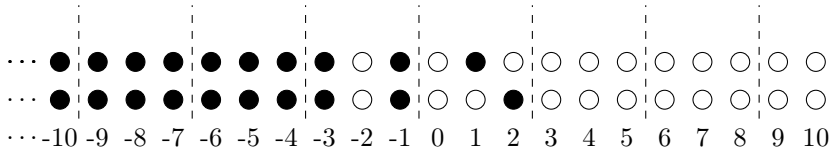
where for all $i = 1, \dots, m$, we have $\beta_{m-i} = \lambda_i - i + m$.

To each l-symbol $\mathbf{X} = (X_0, \dots, X_{l-1})$, we can associate an l-partition attached to this symbol together with a multicharge which is the multicharge of the symbol. Reciprocally, to each multipartition $\boldsymbol{\lambda}^l = (\lambda^0, \dots, \lambda^{l-1})$ and multicharge $\mathbf{s}^l = (s_0, \dots, s_{l-1})$, one can attach an l-symbol $\mathbf{X}^{\mathbf{s}^l}(\boldsymbol{\lambda}^l) = (X^{s_0}(\lambda^0), \dots, X^{s_{l-1}}(\lambda^{l-1}))$.

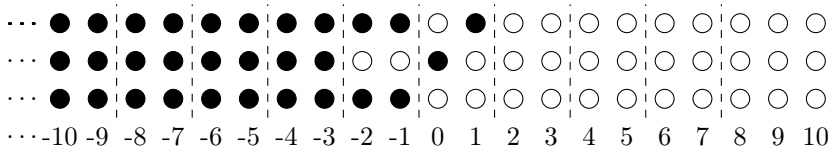
2.3. **Core and quotient.** Assume that $s \in \mathbb{Z}$. We fix $\mathbf{s}^l \in \mathbb{Z}^l[s]$, $e \in \mathbb{N}$ and let $n \in \mathbb{N}$. Let $\boldsymbol{\lambda}^l \in \Pi^l(n)$. We associate to this datum an e-partition $\boldsymbol{\lambda}_e$ together with a multicharge $\mathbf{s}_e \in \mathbb{Z}^e[s]$ as follows. To do that, start with the abacus associated to $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$:

- We define a rectangle on the l-abacus, containing e beads in each abacus. This rectangle starts with the beads numbered with 0 and finishes with the beads numbered with $e-1$. We get a rectangle with el beads, then again define a second rectangle with the beads numbered with e to the beads numbered by $2e-1$ an so on, even with the beads marked with negative integers.
- Rotate each rectangle 90 degree anticlockwise.
- We get a new e-abacus, which is the e-abacus of $(\boldsymbol{\lambda}_e, \mathbf{s}_e)$.

Example 3. Assume that $l = 2$, $\boldsymbol{\lambda}^l = ((3,1),(2,1))$ and $e = 3$ with $\mathbf{s}^l = (0,0) \in \mathbb{Z}^2[0]$. The l-abacus associated with $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$:



and then after rotation, the e-abacus associated to $(\boldsymbol{\lambda}_e, \mathbf{s}_e)$



So we have $\boldsymbol{\lambda}_e = (\emptyset, (2), (1))$ and $\mathbf{s}_e = (0, -1, 1)$.

It is clear that we have here a bijection

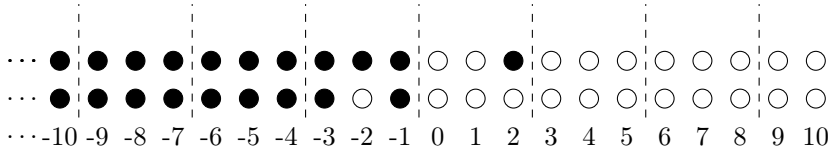
$$\begin{array}{rccc} \varphi: & \Pi^l \times \mathbb{Z}^l[s] & \to & \Pi^e \times \mathbb{Z}^e[s] \\ & (\boldsymbol{\lambda}^l, \mathbf{s}^l) & \mapsto & (\boldsymbol{\lambda}_e, \mathbf{s}_e) \end{array}$$

- The e-partition $\boldsymbol{\lambda}_e$ is the *e-quotient* of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$.
- The pair $(\nu^l, \mathbf{s}'^l) \in \Pi^l \times \mathbb{Z}^l[s]$ such that $\varphi(\nu^l, \mathbf{s}'^l) = (\emptyset, \mathbf{s}_e)$ is called the *e-core* of $(\boldsymbol{\lambda}, \mathbf{s}^l)$. It is uniquely determined by $\mathbf{s}_e$ which is called the *e-core multicharge* of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ (in the following, we will often says that the e-core $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is associated to $\mathbf{s}_e$).

We say that a pair $(\nu^l, \mathbf{s}'^l) \in \Pi^l \in \mathbb{Z}^l[s]$ is an *e-core* if $\varphi(\nu^l, \mathbf{s}'^l) = (\emptyset, \mathbf{s}_e)$ for an e-tuple $\mathbf{s}_e \in \mathbb{Z}^e[s]$. One can easily see if a pair $(\nu^l, \mathbf{s}'^l) \in \Pi^l \in \mathbb{Z}^l[s]$ is an e-core looking at its l-abacus. In this l-abacus, for each black bead in position i, there must be a black bead at the top of it (that is in the abacus juste above) in the same position and in the same abacus at the position $i - e$.

If $l = 1$, the e-quotient defined above agrees with the usual notion of quotient. The e-core of a pair (λ, s) is always of the form (ν, s). We denote $\lambda^{\circ_e} := \nu$. This is the usual e-core partition associated to λ.

Example 4. Keeping the above example the e-core of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ have the following l-abacus:



The e-core is $(((1),(2)),(-1,1))$ and the e-core multicharge is $(0,-1,1)$.

3. Comparing the size of the l-partition with the one of its e-quotient

The first main result of this note is the following Theorem:

Theorem 5. *Let $s \in \mathbb{Z}$, $\boldsymbol{\lambda}^l \in \Pi^l$, $\boldsymbol{\mu}^l \in \Pi^l$ and $\mathbf{s}^l \in \mathbb{Z}^l[s]$. Denote $(\boldsymbol{\lambda}_e, \mathbf{s}_e) := \varphi(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ and $(\boldsymbol{\mu}_e, \mathbf{s}_e) := \varphi(\boldsymbol{\mu}^l, \mathbf{s}^l)$. Assume that $|\boldsymbol{\mu}^l| = |\boldsymbol{\lambda}^l|$. Then we have $|\boldsymbol{\mu}_e| = |\boldsymbol{\lambda}_e|$.*

This theorem is in fact a direct consequence of the following lemma:

Lemma 6. *Let $s \in \mathbb{Z}$, $\boldsymbol{\lambda}^l \in \Pi^l$, $\boldsymbol{\mu}^l \in \Pi^l$ and $\mathbf{s}^l \in \mathbb{Z}^l[s]$. Denote $\boldsymbol{\lambda}^l = (\lambda^1, \dots, \lambda^l)$ and $\boldsymbol{\lambda}_e = (\nu^1, \dots, \nu^e)$. Then the integer $l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e|$ only depends on the multicharges $\mathbf{s}_e$ and $\mathbf{s}^l$ (where $\mathbf{s}_e$ is the e-core multicharge of $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$).*

Proof. Assume that $\boldsymbol{\lambda}^l \neq \boldsymbol{\mu}^l$ and take the two l-abacus associated with the same multicharge $\mathbf{s}^l$. Assume in addition that the e-core multicharge $\mathbf{s}_e$ is the same in both case. As $\boldsymbol{\lambda}^l \neq \boldsymbol{\mu}^l$, this means that there exists a runner i_1 and a position $i + k_1 e$ with $i \in \{0, \dots, e-1\}$ such that :

- There is a black bead in this position in the l-abacus of $\boldsymbol{\lambda}^l$,
- There is a white bead in this position in the l-abacus of $\boldsymbol{\mu}^l$.

As a consequence, as the l-multicharges are the same in both cases, there exists a position $j + k_2 e$ in runner i_1 with $j \in \{0, \dots, e-1\}$ such that :

- There is a black bead in this position in the l-abacus of $\boldsymbol{\mu}^l$,
- There is a white bead in this position in the l-abacus of $\boldsymbol{\lambda}^l$.

Now, as the e-multicharges are the same in both cases, there are two cases to consider : $i = j$, or $i \neq j$. In this later case there must exists $(k_3, k_4) \in \mathbb{Z}^2$ such that :

- There is a black bead in runner i_2 position $j + k_4 e$ in $\boldsymbol{\lambda}^l$ and white bead in the same position in $\boldsymbol{\mu}$.
- There is a black bead in runner i_2 position $i + k_3 e$ in $\boldsymbol{\mu}^l$ and white bead in $\boldsymbol{\lambda}^l$.

So we have the following picture:

- for the l-abacus associated to $\boldsymbol{\lambda}^l$ and $\mathbf{s}^l$:

i_2	$\cdots$	$\cdots$	○	●
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
i_1	●	○	$\cdots$	$\cdots$
	$i + k_1 e$	$j + k_2 e$	$i + k_3 e$	$j + k_4 e$

- for the l-abacus associated to $\boldsymbol{\mu}^l$ and $\mathbf{s}^l$:

i_2	$\cdots$	$\cdots$	●	○
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
i_1	○	●	$\cdots$	$\cdots$
	$i + k_1 e$	$j + k_2 e$	$i + k_3 e$	$j + k_4 e$

Now, taking the l-abacus of $\boldsymbol{\lambda}^l$, we exchange the beads :

(1) in runner i_1 position $i + k_1 e$ with the one in runner i_2, position $i + k_3 e$,
(2) in runner i_1 position $j + k_2 e$ with the one in runner i_2, position $j + k_2 e$.

We obtain the l-abacus of a new l-partition $\boldsymbol{\nu}^l$. Now, in a l-abacus, if we switch a black bead with a white one from a runner i to a runner $i+1$, the size of the associated l-partition is modified by the addition of $s_i - s_{i+1} - 1$. If we slide a black bead in position i to a position $i + e$ in the same runner, the size of the associated l-partition is increased by e. We have that:

$$\begin{aligned} |\boldsymbol{\nu}^l| &= |\boldsymbol{\lambda}^l| + (s_{i_1} - s_{i_2}) + (i_1 - i_2) + (s_{i_2} - s_{i_1}) + (i_2 - i_1) + (k_3 - k_1)e - (k_2 - k_4)e \\ &= |\boldsymbol{\lambda}^l| + (k_3 - k_1 - k_2 + k_4)e \end{aligned}$$

whereas,

$$\begin{aligned} |\boldsymbol{\nu}_e| &= |\boldsymbol{\lambda}^l| + (i_1 - i_2) + (i_2 - i_1) + (k_3 - k_1)l - (k_2 - k_4)l \\ &= |\boldsymbol{\lambda}_e| + (k_3 - k_1 - k_2 + k_4)l \end{aligned}$$

We have that:

$$l|\boldsymbol{\nu}^l| - e|\boldsymbol{\nu}_e| = l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e|$$

and we may conclude by induction on the number of black beads which are in different position in $\boldsymbol{\lambda}^l$ and $\boldsymbol{\nu}^l$. □

One easy consequence is the following proposition.

Proposition 1. *Let $\boldsymbol{\lambda}^l = (\lambda^0, \dots, \lambda^{l-1})$ be an l-partition and $\mathbf{s}^l \in \mathbb{Z}^l[s]$. Set $(\boldsymbol{\lambda}_e, \mathbf{s}_e) := \varphi(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ and denote $(\nu^0, \dots, \nu^{l-1}) = \boldsymbol{\lambda}_e$. Then we have:*

$$|\boldsymbol{\lambda}_e| = \sum_{0 \leq i \leq e-1} |(\nu^i)^{\circ_l}| + \frac{l}{e}\left(|\boldsymbol{\lambda}^l| - \sum_{0 \leq i \leq l-1} |(\lambda^i)^{\circ_e}|\right)$$

Proof. Set $\boldsymbol{\mu}^l := ((\lambda^0)^{\circ_e}, \dots, (\lambda^{l-1})^{\circ_e})$ where for each $i = 0, \dots, l-1$, $(\lambda^i)^{\circ_e}$ denotes the e-core of λ^i. Then it is easy to see that the e-core multicharge of $(\boldsymbol{\mu}^l, \mathbf{s}^l)$ is still $\mathbf{s}_e$. In addition, in the associated l-abacus, for each black bead in position x in a given runner, we have a black bead in the same runner in position $x - e$. As a consequence, by construction, in the e-abacus associated to $(\boldsymbol{\mu}_e, \mathbf{s}_e)$, for each black bead in position x in a given runner, we have a black bead in the same runner in position $x - l$. Thus we have that:

$$\boldsymbol{\mu}_e := ((\nu^1)^{\circ_l}, \dots, (\nu^e)^{\circ_l}).$$

Thus by our proposition we must have

$$l|\boldsymbol{\mu}^l| - e|\boldsymbol{\mu}_e| = l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e|$$

which is what we wanted. □

We give a more conceptual proof of the above proposition which is independant of the above Theorem. Write the l-abacus of $\boldsymbol{\lambda}^l$ as follows:

$\cdots$	X_{l-1}^{-2}	X_{l-1}^{-1}	X_{l-1}^{0}	X_{l-1}^{1}	X_{l-1}^{2}	X_{l-1}^{3}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	X_{1}^{-2}	X_{1}^{-1}	X_{1}^{0}	X_{1}^{1}	X_{1}^{2}	X_{1}^{3}
$\cdots$	X_{0}^{-2}	X_{0}^{-1}	X_{0}^{0}	X_{0}^{1}	X_{0}^{2}	X_{0}^{3}

Each X_j^k corresponds to a serie of e beads in the kth abacus numerated by $i + k.e$ with $i = 1, \dots, e$. The abacus of λ then writes:

$\cdots$	X_{l-1}^{0}	X_{l-2}^{0}	$\cdots$	X_{0}^{0}	X_{l-1}^{1}	$\cdots$

Let us now consider the el-abacus given as follows (the notation ^{t} denotes the transpose)

$\cdots$	${}^tX_{l-1}^{-2}$	${}^tX_{l-1}^{-1}$	${}^tX_{l-1}^{0}$	${}^tX_{l-1}^{1}$	${}^tX_{l-1}^{2}$	${}^tX_{l-1}^{3}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	${}^tX_{1}^{-2}$	${}^tX_{1}^{-1}$	${}^tX_{1}^{0}$	${}^tX_{1}^{1}$	${}^tX_{1}^{2}$	${}^tX_{1}^{3}$
$\cdots$	${}^tX_{0}^{-2}$	${}^tX_{0}^{-1}$	${}^tX_{0}^{0}$	${}^tX_{0}^{1}$	${}^tX_{0}^{2}$	${}^tX_{0}^{3}$

Then for each $i = 0, \dots, l-1$, the e-partition associated to the e-abacus:

$\cdots$	${}^tX_i - 1$	${}^tX_i^0$	${}^tX_i^1$	${}^tX_i^2$	$\cdots$	${}^tX_i^k$

is the e-quotient of λ^i. We denote it by $Q_i := (\lambda^i[0], \dots, \lambda^i[e-1])$. We have that

$$|\lambda^i| = |(\lambda^i)^{\circ_e}| + e|Q_i|$$

and thus, summing on all $0 \leq i \leq l-1$, we obtain:

$$|\boldsymbol{\lambda}^l| = \sum_{0 \leq i \leq l-1} |(\lambda^i)^{\circ_e}| + e \sum_{0 \leq i \leq l-1} |Q_i|$$

On the other hand, we know that $\boldsymbol{\lambda}^e$ is the e-quotient of λ with associated e-abacus:

$$\cdots \quad \vdots \quad {}^tX_l^0 \quad \vdots \quad {}^tX_{l-1}^0 \quad \vdots \quad \cdots \quad \vdots \quad {}^tX_1^0 \quad \vdots \quad {}^tX_l^1 \quad \vdots \quad \cdots \quad \vdots$$

Thus for each $0 \leq i \leq e-1$, the l-quotient of ν^i is $Q'_i := (\lambda^{l-1}[i], \ldots, \lambda^0[i])$. Thus we obtain

$$|\nu^i| = |(\nu^i)^{\circ_l}| + |Q'_i|.$$

Summing over all $0 \leq i \leq e-1$, we get:

$$|\boldsymbol{\lambda}_e| = \sum_{0 \leq i \leq e-1} |(\nu^i)^{\circ_l}| + l \sum_{0 \leq i \leq e-1} |Q'_i|.$$

Now we have that

$$\sum_{0 \leq i \leq l-1} |Q_i| = \sum_{0 \leq i \leq e-1} |Q'_i|$$

So we obtain

$$\frac{\boldsymbol{\lambda}_e - \sum_{0 \leq i \leq e-1} |(\nu^i)^{\circ_l}|}{l} = \frac{\boldsymbol{\lambda}^l - \sum_{0 \leq i \leq l-1} |(\lambda^i)^{\circ_e}|}{e}$$

as desired.

4. Explicit computation

Given $s \in \mathbb{Z}$, $\boldsymbol{\lambda}^l \in \Pi^l$ and $\mathbf{s}^l \in \mathbb{Z}^l[s]$, we now want to compute explicitly the number $l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e|$. To do this, we will first use the following result from [4, Bijection 2] which allows to compute the size of the e-core partition from the datum of its e-core multicharge when $s = 0$.

Theorem 1 (Garvan-Kim-Stanton)**.** *Assume that λ is an e-core with associated multicharge $\mathbf{s}_e = (s_0, \ldots, s_{e-1})$. Assume that $s = \sum_{0 \leq i \leq e-1} s_i = 0$. Then we have*

$$|\lambda| = \frac{e}{2} \sum_{0 \leq i \leq e-1} s_i^2 + \sum_{0 \leq i \leq e-1} i s_i.$$

We will begin by slightly generalize this result to an arbitrary multicharge with possible non zero sum.

Corollary 1. *Assume that λ is an e-core with associated multicharge $\mathbf{s}_e = (s_0, \ldots, s_{e-1})$. Assume that $s = \sum_{0 \leq i \leq e-1} s_i = s$. Then we have*

$$|\lambda| = \frac{e}{2} \sum_{0 \leq i \leq e-1} s_i^2 + \sum_{0 \leq i \leq e-1} i s_i - \frac{s}{2}(e-1) - \frac{s^2}{2}$$

Proof. Denote by:

$$N(\mathbf{s}_e) := \frac{e}{2} \sum_{0 \leq i \leq e-1} s_i^2 + \sum_{0 \leq i \leq e-1} i s_i.$$

Then, from the definition of the e-core multicharge, we have that the e-core partition associated to $\mathbf{s}_e$ or $\widetilde{\mathbf{s}}_e = (s_1, s_2, \ldots, s_{e-1}, s_0 - 1)$ is the same. Now we have that

$$N(\widetilde{\mathbf{s}}_e) = \frac{e}{2} \sum_{0 \leq i \leq e-1} s_i^2 - s_0 e + \frac{e}{2} + s_2 + 2s_3 + \ldots + (e-2)s_{e-1} + (e-1)(s_0 - 1).$$

We obtain

$$N(\widetilde{\mathbf{s}}_e) = N(\mathbf{s}_e) + 1 - \frac{e}{2} + s.$$

We can now perform this procedure s times and use the above Theorem 1:

$$N(\widetilde{\mathbf{s}}_e) = |\lambda| + s(1 - \frac{e}{2}) + \sum_{0 \leq i \leq s} i.$$

We obtain the desired result. □

Proposition 2. *Let $\boldsymbol{\lambda}^l \in \Pi^l$ and $\mathbf{s}^l = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l[s]$. Denote $(\boldsymbol{\lambda}_e, \mathbf{s}_e) = \varphi(\boldsymbol{\lambda}^l, \mathbf{s}^l)$. Then we have:*

$$l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e| = l(\sum_{0 \leq i \leq e-1} i t_i - \frac{1}{2} \sum_{0 \leq i \leq l-1} s_i^2) - e(\sum_{0 \leq i \leq l-1} i s_{l-1-i} - \frac{1}{2} \sum_{0 \leq i \leq e-1} t_i^2) + \frac{s}{2}(l-e)$$

Proof. By Proposition 1, we know one can assume that $\boldsymbol{\lambda}^l = (\lambda^0, \dots, \lambda^{l-1})$ and that each λ^k is an e-core. In this case, if we denote $\boldsymbol{\lambda}_e = (\mu^0, \dots, \mu^{e-1})$ then each μ^k is also an l-core. We can now use the last result. For each $0 \leq i \leq e-1$, one can associate to the e-core partition λ^i, its e-core multicharge $(t_0^i, \dots, t_{e-1}^i)$, which is of sum s_i and we thus have:

$$|\lambda^i| = \frac{e}{2} \sum_{0 \leq j \leq e-1} (t_j^i)^2 + \sum_{0 \leq j \leq e-1} i t_j^i - \frac{s_i}{2}(e-1) - \frac{s_i^2}{2}$$

and we can do the same for the l-cores μ^j: there exist l-core multicharges $(s_0^j, \dots, s_{l-1}^j)$ for $j = 0, \dots, e-1$ such that:

$$|\mu^i| = \frac{l}{2} \sum_{0 \leq j \leq l-1} (s_j^i)^2 + \sum_{0 \leq j \leq l-1} i s_j^i - \frac{t_i}{2}(l-1) - \frac{t_i^2}{2}$$

We have that

$$l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e| = l \sum_{0 \leq i \leq l-1} |\lambda^i| - e \sum_{0 \leq i \leq e-1} |\mu^i|$$

Now we have that for all $0 \leq j \leq l-1$ and $0 \leq i \leq e-1$, $t_i^j = s_{l-1-j}^i$, so when summing, the quadrating terms in the s_j^i and t_j^i vanish and we obtain:

$$l|\boldsymbol{\lambda}^l| - e|\boldsymbol{\lambda}_e| = l.\Big(\sum_{0 \leq j \leq e-1} j t_j - \frac{s}{2}(e-1) - \frac{1}{2} \sum_{0 \leq i \leq l-1} s_i^2\Big) - e\Big(\sum_{0 \leq j \leq l-1} j s_{l-1-j} - \frac{s}{2}(l-1) - \frac{1}{2} \sum_{0 \leq i \leq e-1} t_i^2\Big)$$

which gives the desired result. □

Last, one can deduce a generalization of Theorem 1 for the set of e-core multipartitions associated with a multicharge $\mathbf{s}_e$. First note that the l-multicharge $\mathbf{s}^l := (s_0, \dots, s_{l-1})$ of the e-core multipartition associated to $\mathbf{s}_e = (t_0, \dots, t_{e-1})$ may be easily deduced, we have for each $j = 0, \dots, l-1$:

$$s_i := \sharp\{0 \leq j \leq e-1 \mid t_j \equiv e - i\}$$

Now, if $(\boldsymbol{\lambda}^l, \mathbf{s}^l)$ is an e-core, then $\boldsymbol{\lambda}_e = \emptyset$ and we thus have:

$$|\boldsymbol{\lambda}^l| = \Big(\sum_{0 \leq i \leq e-1} i t_i - \frac{1}{2} \sum_{0 \leq i \leq l-1} s_i^2\Big) - (e/l)\Big(\sum_{0 \leq i \leq l-1} i s_{l-1-i} - \frac{1}{2} \sum_{0 \leq i \leq e-1} t_i^2\Big) + \frac{s}{2l}(l-e)$$

Assume that $\boldsymbol{\lambda}^l$ is an l-partition and that the associated e-core is $(\boldsymbol{\nu}, \mathbf{s}')$. Then we known that:

$$|\boldsymbol{\lambda}_e| = (1/e)(l|\boldsymbol{\lambda}^l| - l\Big(\sum_{0 \leq i \leq e-1} i t_i - \frac{1}{2} \sum_{0 \leq i \leq l-1} s_i^2\Big) + e\Big(\sum_{0 \leq i \leq l-1} i s_{l-1-i} - \frac{1}{2} \sum_{0 \leq i \leq e-1} t_i^2\Big) - \frac{s}{2}(l-e))$$

But we also have

$$|\boldsymbol{\nu}^l| = \Big(\sum_{0 \leq i \leq e-1} i t_i - \frac{1}{2} \sum_{0 \leq i \leq l-1} {s'_i}^2\Big) - (e/l)\Big(\sum_{0 \leq i \leq l-1} i s'_{l-1-i} - \frac{1}{2} \sum_{0 \leq i \leq e-1} t_i^2\Big) + \frac{s}{2l}(l-e)$$

so we have

$$|\boldsymbol{\nu}^l| + \frac{1}{2} \sum_{0 \leq i \leq l-1} {s'_i}^2 + \frac{e}{l} \sum_{0 \leq i \leq l-1} i s'_{l-1-i} = \sum_{0 \leq i \leq e-1} i t_i + \frac{e}{2l} \sum_{0 \leq i \leq e-1} t_i^2 + \frac{s}{2l}(l-e)$$

So we deduce:

Corollary 2. *Assume that $\boldsymbol{\lambda}^l \in \Pi^l$ and $\mathbf{s}^l = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l[s]$. Let $(\boldsymbol{\nu}, \mathbf{s}'_l)$ be the associated e-core. Then we have:*

$$|\boldsymbol{\lambda}_e| = \frac{1}{e}(l|\boldsymbol{\lambda}^l| - l|\boldsymbol{\nu}^l| + \frac{l}{2} \sum_{0 \leq i \leq l-1} (s_i^2 - {s'_i}^2) + e \sum_{0 \leq i \leq l-1} i(s_{l-1-i} - s'_{l-1-i}))$$

If $l = 1$, that is if λ is a partition with e-quotient $\boldsymbol{\lambda}_e$ and e-core λ° we obtain:

$$|\boldsymbol{\lambda}_e| = \frac{1}{e}(l|\lambda| - l\lambda^\circ|)$$

which is indeed a welle known formula.

In the general case, $|\boldsymbol{\lambda}_e|$ is nothing but the e-weight in the sense of [3] so the above formula permits to compute the e-weight of an l-partition with a multicharge from the datum of its e-core in the sense of [5].

Example 7. We come back with the datum of Example 3. We have that the second term of the above corollary is:

$$(1/3)(2 \times 7 - 2 \times 3 + \frac{2}{2}(0 - 1 + 0 - 1) + 3 \times (0 - (-1)) = 3$$

which is indeed the size of $\boldsymbol{\lambda}_e$.

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