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Nested Saturation Proportional Derivative Control for Conservative PDE-ODE Interconnections: the Gantry Crane Example

L. Ma^{a,c}, N. Vanspranghe^b, D. Astolfi^c, V. Andrieu^c, M. Bajodek^c, X. Lou^a

^a*Institute of System Engineering, Jiangnan University, Wuxi 214122, China.*

^b*Faculty of Information Technology and Communication Sciences, Tampere University, P.O. Box 692, 33101 Tampere, Finland.*

^c*Université Lyon 1, Villeurbanne, France – CNRS, UMR 5007, LAGEPP, France.*

Abstract

This paper addresses the feedback stabilization problem for a gantry crane system with input constraints. Such a system is described by a wave equation interconnected at the boundary conditions with a double integrator, which represents the top cart's position and its speed. We propose a simple nested-saturation proportional derivative feedback which ensures that the control inputs remain within certain given limits. Global asymptotic stability of the origin of the closed-loop system is established. To this end, a new weak Lyapunov functional and a new methodology to study pre-compactness of solutions are introduced. Numerical simulations are presented to illustrate the effectiveness of the proposed control method.

Keywords: Gantry crane, asymptotic stabilization, nested saturation, boundary control.

1. Introduction

The gantry crane system consists of three elements: a top cart, a flexible cable, and a bottom payload. This system is practical for transporting the bottom payload to the desired position and is widely used in various industrial settings. Due to the flexibility of the cable, this system is more accurately modeled by PDEs. For a complete analysis of the full dynamics of the gantry crane system, see, for example, [1]. The development of effective control strategies for this system has been extensively studied in many research papers over the years, see, e.g., [2–7] and various properties under different setting and conditions have been studied. However, none of these work studied the problem of feedback design in the presence of control input limitations. As a matter of fact, in practical gantry crane systems, actuators often operate within physical limits, meaning the control input signals are naturally constrained. These limitations, known as actuator saturation, are critical to account for in control design to prevent excessive forces or torques that could cause instability, equipment damage, or even accidents. By considering the saturation problem, we can ensure that control

inputs remain within safe boundaries, ultimately improving the reliability and safety of the system. The gantry crane control problem in the presence of input saturation remains, to the best of author's knowledge, an open problem. Among the articles which tried to achieve some progresses in such a direction, we acknowledge the following: the recent article [8], in which a PID-like controller is employed, where the cable-dynamics is modeled as an ODE, thus simplifying the problem from a mathematical point of view; the work [9, 10] where a nonlinear function is used to constrain the speed of the top cart but neglecting the dynamics of the top cart or of the payload; and [11], in which only the speed signal is used in the feedback, and as a result, the cart can only stop at a position determined by the initial conditions (i.e., is not controlled).

We highlight that in recent decades, the control problem of PDEs and abstract systems in the presence of input saturation and nonlinearities has done significant progress. See, for instance, the following (non exhaustive) list of works: [12–21]. The main strategies rely on leveraging the dissipativity property of the open-loop system [12–15], or forwarding-based approach [19–21]. However, we emphasize that none of those approaches directly apply to the particular context of gantry crane at hand. The main reason lies in the presence of a double integrator dynamics coupled with a conservative PDE. We remark that it is not even clear if a simple linear feedback can stabilize the overall dynamics due to the presence of multiple eigenvalues on the imaginary axes, as well known in the finite-dimensional literature, see, e.g., [22, 23].

In this paper, we suppose to measure the position and speed of the top cart and propose a simple boundary pro-

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Email addresses: lingma@stu.jiangnan.edu.cn (L. Ma), nicolas.vanspranghe@tuni.fi (N. Vanspranghe), danielle.astolfi@univ-lyon1.fr (D. Astolfi), vincent.andrieu@gmail.com (V. Andrieu), mathieu.bajodek@cpe.fr (M. Bajodek), Louxy@jiangnan.edu.cn (X. Lou)

portional derivative feedback within a nested saturation framework. With this boundary controller, the cart can be stopped at a prescribed position. Furthermore, we introduce a new weak Lyapunov functional and an innovative approach to study the pre-compactness of solutions. These elements allow to apply LaSalle's Invariance Principle to analyze the stability properties of the origin of the closed-loop dynamics.

The paper is structured as follows. In Section 2 we present the main new ideas of this article. To this end, the discussion is developed in the more general context of conservative PDE-ODE interconnected systems. The developments are formal but allow to easily convey the new key elements and insights characterizing this article. As a matter of fact, we believe that the proposed principles may be useful to solve stabilization problems for a class of systems of physical interest that goes beyond the problem of gantry crane systems. We refer, in particular, to impedance passive or Port-Hamiltonian systems (see, e.g., [24–26]). In Section 3, we study the gantry crane system and present the main results of this article: its model, the new feedback law, and the statement of the main theorems concerning well-posedness and stability properties. The proofs of the main results are outlined in Sections 4 and 5. Simulation results are presented in Section 6, followed by conclusions and perspectives in Section 7.

Notation: In this article, we set $\mathbb{R}_+ = [0, \infty)$ and $|\cdot|$ denotes the standard Euclidean norm in \mathbb{R}^n . For a real-valued function w defined over $\mathbb{R}_+ \times [0, 1]$, w_t (resp. w_x) denotes the partial derivative of w with respect to t (resp. with respect to x). Let $L^2(0, 1)$ be the Hilbert space of real-valued square-integrable (classes of) functions over the interval $(0, 1)$. Let $H^1(0, 1) \subset L^2(0, 1)$ be the Hilbert space of real-valued absolutely continuous functions over $[0, 1]$ with square-integrable derivative. The symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ indicate inner products and norms respectively. Finally, given $\gamma > 0$, we define the standard saturation function $\text{sat}_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\text{sat}_\gamma(s) = \begin{cases} s, & |s| \leq \gamma, \\ \gamma, & s \geq \gamma, \\ -\gamma, & s \leq -\gamma, \end{cases} \quad (1)$$

for any $s \in \mathbb{R}$. We say that $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a class- \mathcal{K}_∞ function if it is strictly increasing, $\alpha(0) = 0$ and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$.

2. General methodology

In this section, we formally present a general methodology for stabilizing systems that involve ODE-PDE in an abstract framework, highlighting the main novelties and ideas of this article. This approach provides essential insights for addressing the stabilization challenges specific to the gantry crane system. The general abstract form

of the conservative ODE-PDE interconnected systems we consider is given as follows:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \text{sat}_{\gamma_0}(u) + Cy, \\ \dot{y} &= Ay + Bz_2, \end{aligned} \quad (2)$$

where $z = (z_1, z_2) \in \mathbb{R}^2$ is the state of the ODE system, $u \in \mathbb{R}$ is the control input, $\gamma_0 > 0$ is the saturation level of the control, and $y \in \mathcal{H}$ is the state of a conservative PDE living in a Hilbert space \mathcal{H} with associated norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$ satisfying the following passivity condition: for some strictly positive operator P , the following conditions hold

$$\langle PAy, y \rangle = 0, \quad \langle Py, Bz_2 \rangle + z_2 Cy = 0,$$

for any $y \in \mathcal{H}$ and $z_2 \in \mathbb{R}$. Such a class of systems naturally appears in many mechanical systems, in which z_1 , resp. z_2 , describes the position (or an angle), resp. the speed (an angular speed) and y represents some vibrational mode, see, e.g., [24, 27, 28]. Typically, the energy is used as a Lyapunov function for y , namely P coincides with the identity element of the Hilbert space \mathcal{H} , and the y -dynamics with input operator B and output operator C satisfies an input-output passivity property. This has been well studied in the context of impedance passive systems (e.g., [25]) or Port-Hamiltonian systems (e.g., [26, Page 22]). As a matter of fact, one can verify the derivative of the function

$$2E(y, z_2) = \langle Py, y \rangle + z_2^2 \quad (3)$$

along solutions to (2) verifies

$$\dot{E} = z_2 \text{sat}_{\gamma_0}(u).$$

It is immediately seen that a feedback law $u = -kz_2$ with positive controller gain k guarantees the function E to converge to zero, but without any guarantee on the asymptotic equilibrium of the variable z_1 .

Our goal is to design a feedback law in order to stabilize the origin of the system (2). To this end, relying on the conservative properties of the y -dynamics the main idea is to use a feedback that depends only on z and ignore the effect of y . In particular, the proposed control law, inspired by the nested saturation approaches and forwarding approaches (see, e.g., [22, 23]), is selected as

$$u = -k_d z_2 - \text{sat}_{\gamma_1}(k_p z_1), \quad (4)$$

where the proportional and derivative gains k_d and k_p are any positive constants and γ_1 is a saturation level satisfying $0 < \gamma_1 < \gamma_0$. Formally, we can show that the origin of the system (2) with controller (4) is globally asymptotically stable under an observability assumption of the pair (A, C) . However, we highlight that the presence of the saturation function concerning the proportional action $\text{sat}_{\gamma_1}(k_p z_1)$ is crucial; without it, it is not clear if the origin of the closed-loop dynamics is asymptotically stable.

Furthermore, if the y -dynamics was an ODE, the result would directly follow by application of [23, Proposition 1]. In the more general PDE context, however, this approach cannot be applied because constructing an ISS-Lyapunov function for the subsystem (z_2, y) with feedback $u = -kz_2$ is challenging. For instance, in the case of a gantry-crane system, it is not even clear whether such a function exists.

As a consequence, in order to develop the stability analysis, we first modify the function E defined in (3) by constructing a new Lyapunov functional as follows:

$$\begin{aligned} V(z, y) &= E(y, z_2) + U(z_1), \\ U(z_1) &= \frac{1}{k_p} \int_0^{-k_p z_1} \text{sat}_{\gamma_1}(s) ds. \end{aligned} \quad (5)$$

Such a Lyapunov functional V (5) is inspired by a Lure-type Lyapunov function in which the integral of the non-linear function is summed to a quadratic function, see, e.g., [29, Page 403], and, to the best of our knowledge, is new also in the context of finite-dimensional systems (compare, for instance, to [22, 23]). With some computations (that will be better developed in Section 5), it can be shown that $\dot{V} \leq -\alpha(|z_2|)$ where α is a class- \mathcal{K}_∞ function. As a consequence, the Lyapunov functional V is not strict, namely its derivative is negative only in z_2 and not in the full state (y, z) . For this reason, in order to conclude the asymptotic stability of the origin, LaSalle's like arguments need to be employed.

We highlight, however, that in order to use such arguments, typically, pre-compactness of solutions is required. However, because of the presence of a double integrator and saturated control, the overall closed-loop dynamics don't satisfy typical contraction and monotonicity conditions, that is m -dissipative, see, e.g., [30]. As a consequence, we follow here a different route. In particular, we first observe that it is possible to prove pre-compactness of solutions by showing that not only the state (y, z) is bounded for any initial condition in some considered Hilbert space, but also its derivative, that is, (\dot{y}, \dot{z}) . Formally, the main idea we follow in this article is to look for a Lyapunov functional W that depends on $(y, z, \text{sat}_{\gamma_0}(u) + Cy, Ay + Bz_2)$, that is, a Lyapunov functional W satisfying

$$\underline{\alpha}(\|\dot{y}\| + |\dot{z}|) \leq W(z, y, \dot{z}_2, \dot{y}), \quad \dot{W} \leq 0$$

with $\underline{\alpha}$ being a class- \mathcal{K}_∞ function, allowing to conclude the desired result of pre-compactness. We highlight that formally, the energy function E in (3) is also a good candidate for (\dot{y}, \dot{z}_2) when $u = 0$. Indeed, some simple computations show that taking

$$E_d = \langle P(Ay + Bz_2), (Ay + Bz_2) \rangle + (Cy)^2$$

its derivative satisfies $\dot{E}_d = 0$. In the context of the gantry crane system, we will consider a Lyapunov functional of the form $W = V + \dot{z}_2^2 + \langle P\dot{y}, \dot{y} \rangle$. Note, however, that the

resulting Lyapunov functional is not continuously differentiable (i.e., it is not C^1), which prevents the direct application of standard Lyapunov stability arguments. Consequently, alternative techniques are required to demonstrate that the functional W is indeed decreasing along system trajectories.

In conclusion, the main contributions of this article can be summarized as follows:

1. the use of a simple Proportional Derivative controller with nested saturation for the class of conservative PDE-ODE systems (2);
2. the construction of a new (non-quadratic) weak Lyapunov functional (5) for the overall closed-loop dynamics;
3. the use of non smooth Lyapunov functional of the state and its derivative to study pre-compactness of solutions.

All these items will be developed for the particular problem of control of gantry crane. Nonetheless, we believe that the same principles and ideas can be used in many other contexts, as formally shown in this section.

3. The gantry crane system: main results

3.1. Problem statement

The gantry crane system is represented by a coupled PDE-ODE system, as detailed in [1, Page 12] and given by the following equation:

$$\begin{cases} \rho y_{tt}(x, t) = (S(x)y_x(x, t))_x, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ y_t(0, t) = v(t), \\ M\dot{v}(t) = \text{sat}_{\gamma_0}(u(t)) + S(0)y_x(0, t), \\ y_t(L, t) = w(t), \\ m\dot{w}(t) = -S(L)y_x(L, t), \end{cases} \quad (6)$$

with initial conditions

$$\begin{aligned} (y(\cdot, 0), y_t(\cdot, 0)) &= (y_0, y_{t0}) \in H^1(0, L) \times L^2(0, L), \\ (v(0), w(0)) &= (y_{t0}(0), y_{t0}(L)). \end{aligned}$$

In the above system, $y : (0, L) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is the transversal displacement of the cable, M is the mass of the top cart, ρ is the mass per unit length of the flexible cable with length L , and $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the control force applied to the top cart to guide its motion. The tension S that distributes through the cable is given by $S(x) = mg + \rho g(L - x)$, where m represents the mass of the bottom payload and g is the gravitational acceleration. Based on the physical parameters, we know that $S(x) \geq a_0 > 0$ for some constant a_0 . Finally, γ_0 is the level of the input saturation which represents the physical force limits which have to be satisfied.

The first key aspect to consider is the system's energy function E , which represents the total mechanical energy

of the gantry crane system, excluding the cart's position $y(0, t)$, and defined as

$$E(y, y_t, v, w) = \frac{1}{2} \int_0^L (S(x)y_x(\cdot, x)^2 + \rho y_t(\cdot, x)^2) dx + \frac{M}{2}v^2 + \frac{m}{2}w^2, \quad (7)$$

which is the sum of the elastic potential energy stored in the flexible cable, and kinetic energies. With some abuse of notation, for a fixed state trajectory we define the energy at time t as $E(t) := E(y(\cdot, t), y_t(\cdot, t), v(t), w(t))$. The same convention will be used in the sequel for other Lyapunov-type functions.

When the control input is zero, $u(t) = 0$, the energy function $E(t)$ remains constant, meaning that the total energy is conserved along the trajectories of the system (6). However, when the control input is nonzero, the rate of change of the energy $E(t)$ for solutions of system (6) is given by:

$$\dot{E}(t) = v(t) \text{sat}_{\gamma_0}(u(t)).$$

A simple feedback law $u(t) = -k_d v(t)$, where $k_d > 0$ is a feedback gain, could be used to stabilize the cart's velocity by dissipating energy. This control law would reduce the kinetic energy over time, driving $E(t) \rightarrow 0$. However, this approach alone does not guarantee stabilization of the cart's position $y(0, t)$ at a desired equilibrium point. The cart may stop, but its final position would depend on the initial conditions. This limitation is highlighted in [11], where a boundary controller that only uses the velocity feedback is unable to fully control the position of the cart.

To address this issue and ensure that the cart stabilizes at a prescribed position, a more sophisticated control strategy is required. In particular, we propose employing a nested saturation approach, inspired by the methods developed in finite-dimensional systems in [22, 23].

3.2. Feedback design

We propose a feedback law given by the equation:

$$u(t) = -k_d v(t) - \text{sat}_{\gamma_1}(k_p y(0, t)), \quad (8)$$

where k_d , k_p and γ_1 are positive values to be chosen.

Remark 1. For a small reference value y_r , the term $k_p y(0, t)$ can be replaced by $k_p(y(0, t) - y_r)$, allowing the cart to be guided to the desired position y_r . Without loss of generality, we consider $y_r = 0$. Additionally, because the controller is constrained by a saturation function, the reference y_r should relate to the first saturation level γ_0 .

We now define the Hilbert spaces \mathcal{H} as

$$\mathcal{H} = \{(y, y_t, v, w) \in H^1(0, L) \times L^2(0, L) \times \mathbb{R}^2\}.$$

Then, we also define the inner product in \mathcal{H} as follows:

$$\langle (y, y_t, v, w), (\tilde{y}, \tilde{y}_t, \tilde{v}, \tilde{w}) \rangle_{\mathcal{H}} = \int_0^L (y_x S \tilde{y}_x + \rho y_t \tilde{y}_t) dx + Mv\tilde{v} + mw\tilde{w} + k_p y(0)\tilde{y}(0), \quad (9)$$

that is equivalent to the standard inner product in \mathcal{H} .

With the controller (8), the closed-loop control system is given as follows:

$$\dot{\eta}(t) = \mathcal{A}\eta(t) + \mathcal{B}(\eta(t)), \quad \eta = (y, y_t, v, w), \quad (10)$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{H}$ are defined as

$$\mathcal{A}\eta = \begin{pmatrix} y_t \\ \frac{1}{\rho}(S y_x)_x \\ \frac{S(0)}{M}y_x(0) - \frac{1}{M}(k_d v + k_p y(0)) \\ -\frac{S(L)y_x(L)}{m} \end{pmatrix}, \quad \mathcal{B}(\eta) = \begin{pmatrix} 0 \\ 0 \\ b(y(0), v) \\ 0 \end{pmatrix},$$

with $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$b(y(0), v) = \frac{1}{M}(k_d v + k_p y(0)) + \frac{1}{M} \text{sat}_{\gamma_0}(-k_d v - \text{sat}_{\gamma_1}(k_p y(0))). \quad (11)$$

with the domain of the operator \mathcal{A} as

$$D(\mathcal{A}) = \left\{ (y, y_t, v, w) \in H^2(0, L) \times H^1(0, L) \times \mathbb{R}^2; \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} y_t(0) \\ y_t(L) \end{pmatrix} \right\},$$

and with initial condition $\eta(0) = \eta_0 \in \mathcal{H}$.

3.3. Statement of the main results

First, we show that the solution of the closed-loop system are well posed.

Theorem 1 (Well-posedness). Consider the closed-loop system (10) and let the controller gains in (8) satisfy $k_p > 0$, $k_d > 0$, $\gamma_0 > 0$ and $\gamma_1 > 0$. For any initial condition $\eta_0 \in \mathcal{H}$ (resp., $\eta_0 \in D(\mathcal{A})$), there exists a unique mild (resp., classical) solution $\eta \in C^0(\mathbb{R}_+; \mathcal{H})$ (resp., $\eta \in C^1(\mathbb{R}_+; \mathcal{H}) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$).

To establish the well-posedness of the closed-loop system (10), we first show that the operator \mathcal{A} is m -dissipative. The existence and uniqueness of solutions are then guaranteed by applying a perturbation theorem, which takes advantage of the fact that the operator \mathcal{B} is Lipschitz continuous. The proof is deferred to Section 4.

Next, we show that if the saturation levels γ_0 and γ_1 are correctly chosen, one obtains also asymptotic stability of the origin of system (10).

Theorem 2 (Lyapunov Stability and Attractivity).

Consider the closed-loop system (10) and let the controller gains in (8) satisfy $k_p > 0$ and $k_d > 0$ and the saturation levels satisfy $\gamma_0 > \gamma_1 > 0$. Then, there exists a class- \mathcal{K}_∞ function α_V such that for all $\eta_0 \in \mathcal{H}$,

$$\|\eta(t)\|_{\mathcal{H}} \leq \alpha_V(\|\eta_0\|_{\mathcal{H}}), \quad \forall t \geq 0. \quad (12)$$

Moreover, the origin of the closed-loop system (10) is globally attractive in \mathcal{H} for classical solutions. Specifically, for any initial condition $\eta_0 \in D(\mathcal{A})$, the solution satisfies

$$\lim_{t \rightarrow +\infty} \|\eta(t)\|_{\mathcal{H}} = 0.$$

To establish both Lyapunov stability and asymptotic stability of the closed-loop system (10), we first define a weak Lyapunov functional, which will allow us to prove Lyapunov stability. We then introduce a second Lyapunov functional to show the pre-compactness of the solution. Finally, by applying LaSalle's Invariance Principle and utilizing an observability property, we conclude the proof of global attractivity of the origin. Details of the proof are deferred to Section 5.

Remark 2. *The result of Theorem 2 guarantees convergence to the origin only for initial conditions in $D(\mathcal{A})$, which corresponds to classical solutions. In the absence of contractivity or (more generally) uniform Lipschitz continuity of the feedback semigroup, it remains unclear whether the same holds for weak solutions.*

4. Proof of Theorem 1 (well-posedness)

Before providing the elements of the proof of Theorem 1, we need a preliminary result establishing that the operator \mathcal{A} is m -dissipative.

Lemma 1. *The operator \mathcal{A} generates a C_0 -semigroup of contractions on \mathcal{H} .*

Proof. According to the Lumer-Phillips theorem [30], establishing that the operator \mathcal{A} is m -dissipative requires verifying two conditions: the dissipativity property and the range condition, i.e., $\text{Ran}(\mathcal{I}_{\mathcal{H}} - \mathcal{A}) = \mathcal{H}$.

For all η in $D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}\eta, \eta \rangle_{\mathcal{H}} &= \int_0^L ((y_t)_x S y_x + y_t (S y_x)_x) dx \\ &\quad + S(0)y_x(0)v - wS(L)y_x(L) - k_d v^2 \\ &\leq -k_d v^2 \leq 0, \end{aligned} \quad (13)$$

demonstrating that the operator \mathcal{A} is dissipative.

To verify the range condition, one needs to ensure that, pick any $\bar{\eta} \in \mathcal{H}$, there exists $\eta \in D(\mathcal{A})$ such that $\eta - \mathcal{A}\eta = \bar{\eta}$. To this end one can follow very similar arguments as in [11, Lemma 2.2]. \square

We are now ready to draw our proof of the first result regarding the well-posedness of the closed-loop system (10).

Proof of Theorem 1 According to Lemma 1, we have already established that the operator \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions on \mathcal{H} . Now, we need to demonstrate that $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{H}$ is globally Lipschitz.

First, note that with the notation $\bar{v} = k_d v$, $\bar{y} = k_p y(0)$, the function b in (11), can be expressed as

$$b(y(0), v) = \frac{1}{M} \left(\bar{v} + \bar{y} + \text{sat}_{\gamma_0}(-\bar{v} - \text{sat}_{\gamma_1}(\bar{y})) \right).$$

Recalling the Lipschitz condition for the saturation function $|\text{sat}_{\gamma}(x) - \text{sat}_{\gamma}(y)| \leq |x - y|$, for any $\gamma > 0$, and using triangular inequality, we obtain

$$|b(y(0), v) - b(\hat{y}(0), \hat{v})| \leq 3 \frac{k_d + k_p}{M} |(y(0), v) - (\hat{y}(0), \hat{v})|$$

for any pair of elements $(y(0), v)$ and $(\hat{y}(0), \hat{v})$. Hence, given any vector $\eta = (y, y_t, v, w)$ and $\hat{\eta} = (\hat{y}, \hat{y}_t, \hat{v}, \hat{w})$, we obtain

$$\begin{aligned} M \|\mathcal{B}(\eta) - \mathcal{B}(\hat{\eta})\|_{\mathcal{H}}^2 &= M |b(y(0), v) - b(\hat{y}(0), \hat{v})|^2 \\ &\leq \ell \|\eta - \hat{\eta}\|_{\mathcal{H}}^2, \end{aligned}$$

for some $\ell > 0$. Thus, according to [31, Theorem 11.1.5], equation (10) possesses a unique global mild solution in \mathcal{H} for every initial condition in \mathcal{H} , and for each η_0 in $D(\mathcal{A})$, it is a classical solution. This completes the proof of Theorem 1. \square

5. Proof of Theorem 2 (stability)

In this proof, the following steps are employed to show the deired result.

Step 1. Introduce a weak Lyapunov functional, as described in Section 5.1, ensuring it is a coercive functional. Establish the Lyapunov stability of the system (10).

Step 2. Based on this weak Lyapunov functional, introduce a new Lyapunov functional in Section 5.2 and prove that the solution of the system (10) is precompact.

Step 3. In Section 5.3, use LaSalle's Invariance Principle along with an observability argument to prove the asymptotic stability of the system (10).

5.1. Weak Lyapunov functional

In this section, we investigate the stability properties of the closed-loop system (10). Drawing inspiration from Lyapunov functions commonly used in Lur'e systems (e.g., [32]), by setting $E(\eta) = E(y, y_t, v, w)$ as defined in (7) we define the following Lyapunov functional:

$$\begin{aligned} V(\eta) &= E(\eta) + U(y(0)), \\ U(p) &= \frac{1}{k_p} \int_0^{-k_p p} \text{sat}_{\gamma_1}(s) ds. \end{aligned} \quad (14)$$

The function $U : \mathbb{R} \rightarrow \mathbb{R}_+$ can be expressed explicitly as:

$$U(p) = \begin{cases} \frac{\gamma_1}{k_p} (k_p p - \frac{\gamma_1}{2}), & \text{if } k_p p \geq \gamma_1, \\ \frac{k_p}{2} p^2, & \text{if } |k_p p| \leq \gamma_1, \\ \frac{\gamma_1}{k_p} (-\frac{\gamma_1}{2} - k_p p), & \text{if } k_p p \leq -\gamma_1. \end{cases} \quad (15)$$

Thus, U is a positive definite and proper function. Moreover, there exist class- \mathcal{K}_{∞} functions $\bar{\alpha}_U$ and $\underline{\alpha}_U$ such that:

$$\underline{\alpha}_U(|p|) \leq U(p) \leq \bar{\alpha}_U(|p|), \quad \forall p \in \mathbb{R},$$

by [29, Lemma 4.3]. Next, we define two class- \mathcal{K}_∞ functions, $\underline{\alpha}_V$ and $\bar{\alpha}_V$, for the Lyapunov function V as follows:

$$\underline{\alpha}_V(s) = \min_{(s_1, s_2) \in \mathbb{R}_+^2, s_1 + k_p s_2^2 = s} s_1 + \underline{\alpha}_U(s_2), \quad (16)$$

$$\bar{\alpha}_V(s) = \max_{(s_1, s_2) \in \mathbb{R}_+^2, s_1 + k_p s_2^2 = s} s_1 + \bar{\alpha}_U(s_2). \quad (17)$$

For all $\eta \in \mathcal{H}$ the following inequalities hold:

$$\begin{aligned} V(\eta) &= E(\eta) + U(y(0)) \geq E(\eta) + \underline{\alpha}_U(\|y(0)\|) \\ &\geq \underline{\alpha}_V(E(\eta) + k_p \|y(0)\|^2) = \underline{\alpha}_V(\|\eta\|_{\mathcal{H}}). \end{aligned}$$

Similarly, we have $V(\eta) \leq \bar{\alpha}_V(\|\eta\|_{\mathcal{H}})$, showing that

$$\underline{\alpha}_V(\|\eta\|_{\mathcal{H}}) \leq V(\eta) \leq \bar{\alpha}_V(\|\eta\|_{\mathcal{H}}).$$

To demonstrate Lyapunov stability, let $\eta_0 \in D(\mathcal{A})$. The energy function E is differentiable, and by using the control law (8), we obtain, for all $t \geq 0$:

$$\dot{E}(t) = v(t) \text{sat}_{\gamma_0}(-k_d v(t) + \text{sat}_{\gamma_1}(-k_p y(0, t))). \quad (18)$$

Similarly, the time derivative of U is given by:

$$\dot{U}(t) = -v(t) \text{sat}_{\gamma_1}(-k_p y(0, t)). \quad (19)$$

To analyze $\dot{V}(t)$, we consider two cases based on the value of the control input $u(t)$:

Case 1: $|u(t)| \geq \gamma_0$

In this case, since $\gamma_0 > \gamma_1$ and $k_d > 0$, we have $\text{sat}_{\gamma_0}(u(t)) = -\text{sign}(v(t))\gamma_0$ and $|\text{sat}_{\gamma_1}(-k_p y(0, t))| \leq \gamma_1$. Therefore, recalling (18) and (19), we obtain

$$\dot{V}(t) \leq |v(t)|(-\gamma_0 + \gamma_1). \quad (20)$$

Case 2: $|u(t)| < \gamma_0$

In this case, the control input operates in the linear region, implying that $u(t) = -k_d v(t) + \text{sat}_{\gamma_1}(-k_p y(0, t))$. Since $|\text{sat}_{\gamma_1}(-k_p y(0, t))| \leq \gamma_1$, we have:

$$\dot{V}(t) = -k_d v^2(t). \quad (21)$$

As a result, for any initial condition $\eta_0 \in D(\mathcal{A})$, the time derivative $\dot{V}(t)$ is nonpositive for all t . Hence, V is a non-increasing function of time. Consequently, for all $t \geq 0$, we have

$$\underline{\alpha}_V(\|\eta(t)\|_{\mathcal{H}}) \leq V(t) \leq V(0) \leq \bar{\alpha}_V(\|\eta_0\|_{\mathcal{H}}),$$

which further implies:

$$\|\eta(t)\|_{\mathcal{H}} \leq \underline{\alpha}_V^{-1}(\bar{\alpha}_V(\|\eta_0\|_{\mathcal{H}})).$$

This property is also valid for weak solutions. Thus, the origin of the system (10) is Lyapunov stable, completing the first part of Theorem 2.

5.2. Precompactness of solution

To establish the global asymptotic stability of the origin of (10), we need to employ Lasalle's Invariance Principle. This requires verifying that the set of solutions is precompact, as detailed in [10, 33]. Therefore, we first establish the following lemma.

Lemma 2. *The canonical embedding from $D(\mathcal{A})$ equipped with the graph norm, into \mathcal{H} is compact.*

Proof. Let's recall the definition of the graph norm associated to the operator \mathcal{A}

$$\begin{aligned} \|\eta\|_{\text{gr}}^2 &:= \|\eta\|_{\mathcal{H}}^2 + \|\mathcal{A}\eta\|_{\mathcal{H}}^2 \\ &= \int_0^L (S(x)y_x^2 + \rho y_t^2) dx + Mv^2 + mw^2 + k_p y(0)^2 \\ &\quad + \int_0^L \left(S(x)(y_t)_x^2 + \frac{1}{\rho} (S(x)y_x)_x^2 \right) dx \\ &\quad + M \left(\frac{S(0)}{M} y_x(0) - \frac{1}{M} (k_d v + k_p y(0)) \right)^2 \\ &\quad + \frac{1}{m} (S(L)y_x(L))^2 + k_p v^2. \end{aligned}$$

To prove the lemma, we need to establish the following two results: first, there exists a positive scalar \bar{C} such that $\|\eta\|_{D(\mathcal{A})}^2 \leq \bar{C} \|\eta\|_{\text{gr}}^2$, for all $\eta \in D(\mathcal{A})$; second, each bounded sequence in $D(\mathcal{A})$ is precompact in \mathcal{H} .

Observing the right-hand side of the above equation, we can see that

$$\|\eta\|_{D(\mathcal{A})}^2 \leq \bar{C} \|\eta\|_{\text{gr}}^2. \quad (22)$$

Now, considering a sequence $\{\eta_k\}_{k \in \mathbb{N}}$ in $D(\mathcal{A})$ bounded with the graph norm. Based on (22), we find that the sequence $\{\eta_k = (y_k, y_{t_k}, v_k, w_k)\}_{k \in \mathbb{N}}$ is bounded in the space $D(\mathcal{A})$ and therefore in $H^2(0, L) \times H^1(0, L) \times \mathbb{R}^2$. Furthermore, by [34, Theorem 9.16], we know that the canonical embedding from H^2 to H^1 (resp. H^1 to L^2) is compact. As a result, $\{\eta_k\}_{k \in \mathbb{N}}$ possesses a sublimit that converges in $\mathcal{H} = H^1(0, L) \times L^2(0, L) \times \mathbb{R}^2$, as required. \square

Given that the operator $\eta \in D(\mathcal{A}) \mapsto \mathcal{A}\eta + \mathcal{B}(\eta) \in \mathcal{H}$ is not m -dissipative operator, unlike the situation in [10], it is not straightforward to establish boundedness of the solution in the graph norm associated to this operator. This is however established in the following result.

Lemma 3. *For all initial condition η_0 in $D(\mathcal{A})$, the function $t \in \mathbb{R}_+ \mapsto \|\mathcal{A}\eta(t) + \mathcal{B}(\eta(t))\|_{\mathcal{H}}$ is bounded.*

Proof. To begin with, we define the new function W as follows:

$$W(t, s) = cV(t) + E \left(\frac{\eta(t+s) - \eta(t)}{s} \right), \quad (23)$$

for all $t \geq 0$, s is a positive parameter and E is defined in (7) and c is a positive real number. For each t , note that

$$\bar{W}(t) = \lim_{s \rightarrow 0^+} W(t, s) = cV(t) + E(\mathcal{A}\eta(t) + \mathcal{B}(\eta(t))), \quad (24)$$

(recall that classical solutions are C^1) and it can be checked similarly as before that there exists a class- \mathcal{K}_∞ function such that

$$\bar{W}(t) \geq \alpha_W(\|\mathcal{A}\eta(t) + \mathcal{B}(\eta(t))\|_{\mathcal{H}}), \quad \forall t \geq 0. \quad (25)$$

We take the time derivative along the trajectories of system (10) and get

$$\begin{aligned} \frac{\partial W}{\partial t}(t, s) &= c\dot{V}(t) \\ &+ \int_0^L S(x) \frac{y_x(x, t+s) - y_x(x, t)}{s} \frac{y_{xt}(x, t+s) - y_{xt}(x, t)}{s} dx \\ &+ \int_0^L \frac{y_t(x, t+s) - y_t(x, t)}{s} \left(S(x) \frac{y_{xt}(x, t+s) - y_{xt}(x, t)}{s} \right) dx \\ &+ S(0) \frac{y_x(0, t+s) - y_x(0, t)}{s} \frac{v(t+s) - v(t)}{s} \\ &+ \frac{v(t+s) - v(t)}{s} \frac{\text{sat}_{\gamma_0}(u(t+s)) - \text{sat}_{\gamma_0}(u(t))}{s} \\ &- S(L) \frac{w(t+s) - w(t)}{s} \frac{y_x(L, t+s) - y_x(L, t)}{s}. \end{aligned}$$

By using integration by parts for the above equality, we further obtain that the time derivative of this function is

$$\frac{\partial W}{\partial t}(t, s) = R(t, s), \quad (26)$$

where

$$\begin{aligned} R(t, s) &= c\dot{V}(t) + \left(\frac{v(t+s) - v(t)}{s} \right) \\ &\quad \times \left(\frac{\text{sat}_{\gamma_0}(u(t+s)) - \text{sat}_{\gamma_0}(u(t))}{s} \right). \end{aligned}$$

From (26), we know that $W(\cdot, s)$ is C^1 and (26) can be rewritten as

$$W(t, s) - W(0, s) = \int_0^t R(r, s) dr. \quad (27)$$

Let us first show that for each fixed r in $[0, t]$, R is upper bounded. Several cases may be distinguished.

If $(|u(r)| > \gamma_0)$: By continuity of the function u , there exists s^* such that for all s in $(0, s^*)$. Since $\text{sat}_{\gamma_0}(u(r)) = \text{sat}_{\gamma_0}(u(r+s)) = \gamma_0$, from (26), we have

$$R(r, s) = c\dot{V}(r) \leq 0, \quad \forall s \in (0, s^*).$$

If $(|u(r)| < \gamma_0)$: There exists s^* such that for all s in $(0, s^*)$, $u(r+s) < \gamma_0$ and $r+s^* < t$. In this case, both $u(r)$ and $u(r+s)$ lie in the linear region. There-

fore, R satisfies

$$\begin{aligned} R(r, s) &\leq c\dot{V}(r) - k_d \left(\frac{v(r+s) - v(r)}{s} \right)^2 \\ &\quad + \left| \frac{v(r+s) - v(r)}{s} \right| \times \\ &\quad \left| \frac{\min\{|k_p y(0, r+s)|, \gamma_1\} - \min\{|k_p y(0, r)|, \gamma_1\}}{s} \right| \\ &\leq -ck_d v^2(r) \\ &\quad + \frac{1}{4k_d} \left| \frac{\min\{|k_p y(0, r+s)|, \gamma_1\} - \min\{|k_p y(0, r)|, \gamma_1\}}{s} \right|^2 \\ &\leq -ck_d v^2(r) + \frac{k_p^2}{4k_d} \left(\frac{y(0, r) - y(0, r+s)}{s} \right)^2. \end{aligned}$$

However, we have from equation (12)

$$\left| \frac{y(0, r) - y(0, r+s)}{s} \right| \leq \max_{\tau \in [0, t]} |v(\tau)| < \infty.$$

This implies that

$$R(r, s) \leq \frac{k_p^2}{4k_d} (\max_{\tau \in [0, t]} |v(\tau)|)^2, \quad \forall s \in (0, s^*). \quad (28)$$

If $(|u(r)| = \gamma_0)$: Let s^* be such that $r+s^* < t$. For each s in $[0, s^*)$, depending on the value of $u(r+s)$, one of the former two case may be considered and it yields

$$R(r, s) \leq \frac{k_p^2}{4k_d} (\max_{\tau \in [0, t]} |v(\tau)|)^2, \quad \forall s \in (0, s^*). \quad (29)$$

Hence, for each r , $R(r, s)$ is upper bounded for s small enough, the bound being independent on r . With (reverse) Fatou's lemma, we have

$$\lim_{s \rightarrow 0^+} \int_0^t R(r, s) dr \leq \int_0^t \limsup_{s \rightarrow 0^+} R(r, s) dr.$$

Again, we can distinguish several cases,

Case 1 $(|u(r)| > \gamma_0)$:

From the former computation,

$$\limsup_{s \rightarrow 0^+} R(r, s) = c\dot{V}(r) \leq 0, \quad \forall s \in [0, s^*).$$

Case 2 $(|u(r)| < \gamma_0)$: We obtain the bound

$$\limsup_{s \rightarrow 0} R(r, s) = \left(-ck_d + \frac{k_p^2}{4k_d} \right) v^2(r).$$

Hence, picking c sufficiently large such that $-ck_d + \frac{k_p^2}{4k_d} < 0$, it yields

$$\limsup_{s \rightarrow 0} R(r, s) \leq 0.$$

Case 3 ($|u(r)| = \gamma_0$): This simply gives the max of the former two possibilities which is negative.

And consequently, for each t , we showed that

$$\lim_{s \rightarrow 0^+} \int_0^t R(r, s) dr \leq 0.$$

This yields with (24) and (27)

$$\bar{W}(t) \leq \bar{W}(0).$$

and consequently, from (25), we get the result. \square

Lemma 4. *Given η_0 in $D(\mathcal{A})$, the trajectory $\{\eta(t), t \geq 0\}$ is a precompact subset of \mathcal{H} .*

Proof. First of all, given η_0 in $D(\mathcal{A})$ the following inequality holds:

$$\|\mathcal{A}\eta(t)\|_{\mathcal{H}} \leq \|\mathcal{A}\eta(t) + \mathcal{B}(\eta(t))\|_{\mathcal{H}} + \|\mathcal{B}(\eta(t))\|_{\mathcal{H}}, \forall t \geq 0.$$

By Lemma 3, $t \in \mathbb{R}_+ \mapsto \|\mathcal{A}\eta(t) + \mathcal{B}(\eta(t))\|_{\mathcal{H}}$ is bounded. Moreover, \mathcal{B} is Lipschitz in \mathcal{H} and by Equation (12), $t \in \mathbb{R}_+ \mapsto \|\eta(t)\|_{\mathcal{H}}$ is bounded, we get that $t \in \mathbb{R}_+ \mapsto \|\mathcal{A}\eta(t)\|_{\mathcal{H}}$ is bounded. According to Lemma 2 it yields that the trajectory $\{\eta(t), t \geq 0\}$ is precompact in \mathcal{H} . \square

5.3. Asymptotic stability

In this part of the proof we show the asymptotic convergence to the equilibrium as stated in the Theorem. According to [35, Théorème 1.1.8] or [36, Proposition 2.1], the ω -limit set $\omega(\eta_0)$ is a nonempty invariant subset of \mathcal{H} that attracts (in the norm of \mathcal{H}) the state trajectory η originating from η_0 . Furthermore, as seen in the proof of Lemma 4, the $D(\mathcal{A})$ -norm of η remains bounded, which implies with a closedness argument that $\omega(\eta_0) \subset D(\mathcal{A})$. Now, following LaSalle's Invariance Principle, proving asymptotic convergence to the origin amounts to showing that $\omega(\eta_0) = \{0\}$. To do so, we consider a classical solution (with an abuse of notation, denoted by the same symbols) such that $\dot{V}(t) = 0$ for all $t \geq 0$. This implies $v(t) = 0$ for all $t \geq 0$ and yields the following conditions:

$$\begin{aligned} \rho y_{tt}(x, t) - (S(x)y_x(x, t))_x &= 0, \\ y_t(0, t) &= 0, \quad S(0)y_x(0, t) + \text{sat}_{\gamma_1}(-k_p y(0, t)) = 0, \\ y_t(L, t) &= w(t), \quad m\dot{w}(t) + S(L)y_x(L, t) = 0. \end{aligned} \quad (30)$$

We now show that the only possible solution to (30) is $y(x, t) = 0$.

From $y_t(0, t) = 0$ of (30), we know that $y(0, t)$ is a constant. We integrate the first line of (30) with respect to x from 0 to L and t from 0 to τ :

$$\rho \int_0^\tau \int_0^L y_{tt}(x, t) dx d\tau = \int_0^\tau \int_0^L (S(x)y_x(x, t))_x dx d\tau. \quad (31)$$

Furthermore, using integration by parts, the left-hand side of (31) becomes

$$\rho \int_0^\tau \int_0^L y_{tt}(x, t) dx d\tau = \rho \int_0^L (y_t(x, \tau) - y_t(x, 0)) dx. \quad (32)$$

By substituting the two boundary conditions in (30), the right-hand side of (31) becomes

$$\begin{aligned} & \int_0^\tau [S(x)y_x(x, t)]_0^L dt \\ &= \int_0^\tau (S(L)y_x(L, t) - S(0)y_x(0, t)) dt \\ &= \int_0^\tau (-my_{tt}(L, t) + \text{sat}_{\gamma_1}(-k_p y(0, t))) dt \\ &= [-my_t(L, t)]_0^\tau + \int_0^\tau \text{sat}_{\gamma_1}(-k_p y(0, t)) dt \\ &= -mw(\tau) + mw(0) + \tau \times \text{constant}, \end{aligned} \quad (33)$$

where in the last step, we use the fact that $y(0, t)$ does not depend on time t . Combining (32) and (33), we get

$$\tau \times \text{constant} = \rho \int_0^L (y_t(x, \tau) - y_t(x, 0)) dx + mw(\tau) - mw(0). \quad (34)$$

By applying Young's inequality, one gets that

$$\begin{aligned} & \rho \int_0^L y_t(x, \tau) dx + mw(\tau) \\ & \leq \left| \rho \int_0^L y_t(x, \tau) dx + mw(\tau) \right| \\ & \leq \rho \int_0^L |y_t(x, \tau)| dx + |mw(\tau)| \\ & \leq \frac{\rho L}{2} + \frac{\rho}{2} \int_0^L |y_t(x, \tau)|^2 dx + \frac{m}{2} + \frac{m}{2} w(\tau)^2 \\ & \leq \frac{1}{2}(m + \rho L) + V(\tau). \end{aligned}$$

Similarly, we can get

$$\begin{aligned} -\rho \int_0^L y_t(x, 0) dx - mw(0) & \leq \left| -\rho \int_0^L y_t(x, 0) dx - mw(0) \right| \\ & \leq \frac{1}{2}(m + \rho L) + V(0). \end{aligned}$$

Then, it gives that, for all $\tau \in \mathbb{R}_+$,

$$\tau \times \text{constant} \leq (m + \rho L) + 2V(0), \quad (35)$$

which means that this constant should be 0 and yields $y(0, t) = 0$. The end of the proof follows [37, Lemma 2.3] or [38, Lemma 3.2]. Finally, we can get the only trivial solution to system (30). Hence, by LaSalle Invariance Principle, the conclusion of the Theorem holds.

6. Simulations

In this section, we demonstrate the effectiveness of the proposed boundary controller (8) for the system described by (6), subjected to control constraints. The parameters of the gantry crane system are specified as follows: $M = 2.1$ kg, $m = 10$ kg, $\rho = 0.2$ kg/m, $g = 9.8$ m/s², and $L = 1$ m. The two saturation levels are $\gamma_0 = 1$, and $\gamma_1 = 0.5$, and the controller gains are $k_p = 0.5$ and $k_d = 3$. In the numerical simulation part, we primarily utilize the finite difference method to approximate the solution to the closed loop system (10). We discretize both the space and time domains using grid spaces of $\Delta x = 0.01$ and $\Delta t = 0.0001$, respectively. The initial conditions are given as $y(x, 0) = 1 + 0.1 \sin(\frac{\pi}{2}x)$ and $y_t(x, 0) = 0$.

Figure 1(a) illustrates the evolution of the displacement $y(x, t)$ over time. The cart's position $y(0, t)$ stabilizes at zero around $t = 30$, as shown in 1(b). The saturation controller response, which saturates at $\gamma_1 = 0.2$, is depicted in Figure 1(c), while the L^2 -norm $\|y(\cdot, t)\|_{L^2}$ is presented in Figure 1(d). These simulation results demonstrate the efficacy of the proposed control method for the gantry crane system.

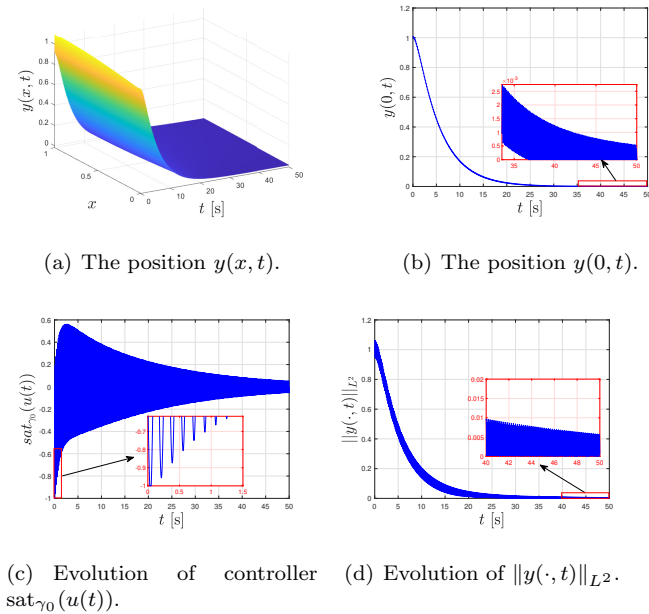


Figure 1: The transient dynamics of the gantry crane system with controller (8).

7. Conclusions

This paper addresses the stabilization problem of a gantry crane system modeled by a PDE-ODE, constrained by input saturation. The controller is a simple proportional derivative controller with a nested saturation, using the displacement and speed of the cart. The well-posedness of the problem is resolved using Lipschitz per-

turbations, and the stability of the origin is demonstrated through Lyapunov theory and LaSalle-like arguments.

For future work, it would be worthwhile to explore the addition of an integral action so that to obtain a PID-like controller to improve system's performance and robustness against disturbances and model uncertainties.

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