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SOME INSIGHTS INTO PARTIAL POLE PLACEMENT METHOD IN OBSERVERS DESIGN FOR TIME-DELAY SYSTEMS

AHLEM SASSI¹, ISLAM BOUSSAADA², AND SILVIU-IULIAN NICULESCU³

ABSTRACT. This contribution focuses on the use of the pole placement approach in the design of observers for certain classes of *Linear-Time-Invariant* (LTI) systems with time delay. Specifically, depending on how the delay appears in the system dynamics, two classes of dynamic systems are considered: state-delayed systems and input-delayed systems, respectively. First, we address the problem of designing a full-order Luenberger observer for the considered systems using partial placement of the error poles. Namely, we exploit the *multiplicity-induced-dominancy* (MID) property of the characteristic root with the maximal admissible multiplicity of the characteristic function corresponding to the system's error. After giving the existing condition for the proposed observer, we use the same MID property but in the so-called *generic* case. The performance and effectiveness of the proposed observers are highlighted through several illustrative examples.

Keywords. Dynamical systems, Delay systems, Observer design, Exponential stability.

MSC 2020. 39B05, 93D23, 93C23, 93B53, 93B55.

1. INTRODUCTION

Transport, propagation and communication are the main ways in which time-delay appear in natural phenomena and engineering processes. For instance, delays are encountered in the modeling of heat exchanges, distillation units, mining processes, steel manufacturing, but also in describing heredity in population dynamics or commodity markets in economics. For further examples and discussions on related applications, we refer to [23, 37, 39, 49], and the references therein. Such dynamical systems belong to the class of infinite-dimensional systems and a distinctive feature of such systems is that their rate of evolution can be described by differential equations including information about the *past history* of the system. There are several ways to represent such dynamical behaviours and one of the simplest way is by using differential equations on functional spaces. For an introduction to the theory of functional differential equations, we refer to [7, 26].

As discussed in [48], the presence of time-delay in a system model may alter its performances and may even be a source of instability. However, there are cases,

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less intuitive, when “large” delays can help stabilize systems while “small” delays cannot, as highlighted by [22, 23, 43, 48], and the references therein.

In most of the references cited above, it is assumed that the system “state” (“piece-of-trajectory”) is fully known and accessible. However, in most of practical applications, we have only partial information and the state is not fully accessible, and thus, one needs to estimate the “state”¹ or at least the “state-vector” at a given (well-defined) time (moment). It is interesting to mention that the second angle (i.e. “state-vector”) opens the perspective to extend ideas from finite-dimensional systems to our class of delay systems by an appropriate interpretation of the system trajectory over the time-delay interval.

Roughly speaking, in finite-dimension, the main idea of the state estimation is to provide an estimate of the internal “state” of a given system based on the system model and the measured output signal. So far, the state estimation problem has found successful applications in a broad range of fields, such as aircraft tracking, smart grids, and bio-processes monitoring. Among a variety of available state estimation approaches, the most classic one is the Luenberger-type observer which has shown suitable results in state estimation from early works of [29, 30, 31] to the more recent ones [25, 27, 52, 54] extended to many type of dynamical systems. It is important to emphasize that the Luenberger-type observer is particularly interesting due to its easy implementation because it depends on only one (single) tuning parameter, which requires a specific condition, to ensure exponential convergence. Despite this simplicity, this observer is far from being a perfect solution to state estimation. Since the number of state variables in a *reduced-order observer* is less than the order of the considered system, the reduced-order observer is parsimonious, often a desirable engineering quality. But, in addition, a reduced-order observer may have better properties than a full-order observer, especially with respect to robustness of a control system which uses an observer to implement the control algorithm in an “observer-based” control design. Reduced-order observers are well recognized for having faster convergence rates and lower computational burden, as the only state variables to be estimated are the ones which are actually not measured (see for instance, [14, 18, 19, 24, 50, 51]).

As emphasized above, the idea of observer design is to reconstruct the trajectory of states of a dynamical system. More precisely, an observer can be interpreted as an auxiliary system such that the error between the observer state and the system state decreases asymptotically towards zero. In this framework, stability analysis methods are necessary in the design of observers.

In the delay case, there exists a lot of extensions that exploits the estimation of the “state-vector” at a given (well-defined) time (moment). If in such cases, the construction follows closely ideas from finite-dimensional cases, however, the stability analysis methods are based on the stability of the trivial solution of appropriate functional differential equations that appear when defining the overall system dynamics. Without being exhaustive, a lot of works in the design of observers mostly use the Lyapunov approaches (Krasovskii method of Lyapunov functionals and the Razumikhin method of Lyapunov functions for time delay systems) in the stability analysis of the error system. Those methods usually lead to finding solutions of some appropriate Linear Matrix Inequalities (LMIs) (see [1, 16, 18, 20, 24, 40, 52] and references there-in). Excepting Lyapunov approaches, among the existing

¹Delay systems are infinite-dimensional systems defined on some appropriate functional spaces

methods, eigenvalue-based methods have become popular in the stabilization of linear time-invariant (LTI) time-delay systems [37] and recently in observers design [21, 44, 45]. A recent pole placement analytical paradigm, called *Partial Poles Placement* (PPP), has been introduced in [13, 11, 34, 36, 10]. It derives from two properties called respectively *multiplicity-induced-dominancy* (MID) and *coexistent-real-roots-inducing-dominancy* (CRRID), see for instance [2, 6, 8, 46]. It ensues from an observation on the effect of multiple spectral values on the stability of *Delay Differential Equations* (DDE). Indeed, some works have shown that, for some classes of time-delay systems, a real root of maximal multiplicity is necessarily the rightmost root, a property we call generic multiplicity-induced-dominancy, or GMID for short. This link between maximal multiplicity and dominance has been suggested in [41] after the study of some simple, low-order cases, but without any attempt to address the general case. The GMID idea may be particularly adapted for tuning low-complexity controllers, i.e., controllers including a small number of parameters (including also the delay among the parameters) with a guarantee on the location of the remaining characteristic roots for the closed-loop system, see also [38, 12, 15, 4] for further insights on the use of the partial-poles-placement in concrete applications.

The remaining of this chapter is organised as follows. Section 2 presents some prerequisites in complex analysis : Starting by the class of system under interest and its formulation in the frequency domain. We recall some properties of quasipolynomials and state some technical results needed in the study of the stability of the error systems. In Section 3, the problem is formulated and the classes of systems under consideration are presented. The Luenberger-type observer is then given for each class and its convergence is analyzed using the MID property, following by simulation results which highlight the performances of the proposed observers. Another observer design type is proposed in section 4 for the same classes of systems considered before. The convergence analysis is detailed using the GMID property and two simulation examples are given to verify the efficiency of the proposed method. Some concluding remarks (Section 5) ends the chapter.

2. PRELIMINARIES AND PREREQUISITES

Consider a generic dynamical system with a single delay described by the DDE:

$$(2.1) \quad \dot{X}(t) = AX(t) + A_d X(t - \tau),$$

under appropriate initial conditions, where $X \in \mathbb{R}^n$ is the state vector, τ is a positive constant delay. The matrices $A \in \mathbb{R}^{n \times n}$ and $A_d \in \mathbb{R}^{n \times n}$ are known constant matrices.

It is well known that the asymptotic behavior of the solutions of (2.1) is determined from its spectrum (see, e.g. [7]), that is the set of the (characteristic) roots of the associated characteristic function (denoted $\Delta(s, \tau)$ in the sequel).

The characteristic function $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ corresponding to (2.1) writes as follows:

$$(2.2) \quad \Delta(s, \tau) = \det(sI_n - A - A_d e^{-\tau s})$$

A generic result on the location of spectral values corresponding to (2.2) is given by the following:

Proposition 1. [37] If s is a characteristic root of system (2.1), then it satisfies

$$(2.3) \quad |s| \leq \|A + A_d e^{-\tau s}\|_2$$

This result combined with the triangular inequality provides a generic envelope curve around the characteristic roots corresponding to system (2.1).

In the sequel, we are interested in the study of a class of LTI delay systems characterized by the quasipolynomial function of the form:

$$(2.4) \quad \Delta(s, \tau) = P_0(s) + P_1(s)e^{-\tau s}$$

where $\deg(P_0) > \deg(P_1)$. More precisely, the problem of the analytical characterization of its rightmost root will be an essential ingredient in deriving our results.

2.1. Partial pole placement in delay systems. One of the most natural and classical ways to stabilize a dynamical LTI delay system is to select the free parameters of the controller in order to choose the location of finitely many roots while also guaranteeing that the *dominant root*² is among the chosen ones. This has been the subject of several recent works, such as [6, 11, 34]. Contrary to the strategy of FPP used, e.g., in [33], the controllers designed using these techniques do not render the closed-loop system finite-dimensional, but instead control its rightmost spectral value. These methods also extend to some partial differential equations, as detailed, for example, in [35]. In particular, in the sequel, we shall use the MID property as described in [5]. For linear dynamical systems including delays in their model representation, spectral methods can be used to understand the asymptotic behavior of solutions by considering the roots of some characteristic function (see, e.g., [26, 37, 7, 17, 49, 53]) which, for (2.1), is the function $\Delta : \mathbb{C} \rightarrow \mathbb{C}$ defined for $s \in \mathbb{C}$ by

$$(2.5) \quad \Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^m \alpha_k s^k.$$

More precisely, the exponential behavior of solutions of (2.1) is given by the real number $\gamma_0 = \sup\{\Re(s)/s \in \mathbb{C}, \Delta(s) = 0\}$, called the *spectral abscissa* of Δ , in the sense that, for every $\varepsilon > 0$, there exists $C > 0$ such that, for every solution e of (2.1), one has $|e(t)| \leq C e^{(\gamma_0 + \varepsilon)t} \max_{\theta \in [-\tau, 0]} |e(\theta)|$ [26]. Moreover, all solutions of (2.1) converge exponentially to 0 if and only if $\gamma_0 < 0$. An important difficulty in the analysis of the asymptotic behavior of (2.1) is that, contrary to the delay-free case, the corresponding characteristic function Δ has infinitely many roots.

Theorem 2.1 ([9]). *Consider the quasipolynomial Δ given by (2.5). Let $s_0 \in \mathbb{R}$ be a root of Δ with maximal multiplicity i.e. $M(s_0) = \deg(\Delta)$, then,*

- (1) (Retarded) *If $m < n$, then s_0 is a strictly dominant root of Δ .*
- (2) (Neutral) *If $m = n$, then s_0 is a dominant root of Δ and, for every other complex root s of Δ , one has $\Re(s) = s_0$.*

Notice that the GMID consists in forcing a root to reach its maximal multiplicity, which does not allow any degree of freedom in assigning s_0 . In order to allow for some additional freedom when assigning s_0 , one can relax such a constraint by forcing the root s_0 to have a multiplicity lower than the maximal.

²the rightmost characteristic root in the complex plane

3. MID PROPERTY IN THE DESIGN OF FULL-ORDER OBSERVER DESIGN

3.1. Problem Formulation.

3.1.1. *System with delayed states.* Consider the system with delayed state described by the following state space model.

$$(3.1a) \quad \dot{x}(t) = Ax(t) + A_d x(t - \tau) + Bu(t)$$

$$(3.1b) \quad y(t) = Cx(t)$$

$$(3.1c) \quad x(t) = \phi(t) \quad t \in [0, \tau]$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are, respectively, the state vector, the control input and the measurements vector. The non-negative scalar τ is a known constant delay. Finally, $\phi(t)$ is a continuous function defined as the initial condition. Matrices $A \in \mathbb{R}^{n \times n}$, $A_d \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are known constant matrices.

For system (2.1), we propose to design a Luenberger-type observer, allowing the estimation of all the system's state vector $x(t)$, described by:

$$(3.2) \quad \begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + A_d\hat{x}(t - \tau) + Bu(t) + L(y(t) - C\hat{x}(t)) \\ &\quad + T(y(t - \tau) - C\hat{x}(t - \tau)) \end{aligned}$$

where $\hat{x} \in \mathbb{R}^n$ denotes the estimates of x . Matrices L and T are unknown and shall be determined so that the estimation of the states \hat{x} converges to its real values x .

3.1.2. *System with delayed inputs.* Now, we consider a system with a known time delay in the input as follows

$$(3.3a) \quad \dot{x}(t) = Ax(t) + Bu(t - \tau),$$

$$(3.3b) \quad y(t) = Cx(t),$$

and for which, we propose the following observer dynamics.

$$(3.4) \quad \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t - \tau) + L(y(t) - C\hat{x}(t)) + T(y(t - \tau) - C\hat{x}(t - \tau))$$

3.1.3. *The error equation.* Let us denote by $e(t) = x(t) - \hat{x}(t)$ the estimation error vector. In both cases, the dynamic of the estimation error is described by the following system.

$$(3.5) \quad \dot{e}(t) = (A - LC)e(t) + (A_d - TC)e(t - \tau).$$

Notice that such an error equation applies in both cases, input-delay or state-delay. In the sequel, we shall use the notations $\bar{A} = A - LC$ and $\bar{A}_d = A_d - TC$.

Proposition 2. The dynamical system (3.2) (system (3.4)) represents an observer for the time-delay system described by (3.1) (system (3.3)) if, and only if, the dynamic error system given by (3.5) is asymptotically stable.

Proof 1. Computing the dynamics of the estimation error $\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$, one obtains the system given by (3.5) (with $A_d = 0$ for the second case). This system needs to be stable, to ensure the convergence of the estimation error $e(t)$ to zero. \square

3.2. Main results. Let us consider system (3.5) written in the following form.

$$(3.6) \quad \dot{e}(t) - \bar{A}e(t) - \bar{A}_d e(t - \tau) = 0.$$

Its characteristic function is $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ is given by

$$(3.7) \quad \Delta(s, \tau) = |sI_n - \bar{A} - \bar{A}_d e^{-\tau s}|$$

Hence, the error system is characterised by the quasipolynomial function with the form of equation (2.4) where the polynomial P_0 is given by:

$$P_0(s) = |sI_n - \bar{A}| = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

The following assumptions are considered:

Assumption 1. $\text{rank}(\bar{A}_d) = 1$.

Assumption 2. The polynomial P_0 is real-rooted.

Remark 3.1. Under the above assumptions we are able to investigate the stability of the estimation error. Notice that Assumption 1 is a sufficient condition to guarantee that the quasipolynomial Δ involves a single delay. The assumption 2 is requested since we shall apply Theorem 1 in [5].

Let us note s_0 a real root with multiplicity at least n of the characteristic function (2.4). It is known that if $s_0 < 0$ is the corresponding rightmost root then the zero solution of system (3.6) is asymptotically stable and consequently the estimation error is stable. The following proposition from [5] gives explicitly the integral representation of the quasipolynomial.

Proposition 1. If the quasipolynomial (2.4) has a real root s_0 with multiplicity at least n then it can be written as

$$\Delta(s) = (s - s_0)^n \left(a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0, \tau t)}{(n-1)!} dt \right),$$

where the family of polynomials $R_k(s, \tau)$ is defined as

$$(3.8) \quad R_k(s, \tau) = \sum_{i=0}^k \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \quad k \in \mathbb{N}^*.$$

The following proposition, also from [5], provides sufficient conditions for the dominance of the multiple spectral value.

Proposition 2. Let $P(s)$ be real-rooted and s_a be the average of its roots. Then system (2.4) is γ -stabilizable with $\gamma \leq s_a$ if, and only if, $\tau \in]0, \tau_\gamma[$, where τ_γ is the smallest positive root of $R_n(\gamma, \tau)$ for τ .

Now we are able to formulate and prove the following result.

Theorem 3.2. *Consider that Assumptions 1 and 2 hold and the quasipolynomial (4.14) admits a root at some complex number s_0 with multiplicity $n + 1$. Then, s_0 is necessarily algebraic and the corresponding characteristic polynomial denoted \mathbb{P} (called in the sequel Elimination-produced polynomial) is real-rooted. Moreover, if \mathbb{P} is the minimal-degree polynomial such that $\mathbb{P}(s_0) = 0$ and that s_0 is the spectral abscissa of \mathbb{P} , then s_0 is the spectral abscissa of (4.14). Furthermore, if \mathbb{P} is Hurwitz then the estimation error described by (3.5) is exponentially stable with s_0 as decay rate and the estimation states vector converges to its real value (in both cases).*

Proof 2. The vanishing of the characteristic function Δ is equivalent to $\exp(-s\tau) = -P_0(s)/P_1(s)$. Then, one considers the algebraic system of n equations $\Delta'(s) = 0, \dots, \Delta^{(n)}(s) = 0$. First, owing to the linearity of the system, one has $\Delta'(s) = 0, \dots, \Delta^{(n-1)}(s) = 0$ with respect to the coefficients of P_1 , then one proceeds by eliminating them order by order. Next, the values of these coefficients are substituted in the remaining equation $\Delta^{(n)}(s) = 0$ which gives \mathbb{P} the so-called elimination-produced polynomial in the variable s , the delay τ and the coefficients of P_0 . Next, by assuming that assumptions 1 and 2 hold, the proof of the theorem is based on proposition 2 since the quasipolynomial associated to the error system (3.5) is given by equation (2.4). \square

3.3. Illustrative examples. In this section, we provide some illustrative examples in both cases: input-delay and state-delay through the problems of Mach number control in a wind tunnel and an inverted pendulum on a cart.

3.3.1. System with state delay : Transonic flow control. We revisit in this section the problem of control of a transonic flow in a wind tunnel discussed in [32]. The analysis of transonic flows is a challenging problem in compressible fluid dynamics, since a full model of the flow would involve the Navier–Stokes equations in a three-dimensional domain and boundary controls for temperature and pressure regulation. A further simplified model was presented in [3] in order to analyze the response of the Mach number of the flow to changes in the guide vane angle. Instead of using partial differential equations, propagation phenomena are modeled in [3] through a time-delay, leading to the time-delay system

$$(3.9) \quad \begin{cases} \kappa m'(t) + m(t) = k\vartheta(t - \tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases}$$

in which m , ϑ , and u represent, respectively, perturbations of the Mach number of the flow, the guide vane angle, and the input of the guide vane actuator, with respect to steady-state values. The parameters κ and k depend on the steady-state operating point and are assumed to be constant as long as m , ϑ , and u remain small, and satisfying $\kappa > 0$ and $k < 0$. The parameters $\zeta \in (0, 1)$ and $\omega > 0$ come from the design of the guide vane angle actuator and are thus independent from the operating point. The time-delay τ_0 is assumed to depend only on the temperature of the flow. In the absence of control ($u(t) = 0$), the open-loop system (3.9) is exponentially stable.

Equation (3.9) may be written under a state space model as follows

$$(3.10) \quad \dot{X}(t) = AX(t) + A_d X(t - \tau) + Bu(t),$$

where $X(t) = [\vartheta(t) \ \vartheta'(t) \ m(t)]^T$, matrices A , A_d and B are given by

$$A = \begin{bmatrix} -2\zeta\omega & -\omega^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{\kappa} \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{k}{\kappa} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \omega^2 \\ 0 \\ 0 \end{bmatrix}.$$

The available measurement vector is described by

$$y(t) = CX(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X(t).$$

As the obtained system fits with system (3.1), in order to estimate the corresponding state vector $X(t)$, we consider an observer in the form (3.2). In this

example, the error dynamics obey to system (3.5). By choosing the observer matrices in the following fashion

$$L = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & 0 \\ 1 & \alpha_{3,2} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & \beta_{1,2} \\ 0 & \beta_{2,2} \\ \frac{k}{\kappa} & \beta_{3,2} \end{bmatrix}$$

the rank condition (Assumption 1) is satisfied and the associated characteristic function (2.4) is defined by

$$\begin{aligned} P_0(s) = & s^3 + \left(2\xi\omega + \frac{\alpha_{3,2}\kappa + 1}{\kappa}\right) s^2 \\ & + \left(\omega^2 + \frac{(2\kappa\xi\alpha_{3,2} + 2\xi)\omega}{\kappa} + \alpha_{1,1}\right) s \\ & + \frac{(\alpha_{3,2}\kappa + 1)\omega^2}{\kappa} + \frac{\kappa\alpha_{1,1}\alpha_{3,2} - \kappa\alpha_{1,2} + \alpha_{1,1}}{\kappa}, \end{aligned} \quad (3.11a)$$

$$\begin{aligned} P_1(s) = & \beta_{3,2}s^2 + (2\omega\beta_{3,2}\xi - \beta_{2,2})s + \omega^2\beta_{3,2} - 2\omega\beta_{2,2}\xi \\ & + \alpha_{1,1}\beta_{3,2} - \beta_{1,2}. \end{aligned} \quad (3.11b)$$

By forcing the multiplicity, one obtains the remaining parameters $\beta_{i,j}$ and concludes that s_0 is a root with multiplicity $n + 1 = 4$ if, and only if, s_0 is a root of *the elimination-produced polynomial*

$$\begin{aligned} \mathbb{P}(s) = & \kappa s^3 \tau^3 + \left((1 + (2\xi\omega + \alpha_{3,2})\kappa)\tau^3 + 9\kappa\tau^2\right) s^2 \\ & + \left(\left((2\omega\xi\alpha_{3,2} + \omega^2 + \alpha_{1,1})\kappa + 2\xi\omega\right)\tau^3\right) s \\ & + \left((6 + (12\xi\omega + 6\alpha_{3,2})\kappa)\tau^2 + 18\kappa\tau\right) s \\ & + \left((\alpha_{3,2}\omega^2 + \alpha_{3,2}\alpha_{1,1} - \alpha_{1,2})\kappa + \omega^2 + \alpha_{1,1}\right)\tau^3 \\ & + \left((6\omega\xi\alpha_{3,2} + 3\omega^2 + 3\alpha_{1,1})\kappa + 6\xi\omega\right)\tau^2 \\ & + (6 + (12\xi\omega + 6\alpha_{3,2})\kappa)\tau + 6\kappa. \end{aligned}$$

Now, considering the system (3.9) with parameter values as the ones proposed in [32] where $\kappa = 2$ s, $k = -0.67036$ rad⁻¹, $\xi = 1/3$, $\omega = 5$ rad/s and $\tau_0 = 1/3$, one obtains $\alpha_{1,1} = \frac{316}{9}$, $\alpha_{1,2} = \frac{2294}{27}$, $\alpha_{3,2} = -\frac{5}{6}$ guaranteeing the realrootedness of P_0 as well as that of polynomial \mathbb{P} .

In particular, such a choice of $\alpha_{i,j}$ leads to the following observer matrices

$$L = \begin{bmatrix} \frac{316}{9} & \frac{2294}{27} \\ 0 & 0 \\ 1 & -\frac{5}{6} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & -\frac{3536 e^{-\frac{5}{3}}}{9} \\ 0 & -\frac{554 e^{-\frac{5}{3}}}{9} \\ \frac{k}{\kappa} & \frac{7 e^{-\frac{5}{3}}}{3} \end{bmatrix},$$

which guarantees that the assignable rightmost multiple root can be achieved at $s_0 = -5$ as illustrated through Figure 1.

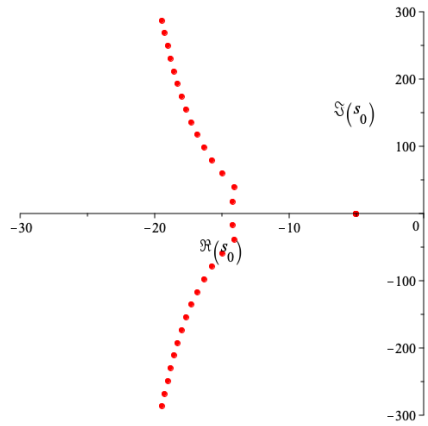


FIGURE 1. The spectrum distribution of the quasipolynomial (2.4) with polynomials P_0 and P_1 defined in (3.11).

Figure 2 provides the evolution of the Mach number $m(t)$, the guide vane position $\vartheta(t)$, its derivative and their estimation, in the case where the initial conditions are given by $m(t) = -0.1$, $\vartheta(0) = \frac{m(0)}{k} = 0.15\text{rad}$, and $\dot{\vartheta}(t) = 0$.

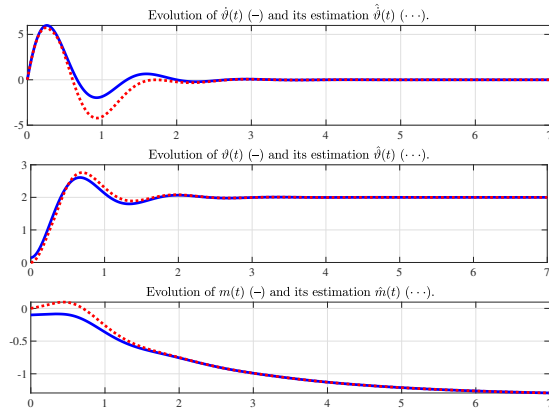
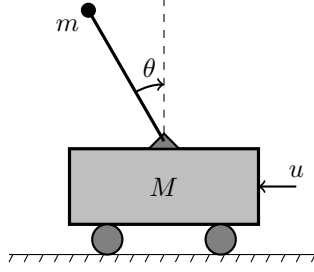


FIGURE 2. Evolution of the state vector $X(t)$ (—) and its estimation $\hat{X}(t)$ (···).

3.3.2. *Systems with input delay : Inverted Pendulum.* In this section, we shall give an example to show the efficiency and the feasibility of the results obtained in this paper. Indeed, let us consider a dynamical system modeling a friction-free inverted pendulum on a cart.



The model adopted here was discussed in [47]. The dynamics of the inverted pendulum on a cart can be modelled by a second-order differential equation for the angular displacement θ of the tip of the pendulum

$$(3.12) \quad \left(1 - \frac{3m}{4(m+M)} \cos^2(\theta)\right) \ddot{\theta} + \frac{3m}{8(m+M)} \dot{\theta}^2 \sin(2\theta) + \sin \theta + u \cos \theta = 0,$$

where M is the mass of the cart, m is the mass of the pendulum, and u represents the control law, which is the horizontal driving force.

We assume that the system is controlled by a delayed input $u(t - \tau)$. For small θ , we can write $\sin(\theta) = \theta$ and $\cos(\theta) = 1$. Also dropping all non-linear components ($\dot{\theta}^2 = 0$), equation (3.12) may be rewritten as

$$(3.13) \quad \left(1 - \frac{3m}{4(m+M)}\right) \ddot{\theta}(t) + \theta(t) + u(t - \tau) = 0.$$

Next, We denote by $X(t) = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix}^T$, so system (3.13) may be rewritten in the following state space model

$$(3.14) \quad \dot{X}(t) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} X(t) + \begin{bmatrix} -a \\ 0 \end{bmatrix} u(t - \tau)$$

where $a = \left(1 - \frac{3m}{4(m+M)}\right)^{-1}$. The available measurement is given by $y(t) = [0 \ 1]X(t)$.

In this case the rank condition (Assumption 1) is satisfied and the associated characteristic quasipolynomial is given by:

$$(3.15) \quad \Delta(s) = (s\beta_{2,1} + \beta_{1,1}) e^{-s\tau} + s^2 + s\alpha_{2,1} + a + \alpha_{1,1}.$$

Next, the realrootedness of the corresponding P_0 (Assumption 2) is guaranteed by the positivity of its discriminant, that is, $\delta = \alpha_{2,1}^2 - 4a - 4\alpha_{1,1}$. As shown in [11], if the discriminant $\delta \geq 0$, then the multiplicity of any given root of the quasipolynomial function (3.15) is bounded by 3. For an arbitrary positive delay τ , the quasipolynomial (3.15) admits a real spectral value at $s = s_{\pm}$ with algebraic multiplicity 3 if, and only if,

$$s_{\pm} = \frac{-\tau \alpha_{2,1} - 4 \pm \sqrt{8 + \tau^2 \delta}}{2\tau},$$

and the system parameters satisfy

$$(\star_{\pm}) \quad \begin{cases} \beta_{1,1} = \left(\alpha_{2,1}s_{\pm} + \frac{\alpha_{2,1}^2}{2} - \frac{\delta}{2} + \frac{6\alpha_{2,1} + 10s_{\pm}}{\tau} + \frac{6}{\tau^2}\right) e^{\tau s_{\pm}}, \\ \beta_{2,1} = \left(2s_{\pm} + \alpha_{2,1} + \frac{2}{\tau}\right) e^{\tau s_{\pm}}. \end{cases}$$

If (\star_+) is satisfied then $s = s_+$ is the spectral abscissa corresponding to (3.15). Furthermore, the trivial solution is asymptotically stable if, and only if, τ satisfies the following conditions

$$\begin{cases} \tau \in]0, \tau_-[& \text{when } \alpha_{1,1} < -a, \\ \text{or} \\ \tau \in]0, \tau_-[\cup]\tau_+, \infty[& \text{when } \alpha_{1,1} > -a \text{ and } \alpha_{2,1} < 0, \end{cases}$$

where $\tau_{\pm} = \frac{-\alpha_{2,1} \pm \sqrt{\delta + 2\alpha_{1,1} + a}}{\alpha_{1,1} + a}$.

For instance, choosing $\alpha_{1,1} = -a - 1$ and $\alpha_{2,1} = 1$, we define the rightmost root of (2.4) as a function of the delay τ as illustrated in figure 3.

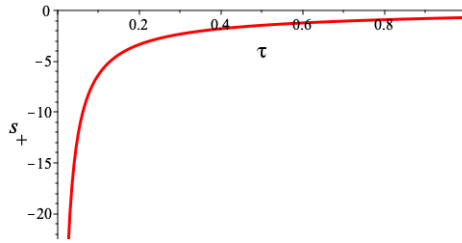


FIGURE 3. The spectral abscissa of (2.4) as a function of the delay τ where $\alpha_{1,1} = -a - 1$ and $\alpha_{2,1} = 1$.

For further discussions on the MID property in second-order delay, the reader may consider [11].

As a numerical example, we choose $m = 0.5$ Kg and $M = 5$ Kg and a time delay $\tau = 0.1$ s. By setting the observer matrices to the latter computation, we obtain

$$L = \begin{bmatrix} -2.0732 \\ 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 10.803 \\ 4.453 \end{bmatrix}$$

The initial conditions are given by $X(0) = [0 \ 0.5]^T$. Figure 4 shows the convergence of the estimation state vector to its real values.

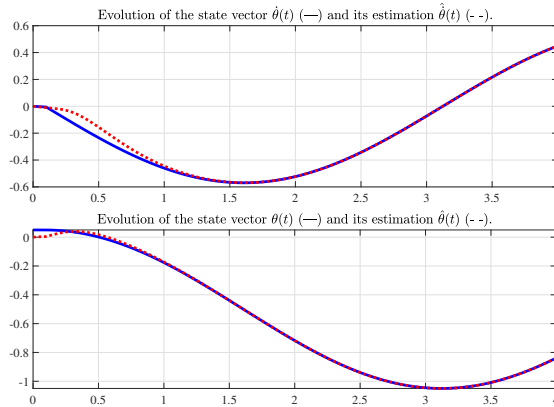


FIGURE 4. Evolution of the state vector $X(t)$ (—) and its estimation $\hat{X}(t)$ (- -) with $\tau = 0.1$ s.

4. GMID IN THE DESIGN OF REDUCED ORDER OBSERVER DESIGN

4.1. Problem Formulation. For both systems (3.1) and (3.3), we propose to design a reduced order functional observer, allowing to estimate the states $z(t)$ and described by the following system

$$(4.1a) \quad \dot{\eta}(t) = N\eta(t) + N_d\eta(t - \tau) + My(t) + M_dy(t - \tau) + Fu(t)$$

$$(4.1b) \quad \hat{z}(t) = \eta(t) + Ey(t)$$

where $\eta \in \mathbb{R}^r$. $\hat{z} \in \mathbb{R}^r$ is the estimate of $z = Lx(t)$. Without loss of generality, we assume that $\text{rank}(L) = r \leq n$. The matrices N , M , N_d , M_d , F and E are unknown, which should be determined later such that the estimation of the states \hat{z} converge to its real values z .

Notice that, in the case of system (3.3), the term $Fu(t)$ in the observer dynamic system becomes $Fu(t - \tau)$ with the aim to guarantee the unbiasedness of the error dynamics in the sequel.

Let us note $e(t) = z(t) - \hat{z}(t) = \Psi x(t) - \eta(t)$, where $\Psi = L - EC$. The estimation error dynamics is described by the following system

$$(4.2) \quad \begin{aligned} \dot{e}(t) = & Ne(t) + N_de(t - \tau) + (\Psi A - N\Psi - MC)x(t) \\ & + (\Psi A_d - N_d\Psi - M_dC)x(t - \tau) + (F - \Psi B)u(t) \end{aligned}$$

Proposition 3. The dynamical system (4.1) represents an observer for both time-delay systems described by (3.1) and (3.3), if and only if the dynamic error system given by

$$(4.3) \quad \dot{e}(t) = Ne(t) + N_de(t - \tau)$$

is asymptotically stable and the following equations hold

$$\begin{aligned} F - \Psi B &= 0 \\ \Psi A - N\Psi - MC &= 0 \\ \Psi A_d - N_d\Psi - M_dC &= 0 \end{aligned}$$

Proof 3. Let us compute the dynamics of the estimation error, as follows:

$$\begin{aligned} \dot{e}(t) &= \Psi \dot{x}(t) - \dot{\eta}(t) \\ &= \Psi Ax(t) + \Psi A_dx(t - \tau) + \Psi Bu(t) - N\eta(t) \\ &\quad - N_d\eta(t - \tau) - My(t) - M_dy(t - \tau) - Fu(t) \end{aligned}$$

By adding and subtracting the terms $\pm N\Psi x(t) \pm N_d\Psi x(t - \tau)$, system (4.2) is obtained. By cancelling terms which depend on the state vector $x(t)$ and of the input vector $u(t)$, one can obtain system (4.3). This latter system needs to be stable, to ensure the convergence of the estimation error $e(t)$ to zero. \square

The first step for the stability analysis of the estimation error is to guarantee the unbiasedness of system (4.2). In other words, we have to put the terms which do not depend on the error equal to zero. Thus, the matrix F had to be chosen as

$$(4.4) \quad F = \Psi B$$

and the following Sylvester equations must hold

$$(4.5a) \quad \Psi A - N\Psi - MC = 0$$

$$(4.5b) \quad \Psi A_d - N_d\Psi - M_dC = 0$$

4.2. Existing condition of the proposed observer. To ensure the unbiasedness of the observer, the Sylvester equations described by (4.5) should be verified.

Using the expression $\Psi = L - EC$, the Sylvester equations are rewritten as

$$(4.6a) \quad LA = NL + KC + ECA$$

$$(4.6b) \quad LA_d = N_dL + K_dC + ECA_d$$

where $K = M - NE$, and $K_d = M_d - N_dE$.

It's clear that these Sylvester equations (4.6) can be written in the following compact form

$$(4.7) \quad V = \bar{X}W$$

where V and W contain the known matrices such as

$$V = [LA, LA_d]$$

$$W = \begin{bmatrix} L & 0 \\ 0 & L \\ C & 0 \\ 0 & C \\ CA & CA_d \end{bmatrix}$$

and \bar{X} is the matrix of the unknown observer's matrices to be designed, described as follows

$$\bar{X} = [N, N_d, K, K_d, E]$$

Equation (4.7) admits a solution if and only if the following rank condition is satisfied

$$(4.8) \quad \text{rank}(W) = \text{rank}([V^T, W^T]^T)$$

and, by defining $\ell = 2(r + p) + p$, the general solution is given by

$$(4.9) \quad \bar{X} = VW^\dagger + Z(I_\ell - WW^\dagger)$$

where Z is an arbitrary matrix with appropriate dimension, which will be chosen in order to satisfy the convergence of the estimation errors and W^\dagger is any generalised inverse of W , which fulfils the following equation [42]

$$(4.10) \quad W = WW^\dagger W$$

Then, the observers matrices can be expressed through a single gain matrix Z as

$$(4.11) \quad \underbrace{\begin{bmatrix} N & N_d & K & K_d & E \end{bmatrix}}_{\bar{X}} = \underbrace{\begin{bmatrix} N_1 & N_{d_1} & K_1 & K_{d_1} & E_1 \end{bmatrix}}_{VW^\dagger} + Z \underbrace{\begin{bmatrix} N_2 & N_{d_2} & K_2 & K_{d_2} & E_2 \end{bmatrix}}_{I_\ell - WW^\dagger}$$

In other words:

$$(4.12a) \quad N = N_1 + ZN_2 = VW^\dagger R_N + Z(I_\ell - WW^\dagger)R_N$$

$$(4.12b) \quad N_d = N_{d1} + ZN_{d2} = VW^\dagger R_{N_d} + Z(I_\ell - WW^\dagger)R_{N_d}$$

$$(4.12c) \quad K = K_1 + ZK_2 = VW^\dagger R_K + Z(I_\ell - WW^\dagger)R_K$$

$$(4.12d) \quad K_d = K_{d1} + ZK_{d2} = VW^\dagger R_{K_d} + Z(I_\ell - WW^\dagger)R_{K_d}$$

$$(4.12e) \quad E = E_1 + ZE_2 = VW^\dagger R_E + Z(I_\ell - WW^\dagger)R_E$$

with

$$\begin{aligned} R_N^T &= [I_r \ 0_{r \times r} \ 0_{p \times r} \ 0_{p \times r} \ 0_{p \times r}] \\ R_{N_d}^T &= [0_{r \times r} \ I_r \ 0_{p \times r} \ 0_{p \times r} \ 0_{p \times r}] \\ R_K^T &= [0_{r \times p} \ 0_{r \times p} \ I_p \ 0_{p \times p} \ 0_{p \times p}] \\ R_{K_d}^T &= [0_{r \times p} \ 0_{r \times p} \ 0_{p \times p} \ I_p \ 0_{p \times p}] \\ R_E^T &= [0_{r \times p} \ 0_{r \times p} \ 0_{p \times p} \ 0_{p \times p} \ I_p] \end{aligned}$$

4.3. Stability of the estimation error. Since the unbiasedness conditions given in section 4.2 are satisfied, the dynamic of the estimation error satisfies equation (4.3) and may be rewritten in the following form

$$(4.13) \quad \dot{e}(t) - Ne(t) - N_d e(t - \tau) = 0$$

So that the characteristic function associated $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ is given by

$$(4.14) \quad \Delta(s, \tau) = |sI_n - N - N_d e^{-\tau s}|$$

Hence, the error system is characterised by the quasipolynomial function with the form of equation (2.4) where the polynomial P_0 is given by:

$$P_0(s) = |sI_n - N| = n_n s^n + n_{n-1} s^{n-1} + \dots + n_1 s + n_0$$

In order to check the stability of the estimation error, we assume that

Assumption 3. $\text{rank}(\overline{N}_d) = 1$.

Notice that this latter assumption is similar to assumption 1, which ensure that the quasipolynomial Δ (described by (4.14)) involves a single delay.

In section 3, the intermediate MID property has been exploited for the design of observer through the result of [5] restricting the observer design to the following additional assumption 2. The use of the GMID property allows to relax such an assumption by using the result from [9].

Now we are able to formulate and prove the following result.

Theorem 4.1. *Consider that Assumption 3 holds and let a negative s_0 be a multiple root of (4.14) with the maximal multiplicity, then the estimation error described by (4.13) is exponentially stable with s_0 as decay rate and the estimation states vector converge to its real value.*

Proof 4. Since condition 4.8 holds, the error dynamics is written under equation (4.3). Under the assumption 1, the quasipolynomial (4.14) is a particular case of (2.5). Thus, following Theorem 2.1, a root s_0 of Δ is necessarily dominant. So that, it corresponds to the exponential decay of (4.13). \square

Algorithm 1: Reduced order observer design

-
- Compute the matrices N_1, N_2, N_{d1}, N_{d2} using equations (4.12)
 - Compute the characteristic polynomial corresponding to the error system ,
 - Consider assumption 1,
 - Solve the characteristic polynomial in function of the matrix Z ,
 - Tune the matrix Z ,
 - Deduce the observer matrices N, N_d, M, M_d, F and E .
-

4.4. Illustrative examples. We illustrate our reduced observer design for the estimation of the intake manifold pressure and the compressor power in a diesel engine described with a LTI system with delay in the state and the estimation of the speed in a two-finger robot hand described with a LTI system with input delay.

4.4.1. Systems with state delay : Diesel Engine. A diesel engine with an exhaust gas recirculation (EGR) valve and a turbo-compressor with a variable geometry turbine (VGT) was modeled in [28]. The system is subjected to intake-to-exhaust transport delay. A LTI time-delay model is obtained by linearizing the original non-linear model under the assumption that the engine operates at a constant speed $N = 1500$ RPM. This model is described as

$$(4.15a) \quad \dot{x}(t) = A_0x(t) + A_1x(t - \tau) + Bu(t)$$

$$(4.15b) \quad y(t) = Cx(t) = [0 \ 1 \ 0]x(t)$$

$$\text{where } A_0 = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \\ 0 & 0.18 \end{bmatrix} \text{ and } \tau \text{ is given as } 0.06\text{s at this particular operating point.}$$

The state variables are defined as intake manifold pressure (x_1), exhaust manifold pressure (x_2), and compressor power (x_3). Furthermore, the system has two control inputs in which u_1 is an input for the EGR valve opening and u_2 is an input for the VGT mass flow rate.

The model includes intake-to-exhaust transport delay $\tau = 60\text{ms}$ when engine speed is 1500 RPM. Due to the time-delay assignable to the transport time of the gas from intake to exhaust manifold, the system is represented a DDE as in Equation (4.15). Since the observability matrix is full rank matrix, system (4.15) is observable and the corresponding quasipolynomial function is given by

$$(4.16) \quad \begin{aligned} \Delta(s, \tau) &= \det(sI_r - N - N_d e^{-\tau s}) \\ &= \det(sI_r - (N_1 + ZN_2)) - (N_{d1} + ZN_{d2})e^{-\tau s} \end{aligned}$$

where $r = 2$, $P_0(s)$ is with order 1 and $P_1(s)$ is with order 2. The matrix Z is tuned to get assumption 1.

For instance, choosing Z as :

$$Z = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & z \end{bmatrix}$$

one can define the rightmost root s_0 of (4.16), with a maximal multiplicity 4, as a function of the delay τ as shown in Figure 5. To show the convergence of the states

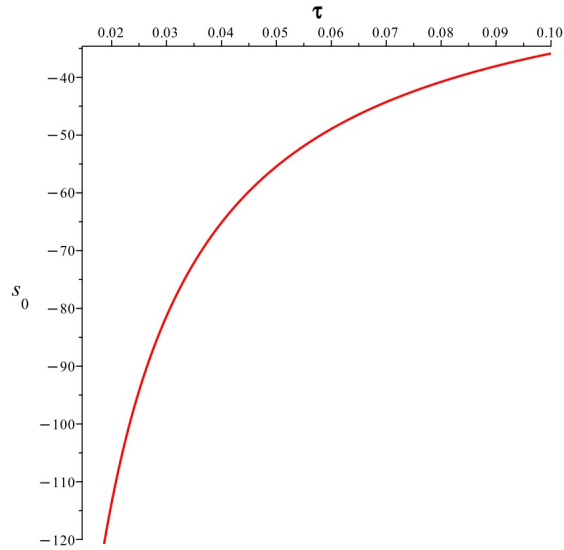


FIGURE 5. The spectral abscissa of (4.16) as a function of the delay τ .

estimation to their real values, we choose Z with the following numerical values :

$$Z = \begin{bmatrix} -170 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -150 & 0 & 0 & 0 & 100 \end{bmatrix}$$

For the simulation, the initial states are chosen as $x(0) = [0.3 \ 0.2 \ -0.3]^T$ and $z(0) = [0 \ 0]^T$. As shown in Figure 6, the error $e(t)$ of the states and their estimation converge to zero in a short span of time. Thus, the observer reveals satisfactory performance.

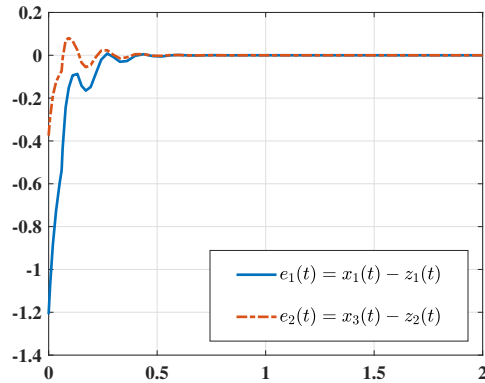


FIGURE 6. Evolution of the errors between the system's states and their estimations.

4.4.2. *Systems with input delay : Two-finger robot hand.* This example considers a practical master-slave tele-operated system by a two-finger robot hand, where the fingers are driven by DC servomotors and a belt-pulley mechanism. Due to the transmission delay between the master and the slave in the forward path, the control input at the slave side is subject to a delay $\tau = 1$ s. By ignoring the disturbance effects coming from the spring and the delay in the backward path, the mathematical model of the two-finger robot hand is given as follows

$$(4.17a) \quad \dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 0 & 26 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 4.5 \end{bmatrix} u(t - \tau)$$

$$(4.17b) \quad y(t) = Cx(t) = [1 \ 0]x(t)$$

The DC servomotor rotation angle is measured by a rotary encoder, and the finger displacement $x_1(t)$ can be obtained by multiplying the pulley radius by the rotation angle. However, in teleoperation of the control system, we need to have also information of the slave finger velocity. Thus, we call the reduced order observer to estimate the speed $x_2(t) = \dot{x}_1(t)$ of the finger. The resolution of the characteristic function associated to the error system gives the spectrum distribution in Figure 7 with a choice of a matrix $Z = [0, z_1, 0, 0, z_2]$. By tuning $z_1 = -100$ et $z_2 = 25$, the obtained observer is given by

$$\dot{\eta}(t) = -25.5\eta(t) - 2.210^{-14}\eta(t - \tau) + 12.75y(t) + 1.110^{-14}y(t - \tau) + 4.5u(t - \tau)$$

$$\hat{z}(t) = \eta(t) - 0.5y(t)$$

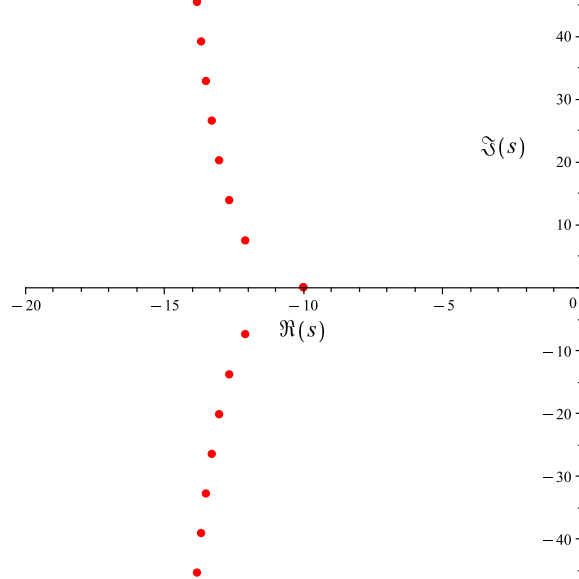


FIGURE 7. Evolution of the errors between the system's states and their estimations.

The simulation of both the observer and the system is given in Figure 8, which shows the convergence of the estimated speed of the finger $\hat{z}(t)$ exponentially to the actual speed $x_2(t) = \dot{x}_1(t)$. The control input chosen as $u(t) = 0.45 + 0.01 \sin(t)$ with

transport delay $\tau = 1$ s. The initial conditions are chosen such that $x(0) = [0.5, 0]$ and $\eta(0) = 0$.

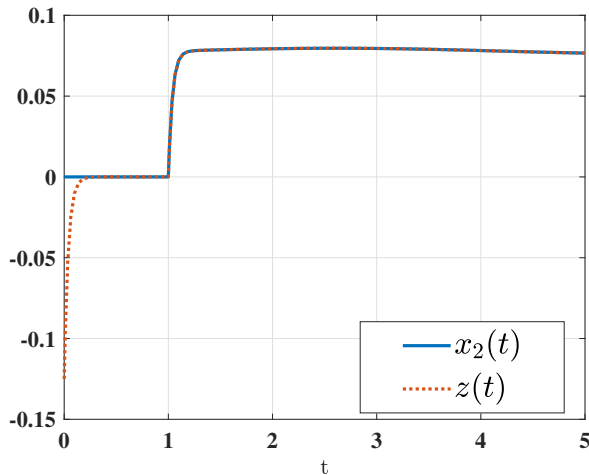


FIGURE 8. Evolution of the errors between the system's states and their estimations.

5. CONCLUSION

In this paper, The use of partial pole placement in the design of observers has been developed. Two types of delayed systems have been considered, systems with delayed states and systems with delayed inputs. Under sufficient conditions, the stability of the estimation error systems has been proved in both cases exploiting the MID and the GMID property. To summarize, the GMID idea may be particularly adapted for tuning low-complexity controllers, i.e., controllers including a small number of parameters (including also the delay among the parameters) with a guarantee on the location of the remaining characteristic roots for the closed-loop system. Finally, numerical examples are provided to illustrate the design procedure, practicality and efficiency of the proposed approach for each study case.

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