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Boundedness and Local Stability of Oscillation in a Class of Piecewise Affine Systems [★]

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Abstract

This paper investigates self-oscillation behaviours occurring in a class of piecewise affine (PWA) systems. We address a particular case of PWA systems with two subsystems and a linear switching surface. Conditions are given for characterizing forward invariant sets containing self-oscillating solutions and domains of attraction. Rather than relying on purely computational tools, we provide conditions taking into account the structure of the system.

Key words: oscillations; piecewise affine system; forward invariance; local stability.

1 Introduction

Piecewise affine (PWA) systems represent an important class of hybrid dynamical systems [18, 29]. They are characterized by a set of affine dynamics together with a state-dependent switching law. Such systems can be found in several engineering applications: power converters [31], robotics [23], relay control systems [6, 27], electromechanical systems with non-smooth phenomena [24]. PWA models are also interesting since they allow to approximate complex nonlinear dynamics [15]. Analysis of PWA systems is a classical topic in hybrid control theory [33, 40]. Although they are rather simple in structure, the analysis of such systems is a complex problem. For such systems, one needs to handle phenomena such as sliding dynamics [5], non-common equilibria [28], non-uniqueness of solutions [22, 39] and self-oscillations [5, 19]. In the literatures, several numerical methods have been provided for investigating global stability of the null equilibria [9, 26, 28, 33, 38]. The provided conditions generally lead to investigating

the feasibility of a set of Linear Matrix Inequalities (LMIs) [7]. Alternatively, some necessary and sufficient analytic conditions for checking the quadratic stability of null equilibrium can be found in [12, 37].

The studies concerning the stability of the null equilibrium are relevant in several practical domains. However, a theoretical challenging and interesting case is provided by PWA systems with oscillating behaviors [3]. The study of oscillating behaviours in PWA systems is precisely the focus of our paper. Such oscillating behaviors may include limit cycles [2, 19], but also aperiodic oscillations where multi-sliding bifurcations may occur [3, 25]. In the literature, pioneering results concerning the stability of limit cycle for planar PWA systems can be found in [17, 30]. Stability conditions have been provided by investigating Poincaré map [41] and Lyapunov function on switching surfaces [34]. Few results exist for PWA systems beyond the two-dimensional case. In [14], the author considered the the case of three-dimensional piecewise linear systems and characterized bifurcations. A canonical transformation for generic n-dimensional bimodal piecewise linear-continuous vector fields was proposed in [4]. In [19], a method for estimating the region of attraction around a limit cycle have been proposed based on the use of invariant ellipsoids on the switching surfaces. Nevertheless, the provided conditions lead to checking an infinite number of matrices inequalities parameterized by the

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switching times. Although not directly connected the mathematical study of oscillations in continuous-time PWA systems, this topic can also be related to the works concerning switched affine systems with time-dependent switching [11], convergent PWA systems [32] and discrete-time PWA systems [20, 36]. In general, investigating the stability and characterizing the domain of attraction around the oscillating solution remain difficult problems.

In this paper, we focus on bimodal PWA systems with linear switching surfaces. Conditions for characterizing forward invariant sets and domains of attraction are proposed. Rather than relying on purely computational tools, the conditions provided in this work take into account system structures. More precisely, we rewrite the PWA system as a system with a relay-like input and we exploit passivity [8] and output stabilizability property of the system with respect to an artificial output corresponding to switching surface. As a side result, we provide some new conditions allowing to check the local asymptotic stability of the null equilibrium.

This paper is organized as follows. Section 2 describes the problem to be solved and gives the concepts of solutions and the stability notions. Section 3 presents the main result of this paper followed by numerical examples in Section 4. Finally, a conclusion is stated in Section 5.

Notations: The Euclidean vector norm in \mathbb{R}^n is denoted by $\|\cdot\|$. \mathbb{R}^+ denotes the set of positive reals. For a set \mathcal{S} , the symbol $\text{Conv}\{\mathcal{S}\}$ denotes the closed convex hull of the set and $\text{Int}(\mathcal{S})$ denotes the interior of \mathcal{S} . $\mathcal{B}(x, \delta)$ is the open ball centered at x of radius $\sqrt{\delta}$. In a matrix, \star denotes the elements that can be inferred by symmetry.

2 Preliminaries

Consider the following piecewise affine system with two modes and a linear partitioning of the state space:

$$\dot{x} = \mathcal{X}(x) := \begin{cases} A_1x + b_1, & Cx < 0 \\ A_2x + b_2, & Cx \geq 0 \end{cases} \quad (1)$$

where $\mathcal{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \in \mathbb{R}^n$ is the vector of the state variables and Cx is a switching hyperplane. $A_1, A_2 \in \mathbb{R}^{n \times n}$, $b_1, b_2 \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ are known matrices. The system under study is a differential equation with discontinuous right-hand side [10]. In order to define the system solutions and take into account sliding dynamics, we use the Filippov regularization procedure [13].

Definition 1 [13]: Consider system $\dot{x} = \mathcal{X}(x)$ with $\mathcal{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ locally bounded. A Filippov solution over the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous map

$\varsigma : [t_a, t_b] \rightarrow \mathbb{R}^n$ satisfying: $\dot{\varsigma}(t) \in \mathcal{F}[\mathcal{X}](\varsigma(t))$ for almost all $t \in [t_a, t_b]$ with

$$\mathcal{F}[\mathcal{X}](x) = \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S})=0} \text{Conv}\{\mathcal{X}(\mathcal{B}(x, \delta) \setminus \mathcal{S})\}, \quad (2)$$

with $\mu(\mathcal{S})$ being the measure, in the sense of Lebesgue, of the set \mathcal{S} .

Basic conditions for the existence of solutions can be found in [13]. For system (1), Filippov solutions are always defined on $[0, +\infty)$ (see [35] or [16], page 169). Note that for differential equations with discontinuous right-hand side, other concepts of system solution can be used [10]. Since the system is piecewise affine with the particular structure (1), the results in the paper also hold when the Krasovskii regularization procedure is used. We recall [1, 18] as follows some properties of sets with respect to solutions of discontinuous systems.

Definition 2 A compact set $\mathcal{S} \subset \mathbb{R}^n$ is forward invariant for (1), if for every Filippov solution $x(\cdot)$ to (1) with $x(0) \in \mathcal{S}$, $x(t) \in \mathcal{S}$ for all $t \in [0, \infty)$.

Remark 1 Let us recall that a point $x^* \in \mathbb{R}^n$ is an equilibrium point for system $\dot{x} = \mathcal{X}(x)$ when solutions are considered in the sense of Filippov, if $0 \in \mathcal{F}[\mathcal{X}](x^*)$.

Definition 3 The equilibrium $x^* = 0$ of system (1) is said to be locally asymptotically stable if (a) for all $\varepsilon > 0$, there exists $\phi > 0$ such that for any Filippov solutions $x(\cdot)$ with $\|x(0)\| \leq \phi$, $\|x(t)\| < \varepsilon$ for all $t > 0$; (b) there exists $\delta > 0$ such that any Filippov solution $x(\cdot)$ with $\|x(0)\| \leq \delta$, $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

Consider a positive definite \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and the sets $\mathcal{D}_A := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$ and $\mathcal{S} := \{x \in \mathbb{R}^n : V(x) \leq \varepsilon\}$ for positive scalars $\varepsilon < \gamma$. We recall [1, 18] that for a system $\dot{x} = \mathcal{X}(x)$, $x(0) \in \mathbb{R}^n$, with $\mathcal{X} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ locally bounded, if $\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma \leq -\alpha V(x)$, $\forall x \in \mathcal{D}_A \setminus \text{Int}(\mathcal{S})$, and for some $\alpha > 0$, then the set \mathcal{S} is forward invariant for the system. In addition, the set \mathcal{S} is attractive with a domain of attraction \mathcal{D}_A in the sense that the solutions $x(\cdot)$ of the systems with $x(0) \in \mathcal{D}_A$ converge to \mathcal{S} at most in $\frac{1}{\alpha} \ln \frac{V(x_0)}{\varepsilon}$ time. Global/local asymptotic stability/attractivity can be obtained by modifying the sets \mathcal{D}_A and \mathcal{S} .

3 Main results

In this section, we present our main results. First, in Section 3.1, conditions are provided for estimating a forward invariant set which is globally attracting oscillating solutions. Next, in Section 3.2, as a side result, we provide local asymptotic stability conditions for the null equilibrium. At last, in Section 3.3, we built upon the results provided in Sections 3.1 and 3.2 to

provide conditions for characterizing forward invariant sets locally attracting oscillating solutions.

In the approach considered in this paper, oscillations are studied as the result of perturbations with respect to systems that exhibit asymptotic convergence of solutions towards the origin. The following assumptions will be considered:

Assumption 1 *None of the subsystems have an equilibrium at the origin, i.e. $b_i \neq 0, \forall i \in \{1, 2\}$.*

Assumption 2 *The origin $x^* = 0$ is a Filippov equilibrium, i.e. $0 \in \mathcal{F}[\mathcal{X}](x^*)$.*

Remark 2 *The results may also be used for systems*

$$\dot{z} = \mathcal{H}(z) := \begin{cases} A_1 z + b_1, & Cz < d \\ A_2 z + b_2, & Cz \geq d \end{cases} \quad (3)$$

where $A_1, A_2 \in \mathbb{R}^{n \times n}$, $b_1, b_2 \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and d is a scalar, provided that there exists a non-trivial Filippov equilibrium somewhere on the switching surface: $\exists z^* \in \mathbb{R}^n$ s.t. $0 \in \mathcal{F}[\mathcal{H}](z^*)$ while $A_i z^* + b_i \neq 0, \forall i \in \{1, 2\}$. This leads to $\exists (z^*, \alpha^*) \in \mathbb{R}^n \times (0, 1)$ s.t.

$$A(\alpha^*)z^* + b(\alpha^*) = 0, \quad Cz^* = d \quad (4)$$

where $A(\alpha^*) := \alpha^* A_1 + (1 - \alpha^*) A_2$, $b(\alpha^*) := \alpha^* b_1 + (1 - \alpha^*) b_2$. Then using $\eta = z - z^*$, we obtain a system

$$\dot{\eta} = \begin{cases} A_1 \eta + \bar{b}_1, & C\eta < 0 \\ A_2 \eta + \bar{b}_2, & C\eta \geq 0 \end{cases} \quad (5)$$

where $\bar{b}_1 := A_1 z^* + b_1$, $\bar{b}_2 := A_2 z^* + b_2$, which is a system of the form (1) that satisfies Assumptions 1 and 2. The study of system (3) can be expressed as one for system (1), modulo a translation of the state space. For the particular case of system (1), using similar arguments, Assumptions 1, 2 lead to the following property:

Property 1 *Consider Assumptions 1, 2. Then there exist $\alpha^* \in (0, 1)$, s.t. $b_1 \alpha^* + b_2 (1 - \alpha^*) = 0$.*

3.1 Global results

In order to take into account the relation between the terms b_1, b_2 and the switching surface, we rewrite system (1) as a model with a relay-like input.

Lemma 1 *Consider system (1) and Property 1. Denote $A_0 = \alpha^* A_1 + (1 - \alpha^*) A_2$, $N = A_1 - A_2$ and $B = b_1 - b_2$. Then system (1) can be expressed as*

$$\dot{x} = \mathcal{X}(x) = A(u(x))x + Bu(x), \quad (6)$$

with $A(\theta) = A_0 + N\theta$ for all $\theta \in \{-\alpha^*, 1 - \alpha^*\}$ and

$$u(x) = \begin{cases} 1 - \alpha^*, & Cx < 0, \\ -\alpha^*, & Cx \geq 0. \end{cases} \quad (7)$$

Proof: Consider the function $\alpha(x) = 1$ if $Cx < 0$ and $\alpha(x) = 0$ otherwise. System (1) is represented as

$$\dot{x} = (A_1 - A_2)x\alpha(x) + A_2x + (b_1 - b_2)\alpha(x) + b_2. \quad (8)$$

From Property 1, $\exists \alpha^* \in (0, 1)$, s.t. $(b_1 - b_2)\alpha^* + b_2 = 0$. Remark that $u(x) = \alpha(x) - \alpha^*$. Using $\alpha(x) = u(x) + \alpha^*$ in (8), we have $\dot{x} = \mathcal{X}(x) = A_0x + (Nx + B)u(x) + B\alpha^* + b_2$. Note that $B\alpha^* + b_2 = (b_1 - b_2)\alpha^* + b_2 = 0$. Then, (1) is equivalent to (6) and (7). ■

In the following proposition, we present simple conditions for computing a forward invariant set which is globally attractive for system's (1) solutions.

Proposition 1 *Consider system (1) and Property 1. Denote $v_1 = 1 - \alpha^*$, $v_2 = -\alpha^*$.*

I. *Let there exist $P^\top = P \succ 0$ and $\alpha > 0$ s.t.*

$$A_i^\top P + PA_i \prec -\alpha P, \quad i \in \{1, 2\}. \quad (9)$$

Then for all $\delta \in (0, \alpha)$, the set $\mathcal{S} := \{x \in \mathbb{R}^n : x^\top P x \leq \varepsilon\}$ with

$$\varepsilon > \max_{v \in \{v_1, v_2\}} \rho(v), \quad (10)$$

where $\rho(v) = \frac{4(C - (b_1 - b_2)^\top P)P^{-1} \left(C^\top - P(b_1 - b_2) \right) v^2}{(\alpha - \delta)^2}$ is forward invariant for system (1) and globally attractive.

II. *If P can be taken such that (9) holds and $(b_1 - b_2)^\top P = C$, then system (1) is globally asymptotically stable at the origin.*

Proof of Proposition 1. Using Lemma 1, system (1) is represented in the form (6) and (7). Note that $A(v_i) = A_i, i = 1, 2$. Then the Filippov set-valued map (2) associated with (6), (7) is:

$$\mathcal{F}[\mathcal{X}](x) = \begin{cases} \{A_1 x + Bv_1\}, & \text{if } Cx < 0, \\ \{A_2 x + Bv_2\}, & \text{if } Cx > 0, \\ \text{Conv} \left\{ A_i x + Bv_i, i \in \{1, 2\} \right\}, & \text{if } Cx = 0, \end{cases} \quad (11)$$

Consider $V(x) = x^\top Px$ with $P = P^\top \succ 0$ satisfying (9). We want to show that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -\delta V(x), \quad \forall x \in \mathbb{R}^n \setminus \text{Int}(\mathcal{S}). \quad (12)$$

According to (11), we get the following three cases:

1. If $Cx < 0$, from (6)-(7), we obtain

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma &= x^\top (A_1^\top P + PA_1) x + 2x^\top PBv_1 \\ &+ 2x^\top (C^\top - PB)v_1 - 2x^\top (C^\top - PB)v_1. \end{aligned} \quad (13)$$

Since $v_1 = 1 - \alpha^* > 0$ we have $2x^\top C^\top v_1 < 0$. Then

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -2x^\top (C^\top - PB)v_1 - \alpha x^\top Px, \quad (14)$$

where (9) has been used for $i = 1$.

2. If $Cx > 0$, then using similar arguments we have

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -2x^\top (C^\top - PB)v_2 - \alpha x^\top Px. \quad (15)$$

3. If $Cx = 0$, by using a continuity argument and considering that (9) holds for $i \in \{1, 2\}$, we have

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma &< \max_{i \in \{1, 2\}} \{-2x^\top (C^\top - PB)v_i\} \\ &- \alpha x^\top Px. \end{aligned} \quad (16)$$

Since (10) holds, we have

$$\varepsilon > \frac{(C - B^\top P)P^{-1}(C^\top - PB)v^2}{(\alpha - \delta - \alpha_1)\alpha_1} \quad (17)$$

for all $v \in \{v_1, v_2\}$ and $\alpha_1 = \frac{\alpha - \delta}{2}$. By using the Schur complement on (17), we have

$$\begin{bmatrix} (\alpha - \delta - \alpha_1)P & (C^\top - PB)v \\ \star & \alpha_1 \varepsilon I \end{bmatrix} \succ 0, \quad (18)$$

for all $v \in \{v_1, v_2\}$. Therefore,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^\top \begin{bmatrix} (\alpha - \delta - \alpha_1)P & (C^\top - PB)v \\ \star & \alpha_1 \varepsilon I \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0, \quad (19)$$

for all $v \in \{v_1, v_2\}$, which is the same as

$$(\alpha - \delta)x^\top Px + 2x^\top (C^\top - PB)v + \alpha_1(\varepsilon - x^\top Px) > 0,$$

with $\alpha_1 = \frac{\alpha - \delta}{2} > 0$. When $x^\top Px \geq \varepsilon$, we get

$$(\alpha - \delta)x^\top Px + 2x^\top (C^\top - PB)v > 0, \quad (20)$$

for all $v \in \{v_1, v_2\}$. From (14), (15), (16) and (20), we have $\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -\delta V(x)$, whenever $x^\top Px \geq \varepsilon$. Hence, (12) is guaranteed whenever ε is taken such that (10) holds, which proves **statement I**.

Proof of statement II. The proof follows the same steps. One can directly see that if P is such that $(b_1 - b_2)^\top P = C$, then in (14), (15) and (16) we get $\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -\delta V(x)$, for all $x \in \mathbb{R}^n$. ■

Remark 3 Proposition 1 provides conditions for characterizing a forward invariant set \mathcal{S} for system (1) which is attractive. Some similarities can be found between this result and the result in Proposition 1 from [11] for the case of switched affine systems with time-dependent switching. However, here the set is characterized by an ε level-set of the Lyapunov function $V(x) = x^\top Px$ while taking into account the switching hyperplane. In Proposition 1, the scalar α can be interpreted as an estimation of the decay rate for the set of linear systems $\dot{x} = A_i x$, $i \in \{1, 2\}$. The parameter δ corresponds to an estimation of the decay rate towards the set \mathcal{S} . Note that by using the Schur complement, (10) can be expressed as

$$\begin{bmatrix} \left(\frac{\alpha - \delta}{2}\right)P & (C^\top - P(b_1 - b_2))v \\ \star & \frac{\alpha - \delta}{2}\varepsilon I \end{bmatrix} \succ 0, \quad v \in \{v_1, v_2\}. \quad (21)$$

Then for prescribed α and δ , (9) and (21) can be solved jointly as LMI problems in P and ε . Minimizing ε for prescribed decay rates α and δ , allows to increase the accuracy of the forward invariant set \mathcal{S} . Based on this LMI problem formulation, we may imagine various computational procedures for optimizing the estimation of the forward invariant set. For some particular cases, we can also prove asymptotic stability of the origin (as seen in point **II** in Proposition 1). The intuition is that when $(b_1 - b_2)^\top P = C$, the bound ε characterizing the forward invariant set can be chosen arbitrary small.

Remark 4 We present as follows some interpretations of the conditions of Proposition 1. Let us first remark that the representation of system (6), (7) (in Lemma 1) can be seen as an interconnection between the system

$$\begin{cases} \dot{x} = A(u(x))x + Bu_p, \\ y = Cx \end{cases} \quad (22)$$

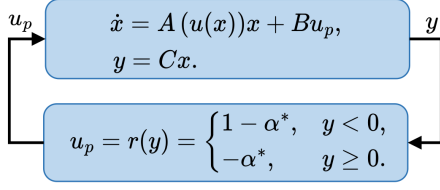


Fig. 1. Representation of system (6), (7) as an interconnection.

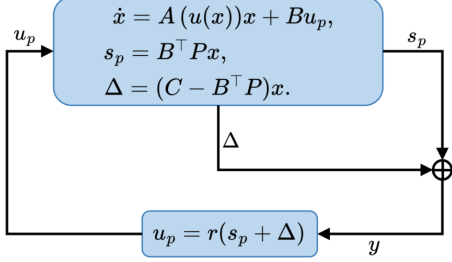


Fig. 2. Representation as a system controlled by a perturbed relay.

with $u(x)$ as in (7) and an artificial input u_p connected to a scalar relay-like function $r(\cdot)$ defined by $r(y) = 1 - \alpha^*$ if $y < 0$, and $r(y) = -\alpha^*$ otherwise. (see Fig. 1). When we are able to show the asymptotic stability of the origin (point **II** in Proposition 1), we use the fact that system (22) is passive (see [8, 21]) w.r.t. the artificial input u_p and the output $y = Cx$ (characterising the switching surface). This is reflected by requiring $B^\top P = (b_1 - b_2)^\top P = C$. In the first case, when studying the forward invariance, the switching surface $y = Cx$ is considered as a perturbed output with respect to a nominal passive output $s_p = B^\top Px$ (see Fig. 2). If the state is sufficiently far from the origin ($V(x) \geq \varepsilon$), then the difference ($\Delta = y - s_p$ in Fig. 2) between the perturbed output $y = Cx$ and the passive output $s_p = B^\top Px$ can be neglected. This is reflected in condition (20) in the proof. More precisely, the forward invariant set \mathcal{S} is found such that the function $g(x, v) = \alpha x^\top Px + 2x^\top (C^\top - PB)v = \alpha x^\top Px + 2(y - s_p)^\top v$ is positive for $x \notin \text{Int}(\mathcal{S})$ and for all inputs $v \in \{v_1, v_2\}$. Note that in this case Property 1 is a technical artifact, since the size of the set \mathcal{S} is computed by considering the system under study as a perturbation to a passive system. Let us remark that in the previous proposition, the matrices A_1, A_2 are required to be Hurwitz and to share a common Lyapunov matrix P . In addition, for asymptotic stability, the passivity condition $B^\top P = (b_1 - b_2)^\top P = C$ is required. In the next sections, we will see that these conditions can be alleviated when considering local stability properties.

3.2 Local Asymptotic Stability of the null equilibrium

As follows, we provide conditions for checking the local asymptotic stability. Before introducing the main result, we provide a useful lemma as follows:

Lemma 2 Consider system (1), Property 1 and the notations $v_1 = 1 - \alpha^*$, $v_2 = -\alpha^*$, $A_0 = \alpha^* A_1 + (1 - \alpha^*) A_2$, $N = A_1 - A_2$, $A(\theta) = A_0 + N\theta$ for $\theta \in \{v_1, v_2\}$, and $B = b_1 - b_2$. Let k be a positive scalar. System (1) can be expressed as the interconnection between a plant

$$G : \begin{cases} \dot{x} = A(u(x))x + Bu_c(y) + B\omega, \\ u_c(y) = -ky, \\ y = Cx, \end{cases} \quad (23)$$

with $u(x) := v_1$ if $Cx < 0$, $u(x) := v_2$ if $Cx \geq 0$, and a nonlinearity $\omega = z(y)$ with

$$z(y) = \begin{cases} ky + v_1, & y < 0, \\ ky + v_2, & y \geq 0. \end{cases} \quad (24)$$

Proof: Consider the function $\sigma(x) := 1$ if $Cx < 0$ and $\sigma(x) := 2$ if $Cx \geq 0$. Using Lemma 1, system (1) can be expressed in the form

$$\dot{x} = \mathcal{X}(x) := A(u(x))x + Bv_{\sigma(x)}. \quad (25)$$

Adding $-kBCx + kBCx$ to the right side of system (25), we have

$$\dot{x} = (A(u(x)) - kBC)x + kBCx + Bv_{\sigma(x)}. \quad (26)$$

Then (26) can be further rewritten as follows

$$\dot{x} = A(u(x))x + Bu_c(Cx) + Bz(Cx), \quad (27)$$

where $z(Cx) = kCx + v_{\sigma(x)}$. ■

Remark 5 In Lemma 2 the perturbation $z(y)$ represents the difference between the relay-like function $r(\cdot)$ defined by $r(y) = 1 - \alpha^*$ if $y < 0$, and $r(y) = -\alpha^*$ otherwise, and a linear static output feedback $u_c(y) = -ky$, that is $z(y) = r(y) - u_c(y)$. Taking into account the properties of this perturbation $\omega = z(y)$ allows us to derive conditions for checking local stability of system (1).

Lemma 3 Consider a function $z : \mathbb{R} \rightarrow \mathbb{R}$ defined as in (24) where $k > 0$, $v_1 > 0$, $v_2 < 0$. Then we have $z(y) < 0$ if $y \in (0, \frac{-v_2}{k})$ and $z(y) > 0$ if $y \in (-\frac{v_1}{k}, 0)$.

Proof. The proof uses simple arguments and is omitted. ■

Lemma 3 shows that the function z presents a local anti-passivity property. More precisely, when $|y| < \min\{|\frac{-v_2}{k}|, |\frac{-v_1}{k}|\}$ and $y \neq 0$, we have $z(y)y < 0$. In the following proposition, we use Lemmas 2 and 3 to provide conditions for checking the local asymptotic stability.

Proposition 2 Consider system (1) and Property 1. Assume that there exist $k > 0$ and $P^\top = P \succ 0$ satisfying

$$(A_i - k(b_1 - b_2)C)^\top P + P(A_i - k(b_1 - b_2)C) \prec 0, \quad i \in \{1, 2\}, \quad (28)$$

$$(b_1 - b_2)^\top P = C. \quad (29)$$

Then, system (1) is locally asymptotically stable at the origin. An estimation of the domain of attraction is given by $\mathcal{D}_A = \{x \in \mathbb{R}^n : x^\top P x \leq \gamma\}$, where $\gamma < \min_{v \in \{v_1, v_2\}} \left(\frac{v}{k}\right)^2 \frac{1}{CP^{-1}C^\top}$.

Proof. Based on Lemma 2, the piecewise affine system (1) can be written as the interconnection between the plant G in (23) and a perturbation $\omega = z(y)$ defined in (24) (see also Fig. 3). Considering this model and the particular form of the perturbation $\omega = z(y)$, we are able to provide local stability conditions. In Step 1, we consider a Lyapunov function $V(x) = x^\top P x$ and we compute a level-set γ around the origin such that $x^\top P x \leq \gamma$ implies $z(y)y < 0$ when $y \neq 0$. In Step 3, the Lyapunov function V is shown to be decreasing along the directions of the Filippov inclusion computed in Step 2. This is done by using a passivity relation between the plant G and the interconnection ω .

Step 1: Estimation of the domain of attraction.

We show that for $\gamma < \min_{v \in \{v_1, v_2\}} \left(\frac{v}{k}\right)^2 \frac{1}{CP^{-1}C^\top}$, $x^\top P x \leq \gamma$ implies $y = Cx \in (-\frac{v_1}{k}, -\frac{v_2}{k})$. Let us denote $\mu := \min_{v \in \{v_1, v_2\}} \left\{ \left(\frac{v}{k}\right)^2 \right\}$. From $\gamma < \frac{\mu}{CP^{-1}C^\top}$, we have $\mu - C(P/\gamma)^{-1}C^\top > 0$. Using the Schur complement, we have $\begin{bmatrix} \mu & C \\ C^\top & P/\gamma \end{bmatrix} \succ 0$. Using again the Schur complement $P/\gamma - C^\top C/\mu \succ 0$, that is $\frac{P}{\gamma} \succ \frac{C^\top C}{\mu}$. Then, if $x^\top \frac{P}{\gamma} x \leq 1$, we have $\frac{x^\top C^\top C x}{\mu} < 1$, that is $x^\top C^\top C x < \mu$. This means that when $x^\top P x \leq \gamma$, we have $|Cx| < \sqrt{\mu}$, which proves the statement.

Step 2: Filippov differential inclusion. Consider system (23), the nonlinearity $z(y)$ in (24), the function $\sigma(x) := 1$ if $Cx < 0$ and $\sigma(x) := 2$ otherwise, and the notation $\tilde{A}_i := A_i - kBC$, $i \in \{1, 2\}$. Using simple manipulations the closed system (23), (24) can be expressed as

$$\dot{x} = \mathcal{X}(x) := \tilde{A}_{\sigma(x)} x + Bz(Cx). \quad (30)$$

The set valued map of the differential inclusion associ-

ated with (23), (24) is given by

$$\mathcal{F}[\mathcal{X}](x) = \begin{cases} \left\{ \begin{array}{l} \tilde{A}_1 x + Bz(Cx) \\ \tilde{A}_2 x + Bz(Cx) \end{array} \right\}, & \text{if } Cx < 0, \\ \left\{ \begin{array}{l} \tilde{A}_1 x + Bv_1 \\ \tilde{A}_2 x + Bv_2 \end{array} \right\}, & \text{if } Cx = 0. \end{cases} \quad (31)$$

Step 3: Lyapunov analysis. Consider the candidate Lyapunov function $V(x) = x^\top P x$. We want to show that

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < 0, \quad \forall x \in \mathcal{D}_A \setminus \{0\}. \quad (32)$$

Define $\mathcal{D}_1 = \{x \in \mathbb{R}^n : x^\top P x \leq \gamma, Cx \neq 0\}$ and $\mathcal{D}_0 = \{x \in \mathbb{R}^n : x^\top P x \leq \gamma, Cx = 0\}$. Then we get the following two cases:

1. If $y = Cx \in (-\frac{v_1}{k}, 0) \cup (0, -\frac{v_2}{k})$, from (31) we obtain

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma = 2x^\top \tilde{A}_{\sigma(x)}^\top P x + 2z^\top(Cx) B^\top P x. \quad (33)$$

According to (29), $2z^\top(Cx) B^\top P x = 2z^\top(Cx) Cx$. Using Lemma 3, we have $z^\top(Cx) Cx = z^\top(y) y < 0$ whenever $Cx \in (-\frac{v_1}{k}, 0) \cup (0, -\frac{v_2}{k})$. From Step 1, $x^\top P x \leq \gamma$ implies $Cx \in (-\frac{v_1}{k}, -\frac{v_2}{k})$. Using (28) and (24), (33), it yields

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < 0, \quad \forall x \in \mathcal{D}_1 \setminus \{0\}. \quad (34)$$

2. If $y = Cx = 0$, from (31), we have

$$\mathcal{F}[\mathcal{X}](x) = \text{Conv} \left\{ \tilde{A}_i x + Bv_i, i \in \{1, 2\} \right\}. \quad (35)$$

Using $2z^\top(Cx) B^\top P x = 2z^\top(Cx) Cx$ and (29), it holds

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma = \max_{i \in \{1, 2\}} \left\{ 2x^\top \tilde{A}_i^\top P x + 2v_i Cx \right\}. \quad (36)$$

Since $Cx = 0$ and $\tilde{A}_i = A_i - k(b_1 - b_2)C$ satisfying (28), we have

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma = \max_{i \in \{1, 2\}} \left\{ 2x^\top \tilde{A}_i^\top P x \right\} < 0 \quad (37)$$

for all $x \in \mathcal{D}_0 \setminus \{0\}$. Since $\mathcal{D}_0 \cup \mathcal{D}_1 = \mathcal{D}_A$, from (34) and (37) the inequality (32) is verified. \blacksquare

Remark 6 Proposition 2 provides simple conditions for checking the local asymptotic stability of the piecewise affine system (1). The result is based on rewriting the

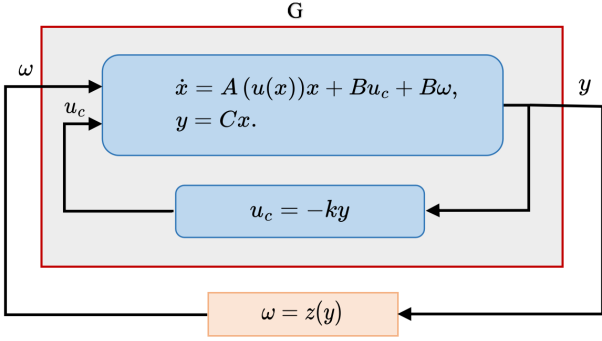


Fig. 3. Representation as a perturbed system.

initial piecewise affine system (1) as an interconnection between a nominal switched linear plant G , and a nonlinear perturbation $z(y)$ (see Lemma 2 and Fig. 3). This allows to take into account the case of non-Hurwitz matrices $A_i, i \in \{1, 2\}$. Here $y = Cx$ (the system switching surface) is seen as an artificial output. In the conditions of Proposition 2, the parameter k can be interpreted as the gain of a linear static output feedback controller ($u_c = -ky$). Condition (28) corresponds to the fact that the system $\dot{x} = A(u(x))x + Bu_c$ with $B = b_1 - b_2$ is stabilized by the static output feedback $u_c = -ky$. Condition (29) can be interpreted as a passivity condition [8, 21]. On the other hand, Lemma 3 is used to characterize the local anti-passivity property of the nonlinearity z . Using these two passivity-like properties, we are able to show the local asymptotic stability of the interconnection. In summary, the proposition shows that system (1) is locally asymptotically stable if one can find a stabilizing output feedback $u_c = -ky$ for the piecewise linear system G in (23) such that the closed-loop system is passive with respect to the output $y = Cx$ (the switching surface) and the input ω . When $A_1 = A_2 = A$, (28) and (29) are equivalent to the fact that $C(b_1 - b_2) > 0$ and the zero dynamics of the system $\dot{x} = Ax + (b_1 - b_2)\omega$, $y = Cx$ is asymptotically stable (see Th. 3.35 in [8]), which is a very simple condition for making an LTI system passive.

Remark 7 Let $Q = P^{-1}$. Then, (28)-(29) lead to

$$QA_i^\top + A_iQ - 2k(b_1 - b_2)(b_1 - b_2)^\top \prec 0, \quad i \in \{1, 2\}, \quad (38)$$

$$(b_1 - b_2)^\top = CQ. \quad (39)$$

Therefore, conditions (28)-(29) can be transformed into simple matrix constraints that can be checked numerically using convex optimization procedures with variables $Q \succ 0$ and $k > 0$. The estimation of the domain of attraction is given with $\gamma = \min_{v \in \{v_1, v_2\}} \left(\frac{v}{k}\right)^2 \frac{1}{CQC^\top}$. We may remark that minimizing k (the output feedback gain in system (23)) such that (38) and (39) hold, allows to enlarge the estimate of the domain of attraction.

3.3 Locally attractive forward invariant sets

Using the ideas in Proposition 1 and 2, we now provide sufficient conditions allowing to characterize forward invariant sets for system (1) which are locally attractive.

Proposition 3 Consider system (1) and Property 1. Let $v_1 = 1 - \alpha^*$ and $v_2 = -\alpha^*$. Assume that there exist $k, \alpha > 0$ and $P^\top = P \succ 0$ satisfying

$$\begin{aligned} & (A_i - k(b_1 - b_2)C)^\top P + P(A_i - k(b_1 - b_2)C) \\ & \prec -\alpha P, i \in \{1, 2\}, \end{aligned} \quad (40)$$

Let $\gamma = \min_{v \in \{v_1, v_2\}} \left(\frac{v}{k}\right)^2 \frac{1}{CP^{-1}C^\top}$ and

$$\varepsilon > \max_{v \in \{v_1, v_2\}} \rho(v), \quad (41)$$

where $\rho(v) = \frac{4(C - (b_1 - b_2)^\top P)P^{-1}(C^\top - P(b_1 - b_2))v^2}{(\alpha - \delta)^2}$ for some $\delta \in (0, \alpha)$. If $\varepsilon < \gamma$, then the set $\mathcal{S} := \{x \in \mathbb{R}^n : x^\top Px \leq \varepsilon\}$ is forward invariant for system (1) and locally attractive. An estimate of the domain of attraction is given by $\mathcal{D}_A = \{x \in \mathbb{R}^n : x^\top Px \leq \gamma\}$.

Proof: Consider the function $V(x) = x^\top Px$ and the set valued map (31). We want to show that $\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -\delta V(x)$, $\forall x \in \mathcal{D}_A \setminus \text{Int}(\mathcal{S})$. Using similar arguments as in Step 1 of Proposition 2, $x \in \mathcal{D}_A$ implies $Cx \in (-\frac{v_1}{k}, -\frac{v_2}{k})$. According to (31), we get the following two cases:

1. If $y = Cx \in (-\frac{v_1}{k}, 0) \cup (0, -\frac{v_2}{k})$, from (31) we obtain

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma &= 2x^\top \tilde{A}_{\sigma(x)}^\top Px + 2z^\top(Cx)B^\top Px \\ &+ 2z^\top(Cx)(C - B^\top P)x - 2z^\top(Cx)(C - B^\top P)x. \end{aligned}$$

Using Lemma 3, we have $2z^\top(Cx)Cx = 2z^\top(y)y < 0$ whenever $Cx \in (-\frac{v_1}{k}, 0) \cup (0, -\frac{v_2}{k})$. By using condition (40), it yields

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma &= 2x^\top \tilde{A}_{\sigma(x)}^\top Px + 2z^\top(Cx)Cx \\ &- 2z^\top(Cx)(C - B^\top P)x \\ &< -2z^\top(Cx)(C - B^\top P)x - \alpha x^\top Px. \end{aligned} \quad (42)$$

From the definition of z , that $z(Cx) \in [v_2, v_1]$ when $Cx \in (-\frac{v_1}{k}, -\frac{v_2}{k})$. Therefore,

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < \max_{v \in \{v_1, v_2\}} \{-2v(C - B^\top P)x\} - \alpha x^\top Px \quad (43)$$

2. If $y = Cx = 0$, from (31), we have

$$\mathcal{F}[\mathcal{X}](x) = \text{Conv} \left\{ \tilde{A}_i x + Bv_i, i \in \{1, 2\} \right\}. \quad (44)$$

Then, we have

$$\begin{aligned} \sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma &= \sup_{i \in \{1, 2\}} \left\{ 2x^\top \tilde{A}_i^\top Px + 2v_i B^\top Px \right\} \\ &= \sup_{i \in \{1, 2\}} \left\{ 2x^\top \tilde{A}_i^\top Px + 2v_i B^\top Px + 2v_i (C - B^\top P)x \right. \\ &\quad \left. - 2v_i (C - B^\top P)x \right\} \\ &= \sup_{i \in \{1, 2\}} \left\{ 2x^\top \tilde{A}_i^\top Px + 2v_i Cx - 2v_i (C - B^\top P)x \right\}. \end{aligned}$$

Since $Cx = 0$ and (40) holds for $i \in \{1, 2\}$, we have

$$\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < \max_{i \in \{1, 2\}} \left\{ -2v_i (C - B^\top P)x \right\} - \alpha x^\top Px. \quad (45)$$

Using the same arguments as in Proposition 1, when $x^\top Px \geq \varepsilon$, we get

$$(\alpha - \delta)x^\top Px + 2x^\top (C^\top - PB)v > 0, \quad (46)$$

for all $v \in \{v_1, v_2\}$. Using (46), together with (43) and (45), we have $\sup_{\varsigma \in \mathcal{F}[\mathcal{X}](x)} \frac{\partial V}{\partial x} \varsigma < -\delta V(x), \forall x \in \mathcal{D}_A \setminus \text{Int}(\mathcal{S})$.

Remark 8 Proposition 3 provides conditions for characterizing forward invariant sets \mathcal{S} which are locally attractive for (1). An estimate of the domain of attraction is characterized by \mathcal{D}_A . The sets \mathcal{S} and \mathcal{D}_A are characterized as the level sets of the Lyapunov function $V(x) = x^\top Px$ where P is a matrix to be found using the conditions of Proposition 3. Let $Q = P^{-1}$. Then, using the Schur complement, (40) and (41) are equivalent to

$$\begin{aligned} QA_i^\top + A_i Q - kQC^\top (b_1 - b_2)^\top - k(b_1 - b_2)CQ \\ < -\alpha Q, i \in \{1, 2\}, \end{aligned} \quad (47)$$

$$\begin{bmatrix} \left(\frac{\alpha - \delta}{2}\right) Q & (QC^\top - b_1 + b_2)v \\ \star & \frac{\alpha - \delta}{2} \varepsilon I \end{bmatrix} \succ 0, v \in \{v_1, v_2\}. \quad (48)$$

Therefore, for given k (static output feedback gain in system (23)), α (decay rate of system (23) with the given gain k) and δ (estimation of the decay rate), the constraints (40)-(41) are transformed into LMI constraints with variables $Q \succ 0$ and $\varepsilon > 0$. Then minimizing ε allows to minimize the forward invariant set \mathcal{S} . The estimation of the domain of attraction is

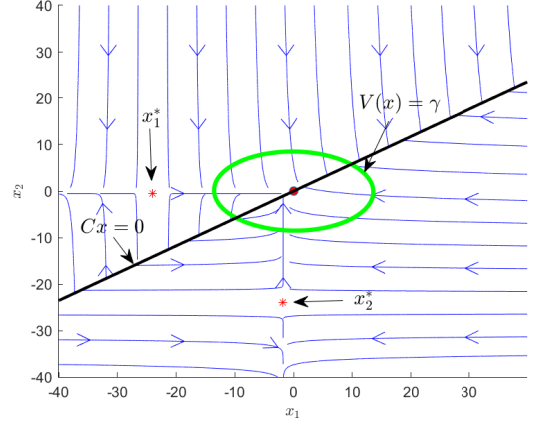


Fig. 4. Phase plot for the system in Example 1.

$\gamma = \min_{v \in \{v_1, v_2\}} \left(\frac{v}{k}\right)^2 \frac{1}{CQC^\top}$. From this expression, we may remark that minimizing k such that (47) and (48) hold, allows to enlarge the estimation of the domain of attraction. Note also that by minimizing k , the allowable decay rate α of the piecewise linear systems (23) is decreasing and therefore the upper bound ε (the level set characterizing the forward invariant set \mathcal{S}) is increasing. Based on the relations between these parameters in (40) and (41), we can imagine various iterative procedures to minimize the estimation of the forward invariant set or to maximize the estimation of the domain of attraction.

Remark 9 In order to take into account oscillating behaviours, the equality condition (29) in Proposition 2 is replaced with the condition (41). This condition takes into account the difference between the parameter C of switching surface and $B^\top P$ (characterizing the passive output $s_p = B^\top Px$). We emphasize that this approach does not require the matrices A_1 and A_2 to be Hurwitz.

4 Numerical examples

In this section, we present some numerical examples to illustrate the criteria we proposed in Section 3.

Example 1. Consider system (1) with $A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0.1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2.4 & -1.6 \end{bmatrix}^\top$, $b_2 = \begin{bmatrix} -3.6 & 2.4 \end{bmatrix}^\top$, and a switching surface described by $C = \begin{bmatrix} 0.5 & -0.85 \end{bmatrix}$. The system phase plane is illustrated in Figure 4. The system presents two unstable equilibrium points $x_1^* = -A_1^{-1}b_1 = [-24, -0.53]^\top$ and $x_2^* = -A_2^{-1}b_2 = [-1.8, -24]^\top$ (marked with red stars in Figure 4). Property 1 holds with $\alpha^* = 0.6$. We apply Proposition 2 to estimate the domain of attraction and to prove the stability of the null equilibrium obtained

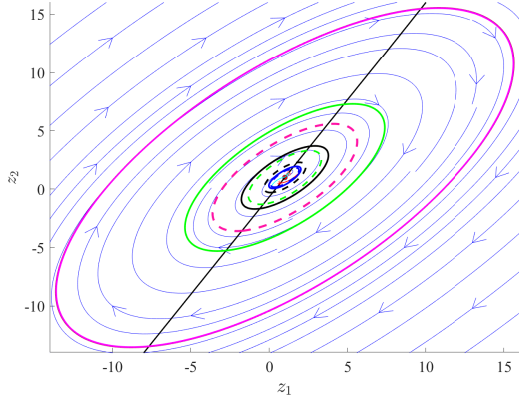


Fig. 5. Phase plot for the system in Example 2 and estimates obtained based on several values of k ($k = 1.3 \times 10^{-3}$ - magenta, $k = 0.003$ - green, $k = 0.007$ - black, $k = 0.02$ - blue; solid line - \mathcal{D}_A , dashed line - \mathcal{S}).

by sliding dynamics. The conditions of Proposition 2 are satisfied with a Lyapunov function characterized by

$$P = \begin{bmatrix} 8.5 & 0.25 \\ \star & 21.6 \end{bmatrix} \times 10^{-2}$$

and $k = 0.04$. These parameters were found by solving jointly the matrix inequalities (38) and (39) in Remark 7 while minimizing k . The domain of attraction (the green ellipsoid in Figure 4) is characterized by $\gamma = 15.6$. From Figure 4 we can see that starting from the domain $\mathcal{D}_A := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$, the trajectories of the switched system converge to the origin. The two red stars (*) in Figure 4 correspond to the two unstable equilibrium points of the two subsystems. The distance between the unstable equilibria and the bound of \mathcal{D}_A illustrates the conservatism of our method.

Example 2. Consider system (3) with the matrices

$$A_1 = \begin{bmatrix} -2 & 3 \\ -3 & 2 \end{bmatrix}, A_2 = 0.95 \cdot A_1, b_1 = \begin{bmatrix} -0.154 & 1 \end{bmatrix}^\top,$$

$b_2 = \begin{bmatrix} -1.514 & 0.95 \end{bmatrix}^\top$, and $C = \begin{bmatrix} 28.28 & -16.97 \end{bmatrix}$, $d = 11.31$. Figures 5 and 6 show the phase plot of the PWA system. Both of the matrices A_1, A_2 have purely complex eigenvalues. The system has two stable equilibria $z_1^* = [0.66, 0.49]^\top$, $z_2^* = [1.24, 1.36]^\top$ and an unstable (Filippov) equilibrium at $z^* = [1, 1]^\top$. According to initial conditions, solutions may reach the switching surface and next oscillate either around z_1^* or z_2^* (sometimes after sliding on the switching surface - see Figure 6). We want to characterize the forward invariant set \mathcal{S} containing both of the oscillating behaviours. In addition, we intend to estimate the domain of attraction \mathcal{D}_A of solutions that converge towards \mathcal{S} with a decay rate of at least of $\delta = 10^{-5}$. Condition (4) is satisfied with

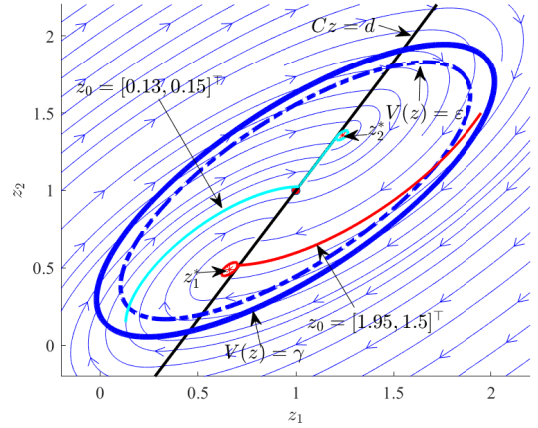


Fig. 6. Zoom on the phase plot for the system in Example 2 and estimates obtained based on $k = 0.02$ and $\alpha = 0.77$. The cyan line is the solution with an initial value $z_0 = [0.13, 0.15]^\top$ and the red line is the solution with an initial value $z_0 = [1.95, 1.5]^\top$. z_1^*, z_2^* are the equilibriums of the two subsystems.

$\alpha^* = 0.4$ and $z^* = [1, 1]^\top$. Based on the condition (5), we obtain a system of the form (1) satisfying Property 1. Condition (40)-(41) in Proposition 3 were tested for various values of k (the gain of the linear static output feedback controller $u_c = -ky$ for the transformed system (23)). For $k = 1.3 \times 10^{-3}$ and $\alpha = 0.05$, we obtain $\varepsilon = 237.11$ characterizing a forward invariant set and $\gamma = 2.35 \times 10^3$ characterizing the domain of attraction based on a Lyapunov function with

$$P = \begin{bmatrix} 20.1 & -13.57 \\ \star & 20.31 \end{bmatrix}.$$

The domain of attraction and the forward invariant set are represented in magenta in Figure 5 (solid and dashed lines, respectively). Choosing a stronger decay rate $\alpha = 0.77$ (based on $k = 0.02$), we have $\varepsilon = 7.46, \gamma = 9.66$ with

$$P = \begin{bmatrix} 20.83 & -16.71 \\ \star & 24.21 \end{bmatrix}.$$

A zoom of the phase plot is illustrated in Figure 6. As expected, imposing a stronger gain k (stronger decay rate α for the of piecewise linear systems) allows a tighter estimates of the forward invariant set. However, it leads to a smaller estimated of the domain of attraction (to be compared with the other estimates provided in Figure 5). Patching together the different estimates allows to get an idea about the actual domain of attraction and forward invariant set.

5 Conclusion

In this paper, we proposed methods for analyzing the stability properties of self-oscillating solutions for

a class of PWA systems with two subsystems and linear switching surfaces. Conditions were provided to characterise attractive forward invariant sets and domains of attraction. For some particular cases, we have given, as a side result, new methods for checking the local asymptotic stability of the origin. The approach provides some analytical insights on the properties of PWA systems. The proposed conditions are based on passivity and output stabilizability of a relay-like system with respect to an artificial output corresponding to the switching surface. In the future, we plan to extend our methods to PWA systems with several subsystems. In addition, since the approach is essentially based on quadratic Lyapunov functions, it would be interesting to investigate the extension to more general classes of Lyapunov functions. This perspective might be interesting for reducing the conservatism and for enlarging the estimates of the domains of attraction.

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