Jump Conditions for Boussinesq Equations Due to an Abrupt Depth Transition

Eduardo Monsalve, Kim Pham, Agnès Maurel

JUMP CONDITIONS FOR BOUSSINESQ EQUATIONS DUE TO AN ABRUPT DEPTH TRANSITION*

EDUARDO MONSALVE †, KIM PHAM †, AND AGNÈS MAUREL ‡

Abstract. We revisit the problem of nonlinear water wave propagation in the presence of an abrupt depth transition. To this end, we use an asymptotic approach conducted to order 3 with respect to the shallowness parameter, in order to capture the first nonlinear and dispersive contributions. However, the discontinuity of bathymetry, as opposed to slowly varying bathymetry, requires the use of a consistent three-scale analysis framework and the consideration of different regions, far from the step and free surface, near the free surface and near the step. This framework enables consistent navigation, ultimately providing Boussinesq equations supplemented by jump conditions at the depth discontinuity that encompass the effect of step on wave propagation.

Key words. non-linear water waves; depth transition; bathymetry; Boussinesq equations; jump conditions; asymptotic techniques; matched asymptotic expansions; boundary layers

AMS subject classifications. 34E13, 35B27, 35L05, 76B15

1. Introduction.

The origin of large waves, so-called rogue or freak waves, and the reasons for their appearance has been studied for many years [18, 12, 2, 3, 45, 1]. In the case of shallow or finite water depth, the presence of depth transitions has been shown to play a role in the formation and dynamics of these extreme waves [19, 52, 13, 58, 57, 59]. In many previous studies, the case of a smooth bathymetry, with a continuous depth transition over a non-zero distance L, has been considered [52, 13, 58, 51, 57, 20, 21, 22]. The influence of slope has been considered in [13], [60] and [20] and studied more systematically in [40] who show that higher slopes foster the appearance of rogue waves up to a critical value beyond which its effect is saturated. These results suggested the interest of considering waves passing an abrupt transition ($L = 0$, see figure 1), *i.e.* a step, as this avoids discussions about the transition region (its shape and value of the distance L) and provides a reference limiting case. After the pionneering work of [14] this problem has recently been addressed in a series of papers by Li and co-workers [23, 24, 25, 10, 26, 27] where the authors examine on the generation of free and bound waves, which are waves corresponding to harmonics generated by non-linearities, and show the crucial influence of these second order effects in the occurrence of extreme events, see also [42, 31, 32, 60, 11, 43, 50, 56].

The interest in this limiting case is also of a fundamental nature, as it poses specific problems, both numerically and theoretically. While linear wave propagation in the presence of an abrupt depth transition is simple and well known, the extension to non-linear (even weakly) propagation remains in part an open subject. Let us start with numerical methods; among them, the simplest semi-analytical multimodal analysis easily handles, in the linear case, the reflected and transmitted fields accounting for the evanescent field triggered at the depth transition, see *e.g.* [44] and [41]. A classical reference to the extension of this method for weakly non-linear propagation is

*Submitted to the editors DATE.

Funding: K.P. and E.M. acknowledge support from the Agence de l'Innovation de Défense (AID) from the Direction Générale de l'Armement (DGA), under grant no. 2019 65 0070 and 2019 65 0042, and the Agence Nationale de la Recherche under grant ANR-19-CE08-0006. A.M. acknowledges the support of the Agence Nationale de la Recherche (ANR) under grant 243560 CoProMM.

†LMI, ENSTA Paris, Institut Polytechnique de Paris, 91120, Palaiseau, France, (eduardo.monsalve@ensta-paris.fr, kim.pham@ensta-paris.fr)

‡ESPCI Paris, PSL University, CNRS, Institut Langevin, France, (agnes.maurel@espci.fr)

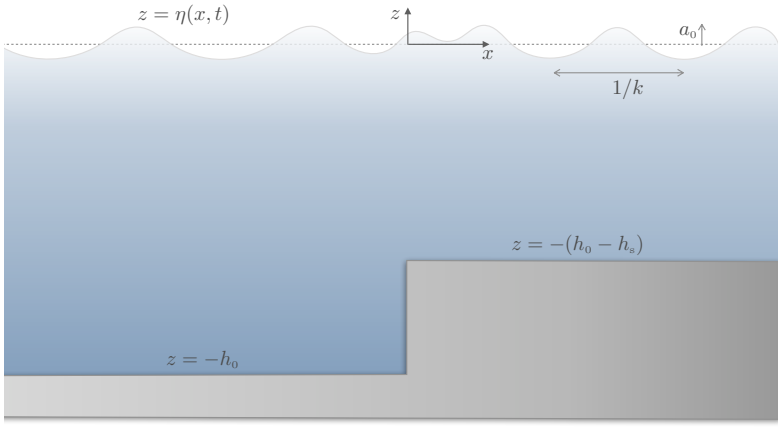

FIG. 1. *Non-linear propagation over an abrupt depth transition between two regions of constant depth* h_0 *and* $(h_0 - h_s)$.

that of [35, 34], see also [5]; the method relies on the use of a small parameter, namely the ratio between wave amplitude and wavelength, and on an iterative process thanks to asymptotic expansion of the solution. However, this model suffers from inherent divergences due to evanescent terms in the second oder solution, and this has been circumvented at the cost of ad-hoc procedures, using specific truncations [42, 43] or neglecting these nonlinear evanescent contributions [47, 25]. From a theoretical perspective, which is the focus of the present study, asymptotic techniques in the linear regime are a powerful tool for reducing the problem of water wave propagation in the presence of a scattering region of dimension smaller than the wavelength. This technique has a long history, which began in the 1960s [4, 53, 54, 55, 49], see also [33], [38], [16] and [39], Ch. 4 and 5. In shallow or finite-depth waters, the procedure consists in eliminating the z-vertical dependence of the solution and encapsulating the effect of the (small) scatterer in effective jump conditions. This dimensional reduction has been extended to linear [36, 37] and nonlinear [48] propagation to determine the evolution of surface waves propagating over periodic bottom irregularities. The present study aims to use such asymptotic procedure conducted at high order to recover a Boussinesq-type equation, with dispersion and nonlinearity, that is classical [28, 30, 9], and to supplement this equation with effective jump conditions that encapsulate the effect of rapid variations in the evanescent field in the vicinity of the abrupt depth transition.

The paper is organized as follows. In Section 2, we pose the direct problem of water wave propagation in the presence of an abrupt depth transition in two dimensions and give the result of the study *i.e.* the effective problem comprising the Boussinesq equations and associated jump conditions. The rest of the paper is devoted to the derivation of this effective model, starting, in Section 3, with the derivation of the effective wave equations far from the bathymetry discontinuity. Then, in Section 4, we present the derivation of the jump conditions across the discontinuity, which is the main result of the analysis; we establish the corresponding boundary layer problems which define the effective parameters entering the jumps (explicit expressions for these coefficients, which depend only on the depth ratio, are given in the appendix A). Finally, in Section 5, we combine the results from the previous sections to reconstruct

the Boussinesq equations and the associated jump conditions.

2. Sum-up of the main result: Boussinesq approximation for an abrupt depth transition.

2.1. The actual and effective problems. We begin with the equations governing the actual problem. Using the assumptions of an inviscid and incompressible two-dimensional fluid in irrotational motion, the velocity $\boldsymbol{u}(x,z,t)$ and the associated velocity potential $\varphi(x,z,t)$ satisfy

$$\operatorname{div}\boldsymbol{u}=0, \quad \boldsymbol{u}=\boldsymbol{\nabla}\varphi, \tag{2.1}$$

along with the dynamic and kinematic boundary conditions at the free surface, *i.e.*

$$\frac{\partial\varphi}{\partial t}+\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}+g\eta=0, \qquad u_z=\frac{\partial\eta}{\partial t}+u_x\frac{\partial\eta}{\partial x} \quad \text{at } z=\eta(x,t). \tag{2.2}$$

The depth transition results in a discontinuous bathymetry with water depth $h(x)$ being a piecewise constant function such that

$$h(x<0)=h_0, \qquad h(x>0)=h_0-h_s, \tag{2.3}$$

(with $h_s<h_0$) so the vanishing normal velocity condition applies on all rigid boundaries (noted Γ)

$$\boldsymbol{u}\cdot\mathbf{n}=0 \quad \text{on} \quad \Gamma. \tag{2.4}$$

In relatively shallow waters and in the weakly nonlinear regime, we will establish that the two-dimensional problem (2.1)-(2.4) can be reduced to the classical one-dimensional Boussinesq equations [8, 46] which, in a conservative form, are expressed as follows

$$\begin{cases} \dfrac{\partial}{\partial t}\left(U_x-\dfrac{h^2}{3}\dfrac{\partial^2U_x}{\partial x^2}\right)+\dfrac{\partial}{\partial x}\left(g\eta+\dfrac{U_x^2}{2}\right)=0, \\ \dfrac{\partial\eta}{\partial t}+\dfrac{\partial}{\partial x}\big((h+\eta)U_x\big)=0, \end{cases} \tag{2.5}$$

where the average horizontal velocity field U_x is defined as

$$U_x(x,t)=\frac{1}{h(x)}\int_{-h(x)}^{0}u_x(x,z,t)\,\mathrm{d}z.$$

The Boussinesq equations are only valid for $x\neq 0$. At $x=0$, given the presence of the bathymetry discontinuity, the motion of the free surface η and the velocity U_x lose their regularity. The main result of this paper is to establish the jump conditions that the fluxes in the Boussinesq equations (2.5) must satisfy at $x=0$, specifically

$$\begin{cases} \left[\!\left[g\eta+\dfrac{U_x^2}{2}\right]\!\right]=-\dfrac{\partial}{\partial t}\left(\mathcal{B}_1\overline{hU_x}+h_0\mathcal{B}_2\dfrac{\partial\overline{\eta}}{\partial t}+\dfrac{h_0\mathcal{B}_3}{g}\dfrac{\partial^2\overline{hU_x}}{\partial t^2}\right), \\ [\![(h+\eta)U_x]\!]=\dfrac{1}{g}\dfrac{\partial^2}{\partial t^2}\left(h_0\mathcal{C}_1\overline{hU_x}+h_0^2\mathcal{C}_2\dfrac{\partial\overline{\eta}}{\partial t}\right), \end{cases} \tag{2.6}$$

with the jump $[\![f]\!]$ and the average $\overline{f}$ of a quantity $f(x,t)$ at $x=0$ defined as follows

$$[\![f]\!](t)=f(0^+,t)-f(0^-,t), \qquad \overline{f}(t)=\frac{f(0^+,t)+f(0^-,t)}{2}. \tag{2.7}$$

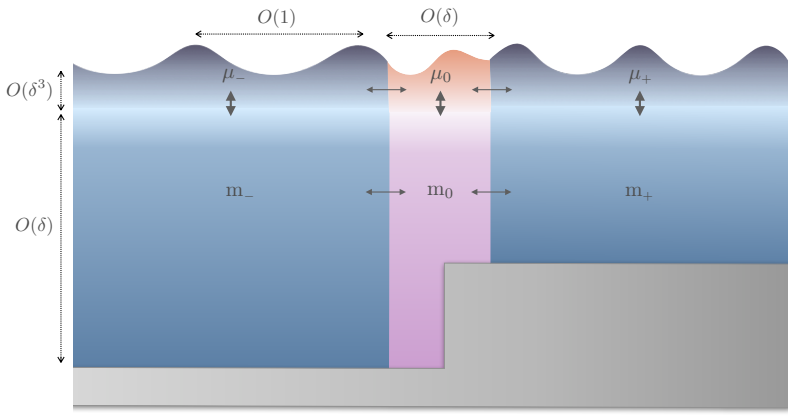

FIG. 2. *The different domains in which the analysis is performed. The $\mu_\pm$ and m_0 regions correspond to boundary layers that aim to disappear; in the $m_\pm$ regions, a reduction along the vertical direction is performed. Solutions in connected regions are matched vertically (large arrows) or horizontally (small arrows).*

In (2.6), $(\mathcal{B}_i)$ and $(\mathcal{C}_i)$ are non-dimensional boundary layer coefficients, whose values depend only on h_s/h_0, see Appendix A. It is worth noting that the quantities $(g\eta+\frac{U_x^2}{2})$ and $(h+\eta)U_x$ are continuous elsewhere, specifically for $x \neq 0$. We observe that these jumps are overlooked in several studies investigating the impact of a step on the propagation of shock waves [17] or solitons [31, 32]. From a physical standpoint, it can be argued that these jumps, arising from evanescent fields triggered in the vicinity of the step, may have a weak effect under certain conditions. However, the derivation of the shallow water equations in these references stems from an asymptotic analysis conducted to high orders, aimed at capturing non-linear and dispersive contributions. Accordingly, neglecting the contribution of these higher order terms in the jump conditions raises mathematical concerns, potentially impacting the stability of the resulting models.

2.2. Summary of the asymptotic procedure. To capture the first contributions due to non-linearities and dispersion, we use classical scalings for characteristic wavelength $1/k$, water depth h_0 and wave amplitude a_0. Specifically, with

$$\varepsilon = \frac{a_0}{h_0} \ll 1, \quad \delta = kh_0 \ll 1, \tag{2.8}$$

the non-linearity and shallowness parameters, we consider

$$\varepsilon = \alpha\delta^2, \qquad \text{with } \alpha = O(1). \tag{2.9}$$

In the following sections, we will conduct the asymptotic analysis in the different regions shown in figure 2 and, for simplicity, we will consider the non-dimensional form of (2.1)-(2.4). Introducing $\omega = \sqrt{gh_0}\,k$, we use

$$t \to \omega t, \quad x \to kx, \quad z \to z_{\rm m} = z/h_0, \quad \eta \to \eta/a_0, \quad \boldsymbol{u} \to \boldsymbol{u}/(\omega a_0), \quad \varphi \to k\varphi/(\omega a_0), \tag{2.10}$$

which provide the non-dimensional form of (2.1) (incompressibility Inc, and irrotationality Rot)

$$(\mathsf{Inc}): \ \frac{\partial u_x}{\partial x} + \frac{1}{\delta}\frac{\partial u_z}{\partial z_{\rm m}} = 0, \qquad (\mathsf{Rot}): \ u_x = \frac{\partial \varphi}{\partial x}, \quad u_z = \frac{1}{\delta}\frac{\partial \varphi}{\partial z_{\rm m}}, \tag{2.11}$$

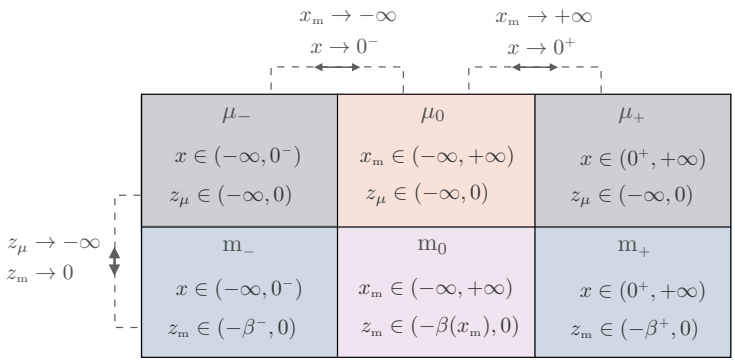

FIG. 3. *Diagram summarizing the asymptotic procedure: in each region of figure 2, the problem is set in proper coordinate (see (2.13)) in order that the rescaled geometry is independent of δ. The solutions of these problems are then matched asymptotically.*

and (2.2) and (2.4) (dynamic DC and kinematic KC conditions at the free surface and vanishing normal velocity RC on the rigid walls formed by the seabed at $z_{\mathrm{m}} = -\beta_\pm = -h/h_0$ and the vertical wall of the step)

$$(2.12)\qquad \mathsf{(DC)}:\ \frac{\partial \varphi}{\partial t} + \varepsilon\delta\,\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2} + \delta^{-1}\eta = 0, \mathsf{(KC)}:\ u_z = \frac{\partial \eta}{\partial t} + \varepsilon\delta\, u_x \frac{\partial \eta}{\partial x},\ \text{at } z_{\mathrm{m}} = \varepsilon\eta(x,t),$$
$$\mathsf{(RC)}:\ \boldsymbol{u}\cdot\mathbf{n} = 0 \quad \text{on the walls,}$$

The above equations explicitly account for field variations horizontally on the wavelength scale (via x) and vertically on the depth scale (via z_{m}); this is appropriate in $\mathrm{m}_\pm$ *i.e.* far from the step and the free surface. Near the step, however, the fields vary horizontally on the depth scale; near the free surface, they vary vertically on the wave amplitude scale. Based on these considerations, we introduce rescaled coordinates

$$(2.13)\qquad \text{macro}: x, \qquad \text{meso}: x_{\mathrm{m}} = \frac{x}{\delta}, \quad z_{\mathrm{m}}, \qquad \text{micro}: z_\mu = \frac{z_{\mathrm{m}} - \varepsilon\eta}{\varepsilon\eta},$$

and in the final model, only the x spatial coordinate will be retained. In each region, the procedure consists in using the appropriate spatial coordinate so that the geometry does not depend on δ or ε (it is $O(1)$ in both directions) and the dependence on the small parameter appears explicitly in a new system of equations. The different regions are then connected by asymptotically matching their solutions; this procedure is summarized in figure 3.

In the linear regime ($\mu_\pm$ and μ_0 do not exist), the horizontal matching between the regions $\mathrm{m}_\pm$ and m_0 is classical and makes it possible to reduce the effect of the *boundary layer* in m_0 to jump conditions at $x = 0^\pm$ [55]. In the non-linear regime, boundary layers in $\mu_\pm$ and μ_0 must be taken into account due to the small but non-zero amplitude of the wave and are treated in the same way, namely a horizontal matching between $\mu_\pm$ and μ_0 and a vertical matching between $\mu_\pm$ and $\mathrm{m}_\pm$. We stress that for a flat or smoothly varying bathymetry, m_0 does not exist since it is implicitly assumed that the bathymetry varies at the wavelength scale x; whereas strictly speaking, the asymptotic analysis should be conducted by considering the boundary layer region μ with $z_\mu \in (-\infty, 0)$ and the region m with $z_{\mathrm{m}} \in (-\beta(x), 0)$ (with a single vertical matching between them), the same result is obtained, namely

a Boussinesq-type equation, considering a single region with $z_{\mathrm{m}} \in (-\beta(x), \varepsilon\eta(x,t))$ [46, 6].

3. Asymptotic analysis in $\mathrm{m}_\pm$ and $\mu_\pm$, far from the depth transition. In this section, we derive the Boussinesq equations in their classical form due to [46] in the two constant depth regions far from depth transition (the step). The procedure is the same on both sides of the step, and we will consider μ_+, m_+.

3.1. Setting of the asymptotic procedure.

3.1.1. The mesoscopic in m_+ and the microscopic in μ_+ problems. In the *mesoscopic region* m_+, the problem is set in (x, z_{m}) coordinates, with $x \in (0^+, +\infty)$, $z_{\mathrm{m}} \in (-\beta_+, 0)$, and the governing equations are, using (2.11)-(2.12),

$$
\begin{aligned}
&{}^{\mathrm{m}}(\mathsf{Inc}): \quad \frac{\partial u_x}{\partial x} + \frac{1}{\delta}\frac{\partial u_z}{\partial z_{\mathrm{m}}} = 0, \qquad {}^{\mathrm{m}}(\mathsf{Rot}): \quad u_x = \frac{\partial \varphi}{\partial x}, \quad u_z = \frac{1}{\delta}\frac{\partial \varphi}{\partial z_{\mathrm{m}}},\\
&{}^{\mathrm{m}}(\mathsf{RC}): \quad u_z(x, z_{\mathrm{m}} = -\beta_+, t) = 0,
\end{aligned} \tag{3.1}
$$

(at this stage, boundary conditions at $z_{\mathrm{m}} = 0$ are missing). The solutions of (3.1) are expanded in terms of series of the small parameter δ

$$
u_x = \sum_{n\geq 0} \delta^n u_x^{(n)}(x, z_{\mathrm{m}}, t), \quad u_z = \delta \sum_{n\geq 0} \delta^n u_z^{(n)}(x, z_{\mathrm{m}}, t), \quad \varphi = \sum_{n\geq 0} \delta^n \varphi^{(n)}(x, z_{\mathrm{m}}, t). \tag{3.2}
$$

where we anticipate that $u_x = O(1)$ imposes $u_z = O(\delta)$ from ${}^{\mathrm{m}}(\mathsf{Inc})$ in (3.1).

In the *microscopic region* μ_+, the problem is set in (x, z_μ) coordinate, with $x \in (0^+, +\infty)$, $z_\mu \in (-\infty, 0)$, and with z_μ defined in (2.13) (the free surface is now fixed at $z_\mu = 0$). Consequently, the governing equations are, from (2.11)-(2.12) and using (2.13) with $\varepsilon = \alpha\delta^2$,

$$
\begin{aligned}
&{}^{\mu}(\mathsf{Inc}): \ \eta\frac{\partial u_x}{\partial x} + \frac{1}{\alpha\delta^3}\frac{\partial u_z}{\partial z_\mu} = 0, \qquad {}^{\mu}(\mathsf{Rot}): \ u_x = \frac{\partial \varphi}{\partial x}, \quad \eta u_z = \frac{1}{\alpha\delta^3}\frac{\partial \varphi}{\partial z_\mu},\\
&{}^{\mu}(\mathsf{DC}): \ \frac{\partial \varphi}{\partial t} + \alpha\delta^3\,\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2} + \frac{1}{\delta}\eta = 0, \quad {}^{\mu}(\mathsf{KC}): \ u_z = \frac{\partial \eta}{\partial t} + \alpha\delta^3\, u_x\frac{\partial \eta}{\partial x} \quad \text{at } z_\mu = 0.
\end{aligned} \tag{3.3}
$$

(at this stage, boundary conditions are missing when $z_\mu \to -\infty$). The solutions of (3.3) are expanded in terms of series of the small parameter δ,

$$
\begin{aligned}
&u_x = \sum_{n\geq 0} \delta^n v_x^{(n)}(x, z_\mu, t), \quad u_z = \delta \sum_{n\geq 0} \delta^n v_z^{(n)}(x, z_\mu, t), \quad \varphi = \sum_{n\geq 0} \delta^n \psi^{(n)}(x, z_\mu, t),\\
&\eta = \delta \sum_{n\geq 0} \delta^n \eta^{(n)}(x, t),
\end{aligned} \tag{3.4}
$$

where we anticipate that $u_x, \varphi = O(1)$ imposes $u_z, \eta = O(\delta)$, according to ${}^{\mu}(\mathsf{DC})$-${}^{\mu}(\mathsf{KC})$ in (3.3).

3.1.2. The hierarchies of equations. In m_+, by injecting (3.2) into (3.1), we obtain a hierarchy of equations, namely $\forall\, n$

$$
\begin{aligned}
&{}^{\mathrm{m}}(\mathsf{Inc})^n: \quad \frac{\partial u_x^{(n)}}{\partial x} + \frac{\partial u_z^{(n)}}{\partial z_{\mathrm{m}}} = 0,\\
&{}^{\mathrm{m}}(\mathsf{Rot})_x^n: \quad u_x^{(n)} = \frac{\partial \varphi^{(n)}}{\partial x}, \quad {}^{\mathrm{m}}(\mathsf{Rot})_z^n: \quad u_z^{(n)} = \frac{\partial \varphi^{(n+2)}}{\partial z_{\mathrm{m}}},\\
&{}^{\mathrm{m}}(\mathsf{RC})^n: \quad u_z^{(n)}(x, z_{\mathrm{m}} = -\beta_+, t) = 0,
\end{aligned} \tag{3.5}
$$

with by convention $u_z^{(n)} = 0$ for $n < 0$. In μ_+, by injecting (3.4) into (3.3), we obtain

$$
\begin{aligned}
&{}^{\mu}(\mathsf{Inc})^n: \quad \frac{\partial v_z^{(n)}}{\partial z_\mu} = 0, \qquad n \le 2, \qquad \frac{\partial v_z^{(3)}}{\partial z_\mu} = -\alpha\,\eta^{(0)}\frac{\partial v_x^{(0)}}{\partial x},\\
&{}^{\mu}(\mathsf{Rot})_x^n: \quad v_x^{(n)} = \frac{\partial \psi^{(n)}}{\partial x}, \qquad n \ge 0, \qquad {}^{\mu}(\mathsf{Rot})_z^n: \quad \frac{\partial \psi^{(n)}}{\partial z_\mu} = 0, \quad n \le 4,
\end{aligned}
\tag{3.6}
$$

along with the boundary conditions at the free surface $z_\mu = 0$,

$$
\begin{aligned}
&{}^{\mu}(\mathsf{DC})^n: \quad \frac{\partial \psi^{(n)}}{\partial t} + \eta^{(n)} = 0, \qquad n \le 2, \qquad \frac{\partial \psi^{(3)}}{\partial t} + \frac{\alpha}{2}\big(v_x^{(0)}\big)^2 + \eta^{(3)} = 0,\\
&{}^{\mu}(\mathsf{KC})^n: \quad v_z^{(n)}|_{z_\mu=0} = \frac{\partial \eta^{(n)}}{\partial t}, \qquad n \le 2, \qquad v_z^{(3)} = \frac{\partial \eta^{(3)}}{\partial t} + \alpha\, v_x^{(0)}\frac{\partial \eta^{(0)}}{\partial x}.
\end{aligned}
\tag{3.7}
$$

3.1.3. Vertical matching conditions between $\mathbf{m}_+$ and μ_+. Solutions in m_+ and μ_+ must match in an intermediate region when $z_\mathrm{m} \to 0$ and $z_\mu \to -\infty$ (figure 4). Specifically, for φ (for example), we need to have

$$
\varphi^{(0)}(x, z_\mathrm{m}, t) + \delta\varphi^{(1)}(x, z_\mathrm{m}, t) + \cdots \sim \psi^{(0)}(x, z_\mu, t) + \delta\psi^{(1)}(x, z_\mu, t) + \cdots,
\tag{3.8}
$$

Taking into account (2.13) and using Taylor expansion

$$
f(x, z_\mathrm{m}, t) = f(x, 0, t) + \alpha\delta^2(z_\mu + 1)\eta(x, t)\frac{\partial f}{\partial z_\mathrm{m}}(x, 0, t) + \cdots
\tag{3.9}
$$

we identify in (3.8) the terms of the same power of δ, which gives

$$
\begin{cases}
\varphi^{(n)}(x, 0, t) = \lim\limits_{z_\mu \to -\infty} \psi^{(n)}(x, z_\mu, t), \qquad n \le 2,\\
\varphi^{(3)}(x, 0, t) = \lim\limits_{z_\mu \to -\infty} \left(\psi^{(3)}(x, z_\mu, t) - \alpha(z_\mu + 1)\eta^{(0)}(x, t)\dfrac{\partial \varphi^{(0)}}{\partial z_\mathrm{m}}(x, 0, t)\right).
\end{cases}
\tag{3.10}
$$

The same relations are obtained by replacing $\varphi^{(n)}$ by $u_x^{(n)}$ or $u_z^{(n)}$ and $\psi^{(n)}$ by $v_x^{(n)}$ or $v_z^{(n)}$.

3.2. Effective wave equation at order $n = 0, 1$. At the first two orders, the effective wave equation is linear and non-dispersive.

PROPOSITION 3.1. *The effective wave equation reads at the first two orders $n = 0$, 1:*

$$
(\mathsf{EW})_1^n: \quad \frac{\partial U_x^{(n)}}{\partial t} + \frac{\partial \eta^{(n)}}{\partial x} = 0,
\tag{3.11}
$$

$$
(\mathsf{EW})_2^n: \quad \beta\frac{\partial U_x^{(n)}}{\partial x} + \frac{\partial \eta^{(n)}}{\partial t} = 0,
\tag{3.12}
$$

with $\beta = \beta_+$ or β_-.

3.2.1. Derivation of $(\mathsf{EW})_1^n$ for $n = 0, 1$. From ${}^{\mathrm{m}}(\mathsf{Rot})_z^{-2}$ and ${}^{\mathrm{m}}(\mathsf{Rot})_z^{-1}$ in (3.5), $\varphi^{(n)}$ does not depend on z_m for $n = 0, 1$ hence

$$
\varphi^{(n)} = \phi^{(n)}(x, t), \qquad n = 0, 1,
\tag{3.13}
$$

and from ${}^{\mathrm{m}}(\mathsf{Rot})_x^0$ and ${}^{\mathrm{m}}(\mathsf{Rot})_x^1$ in (3.5), we also have

$$
u_x^{(n)} = U_x^{(n)}(x, t) = \frac{\partial \phi^{(n)}}{\partial x}(x, t), \qquad n = 0, 1,
\tag{3.14}
$$

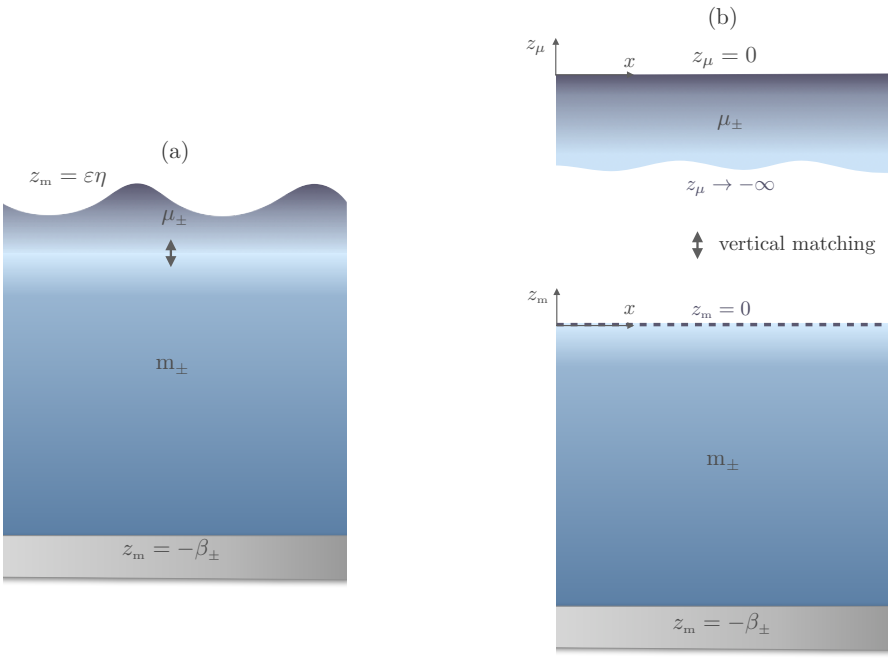

FIG. 4. *(a) A region of constant depth $\beta = \beta_\pm = O(1)$ over which waves of amplitude $O(\varepsilon)$ propagate; (b) Asymptotically, the analysis is performed in the regions $\mu = \mu_\pm$ (in coordinate (x, z_μ)) and $m_\pm$ (in coordinate (x, z_m)) and matching conditions apply when $z_\mu \to -\infty$, $z_m = 0$.*

with $U_x^{(n)}$ the mean velocity field, and the associated flux, defined as

$$\forall\, n, \quad U_x^{(n)}(x,t) = \frac{1}{\beta}\int_{-\beta}^{0} u_x^{(n)}(x, z_{\rm m}, t)\,{\rm d}z_{\rm m}, \quad q^{(n)}(x,t) = \beta U_x^{(n)}(x,t), \tag{3.15}$$

for $\beta = \beta_\pm$. Note that (3.13) is consistent with ${}^{\mu}(\mathsf{Rot})_z^0$ and ${}^{\mu}(\mathsf{Rot})_z^1$ in (3.6), which tell us that, for $n = 0, 1$, $\psi^{(n)}$ does not depend on z_μ, hence $\psi^{(n)} = \phi^{(n)}(x,t)$ from the matching conditions (3.10) and $v_x^{(n)} = U_x^{(n)}$. It remains to use ${}^{\mu}(\mathsf{DC})^n$ in (3.7) which provides

$$\frac{\partial \phi^{(n)}}{\partial t}(x,t) + \eta^{(n)}(x,t) = 0, \qquad n = 0, 1, \tag{3.16}$$

which we differentiate with respect to x to obtain (3.11).

3.2.2. Derivation of $(\mathsf{EW})_2^n$ for $n = 0, 1$. We now use ${}^{\rm m}(\mathsf{Inc})^n$ in (3.5) and $u_x^{(n)}$ in (3.14). Taking into account the vanishing vertical velocity condition at $z_{\rm m} = -\beta_+$, we obtain

$$u_z^{(n)}(x, z_{\rm m}, t) = -(z_{\rm m} + \beta_+)\frac{\partial U_x^{(n)}}{\partial x}(x,t), \qquad n = 0, 1. \tag{3.17}$$

Then, according to ${}^{\mu}(\mathsf{Inc})^n$ in (3.6), v_z^n does not depend on z_μ. By combining ${}^{\mu}(\mathsf{KC})^n$ in (3.7) with the matching condition (3.10), we conclude that $u_z^{(n)}(x, 0, t) = v_z^{(n)}(x,t) = \partial_t \eta^{(n)}(x,t)$. Finally, returning to (3.17) and setting $z_{\rm m} = 0$, we obtain (3.12).

3.3. Effective wave equation at order $n = 2$. At order 2, a dispersive term appears in the effective wave equation.

PROPOSITION 3.2. *The effective wave equation at order $n = 2$ reads :*

$$(\mathsf{EW})^2_1: \quad \frac{\partial U_x^{(2)}}{\partial t} + \frac{\partial \eta^{(2)}}{\partial x} - \frac{\beta^2}{3}\frac{\partial^3 U_x^{(0)}}{\partial t \partial x^2} = 0, \tag{3.18}$$

$$(\mathsf{EW})^2_2: \quad \beta\frac{\partial U_x^{(2)}}{\partial x} + \frac{\partial \eta^{(2)}}{\partial t} = 0, \tag{3.19}$$

with $\beta = \beta_+$ or β_-.

3.3.1. Derivation of $(\mathsf{EW})^2_1$. We start with ${}^{\mathrm{m}}(\mathsf{Rot})^0_z$ in (3.5) which, after integrating (3.17), yields

$$\varphi^{(2)}(x, z_{\mathrm{m}}, t) = -\left(\frac{z_{\mathrm{m}}^2}{2} + \beta_+ z_{\mathrm{m}}\right)\frac{\partial U_x^{(0)}}{\partial x}(x,t) + \phi^{(2)}(x,t). \tag{3.20}$$

Then, from ${}^{\mu}(\mathsf{Rot})^2_z$ in (3.6), we deduce as before that $\psi^{(2)}$ is independent of z_μ. Using ${}^{\mu}(\mathsf{DC})^2$ in (3.7) and taking into account the matching condition (3.10), we obtain

$$\frac{\partial \phi^{(2)}}{\partial t}(x,t) + \eta^{(2)}(x,t) = 0. \tag{3.21}$$

Injecting (3.20) in ${}^{\mathrm{m}}(\mathsf{Rot})^2_x$ from (3.5) yields

$$u_x^{(2)}(x, z_{\mathrm{m}}, t) = -\left(\frac{z_{\mathrm{m}}^2}{2} + \beta_+ z_{\mathrm{m}}\right)\frac{\partial^2 U_x^{(0)}}{\partial x^2}(x,t) + \frac{\partial \phi^{(2)}}{\partial x}(x,t), \tag{3.22}$$

which, after taking the average with respect to z_{m}, leads to

$$U_x^{(2)}(x,t) = \frac{\beta_+^2}{3}\frac{\partial^2 U_x^{(0)}}{\partial x^2}(x,t) + \frac{\partial \phi^{(2)}}{\partial x}(x,t). \tag{3.23}$$

Finally, differentiating (3.23) with respect to t and (3.21) with respect to x provides (3.18) after combination.

3.3.2. Derivation of $(\mathsf{EW})^2_2$. As before, we use ${}^{\mathrm{m}}(\mathsf{Inc})^2$ in (3.5) with $u_x^{(2)}$ given bt (3.22). After integration, we obtain

$$u_z^{(2)}(x, z_{\mathrm{m}}, t) = \left(\frac{z_{\mathrm{m}}^3}{6} + \frac{\beta_+ z_{\mathrm{m}}^2}{2} - \frac{\beta_+^3}{3}\right)\frac{\partial^3 U_x^{(0)}}{\partial x^3}(x,t) - (z_{\mathrm{m}} + \beta_+)\frac{\partial^2 \phi^{(2)}}{\partial x^2}(x,t),$$

which satisfies the boundary condition ${}^{\mathrm{m}}(\mathsf{RC})^2$ on the rigid wall at $z_{\mathrm{m}} = -\beta_+$. The matching condition reads $u_z^{(2)}(x,0,t) = v_z^{(2)}(x,t)$ since $v_z^{(2)}$ is independent of z_μ, according to ${}^{\mu}(\mathsf{Inc})^2$ in (3.6). Simply combine with ${}^{\mu}(\mathsf{KC})^2$ in (3.7) to obtain

$$u_z^{(2)}(x,0,t) = -\beta_+\frac{\partial}{\partial x}\left(\frac{\beta_+^2}{3}\frac{\partial^3 U_x^{(0)}}{\partial x^3}(x,t) + \frac{\partial \phi^{(2)}}{\partial x}(x,t)\right) = \frac{\partial \eta^{(2)}}{\partial t}(x,t).$$

The term in the parentheses is none than $U_x^{(2)}$, see (3.23), hence (3.19).

3.4. Effective wave equation at order $n = 3$. At order $n = 3$ a non-linear contribution appears.

PROPOSITION 3.3. *The effective wave equation reads at order $n = 3$:*

$$(\mathsf{EW})^3_1: \quad \frac{\partial U_x^{(3)}}{\partial t} + \frac{\partial \eta^{(3)}}{\partial x} - \frac{\beta^2}{3}\frac{\partial^3 U_x^{(1)}}{\partial t \partial x^2} + \alpha U_x^{(0)} \frac{\partial U_x^{(0)}}{\partial x} = 0, \tag{3.24}$$

$$(\mathsf{EW})^3_2: \quad \beta \frac{\partial U_x^{(3)}}{\partial x} + \frac{\partial \eta^{(3)}}{\partial t} + \alpha \frac{\partial}{\partial x}\left(U_x^{(0)} \eta^{(0)}\right) = 0, \tag{3.25}$$

with $\beta = \beta_+$ or β_-.

3.4.1. Derivation of $(\mathsf{EW})^3_1$. We start with ${}^{\mathrm{m}}(\mathsf{Rot})^1_z$ in (3.5), *i.e.* $u_z^{(1)} = \partial_{z_{\mathrm{m}}} \varphi^{(3)}$. Using (3.17) we obtain

$$\varphi^{(3)}(x, z_{\mathrm{m}}, t) = -\left(\frac{z_{\mathrm{m}}^2}{2} + \beta_+ z_{\mathrm{m}}\right) \frac{\partial U_x^{(1)}}{\partial x}(x,t) + \phi^{(3)}(x,t). \tag{3.26}$$

From ${}^{\mu}(\mathsf{Rot})^3_z$ in (3.6) we still have $\psi^{(3)}(x,t) = \varphi^{(3)}(x,0,t) = \phi^{(3)}(x,t)$ independent of z_μ. But now ${}^{\mu}(\mathsf{DC})^3$ in (3.7) reveals a non-linear contribution, with

$$\frac{\partial \phi^{(3)}}{\partial t}(x,t) + \frac{\alpha}{2}(U_x^{(0)}(x,t))^2 + \eta^{(3)}(x,t) = 0, \tag{3.27}$$

since $u_x^{(0)} = U_x^{(0)} = v_x^{(0)}$ from the matchings (3.10). As we did previously, by injecting (3.26) into ${}^{\mathrm{m}}(\mathsf{Rot})^3_x$ in (3.5), we obtain

$$u_x^{(3)}(x, z_{\mathrm{m}}, t) = -\left(\frac{z_{\mathrm{m}}^2}{2} + \beta_+ z_{\mathrm{m}}\right) \frac{\partial^2 U_x^{(1)}}{\partial x^2}(x,t) + \frac{\partial \phi^{(3)}}{\partial x}(x,t), \tag{3.28}$$

which, after taking the average with respect to z_{m}, leads to

$$U_x^{(3)}(x,t) = \frac{\beta^2}{3}\frac{\partial^2 U_x^{(1)}}{\partial x^2}(x,t) + \frac{\partial \phi^{(3)}}{\partial x}(x,t). \tag{3.29}$$

The differentiation of (3.29) with respect to t and (3.27) with respect to x gives, after combination, the equation (3.24).

3.4.2. Derivation of $(\mathsf{EW})^3_2$. We start by combining ${}^{\mathrm{m}}(\mathsf{Inc})^3$ in (3.5) with $u_x^{(3)}$ in (3.28). After integration, we obtain

$$u_z^{(3)}(x, z_{\mathrm{m}}, t) = \left(\frac{z_{\mathrm{m}}^3}{6} + \frac{\beta_+ z_{\mathrm{m}}^2}{2} - \frac{\beta_+^3}{3}\right) \frac{\partial^3 U_x^{(1)}}{\partial x^3}(x,t) - (z_{\mathrm{m}} + \beta_+) \frac{\partial^2 \phi^{(3)}}{\partial x^2}(x,t) \tag{3.30}$$

which satisfies the boundary condition ${}^{\mathrm{m}}(\mathsf{RC})^3$ of vanishing vertical velocity at $z_{\mathrm{m}} = -\beta_+$. The matching condition for $n = 3$ can be read from the second relation in (3.10),

$$u_z^{(3)}(x,0,t) = \lim_{z_\mu \to -\infty} \left(v_z^{(3)}(x, z_\mu, t) - \alpha(z_\mu + 1)\eta^{(0)}(x,t) \frac{\partial u_z^{(0)}}{\partial z_{\mathrm{m}}}(x,0,t) \right). \tag{3.31}$$

To determine the form of $v_z^{(3)}$, we integrate ${}^{\mu}(\mathsf{Inc})^3$ in (3.6) and use the boundary condition ${}^{\mu}(\mathsf{KC})^3$ in (3.7) (remembering that $v_x^{(0)} = U_x^{(0)}$), we have

$$v_z^{(3)}(x, z_\mu, t) = \frac{\partial \eta^{(3)}}{\partial t}(x,t) + \alpha U_x^{(0)}(x,t) \frac{\partial \eta^{(0)}}{\partial x}(x,t) - \alpha z_\mu \eta^{(0)}(x,t) \frac{\partial U_x^{(0)}}{\partial x}(x,t).$$

Using the above relation in the matching condition (3.31) and taking into account ${}^{\mathrm{m}}(\mathsf{Inc})^0$ in (3.5), we obtain

$$u_z^{(3)}(x,0,t) = \frac{\partial \eta^{(3)}}{\partial t}(x,t) + \alpha \frac{\partial}{\partial x}\left(U_x^{(0)}(x,t)\eta^{(0)}(x,t)\right).$$

Now, we just need to identify the above relation with (3.30) for $z_{\mathrm{m}} = 0$, *i.e.*

$$u_z^{(3)}(x,0,t) \quad = -\beta \frac{\partial}{\partial x}\left(\frac{\beta^2}{3}\frac{\partial^2 U_x^{(1)}}{\partial x^2}(x,t) - \beta \frac{\partial \phi^{(3)}}{\partial x}(x,t)\right).$$

The term in parentheses being $U_x^{(3)}$ (see (3.29)), we deduce (3.25).

4. Asymptotics near the depth transition.

4.1. Setting of the asymptotic procedure. In this section, we analyse the solutions in m_0 and μ_0, their vertical matchings as in the previous section, and their horizontal matchings with the solutions in $\mathrm{m}_\pm$ and $\mu_\pm$ (see figures 3 and 5).

4.1.1. The mesoscopic and the microscopic problems. In the *mesoscopic region* m_0, the problem is set in $(x_{\mathrm{m}}, z_{\mathrm{m}})$ coordinates, with $x_{\mathrm{m}} \in (-\infty, \infty)$, $z_{\mathrm{m}} \in (-\beta(x_{\mathrm{m}}), 0)$. From (2.11)-(2.12), it can be written as

$$\begin{aligned} &{}^{\mathrm{m}}_{0}(\mathsf{Inc}): \ \mathrm{div}_{\mathrm{m}}\boldsymbol{u} = \frac{\partial u_x}{\partial x_{\mathrm{m}}} + \frac{\partial u_z}{\partial z_{\mathrm{m}}} = 0, \qquad {}^{\mathrm{m}}_{0}(\mathsf{Rot}): \ \boldsymbol{u} = \frac{1}{\delta}\boldsymbol{\nabla}_{\mathrm{m}}\varphi, \\ &{}^{\mathrm{m}}_{0}(\mathsf{RC}): \ \boldsymbol{u}\cdot\mathbf{n} = 0 \quad \text{on the walls,} \end{aligned} \tag{4.1}$$

(with missing conditions at $z_{\mathrm{m}} = 0$, as before) and the solutions of (4.1) are expanded in terms of series

$$u_x = \sum_{n\geq 0}\delta^n \tilde{u}_x^{(n)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t), \quad u_z = \sum_{n\geq 0}\delta^n \tilde{u}_z^{(n)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t), \quad \varphi = \sum_{n\geq 0}\delta^n \tilde{\varphi}^{(n)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t). \tag{4.2}$$

Note that, unlike (3.2), (3.4) and the forthcoming (4.4), we consider $u_z = O(1)$; the reason for this will become apparent later in the asymptotic analysis.

In the *microscopic region* μ_0, the problem is set in (x_{m}, z_μ) coordinate, with $x_{\mathrm{m}} \in (-\infty, \infty)$, $z_\mu \in (-\infty, 0)$ with z_μ in (2.13), as in the previous section. Consequently, starting from (2.11)-(2.12) and using (2.13) with $\varepsilon = \alpha\delta^2$, the problem is written as

$$\begin{aligned} &{}^{\mu}_{0}(\mathsf{Inc}): \ \eta\frac{\partial u_x}{\partial x_{\mathrm{m}}} + \frac{1}{\alpha\delta^2}\frac{\partial u_z}{\partial z_\mu} = 0, \qquad {}^{\mu}_{0}(\mathsf{Rot}): \ u_x = \frac{1}{\delta}\frac{\partial \varphi}{\partial x_{\mathrm{m}}}, \quad \eta u_z = \frac{1}{\alpha\delta^3}\frac{\partial \varphi}{\partial z_\mu}, \\ &{}^{\mu}_{0}(\mathsf{DC}): \ \frac{\partial \varphi}{\partial t} + \alpha\delta^3\frac{\boldsymbol{u}^2}{2} + \frac{1}{\delta}\eta = 0, \quad {}^{\mu}_{0}(\mathsf{KC}): \ u_z = \frac{\partial \eta}{\partial t} + \alpha\delta^2 u_x\frac{\partial \eta}{\partial x_{\mathrm{m}}} \ \text{ at } z_\mu = 0. \end{aligned} \tag{4.3}$$

(and boundary conditions are still missing when $z_\mu \to -\infty$). The solutions of (4.3) are expanded in terms of series

$$\begin{aligned} &u_x = \sum_{n\geq 0}\delta^n \tilde{v}_x^{(n)}(x_{\mathrm{m}}, z_\mu, t), \quad u_z = \delta\sum_{n\geq 0}\delta^n \tilde{v}_z^{(n)}(x_{\mathrm{m}}, z_\mu, t), \quad \varphi = \sum_{n\geq 0}\delta^n \tilde{\psi}^{(n)}(x_{\mathrm{m}}, z_\mu, t), \\ &\eta = \delta\sum_{n\geq 0}\delta^n \tilde{\eta}^{(n)}(x_{\mathrm{m}}, t), \end{aligned} \tag{4.4}$$

where we anticipate that $\varphi = O(1)$ imposes $\eta, u_z = O(\delta)$ from ${}^{\mu}_{0}(\mathsf{DC})$ - ${}^{\mu}_{0}(\mathsf{KC})$ in (4.3).

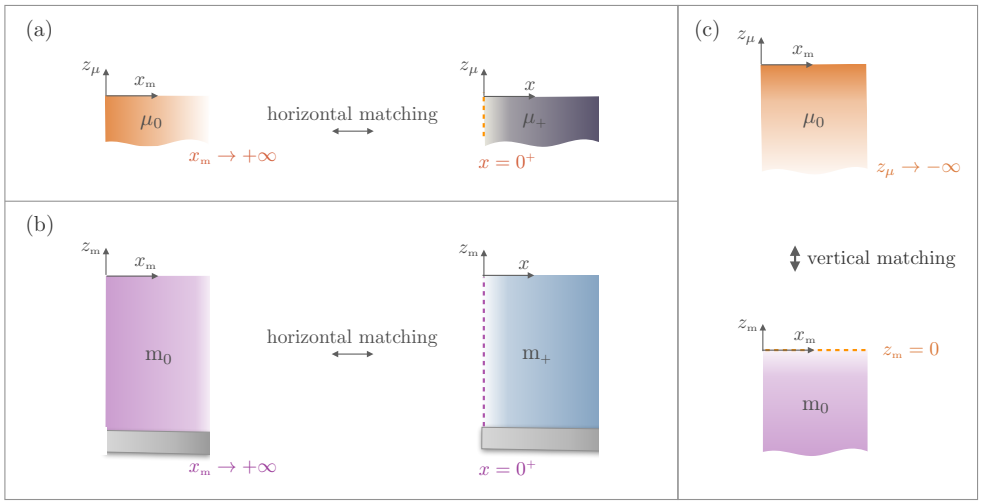

FIG. 5. *The different matchings required for the analysis in the the depth transition regions, according to figure 2. (a) Horizontal matching in the boundary layer at the free surface, near (μ_0) and far ($\mu_\pm$) from the transition; (b) horizontal matching in the constant depth regions $\beta = \beta_\pm = O(1)$ near (m_0) and far (m_+) from the transition; (c) vertical matching, near the transition, between the boundary layer at the free surface (μ_0) and the region of constant depth (m_0).*

4.1.2. The hierarchies of equations. In m_0, we inject (4.2) into (4.1), and we obtain the hierarchy of equations, namely $\forall\, n$

$$
\begin{aligned}
&{}^{\mathrm{m}}_{0}(\mathsf{Inc})^n: \quad \mathrm{div}_\mathrm{m}\tilde{\boldsymbol{u}}^{(n)} = 0, \qquad {}^{\mathrm{m}}_{0}(\mathsf{Rot})^n: \quad \tilde{\boldsymbol{u}}^{(n)} = \boldsymbol{\nabla}_\mathrm{m}\tilde{\varphi}^{(n+1)},\\
&{}^{\mathrm{m}}_{0}(\mathsf{RC})^n: \quad \tilde{\boldsymbol{u}}^{(n)}\cdot\mathbf{n} = 0 \quad \text{on the walls,}
\end{aligned}
\tag{4.5}
$$

(by convention $\tilde{\boldsymbol{u}}^{(n)} = 0$ for $n < 0$). In μ_0, we inject (4.4) into (4.3), and we obtain

$$
\begin{aligned}
&{}^{\mu}_{0}(\mathsf{Inc})^n: \quad \frac{\partial\tilde{v}_z^{(n)}}{\partial z_\mu} = 0, \quad n \le 1, \qquad \frac{\partial\tilde{v}_z^{(2)}}{\partial z_\mu} = -\alpha\,\tilde{\eta}^{(0)}\frac{\partial\tilde{v}_x^{(0)}}{\partial x_\mathrm{m}},\\
&{}^{\mu}_{0}(\mathsf{Rot})^n_x: \quad \tilde{v}_x^{(n)} = \frac{\partial\tilde{\psi}^{(n+1)}}{\partial x_\mathrm{m}}, \qquad {}^{\mu}_{0}(\mathsf{Rot})^n_z: \quad \frac{\partial\tilde{\psi}^{(n)}}{\partial z_\mu} = 0, \quad n \le 5,
\end{aligned}
\tag{4.6}
$$

along with the boundary conditions at the free surface $z_\mu = 0$,

$$
\begin{aligned}
&{}^{\mu}_{0}(\mathsf{DC})^n: \quad \frac{\partial\tilde{\psi}^{(n)}}{\partial t} + \tilde{\eta}^{(n)} = 0, \quad n \le 2, \qquad \frac{\partial\tilde{\psi}^{(3)}}{\partial t} + \frac{\alpha}{2}\big(\tilde{v}_x^{(0)}\big)^2 + \tilde{\eta}^{(3)} = 0,\\
&{}^{\mu}_{0}(\mathsf{KC})^n: \quad \tilde{v}_z^{(n)} = \frac{\partial\tilde{\eta}^{(n)}}{\partial t}, \qquad n \le 1, \qquad \tilde{v}_z^{(3)} = \frac{\partial\tilde{\eta}^{(3)}}{\partial t} + \alpha\,\tilde{v}_x^{(0)}\frac{\partial\tilde{\eta}^{(0)}}{\partial x}.
\end{aligned}
\tag{4.7}
$$

4.1.3. Vertical matching conditions between m_0 and μ_0 . As in the previous section, the solutions in m_0 and μ_0 must match in an intermediate region when $z_\mathrm{m} \to 0$ and $z_\mu \to -\infty$. The result is identical (replacing x by x_m in (3.8)-(3.9)), *i.e.*

$$
\begin{cases}
\tilde{\varphi}^{(n)}(x_\mathrm{m}, 0, t) = \lim\limits_{z_\mu\to-\infty}\tilde{\psi}^{(n)}(x_\mathrm{m}, z_\mu, t), \qquad n \le 2,\\
\tilde{\varphi}^{(3)}(x_\mathrm{m}, 0, t) = \lim\limits_{z_\mu\to-\infty}\left(\tilde{\psi}^{(3)}(x_\mathrm{m}, z_\mu, t) - \alpha(z_\mu+1)\tilde{\eta}^{(0)}(x_\mathrm{m}, t)\dfrac{\partial\tilde{\varphi}^{(0)}}{\partial z_\mathrm{m}}(x_\mathrm{m}, 0, t)\right),
\end{cases}
\tag{4.8}
$$

and the same relations are obtained replacing $\tilde{\varphi}^{(n)}$ by $\tilde{u}_x^{(n)}$ and $\tilde{\psi}^{(n)}$ by $\tilde{v}_x^{(n)}$. However, for $\tilde{v}_z^{(n)}$, we have to take into account a slightly different matching, according to the expansions (4.2) and (4.4), which imposes

$$\tilde{u}_z^{(0)}(x_{\rm m},z_{\rm m},t)+\delta\tilde{u}_z^{(1)}(x_{\rm m},z_{\rm m},t)+\cdots\sim\delta\tilde{v}_z^{(0)}(x_{\rm m},z_\mu,t)+\delta^2\tilde{v}_z^{(1)}(x_{\rm m},z_\mu,t),$$

so using (3.9), we obtain

$$\left\{\begin{array}{l}\tilde{u}_z^{(0)}(x_{\rm m},0,t)=0,\\ \tilde{u}_z^{(n)}(x_{\rm m},0,t)=\displaystyle\lim_{z_\mu\to-\infty}\tilde{v}_z^{(n-1)}(x_{\rm m},z_\mu,t),\qquad n=1,2,\\ \tilde{u}_z^{(3)}(x_{\rm m},0,t)=\displaystyle\lim_{z_\mu\to-\infty}\left(\tilde{v}_z^{(2)}(x_{\rm m},z_\mu,t)-\alpha(z_\mu+1)\tilde{\eta}^{(0)}(x_{\rm m},t)\frac{\partial\tilde{u}_z^{(0)}}{\partial z_{\rm m}}(x_{\rm m},0,t)\right).\end{array}\right. \tag{4.9}$$

4.1.4. Horizontal matching conditions between m_0 and $\mathrm{m}_\pm$ and between μ_0 and $\mu_\pm$. Horizontal matchings are obtained, as for the vertical matchings, by imposing that solutions in m_0 (resp. μ_0) correspond, in an intermediate region, to solutions in $\mathrm{m}_\pm$ (resp. $\mu_\pm$) when $x\to0^\pm$ and $x_{\rm m}\to\pm\infty$, see figures 3 and 5. Between μ_0 and $\mu_\pm$, with

$$\psi^{(0)}(x,z_\mu,t)+\delta\psi^{(1)}(x,z_\mu,t)+\cdots\sim\tilde{\psi}^{(0)}(x_{\rm m},z_\mu,t)+\delta\tilde{\psi}^{(1)}(x_{\rm m},z_\mu,t)+\cdots,$$

and taking into account the Taylor expansion (in the horizontal direction)

$$f(x,z_\mu,t)=f(0,z_\mu,t)+\delta x_{\rm m}\frac{\partial f}{\partial x}(0,z_\mu,t)+\cdots,$$

we obtain, for $n\geq0$ and with $p\geq1$, $m\geq0$,

$$\psi^{(n)}(0^\pm,z_\mu,t)=\lim_{x_{\rm m}\to\pm\infty}\left(\tilde{\psi}^{(n)}(x_{\rm m},z_\mu,t)-\sum_{p+m=n}\frac{x_{\rm m}^p}{p!}\frac{\partial^p\psi^{(m)}}{\partial x^p}(0^\pm,z_\mu,t)\right), \tag{4.10}$$

which remain valid by replacing $\psi^{(n)}$ by $\eta^{(n)}$, $v_x^{(n)}$ or $v_z^{(n)}$, and $\tilde{\psi}^{(n)}$ by $\tilde{\eta}^{(n)}$, $\tilde{v}_x^{(n)}$ or $\tilde{v}_z^{(n)}$ (for $\eta^{(n)}$ and $\tilde{\eta}^{(n)}$, the dependence in z_μ has to be omitted). The same applies between m_0 and $\mathrm{m}_\pm$, *i.e.*

$$\varphi^{(n)}(0^\pm,z_{\rm m},t)=\lim_{x_{\rm m}\to\pm\infty}\left(\tilde{\varphi}^{(n)}(x_{\rm m},z_{\rm m},t)-\sum_{p+m=n}\frac{x_{\rm m}^p}{p!}\frac{\partial^p\varphi^{(m)}}{\partial x^p}(0^\pm,z_{\rm m},t)\right), \tag{4.11}$$

which remain valid by replacing $\varphi^{(n)}$ by $u_x^{(n)}$ and $\tilde{\varphi}^{(n)}$ by $\tilde{u}_x^{(n)}$. For $u_z^{(n)}$, however, we must take into account the shift of the expansions (3.2) and (4.2), which gives

$$\left\{\begin{array}{l}0=\displaystyle\lim_{x_{\rm m}\to\pm\infty}\tilde{u}_z^{(0)}(x_{\rm m},z_\mu,t),\\ u_z^{(n)}(0^\pm,z_{\rm m},t)=\displaystyle\lim_{x_{\rm m}\to\pm\infty}\left(\tilde{u}_z^{(n+1)}(x_{\rm m},z_{\rm m},t)-\sum_{p+m=n}\frac{x_{\rm m}^p}{p!}\frac{\partial^p u_z^{(m)}}{\partial x^p}(0^\pm,z_{\rm m},t)\right).\end{array}\right. \tag{4.12}$$

We are now in position to proceed with the asymptotic analysis of the step discontinuity. The main objective of the forthcoming section is to establish the relations between the surface elevation $\eta^{(n)}$ and the flux $q^{(n)}$ defined in (3.15) satisfy on both sides of the step discontinuity. These relations will involve both jump and average as defined in (2.7).

4.2. Continuity conditions at order $n = 0$. At the leading order, surface elevation and flux are continuous across the step.

PROPOSITION 4.1. *The continuity conditions apply at the dominant order $n = 0$:*

$$(\mathsf{JC})^0_1 : \quad [\![\eta^{(0)}]\!] = 0, \tag{4.13}$$

$$(\mathsf{JC})^0_2 : \quad [\![q^{(0)}]\!] = 0. \tag{4.14}$$

4.2.1. Derivation of $(\mathsf{JC})^0_1$. We start by using ${}^{\mathrm{m}}_{0}(\mathsf{Rot})^{-1}$ in (4.5) and ${}^{\mu}_{0}(\mathsf{Rot})^{-1}_{x}$, ${}^{\mu}_{0}(\mathsf{Rot})^{0}_{z}$ in (4.6) from which we deduce that $\tilde{\varphi}^{(0)}$ and $\tilde{\psi}^{(0)}$ depend only on t. Using the matching conditions (4.8), (4.10) and (4.11), we obtain

$$\psi^{(0)}(0^{\pm}, t) = \tilde{\psi}^{(0)}(t) = \tilde{\varphi}^{(0)}(t) = \phi^{(0)}(0^{\pm}, t). \tag{4.15}$$

Then, taking the derivative with respect to t of (4.15) and using ${}^{\mu}(\mathsf{DC})^{0}$, we find

$$\eta^{(0)}(0^{\pm}, t) = \tilde{\eta}^{(0)}(t) = -\frac{\partial \psi^{(0)}}{\partial t}(0^{\pm}, t) = -\frac{\partial \tilde{\psi}^{(0)}}{\partial t}(t), \tag{4.16}$$

from which we deduce the continuity condition (4.13). We also deduce from ${}^{\mathrm{m}}_{0}(\mathsf{Inc})^{0}$ in (4.6) that $\tilde{v}_z^{(0)}$ does not depend on z_μ, which, combined with ${}^{\mu}_{0}(\mathsf{KC})^{0}$, tells us that $\tilde{v}_z^{(0)}$ also does not depend on x_{m} and satisfies

$$\tilde{v}_z^{(0)}(t) = \frac{\partial \eta^{(0)}}{\partial t}(t). \tag{4.17}$$

4.2.2. Derivation of $(\mathsf{JC})^0_2$. We now use the relation ${}^{\mathrm{m}}_{0}(\mathsf{Inc})^{0}$ in (4.5) which we integrate over

$$\Omega^*_{\mathrm{m}} = \{x_{\mathrm{m}} \in (-x^*_{\mathrm{m}}, x^*_{\mathrm{m}}), z_{\mathrm{m}} \in (-\beta(x_{\mathrm{m}}), 0)\}, \tag{4.18}$$

(with $x^*_{\mathrm{m}} > 0$) to obtain

$$\int_{-\beta_+}^{0} \tilde{u}_x^{(0)}(x^*_{\mathrm{m}}, z_{\mathrm{m}}, t)\mathrm{d}z_{\mathrm{m}} - \int_{-\beta_-}^{0} \tilde{u}_x^{(0)}(-x^*_{\mathrm{m}}, z_{\mathrm{m}}, t)\mathrm{d}z_{\mathrm{m}} + \int_{-x^*_{\mathrm{m}}}^{x^*_{\mathrm{m}}} \tilde{u}_z^{(0)}(x_{\mathrm{m}}, 0, t)\mathrm{d}x_{\mathrm{m}} = 0. \tag{4.19}$$

According to (4.9), $\tilde{u}_z^{(0)}(x_{\mathrm{m}}, 0, t) = 0$, and according to (4.11), $\tilde{u}_x^{(0)}(\pm x^*_{\mathrm{m}}, z_{\mathrm{m}}, t) \to U_x^{(0)}(0^{\pm}, t)$ when x^*_{m} tends to $+\infty$ (see (3.14)). This allows us to conclude that (4.19) simplifies, after passing to the limit $x^*_{\mathrm{m}} \to +\infty$, to $q^{(0)}(0, t) = \beta_{\pm} U_x^{(0)}(0^{\pm}, t)$, hence the continuity of the flux (4.14).

4.3. Jump conditions at order $n = 1$. At order $n = 1$, we have a jump condition of the surface elevation and continuity of the flux.

PROPOSITION 4.2. *Jump condition and continuity condition apply at the order $n = 1$:*

$$(\mathsf{JC})^1_1 : \quad [\![\eta^{(1)}]\!] = -\mathcal{B}_1 \frac{\partial q^{(0)}}{\partial t}(0, t), \tag{4.20}$$

$$(\mathsf{JC})^1_2 : \quad [\![q^{(1)}]\!] = 0, \tag{4.21}$$

where the blockage coefficient $\mathcal{B}_1$ is defined in (4.26).

4.3.1. Derivation of $(\mathsf{JC})^1_1$. To establish this relation, we will consider the problem satisfied by $(\tilde{\boldsymbol{u}}^{(0)}, \tilde{\varphi}^{(1)})$. It consists in ${}^{\mathrm{m}}_{0}(\mathsf{Inc})^0$, ${}^{\mathrm{m}}_{0}(\mathsf{Rot})^0$ and ${}^{\mathrm{m}}_{0}(\mathsf{RC})^0$ in (4.5), and the first equation in (4.9). The behavior of $\tilde{\boldsymbol{u}}^{(0)}$ when $x_{\mathrm{m}} \to \pm\infty$ is provided by the matching conditions, which tell us that $\tilde{u}_x^{(0)}(\pm x_{\mathrm{m}}^*, z_{\mathrm{m}}, t) \to U_x^{(0)}(0^{\pm}, t)$, according to (4.11) and (3.14), and that $\tilde{u}_z^{(0)}(\pm x_{\mathrm{m}}^*, z_{\mathrm{m}}, t) \to 0$ according to (4.12). Recalling (4.14) with (3.15), we can finally establish that $(\tilde{\boldsymbol{u}}^{(0)}, \tilde{\varphi}^{(1)})$ satisfies

$$\left\{\begin{array}{l} \mathrm{div}_{\mathrm{m}}\tilde{\boldsymbol{u}}^{(0)} = 0, \quad \tilde{\boldsymbol{u}}^{(0)} = \boldsymbol{\nabla}_{\mathrm{m}}\tilde{\varphi}^{(1)}, \\ \tilde{\boldsymbol{u}}^{(0)} \cdot \mathbf{n} = 0 \ \text{ on the walls}, \qquad \tilde{u}_z^{(0)}(x_{\mathrm{m}}, 0, t) = 0, \\ \displaystyle\lim_{x_{\mathrm{m}} \to \pm\infty} \tilde{\boldsymbol{u}}^{(0)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) = \frac{q^{(0)}(0,t)}{\beta_{\pm}}\,\mathbf{e}_x. \end{array}\right. \tag{4.22}$$

The above problem is linear with respect to $q^{(0)}(0,t)$, hence we set

$$\begin{aligned} \tilde{\varphi}^{(1)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) &= q^{(0)}(0,t) Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}) + \overline{\phi^{(1)}}(t), \\ \tilde{\boldsymbol{u}}^{(0)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) &= q^{(0)}(0,t) \boldsymbol{\nabla} Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}), \end{aligned} \tag{4.23}$$

where we have defined $\overline{\phi^{(1)}}(t) = \frac{1}{2}\left(\phi^{(1)}(0^-, t) + \phi^{(1)}(0^+, t)\right)$, and where Q_1 is the solution to the elementary problem

$$\left\{\begin{array}{l} \Delta Q_1 = 0, \\ \boldsymbol{\nabla} Q_1 \cdot \mathbf{n} = 0 \ \text{ on the walls}, \quad \dfrac{\partial Q_1}{\partial z_{\mathrm{m}}}(x_{\mathrm{m}}, 0) = 0, \\ \displaystyle\lim_{x_{\mathrm{m}} \to \pm\infty} \boldsymbol{\nabla} Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}) = \frac{\mathbf{e}_x}{\beta_{\pm}}. \end{array}\right. \tag{4.24}$$

Since the solution Q_1 is defined up to a constant, we impose that

$$Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}) = Q_1^{\mathrm{ev}}(x_{\mathrm{m}}, z_{\mathrm{m}}) + \left\{\begin{array}{ll} \dfrac{x_{\mathrm{m}}}{\beta_+} + \dfrac{\mathcal{B}_1}{2}, & \text{if} \quad x_{\mathrm{m}} > 0, \\ \dfrac{x_{\mathrm{m}}}{\beta_-} - \dfrac{\mathcal{B}_1}{2}, & \text{if} \quad x_{\mathrm{m}} < 0, \end{array}\right. , \quad \lim_{x_{\mathrm{m}} \to \pm\infty} Q_1^{\mathrm{ev}} = 0, \tag{4.25}$$

where Q_1^{ev} is an evanescent field and $\mathcal{B}_1$ a blockage coefficient defined as follows

$$\mathcal{B}_1 = \lim_{x_{\mathrm{m}} \to +\infty} \left(Q_1 - \frac{x_{\mathrm{m}}}{\beta_+}\right) - \lim_{x_{\mathrm{m}} \to -\infty} \left(Q_1 - \frac{x_{\mathrm{m}}}{\beta_-}\right). \tag{4.26}$$

Owing to the matching condition (4.11) ($n = 1$), with $\tilde{\varphi}^1$ in (4.23), $\partial_x \varphi^{(0)} = q^{(0)}/\beta_{\pm}$ from (3.13), (3.14) and (4.14), and accounting for (4.25), we obtain

$$\varphi^{(1)}(0^{\pm}, z_{\mathrm{m}}, t) = \pm q^{(0)}(0,t)\frac{\mathcal{B}_1}{2} + \overline{\phi^{(1)}}(t). \tag{4.27}$$

Using (3.16), we finally obtain the jump of $\eta^{(1)}$ announced in (4.20). In particular we have

$$\eta^{(1)}(0^{\pm}, t) = \overline{\eta^{(1)}}(t) \pm \frac{1}{2}[\![\eta^{(1)}]\!] = \overline{\eta^{(1)}}(t) \mp \frac{\mathcal{B}_1}{2}\frac{\partial q^{(0)}}{\partial t}(0,t). \tag{4.28}$$

4.3.2. Derivation of $(\mathsf{JC})^1_2$. As before, we use ${}^{\mathrm{m}}_{0}(\mathsf{Inc})^1$ in (4.5) which we integrate over Ω^*_{m} defined in (4.18). We obtain, after using ${}^{\mathrm{m}}_{0}(\mathsf{RC})^1$ in (4.5),

$$\int_{-\beta_+}^{0} \tilde{u}_x^{(1)}(x^*_{\mathrm{m}}, z_{\mathrm{m}}, t)\mathrm{d}z_{\mathrm{m}} - \int_{-\beta_-}^{0} \tilde{u}_x^{(1)}(-x^*_{\mathrm{m}}, z_{\mathrm{m}}, t)\mathrm{d}z_{\mathrm{m}} + \int_{-x^*_{\mathrm{m}}}^{x^*_{\mathrm{m}}} \tilde{u}_z^{(1)}(x_{\mathrm{m}}, 0, t)\mathrm{d}x_{\mathrm{m}} = 0. \tag{4.29}$$

The first two integrals in (4.29) are given by the matching condition (4.11) with $u_x^{(1)} = U_x^{(1)}(x,t)$ from (3.14). Also using (3.12), we obtain $\int_{-\beta_\pm}^{0} \tilde{u}_x^{(1)}(x^*_{\mathrm{m}}, z_{\mathrm{m}}, t)\mathrm{d}z_{\mathrm{m}} = \beta_\pm U_x^{(1)}(0^\pm, t) \mp x^*_{\mathrm{m}}\partial_t\eta^{(0)}(0,t)$. The third integral in (4.29) is given by the matching condition (4.9) in $\tilde{u}_z^{(1)}$ with $\tilde{v}_z^{(0)}$ in (4.17), and so we have $\int_{-x^*_{\mathrm{m}}}^{x^*_{\mathrm{m}}} \tilde{u}_z^{(1)}(x_{\mathrm{m}}, 0, t)\mathrm{d}x_{\mathrm{m}} = 2x^*_{\mathrm{m}}\partial_t\eta^{(0)}(0,t)$. By summing the contributions, the diverging terms, linear in x^*_{m}, cancel out and we obtain the continuity (4.21) of the flux at order 1 defined by (3.15) with $q^{(1)}(0,t) = \beta_\pm U_x^{(1)}(0^\pm, t)$.

4.4. Jump conditions at order $n = 2$. At order $n = 2$, we have jump conditions on both surface elevation and flux across the step discontinuity.

PROPOSITION 4.3. *Jump conditions apply at the order $n = 2$:*

$$(\mathsf{JC})^2_1: \quad [\![\eta^{(2)}]\!] = -\mathcal{B}_1 \frac{\partial q^{(1)}}{\partial t}(0,t) - \mathcal{B}_2 \frac{\partial^2 \eta^{(0)}}{\partial t^2}(0,t), \tag{4.30}$$

$$(\mathsf{JC})^2_2: \quad [\![q^{(2)}]\!] = \mathcal{C}_1 \frac{\partial^2 q^{(0)}}{\partial t^2}(0,t), \tag{4.31}$$

where the effective coefficients $\mathcal{B}_2$ and $\mathcal{C}_1$ are defined in (4.35) and (4.40).

4.4.1. Derivation of $(\mathsf{JC})^2_1$. We proceed as at order 1, and consider the problem satisfied by $(\tilde{\boldsymbol{u}}^{(1)}, \tilde{\varphi}^{(2)})$ which consists of ${}^{\mathrm{m}}_{0}(\mathsf{Inc})^1$, ${}^{\mathrm{m}}_{0}(\mathsf{Rot})^1$ and ${}^{\mathrm{m}}_{0}(\mathsf{RC})^1$ from (4.5), and the second equation from (4.9) for $n = 1$ with $\tilde{v}_z^{(0)}$ given by (4.17) which provides the boundary condition at $z_{\mathrm{m}} = 0$. To complete the problem, we need to determine the behavior of $\tilde{\boldsymbol{u}}^{(1)}$ when $x_{\mathrm{m}} \to \pm\infty$. By combining the matching condition (4.11) with (3.12), (3.14) and (4.21), we obtain $\tilde{u}_x^{(1)}(\pm x^*_{\mathrm{m}}, z_{\mathrm{m}}, t) \to \left(q^{(1)}(0,t) - x_{\mathrm{m}}\partial_t\eta^{(0)}(0,t)\right)/\beta_\pm$. Then, we have $\tilde{u}_z^{(1)}(\pm x^*_{\mathrm{m}}, z_{\mathrm{m}}, t) \to (z_{\mathrm{m}}/\beta_\pm + 1)\,\partial_t\eta^{(0)}(0,t)$, from the matching (4.12) and (3.17). This allows us to establish the problem satisfied by $(\tilde{\boldsymbol{u}}^{(1)}, \tilde{\varphi}^{(2)})$, *i.e.*

$$\begin{cases} \mathrm{div}_{\mathrm{m}}\tilde{\boldsymbol{u}}^{(1)} = 0, \quad \tilde{\boldsymbol{u}}^{(1)} = \boldsymbol{\nabla}_{\mathrm{m}}\tilde{\varphi}^{(2)}, \\ \tilde{\boldsymbol{u}}^{(1)} \cdot \mathbf{n} = 0 \ \text{ on the walls}, \qquad \tilde{u}_z^{(1)}(x_{\mathrm{m}}, 0, t) = \dfrac{\partial \eta^{(0)}}{\partial t}(0,t), \\ \lim\limits_{x_{\mathrm{m}}\to\pm\infty} \tilde{\boldsymbol{u}}^{(1)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) = \dfrac{1}{\beta_\pm}\left(q^{(1)}(0,t) - x_{\mathrm{m}}\dfrac{\partial \eta^{(0)}}{\partial t}(0,t)\right)\mathbf{e}_x + \left(\dfrac{z_{\mathrm{m}}}{\beta_\pm} + 1\right)\dfrac{\partial \eta^{(0)}}{\partial t}(0,t)\mathbf{e}_z. \end{cases}$$

The above problem is linear with respect to $q^{(1)}(0,t)$ and $\partial_t\eta^{(0)}(0,t)$, so we pose

$$\begin{aligned} \tilde{\varphi}^{(2)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) &= q^{(1)}(0,t)Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}) + \frac{\partial \eta^{(0)}}{\partial t}(0,t)Q_2(x_{\mathrm{m}}, z_{\mathrm{m}}) + \overline{\phi^{(2)}}(t), \\ \tilde{\boldsymbol{u}}^{(1)}(x_{\mathrm{m}}, z_{\mathrm{m}}, t) &= q^{(1)}(0,t)\boldsymbol{\nabla}Q_1(x_{\mathrm{m}}, z_{\mathrm{m}}) + \frac{\partial \eta^{(0)}}{\partial t}(0,t)\boldsymbol{\nabla}Q_2(x_{\mathrm{m}}, z_{\mathrm{m}}), \end{aligned} \tag{4.32}$$

where we have defined $\overline{\phi^{(2)}}(t) = \frac{1}{2}\left(\phi^{(2)}(0^-,t) + \phi^{(2)}(0^+,t)\right)$, and Q_2 is the solution to the elementary problem

$$\text{(4.33)}\qquad \begin{cases} \Delta Q_2 = 0, \\ \boldsymbol{\nabla} Q_2 \cdot \mathbf{n} = 0 \ \text{ on the walls}, \quad \dfrac{\partial Q_2}{\partial z_{\rm m}}(x_{\rm m},0) = 1, \\ \displaystyle\lim_{x_{\rm m}\to\pm\infty} \boldsymbol{\nabla} Q_2(x_{\rm m},z_{\rm m}) = -\frac{x_{\rm m}}{\beta_\pm}\mathbf{e}_x + \left(\frac{z_{\rm m}}{\beta_\pm}+1\right)\mathbf{e}_z. \end{cases}$$

As the solution Q_2 is defined up to a constant, we impose that

$$\text{(4.34)}\qquad Q_2(x_{\rm m},z_{\rm m}) = Q_2^{\rm ev}(x_{\rm m},z_{\rm m}) + \begin{cases} \dfrac{z_{\rm m}^2 - x_{\rm m}^2}{2\beta_+} + z_{\rm m} + \dfrac{\mathcal{B}_2}{2}, & \text{if} \quad x_{\rm m} > 0, \\ \dfrac{z_{\rm m}^2 - x_{\rm m}^2}{2\beta_-} + z_{\rm m} - \dfrac{\mathcal{B}_2}{2}, & \text{if} \quad x_{\rm m} < 0, \end{cases} \quad , \lim_{x_{\rm m}\to\pm\infty} Q_2^{\rm ev} = 0,$$

where $Q_2^{\rm ev}$ is an evanescent field and $\mathcal{B}_2$ is a blockage coefficient defined as follows

$$\text{(4.35)}\qquad \mathcal{B}_2 = \lim_{x_{\rm m}\to+\infty}\left(Q_2 - \frac{z_{\rm m}^2 - x_{\rm m}^2}{2\beta_+} - z_{\rm m}\right) - \lim_{x_{\rm m}\to-\infty}\left(Q_2 - \frac{z_{\rm m}^2 - x_{\rm m}^2}{2\beta_-} - z_{\rm m}\right).$$

We now consider the matching condition (4.11) for $n = 2$, which involves the quantities $\partial_x\varphi^{(1)}(0^\pm,t) = q^{(1)}(0,t)/\beta_\pm$ from (3.13)-(3.14) and (4.21), and $\partial_{xx}\varphi^{(0)}(0^\pm,t) = -\partial_t\eta^{(0)}(0,t)/\beta_\pm$ from (3.12)-(3.14). We thus obtain

$$\varphi^{(2)}(0^\pm,z_{\rm m},t) = \lim_{x_{\rm m}\to\pm\infty}\left(\tilde\varphi^{(2)}(x_{\rm m},z_{\rm m},t) - \frac{x_{\rm m}}{\beta_\pm}q^{(1)}(0,t) + \frac{x_{\rm m}^2}{2\beta_\pm}\frac{\partial\eta^{(0)}}{\partial t}(0,t)\right).$$

Using $\tilde\varphi^{(2)}$ in (4.32) and the asymptotic behaviours of Q_1 (in (4.25)) and Q_2 (in (4.34)), we deduce that

$$\text{(4.36)}\qquad \begin{aligned} \varphi^{(2)}(0^\pm,z_{\rm m},t) &= \pm q^{(1)}(0,t)\frac{\mathcal{B}_1}{2} + \frac{\partial\eta^{(0)}}{\partial t}(0,t)\left(\frac{z_{\rm m}^2}{2\beta_\pm} + z_{\rm m} \pm \frac{\mathcal{B}_2}{2}\right) + \overline{\phi^{(2)}}(t), \\ \phi^{(2)}(0^\pm,t) &= \pm\left(q^{(1)}(0,t)\frac{\mathcal{B}_1}{2} + \frac{\partial\eta^{(0)}}{\partial t}(0,t)\frac{\mathcal{B}_2}{2}\right) + \overline{\phi^{(2)}}(t), \end{aligned}$$

with $\phi^{(2)}(0^\pm,t) = \varphi^{(2)}(x,0,t)$ from (3.20) (note that (4.36) is consistent with (3.20) thanks to (3.12)). Finally, using (3.21), we obtain the jump (4.30) of $\eta^{(2)}$.

4.4.2. Derivation of $(\mathsf{JC})_2^2$. As we did previously, we use ${}_0^{\rm m}(\mathsf{Inc})^2$ in (4.5) which we integrate over $\Omega_{\rm m}^*$, see (4.18). We obtain, after using ${}_0^{\rm m}(\mathsf{RC})^2$ in (4.5),

$$\text{(4.37)}\qquad \int_{-\beta_+}^0 \tilde u_x^{(2)}(x_{\rm m}^*,z_{\rm m},t)\,{\rm d}z_{\rm m} - \int_{-\beta_-}^0 \tilde u_x^{(2)}(-x_{\rm m}^*,z_{\rm m},t)\,{\rm d}z_{\rm m} + \int_{-x_{\rm m}^*}^{x_{\rm m}^*} \tilde u_z^{(2)}(x_{\rm m},0,t)\,{\rm d}x_{\rm m} = 0.$$

For the first two integrals, we use (4.11) with $\int_{-\beta_\pm}^0 u_x^{(2)}(0^\pm,z_{\rm m},t)\,{\rm d}z_{\rm m} = q^{(2)}(0^\pm,t)$ by definition, (3.15), and which involves $\partial_x u_x^{(1)}(0^\pm,t) = \partial_{tt}\left(\pm q^{(0)}(0,t)\frac{\mathcal{B}_1}{2} + \overline{\phi^{(1)}}(t)\right)/\beta_\pm$, from (3.12), (3.14), (3.16) and (4.27), and $\partial_{xx}u_x^{(0)}(0^\pm,t) = \partial_{tt}q^{(0)}(0,t)/\beta_\pm^2$ from (3.11)-(3.12), (3.14) and (4.14). So for $x_{\rm m}^* \to +\infty$ we get

$$\text{(4.38)}\qquad \int_{-\beta_\pm}^0 \tilde u_x^{(2)}(\pm x_{\rm m}^*,z_{\rm m},t){\rm d}z_{\rm m} \underset{x_{\rm m}^*\to+\infty}{\sim} q^{(2)}(0^\pm,t) \pm x_{\rm m}^*\frac{\partial^2\overline{\phi^{(1)}}}{\partial t^2}(t) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)\left(\frac{{x_{\rm m}^*}^2}{2\beta_\pm} + x_{\rm m}^*\frac{\mathcal{B}_1}{2}\right). \ \blacksquare$$

For the third integral in (4.37), we use (4.9), so $\tilde{u}_z^{(2)}(x_{\rm m},0,t)=\tilde{v}_z^{(1)}(x_{\rm m},t)$ since $\tilde{v}_z^{(1)}$ and $\tilde{\psi}^{(1)}$ do not depend on z_μ, according to ${}^{\mu}_{0}(\mathsf{Inc})^1$ and ${}^{\mu}_{0}(\mathsf{Rot})^1_z$ in (4.6). Using ${}^{\mu}_{0}(\mathsf{DC})^1$ and ${}^{\mu}_{0}(\mathsf{KC})^1$ in (4.7) (for $n=1$), the matching (4.8) (for $n=1$) and finally the form of $\tilde{\varphi}^{(1)}$ in (4.23), we deduce that $\tilde{u}_z^{(2)}(x_{\rm m},0,t)=-\partial_{tt}\left(q^{(0)}(0,t)Q_1(x_{\rm m},0)+\overline{\phi^{(1)}}(t)\right)$. Finally, using (4.25), we have, for $x_{\rm m}^*\to+\infty$,

$$\int_{-x_{\rm m}^*}^{x_{\rm m}^*}\tilde{u}_z^{(2)}(x_{\rm m},0,t)\mathrm{d}x_{\rm m}\underset{x_{\rm m}^*\to+\infty}{\sim}-\frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)\left(\mathcal{C}_1+\frac{{x_{\rm m}^*}^2}{2\beta_+}-\frac{{x_{\rm m}^*}^2}{2\beta_-}\right)-2x_{\rm m}^*\frac{\partial^2\overline{\phi^{(1)}}}{\partial t^2}(t), \tag{4.39}$$

with

$$\mathcal{C}_1=\int_{-\infty}^{+\infty}Q_1^{\rm ev}(x_{\rm m},0)\mathrm{d}x_{\rm m}=\mathcal{B}_2-\frac{1}{3}(\beta_+-\beta_-), \tag{4.40}$$

and $\mathcal{C}_1$ is finite because $Q_1^{\rm ev}$ vanishes for $x_{\rm m}\to\pm\infty$ according to (4.25) (see appendix A.2 for the relation between $\mathcal{C}_1$ and $\mathcal{B}_2$). By gathering (4.38) and (4.39) in (4.37), the diverging terms, linear and quadratic in $x_{\rm m}^*$ cancel out, leaving us with

$$[\![q^{(2)}]\!]=\mathcal{C}_1\frac{\partial^2 q^{(0)}}{\partial t^2}(0,t),\quad\text{with}\quad q^{(2)}(0^\pm,t)=\beta_\pm U_x^{(2)}(0^\pm,t). \tag{4.41}$$

In particular, we have

$$q^{(2)}(0^\pm,t)=\overline{q^{(2)}}(t)\pm\frac{\mathcal{C}_1}{2}\frac{\partial^2 q^{(0)}}{\partial t^2}(0,t), \tag{4.42}$$

with $\overline{q^{(2)}}(t)=\frac{1}{2}\left(q^{(2)}(0^-,t)+q^{(2)}(0^+,t)\right)$ being the average flux across the discontinuity.

4.5. Jump conditions at order $n=3$. At order $n=3$, non-linear contributions arise in the jump conditions.

PROPOSITION 4.4. *Jump conditions with non-linear contribution apply at the order $n=3$:*

$$(\mathsf{JC})_1^3:\quad [\![\eta^{(3)}]\!]=-\left(\mathcal{B}_1\frac{\partial\overline{q^{(2)}}}{\partial t}+\mathcal{B}_2\frac{\partial^2\overline{\eta^{(1)}}}{\partial t^2}+\mathcal{B}_3\frac{\partial^3 q^{(0)}}{\partial t^3}\right)-\frac{\alpha}{2}(q^{(0)})^2\left(\frac{1}{\beta_+^2}-\frac{1}{\beta_-^2}\right), \tag{4.43}$$

$$(\mathsf{JC})_2^3:\quad [\![q^{(3)}]\!]=\mathcal{C}_1\frac{\partial^2 q^{(1)}}{\partial t^2}+\mathcal{C}_2\frac{\partial^3\eta^{(0)}}{\partial t^3}-\alpha\eta^{(0)}q^{(0)}\left(\frac{1}{\beta_+}-\frac{1}{\beta_-}\right), \tag{4.44}$$

where the effective coefficients $\mathcal{B}_3$ and $\mathcal{C}_2$ are defined in (4.48) and (4.52).

4.5.1. Derivation of $(\mathsf{JC})_1^3$. We repeat the exercice and consider the problem satisfied by $(\tilde{\boldsymbol{u}}^{(2)},\tilde{\varphi}^{(3)})$ which consists in ${}^{\rm m}_{0}(\mathsf{Inc})^2$, ${}^{\rm m}_{0}(\mathsf{Rot})^2$ and ${}^{\rm m}_{0}(\mathsf{RC})^2$ in (4.5). The condition at $z_{\rm m}=0$ is obtained using (4.9), with $\tilde{v}_z^{(1)}$ given by ${}^{\mu}_{0}(\mathsf{KC})^1$, ${}^{\mu}_{0}(\mathsf{DC})^1$ in (4.7) and (4.8) for $n=1$, and finally (4.23) as well as (3.16).
The conditions when $x_{\rm m}\to\pm\infty$ on $\tilde{\boldsymbol{u}}^{(2)}$ are obtained as follows. For $\tilde{u}_x^{(2)}$, we use (4.11) which involves $u_x^{(2)}=U_x^{(2)}-(\frac{z_{\rm m}^2}{2\beta}+z_{\rm m}+\frac{\beta}{3})\partial_{tt}U_x^{(0)}$ from (3.22)-(3.23) and (3.11)-(3.12) (with (3.15) for $n=0,2$ and $q^{(2)}$ given in (4.42)), $\partial_x u_x^{(1)}(x_{\rm m},0^\pm,t)=-\partial_t\eta^{(1)}(x_{\rm m},t)/\beta_\pm$ from (3.12) and (3.14) ($\eta^{(1)}$ given in (4.28)), and $\partial_{xx}u_x^{(0)}(x_{\rm m},0^\pm,t)=\partial_{tt}q^{(0)}(0,t)/\beta_\pm$. For $\tilde{u}_z^{(2)}$, we use (4.12) which involves $u_z^{(1)}=\left(\frac{z_{\rm m}}{\beta_\pm}+1\right)\partial_t\eta^{(1)}$ and

$\partial_x u_z^{(0)} = -\frac{1}{\beta_\pm}\left(\frac{z_{\rm m}}{\beta_\pm}+1\right)\partial_{tt}q^{(0)}$, from (3.17) with (3.11)- (3.12). We obtain

$$\left\{\begin{array}{l} \mathrm{div}_{\rm m}\tilde{\boldsymbol{u}}^{(2)} = 0, \quad \tilde{\boldsymbol{u}}^{(2)} = \boldsymbol{\nabla}_{\rm m}\tilde{\varphi}^{(3)}, \\ \\ \tilde{\boldsymbol{u}}^{(2)}\cdot\mathbf{n} = 0 \ \text{ on the walls}, \qquad \tilde{u}_z^{(2)} = \dfrac{\partial\overline{\eta^{(1)}}}{\partial t} - Q_1\dfrac{\partial^2 q^{(0)}}{\partial t^2} \quad \text{at} \ \ z_{\rm m}=0, \\ \\ \lim\limits_{x_{\rm m}\to\pm\infty}\tilde{\boldsymbol{u}}^{(2)} = \dfrac{1}{\beta_\pm}\left(\overline{q}^{(2)} - x_{\rm m}\dfrac{\partial\overline{\eta^{(1)}}}{\partial t} + \chi_x^\pm\dfrac{\partial^2 q^{(0)}}{\partial t^2}\right)\mathbf{e}_x + \left(\left(\dfrac{z_{\rm m}}{\beta_\pm}+1\right)\dfrac{\partial\overline{\eta^{(1)}}}{\partial t} - \chi_z^\pm\dfrac{\partial^2 q^{(0)}}{\partial t^2}\right)\mathbf{e}_z, \end{array}\right.$$

where we have defined

$$\begin{aligned} \chi_x^\pm(x_{\rm m},z_{\rm m}) = \ & \pm\frac{\mathcal{C}_1}{2} - \frac{\beta_\pm}{3} - \left(\frac{z_{\rm m}^2}{2\beta_\pm}+z_{\rm m}\right) + \frac{x_{\rm m}}{2}\left(\frac{x_{\rm m}}{\beta_\pm}\pm\mathcal{B}_1\right), \\ \chi_z^\pm(x_{\rm m},z_{\rm m}) = \ & \left(\frac{z_{\rm m}}{\beta_\pm}+1\right)\left(\frac{x_{\rm m}}{\beta_\pm}\pm\frac{\mathcal{B}_1}{2}\right), \end{aligned}$$

The above problem is linear with respect to $\overline{q}^{(2)}(t)$, $\partial_t\overline{\eta^{(1)}}(t)$ and $\partial_{tt}q^{(0)}(0,t)$, so we set

(4.45)
$$\begin{aligned} \tilde{\varphi}^{(3)} &= \overline{q}^{(2)}(t)Q_1(x_{\rm m},z_{\rm m}) + \frac{\partial\overline{\eta^{(1)}}}{\partial t}(t)Q_2(x_{\rm m},z_{\rm m}) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)Q_3(x_{\rm m},z_{\rm m}) + \overline{\phi^{(3)}}(t), \\ \tilde{\boldsymbol{u}}^{(2)} &= \overline{q}^{(2)}(t)\boldsymbol{\nabla}Q_1(x_{\rm m},z_{\rm m}) + \frac{\partial\overline{\eta^{(1)}}}{\partial t}(t)\boldsymbol{\nabla}Q_2(x_{\rm m},z_{\rm m}) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)\boldsymbol{\nabla}Q_3(x_{\rm m},z_{\rm m}), \end{aligned}$$

where Q_3 satisfies the elementary problem

(4.46)
$$\left\{\begin{array}{l} \Delta Q_3 = 0, \\ \\ \boldsymbol{\nabla}Q_3\cdot\mathbf{n} = 0 \ \text{ on the walls}, \quad \dfrac{\partial Q_3}{\partial z_{\rm m}}(x_{\rm m},0) = -Q_1(x_{\rm m},0), \\ \\ \lim\limits_{x_{\rm m}\to\pm\infty}\boldsymbol{\nabla}Q_3(x_{\rm m},z_{\rm m}) = \dfrac{1}{\beta_\pm}\chi_x^\pm(x_{\rm m},z_{\rm m})\mathbf{e}_x - \chi_z^\pm(x_{\rm m},z_{\rm m})\mathbf{e}_z. \end{array}\right.$$

As the solution Q_3 is defined up to a constant, we impose

(4.47)
$$Q_3(x_{\rm m},z_{\rm m}) = Q_3^{\rm ev}(x_{\rm m},z_{\rm m}) + \left\{\begin{array}{ll} \zeta^+(x_{\rm m},z_{\rm m}) + \dfrac{\mathcal{B}_3}{2} & \text{if} \quad x_{\rm m}>0, \\ \\ \zeta^-(x_{\rm m},z_{\rm m}) - \dfrac{\mathcal{B}_3}{2} & \text{if} \quad x_{\rm m}<0, \end{array}\right. , \quad \lim_{x_{\rm m}\to\pm\infty} Q_3^{\rm ev} = 0,$$

where $Q_3^{\rm ev}$ is an evanescent field, $\zeta^\pm$ polynomial functions given by

$$\zeta^\pm(x_{\rm m},z_{\rm m}) = x_{\rm m}\left(-\frac{1}{3}\pm\frac{\mathcal{C}_1}{2\beta_\pm}\right) + \frac{x_{\rm m}^2}{2\beta_\pm}\left(\frac{x_{\rm m}}{3\beta_\pm}\pm\frac{\mathcal{B}_1}{2}\right) - \left(\frac{z_{\rm m}^2}{2\beta_\pm}+z_{\rm m}\right)\left(\frac{x_{\rm m}}{\beta_\pm}\pm\frac{\mathcal{B}_1}{2}\right),$$

and $\mathcal{B}_3$ a blockage coefficient defined as follows

(4.48)
$$\mathcal{B}_3 = \lim_{x_{\rm m}\to+\infty}\Big(Q_3 - \zeta^+(x_{\rm m},z_{\rm m})\Big) - \lim_{x_{\rm m}\to-\infty}\Big(Q_3 - \zeta^-(x_{\rm m},z_{\rm m})\Big).$$

To obtain the jump of $\varphi^{(3)}$, we use (4.11) which involves $\tilde{\varphi}^{(3)}$ in (4.45) (with Q_n in (4.25), (4.34), (4.47)), $\partial_x\varphi^{(2)} = \frac{q^{(2)}}{\beta_\pm} - \left(\frac{z_{\rm m}^2}{2\beta_\pm^2}+\frac{z_{\rm m}}{\beta_\pm}+\frac{1}{3}\right)\partial_{tt}q^{(0)}$ from (3.20) and (3.23)

(with $q^{(2)}$ in (4.42)) and, from (3.12)-(3.14), $\partial_{xx}\varphi^{(1)} = -\partial_t\eta^{(1)}/\beta_\pm$ (with $\eta^{(1)}$ in (4.28)) and $\partial_{xxx}\varphi^{(0)} = -\partial_{tt}q^{(0)}/\beta_\pm$. We obtain

$$\varphi^{(3)}(0^\pm, z_{\rm m}, t) = \phi^{(3)}(0^\pm, t) - \left(\frac{z_{\rm m}^2}{2\beta_\pm} + z_{\rm m}\right)\left(-\frac{\partial\overline{\eta^{(1)}}}{\partial t}(t) \pm \frac{\mathcal{B}_1}{2}\frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)\right),$$

with

$$\phi^{(3)}(0^\pm, t) = \pm\left(\frac{\mathcal{B}_1}{2}\overline{q}^{(2)}(t) + \frac{\mathcal{B}_2}{2}\frac{\partial\overline{\eta^{(1)}}}{\partial t}(t) + \frac{\mathcal{B}_3}{2}\frac{\partial^2 q^{(0)}}{\partial t^2}(0,t)\right) + \overline{\phi^{(3)}}(t).$$

Note that the above relation is consistent with (3.26) since $\partial_x U_x^{(1)} = -\partial_t\eta^{(1)}$ with $\eta^{(1)}$ in (4.28). Using (3.27) and (4.14), we deduce that

$$\eta^{(3)}(0^\pm, t) = -\frac{\partial\phi^{(3)}}{\partial t}(0^\pm, t) - \frac{\alpha}{2\beta_\pm^2}(q^{(0)}(t))^2,$$

hence the jump (4.43) of $\eta^{(3)}$.

4.5.2. Derivation of $(\mathsf{JC})^3_2$. We repeat the exercice for the last time, with ${}^{\rm m}_0(\mathsf{Inc})^3$ in (4.5) which we integrate over $\Omega^*_{\rm m}$, see (4.18). We obtain, after using ${}^{\rm m}_0(\mathsf{RC})^3$ in (4.5),

(4.49)
$$\int_{-\beta_+}^0 \tilde{u}_x^{(3)}(x_{\rm m}^*, z_{\rm m}, t)\mathrm{d}z_{\rm m} - \int_{-\beta_-}^0 \tilde{u}_x^{(3)}(-x_{\rm m}^*, z_{\rm m}, t)\mathrm{d}z_{\rm m} + \int_{-x_{\rm m}^*}^{x_{\rm m}^*} \tilde{u}_z^{(3)}(x_{\rm m}, 0, t)\mathrm{d}x_{\rm m} = 0.$$

The first two integrals in (4.49) involve the behavior of $\tilde{u}_x^{(3)}$ which, for $x_{\rm m}^* \to +\infty$, is given by (4.11). It involves several terms, namely $\partial_{x^p}u_x^{(m)}$, $p+m=3$ which after integration with respect to $z_{\rm m}$ provides $\beta_\pm\partial_{x^p}U_x^{(m)}(0^\pm,t)$ (see (3.15)). These contributions are determined as follows.
From (3.19) and (3.21), we have $\beta_\pm\partial_x U_x^{(2)}(0^\pm, z_{\rm m}, t) = \partial_{tt}\phi^{(2)}(0^\pm, t)$. Then, from (3.11) and (3.12), we have $\beta_\pm\partial_{xx}U_x^{(1)}(0^\pm,t) = \partial_{tt}q^{(1)}(0,t)/\beta_\pm$ and $\beta_\pm\partial_{xxx}U_x^{(0)}(0^\pm,t) = -1/\beta_\pm\partial_{ttt}\eta^{(0)}(0,t)$. Consequently, noting $J(\pm x_{\rm m}^*, t) = \int_{-\beta_\pm}^0 \tilde{u}_x^{(3)}(\pm x_{\rm m}^*, z_{\rm m}, t)\mathrm{d}z_{\rm m}$, we have

(4.50)
$$J(\pm x_{\rm m}^*, t) \underset{x_{\rm m}^*\to+\infty}{\sim} q^{(3)}(0^\pm, t) \pm x_{\rm m}^*\frac{\partial^2\phi^{(2)}}{\partial t^2}(0^\pm, t) + \frac{{x_{\rm m}^*}^2}{2\beta_\pm}\frac{\partial^2 q^{(1)}}{\partial t^2}(0,t) \mp \frac{{x_{\rm m}^*}^3}{6\beta_\pm}\frac{\partial^3\eta^{(0)}}{\partial t^3}(0,t).$$

For the third integral in (4.49), we use (4.9) and take into account the following relations. By integrating ${}^\mu_0(\mathsf{Inc})^2$ in (4.6) with respect to z_μ we have $\tilde{v}_z^{(2)}(x_{\rm m}, z_\mu, t) = -\alpha\eta^{(0)}(0,t)\partial_{x_{\rm m}}\tilde{u}_x^{(0)}(x_{\rm m},0,t)z_\mu + \tilde{v}_z^{(2)}(x_{\rm m},0,t)$ with $\tilde{v}_z^{(2)}(x_{\rm m},0,t) = -\partial_{tt}\tilde{\varphi}^{(2)}(x_{\rm m},0,t)$, from ${}^\mu_0(\mathsf{KC})^2$ and ${}^\mu_0(\mathsf{DC})^2$ in (4.7), as well as $\tilde{\psi}^{(2)}(x_{\rm m}, z_\mu, t) = \tilde{\varphi}^{(2)}(x_{\rm m}, 0, t)$ from (4.8) ($\tilde{\psi}^{(2)}$ being independent of z_μ from ${}^\mu_0(\mathsf{Rot})^2_z$ in (4.6)). Then, we have $\tilde{\eta}^{(0)}(t)\,\partial_{z_{\rm m}}\tilde{u}_z^{(0)} = -\eta^{(0)}(0,t)\,\partial_{x_{\rm m}}\tilde{u}_x^{(0)}$ from (4.16) and ${}^{\rm m}_0(\mathsf{Inc})^0$ in (4.5). It follows that

$$\tilde{u}_z^{(3)}(x_{\rm m}, 0, t) = -\frac{\partial^2\tilde{\varphi}^{(2)}}{\partial t^2}(x_{\rm m},0,t) + \alpha\eta^{(0)}(0,t)\frac{\partial\tilde{u}_x^{(0)}}{\partial x_{\rm m}}(x_{\rm m},0,t).$$

Now we just need to use the forms of $\tilde{u}_x^{(0)}$ in (4.23) and of $\tilde{\varphi}^{(2)}$ in (4.32) (as well as

(4.25) and (4.34)) to find that, denoting $I(t) = \int_{-x_{\rm m}^*}^{x_{\rm m}^*} \tilde{u}_z^{(3)}(x_{\rm m}, 0, t)\mathrm{d}x_{\rm m}$, we have

(4.51)
$$I(t) \underset{x_{\rm m}^* \to +\infty}{\sim} -2x_{\rm m}^* \frac{\partial^2 \overline{\phi^{(2)}}}{\partial t^2}(t) - \frac{\partial^2 \overline{q}^1}{\partial t^2}(t)\left(\mathcal{C}_1 + \frac{{x_{\rm m}^*}^2}{2\beta_+} - \frac{{x_{\rm m}^*}^2}{2\beta_-}\right) - \frac{\partial^3 \eta^{(0)}}{\partial t^3}(0,t)\left(\mathcal{C}_2 - \frac{{x_{\rm m}^*}^3}{6\beta_+} - \frac{{x_{\rm m}^*}^3}{6\beta_-}\right) + \alpha\left(\frac{1}{\beta_+} - \frac{1}{\beta_-}\right)\eta^{(0)}(0,t)q^{(0)}(0,t),$$

where the effective parameter $\mathcal{C}_2$ is defined as follows

(4.52)
$$\mathcal{C}_2 = \int_{-\infty}^{+\infty} Q_2^{\rm ev}(x_{\rm m}, 0)\mathrm{d}x_{\rm m},$$

with $Q_2^{\rm ev}$ given by (4.33) and (4.34). By gathering (4.50) and (4.51) in (4.49), we obtain that all the diverging terms, linear, quadratic and quartic with respect to $x_{\rm m}^*$, cancel out, and we get the jump (4.44) in the flux.

5. Construction of a unique model. In this section, we reconstruct the Boussinesq equations and corresponding jump conditions at the depth transition from the results of the asymptotic analysis. Classically, this means combining the equations obtained at orders $n = 0$ to 3 into a single equation. We begin by reconstructing the Boussinesq equations, which is a classical procedure, before repeating the same procedure for the jump conditions.

5.1. The Boussinesq equations. Gathering the results from (3.11), (3.18) and (3.24), we have for $\beta = \beta_\pm$,

(5.1)
$$\begin{cases} \dfrac{\partial U_x^{(n)}}{\partial t} + \dfrac{\partial \eta^{(n)}}{\partial x} = 0, \quad n = 0, 1. \\ \dfrac{\partial U_x^{(2)}}{\partial t} + \dfrac{\partial \eta^{(2)}}{\partial x} - \dfrac{\beta^2}{3}\dfrac{\partial^3 U_x^{(0)}}{\partial t \partial x^2} = 0, \\ \dfrac{\partial U_x^{(3)}}{\partial t} + \dfrac{\partial \eta^{(3)}}{\partial x} - \dfrac{\beta^2}{3}\dfrac{\partial^3 U_x^{(1)}}{\partial t \partial x^2} + \alpha U_x^{(0)}\dfrac{\partial U_x^{(0)}}{\partial x} = 0, \end{cases}$$

and gathering the results (3.12), (3.19) and (3.25), we also have

(5.2)
$$\begin{cases} \beta\dfrac{\partial U_x^{(n)}}{\partial x} + \dfrac{\partial \eta^{(n)}}{\partial t} = 0, \qquad n = 0, 1, 2 \\ \beta\dfrac{\partial U_x^{(3)}}{\partial x} + \dfrac{\partial \eta^{(3)}}{\partial t} + \alpha\dfrac{\partial}{\partial x}\left(\eta^{(0)}U_x^{(0)}\right) = 0. \end{cases}$$

Considering the truncated expansions of the fields, $U_x = \sum_{n=0}^{3}\delta^n U_x^{(n)}$ and $\eta = \delta\sum_{n=0}^{3}\delta^n\eta^{(n)}$ (up to terms $O(\delta^4)$) and summing the results from order $n = 0$ to 3 in (5.1)-(5.2) after multiplication by δ^n we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\delta U_x - \dfrac{\beta^2\delta^3}{3}\dfrac{\partial^2 U_x}{\partial x^2}\right) + \dfrac{\partial}{\partial x}\left(\eta + \alpha\delta^4\dfrac{(U_x)^2}{2}\right) = 0, \\ \dfrac{1}{\delta}\dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}\left((\beta + \alpha\delta^2\eta)U_x\right) = 0. \end{cases}$$

We then simply return to the dimensional form of the problem, using (2.10) and scalings (2.8)-(2.9) to obtain the classical Boussinesq equations (2.5). Note that a

more general family of Boussinesq equations could have been obtained at this stage by using Boussinesq's trick, the idea being that lower-order relations such as (3.11)-(3.12) can be used to modify the higher-order equations without loss of the asymptotic order of approximation, see *e.g.* [6, 7].

5.2. The jump conditions. Gathering the results of (4.13), (4.20), (4.30) and (4.43) we have

$$\begin{cases} [\![\eta^{(0)}]\!] = 0, \\ [\![\eta^{(1)}]\!] = -\mathcal{B}_1 \dfrac{\partial q^{(0)}}{\partial t}(0,t) \\ [\![\eta^{(2)}]\!] = -\mathcal{B}_1 \dfrac{\partial q^{(1)}}{\partial t}(0,t) - \mathcal{B}_2 \dfrac{\partial^2 \eta^{(0)}}{\partial t^2}(0,t), \\ \left[\!\!\left[\eta^{(3)} + \dfrac{\alpha}{2}(U_x^{(0)})^2\right]\!\!\right] = -\mathcal{B}_1 \dfrac{\partial \overline{q^{(2)}}}{\partial t}(t) - \mathcal{B}_2 \dfrac{\partial^2 \overline{\eta^{(1)}}}{\partial t^2}(t) - \mathcal{B}_3 \dfrac{\partial^3 q^{(0)}}{\partial t^3}(0,t), \end{cases} \tag{5.3}$$

and gathering the results of (4.14), (4.21), (4.31) and (4.44), we also have

$$\begin{cases} [\![q^{(n)}]\!] = 0, \qquad n = 0, 1, \\ [\![q^{(2)}]\!] = \mathcal{C}_1 \dfrac{\partial^2 q^{(0)}}{\partial t^2}(0,t), \\ [\![q^{(3)} + \alpha \eta^{(0)} U_x^{(0)}]\!] = \mathcal{C}_1 \dfrac{\partial^2 q^{(1)}}{\partial t^2}(0,t) + \mathcal{C}_2 \dfrac{\partial^3 \eta^{(0)}}{\partial t^3}(0,t). \end{cases} \tag{5.4}$$

Considering again the truncated expansions of the fields $U_x = \sum_{n=0}^{3} \delta^n U_x^{(n)}$ (with $q^{(n)}$ defined in (3.15)) and $\eta = \delta \sum_{n=0}^{3} \delta^n \eta^{(n)}$ and summing the results at order $n = 0$ to 3 given by (5.3)-(5.4) after multiplication by δ^n give

$$\begin{cases} \left[\!\!\left[\eta + \alpha\delta^4 \dfrac{(U_x)^2}{2}\right]\!\!\right] = -\mathcal{B}_1 \delta^2 \dfrac{\partial \overline{\beta U_x}}{\partial t} - \mathcal{B}_2 \delta^2 \dfrac{\partial^2 \overline{\eta}}{\partial t^2} - \mathcal{B}_3 \delta^4 \dfrac{\partial^3 \overline{\beta U_x}}{\partial t^3}, \\ [\![(\beta + \alpha\delta^2 \eta) U_x]\!] = \mathcal{C}_1 \delta^2 \dfrac{\partial^2 \overline{\beta U_x}}{\partial t^2} + \mathcal{C}_2 \delta^2 \dfrac{\partial^3 \overline{\eta}}{\partial t^3}. \end{cases} \tag{5.5}$$

Once again, we only need to return to the dimensional form of the problem using (2.10) and the scalings (2.8)-(2.9) to obtain the jump conditions announced in (2.6).

6. Conclusion. In this work, we have sought to challenge, in the presence of an abrupt depth transition, the classical asymptotic approach used to obtain the Boussinesq equations. Indeed, the classical procedure for a flat or slowly varying sea bottom relies on normalizing the vertical coordinate by the total water depth (including the wave amplitude at the free surface), which obviously cannot be used here since the water depth is a discontinuous function. To overcome this problem, we introduced a new three-scale analysis framework that enables consistent navigation using matching conditions between a macroscopic scale (the incident wavelength), a mesoscopic scale (the water depth at rest or the boundary layer thickness near the discontinuity), and a microscopic scale (the wave amplitude or the boundary layer thickness at the free surface). The asymptotic expansion of the solutions up to the third-order in this three-scale framework led to the Boussinesq equations, supplemented by jump conditions at the depth discontinuity. These jump conditions involve 5 boundary layer

coefficients that depend only on the water depth ratio and for which the (semi-)-analytical formula has been provided.

Some extensions of the present study are straightforward, others less so. On the one hand, the extension of the present analysis to variants of the classical Boussinesq equations [29, 6, 7] is almost immediate, by adapting the procedure for constructing a single model in section 5, including the jump conditions. Also, the derivation of the jump conditions in three-dimensional geometry or for a more general abrupt discontinuity profile present no additional difficulties, since the same asymptotic procedure can be carried out. On the other hand, the theoretical and numerical analysis of the coupled model "Boussinesq and jump conditions" is a more tedious challenge. It would be interesting, for example, to find analytical solutions to Boussinesq-type equations (e.g. solitary waves) in the presence of an abrupt depth transition using these jump conditions. We may think to scattering problems, of solitons by adapting the present model to a Korteweg-de Vries-like form [15, 31, 32], or of shock waves in shallow water [17], and to assess the effect of the boundary layers which has not been considered thus far. It also remains to establish appropriate numerical discretization schemes for these jump conditions. This is a necessary step before considering validation of the jump conditions by comparison with direct numerical simulations or with experimental data.

Appendix A. Effective parameters entering the jump conditions.

A.1. Variations of effective parameters as a function of depth ratio. The effective parameters entering the jump conditions (2.6) depend only on the depth ratio $\nu = 1 + h_{\rm s}/h_0$. They can be calculated by numerically solving the elementary problems (4.24), (4.33) and (4.46) and using their definitions (4.26), (4.35), (4.48) and (4.40), (4.52). We give below approximate expressions that are good estimates, as shown in figure 6.

$$\left\{\begin{array}{l} \mathcal{B}_1 = \frac{1}{\pi}\left(\frac{\nu^2+1}{\nu}\log\frac{\nu+1}{|\nu-1|} - 2\log\frac{4\nu}{|\nu^2-1|}\right), \\ \mathcal{B}_2 \simeq -0.2\nu(1-\nu), \\ \mathcal{B}_3 \simeq 0.1(1-\nu)\mathcal{B}_1, \end{array}\right. \tag{A.1}$$

and

$$\mathcal{C}_1 = \mathcal{B}_2 + \frac{1}{3}(1-\nu), \qquad \mathcal{C}_2 \simeq -0.044\nu^2(1-\nu)^2. \tag{A.2}$$

Note that the expression for $\mathcal{B}_1$ id exact and is due to [55], Eq. (2.12) in this reference, and the relation between $\mathcal{C}_1$ and $\mathcal{B}_2$ is proved in the next section.

A.2. Relations between the parameters. The effective parameters $\mathcal{B}_2$ and $\mathcal{B}_3$ defined in (4.35) and (4.48) can be determined from knowledge of Q_1 only, which is given by (4.24). Indeed, for a step, we have the following relations

$$\mathcal{B}_2 = \mathcal{C}_1 + \frac{1}{3}(\beta_+ - \beta_-), \qquad \mathcal{B}_3 = \mathcal{G}_1 - \frac{\mathcal{B}_1}{6}(\beta_+ + \beta_-), \tag{A.3}$$

with $(\mathcal{B}_1, \mathcal{C}_1)$ given by (4.26) and (4.52) and $\mathcal{G}_1$ defined by

$$\mathcal{G}_1 = -\int_0^{+\infty} Q_+(x_{\rm m})\,{\rm d}x_{\rm m} - \int_{-\infty}^{0} Q_-(x_{\rm m})^2\,{\rm d}x_{\rm m}.$$

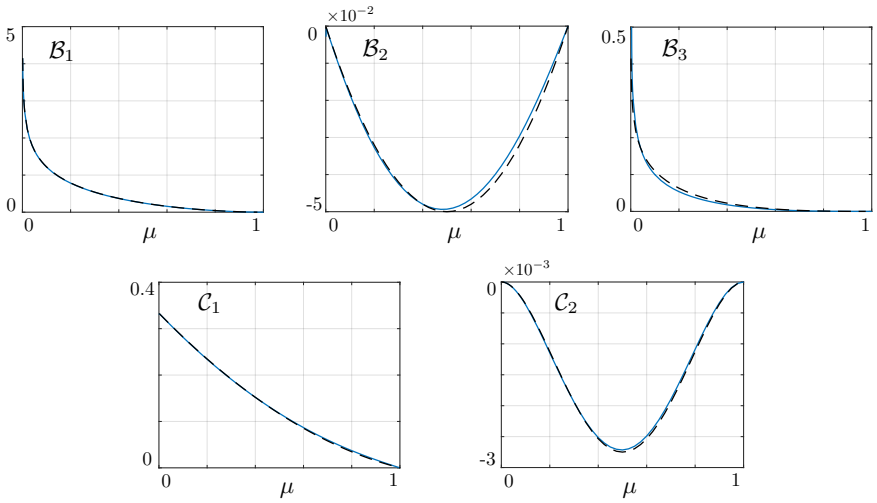

FIG. 6. *Variations of effective parameters, $\mathcal{B}_n$, $n = 1, 2, 3$ and $\mathcal{C}_n$, $n = 1, 2$ entering jump conditions (2.6), as a function of $\nu = 1 - h_s/h_0$. Solid blue lines from direct numerical calculations of elementary solutions Q_n in (4.24), (4.33) and (4.46); dashed black lines from (A.1) and (A.2).*

with

$$Q_{\pm}(x_{\mathrm{m}}) = (Q_1(x_{\mathrm{m}}, 0))^2 - \left(\frac{x_{\mathrm{m}}}{\beta_{\pm}} \pm \frac{\mathcal{B}_1}{2}\right)^2$$

To prove (A.3), simply note that using (4.24), (4.33) and (4.46) for $i = 2, 3$ we have

$$0 = \int_{\Omega_{\mathrm{m}}} (Q_1 \Delta Q_i - Q_i \Delta Q_1) = -\int_{\partial\Omega_{\mathrm{m}}} (Q_1 \boldsymbol{\nabla} Q_i - Q_i \boldsymbol{\nabla} Q_1) \cdot \mathbf{n}.$$

Applying the boundary conditions on (Q_1, Q_2, Q_3) and using the decomposition (4.25), (4.34) and (4.47) we obtain after a few calculations

$$0 = \int_{\partial\Omega_{\mathrm{m}}} (Q_1 \boldsymbol{\nabla} Q_2 - Q_2 \boldsymbol{\nabla} Q_1) \cdot \mathbf{n} = \mathcal{C}_1 + \frac{1}{3}(\beta_+ - \beta_-) - \mathcal{B}_2,$$

$$0 = \int_{\partial\Omega_{\mathrm{m}}} (Q_1 \boldsymbol{\nabla} Q_3 - Q_3 \boldsymbol{\nabla} Q_1) \cdot \mathbf{n} = \mathcal{G}_1 - \frac{\mathcal{B}_1}{6}(\beta_+ + \beta_-) - \mathcal{B}_3.$$

REFERENCES

[1] T. Adcock and P. Taylor, *The physics of anomalous ('rogue') ocean waves*, Reports on Progress in Physics, 77 (2014), p. 105901.
[2] N. Akhmediev, A. Ankiewicz, and M. Taki, *Waves that appear from nowhere and disappear without a trace*, Physics Letters A, 373 (2009), pp. 675–678.
[3] N. Akhmediev, J. Soto-Crespo, and A. Ankiewicz, *Extreme waves that appear from nowhere: on the nature of rogue waves*, Physics Letters A, 373 (2009), pp. 2137–2145.
[4] E. F. Bartholomeusz, *The reflexion of long waves at a step*, in Mathematical Proceedings of the Cambridge Philosophical Society, vol. 54, Cambridge Univ Press, 1958, pp. 106–118.
[5] K. A. Belibassakis and G. A. Athanassoulis, *Extension of second-order stokes theory to variable bathymetry*, Journal of Fluids Mechanics, 464 (2002), pp. 35–80.

[6] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. i: Derivation and linear theory*, Journal of Nonlinear Science, 12 (2002), pp. 283–318.

[7] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: Ii. the nonlinear theory*, Nonlinearity, 17 (2004), p. 925.

[8] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, Journal de mathématiques pures et appliquées, 17 (1872), pp. 55–108.

[9] M. Chen, *Exact traveling-wave solutions to bidirectional wave equations*, International Journal of Theoretical Physics, 37 (1998), pp. 1547–1567.

[10] S. Draycott, Y. Li, P. Stansby, T. Adcock, and T. van den Bremer, *Harmonic-induced wave breaking due to abrupt depth transitions: An experimental and numerical study*, Coastal Engineering, 171 (2022), p. 104041.

[11] G. Ducrozet, A. Slunyaev, and Y. Stepanyants, *Transformation of envelope solitons on a bottom step*, Physics of Fluids, 33 (2021), p. 066606.

[12] K. Dysthe, H. E. Krogstad, and P. Müller, *Oceanic rogue waves*, Annu. Rev. Fluid Mech., 40 (2008), pp. 287–310.

[13] O. Gramstad, H. Zeng, K. Trulsen, and G. Pedersen, *Freak waves in weakly nonlinear unidirectional wave trains over a sloping bottom in shallow water*, Physics of Fluids, 25 (2013), p. 122103.

[14] J. Grue, *Nonlinear water waves at a submerged obstacle or bottom topography*, Journal of Fluid Mechanics, 244 (1992), pp. 455–476.

[15] R. S. Johnson, *A modern introduction to the mathematical theory of water waves*, no. 19, Cambridge university press, 1997.

[16] S. Kakuno and P. L.-F. Liu, *Scattering of water waves by vertical cylinders*, Journal of Waterway, Port, Coastal, and Ocean Engineering, 119 (1993), pp. 302–322.

[17] H. Kalisch, D. Mitrovic, and V. Teyekpiti, *Delta shock waves in shallow water flow*, Physics Letters A, 381 (2017), pp. 1138–1144.

[18] C. Kharif and E. Pelinovsky, *Physical mechanisms of the rogue wave phenomenon*, European Journal of Mechanics-B/Fluids, 22 (2003), pp. 603–634.

[19] C. Kharif, E. Pelinovsky, and A. Slunyaev, *Shallow-water rogue waves*, Rogue Waves in the Ocean, (2009), pp. 173–209.

[20] C. Lawrence, K. Trulsen, and O. Gramstad, *Statistical properties of wave kinematics in long-crested irregular waves propagating over non-uniform bathymetry*, Physics of Fluids, 33 (2021), p. 046601.

[21] C. Lawrence, K. Trulsen, and O. Gramstad, *Extreme wave statistics of surface elevation and velocity field of gravity waves over a two-dimensional bathymetry*, Journal of Fluid Mechanics, 939 (2022), p. A41.

[22] Y. Li and A. Chabchoub, *On the formation of coastal rogue waves in water of variable depth*, Cambridge Prisms: Coastal Futures, 1 (2023), p. e33.

[23] Y. Li, S. Draycott, T. Adcock, and T. Van den Bremer, *Surface wavepackets subject to an abrupt depth change. part 2. experimental analysis*, Journal of Fluid Mechanics, 915 (2021), p. A72.

[24] Y. Li, S. Draycott, Y. Zheng, Z. Lin, T. Adcock, and T. Van den Bremer, *Why rogue waves occur atop abrupt depth transitions*, Journal of Fluid Mechanics, 919 (2021), p. R5.

[25] Y. Li, Y. Zheng, Z. Lin, T. Adcock, and T. Van den Bremer, *Surface wavepackets subject to an abrupt depth change. part 1. second-order theory*, Journal of Fluid Mechanics, 915 (2021), p. A71.

[26] Z. Li, *Numerical simulations of waves passing over a slope*, PhD thesis, University of Oxford, 2022.

[27] Z. Li, T. Tang, Y. Li, S. Draycott, T. Van den Bremer, and T. Adcock, *Wave loads on ocean infrastructure increase as a result of waves passing over abrupt depth transitions*, Journal of Ocean Engineering and Marine Energy, 9 (2023), pp. 309–317.

[28] P. Madsen, R. Murray, and O. Sørensen, *A new form of the boussinesq equations with improved linear dispersion characteristics*, Coastal engineering, 15 (1991), pp. 371–388.

[29] P. Madsen and H. Schäffer, *Higher–order boussinesq–type equations for surface gravity waves: derivation and analysis*, Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 356 (1998), pp. 3123–3181.

[30] P. Madsen and O. Sørensen, *Bound waves and triad interactions in shallow water*, Ocean Engineering, 20 (1993), pp. 359–388.

[31] A. J. Majda, M. Moore, and D. Qi, *Statistical dynamical model to predict extreme events and anomalous features in shallow water waves with abrupt depth change*, Proceedings of the National Academy of Sciences, 116 (2019), pp. 3982–3987.
[32] A. J. Majda and D. Qi, *Statistical phase transitions and extreme events in shallow water waves with an abrupt depth change*, Journal of Statistical Physics, 179 (2020), pp. 1718–1741.
[33] P. A. Martin and R. A. Dalrymple, *Scattering of long waves by cylindrical obstacles and gratings using matched asymptotic expansions*, Journal of Fluid Mechanics, 188 (1988), pp. 465–490.
[34] S. Massel, *Extended refraction-diffraction equation for surface waves*, Coastal Engineering, 19 (1993), pp. 97–126.
[35] S. R. Massel, *Harmonic generation by waves propagating over a submerged step*, Coastal Engineering, 7 (1983), pp. 357–380.
[36] A. Maurel, J.-J. Marigo, P. Cobelli, P. Petitjeans, and V. Pagneux, *Revisiting the anisotropy of metamaterials for water waves*, Physical Review B, 96 (2017), p. 134310.
[37] A. Maurel, K. Pham, and J.-J. Marigo, *Scattering of gravity waves by a periodically structured ridge of finite extent*, Journal of Fluid Mechanics, 871 (2019), pp. 350–376.
[38] P. McIver and C. Linton, *The added mass of bodies heaving at low frequency in water of finite depth*, Applied Ocean Research, 13 (1991), pp. 12–17.
[39] C. C. Mei, M. A. Stiassnie, and D. K.-P. Yue, *Theory and applications of ocean surface waves: Part 1: linear aspects*, World Scientific, 2005.
[40] S. Mendes and J. Kasparian, *Saturation of rogue wave amplification over steep shoals*, Physical Review E, 106 (2022), p. 065101.
[41] J. W. Miles, *Surface-wave scattering matrix for a shelf*, Journal of Fluid Mechanics, 28 (1967), pp. 755–767.
[42] E. Monsalve, *Experimental study of water waves: nonlinear effects and absorption*, PhD thesis, Université Pierre & Marie Curie-Paris 6, 2017.
[43] E. Monsalve, A. Maurel, V. Pagneux, and P. Petitjeans, *Nonlinear waves passing over rectangular obstacles: Multimodal method and experimental validation*, Fluids, 7 (2022), p. 145.
[44] J. N. Newman, *Propagation of water waves over an infinite step*, Journal of Fluid Mechanics, 23 (1965), pp. 399–415.
[45] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. Arecchi, *Rogue waves and their generating mechanisms in different physical contexts*, Physics Reports, 528 (2013), pp. 47–89.
[46] D. Peregrine, *Long waves on a beach*, Journal of fluid mechanics, 27 (1967), pp. 815–827.
[47] J. Rhee, *On the transmission of water waves over a shelf*, Applied Ocean Research, 19 (1997), pp. 161–169.
[48] R. Rosales and G. Papanicolaou, *Gravity waves in a channel with a rough bottom*, Studies in Applied Mathematics, 68 (1983), pp. 89–102.
[49] P. Siew and D. Hurley, *Long surface waves incident on a submerged horizontal plate*, Journal of Fluid Mechanics, 83 (1977), pp. 141–151.
[50] T. Tang, C. Moss, S. Draycott, H. B. Bingham, T. S. Van Den Bremer, Y. Li, and T. A. Adcock, *The influence of directional spreading on rogue waves triggered by abrupt depth transitions*, Journal of Fluid Mechanics, 972 (2023), p. R2.
[51] K. Trulsen, A. Raustøl, S. Jorde, and L. Rye, *Extreme wave statistics of long-crested irregular waves over a shoal*, Journal of Fluid Mechanics, 882 (2020), p. R2.
[52] K. Trulsen, H. Zeng, and O. Gramstad, *Laboratory evidence of freak waves provoked by non-uniform bathymetry*, Physics of Fluids, 24 (2012), p. 097101.
[53] E. Tuck, *Shallow-water flows past slender bodies*, Journal of fluid mechanics, 26 (1966), pp. 81–95.
[54] E. Tuck, *Transmission of water waves through small apertures*, Journal of Fluid Mechanics, 49 (1971), pp. 65–74.
[55] E. Tuck, *Some classical water-wave problems in variable depth*, Waves on Water of Variable Depth, Lecture Notes in Physics, 64 (1976).
[56] Q. Wu, X. Feng, Y. Dong, and F. Dias, *On the behavior of higher harmonics in the evolution of nonlinear water waves in the presence of abrupt depth transitions*, Physics of Fluids, 35 (2023).
[57] J. Zhang and M. Benoit, *Wave–bottom interaction and extreme wave statistics due to shoaling and de-shoaling of irregular long-crested wave trains over steep seabed changes*, Journal of Fluid Mechanics, 912 (2021), p. A28.
[58] J. Zhang, M. Benoit, O. Kimmoun, A. Chabchoub, and H.-C. Hsu, *Statistics of extreme waves in coastal waters: large scale experiments and advanced numerical simulations*,

763 Fluids, 4 (2019), p. 99.
764 [59] J. Zhang, M. Benoit, and Y. Ma, *Equilibration process of out-of-equilibrium sea-states in-*
765 *duced by strong depth variation: Evolution of coastal wave spectrum and representative*
766 *parameters*, Coastal Engineering, 174 (2022), p. 104099.
767 [60] Y. Zheng, Z. Lin, Y. Li, T. Adcock, Y. Li, and T. Van Den Bremer, *Fully nonlinear*
768 *simulations of unidirectional extreme waves provoked by strong depth transitions: the effect*
769 *of slope*, Physical Review Fluids, 5 (2020), p. 064804.