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# JUMP CONDITIONS FOR BOUSSINESQ EQUATIONS DUE TO AN ABRUPT DEPTH TRANSITION\*

EDUARDO MONSALVE <sup>†</sup>, KIM PHAM <sup>†</sup>, AND AGNÈS MAUREL <sup>‡</sup>

**Abstract.** We revisit the problem of nonlinear water wave propagation in the presence of an abrupt depth transition. To this end, we use an asymptotic approach conducted to order 3 with respect to the shallowness parameter, in order to capture the first nonlinear and dispersive contributions. However, the discontinuity of bathymetry, as opposed to slowly varying bathymetry, requires the use of a consistent three-scale analysis framework and the consideration of different regions, far from the step and free surface, near the free surface and near the step. This framework enables consistent navigation, ultimately providing Boussinesq equations supplemented by jump conditions at the depth discontinuity that encompass the effect of step on wave propagation.

**Key words.** non-linear water waves; depth transition; bathymetry; Boussinesq equations; jump conditions; asymptotic techniques; matched asymptotic expansions; boundary layers

**AMS subject classifications.** 34E13, 35B27, 35L05, 76B15

**1. Introduction.** The origin of large waves, so-called rogue or freak waves, and the reasons for their appearance has been studied for many years [18, 12, 2, 3, 45, 1]. In the case of shallow or finite water depth, the presence of depth transitions has been shown to play a role in the formation and dynamics of these extreme waves [19, 52, 13, 58, 57, 59]. In many previous studies, the case of a smooth bathymetry, with a continuous depth transition over a non-zero distance  $L$ , has been considered [52, 13, 58, 51, 57, 20, 21, 22]. The influence of slope has been considered in [13], [60] and [20] and studied more systematically in [40] who show that higher slopes foster the appearance of rogue waves up to a critical value beyond which its effect is saturated. These results suggested the interest of considering waves passing an abrupt transition ( $L = 0$ , see figure 1), *i.e.* a step, as this avoids discussions about the transition region (its shape and value of the distance  $L$ ) and provides a reference limiting case. After the pioneering work of [14] this problem has recently been addressed in a series of papers by Li and co-workers [23, 24, 25, 10, 26, 27] where the authors examine on the generation of free and bound waves, which are waves corresponding to harmonics generated by non-linearities, and show the crucial influence of these second order effects in the occurrence of extreme events, see also [42, 31, 32, 60, 11, 43, 50, 56].

The interest in this limiting case is also of a fundamental nature, as it poses specific problems, both numerically and theoretically. While linear wave propagation in the presence of an abrupt depth transition is simple and well known, the extension to non-linear (even weakly) propagation remains in part an open subject. Let us start with numerical methods; among them, the simplest semi-analytical multimodal analysis easily handles, in the linear case, the reflected and transmitted fields accounting for the evanescent field triggered at the depth transition, see *e.g.* [44] and [41]. A classical reference to the extension of this method for weakly non-linear propagation is

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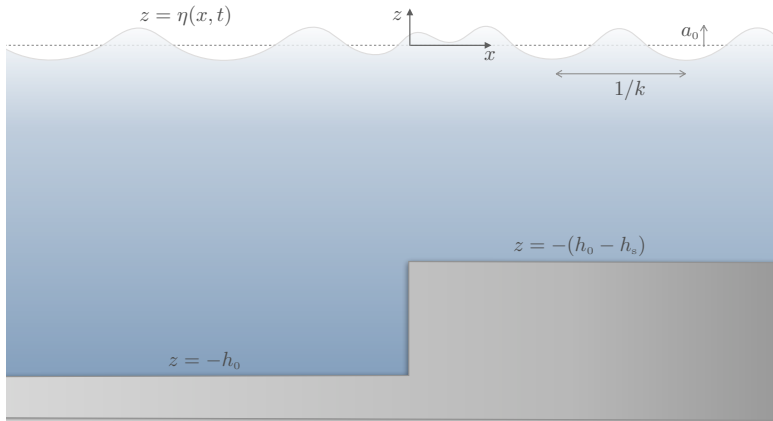


FIG. 1. Non-linear propagation over an abrupt depth transition between two regions of constant depth  $h_0$  and  $(h_0 - h_s)$ .

that of [35, 34], see also [5]; the method relies on the use of a small parameter, namely the ratio between wave amplitude and wavelength, and on an iterative process thanks to asymptotic expansion of the solution. However, this model suffers from inherent divergences due to evanescent terms in the second order solution, and this has been circumvented at the cost of ad-hoc procedures, using specific truncations [42, 43] or neglecting these nonlinear evanescent contributions [47, 25]. From a theoretical perspective, which is the focus of the present study, asymptotic techniques in the linear regime are a powerful tool for reducing the problem of water wave propagation in the presence of a scattering region of dimension smaller than the wavelength. This technique has a long history, which began in the 1960s [4, 53, 54, 55, 49], see also [33], [38], [16] and [39], Ch. 4 and 5. In shallow or finite-depth waters, the procedure consists in eliminating the  $z$ -vertical dependence of the solution and encapsulating the effect of the (small) scatterer in effective jump conditions. This dimensional reduction has been extended to linear [36, 37] and nonlinear [48] propagation to determine the evolution of surface waves propagating over periodic bottom irregularities. The present study aims to use such asymptotic procedure conducted at high order to recover a Boussinesq-type equation, with dispersion and nonlinearity, that is classical [28, 30, 9], and to supplement this equation with effective jump conditions that encapsulate the effect of rapid variations in the evanescent field in the vicinity of the abrupt depth transition.

The paper is organized as follows. In Section 2, we pose the direct problem of water wave propagation in the presence of an abrupt depth transition in two dimensions and give the result of the study *i.e.* the effective problem comprising the Boussinesq equations and associated jump conditions. The rest of the paper is devoted to the derivation of this effective model, starting, in Section 3, with the derivation of the effective wave equations far from the bathymetry discontinuity. Then, in Section 4, we present the derivation of the jump conditions across the discontinuity, which is the main result of the analysis; we establish the corresponding boundary layer problems which define the effective parameters entering the jumps (explicit expressions for these coefficients, which depend only on the depth ratio, are given in the appendix A). Finally, in Section 5, we combine the results from the previous sections to reconstruct

71 the Boussinesq equations and the associated jump conditions.

72 **2. Sum-up of the main result: Boussinesq approximation for an abrupt**  
 73 **depth transition.**

74 **2.1. The actual and effective problems.** We begin with the equations gov-  
 75 erning the actual problem. Using the assumptions of an inviscid and incompressible  
 76 two-dimensional fluid in irrotational motion, the velocity  $\mathbf{u}(x, z, t)$  and the associated  
 77 velocity potential  $\varphi(x, z, t)$  satisfy

78 (2.1) 
$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} = \nabla \varphi,$$

79 along with the dynamic and kinematic boundary conditions at the free surface, *i.e.*

80 (2.2) 
$$\frac{\partial \varphi}{\partial t} + \frac{\mathbf{u} \cdot \mathbf{u}}{2} + g\eta = 0, \quad u_z = \frac{\partial \eta}{\partial t} + u_x \frac{\partial \eta}{\partial x} \quad \text{at } z = \eta(x, t).$$

81 The depth transition results in a discontinuous bathymetry with water depth  $h(x)$   
 82 being a piecewise constant function such that

83 (2.3) 
$$h(x < 0) = h_0, \quad h(x > 0) = h_0 - h_s,$$

84 (with  $h_s < h_0$ ) so the vanishing normal velocity condition applies on all rigid bound-  
 85 aries (noted  $\Gamma$ )

86 (2.4) 
$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

87 In relatively shallow waters and in the weakly nonlinear regime, we will establish  
 88 that the two-dimensional problem (2.1)-(2.4) can be reduced to the classical one-  
 89 dimensional Boussinesq equations [8, 46] which, in a conservative form, are expressed  
 90 as follows

91 (2.5) 
$$\begin{cases} \frac{\partial}{\partial t} \left( U_x - \frac{h^2}{3} \frac{\partial^2 U_x}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( g\eta + \frac{U_x^2}{2} \right) = 0, \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} ((h + \eta)U_x) = 0, \end{cases}$$

92 where the average horizontal velocity field  $U_x$  is defined as

93 
$$U_x(x, t) = \frac{1}{h(x)} \int_{-h(x)}^0 u_x(x, z, t) \, dz.$$

94 The Boussinesq equations are only valid for  $x \neq 0$ . At  $x = 0$ , given the presence of  
 95 the bathymetry discontinuity, the motion of the free surface  $\eta$  and the velocity  $U_x$   
 96 lose their regularity. The main result of this paper is to establish the jump conditions  
 97 that the fluxes in the Boussinesq equations (2.5) must satisfy at  $x = 0$ , specifically

98 (2.6) 
$$\begin{cases} \left[ \left[ g\eta + \frac{U_x^2}{2} \right] \right] = -\frac{\partial}{\partial t} \left( \mathcal{B}_1 \overline{hU_x} + h_0 \mathcal{B}_2 \frac{\partial \bar{\eta}}{\partial t} + \frac{h_0 \mathcal{B}_3}{g} \frac{\partial^2 \overline{hU_x}}{\partial t^2} \right), \\ \left[ \left[ (h + \eta)U_x \right] \right] = \frac{1}{g} \frac{\partial^2}{\partial t^2} \left( h_0 \mathcal{C}_1 \overline{hU_x} + h_0^2 \mathcal{C}_2 \frac{\partial \bar{\eta}}{\partial t} \right), \end{cases}$$

99 with the jump  $\llbracket f \rrbracket$  and the average  $\bar{f}$  of a quantity  $f(x, t)$  at  $x = 0$  defined as follows

100 (2.7) 
$$\llbracket f \rrbracket (t) = f(0^+, t) - f(0^-, t), \quad \bar{f}(t) = \frac{f(0^+, t) + f(0^-, t)}{2}.$$

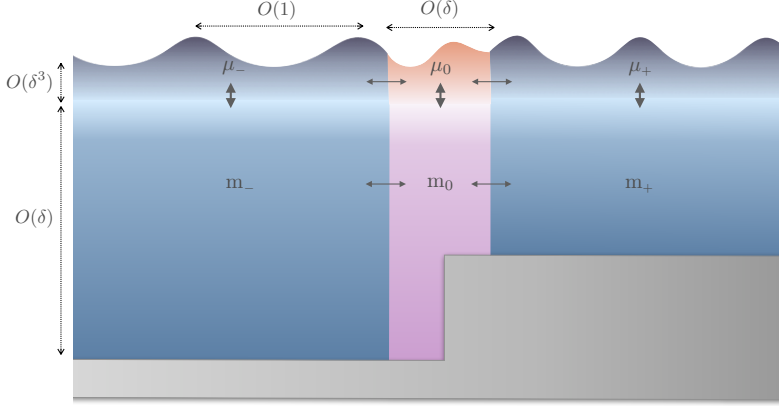


FIG. 2. The different domains in which the analysis is performed. The  $\mu_{\pm}$  and  $m_0$  regions correspond to boundary layers that aim to disappear; in the  $m_{\pm}$  regions, a reduction along the vertical direction is performed. Solutions in connected regions are matched vertically (large arrows) or horizontally (small arrows).

101 In (2.6),  $(\mathcal{B}_i)$  and  $(\mathcal{C}_i)$  are non-dimensional boundary layer coefficients, whose values  
 102 depend only on  $h_s/h_0$ , see Appendix A. It is worth noting that the quantities  $(g\eta + \frac{U_x^2}{2})$   
 103 and  $(h + \eta)U_x$  are continuous elsewhere, specifically for  $x \neq 0$ . We observe that these  
 104 jumps are overlooked in several studies investigating the impact of a step on the  
 105 propagation of shock waves [17] or solitons [31, 32]. From a physical standpoint, it  
 106 can be argued that these jumps, arising from evanescent fields triggered in the vicinity  
 107 of the step, may have a weak effect under certain conditions. However, the derivation  
 108 of the shallow water equations in these references stems from an asymptotic analysis  
 109 conducted to high orders, aimed at capturing non-linear and dispersive contributions.  
 110 Accordingly, neglecting the contribution of these higher order terms in the jump  
 111 conditions raises mathematical concerns, potentially impacting the stability of the  
 112 resulting models.

113 **2.2. Summary of the asymptotic procedure.** To capture the first contribu-  
 114 tions due to non-linearities and dispersion, we use classical scalings for characteristic  
 115 wavelength  $1/k$ , water depth  $h_0$  and wave amplitude  $a_0$ . Specifically, with

$$116 \quad (2.8) \quad \varepsilon = \frac{a_0}{h_0} \ll 1, \quad \delta = kh_0 \ll 1,$$

117 the non-linearity and shallowness parameters, we consider

$$118 \quad (2.9) \quad \varepsilon = \alpha\delta^2, \quad \text{with } \alpha = O(1).$$

119 In the following sections, we will conduct the asymptotic analysis in the different  
 120 regions shown in figure 2 and, for simplicity, we will consider the non-dimensional  
 121 form of (2.1)-(2.4). Introducing  $\omega = \sqrt{gh_0}k$ , we use

$$122 \quad (2.10) \quad t \rightarrow \omega t, \quad x \rightarrow kx, \quad z \rightarrow z_m = z/h_0, \quad \eta \rightarrow \eta/a_0, \quad \mathbf{u} \rightarrow \mathbf{u}/(\omega a_0), \quad \varphi \rightarrow k\varphi/(\omega a_0),$$

123 which provide the non-dimensional form of (2.1) (incompressibility Inc, and irrotationality Rot)

$$125 \quad (2.11) \quad (\text{Inc}) : \frac{\partial u_x}{\partial x} + \frac{1}{\delta} \frac{\partial u_z}{\partial z_m} = 0, \quad (\text{Rot}) : u_x = \frac{\partial \varphi}{\partial x}, \quad u_z = \frac{1}{\delta} \frac{\partial \varphi}{\partial z_m},$$

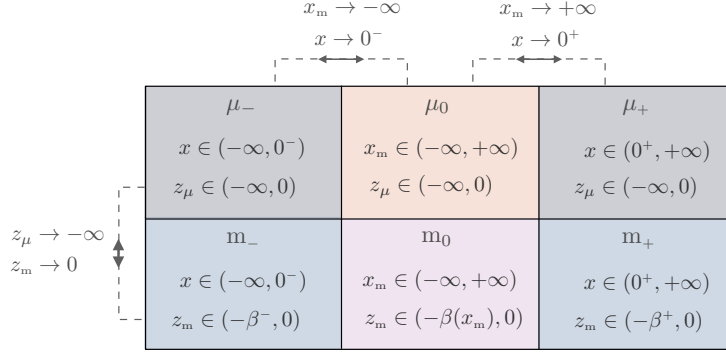


FIG. 3. Diagram summarizing the asymptotic procedure: in each region of figure 2, the problem is set in proper coordinate (see (2.13)) in order that the rescaled geometry is independent of  $\delta$ . The solutions of these problems are then matched asymptotically.

126 and (2.2) and (2.4) (dynamic DC and kinematic KC conditions at the free surface and  
 127 vanishing normal velocity RC on the rigid walls formed by the seabed at  $z_m = -\beta_\pm =$   
 128  $-h/h_0$  and the vertical wall of the step)

(2.12)

129 (DC) :  $\frac{\partial \varphi}{\partial t} + \varepsilon \delta \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \delta^{-1} \eta = 0$ , (KC) :  $u_z = \frac{\partial \eta}{\partial t} + \varepsilon \delta u_x \frac{\partial \eta}{\partial x}$ , at  $z_m = \varepsilon \eta(x, t)$ ,

(RC) :  $\mathbf{u} \cdot \mathbf{n} = 0$  on the walls,

130 The above equations explicitly account for field variations horizontally on the wave-  
 131 length scale (via  $x$ ) and vertically on the depth scale (via  $z_m$ ); this is appropriate in  
 132  $m_\pm$  *i.e.* far from the step and the free surface. Near the step, however, the fields vary  
 133 horizontally on the depth scale; near the free surface, they vary vertically on the wave  
 134 amplitude scale. Based on these considerations, we introduce rescaled coordinates

135 (2.13) macro :  $x$ , meso :  $x_m = \frac{x}{\delta}$ ,  $z_m$ , micro :  $z_\mu = \frac{z_m - \varepsilon \eta}{\varepsilon \eta}$ ,

136 and in the final model, only the  $x$  spatial coordinate will be retained. In each region,  
 137 the procedure consists in using the appropriate spatial coordinate so that the geometry  
 138 does not depend on  $\delta$  or  $\varepsilon$  (it is  $O(1)$  in both directions) and the dependence on  
 139 the small parameter appears explicitly in a new system of equations. The different  
 140 regions are then connected by asymptotically matching their solutions; this procedure  
 141 is summarized in figure 3.

142 In the linear regime ( $\mu_\pm$  and  $\mu_0$  do not exist), the horizontal matching between  
 143 the regions  $m_\pm$  and  $m_0$  is classical and makes it possible to reduce the effect of the  
 144 *boundary layer* in  $m_0$  to jump conditions at  $x = 0^\pm$  [55]. In the non-linear regime,  
 145 boundary layers in  $\mu_\pm$  and  $\mu_0$  must be taken into account due to the small but non-  
 146 zero amplitude of the wave and are treated in the same way, namely a horizontal  
 147 matching between  $\mu_\pm$  and  $\mu_0$  and a vertical matching between  $\mu_\pm$  and  $m_\pm$ . We  
 148 stress that for a flat or smoothly varying bathymetry,  $m_0$  does not exist since it is  
 149 implicitly assumed that the bathymetry varies at the wavelength scale  $x$ ; whereas  
 150 strictly speaking, the asymptotic analysis should be conducted by considering the  
 151 boundary layer region  $\mu$  with  $z_\mu \in (-\infty, 0)$  and the region  $m$  with  $z_m \in (-\beta(x), 0)$   
 152 (with a single vertical matching between them), the same result is obtained, namely

153 a Boussinesq-type equation, considering a single region with  $z_m \in (-\beta(x), \varepsilon\eta(x, t))$   
 154 [46, 6].

155 **3. Asymptotic analysis in  $m_\pm$  and  $\mu_\pm$ , far from the depth transition.** In  
 156 this section, we derive the Boussinesq equations in their classical form due to [46] in  
 157 the two constant depth regions far from depth transition (the step). The procedure  
 158 is the same on both sides of the step, and we will consider  $\mu_+$ ,  $m_+$ .

159 **3.1. Setting of the asymptotic procedure.**

160 **3.1.1. The mesoscopic in  $m_+$  and the microscopic in  $\mu_+$  problems.** In the  
 161 *mesoscopic region*  $m_+$ , the problem is set in  $(x, z_m)$  coordinates, with  $x \in (0^+, +\infty)$ ,  
 162  $z_m \in (-\beta_+, 0)$ , and the governing equations are, using (2.11)-(2.12),

$$163 \quad (3.1) \quad \begin{aligned} {}^m(\text{Inc}) : \quad & \frac{\partial u_x}{\partial x} + \frac{1}{\delta} \frac{\partial u_z}{\partial z_m} = 0, & {}^m(\text{Rot}) : \quad & u_x = \frac{\partial \varphi}{\partial x}, \quad u_z = \frac{1}{\delta} \frac{\partial \varphi}{\partial z_m}, \\ {}^m(\text{RC}) : \quad & u_z(x, z_m = -\beta_+, t) = 0, \end{aligned}$$

164 (at this stage, boundary conditions at  $z_m = 0$  are missing). The solutions of (3.1) are  
 165 expanded in terms of series of the small parameter  $\delta$

$$166 \quad (3.2) \quad u_x = \sum_{n \geq 0} \delta^n u_x^{(n)}(x, z_m, t), \quad u_z = \delta \sum_{n \geq 0} \delta^n u_z^{(n)}(x, z_m, t), \quad \varphi = \sum_{n \geq 0} \delta^n \varphi^{(n)}(x, z_m, t).$$

167 where we anticipate that  $u_x = O(1)$  imposes  $u_z = O(\delta)$  from  ${}^m(\text{Inc})$  in (3.1).

168

169 In the *microscopic region*  $\mu_+$ , the problem is set in  $(x, z_\mu)$  coordinate, with  $x \in$   
 170  $(0^+, +\infty)$ ,  $z_\mu \in (-\infty, 0)$ , and with  $z_\mu$  defined in (2.13) (the free surface is now fixed  
 171 at  $z_\mu = 0$ ). Consequently, the governing equations are, from (2.11)-(2.12) and using  
 172 (2.13) with  $\varepsilon = \alpha\delta^2$ ,

$$(3.3) \quad \begin{aligned} {}^\mu(\text{Inc}) : \quad & \eta \frac{\partial u_x}{\partial x} + \frac{1}{\alpha\delta^3} \frac{\partial u_z}{\partial z_\mu} = 0, & {}^\mu(\text{Rot}) : \quad & u_x = \frac{\partial \varphi}{\partial x}, \quad \eta u_z = \frac{1}{\alpha\delta^3} \frac{\partial \varphi}{\partial z_\mu}, \\ {}^\mu(\text{DC}) : \quad & \frac{\partial \varphi}{\partial t} + \alpha\delta^3 \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{1}{\delta} \eta = 0, & {}^\mu(\text{KC}) : \quad & u_z = \frac{\partial \eta}{\partial t} + \alpha\delta^3 u_x \frac{\partial \eta}{\partial x} \quad \text{at } z_\mu = 0. \end{aligned}$$

174 (at this stage, boundary conditions are missing when  $z_\mu \rightarrow -\infty$ ). The solutions of  
 175 (3.3) are expanded in terms of series of the small parameter  $\delta$ ,

$$(3.4) \quad \begin{aligned} u_x &= \sum_{n \geq 0} \delta^n v_x^{(n)}(x, z_\mu, t), \quad u_z = \delta \sum_{n \geq 0} \delta^n v_z^{(n)}(x, z_\mu, t), \quad \varphi = \sum_{n \geq 0} \delta^n \psi^{(n)}(x, z_\mu, t), \\ \eta &= \delta \sum_{n \geq 0} \delta^n \eta^{(n)}(x, t), \end{aligned}$$

177 where we anticipate that  $u_x, \varphi = O(1)$  imposes  $u_z, \eta = O(\delta)$ , according to  ${}^\mu(\text{DC})$ -  
 178  ${}^\mu(\text{KC})$  in (3.3).

179 **3.1.2. The hierarchies of equations.** In  $m_+$ , by injecting (3.2) into (3.1), we  
 180 obtain a hierarchy of equations, namely  $\forall n$

$$181 \quad (3.5) \quad \begin{aligned} {}^m(\text{Inc})^n : \quad & \frac{\partial u_x^{(n)}}{\partial x} + \frac{\partial u_z^{(n)}}{\partial z_m} = 0, \\ {}^m(\text{Rot})_x^n : \quad & u_x^{(n)} = \frac{\partial \varphi^{(n)}}{\partial x}, \quad {}^m(\text{Rot})_z^n : \quad u_z^{(n)} = \frac{\partial \varphi^{(n+2)}}{\partial z_m}, \\ {}^m(\text{RC})^n : \quad & u_z^{(n)}(x, z_m = -\beta_+, t) = 0, \end{aligned}$$

182 with by convention  $u_z^{(n)} = 0$  for  $n < 0$ . In  $\mu_+$ , by injecting (3.4) into (3.3), we obtain

$$183 \quad (3.6) \quad \begin{aligned} \mu(\text{Inc})^n : \quad \frac{\partial v_z^{(n)}}{\partial z_\mu} = 0, \quad n \leq 2, \quad \frac{\partial v_z^{(3)}}{\partial z_\mu} = -\alpha \eta^{(0)} \frac{\partial v_x^{(0)}}{\partial x}, \\ \mu(\text{Rot})_x^n : \quad v_x^{(n)} = \frac{\partial \psi^{(n)}}{\partial x}, \quad n \geq 0, \quad \mu(\text{Rot})_z^n : \quad \frac{\partial \psi^{(n)}}{\partial z_\mu} = 0, \quad n \leq 4, \end{aligned}$$

184 along with the boundary conditions at the free surface  $z_\mu = 0$ ,

$$185 \quad (3.7) \quad \begin{aligned} \mu(\text{DC})^n : \quad \frac{\partial \psi^{(n)}}{\partial t} + \eta^{(n)} = 0, \quad n \leq 2, \quad \frac{\partial \psi^{(3)}}{\partial t} + \frac{\alpha}{2} (v_x^{(0)})^2 + \eta^{(3)} = 0, \\ \mu(\text{KC})^n : \quad v_z^{(n)}|_{z_\mu=0} = \frac{\partial \eta^{(n)}}{\partial t}, \quad n \leq 2, \quad v_z^{(3)} = \frac{\partial \eta^{(3)}}{\partial t} + \alpha v_x^{(0)} \frac{\partial \eta^{(0)}}{\partial x}. \end{aligned}$$

186 **3.1.3. Vertical matching conditions between  $m_+$  and  $\mu_+$ .** Solutions in  $m_+$   
187 and  $\mu_+$  must match in an intermediate region when  $z_m \rightarrow 0$  and  $z_\mu \rightarrow -\infty$  (figure 4).  
188 Specifically, for  $\varphi$  (for example), we need to have

$$189 \quad (3.8) \quad \varphi^{(0)}(x, z_m, t) + \delta \varphi^{(1)}(x, z_m, t) + \dots \sim \psi^{(0)}(x, z_\mu, t) + \delta \psi^{(1)}(x, z_\mu, t) + \dots,$$

190 Taking into account (2.13) and using Taylor expansion

$$191 \quad (3.9) \quad f(x, z_m, t) = f(x, 0, t) + \alpha \delta^2 (z_\mu + 1) \eta(x, t) \frac{\partial f}{\partial z_m}(x, 0, t) + \dots$$

192 we identify in (3.8) the terms of the same power of  $\delta$ , which gives

$$193 \quad (3.10) \quad \begin{cases} \varphi^{(n)}(x, 0, t) = \lim_{z_\mu \rightarrow -\infty} \psi^{(n)}(x, z_\mu, t), \quad n \leq 2, \\ \varphi^{(3)}(x, 0, t) = \lim_{z_\mu \rightarrow -\infty} \left( \psi^{(3)}(x, z_\mu, t) - \alpha (z_\mu + 1) \eta^{(0)}(x, t) \frac{\partial \varphi^{(0)}}{\partial z_m}(x, 0, t) \right). \end{cases}$$

194 The same relations are obtained by replacing  $\varphi^{(n)}$  by  $u_x^{(n)}$  or  $u_z^{(n)}$  and  $\psi^{(n)}$  by  $v_x^{(n)}$  or  
195  $v_z^{(n)}$ .

196 **3.2. Effective wave equation at order  $n = 0, 1$ .** At the first two orders, the  
197 effective wave equation is linear and non-dispersive.

198 PROPOSITION 3.1. *The effective wave equation reads at the first two orders  $n = 0,$   
199  $1$ :*

$$200 \quad (3.11) \quad (\text{EW})_1^n : \quad \frac{\partial U_x^{(n)}}{\partial t} + \frac{\partial \eta^{(n)}}{\partial x} = 0,$$

$$201 \quad (3.12) \quad (\text{EW})_2^n : \quad \beta \frac{\partial U_x^{(n)}}{\partial x} + \frac{\partial \eta^{(n)}}{\partial t} = 0,$$

202 with  $\beta = \beta_+$  or  $\beta_-$ .

204 **3.2.1. Derivation of  $(\text{EW})_1^n$  for  $n = 0, 1$ .** From  ${}^m(\text{Rot})_z^{-2}$  and  ${}^m(\text{Rot})_z^{-1}$  in (3.5),  
205  $\varphi^{(n)}$  does not depend on  $z_m$  for  $n = 0, 1$  hence

$$206 \quad (3.13) \quad \varphi^{(n)} = \phi^{(n)}(x, t), \quad n = 0, 1,$$

207 and from  ${}^m(\text{Rot})_x^0$  and  ${}^m(\text{Rot})_x^1$  in (3.5), we also have

$$208 \quad (3.14) \quad u_x^{(n)} = U_x^{(n)}(x, t) = \frac{\partial \phi^{(n)}}{\partial x}(x, t), \quad n = 0, 1,$$

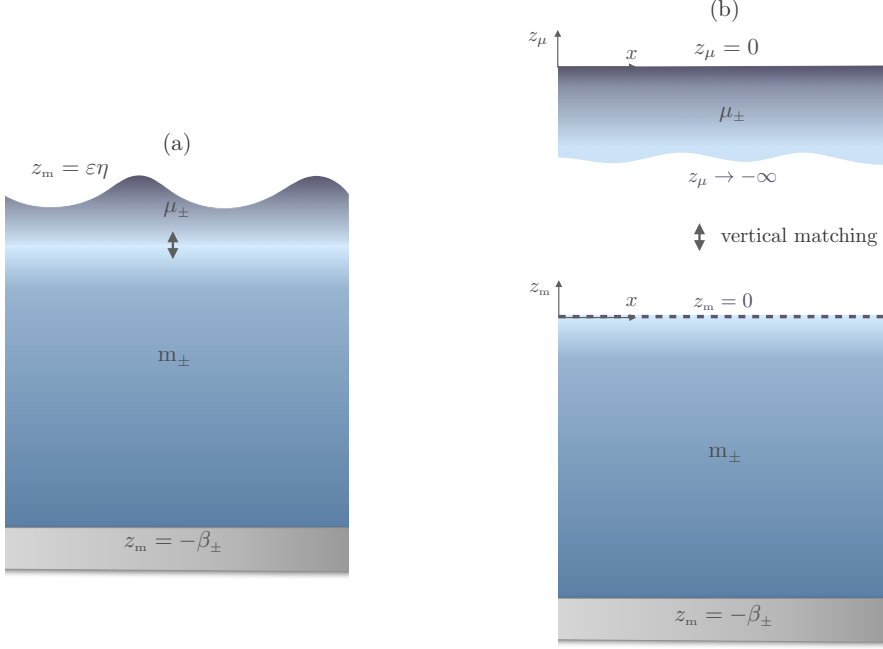


FIG. 4. (a) A region of constant depth  $\beta = \beta_{\pm} = O(1)$  over which waves of amplitude  $O(\varepsilon)$  propagate; (b) Asymptotically, the analysis is performed in the regions  $\mu = \mu_{\pm}$  (in coordinate  $(x, z_{\mu})$ ) and  $m_{\pm}$  (in coordinate  $(x, z_m)$ ) and matching conditions apply when  $z_{\mu} \rightarrow -\infty$ ,  $z_m = 0$ .

209 with  $U_x^{(n)}$  the mean velocity field, and the associated flux, defined as

$$210 \quad (3.15) \quad \forall n, \quad U_x^{(n)}(x, t) = \frac{1}{\beta} \int_{-\beta}^0 u_x^{(n)}(x, z_m, t) dz_m, \quad q^{(n)}(x, t) = \beta U_x^{(n)}(x, t),$$

211 for  $\beta = \beta_{\pm}$ . Note that (3.13) is consistent with  ${}^{\mu}(\text{Rot})_z^0$  and  ${}^{\mu}(\text{Rot})_z^1$  in (3.6), which  
 212 tell us that, for  $n = 0, 1$ ,  $\psi^{(n)}$  does not depend on  $z_{\mu}$ , hence  $\psi^{(n)} = \phi^{(n)}(x, t)$  from the  
 213 matching conditions (3.10) and  $v_x^{(n)} = U_x^{(n)}$ . It remains to use  ${}^{\mu}(\text{DC})^n$  in (3.7) which  
 214 provides

$$215 \quad (3.16) \quad \frac{\partial \phi^{(n)}}{\partial t}(x, t) + \eta^{(n)}(x, t) = 0, \quad n = 0, 1,$$

216 which we differentiate with respect to  $x$  to obtain (3.11).

217 **3.2.2. Derivation of  $(\text{EW})_z^n$  for  $n = 0, 1$ .** We now use  ${}^m(\text{Inc})^n$  in (3.5) and  $u_x^{(n)}$   
 218 in (3.14). Taking into account the vanishing vertical velocity condition at  $z_m = -\beta_+$ ,  
 219 we obtain

$$220 \quad (3.17) \quad u_z^{(n)}(x, z_m, t) = -(z_m + \beta_+) \frac{\partial U_x^{(n)}}{\partial x}(x, t), \quad n = 0, 1.$$

221 Then, according to  ${}^{\mu}(\text{Inc})^n$  in (3.6),  $v_z^n$  does not depend on  $z_{\mu}$ . By combining  ${}^{\mu}(\text{KC})^n$  in  
 222 (3.7) with the matching condition (3.10), we conclude that  $u_z^{(n)}(x, 0, t) = v_z^{(n)}(x, t) =$   
 223  $\partial_t \eta^{(n)}(x, t)$ . Finally, returning to (3.17) and setting  $z_m = 0$ , we obtain (3.12).

224 **3.3. Effective wave equation at order  $n = 2$ .** At order 2, a dispersive term  
 225 appears in the effective wave equation.

226 PROPOSITION 3.2. *The effective wave equation at order  $n = 2$  reads :*

$$227 \quad (3.18) \quad (\mathbf{EW})_1^2 : \quad \frac{\partial U_x^{(2)}}{\partial t} + \frac{\partial \eta^{(2)}}{\partial x} - \frac{\beta^2}{3} \frac{\partial^3 U_x^{(0)}}{\partial t \partial x^2} = 0,$$

$$228 \quad (3.19) \quad (\mathbf{EW})_2^2 : \quad \beta \frac{\partial U_x^{(2)}}{\partial x} + \frac{\partial \eta^{(2)}}{\partial t} = 0,$$

229 with  $\beta = \beta_+$  or  $\beta_-$ .

231 **3.3.1. Derivation of  $(\mathbf{EW})_1^2$ .** We start with  ${}^m(\text{Rot})_z^0$  in (3.5) which, after inte-  
 232 grating (3.17), yields

$$233 \quad (3.20) \quad \varphi^{(2)}(x, z_m, t) = - \left( \frac{z_m^2}{2} + \beta_+ z_m \right) \frac{\partial U_x^{(0)}}{\partial x}(x, t) + \phi^{(2)}(x, t).$$

234 Then, from  ${}^\mu(\text{Rot})_z^2$  in (3.6), we deduce as before that  $\psi^{(2)}$  is independent of  $z_\mu$ . Using  
 235  ${}^\mu(\text{DC})^2$  in (3.7) and taking into account the matching condition (3.10), we obtain

$$236 \quad (3.21) \quad \frac{\partial \phi^{(2)}}{\partial t}(x, t) + \eta^{(2)}(x, t) = 0.$$

237 Injecting (3.20) in  ${}^m(\text{Rot})_x^2$  from (3.5) yields

$$238 \quad (3.22) \quad u_x^{(2)}(x, z_m, t) = - \left( \frac{z_m^2}{2} + \beta_+ z_m \right) \frac{\partial^2 U_x^{(0)}}{\partial x^2}(x, t) + \frac{\partial \phi^{(2)}}{\partial x}(x, t),$$

239 which, after taking the average with respect to  $z_m$ , leads to

$$240 \quad (3.23) \quad U_x^{(2)}(x, t) = \frac{\beta_+^2}{3} \frac{\partial^2 U_x^{(0)}}{\partial x^2}(x, t) + \frac{\partial \phi^{(2)}}{\partial x}(x, t).$$

241 Finally, differentiating (3.23) with respect to  $t$  and (3.21) with respect to  $x$  provides  
 242 (3.18) after combination.

243 **3.3.2. Derivation of  $(\mathbf{EW})_2^2$ .** As before, we use  ${}^m(\text{Inc})^2$  in (3.5) with  $u_x^{(2)}$  given  
 244 bt (3.22). After integration, we obtain

$$245 \quad u_z^{(2)}(x, z_m, t) = \left( \frac{z_m^3}{6} + \frac{\beta_+ z_m^2}{2} - \frac{\beta_+^3}{3} \right) \frac{\partial^3 U_x^{(0)}}{\partial x^3}(x, t) - (z_m + \beta_+) \frac{\partial^2 \phi^{(2)}}{\partial x^2}(x, t),$$

246 which satisfies the boundary condition  ${}^m(\text{RC})^2$  on the rigid wall at  $z_m = -\beta_+$ . The  
 247 matching condition reads  $u_z^{(2)}(x, 0, t) = v_z^{(2)}(x, t)$  since  $v_z^{(2)}$  is independent of  $z_\mu$ , ac-  
 248 cording to  ${}^\mu(\text{Inc})^2$  in (3.6). Simply combine with  ${}^\mu(\text{KC})^2$  in (3.7) to obtain

$$249 \quad u_z^{(2)}(x, 0, t) = -\beta_+ \frac{\partial}{\partial x} \left( \frac{\beta_+^2}{3} \frac{\partial^3 U_x^{(0)}}{\partial x^3}(x, t) + \frac{\partial \phi^{(2)}}{\partial x}(x, t) \right) = \frac{\partial \eta^{(2)}}{\partial t}(x, t).$$

250 The term in the parentheses is none than  $U_x^{(2)}$ , see (3.23), hence (3.19).

251 **3.4. Effective wave equation at order  $n = 3$ .** At order  $n = 3$  a non-linear  
252 contribution appears.

253 PROPOSITION 3.3. *The effective wave equation reads at order  $n = 3$ :*

$$254 \quad (3.24) \quad (\mathbf{EW})_1^3 : \quad \frac{\partial U_x^{(3)}}{\partial t} + \frac{\partial \eta^{(3)}}{\partial x} - \frac{\beta^2}{3} \frac{\partial^3 U_x^{(1)}}{\partial t \partial x^2} + \alpha U_x^{(0)} \frac{\partial U_x^{(0)}}{\partial x} = 0,$$

$$255 \quad (3.25) \quad (\mathbf{EW})_2^3 : \quad \beta \frac{\partial U_x^{(3)}}{\partial x} + \frac{\partial \eta^{(3)}}{\partial t} + \alpha \frac{\partial}{\partial x} (U_x^{(0)} \eta^{(0)}) = 0,$$

256  
257 with  $\beta = \beta_+$  or  $\beta_-$ .

258 **3.4.1. Derivation of  $(\mathbf{EW})_1^3$ .** We start with  ${}^m(\text{Rot})_z^1$  in (3.5), i.e.  $u_z^{(1)} = \partial_{z_m} \varphi^{(3)}$ .  
259 Using (3.17) we obtain

$$260 \quad (3.26) \quad \varphi^{(3)}(x, z_m, t) = - \left( \frac{z_m^2}{2} + \beta_+ z_m \right) \frac{\partial U_x^{(1)}}{\partial x}(x, t) + \phi^{(3)}(x, t).$$

261 From  ${}^\mu(\text{Rot})_z^3$  in (3.6) we still have  $\psi^{(3)}(x, t) = \varphi^{(3)}(x, 0, t) = \phi^{(3)}(x, t)$  independent of  
262  $z_\mu$ . But now  ${}^\mu(\text{DC})^3$  in (3.7) reveals a non-linear contribution, with

$$263 \quad (3.27) \quad \frac{\partial \phi^{(3)}}{\partial t}(x, t) + \frac{\alpha}{2} (U_x^{(0)}(x, t))^2 + \eta^{(3)}(x, t) = 0,$$

264 since  $u_x^{(0)} = U_x^{(0)} = v_x^{(0)}$  from the matchings (3.10). As we did previously, by injecting  
265 (3.26) into  ${}^m(\text{Rot})_x^3$  in (3.5), we obtain

$$266 \quad (3.28) \quad u_x^{(3)}(x, z_m, t) = - \left( \frac{z_m^2}{2} + \beta_+ z_m \right) \frac{\partial^2 U_x^{(1)}}{\partial x^2}(x, t) + \frac{\partial \phi^{(3)}}{\partial x}(x, t),$$

267 which, after taking the average with respect to  $z_m$ , leads to

$$268 \quad (3.29) \quad U_x^{(3)}(x, t) = \frac{\beta^2}{3} \frac{\partial^2 U_x^{(1)}}{\partial x^2}(x, t) + \frac{\partial \phi^{(3)}}{\partial x}(x, t).$$

269 The differentiation of (3.29) with respect to  $t$  and (3.27) with respect to  $x$  gives, after  
270 combination, the equation (3.24).

271 **3.4.2. Derivation of  $(\mathbf{EW})_2^3$ .** We start by combining  ${}^m(\text{Inc})^3$  in (3.5) with  $u_x^{(3)}$   
272 in (3.28). After integration, we obtain

$$273 \quad (3.30) \quad u_z^{(3)}(x, z_m, t) = \left( \frac{z_m^3}{6} + \frac{\beta_+ z_m^2}{2} - \frac{\beta_+^3}{3} \right) \frac{\partial^3 U_x^{(1)}}{\partial x^3}(x, t) - (z_m + \beta_+) \frac{\partial^2 \phi^{(3)}}{\partial x^2}(x, t)$$

274 which satisfies the boundary condition  ${}^m(\text{RC})^3$  of vanishing vertical velocity at  $z_m =$   
275  $-\beta_+$ . The matching condition for  $n = 3$  can be read from the second relation in  
276 (3.10),

$$277 \quad (3.31) \quad u_z^{(3)}(x, 0, t) = \lim_{z_\mu \rightarrow -\infty} \left( v_z^{(3)}(x, z_\mu, t) - \alpha (z_\mu + 1) \eta^{(0)}(x, t) \frac{\partial u_z^{(0)}}{\partial z_m}(x, 0, t) \right).$$

278 To determine the form of  $v_z^{(3)}$ , we integrate  ${}^\mu(\text{Inc})^3$  in (3.6) and use the boundary  
279 condition  ${}^\mu(\text{KC})^3$  in (3.7) (remembering that  $v_x^{(0)} = U_x^{(0)}$ ), we have

$$280 \quad v_z^{(3)}(x, z_\mu, t) = \frac{\partial \eta^{(3)}}{\partial t}(x, t) + \alpha U_x^{(0)}(x, t) \frac{\partial \eta^{(0)}}{\partial x}(x, t) - \alpha z_\mu \eta^{(0)}(x, t) \frac{\partial U_x^{(0)}}{\partial x}(x, t).$$

281 Using the above relation in the matching condition (3.31) and taking into account  
 282  ${}^m(\text{Inc})^0$  in (3.5), we obtain

$$283 \quad u_z^{(3)}(x, 0, t) = \frac{\partial \eta^{(3)}}{\partial t}(x, t) + \alpha \frac{\partial}{\partial x} (U_x^{(0)}(x, t) \eta^{(0)}(x, t)).$$

284 Now, we just need to identify the above relation with (3.30) for  $z_m = 0$ , *i.e.*

$$285 \quad u_z^{(3)}(x, 0, t) = -\beta \frac{\partial}{\partial x} \left( \frac{\beta^2}{3} \frac{\partial^2 U_x^{(1)}}{\partial x^2}(x, t) - \beta \frac{\partial \phi^{(3)}}{\partial x}(x, t) \right).$$

286 The term in parentheses being  $U_x^{(3)}$  (see (3.29)), we deduce (3.25).

#### 287 4. Asymptotics near the depth transition.

288 **4.1. Setting of the asymptotic procedure.** In this section, we analyse the  
 289 solutions in  $m_0$  and  $\mu_0$ , their vertical matchings as in the previous section, and their  
 290 horizontal matchings with the solutions in  $m_{\pm}$  and  $\mu_{\pm}$  (see figures 3 and 5).

291 **4.1.1. The mesoscopic and the microscopic problems.** In the *mesoscopic*  
 292 *region*  $m_0$ , the problem is set in  $(x_m, z_m)$  coordinates, with  $x_m \in (-\infty, \infty)$ ,  $z_m \in$   
 293  $(-\beta(x_m), 0)$ . From (2.11)-(2.12), it can be written as

$$294 \quad (4.1) \quad \begin{aligned} {}^m_0(\text{Inc}) : \quad \text{div}_m \mathbf{u} &= \frac{\partial u_x}{\partial x_m} + \frac{\partial u_z}{\partial z_m} = 0, & {}^m_0(\text{Rot}) : \quad \mathbf{u} &= \frac{1}{\delta} \nabla_m \varphi, \\ {}^m_0(\text{RC}) : \quad \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on the walls,} \end{aligned}$$

295 (with missing conditions at  $z_m = 0$ , as before) and the solutions of (4.1) are expanded  
 296 in terms of series

$$297 \quad (4.2) \quad u_x = \sum_{n \geq 0} \delta^n \tilde{u}_x^{(n)}(x_m, z_m, t), \quad u_z = \sum_{n \geq 0} \delta^n \tilde{u}_z^{(n)}(x_m, z_m, t), \quad \varphi = \sum_{n \geq 0} \delta^n \tilde{\varphi}^{(n)}(x_m, z_m, t).$$

298 Note that, unlike (3.2), (3.4) and the forthcoming (4.4), we consider  $u_z = O(1)$ ; the  
 299 reason for this will become apparent later in the asymptotic analysis.

300 In the *microscopic region*  $\mu_0$ , the problem is set in  $(x_m, z_\mu)$  coordinate, with  $x_m \in$   
 301  $(-\infty, \infty)$ ,  $z_\mu \in (-\infty, 0)$  with  $z_\mu$  in (2.13), as in the previous section. Consequently,  
 302 starting from (2.11)-(2.12) and using (2.13) with  $\varepsilon = \alpha \delta^2$ , the problem is written as  
 (4.3)

$$303 \quad \begin{aligned} {}^\mu_0(\text{Inc}) : \quad \eta \frac{\partial u_x}{\partial x_m} + \frac{1}{\alpha \delta^2} \frac{\partial u_z}{\partial z_\mu} &= 0, & {}^\mu_0(\text{Rot}) : \quad u_x &= \frac{1}{\delta} \frac{\partial \varphi}{\partial x_m}, & \eta u_z &= \frac{1}{\alpha \delta^3} \frac{\partial \varphi}{\partial z_\mu}, \\ {}^\mu_0(\text{DC}) : \quad \frac{\partial \varphi}{\partial t} + \alpha \delta^3 \frac{\mathbf{u}^2}{2} + \frac{1}{\delta} \eta &= 0, & {}^\mu_0(\text{KC}) : \quad u_z &= \frac{\partial \eta}{\partial t} + \alpha \delta^2 u_x \frac{\partial \eta}{\partial x_m} & \text{at } z_\mu = 0. \end{aligned}$$

304 (and boundary conditions are still missing when  $z_\mu \rightarrow -\infty$ ). The solutions of (4.3)  
 305 are expanded in terms of series

$$306 \quad (4.4) \quad \begin{aligned} u_x &= \sum_{n \geq 0} \delta^n \tilde{v}_x^{(n)}(x_m, z_\mu, t), & u_z &= \delta \sum_{n \geq 0} \delta^n \tilde{v}_z^{(n)}(x_m, z_\mu, t), & \varphi &= \sum_{n \geq 0} \delta^n \tilde{\psi}^{(n)}(x_m, z_\mu, t), \\ \eta &= \delta \sum_{n \geq 0} \delta^n \tilde{\eta}^{(n)}(x_m, t), \end{aligned}$$

307 where we anticipate that  $\varphi = O(1)$  imposes  $\eta, u_z = O(\delta)$  from  ${}^\mu_0(\text{DC}) - {}^\mu_0(\text{KC})$  in (4.3).

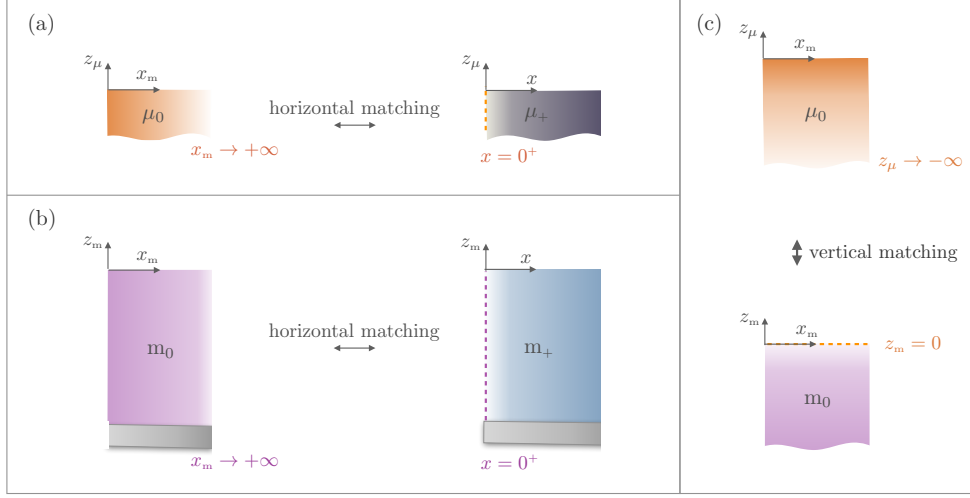


FIG. 5. The different matchings required for the analysis in the the depth transition regions, according to figure 2. (a) Horizontal matching in the boundary layer at the free surface, near  $(\mu_0)$  and far  $(\mu_+)$  from the transition; (b) horizontal matching in the constant depth regions  $\beta = \beta_+ = O(1)$  near  $(m_0)$  and far  $(m_+)$  from the transition; (c) vertical matching, near the transition, between the boundary layer at the free surface  $(\mu_0)$  and the region of constant depth  $(m_0)$ .

308 **4.1.2. The hierarchies of equations.** In  $m_0$ , we inject (4.2) into (4.1), and  
 309 we obtain the hierarchy of equations, namely  $\forall n$

$$310 \quad (4.5) \quad \begin{aligned} {}_0^m(\text{Inc})^n : \quad \text{div}_m \tilde{\mathbf{u}}^{(n)} &= 0, & {}_0^m(\text{Rot})^n : \quad \tilde{\mathbf{u}}^{(n)} &= \nabla_m \tilde{\varphi}^{(n+1)}, \\ {}_0^m(\text{RC})^n : \quad \tilde{\mathbf{u}}^{(n)} \cdot \mathbf{n} &= 0 \quad \text{on the walls,} \end{aligned}$$

311 (by convention  $\tilde{\mathbf{u}}^{(n)} = 0$  for  $n < 0$ ). In  $\mu_0$ , we inject (4.4) into (4.3), and we obtain

$$312 \quad (4.6) \quad \begin{aligned} {}_0^\mu(\text{Inc})^n : \quad \frac{\partial \tilde{v}_z^{(n)}}{\partial z_\mu} &= 0, \quad n \leq 1, & \frac{\partial \tilde{v}_z^{(2)}}{\partial z_\mu} &= -\alpha \tilde{\eta}^{(0)} \frac{\partial \tilde{v}_x^{(0)}}{\partial x_m}, \\ {}_0^\mu(\text{Rot})_x^n : \quad \tilde{v}_x^{(n)} &= \frac{\partial \tilde{\psi}^{(n+1)}}{\partial x_m}, & {}_0^\mu(\text{Rot})_z^n : \quad \frac{\partial \tilde{\psi}^{(n)}}{\partial z_\mu} &= 0, \quad n \leq 5, \end{aligned}$$

313 along with the boundary conditions at the free surface  $z_\mu = 0$ ,

$$314 \quad (4.7) \quad \begin{aligned} {}_0^\mu(\text{DC})^n : \quad \frac{\partial \tilde{\psi}^{(n)}}{\partial t} + \tilde{\eta}^{(n)} &= 0, \quad n \leq 2, & \frac{\partial \tilde{\psi}^{(3)}}{\partial t} + \frac{\alpha}{2} (\tilde{v}_x^{(0)})^2 + \tilde{\eta}^{(3)} &= 0, \\ {}_0^\mu(\text{KC})^n : \quad \tilde{v}_z^{(n)} &= \frac{\partial \tilde{\eta}^{(n)}}{\partial t}, \quad n \leq 1, & \tilde{v}_z^{(3)} &= \frac{\partial \tilde{\eta}^{(3)}}{\partial t} + \alpha \tilde{v}_x^{(0)} \frac{\partial \tilde{\eta}^{(0)}}{\partial x}. \end{aligned}$$

315 **4.1.3. Vertical matching conditions between  $m_0$  and  $\mu_0$ .** As in the previ-  
 316 ous section, the solutions in  $m_0$  and  $\mu_0$  must match in an intermediate region when  
 317  $z_m \rightarrow 0$  and  $z_\mu \rightarrow -\infty$ . The result is identical (replacing  $x$  by  $x_m$  in (3.8)-(3.9)), *i.e.*

$$318 \quad (4.8) \quad \begin{cases} \tilde{\varphi}^{(n)}(x_m, 0, t) = \lim_{z_\mu \rightarrow -\infty} \tilde{\psi}^{(n)}(x_m, z_\mu, t), & n \leq 2, \\ \tilde{\varphi}^{(3)}(x_m, 0, t) = \lim_{z_\mu \rightarrow -\infty} \left( \tilde{\psi}^{(3)}(x_m, z_\mu, t) - \alpha(z_\mu + 1) \tilde{\eta}^{(0)}(x_m, t) \frac{\partial \tilde{\varphi}^{(0)}}{\partial z_m}(x_m, 0, t) \right), \end{cases}$$

319 and the same relations are obtained replacing  $\tilde{\varphi}^{(n)}$  by  $\tilde{u}_x^{(n)}$  and  $\tilde{\psi}^{(n)}$  by  $\tilde{v}_x^{(n)}$ . However,  
 320 for  $\tilde{v}_z^{(n)}$ , we have to take into account a slightly different matching, according to the  
 321 expansions (4.2) and (4.4), which imposes

$$322 \quad \tilde{u}_z^{(0)}(x_m, z_m, t) + \delta \tilde{u}_z^{(1)}(x_m, z_m, t) + \dots \sim \delta \tilde{v}_z^{(0)}(x_m, z_\mu, t) + \delta^2 \tilde{v}_z^{(1)}(x_m, z_\mu, t),$$

323 so using (3.9), we obtain

$$324 \quad (4.9) \quad \begin{cases} \tilde{u}_z^{(0)}(x_m, 0, t) = 0, \\ \tilde{u}_z^{(n)}(x_m, 0, t) = \lim_{z_\mu \rightarrow -\infty} \tilde{v}_z^{(n-1)}(x_m, z_\mu, t), \quad n = 1, 2, \\ \tilde{u}_z^{(3)}(x_m, 0, t) = \lim_{z_\mu \rightarrow -\infty} \left( \tilde{v}_z^{(2)}(x_m, z_\mu, t) - \alpha(z_\mu + 1) \tilde{\eta}^{(0)}(x_m, t) \frac{\partial \tilde{u}_z^{(0)}}{\partial z_m}(x_m, 0, t) \right). \end{cases}$$

325 **4.1.4. Horizontal matching conditions between  $m_0$  and  $m_\pm$  and between**  
 326  **$\mu_0$  and  $\mu_\pm$ .** Horizontal matchings are obtained, as for the vertical matchings, by  
 327 imposing that solutions in  $m_0$  (resp.  $\mu_0$ ) correspond, in an intermediate region, to  
 328 solutions in  $m_\pm$  (resp.  $\mu_\pm$ ) when  $x \rightarrow 0^\pm$  and  $x_m \rightarrow \pm\infty$ , see figures 3 and 5. Between  
 329  $\mu_0$  and  $\mu_\pm$ , with

$$330 \quad \psi^{(0)}(x, z_\mu, t) + \delta \psi^{(1)}(x, z_\mu, t) + \dots \sim \tilde{\psi}^{(0)}(x_m, z_\mu, t) + \delta \tilde{\psi}^{(1)}(x_m, z_\mu, t) + \dots,$$

and taking into account the Taylor expansion (in the horizontal direction)

$$f(x, z_\mu, t) = f(0, z_\mu, t) + \delta x_m \frac{\partial f}{\partial x}(0, z_\mu, t) + \dots,$$

331 we obtain, for  $n \geq 0$  and with  $p \geq 1$ ,  $m \geq 0$ ,

$$332 \quad (4.10) \quad \psi^{(n)}(0^\pm, z_\mu, t) = \lim_{x_m \rightarrow \pm\infty} \left( \tilde{\psi}^{(n)}(x_m, z_\mu, t) - \sum_{p+m=n} \frac{x_m^p}{p!} \frac{\partial^p \psi^{(m)}}{\partial x^p}(0^\pm, z_\mu, t) \right),$$

333 which remain valid by replacing  $\psi^{(n)}$  by  $\eta^{(n)}$ ,  $v_x^{(n)}$  or  $v_z^{(n)}$ , and  $\tilde{\psi}^{(n)}$  by  $\tilde{\eta}^{(n)}$ ,  $\tilde{v}_x^{(n)}$  or  $\tilde{v}_z^{(n)}$   
 334 (for  $\eta^{(n)}$  and  $\tilde{\eta}^{(n)}$ , the dependence in  $z_\mu$  has to be omitted). The same applies between  
 335  $m_0$  and  $m_\pm$ , *i.e.*

$$336 \quad (4.11) \quad \varphi^{(n)}(0^\pm, z_m, t) = \lim_{x_m \rightarrow \pm\infty} \left( \tilde{\varphi}^{(n)}(x_m, z_m, t) - \sum_{p+m=n} \frac{x_m^p}{p!} \frac{\partial^p \varphi^{(m)}}{\partial x^p}(0^\pm, z_m, t) \right),$$

337 which remain valid by replacing  $\varphi^{(n)}$  by  $u_x^{(n)}$  and  $\tilde{\varphi}^{(n)}$  by  $\tilde{u}_x^{(n)}$ . For  $u_z^{(n)}$ , however, we  
 338 must take into account the shift of the expansions (3.2) and (4.2), which gives

$$339 \quad (4.12) \quad \begin{cases} 0 = \lim_{x_m \rightarrow \pm\infty} \tilde{u}_z^{(0)}(x_m, z_\mu, t), \\ u_z^{(n)}(0^\pm, z_m, t) = \lim_{x_m \rightarrow \pm\infty} \left( \tilde{u}_z^{(n+1)}(x_m, z_m, t) - \sum_{p+m=n} \frac{x_m^p}{p!} \frac{\partial^p u_z^{(m)}}{\partial x^p}(0^\pm, z_m, t) \right). \end{cases}$$

340 We are now in position to proceed with the asymptotic analysis of the step dis-  
 341 continuity. The main objective of the forthcoming section is to establish the relations  
 342 between the surface elevation  $\eta^{(n)}$  and the flux  $q^{(n)}$  defined in (3.15) satisfy on both  
 343 sides of the step discontinuity. These relations will involve both jump and average as  
 344 defined in (2.7).

345 **4.2. Continuity conditions at order  $n = 0$ .** At the leading order, surface  
 346 elevation and flux are continuous across the step.

347 PROPOSITION 4.1. *The continuity conditions apply at the dominant order  $n = 0$ :*

$$348 \quad (4.13) \quad (\text{JC})_1^0 : \llbracket \eta^{(0)} \rrbracket = 0,$$

$$349 \quad (4.14) \quad (\text{JC})_2^0 : \llbracket q^{(0)} \rrbracket = 0.$$

351 **4.2.1. Derivation of  $(\text{JC})_1^0$ .** We start by using  ${}^m(\text{Rot})^{-1}$  in (4.5) and  ${}^\mu(\text{Rot})_x^{-1}$ ,  
 352  ${}^\mu(\text{Rot})_z^0$  in (4.6) from which we deduce that  $\tilde{\varphi}^{(0)}$  and  $\tilde{\psi}^{(0)}$  depend only on  $t$ . Using the  
 353 matching conditions (4.8), (4.10) and (4.11), we obtain

$$354 \quad (4.15) \quad \psi^{(0)}(0^\pm, t) = \tilde{\psi}^{(0)}(t) = \tilde{\varphi}^{(0)}(t) = \phi^{(0)}(0^\pm, t).$$

355 Then, taking the derivative with respect to  $t$  of (4.15) and using  ${}^\mu(\text{DC})^0$ , we find

$$356 \quad (4.16) \quad \eta^{(0)}(0^\pm, t) = \tilde{\eta}^{(0)}(t) = -\frac{\partial \psi^{(0)}}{\partial t}(0^\pm, t) = -\frac{\partial \tilde{\psi}^{(0)}}{\partial t}(t),$$

357 from which we deduce the continuity condition (4.13). We also deduce from  ${}^m(\text{Inc})^0$   
 358 in (4.6) that  $\tilde{v}_z^{(0)}$  does not depend on  $z_\mu$ , which, combined with  ${}^\mu(\text{KC})^0$ , tells us that  
 359  $\tilde{v}_z^{(0)}$  also does not depend on  $x_m$  and satisfies

$$360 \quad (4.17) \quad \tilde{v}_z^{(0)}(t) = \frac{\partial \eta^{(0)}}{\partial t}(t).$$

361 **4.2.2. Derivation of  $(\text{JC})_2^0$ .** We now use the relation  ${}^m(\text{Inc})^0$  in (4.5) which we  
 362 integrate over

$$363 \quad (4.18) \quad \Omega_m^* = \{x_m \in (-x_m^*, x_m^*), z_m \in (-\beta(x_m), 0)\},$$

364 (with  $x_m^* > 0$ ) to obtain

$$365 \quad (4.19) \quad \int_{-\beta_+}^0 \tilde{u}_x^{(0)}(x_m^*, z_m, t) dz_m - \int_{-\beta_-}^0 \tilde{u}_x^{(0)}(-x_m^*, z_m, t) dz_m + \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(0)}(x_m, 0, t) dx_m = 0.$$

366 According to (4.9),  $\tilde{u}_z^{(0)}(x_m, 0, t) = 0$ , and according to (4.11),  $\tilde{u}_x^{(0)}(\pm x_m^*, z_m, t) \rightarrow$   
 367  $U_x^{(0)}(0^\pm, t)$  when  $x_m^*$  tends to  $+\infty$  (see (3.14)). This allows us to conclude that (4.19)  
 368 simplifies, after passing to the limit  $x_m^* \rightarrow +\infty$ , to  $q^{(0)}(0, t) = \beta_\pm U_x^{(0)}(0^\pm, t)$ , hence the  
 369 continuity of the flux (4.14).

370 **4.3. Jump conditions at order  $n = 1$ .** At order  $n = 1$ , we have a jump  
 371 condition of the surface elevation and continuity of the flux.

372 PROPOSITION 4.2. *Jump condition and continuity condition apply at the order*  
 373  *$n = 1$ :*

$$374 \quad (4.20) \quad (\text{JC})_1^1 : \llbracket \eta^{(1)} \rrbracket = -\mathcal{B}_1 \frac{\partial q^{(0)}}{\partial t}(0, t),$$

$$375 \quad (4.21) \quad (\text{JC})_2^1 : \llbracket q^{(1)} \rrbracket = 0,$$

377 where the blockage coefficient  $\mathcal{B}_1$  is defined in (4.26).

378 **4.3.1. Derivation of (JC)<sub>1</sub><sup>1</sup>.** To establish this relation, we will consider the  
 379 problem satisfied by  $(\tilde{\mathbf{u}}^{(0)}, \tilde{\varphi}^{(1)})$ . It consists in  ${}^m_0(\text{Inc})^0$ ,  ${}^m_0(\text{Rot})^0$  and  ${}^m_0(\text{RC})^0$  in (4.5),  
 380 and the first equation in (4.9). The behavior of  $\tilde{\mathbf{u}}^{(0)}$  when  $x_m \rightarrow \pm\infty$  is provided by  
 381 the matching conditions, which tell us that  $\tilde{u}_x^{(0)}(\pm x_m^*, z_m, t) \rightarrow U_x^{(0)}(0^\pm, t)$ , according to  
 382 (4.11) and (3.14), and that  $\tilde{u}_z^{(0)}(\pm x_m^*, z_m, t) \rightarrow 0$  according to (4.12). Recalling (4.14)  
 383 with (3.15), we can finally establish that  $(\tilde{\mathbf{u}}^{(0)}, \tilde{\varphi}^{(1)})$  satisfies

$$384 \quad (4.22) \quad \begin{cases} \operatorname{div}_m \tilde{\mathbf{u}}^{(0)} = 0, & \tilde{\mathbf{u}}^{(0)} = \nabla_m \tilde{\varphi}^{(1)}, \\ \tilde{\mathbf{u}}^{(0)} \cdot \mathbf{n} = 0 \text{ on the walls,} & \tilde{u}_z^{(0)}(x_m, 0, t) = 0, \\ \lim_{x_m \rightarrow \pm\infty} \tilde{\mathbf{u}}^{(0)}(x_m, z_m, t) = \frac{q^{(0)}(0, t)}{\beta_\pm} \mathbf{e}_x. \end{cases}$$

385 The above problem is linear with respect to  $q^{(0)}(0, t)$ , hence we set

$$386 \quad (4.23) \quad \begin{aligned} \tilde{\varphi}^{(1)}(x_m, z_m, t) &= q^{(0)}(0, t) Q_1(x_m, z_m) + \overline{\phi^{(1)}}(t), \\ \tilde{\mathbf{u}}^{(0)}(x_m, z_m, t) &= q^{(0)}(0, t) \nabla Q_1(x_m, z_m), \end{aligned}$$

387 where we have defined  $\overline{\phi^{(1)}}(t) = \frac{1}{2} (\phi^{(1)}(0^-, t) + \phi^{(1)}(0^+, t))$ , and where  $Q_1$  is the solu-  
 388 tion to the elementary problem

$$389 \quad (4.24) \quad \begin{cases} \Delta Q_1 = 0, \\ \nabla Q_1 \cdot \mathbf{n} = 0 \text{ on the walls,} & \frac{\partial Q_1}{\partial z_m}(x_m, 0) = 0, \\ \lim_{x_m \rightarrow \pm\infty} \nabla Q_1(x_m, z_m) = \frac{\mathbf{e}_x}{\beta_\pm}. \end{cases}$$

390 Since the solution  $Q_1$  is defined up to a constant, we impose that

$$391 \quad (4.25) \quad Q_1(x_m, z_m) = Q_1^{\text{ev}}(x_m, z_m) + \begin{cases} \frac{x_m}{\beta_+} + \frac{\mathcal{B}_1}{2}, & \text{if } x_m > 0, \\ \frac{x_m}{\beta_-} - \frac{\mathcal{B}_1}{2}, & \text{if } x_m < 0, \end{cases}, \quad \lim_{x_m \rightarrow \pm\infty} Q_1^{\text{ev}} = 0,$$

392 where  $Q_1^{\text{ev}}$  is an evanescent field and  $\mathcal{B}_1$  a blockage coefficient defined as follows

$$393 \quad (4.26) \quad \mathcal{B}_1 = \lim_{x_m \rightarrow +\infty} \left( Q_1 - \frac{x_m}{\beta_+} \right) - \lim_{x_m \rightarrow -\infty} \left( Q_1 - \frac{x_m}{\beta_-} \right).$$

394 Owing to the matching condition (4.11) ( $n = 1$ ), with  $\tilde{\varphi}^1$  in (4.23),  $\partial_x \varphi^{(0)} = q^{(0)}/\beta_\pm$   
 395 from (3.13), (3.14) and (4.14), and accounting for (4.25), we obtain

$$396 \quad (4.27) \quad \varphi^{(1)}(0^\pm, z_m, t) = \pm q^{(0)}(0, t) \frac{\mathcal{B}_1}{2} + \overline{\phi^{(1)}}(t).$$

397 Using (3.16), we finally obtain the jump of  $\eta^{(1)}$  announced in (4.20). In particular we  
 398 have

$$399 \quad (4.28) \quad \eta^{(1)}(0^\pm, t) = \overline{\eta^{(1)}}(t) \pm \frac{1}{2} \llbracket \eta^{(1)} \rrbracket = \overline{\eta^{(1)}}(t) \mp \frac{\mathcal{B}_1}{2} \frac{\partial q^{(0)}}{\partial t}(0, t).$$

400 **4.3.2. Derivation of  $(\mathbf{JC})_2^1$ .** As before, we use  ${}^m_0(\text{Inc})^1$  in (4.5) which we inte-  
 401 grate over  $\Omega_m^*$  defined in (4.18). We obtain, after using  ${}^m_0(\text{RC})^1$  in (4.5),  
 (4.29)

$$402 \int_{-\beta_+}^0 \tilde{u}_x^{(1)}(x_m^*, z_m, t) dz_m - \int_{-\beta_-}^0 \tilde{u}_x^{(1)}(-x_m^*, z_m, t) dz_m + \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(1)}(x_m, 0, t) dx_m = 0.$$

403 The first two integrals in (4.29) are given by the matching condition (4.11) with  
 404  $u_x^{(1)} = U_x^{(1)}(x, t)$  from (3.14). Also using (3.12), we obtain  $\int_{-\beta_\pm}^0 \tilde{u}_x^{(1)}(x_m^*, z_m, t) dz_m =$   
 405  $\beta_\pm U_x^{(1)}(0^\pm, t) \mp x_m^* \partial_t \eta^{(0)}(0, t)$ . The third integral in (4.29) is given by the matching  
 406 condition (4.9) in  $\tilde{u}_z^{(1)}$  with  $\tilde{v}_z^{(0)}$  in (4.17), and so we have  $\int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(1)}(x_m, 0, t) dx_m =$   
 407  $2x_m^* \partial_t \eta^{(0)}(0, t)$ . By summing the contributions, the diverging terms, linear in  $x_m^*$ ,  
 408 cancel out and we obtain the continuity (4.21) of the flux at order 1 defined by (3.15)  
 409 with  $q^{(1)}(0, t) = \beta_\pm U_x^{(1)}(0^\pm, t)$ .

410 **4.4. Jump conditions at order  $n = 2$ .** At order  $n = 2$ , we have jump condi-  
 411 tions on both surface elevation and flux across the step discontinuity.

412 PROPOSITION 4.3. *Jump conditions apply at the order  $n = 2$ :*

$$413 (4.30) \quad (\mathbf{JC})_1^2 : \quad \llbracket \eta^{(2)} \rrbracket = -\mathcal{B}_1 \frac{\partial q^{(1)}}{\partial t}(0, t) - \mathcal{B}_2 \frac{\partial^2 \eta^{(0)}}{\partial t^2}(0, t),$$

$$414 (4.31) \quad (\mathbf{JC})_2^2 : \quad \llbracket q^{(2)} \rrbracket = \mathcal{C}_1 \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t),$$

416 where the effective coefficients  $\mathcal{B}_2$  and  $\mathcal{C}_1$  are defined in (4.35) and (4.40).

417 **4.4.1. Derivation of  $(\mathbf{JC})_1^2$ .** We proceed as at order 1, and consider the problem  
 418 satisfied by  $(\tilde{\mathbf{u}}^{(1)}, \tilde{\varphi}^{(2)})$  which consists of  ${}^m_0(\text{Inc})^1$ ,  ${}^m_0(\text{Rot})^1$  and  ${}^m_0(\text{RC})^1$  from (4.5), and  
 419 the second equation from (4.9) for  $n = 1$  with  $\tilde{v}_z^{(0)}$  given by (4.17) which provides the  
 420 boundary condition at  $z_m = 0$ . To complete the problem, we need to determine the  
 421 behavior of  $\tilde{\mathbf{u}}^{(1)}$  when  $x_m \rightarrow \pm\infty$ . By combining the matching condition (4.11) with  
 422 (3.12), (3.14) and (4.21), we obtain  $\tilde{u}_x^{(1)}(\pm x_m^*, z_m, t) \rightarrow (q^{(1)}(0, t) - x_m \partial_t \eta^{(0)}(0, t)) / \beta_\pm$ .  
 423 Then, we have  $\tilde{u}_z^{(1)}(\pm x_m^*, z_m, t) \rightarrow (z_m / \beta_\pm + 1) \partial_t \eta^{(0)}(0, t)$ , from the matching (4.12)  
 424 and (3.17). This allows us to establish the problem satisfied by  $(\tilde{\mathbf{u}}^{(1)}, \tilde{\varphi}^{(2)})$ , *i.e.*

$$425 \left\{ \begin{array}{l} \text{div}_m \tilde{\mathbf{u}}^{(1)} = 0, \quad \tilde{\mathbf{u}}^{(1)} = \nabla_m \tilde{\varphi}^{(2)}, \\ \tilde{\mathbf{u}}^{(1)} \cdot \mathbf{n} = 0 \text{ on the walls,} \quad \tilde{u}_z^{(1)}(x_m, 0, t) = \frac{\partial \eta^{(0)}}{\partial t}(0, t), \\ \lim_{x_m \rightarrow \pm\infty} \tilde{\mathbf{u}}^{(1)}(x_m, z_m, t) = \frac{1}{\beta_\pm} \left( q^{(1)}(0, t) - x_m \frac{\partial \eta^{(0)}}{\partial t}(0, t) \right) \mathbf{e}_x + \left( \frac{z_m}{\beta_\pm} + 1 \right) \frac{\partial \eta^{(0)}}{\partial t}(0, t) \mathbf{e}_z. \end{array} \right.$$

426 The above problem is linear with respect to  $q^{(1)}(0, t)$  and  $\partial_t \eta^{(0)}(0, t)$ , so we pose

$$427 (4.32) \quad \begin{aligned} \tilde{\varphi}^{(2)}(x_m, z_m, t) &= q^{(1)}(0, t) Q_1(x_m, z_m) + \frac{\partial \eta^{(0)}}{\partial t}(0, t) Q_2(x_m, z_m) + \overline{\phi^{(2)}}(t), \\ \tilde{\mathbf{u}}^{(1)}(x_m, z_m, t) &= q^{(1)}(0, t) \nabla Q_1(x_m, z_m) + \frac{\partial \eta^{(0)}}{\partial t}(0, t) \nabla Q_2(x_m, z_m), \end{aligned}$$

428 where we have defined  $\overline{\phi^{(2)}}(t) = \frac{1}{2}(\phi^{(2)}(0^-, t) + \phi^{(2)}(0^+, t))$ , and  $Q_2$  is the solution to  
 429 the elementary problem

$$430 \quad (4.33) \quad \begin{cases} \Delta Q_2 = 0, \\ \nabla Q_2 \cdot \mathbf{n} = 0 \text{ on the walls, } \frac{\partial Q_2}{\partial z_m}(x_m, 0) = 1, \\ \lim_{x_m \rightarrow \pm\infty} \nabla Q_2(x_m, z_m) = -\frac{x_m}{\beta_{\pm}} \mathbf{e}_x + \left(\frac{z_m}{\beta_{\pm}} + 1\right) \mathbf{e}_z. \end{cases}$$

431 As the solution  $Q_2$  is defined up to a constant, we impose that  
 (4.34)

$$432 \quad Q_2(x_m, z_m) = Q_2^{\text{ev}}(x_m, z_m) + \begin{cases} \frac{z_m^2 - x_m^2}{2\beta_+} + z_m + \frac{\mathcal{B}_2}{2}, & \text{if } x_m > 0, \\ \frac{z_m^2 - x_m^2}{2\beta_-} + z_m - \frac{\mathcal{B}_2}{2}, & \text{if } x_m < 0, \end{cases}, \quad \lim_{x_m \rightarrow \pm\infty} Q_2^{\text{ev}} = 0,$$

433 where  $Q_2^{\text{ev}}$  is an evanescent field and  $\mathcal{B}_2$  is a blockage coefficient defined as follows

$$434 \quad (4.35) \quad \mathcal{B}_2 = \lim_{x_m \rightarrow +\infty} \left( Q_2 - \frac{z_m^2 - x_m^2}{2\beta_+} - z_m \right) - \lim_{x_m \rightarrow -\infty} \left( Q_2 - \frac{z_m^2 - x_m^2}{2\beta_-} - z_m \right).$$

435 We now consider the matching condition (4.11) for  $n = 2$ , which involves the quantities  
 436  $\partial_x \varphi^{(1)}(0^{\pm}, t) = q^{(1)}(0, t)/\beta_{\pm}$  from (3.13)-(3.14) and (4.21), and  $\partial_{xx} \varphi^{(0)}(0^{\pm}, t) =$   
 437  $-\partial_t \eta^{(0)}(0, t)/\beta_{\pm}$  from (3.12)-(3.14). We thus obtain

$$438 \quad \varphi^{(2)}(0^{\pm}, z_m, t) = \lim_{x_m \rightarrow \pm\infty} \left( \tilde{\varphi}^{(2)}(x_m, z_m, t) - \frac{x_m}{\beta_{\pm}} q^{(1)}(0, t) + \frac{x_m^2}{2\beta_{\pm}} \frac{\partial \eta^{(0)}}{\partial t}(0, t) \right).$$

439 Using  $\tilde{\varphi}^{(2)}$  in (4.32) and the asymptotic behaviours of  $Q_1$  (in (4.25)) and  $Q_2$  (in (4.34)),  
 440 we deduce that

$$441 \quad (4.36) \quad \begin{aligned} \varphi^{(2)}(0^{\pm}, z_m, t) &= \pm q^{(1)}(0, t) \frac{\mathcal{B}_1}{2} + \frac{\partial \eta^{(0)}}{\partial t}(0, t) \left( \frac{z_m^2}{2\beta_{\pm}} + z_m \pm \frac{\mathcal{B}_2}{2} \right) + \overline{\phi^{(2)}}(t), \\ \phi^{(2)}(0^{\pm}, t) &= \pm \left( q^{(1)}(0, t) \frac{\mathcal{B}_1}{2} + \frac{\partial \eta^{(0)}}{\partial t}(0, t) \frac{\mathcal{B}_2}{2} \right) + \overline{\phi^{(2)}}(t), \end{aligned}$$

442 with  $\phi^{(2)}(0^{\pm}, t) = \varphi^{(2)}(x, 0, t)$  from (3.20) (note that (4.36) is consistent with (3.20)  
 443 thanks to (3.12)). Finally, using (3.21), we obtain the jump (4.30) of  $\eta^{(2)}$ .

444 **4.4.2. Derivation of (JC)<sub>2</sub><sup>2</sup>.** As we did previously, we use  ${}^m_0(\text{Inc})^2$  in (4.5) which  
 445 we integrate over  $\Omega_m^*$ , see (4.18). We obtain, after using  ${}^m_0(\text{RC})^2$  in (4.5),

$$446 \quad (4.37) \quad \int_{-\beta_+}^0 \tilde{u}_x^{(2)}(x_m^*, z_m, t) dz_m - \int_{-\beta_-}^0 \tilde{u}_x^{(2)}(-x_m^*, z_m, t) dz_m + \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(2)}(x_m, 0, t) dx_m = 0.$$

447 For the first two integrals, we use (4.11) with  $\int_{-\beta_{\pm}}^0 u_x^{(2)}(0^{\pm}, z_m, t) dz_m = q^{(2)}(0^{\pm}, t)$  by  
 448 definition, (3.15), and which involves  $\partial_x u_x^{(1)}(0^{\pm}, t) = \partial_{tt} \left( \pm q^{(0)}(0, t) \frac{\mathcal{B}_1}{2} + \overline{\phi^{(1)}}(t) \right) / \beta_{\pm}$ ,  
 449 from (3.12), (3.14), (3.16) and (4.27), and  $\partial_{xx} u_x^{(0)}(0^{\pm}, t) = \partial_{tt} q^{(0)}(0, t) / \beta_{\pm}^2$  from (3.11)-  
 450 (3.12), (3.14) and (4.14). So for  $x_m^* \rightarrow +\infty$  we get  
 (4.38)

$$451 \quad \int_{-\beta_{\pm}}^0 \tilde{u}_x^{(2)}(\pm x_m^*, z_m, t) dz_m \underset{x_m^* \rightarrow +\infty}{\sim} q^{(2)}(0^{\pm}, t) \pm x_m^* \frac{\partial^2 \overline{\phi^{(1)}}}{\partial t^2}(t) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) \left( \frac{x_m^{*2}}{2\beta_{\pm}} + x_m^* \frac{\mathcal{B}_1}{2} \right).$$

452 For the third integral in (4.37), we use (4.9), so  $\tilde{u}_z^{(2)}(x_m, 0, t) = \tilde{v}_z^{(1)}(x_m, t)$  since  $\tilde{v}_z^{(1)}$  and  
 453  $\tilde{\psi}^{(1)}$  do not depend on  $z_\mu$ , according to  $\mu_0(\text{Inc})^1$  and  $\mu_0(\text{Rot})_z^1$  in (4.6). Using  $\mu_0(\text{DC})^1$  and  
 454  $\mu_0(\text{KC})^1$  in (4.7) (for  $n = 1$ ), the matching (4.8) (for  $n = 1$ ) and finally the form of  $\tilde{\varphi}^{(1)}$   
 455 in (4.23), we deduce that  $\tilde{u}_z^{(2)}(x_m, 0, t) = -\partial_{tt}(q^{(0)}(0, t)Q_1(x_m, 0) + \overline{\phi^{(1)}}(t))$ . Finally,  
 456 using (4.25), we have, for  $x_m^* \rightarrow +\infty$ ,

$$(4.39) \quad \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(2)}(x_m, 0, t) dx_m \underset{x_m^* \rightarrow +\infty}{\sim} -\frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) \left( \mathcal{C}_1 + \frac{x_m^{*2}}{2\beta_+} - \frac{x_m^{*2}}{2\beta_-} \right) - 2x_m^* \frac{\partial^2 \overline{\phi^{(1)}}}{\partial t^2}(t),$$

458 with

$$(4.40) \quad \mathcal{C}_1 = \int_{-\infty}^{+\infty} Q_1^{\text{ev}}(x_m, 0) dx_m = \mathcal{B}_2 - \frac{1}{3}(\beta_+ - \beta_-),$$

460 and  $\mathcal{C}_1$  is finite because  $Q_1^{\text{ev}}$  vanishes for  $x_m \rightarrow \pm\infty$  according to (4.25) (see appendix  
 461 A.2 for the relation between  $\mathcal{C}_1$  and  $\mathcal{B}_2$ ). By gathering (4.38) and (4.39) in (4.37), the  
 462 diverging terms, linear and quadratic in  $x_m^*$  cancel out, leaving us with

$$(4.41) \quad \llbracket q^{(2)} \rrbracket = \mathcal{C}_1 \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t), \quad \text{with} \quad q^{(2)}(0^\pm, t) = \beta_\pm U_x^{(2)}(0^\pm, t).$$

464 In particular, we have

$$(4.42) \quad q^{(2)}(0^\pm, t) = \overline{q^{(2)}}(t) \pm \frac{\mathcal{C}_1}{2} \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t),$$

466 with  $\overline{q^{(2)}}(t) = \frac{1}{2}(q^{(2)}(0^-, t) + q^{(2)}(0^+, t))$  being the average flux across the discontinuity.

467 **4.5. Jump conditions at order  $n = 3$ .** At order  $n = 3$ , non-linear contribu-  
 468 tions arise in the jump conditions.

469 PROPOSITION 4.4. *Jump conditions with non-linear contribution apply at the or-  
 470 der  $n = 3$ :*

(4.43)

$$(4.43) \quad (\text{JC})_1^3: \llbracket \eta^{(3)} \rrbracket = - \left( \mathcal{B}_1 \frac{\partial \overline{q^{(2)}}}{\partial t} + \mathcal{B}_2 \frac{\partial^2 \overline{\eta^{(1)}}}{\partial t^2} + \mathcal{B}_3 \frac{\partial^3 q^{(0)}}{\partial t^3} \right) - \frac{\alpha}{2} (q^{(0)})^2 \left( \frac{1}{\beta_+^2} - \frac{1}{\beta_-^2} \right),$$

(4.44)

$$(4.44) \quad (\text{JC})_2^3: \llbracket q^{(3)} \rrbracket = \mathcal{C}_1 \frac{\partial^2 q^{(1)}}{\partial t^2} + \mathcal{C}_2 \frac{\partial^3 \eta^{(0)}}{\partial t^3} - \alpha \eta^{(0)} q^{(0)} \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right),$$

474 where the effective coefficients  $\mathcal{B}_3$  and  $\mathcal{C}_2$  are defined in (4.48) and (4.52).

475 **4.5.1. Derivation of  $(\text{JC})_1^3$ .** We repeat the exercise and consider the problem  
 476 satisfied by  $(\tilde{u}^{(2)}, \tilde{\varphi}^{(3)})$  which consists in  $\mu_0(\text{Inc})^2$ ,  $\mu_0(\text{Rot})^2$  and  $\mu_0(\text{RC})^2$  in (4.5). The  
 477 condition at  $z_m = 0$  is obtained using (4.9), with  $\tilde{v}_z^{(1)}$  given by  $\mu_0(\text{KC})^1$ ,  $\mu_0(\text{DC})^1$  in (4.7)  
 478 and (4.8) for  $n = 1$ , and finally (4.23) as well as (3.16).

479 The conditions when  $x_m \rightarrow \pm\infty$  on  $\tilde{u}^{(2)}$  are obtained as follows. For  $\tilde{u}_x^{(2)}$ , we use  
 480 (4.11) which involves  $u_x^{(2)} = U_x^{(2)} - \left( \frac{z_m^2}{2\beta} + z_m + \frac{\beta}{3} \right) \partial_{tt} U_x^{(0)}$  from (3.22)-(3.23) and  
 481 (3.11)-(3.12) (with (3.15) for  $n = 0, 2$  and  $q^{(2)}$  given in (4.42)),  $\partial_x u_x^{(1)}(x_m, 0^\pm, t) =$   
 482  $-\partial_t \eta^{(1)}(x_m, t) / \beta_\pm$  from (3.12) and (3.14) ( $\eta^{(1)}$  given in (4.28)), and  $\partial_{xx} u_x^{(0)}(x_m, 0^\pm, t) =$   
 483  $\partial_{tt} q^{(0)}(0, t) / \beta_\pm$ . For  $\tilde{u}_z^{(2)}$ , we use (4.12) which involves  $u_z^{(1)} = \left( \frac{z_m}{\beta_\pm} + 1 \right) \partial_t \eta^{(1)}$  and

484  $\partial_x u_z^{(0)} = -\frac{1}{\beta_{\pm}} \left( \frac{z_m}{\beta_{\pm}} + 1 \right) \partial_{tt} q^{(0)}$ , from (3.17) with (3.11)- (3.12). We obtain

$$485 \left\{ \begin{array}{l} \operatorname{div}_m \tilde{\mathbf{u}}^{(2)} = 0, \quad \tilde{\mathbf{u}}^{(2)} = \nabla_m \tilde{\varphi}^{(3)}, \\ \tilde{\mathbf{u}}^{(2)} \cdot \mathbf{n} = 0 \text{ on the walls,} \quad \tilde{u}_z^{(2)} = \frac{\partial \overline{\eta}^{(1)}}{\partial t} - Q_1 \frac{\partial^2 q^{(0)}}{\partial t^2} \text{ at } z_m = 0, \\ \lim_{x_m \rightarrow \pm\infty} \tilde{\mathbf{u}}^{(2)} = \frac{1}{\beta_{\pm}} \left( \overline{q}^{(2)} - x_m \frac{\partial \overline{\eta}^{(1)}}{\partial t} + \chi_x^{\pm} \frac{\partial^2 q^{(0)}}{\partial t^2} \right) \mathbf{e}_x + \left( \left( \frac{z_m}{\beta_{\pm}} + 1 \right) \frac{\partial \overline{\eta}^{(1)}}{\partial t} - \chi_z^{\pm} \frac{\partial^2 q^{(0)}}{\partial t^2} \right) \mathbf{e}_z, \end{array} \right.$$

486 where we have defined

$$487 \begin{aligned} \chi_x^{\pm}(x_m, z_m) &= \pm \frac{C_1}{2} - \frac{\beta_{\pm}}{3} - \left( \frac{z_m^2}{2\beta_{\pm}} + z_m \right) + \frac{x_m}{2} \left( \frac{x_m}{\beta_{\pm}} \pm \mathcal{B}_1 \right), \\ \chi_z^{\pm}(x_m, z_m) &= \left( \frac{z_m}{\beta_{\pm}} + 1 \right) \left( \frac{x_m}{\beta_{\pm}} \pm \frac{\mathcal{B}_1}{2} \right), \end{aligned}$$

488 The above problem is linear with respect to  $\overline{q}^{(2)}(t)$ ,  $\partial_t \overline{\eta}^{(1)}(t)$  and  $\partial_{tt} q^{(0)}(0, t)$ , so we set (4.45)

$$489 \begin{aligned} \tilde{\varphi}^{(3)} &= \overline{q}^{(2)}(t) Q_1(x_m, z_m) + \frac{\partial \overline{\eta}^{(1)}}{\partial t}(t) Q_2(x_m, z_m) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) Q_3(x_m, z_m) + \overline{\phi}^{(3)}(t), \\ \tilde{\mathbf{u}}^{(2)} &= \overline{q}^{(2)}(t) \nabla Q_1(x_m, z_m) + \frac{\partial \overline{\eta}^{(1)}}{\partial t}(t) \nabla Q_2(x_m, z_m) + \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) \nabla Q_3(x_m, z_m), \end{aligned}$$

490 where  $Q_3$  satisfies the elementary problem

$$491 (4.46) \quad \left\{ \begin{array}{l} \Delta Q_3 = 0, \\ \nabla Q_3 \cdot \mathbf{n} = 0 \text{ on the walls,} \quad \frac{\partial Q_3}{\partial z_m}(x_m, 0) = -Q_1(x_m, 0), \\ \lim_{x_m \rightarrow \pm\infty} \nabla Q_3(x_m, z_m) = \frac{1}{\beta_{\pm}} \chi_x^{\pm}(x_m, z_m) \mathbf{e}_x - \chi_z^{\pm}(x_m, z_m) \mathbf{e}_z. \end{array} \right.$$

492 As the solution  $Q_3$  is defined up to a constant, we impose (4.47)

$$493 \quad Q_3(x_m, z_m) = Q_3^{\text{ev}}(x_m, z_m) + \begin{cases} \zeta^+(x_m, z_m) + \frac{\mathcal{B}_3}{2} & \text{if } x_m > 0, \\ \zeta^-(x_m, z_m) - \frac{\mathcal{B}_3}{2} & \text{if } x_m < 0, \end{cases}, \quad \lim_{x_m \rightarrow \pm\infty} Q_3^{\text{ev}} = 0,$$

494 where  $Q_3^{\text{ev}}$  is an evanescent field,  $\zeta^{\pm}$  polynomial functions given by

$$495 \quad \zeta^{\pm}(x_m, z_m) = x_m \left( -\frac{1}{3} \pm \frac{C_1}{2\beta_{\pm}} \right) + \frac{x_m^2}{2\beta_{\pm}} \left( \frac{x_m}{3\beta_{\pm}} \pm \frac{\mathcal{B}_1}{2} \right) - \left( \frac{z_m^2}{2\beta_{\pm}} + z_m \right) \left( \frac{x_m}{\beta_{\pm}} \pm \frac{\mathcal{B}_1}{2} \right),$$

496 and  $\mathcal{B}_3$  a blockage coefficient defined as follows

$$497 (4.48) \quad \mathcal{B}_3 = \lim_{x_m \rightarrow +\infty} \left( Q_3 - \zeta^+(x_m, z_m) \right) - \lim_{x_m \rightarrow -\infty} \left( Q_3 - \zeta^-(x_m, z_m) \right).$$

499 To obtain the jump of  $\varphi^{(3)}$ , we use (4.11) which involves  $\tilde{\varphi}^{(3)}$  in (4.45) (with  $Q_n$  in (4.25), (4.34), (4.47)),  $\partial_x \varphi^{(2)} = \frac{q^{(2)}}{\beta_{\pm}} - \left( \frac{z_m^2}{2\beta_{\pm}} + \frac{z_m}{\beta_{\pm}} + \frac{1}{3} \right) \partial_{tt} q^{(0)}$  from (3.20) and (3.23)

501 (with  $q^{(2)}$  in (4.42)) and, from (3.12)-(3.14),  $\partial_{xx}\varphi^{(1)} = -\partial_t\eta^{(1)}/\beta_{\pm}$  (with  $\eta^{(1)}$  in (4.28))  
 502 and  $\partial_{xxx}\varphi^{(0)} = -\partial_{tt}q^{(0)}/\beta_{\pm}$ . We obtain

$$503 \quad \varphi^{(3)}(0^{\pm}, z_m, t) = \phi^{(3)}(0^{\pm}, t) - \left( \frac{z_m^2}{2\beta_{\pm}} + z_m \right) \left( -\frac{\partial\overline{\eta^{(1)}}}{\partial t}(t) \pm \frac{\mathcal{B}_1}{2} \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) \right),$$

504 with

$$505 \quad \phi^{(3)}(0^{\pm}, t) = \pm \left( \frac{\mathcal{B}_1}{2} \overline{q^{(2)}}(t) + \frac{\mathcal{B}_2}{2} \frac{\partial\overline{\eta^{(1)}}}{\partial t}(t) + \frac{\mathcal{B}_3}{2} \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t) \right) + \overline{\phi^{(3)}}(t).$$

506 Note that the above relation is consistent with (3.26) since  $\partial_x U_x^{(1)} = -\partial_t\eta^{(1)}$  with  $\eta^{(1)}$   
 507 in (4.28). Using (3.27) and (4.14), we deduce that

$$508 \quad \eta^{(3)}(0^{\pm}, t) = -\frac{\partial\phi^{(3)}}{\partial t}(0^{\pm}, t) - \frac{\alpha}{2\beta_{\pm}^2} (q^{(0)}(t))^2,$$

509 hence the jump (4.43) of  $\eta^{(3)}$ .

510 **4.5.2. Derivation of  $(\mathbf{JC})_2^3$ .** We repeat the exercise for the last time, with  
 511  ${}^m_0(\text{Inc})^3$  in (4.5) which we integrate over  $\Omega_m^*$ , see (4.18). We obtain, after using  ${}^m_0(\text{RC})^3$   
 512 in (4.5),  
 (4.49)

$$513 \quad \int_{-\beta_+}^0 \tilde{u}_x^{(3)}(x_m^*, z_m, t) dz_m - \int_{-\beta_-}^0 \tilde{u}_x^{(3)}(-x_m^*, z_m, t) dz_m + \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(3)}(x_m, 0, t) dx_m = 0.$$

514 The first two integrals in (4.49) involve the behavior of  $\tilde{u}_x^{(3)}$  which, for  $x_m^* \rightarrow +\infty$ ,  
 515 is given by (4.11). It involves several terms, namely  $\partial_{x^p} u_x^{(m)}$ ,  $p + m = 3$  which  
 516 after integration with respect to  $z_m$  provides  $\beta_{\pm} \partial_{x^p} U_x^{(m)}(0^{\pm}, t)$  (see (3.15)). These  
 517 contributions are determined as follows.

518 From (3.19) and (3.21), we have  $\beta_{\pm} \partial_x U_x^{(2)}(0^{\pm}, z_m, t) = \partial_{tt}\phi^{(2)}(0^{\pm}, t)$ . Then, from  
 519 (3.11) and (3.12), we have  $\beta_{\pm} \partial_{xx} U_x^{(1)}(0^{\pm}, t) = \partial_{tt}q^{(1)}(0, t)/\beta_{\pm}$  and  $\beta_{\pm} \partial_{xxx} U_x^{(0)}(0^{\pm}, t) =$   
 520  $-1/\beta_{\pm} \partial_{ttt}\eta^{(0)}(0, t)$ . Consequently, noting  $J(\pm x_m^*, t) = \int_{-\beta_{\pm}}^0 \tilde{u}_x^{(3)}(\pm x_m^*, z_m, t) dz_m$ , we  
 521 have  
 (4.50)

$$522 \quad J(\pm x_m^*, t) \underset{x_m^* \rightarrow +\infty}{\sim} q^{(3)}(0^{\pm}, t) \pm x_m^* \frac{\partial^2 \phi^{(2)}}{\partial t^2}(0^{\pm}, t) + \frac{x_m^{*2}}{2\beta_{\pm}} \frac{\partial^2 q^{(1)}}{\partial t^2}(0, t) \mp \frac{x_m^{*3}}{6\beta_{\pm}} \frac{\partial^3 \eta^{(0)}}{\partial t^3}(0, t).$$

523 For the third integral in (4.49), we use (4.9) and take into account the following rela-  
 524 tions. By integrating  ${}^m_0(\text{Inc})^2$  in (4.6) with respect to  $z_{\mu}$  we have  $\tilde{v}_z^{(2)}(x_m, z_{\mu}, t) =$   
 525  $-\alpha\eta^{(0)}(0, t) \partial_{x_m} \tilde{u}_x^{(0)}(x_m, 0, t) z_{\mu} + \tilde{v}_z^{(2)}(x_m, 0, t)$  with  $\tilde{v}_z^{(2)}(x_m, 0, t) = -\partial_{tt}\tilde{\varphi}^{(2)}(x_m, 0, t)$ ,  
 526 from  ${}^m_0(\text{KC})^2$  and  ${}^m_0(\text{DC})^2$  in (4.7), as well as  $\tilde{\psi}^{(2)}(x_m, z_{\mu}, t) = \tilde{\varphi}^{(2)}(x_m, 0, t)$  from (4.8)  
 527 ( $\tilde{\psi}^{(2)}$  being independent of  $z_{\mu}$  from  ${}^m_0(\text{Rot})_z^2$  in (4.6)). Then, we have  $\tilde{\eta}^{(0)}(t) \partial_{z_m} \tilde{u}_z^{(0)} =$   
 528  $-\eta^{(0)}(0, t) \partial_{x_m} \tilde{u}_x^{(0)}$  from (4.16) and  ${}^m_0(\text{Inc})^0$  in (4.5). It follows that

$$529 \quad \tilde{u}_z^{(3)}(x_m, 0, t) = -\frac{\partial^2 \tilde{\varphi}^{(2)}}{\partial t^2}(x_m, 0, t) + \alpha\eta^{(0)}(0, t) \frac{\partial \tilde{u}_x^{(0)}}{\partial x_m}(x_m, 0, t).$$

530 Now we just need to use the forms of  $\tilde{u}_x^{(0)}$  in (4.23) and of  $\tilde{\varphi}^{(2)}$  in (4.32) (as well as

531 (4.25) and (4.34)) to find that, denoting  $I(t) = \int_{-x_m^*}^{x_m^*} \tilde{u}_z^{(3)}(x_m, 0, t) dx_m$ , we have  
 (4.51)

$$532 \quad I(t) \underset{x_m^* \rightarrow +\infty}{\sim} -2x_m^* \frac{\partial^2 \overline{\phi^{(2)}}}{\partial t^2}(t) - \frac{\partial^2 \overline{q^1}}{\partial t^2}(t) \left( \mathcal{C}_1 + \frac{x_m^{*2}}{2\beta_+} - \frac{x_m^{*2}}{2\beta_-} \right) \\ - \frac{\partial^3 \eta^{(0)}}{\partial t^3}(0, t) \left( \mathcal{C}_2 - \frac{x_m^{*3}}{6\beta_+} - \frac{x_m^{*3}}{6\beta_-} \right) + \alpha \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) \eta^{(0)}(0, t) q^{(0)}(0, t),$$

533 where the effective parameter  $\mathcal{C}_2$  is defined as follows

$$534 \quad (4.52) \quad \mathcal{C}_2 = \int_{-\infty}^{+\infty} Q_2^{\text{ev}}(x_m, 0) dx_m,$$

535 with  $Q_2^{\text{ev}}$  given by (4.33) and (4.34). By gathering (4.50) and (4.51) in (4.49), we  
 536 obtain that all the diverging terms, linear, quadratic and quartic with respect to  $x_m^*$ ,  
 537 cancel out, and we get the jump (4.44) in the flux.

538 **5. Construction of a unique model.** In this section, we reconstruct the  
 539 Boussinesq equations and corresponding jump conditions at the depth transition from  
 540 the results of the asymptotic analysis. Classically, this means combining the equa-  
 541 tions obtained at orders  $n = 0$  to 3 into a single equation. We begin by reconstructing  
 542 the Boussinesq equations, which is a classical procedure, before repeating the same  
 543 procedure for the jump conditions.

544 **5.1. The Boussinesq equations.** Gathering the results from (3.11), (3.18) and  
 545 (3.24), we have for  $\beta = \beta_{\pm}$ ,

$$546 \quad (5.1) \quad \begin{cases} \frac{\partial U_x^{(n)}}{\partial t} + \frac{\partial \eta^{(n)}}{\partial x} = 0, & n = 0, 1, \\ \frac{\partial U_x^{(2)}}{\partial t} + \frac{\partial \eta^{(2)}}{\partial x} - \frac{\beta^2}{3} \frac{\partial^3 U_x^{(0)}}{\partial t \partial x^2} = 0, \\ \frac{\partial U_x^{(3)}}{\partial t} + \frac{\partial \eta^{(3)}}{\partial x} - \frac{\beta^2}{3} \frac{\partial^3 U_x^{(1)}}{\partial t \partial x^2} + \alpha U_x^{(0)} \frac{\partial U_x^{(0)}}{\partial x} = 0, \end{cases}$$

547 and gathering the results (3.12), (3.19) and (3.25), we also have

$$548 \quad (5.2) \quad \begin{cases} \beta \frac{\partial U_x^{(n)}}{\partial x} + \frac{\partial \eta^{(n)}}{\partial t} = 0, & n = 0, 1, 2 \\ \beta \frac{\partial U_x^{(3)}}{\partial x} + \frac{\partial \eta^{(3)}}{\partial t} + \alpha \frac{\partial}{\partial x} (\eta^{(0)} U_x^{(0)}) = 0. \end{cases}$$

549 Considering the truncated expansions of the fields,  $U_x = \sum_{n=0}^3 \delta^n U_x^{(n)}$  and  $\eta =$   
 550  $\delta \sum_{n=0}^3 \delta^n \eta^{(n)}$  (up to terms  $O(\delta^4)$ ) and summing the results from order  $n = 0$  to  
 551 3 in (5.1)-(5.2) after multiplication by  $\delta^n$  we obtain

$$552 \quad \begin{cases} \frac{\partial}{\partial t} \left( \delta U_x - \frac{\beta^2 \delta^3}{3} \frac{\partial^2 U_x}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( \eta + \alpha \delta^4 \frac{(U_x)^2}{2} \right) = 0, \\ \frac{1}{\delta} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} ((\beta + \alpha \delta^2 \eta) U_x) = 0. \end{cases}$$

553 We then simply return to the dimensional form of the problem, using (2.10) and  
 554 scalings (2.8)-(2.9) to obtain the classical Boussinesq equations (2.5). Note that a

555 more general family of Boussinesq equations could have been obtained at this stage  
 556 by using Boussinesq's trick, the idea being that lower-order relations such as (3.11)-  
 557 (3.12) can be used to modify the higher-order equations without loss of the asymptotic  
 558 order of approximation, see *e.g.* [6, 7].

559 **5.2. The jump conditions.** Gathering the results of (4.13), (4.20), (4.30) and  
 560 (4.43) we have

$$561 \quad (5.3) \quad \left\{ \begin{array}{l} \llbracket \eta^{(0)} \rrbracket = 0, \\ \llbracket \eta^{(1)} \rrbracket = -\mathcal{B}_1 \frac{\partial q^{(0)}}{\partial t}(0, t) \\ \llbracket \eta^{(2)} \rrbracket = -\mathcal{B}_1 \frac{\partial q^{(1)}}{\partial t}(0, t) - \mathcal{B}_2 \frac{\partial^2 \eta^{(0)}}{\partial t^2}(0, t), \\ \llbracket \eta^{(3)} + \frac{\alpha}{2} (U_x^{(0)})^2 \rrbracket = -\mathcal{B}_1 \frac{\partial \overline{q^{(2)}}}{\partial t}(t) - \mathcal{B}_2 \frac{\partial^2 \overline{\eta^{(1)}}}{\partial t^2}(t) - \mathcal{B}_3 \frac{\partial^3 q^{(0)}}{\partial t^3}(0, t), \end{array} \right.$$

562 and gathering the results of (4.14), (4.21), (4.31) and (4.44), we also have

$$563 \quad (5.4) \quad \left\{ \begin{array}{l} \llbracket q^{(n)} \rrbracket = 0, \quad n = 0, 1, \\ \llbracket q^{(2)} \rrbracket = \mathcal{C}_1 \frac{\partial^2 q^{(0)}}{\partial t^2}(0, t), \\ \llbracket q^{(3)} + \alpha \eta^{(0)} U_x^{(0)} \rrbracket = \mathcal{C}_1 \frac{\partial^2 q^{(1)}}{\partial t^2}(0, t) + \mathcal{C}_2 \frac{\partial^3 \eta^{(0)}}{\partial t^3}(0, t). \end{array} \right.$$

564 Considering again the truncated expansions of the fields  $U_x = \sum_{n=0}^3 \delta^n U_x^{(n)}$  (with  $q^{(n)}$   
 565 defined in (3.15)) and  $\eta = \delta \sum_{n=0}^3 \delta^n \eta^{(n)}$  and summing the results at order  $n = 0$  to  
 566 3 given by (5.3)-(5.4) after multiplication by  $\delta^n$  give

$$567 \quad \left\{ \begin{array}{l} \llbracket \eta + \alpha \delta^4 \frac{(U_x)^2}{2} \rrbracket = -\mathcal{B}_1 \delta^2 \frac{\partial \overline{\beta U_x}}{\partial t} - \mathcal{B}_2 \delta^2 \frac{\partial^2 \overline{\eta}}{\partial t^2} - \mathcal{B}_3 \delta^4 \frac{\partial^3 \overline{\beta U_x}}{\partial t^3}, \\ \llbracket (\beta + \alpha \delta^2 \eta) U_x \rrbracket = \mathcal{C}_1 \delta^2 \frac{\partial^2 \overline{\beta U_x}}{\partial t^2} + \mathcal{C}_2 \delta^2 \frac{\partial^3 \overline{\eta}}{\partial t^3}. \end{array} \right.$$

568 Once again, we only need to return to the dimensional form of the problem using  
 569 (2.10) and the scalings (2.8)-(2.9) to obtain the jump conditions announced in (2.6).

570 **6. Conclusion.** In this work, we have sought to challenge, in the presence of an  
 571 abrupt depth transition, the classical asymptotic approach used to obtain the Boussi-  
 572 nesq equations. Indeed, the classical procedure for a flat or slowly varying sea bottom  
 573 relies on normalizing the vertical coordinate by the total water depth (including the  
 574 wave amplitude at the free surface), which obviously cannot be used here since the  
 575 water depth is a discontinuous function. To overcome this problem, we introduced a  
 576 new three-scale analysis framework that enables consistent navigation using matching  
 577 conditions between a macroscopic scale (the incident wavelength), a mesoscopic scale  
 578 (the water depth at rest or the boundary layer thickness near the discontinuity), and  
 579 a microscopic scale (the wave amplitude or the boundary layer thickness at the free  
 580 surface). The asymptotic expansion of the solutions up to the third-order in this  
 581 three-scale framework led to the Boussinesq equations, supplemented by jump con-  
 582 ditions at the depth discontinuity. These jump conditions involve 5 boundary layer

583 coefficients that depend only on the water depth ratio and for which the (semi)-  
 584 analytical formula has been provided.

585

586 Some extensions of the present study are straightforward, others less so. On the  
 587 one hand, the extension of the present analysis to variants of the classical Boussinesq  
 588 equations [29, 6, 7] is almost immediate, by adapting the procedure for construct-  
 589 ing a single model in section 5, including the jump conditions. Also, the derivation  
 590 of the jump conditions in three-dimensional geometry or for a more general abrupt  
 591 discontinuity profile present no additional difficulties, since the same asymptotic pro-  
 592 cedure can be carried out. On the other hand, the theoretical and numerical analysis  
 593 of the coupled model “Boussinesq and jump conditions” is a more tedious challenge.  
 594 It would be interesting, for example, to find analytical solutions to Boussinesq-type  
 595 equations (e.g. solitary waves) in the presence of an abrupt depth transition using  
 596 these jump conditions. We may think to scattering problems, of solitons by adapting  
 597 the present model to a Korteweg-de Vries-like form [15, 31, 32], or of shock waves in  
 598 shallow water [17], and to assess the effect of the boundary layers which has not been  
 599 considered thus far. It also remains to establish appropriate numerical discretization  
 600 schemes for these jump conditions. This is a necessary step before considering valida-  
 601 tion of the jump conditions by comparison with direct numerical simulations or with  
 602 experimental data.

603 **Appendix A. Effective parameters entering the jump conditions.**

604 **A.1. Variations of effective parameters as a function of depth ratio.**

605 The effective parameters entering the jump conditions (2.6) depend only on the depth  
 606 ratio  $\nu = 1 + h_s/h_0$ . They can be calculated by numerically solving the elementary  
 607 problems (4.24), (4.33) and (4.46) and using their definitions (4.26), (4.35), (4.48)  
 608 and (4.40), (4.52). We give below approximate expressions that are good estimates,  
 609 as shown in figure 6.

$$610 \quad (A.1) \quad \begin{cases} \mathcal{B}_1 = \frac{1}{\pi} \left( \frac{\nu^2+1}{\nu} \log \frac{\nu+1}{|\nu-1|} - 2 \log \frac{4\nu}{|\nu^2-1|} \right), \\ \mathcal{B}_2 \simeq -0.2\nu(1-\nu), \\ \mathcal{B}_3 \simeq 0.1(1-\nu)\mathcal{B}_1, \end{cases}$$

611 and

$$612 \quad (A.2) \quad \mathcal{C}_1 = \mathcal{B}_2 + \frac{1}{3}(1-\nu), \quad \mathcal{C}_2 \simeq -0.044\nu^2(1-\nu)^2.$$

613 Note that the expression for  $\mathcal{B}_1$  is exact and is due to [55], Eq. (2.12) in this reference,  
 614 and the relation between  $\mathcal{C}_1$  and  $\mathcal{B}_2$  is proved in the next section.

615 **A.2. Relations between the parameters.** The effective parameters  $\mathcal{B}_2$  and  
 616  $\mathcal{B}_3$  defined in (4.35) and (4.48) can be determined from knowledge of  $Q_1$  only, which  
 617 is given by (4.24). Indeed, for a step, we have the following relations

$$618 \quad (A.3) \quad \mathcal{B}_2 = \mathcal{C}_1 + \frac{1}{3}(\beta_+ - \beta_-), \quad \mathcal{B}_3 = \mathcal{G}_1 - \frac{\mathcal{B}_1}{6}(\beta_+ + \beta_-),$$

619 with  $(\mathcal{B}_1, \mathcal{C}_1)$  given by (4.26) and (4.52) and  $\mathcal{G}_1$  defined by

$$620 \quad \mathcal{G}_1 = - \int_0^{+\infty} Q_+(x_m) dx_m - \int_{-\infty}^0 Q_-(x_m)^2 dx_m.$$

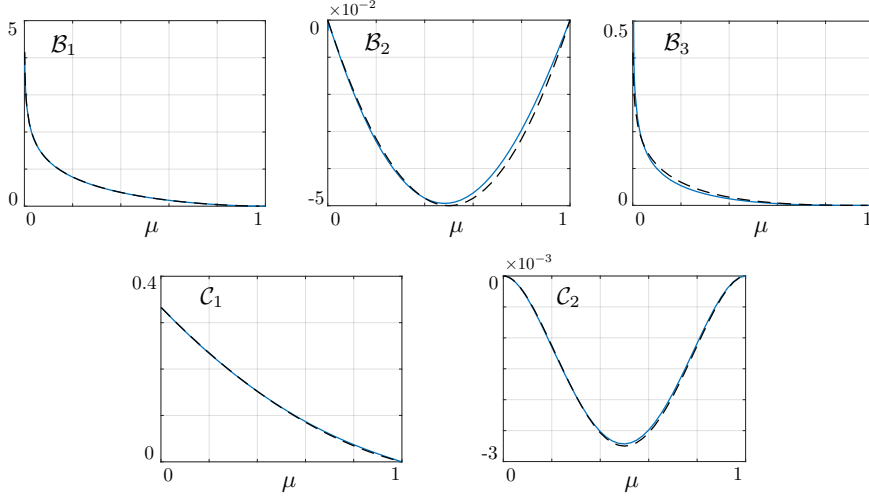


FIG. 6. Variations of effective parameters,  $\mathcal{B}_n$ ,  $n = 1, 2, 3$  and  $\mathcal{C}_n$ ,  $n = 1, 2$  entering jump conditions (2.6), as a function of  $\nu = 1 - h_s/h_0$ . Solid blue lines from direct numerical calculations of elementary solutions  $Q_n$  in (4.24), (4.33) and (4.46); dashed black lines from (A.1) and (A.2).

621 with

$$622 \quad Q_{\pm}(x_m) = (Q_1(x_m, 0))^2 - \left( \frac{x_m \pm \mathcal{B}_1}{\beta_{\pm}} \right)^2$$

623 To prove (A.3), simply note that using (4.24), (4.33) and (4.46) for  $i = 2, 3$  we have

$$624 \quad 0 = \int_{\Omega_m} (Q_1 \Delta Q_i - Q_i \Delta Q_1) = - \int_{\partial \Omega_m} (Q_1 \nabla Q_i - Q_i \nabla Q_1) \cdot \mathbf{n}.$$

625 Applying the boundary conditions on  $(Q_1, Q_2, Q_3)$  and using the decomposition (4.25),  
626 (4.34) and (4.47) we obtain after a few calculations

$$627 \quad 0 = \int_{\partial \Omega_m} (Q_1 \nabla Q_2 - Q_2 \nabla Q_1) \cdot \mathbf{n} = \mathcal{C}_1 + \frac{1}{3}(\beta_+ - \beta_-) - \mathcal{B}_2,$$

$$0 = \int_{\partial \Omega_m} (Q_1 \nabla Q_3 - Q_3 \nabla Q_1) \cdot \mathbf{n} = \mathcal{G}_1 - \frac{\mathcal{B}_1}{6}(\beta_+ + \beta_-) - \mathcal{B}_3.$$

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