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Review on water-hammer waves mechanical and theoretical foundations

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Abstract

Water-hammer waves propagation is an important phenomenon arising in numerous applications. It is also a long-standing topic in the fields of mechanics, mechanical engineering and civil engineering. This review first presents the basic mechanism associated with water-hammer waves as well as a brief historical survey of the topic. It then develops along the twentieth century progress both regarding the Fluid-Structure-Interaction (FSI) influence and wave dissipation modeling. The second part of the review presents recent developments shading new lights on some aspects of the wave propagation with a fluid mechanical viewpoint. This review covers various aspects related to the influence of visco-elastic properties of the pipe's wall, asymptotic analysis as well as wave propagation within networks. Albeit discursive in many places, this review also tries to establish and derive many of the presented results from first principles, as well as emphasize the theoretical understanding of the topic.

Keywords: Water-hammer, Fluid–Structure–Interactions, acoustic waves in pipes, multiple time–scale analysis, asymptotic matching, dispersive waves, visco-elasticity, water hammer, blood hammer, Lamé–Clapeyron equations, wave propagation in networks, viscous dissipation, quantum graph

1. Introduction

1.1. Context of water-hammer wave

Water distribution networks, vascular system, irrigation networks, power-plant hydraulic circuits, geothermal extracting heat pipes, etc., are all complex hydraulic systems dedicated to flow distribution for their specific purpose. Within all these applications an incoming flow is needed to feed the needs. The incoming flow could be time-dependent so as to accommodate the possibly varying needs at various places, but the time-variation of the needs is not short compared to flow adaptation one. In other words, it is possible, in these systems to almost instantaneously adapt supply to demand. These systems are quasi-steady ones in normal use.

Nevertheless, if, for some reasons, either expected or not, a sudden change arises in the supply flow (whether due to pipe breakage, operational valve opening, heart valve closure, gate operation, incidental junction damage, etc..) then, a fast transient pressure wave is generated from it (Cf figure 1 for a real water-hammer wave signature inside a water distribution network). Since liquid incompressibility imposes flow conservation, any adjustment of a sudden flow change can only be due to compressible effects. Because liquids are much less compressible than gases, these sudden changes in fluid flow are much faster and much more intense in liquids than in gases even though also existing in the later. The less compressible the liquid, the faster the pressure wave having the highest amplitude. And since most liquids are very close to incompressible, the resulting pressure wave is very large in many cases, dramatic in some. This detrimental effect of water-hammer pressure waves has attracted interests in many applied areas in mechanical engineering and civil engineering [2, 3]. This is also why the first known scientists of the topic such as Résal [4], Michaud, [5], Korteweg [6], Joukowski [7], belonged to the hydraulic's community. But this is only one part of the water-hammer story. Next section will provide much more details about the various modelling and analysis of it, but, since we believe the reader is, at this point, juvenile on the topic, we will pursue this discursive introduction by

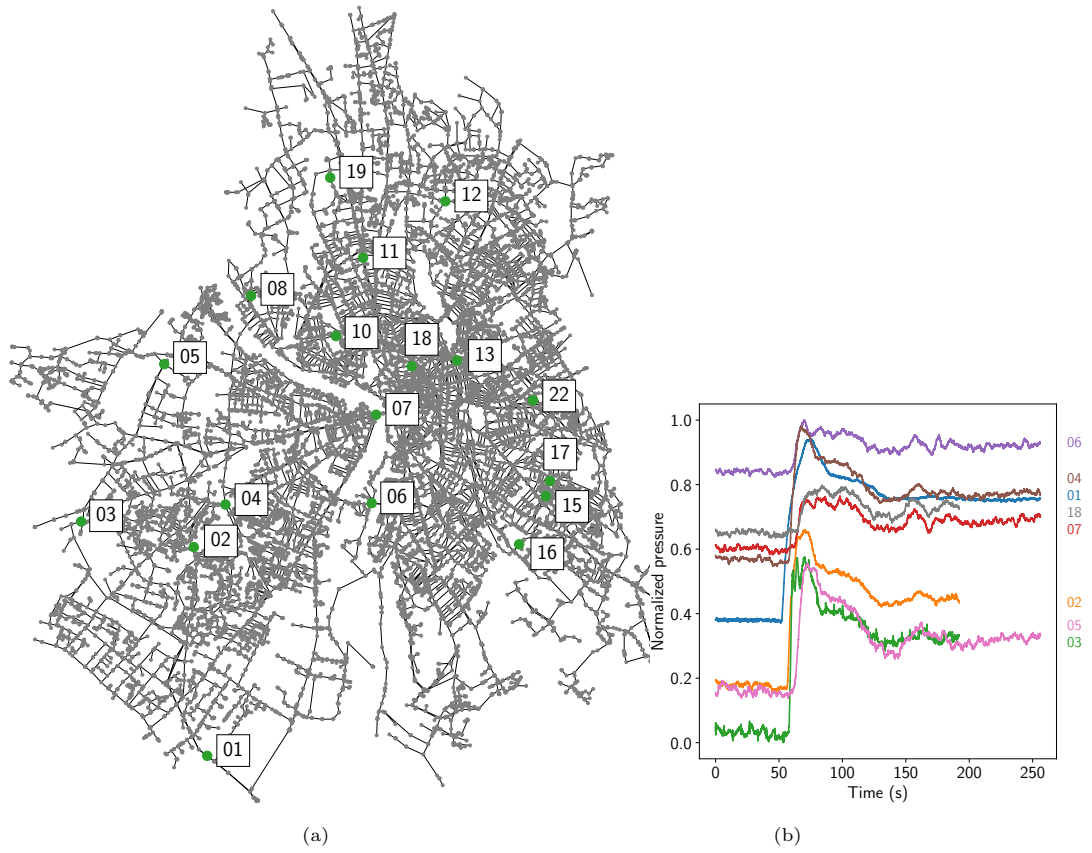


Figure 1: (a) Illustration of a the water distribution network of Toulouse's town from [1] where pressure sensors have been distributed. (b) Water-hammer wave front passage signature at sensors placed in various locations.

31 some basic issues to introduce a second important aspect. As just mentioned,
 32 not only the pressure amplitude increases from the low compressibility of liq-
 33 uids. Also the wave's speed. In every fluid the isentropic compression law of
 34 fluid density variations ρ_f^* with fluid pressure P_f^* is given by $\partial\rho_f^*/\partial P_f^* = \rho_f^*/\mathcal{K}_f^*$.
 35 In the limit of infinite liquid bulk compressibility coefficient, i.e $\mathcal{K}_f^* \rightarrow \infty$, the
 36 incompressible limit, the water-hammer wave speeds goes to infinity. Never-
 37 theless, this velocity cannot be infinite even for almost incompressible liquids.
 38 It is not necessary to invoke light's finite speed to avoid this inextinguishable
 39 sky, but another more down-to-earth fact that this pressure wave speed does
 40 not propagate freely into the liquid, being, on the contrary, confined within
 41 solid walls. Since the tube wall can expand —think of a blood vessel—, the
 42 pipe breathing resulting from the liquid over-pressure has a strong influence
 43 on the wave velocity propagation. Stress waves propagating within solids are
 44 known from a long-time to result from its elastic deformation. For pressure
 45 waves associated with compressive density waves inside one material, the speed
 46 is proportional to the elastic Young modulus. The harder the solid, the faster
 47 its compressive wave. Split-Hopkinson pressure tests are indeed using this ef-
 48 fect to deduce the material property from the wave-speed measurement [8, 9].
 49 Going back to the water-hammer pressure wave, since it is confined within solid
 50 walls, it can not move freely in the transverse direction of its propagation. The
 51 wave's pressure is indeed deforming the solid walls. This results in a coupling
 52 between the liquid pressure wave, and the solid one. This is the second key
 53 aspect of the water-hammer waves : they result from a Fluid-Structure Inter-
 54 action (FSI) between the acoustic wave propagating within the fluid bulk and
 55 the elastic deformation of the solid surrounding the liquid. Furthermore, and
 56 not intuitively, FSI produces a double effect on water-hammer waves: first, as
 57 already mentioned, the propagating speed of the liquid pressure wave both de-
 58 pends on the liquid isentropic bulk modulus, i.e the compressibility coefficient
 59 of the fluid and the Young modulus of the solid. More precisely, on the ratio of
 60 both. Second, this water-hammer wave train hides another : a solid compressive
 61 elastic wave. Because it is faster in most cases, this secondary wave comes first,

62 and has been called the precursive wave [10, 11]. In some applications such
63 as water distribution networks, and many other hydraulic contexts, this elastic
64 wave is disregarded for being of small influence. This generally results from
65 anchored mechanical conditions preventing the free-motion of the external solid
66 wall (cemented, buried, or fit-in) weakening the elastic wave contribution. This
67 is why most hydraulic modeling of water-hammer do not consider the coupled
68 elastic wave. However, this coupling is very important when the solid walls are
69 soft (e.g vascular system), when the conducts are free to move (e.g suspended),
70 or when the solid walls are very thin.

71 Having now set the scene of the two main mechanical effects rooting water-
72 hammer, i.e liquid compressibility and FSI, it is now appropriate to provide
73 an historical survey of how modern understanding has emerged. Nevertheless
74 before this, it is important to mention that various interesting reviews have pre-
75 viously discussed water-hammer [12, 13, 14, 15, 16, 17, 18]. Among those [12, 15]
76 have focused their review on FSI aspects in simple pipes configurations. [14]
77 has more focussed its interest into the historical perspective of the topic. On
78 the other hand [13, 16] discuss hydraulic applications where FSI effects are gen-
79 erally neglected, whilst concentrating on water distribution networks purposes.
80 [18]'s review consider more specifically blood-hammer within the arterial sys-
81 tem whereby issues related to the visco-elastic response of arteries are deepened.
82 Finally, it is important to stress what material is covered by this review and
83 the about-face. This review focuses on water-hammer wave propagation sur-
84 rounded by possibly complex solids including visco-elastic ones. This topic is
85 of major practical relevance, since water-hammer waves are strongly dependent
86 on the solid visco-elastic response, because these solids are encountered in many
87 contexts. Concerning the fluid, this review considers water-hammer waves aris-
88 ing inside Newtonian compressive liquids, including viscous effects in the fluid.
89 Nevertheless this review will not cover several aspects of water-hammer : cav-
90 itation [19], water-hammer in non-Newtonian liquids (e.g [20]), condensation-
91 induced water-hammer (e.g [21]), thermally coupled water-hammer waves (e.g
92 [22]), arc-discharge induced water hammer (e.g. [23]). Also, this review is not

93 going to discuss transient-based defect detection and water-hammer use for in-
94 verse problems which have been vivid areas of investigations over the last fifteen
95 years, but deserve a review on their own. However it covers (some aspects) of
96 water-hammer wave propagation in complex pipe networks. The specificity of
97 this review, is to focus on water-hammer’s mechanics, so as to emphasize how
98 modern understanding can help in modeling simplification, concepts generality,
99 and also, questioning and/or revisiting practical use. This review does not have
100 the ambition to be exhaustive in every aspect, reflecting that the author’s best
101 knowledge of the topic is unfortunately neither objective nor complete. It also
102 reflects the author’s subjective interests on the topic, oriented toward future
103 research directions dedicated toward tackling and simplifying various aspects of
104 water-hammer complexity.

105 This review is organized as follows. Section (1.2) provides a brief historical
106 description of the first understanding and modelling attempts of water-hammer.

107 Section (2) then provides a summary of the twentieth’s century progress on
108 the topic. Albeit discursive, this section also tries to establish many of the
109 presented results from first principles.

110 The two following sections are devoted to the review of more modern issues.
111 Section 3 is mostly concerned about modeling the influence of pipe’s visco-elastic
112 behavior onto the water hammer wave.

113 Section (4) develops on the twenty first century advances in the field. It
114 covers both time and spatial asymptotic analysis, giving new perspective to the
115 underlying understanding of yet unsolved water-hammer’s specific aspects. Fur-
116 thermore section (4) also covers recent modern issues associated with theoretical
117 developments for water-hammer wave propagation within networks.

118 *1.2. A brief historical survey of first water-hammer understanding*

119 As previously mentioned the water-hammer research started from engineer-
120 ing concerns and viewpoints, at the end of the nineteenth century. The first
121 noteworthy contribution can be credited to L. Menabrea [24] reporting the joint
122 influence of fluid bulk modulus \mathcal{K}_f^* , and the solid wall Young’s modulus E^*

123 in water-hammer waves. Thereafter Michaud [5] developed some protection de-
 124 vices guidelines for hydraulic systems. He focused his interest on plans for safety
 125 valves and water-hammer balloons but meanwhile, derives one of the first ex-
 126 pressions for the propagation speed of water hammer waves, c_p^* (in the following
 127 all p subscript on all subsequent c_p^* velocities refers to "pulsed" wave speeds, not
 128 to be confused with possibly pipe's one; this pulse wave speed is a bulk liquid
 129 pressure wave), the same year as Korteweg [6]. Considering the solid wall as a
 130 successive axially independent concentric elastic rings, these authors found that

$$c_{p,K}^* = \frac{c_0^*}{\sqrt{1 + \frac{2\mathcal{K}_f^*}{\alpha E^*}}}, \text{ with } c_0^{*2} = \frac{\mathcal{K}_f^*}{\rho_f^*}, \quad (1)$$

131 c_0^* being the sound speed in the liquid bulk, $\alpha = e^*/R_0^*$ the dimensionless pipe
 132 wall thickness associated with solid tickness e^* having inner radius R_0^* and ρ_f^*
 133 the fluid density (*'s quantities are dimensional in the following, whilst on the
 134 contrary, dimensionless quantities do not have *). The approximate theory (1)
 135 brings out for the first time the cornerstone contribution of the liquid's bulk
 136 modulus to Young modulus ratio, i.e. \mathcal{K}_f^*/E^* , into water-hammer wave c_p^* .
 137 Since (1)'s denominator is larger than unity, the water-hammer wave speed
 138 is lower than the sound speed in bulk liquid c_0^* , as expected from the energy
 139 transfer of the wave into solid elastic deformation.

It is interesting to discuss the comparison of Korteweg and Michaud predic-
 tion with their contemporary Résal [4] who proposed a simpler expression for
 this velocity being based upon the wall's elastic wave speed $\sqrt{\alpha E^*/2\rho_f^*}$. As
 discussed in [19, 25] reorganizing (1) leads to

$$\frac{1}{c_{p,K}^{*2}} = \frac{1}{c_0^{*2}} + \frac{2\rho_f^*}{\alpha E^*}. \quad (2)$$

Result (2) can be interpreted as equivalent resistance of parallel electrical cir-
 cuits or in a more mechanical frame, the equivalent stiffness of two successive
 springs. Korteweg's [6] wave velocity is the equivalent wave speed of the liquid
 bulk sound speed c_0^* , in parallel with the elastic wave speed $\sqrt{\alpha E^*/2\rho_f^*}$ (this
 point is discussed in more details in [25]). As mentioned in the introduction not

only the wave speed, but also the pressure amplitude is of major engineering importance. Few years after Korteweg, Joukowski [7] focused on the mass equilibrium occurring during a transient event, so as to find the maximal reachable over pressure ΔP^* , which is related to the longitudinal velocity variation ΔW^* associated with the flow sudden change $\Delta Q^* = S^* \Delta W^*$ (in the following W_0^* will denote the steady-state reference velocity and ΔW^* will be considered as an order-one fraction of W_0^*)

$$\Delta P^* = \rho_f^* c_p^* \Delta W^*. \quad (3)$$

This fundamental relationship permits to improve the security design efficiency of water plants. The derivation of (3) is now discussed. Let us consider the mass

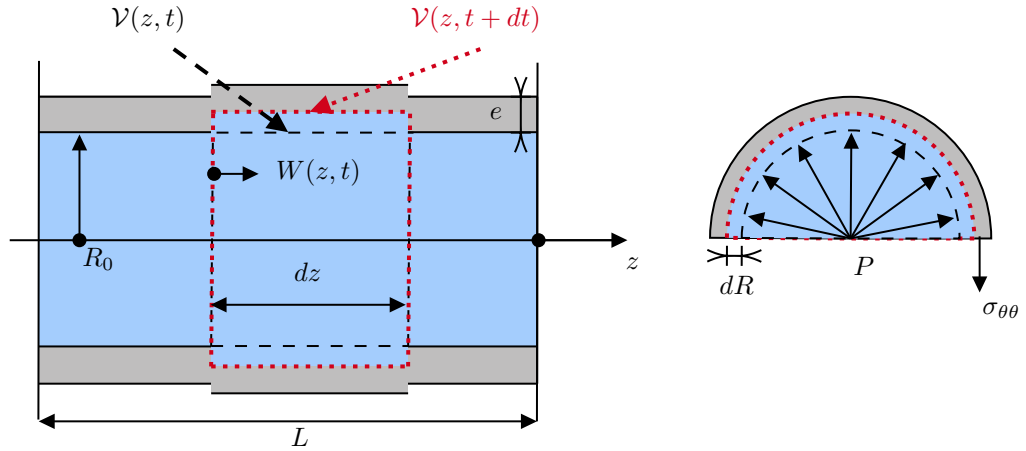


Figure 2: Fluid mass conservation during a hydraulic transient

balance in the elementary volume \mathcal{V}^* , having external surface S^* , as defined in Figure 2

$$\partial_{t^*} (\rho_f^* S^*) + \partial_{z^*} (\rho_f^* S^* W^*) = 0, \quad (4)$$

Introducing Lagrangian derivative $d/dt^* \equiv \partial_{t^*} + W^* \partial_{z^*}$, (4) can be decomposed into three terms

$$\underbrace{\frac{1}{\rho_f^*} \frac{d\rho_f^*}{dt^*}}_{\text{Liquid compression}} + \underbrace{\frac{1}{S^*} \frac{dS^*}{dt^*}}_{\text{Pipe dilatation}} + \underbrace{\partial_{z^*} W^*}_{\text{Velocity gradient}} = 0. \quad (5)$$

Using the previously mentioned isentropic compression law, the fluid density gradient are related to pressure ones

$$\frac{1}{\rho_f^*} \frac{d\rho_f^*}{dt^*} = \frac{1}{\mathcal{K}_f^*} \frac{dP^*}{dt^*}. \quad (6)$$

The pipe dilatation term of (5) requires a little more inspection. It is related to what is called the *pipe breathing phenomenon*, i.e the deformation arising when an axi-symmetric pressure wave propagates into a cylindrical solid, deforming it radially and longitudinally whilst preserving the axi-symmetry. Considering the instantaneous elastic solid response of the azimuthal normal stress, i.e the hoop stress $\sigma_{\theta\theta}^*$, integrated along one pipe's mid-plane section having normal direction \mathbf{e}_θ and longitudinal length L^* , provides a contribution into the solid region only, whose thickness is e^* , so that a an elastic force $2L^*e^*\sigma_{\theta\theta}^*$ is built to balance the fluid pressure P^* integrated along the same pipe's mid-plane section $2R_0^*L^*P^*$, so that

$$\sigma_{\theta\theta}^* = \frac{R_0^*}{e^*} P^* = \frac{1}{\alpha} P^*. \quad (7)$$

Furthermore, from elasticity theory, the relative strain of the pipe is proportional to its stress such that

$$d\sigma_{\theta\theta}^* = E^* \frac{dR^*}{R^*}. \quad (8)$$

From (7) in (8) and since, $S^* = \pi R^{*2}$, one finds

$$\frac{1}{S^*} \frac{dS^*}{dt^*} = \frac{2}{\alpha E^*} \frac{dP^*}{dt^*}. \quad (9)$$

Finally, combining (6), (9) and (5), yields to the following mass conservation equation

$$\frac{1}{\rho_f^*} \frac{dP^*}{dt^*} + \frac{c_0^{*2}}{1 + \frac{2\mathcal{K}_f^*}{\alpha E^*}} \partial_{z^*} W^* = \frac{1}{\rho_f^*} \frac{dP^*}{dt^*} + c_{p,K}^{*2} \partial_{z^*} W^* = 0, \quad (10)$$

140 consistently with (1) found by [6]. It is also important to stress that (10) leads
141 to the Joukowski's scaling (3) for the pressure [7].

142 Although sufficient for many engineering applications, these theoretical out-
143 lines remain too limited in scope as they do not provide an in-depth compre-
144 hensive understanding of hydraulic transients. The relation (8) highlights the

145 weaknesses of these early models. The solid rheology is highly simplified which
146 results in a degraded representation of shear and strain distribution in the pipe
147 (which however might turn-out to be enough for some engineering issues). Fur-
148 thermore, the fluid is supposed inviscid and the long-time dynamics of the pres-
149 sure trend thereby remains unknown. This is why this simplified picture has
150 been dramatically improved during the twentieth's century.

151 **2. Water hammer : a dual phenomenon intertwined with three cou-** 152 **plings**

153 This section now presents the modern view of water-hammer progressively
154 emerging from 20th century developments. This view is very synthetically pre-
155 sented in [19, 25] resulting from the infusion of many previous studies as also
156 discussed in [26] from which this section is inspired. Numerous deep analy-
157 sis of great theoretical mechanics have progressively permit to understand that
158 various fundamental mechanisms were acting together, on the water-hammer
159 wave, in a subtle way. Moreover, and surprisingly enough, as the complexity
160 increased, the ability to synthetically produce analytical results also permit-
161 ted a more compact and clear presentation of the phenomenon to simplify its
162 theoretical understanding. This back-and-forth (hopefully creative) competi-
163 tion between tackling complexity and idealistic simplification is still present at
164 the fore-front of the phenomenon's understanding, as discussed in Section 4
165 References [19, 25] discuss how, resulting from both compressibility and FSI
166 effects, the water-hammer wave is influenced by by three, possibly intertwined,
167 couplings:

- 168 • *Poisson's coupling* which refers to solid axial vibrations arising from radial
169 ones. These couplings depend on the mechanical properties of the solid,
170 as well as on the possibly complex vibrating modes (flexural, torsional,
171 etc...). It is thus related to the possibly complex solid response to the
172 fluid pressure wave.

- 173 • *Junction couplings* which refers to pipe's ends couplings either resulting
 174 from connections with other pipes, or to boundary conditions imposed at
 175 pipe's end. This effect thus results in the boundary conditions influence
 176 on the propagating waves and selected modes.
- 177 • *Friction couplings* which refers to the shear stress couplings between the
 178 fluid and the solid, i.e. the viscous friction coupling, and between the solid
 179 and the surrounding medium, e.g. dry Coulomb's friction if the pipe is
 180 buried.

181 According to [27], Poisson and junction couplings are shaping the system's dy-
 182 namics, i.e. determine the selected wave's modes as well as their structure,
 183 whilst friction couplings induce wave's attenuation due to energetic losses. Fur-
 184 thermore, from a mechanical viewpoint, any pipe's degree of freedom is expected
 185 to interact with the fluid dynamics, generating non-trivial additional couplings,
 186 [27]. We now detail each coupling in the forthcoming sections.

187 2.1. Poisson coupling and Fluid Structure Interaction (FSI) analysis

188 FSI describes couplings occurring in liquid-filled pipe systems. Let us first
 189 consider an homogeneous, elastic and isotropic pipe having density ρ_s^* . Let
 190 us denote the radial displacement ξ^* and axial one ζ^* , being the two compo-
 191 nents of the displacement vector $\boldsymbol{\xi}^* = (\xi^*, \zeta^*)$ —the only relevant ones for axi-
 192 symmetric pipe breathing mode of deformation—, related to the strain tensor
 193 $\boldsymbol{\epsilon}^* = 1/2(\nabla\boldsymbol{\xi}^* + \nabla\boldsymbol{\xi}^{*T})$.

194 One of the first major contribution to pressure wave propagation studies
 195 in pipes taking into account (FSI) can be attributed to [28]. Lamb extends
 196 [6]'s work from taking into account the Poisson's coupling effect. Based on the
 197 second Newton's law for the solid equilibrium, and a radial dependent pressure

198 wave for the fluid, he found

$$\underbrace{\rho_s^* e^* \partial_{t^*}^2 \zeta^*}_{\text{Axial inertia}} = \underbrace{\frac{\alpha E^*}{1 - \nu_s^2} (\nu_s \partial_{z^*} \xi^* + R_0^* \partial_{z^*}^2 \zeta^*)}_{\text{Axial tension}}, \quad (11)$$

$$\underbrace{\rho_s^* e^* \partial_{t^*}^2 \zeta^*}_{\text{Radial inertia}} = - \underbrace{\frac{\alpha E^*}{1 - \nu_s^2} \left(\frac{\xi^*}{R_0^*} + \nu_s \partial_{z^*} \zeta^* \right)}_{\text{Radial tension}} + \underbrace{P^*}_{\text{Dynamic loading}}, \quad (12)$$

$$\partial_{t^*}^2 P = c_0^{*2} \left(\partial_{z^*}^2 + \frac{\partial_{r^*}}{r^*} (r^* \partial_{r^*}) \right) P^*. \quad (13)$$

Under plane-wave long-wavelength framework assumption, Sr. H. Lamb determines the radial pressure variations from radial Bessel function. Furthermore, ensuring the kinematic continuity conditions at the pipe's inner wall, he spells out a cubic (in c^{*2}) dispersion relation for the wave speeds

$$\frac{c^{*2} - c_0^{*2}}{c_0^{*2}} \underbrace{\left[c^{*4} - \left(1 + \frac{\lambda^{*2}}{4\pi^2 R_0^{*2}} \right) \frac{E^*}{\rho_s^* (1 - \nu_s^2)} c^{*2} + \frac{(1 - \nu_s^2) \lambda^{*2}}{4\pi^2 R_0^{*2}} \left(\frac{E^*}{\rho_s^* (1 - \nu_s^2)} \right)^2 \right]}_{\text{Dispersion eq. for } P = 0 \text{ in (12)}} - \frac{2\mathcal{D}}{\alpha} \frac{\lambda^{*2} c^{*2}}{4\pi^2 R_0^{*2}} \left(c^{*2} - \frac{E^*}{\rho_s^* (1 - \nu_s^2)} \right) = 0, \quad (14)$$

with the density ratio

$$\mathcal{D} = \frac{\rho_f^*}{\rho_s^*}, \quad (15)$$

λ^* being the wavelength. The cubic structure of the dispersion relation thus provides a set of three modes of propagation, being in increased order of frequency (decreasing order of wavelength) comparable to corrections upon the fluid wave speed, solid axial wave speed and solid radial wave speed, respectively. Under the long wavelength hypothesis framework, i.e. $\lambda^*/R_0^* \gg 1$, an important set of results can be found. If the dynamic loading term is neglected in (12), i.e. if P^* is set to zero, Lamb derives the compressible axial solid wave speed (Cf. terms in brackets in (14))

$$c_s^{*2} = \frac{E^*}{\rho_s^*}, \quad (\text{Hyp: } P^* = 0). \quad (16)$$

Furthermore, if the solid instantaneously responds to the fluid dynamic load, i.e. neglecting time derivatives in (11)-(12), Lamb proves that his theory leads

to [6]’s one (Cf. (1)) so that no proper (FSI) occurs then

$$c_{p,Lamb}^{*2} = c_{p,K}^{*2} \quad , \quad (\text{Hyp: } \partial_{t^*}^2 \xi^* = \mathbf{0}). \quad (17)$$

If one considers a highly deformable tube, i.e. $\mathcal{K}_f^* \gg E^*$, Lamb’s theory merges with the [4]’s one (Cf. (2))

$$c_{p,Lamb}^{*2} = \frac{\alpha E^*}{2\rho_f^*} \quad , \quad \text{and,} \quad c_{s,Lamb}^{*2} = \frac{E^*}{\rho_s^* (1 - \nu_s^2)} \quad , \quad (\text{Hyp: } \mathcal{K}_f^* \gg E^*). \quad (18)$$

199 Finally, Taylor-expanding the dispersion relation (14) with respect to the ra-
 200 dius per wavelength ratio, i.e. R_0^*/λ^* –long-wavelength asymptotic limit— H.
 201 Lamb finds an analytical formulation for the fluid and axial solid wave speed
 202 corrections due to FSI

$$c_{\pm,Lamb}^{*2} = \frac{1}{2} \left[\bar{c}_{Lamb}^{*2} \pm \sqrt{\bar{c}_{Lamb}^{*4} - \frac{4(c_0^* c_s^*)^2}{1 + \frac{2(1-\nu_s^2)\mathcal{K}_f^*}{\alpha E^*}}} \right], \quad (19)$$

$$\bar{c}_{Lamb}^{*2} = \frac{c_s^{*2} + c_0^{*2} \left(1 + \frac{2\mathcal{D}}{\alpha}\right)}{1 + \frac{2(1-\nu_s^2)\mathcal{K}_f^*}{\alpha E^*}}, \quad (20)$$

where the negative mode c_-^* holds for the fluid wave speed correction, whilst the positive mode c_+^* stands for the axial solid wave speed correction one. Half a century later [10, 11] extends the brilliant contribution of [28]. In a sterling paper, Skalak derives an in-depth analysis of the coupling mechanisms occurring between an elastic shell pipe and the liquid. He considers both rotatory radial inertia and the bending moment of the solid. Skalak’s shell model (hereby slightly re-organized) reads as follows, [29, 30]

$$\underbrace{\rho_s^* e^* \partial_{t^*}^2 \zeta^*}_{\text{Axial inertia}} = \underbrace{\frac{\alpha E^*}{1 - \nu_s^2} (\nu_s \partial_{z^*} \xi^* + R_0^* \partial_{z^*}^2 \zeta^*)}_{\text{[28]’s axial tension}} - \underbrace{\frac{\alpha e^{*2} E^*}{12(1 - \nu_s^2)} \partial_{z^*}^3 \xi^*}_{\text{Bending axial tension}}, \quad (21)$$

and

$$\underbrace{\rho_s^* e^* \partial_{t^*}^2 \xi^*}_{\text{Radial inertia}} - \underbrace{\frac{\rho_s^* e^{*3}}{12} \partial_{z^*}^2 \partial_{t^*}^2 \xi^*}_{\text{Rotatory inertia}} = - \underbrace{\frac{\alpha E^*}{1 - \nu_s^2} \left[\left(1 + \frac{\alpha^2}{12}\right) \frac{\xi^*}{R_0^*} + \nu_s \partial_{z^*} \zeta^* \right]}_{\text{[28]’s modified radial tension}} \quad (22)$$

The continuity conditions at the fluid solid interface were ensured and Skalak overcomes the system resolution by performing a conjugate Fourier (upon space) and Laplace (upon time) analysis. The in-depth investigations of the radial solid displacement field reveals a discrete infinite set of resonance frequencies governed by a transcendental equation. As in [28], [10, 11] then analyzes the solution in the long-wavelength asymptotic limit, i.e. when frequency goes to zero. In this limit, two propagating modes remain and: *"a physical interpretation [...] is that only these two lowest modes have finite phase velocities as the wavelength increases indefinitely.* The two wave speeds, in the infinite wavelength framework, then converges toward the [28]'s one

$$\lim_{\lambda \rightarrow \infty} c_{\pm, Sk}^{*2} \equiv c_{\pm, Lamb}^{*2}. \quad (23)$$

203 R. Skalak [10, 11] is nevertheless the first to mention the *"precursor wave"*
 204 associated with the axial pipe dynamics, although the used of shell approxi-
 205 mation remained a limitation for practical analysis. Whereas it turns out that
 206 the precursor wave prediction was indeed a robust prediction out of the shell
 207 approximation context, Skalak's model also neglects the wave speed dispersion
 208 arising from radial inertia. R. Skalak achieved the first known derivation of
 209 (FSI) four-equations of liquid-filled water-hammer

$$\frac{1}{\mathcal{K}_f^*} \partial_{t^*} P^* + \partial_{z^*} W^* = -\frac{2}{R_0^*} \partial_{t^*} \xi^*, \quad (24)$$

$$\rho_f^* \partial_{t^*} W^* + \partial_{z^*} P^* = 0, \quad (25)$$

$$\underbrace{\rho_s^* e^* \partial_{t^*}^2 \zeta^*}_{\text{Axial inertia}} = \underbrace{\frac{\alpha E^*}{1 - \nu_s^2} (\nu_s \partial_{z^*} \xi^* + R_0^* \partial_{z^*}^2 \zeta^*)}_{\text{[28]'s axial tension}} \quad (26)$$

$$\underbrace{\frac{\alpha E^*}{1 - \nu_s^2} \left[\frac{\xi^*}{R_0^*} + \nu_s \partial_{z^*} \zeta^* \right]}_{\text{[28]'s radial tension}} = P^*, \quad (27)$$

The r.h.s of (24) results from kinematic boundary conditions associated with the solid wall motion influence. It can be derived from Reynolds transport theorem, as discussed in [13, 18] and mentioned in [31] that we now detail. Consider a domain $\Omega(t)$, having boundary $\partial\Omega(t)$. Inside $\Omega(t)$ any field Ψ (scalar, vector,

tensor) fulfills Reynolds transport theorem

$$\frac{d}{dt} \int_{\Omega(t)} \Psi d\Omega = \int_{\Omega(t)} \frac{\partial \Psi}{\partial t} d\Omega + \int_{\partial\Omega(t)} \Psi v_n d\partial\Omega, \quad (28)$$

v_n is the outer-normal velocity of the boundary $\partial\Omega(t)$ motion. This equality can also be rewritten in a usefull way with the use of the divergence theorem on the last term of (28) so that

$$\frac{d}{dt} \int_{\Omega(t)} \Psi d\Omega = \int_{\Omega(t)} \left(\frac{\partial \Psi}{\partial t} + \nabla \cdot (\Psi \mathbf{v}) \right) d\Omega = \int_{\Omega(t)} \left(\frac{d\Psi}{dt} + \Psi \nabla \cdot \mathbf{v} \right) d\Omega \quad (29)$$

When considering the domain $\Omega(t)$ as the pipe section $S^*(t)$ whose boundary is the moving circle $\mathcal{C}^*(t)$ having radius R^* , and considering the unit scalar $\Psi = 1$, then (28), associated with the kinematic boundary condition $u_n = \partial\xi^*/\partial t$ leads to

$$\frac{d}{dt} S^*(t) = \int_{\mathcal{C}^*(t)} \frac{\partial \xi^*}{\partial t} d\mathcal{C} = 2\pi R^* \frac{\partial \xi^*}{\partial t}, \quad (30)$$

210 Using (30) in (5) whilst using (9) leads to (24).

211 These coupled first-order hyperbolic (FSI) four-equations receive a more
212 compact formulation in the form of two coupled propagating waves operators

$$\left[\partial_{t^*}^2 - \begin{pmatrix} c_{p,Sk}^{*2} & 0 \\ 0 & c_s^{*2} \end{pmatrix} \partial_{z^*}^2 - \begin{pmatrix} 0 & 2\nu_s \rho_f^* c_{p,Sk}^{*2} \\ \frac{\nu_s c_s^2}{\alpha E^*} & 0 \end{pmatrix} \partial_{z^*} \partial_{t^*} \right] \begin{pmatrix} P^* \\ \partial_{t^*} \zeta^* \end{pmatrix} = \mathbf{0}, \quad (31)$$

with

$$c_{p,Sk}^{*2} = \frac{c_0^2}{1 + \frac{2\mathcal{K}_f^*(1-\nu_s^2)}{\alpha E^*}}. \quad (32)$$

213 Two years after, [32] followed up [10, 11]'s analysis by (i) integrating the trans-
214 verse shear force contribution in the radial solid momentum conservation (27)
215 (ii) considering the azymutal displacement. Using the same long-wavelength ap-
216 proximation framework, they also carried out a frequency-domain analysis and
217 converged toward [28] and [10, 11] results. Although precursor waves have been
218 theoretically predicted for a long time, there were experimentally confirmed in
219 1969 only [33] for elastic steel (aluminum alloyed) and visco-elastic (PE) pipes
220 by Thorley. Whilst the predictive trend for the wave propagation speeds was
221 inconclusive for visco-elastic materials, the order of magnitude for elastic solids

222 was consistent. Thorley attributed these discrepancies to the temperature sen-
 223 sitivity of visco-elastic rheology as illustrated in Figure 3. [34] carried out a

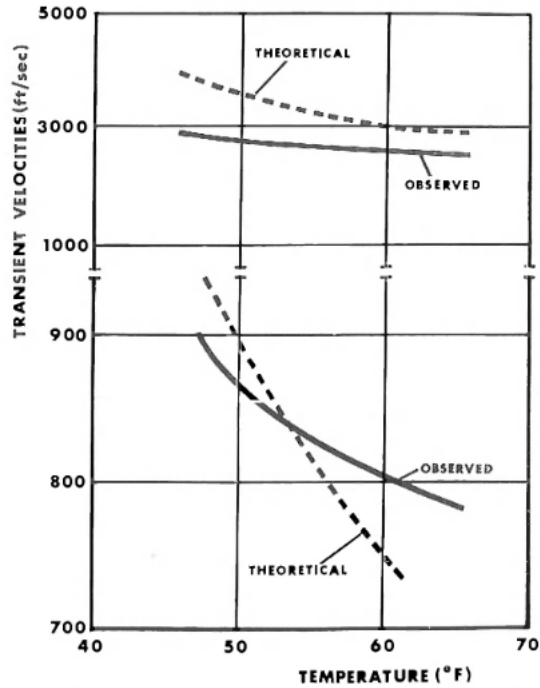


Figure 3: Experimental results of [33]. The upper curve represents the investigated elastic material whilst the lower curve holds for the visco-elastic material.

223
 224 complete study extending the analysis of [32] to viscous fluid. The kinematic
 225 continuity conditions at the pipe's wall were spelled out and ensured, whilst
 226 the set of constitutive equations was analyzed under the scope of the plane
 227 wave framework. The authors then concluded that the: "frequency dependence
 228 of the zeroth mode phase velocity is primarily a result of the tube constraint
 229 at high frequencies and viscosity at low frequencies." Finally, [34] were able to
 230 propose an order of magnitude estimate for the transverse solid shear force,
 231 i.e. $(D/\alpha) (c_0^*/c_s^*)^2 \tau_w^*$ proportional to the fluid wall shear-stress τ_w . [35] car-
 232 ried out experimental tests similar to those of [33] on polymer materials (ABS,
 233 PVC) and steel for flexible and rigid configurations, i.e. unstressed and axi-
 234 ally stressed pipes. Despite Thorley encountered difficulties in discarding the

235 effects of the junction coupling arising from his downstream solenoid valve, he
 236 clearly observed and identified the presence of precursor perturbations resulting
 237 from Poisson's coupling. [35] further noted: "*that mechanical damping can be*
 238 *more important for water-hammer decay than viscous friction*". This remark
 239 is meaningful when the radial inertia of the pipe is preponderant or when the
 240 rheology is inelastic, as is the case for visco-elastic materials. [36, 37] delivered
 241 a complete work emphasizing the previous contributions from [28] to [38]. The
 242 authors

243 considered the rigid, elastic and visco-elastic behavior of a pipe together with
 244 the viscous, or inviscid, behavior of the fluid. Their pipe model furthermore
 245 accounts for the radial thickness influence. [36, 37] derive a complete set of
 246 dispersion relations and studied the frequency dependence of the propagation
 247 wave speeds.

248 [39, 40, 41] carried out an outstanding and complete analysis within solid's
 249 shell approximation taking into account thermal and fluid viscosity effects.

250 The work of [41] provides a comprehensive overview of the main models,
 251 assumptions and results of the early researches on hydraulic transients in pipes.
 252 For the solid, the momentum conservation equation is integrated in the radial
 253 direction leading to an axial dependent problem, whilst the bending effects
 254 were neglected. [19]'s analysis is a breakthrough in the liquid-filled pipe re-
 255 search area. A. Tijsseling combined both the Navier-Stokes equations, averaged
 256 over the pipe's section, with the solid momentum conservation equations (also
 257 called Lamé-Clapeyron equations, [42]) to derive a set of four-(FSI) hyperbolic
 258 equations [19]

$$\begin{aligned} \left[\begin{pmatrix} \frac{c_0^{*2}}{\kappa_f^* c_{p,T}^{*2}} + \frac{4\nu_s^2}{\alpha(2+\alpha)E^*} & 0 \\ 0 & 1 \end{pmatrix} \partial_{t^*} + \begin{pmatrix} 0 & 1 \\ \frac{1}{\rho_f^*} & 0 \end{pmatrix} \partial_{z^*} \right] \begin{pmatrix} P^* \\ W^* \end{pmatrix} &= \frac{2\nu_s}{E^*} \begin{pmatrix} \partial_{t^*} \sigma_{zz}^* \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{2\tau_w^*}{\rho_f^* R_0^*} \end{pmatrix}, \quad (33) \\ \left[\partial_{t^*} - \begin{pmatrix} 0 & E^* \\ \frac{1}{\rho_s^*} & 0 \end{pmatrix} \partial_{z^*} \right] \begin{pmatrix} \sigma_{zz}^* \\ \dot{\zeta}^* \end{pmatrix} &= \frac{2\nu_s}{\alpha(2+\alpha)} \begin{pmatrix} \partial_{t^*} P^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2\tau_w^*}{\rho_s^* c^*(2+\alpha)} \end{pmatrix} \quad (34) \end{aligned}$$

259 with $\dot{\zeta}^* = \partial_{t^*} \zeta^*$ the longitudinal solid velocity and

$$c_{p,T}^{*2} = \frac{c_0^{*2}}{1 + \frac{2\mathcal{K}_f^*}{\alpha E^*} \left(\frac{2(1-\nu_s^2)}{2+\alpha} + \alpha(1+\nu_s) \right)}. \quad (35)$$

As in [28, 10, 32] a set of two coupled wave speeds can be found from (33)-(34)

$$c_{\pm,T}^{*2} = \frac{1}{2} \left[\bar{c}_T^{*2} \pm \sqrt{\bar{c}_T^{*4} - 4c_s^{*2} c_{p,T}^{*2}} \right], \quad \text{with } \bar{c}_T^{*2} = c_s^{*2} + \left(1 + \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} \right) c_{p,T}^{*2}. \quad (36)$$

260 The last term of the (33)'s r.h.s involves a dissipative (a sink term) associated
 261 with the wall shear-rate τ_w^* the derivation of which can again be obtained us-
 262 ing the Reynolds transport theorem as further discussed in Section 2.3. An
 263 undamped simplified derivation of (33)-(34) neglecting wall shear rate τ_w^* , thus
 264 neglecting dissipation, can also be found in [43]. Further information on the de-
 265 velopment of liquid-filled pipe models during the 20th century can also be gath-
 266 ered from the review of [12]. As also discussed in many places [12, 41, 18, 13]
 267 the influence of a body force (e.g gravity) can easily be added into those FSI
 268 four-equations. The wave speeds and the corresponding corrective coupled wave
 269 speeds, are depicted in Figure 4. The Skalak's pulse wave speed model converges,
 270 in the limit α tends to one, to the D. Korteweg's one, whilst as expected, the
 271 A.S. Tijsseling's model differs for thick tubes (Cf. Fig. 4a&4b). In addition, for
 272 very thin pipes, the models strongly differ in the prediction of the coupled wave
 273 speeds as depicted in Figure 4d. To provide the pressure dynamic predictions
 274 the four-(FSI) equations system (33)-(34) nevertheless requires a closure wall
 275 shear stress model for τ_w^* [44]. This necessitates a rather developed discussion
 276 which is postponed to section 2.3. Un-damped wave propagation, is nevertheless
 277 interesting to compute considering $\tau_w^* = 0$ in the coupled hyperbolic problems
 278 (33)-(34) which is strongly dependent on boundary conditions, an issue related
 279 to *junction coupling* which is now considered.

280 2.2. *Junction coupling*

281 2.2.1. *Junction coupling within simple domains*

282 This section first considers the simple pipe's configurations and later-on dis-
 283 cuss more complex ones, such as networks. Coupled FSI propagative hyperbolic
 284 problems are determined by their boundary conditions. Within simple config-
 285 urations, since these boundary conditions might differ, it is possible to propa-
 286 gate one formal, yet un-determined solution —satisfying one of the boundary
 287 conditions— to the other additional boundary conditions. This is how transfer
 288 function matrices are used in free oscillation theory. It is also the way Transfer
 289 Matrix Method (TMM) handles the influence of boundary conditions. One ad-

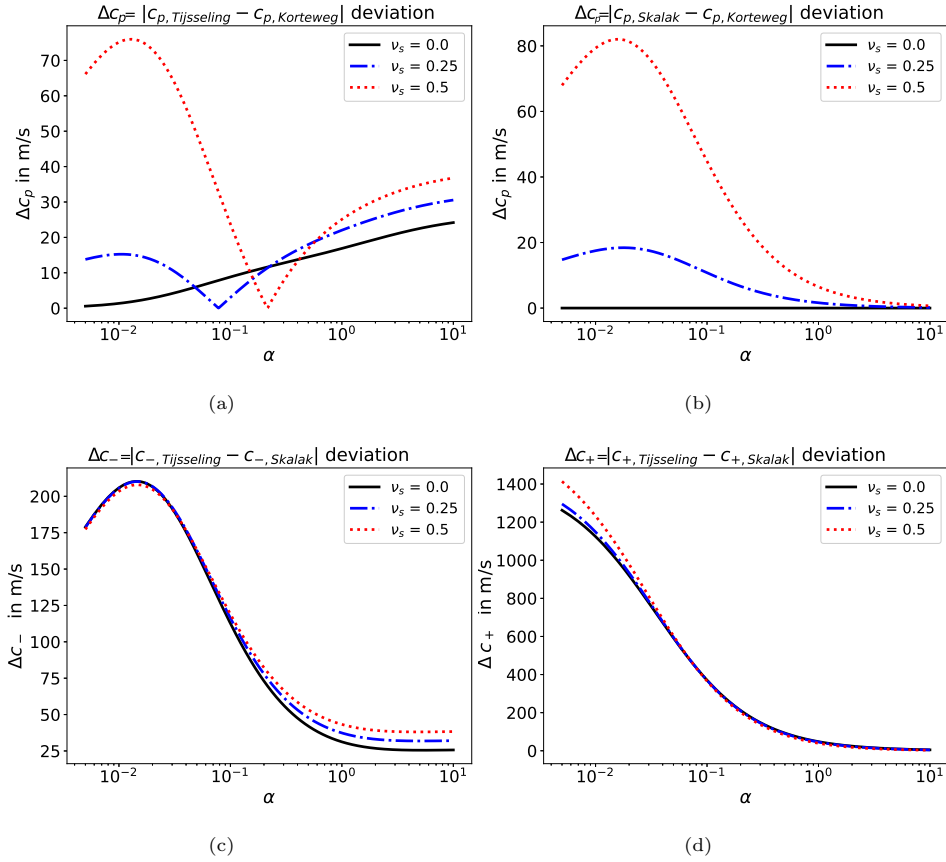


Figure 4: Pulse and (FSI)-corrective wave speeds deviation analysis for the models of [6, 10, 11, 43] for solid/fluid velocity ratio $C_s = 4.93$, and fluid/solid density ratio $\mathcal{D} = 0.126$.

290 vantage of TMM is to be able to perform an explicit dependence of the solution
 291 to the prescribed boundary conditions for general sets of (linear) boundary con-
 292 ditions, either for FSI water-hammer [45, 46, 47, 48] or even in visco-elastic solid
 293 response [49]. TMM can either be expressed in time domain [50] or frequency
 294 domain [51, 52].

295 TMM has permitted explicit analytical solutions in restrained configurations.
 296 Non exhaustively among a vast literature TMM solutions have been developed
 297 for single pipe [47, 53], single pipe with elastic constraints [54], series pipe system
 298 [55], curved circular pipe [56], curved pipe with various degrees of freedom

299 [57], extended blockage [58], simple tree-like metric graphs [59], single pipe of
 300 arbitrary transverse shape approximated by Euler beam theory [60, 61] with
 301 various degrees of freedom or with visco-elastic solids [62, 63] in Laplace domain.
 302 In this section we first discuss the explicit way a transfer matrix condition is set
 303 for general linear boundary conditions. In the following some useful notation is
 304 considered, such as the dimensionless elastic velocity ratio

$$305 \quad \mathcal{C}_s = \frac{c_s^*}{c_p^*}. \quad (37)$$

306 The physical time t^* is furthermore re-scaled with respect to the fluid acoustic
 307 advective time scale, i.e. $\tau = tL^*/c_p^*$, whereas the axial coordinate is non-
 308 sionalized by the pipe's length, i.e. $Z = z/L$. The perturbed fluid pressure P^* ,
 309 and axial solid stress component σ_{zz}^* , are re-scaled by the [64]'s over-pressure,
 310 i.e. $O(\rho_f c_p W_0)$ where W_0 is the flow variation applied within the pipes, so that
 311 their dimensionless counterparts are denoted

$$312 \quad P = \frac{P^*}{\rho_f^* c_p^* W_0^*}, \text{ and, } \sigma_{zz} = \frac{\sigma_{zz}^*}{\rho_f^* c_p^* W_0^*}. \quad (38)$$

313 Note that, on the denominator of the right side of (38) the dimensionless stress
 314 σ_{zz} has been obtained from using Joukowsky's overpressure, anticipating the
 315 normal stress continuity between the fluid and the solid compartment. The di-
 316 mensionless water-hammer four FSI hyperbolic equations can be recasted in the
 317 following two-wave coupled equation system (33)-(34) without friction coupling
 318 (i.e when $\tau_w^* = 0$)

$$319 \quad (\partial_\tau^2 - \mathbf{C}_{\mathbf{P}}^2 \partial_Z^2) \mathbf{P} = \mathbf{0}, \quad (39)$$

320 where

$$321 \quad \mathbf{C}_{\mathbf{P}}^2 = \begin{pmatrix} 1 & 2\nu_s \mathcal{D} \\ \frac{2\nu_s}{\alpha(2+\alpha)} & \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} + \mathcal{C}_s^2 \end{pmatrix}, \text{ and, } \mathbf{P} = \begin{pmatrix} P \\ \sigma_{zz} \end{pmatrix}. \quad (40)$$

322 Off-diagonal terms of matrix $\mathbf{C}_{\mathbf{P}}^2$ are proportional to the Poisson coefficient ν_s
 323 so that the fluid pressure and the solid stress decouple as $\nu_s \rightarrow 0$. Furthermore,
 324 as $\nu_s \rightarrow 0$ the remaining diagonal terms are 1 and \mathcal{C}_s^2 , the two eigenvalues of
 325 the resulting diagonal matrix. These eigenvalues are providing the two distinct
 326 wave-velocities of the uncoupled limit: 1 which is the dimensionless pressure

327 pulse velocity c_p whereas \mathcal{C}_s^2 is the dimensionless elastic wave solid one. When
 328 $\nu_s \neq 0$ the eigenvalues of matrix $\mathbf{C}_{\mathbf{P}}^2$ provide the velocities of the coupled system.
 329 The vector homogeneous wave-equation resolution will be handled within the
 330 eigenvectors basis of $\mathbf{C}_{\mathbf{P}}^2$ as in [47]. The eigenvalues of $\mathbf{C}_{\mathbf{P}}^2$, denoted $c_{\pm}^2 > 0$,
 331 associated with diagonalized matrix $\mathbf{C}_{\mathcal{P}}^2$ correspond to the wave speed mode
 332 propagation. They are the solution of the following polynomial characteristic
 333 problem

$$334 \quad c_{\pm}^4 - \left[1 + \mathcal{C}_s^2 + \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} \right] c_{\pm}^2 + \mathcal{C}_s^2 = 0, \quad (41)$$

335 the solutions of which are

$$336 \quad c_{\pm}^2 = \frac{1 + \mathcal{C}_s^2 + \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} \pm \sqrt{\left(1 + \mathcal{C}_s^2 + \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} \right)^2 - 4\mathcal{C}_s^2}}{2}. \quad (42)$$

337 (42) is the dimensionless version of (36). The asymptotic behavior with respect
 338 to α parameter of all dimensionless velocity is illustrated in figure (5) from [65].
 339 As α increases or $\nu_s \rightarrow 0$, the dimensionless positive and negative wave speed
 340 mode, c_+ tend to \mathcal{C}_s and c_- tends to one. The various asymptotic limits of other
 341 dimensionless velocities are also reported for completeness in figure 5. The fluid
 342 pressure and the axial solid stress as well as their respective time-derivatives are
 343 assumed initially at rest so that

$$344 \quad \mathbf{P}(Z, 0) = \mathbf{0} \quad , \quad \partial_{\tau} \mathbf{P}(Z, 0) = \mathbf{0}. \quad (43)$$

345 In the diagonal base of matrix $\mathbf{C}_{\mathbf{P}}^2$ the system becomes

$$346 \quad (\partial_{\tau}^2 - \mathbf{C}_{\mathcal{P}}^2 \partial_Z^2) \mathcal{P} = \mathbf{0} \quad , \quad \text{with,} \quad \mathcal{P}(Z, 0) = \partial_{\tau} \mathcal{P}(Z, 0) = \mathbf{0}, \quad (44)$$

347 where the change of basis

$$348 \quad \mathbf{\Pi} = \begin{pmatrix} \frac{2\nu_s \mathcal{D}}{c_-^2 - 1} & \frac{2\nu_s \mathcal{D}}{c_+^2 - 1} \\ 1 & 1 \end{pmatrix}, \quad \mathbf{C}_{\mathcal{P}}^2 = \begin{pmatrix} c_-^2 & 0 \\ 0 & c_+^2 \end{pmatrix} \equiv \mathbf{\Pi} \mathbf{C}_{\mathbf{P}}^2 \mathbf{\Pi}^{-1} \quad , \quad \text{and,} \quad \mathcal{P} = \mathbf{\Pi}^{-1} \mathbf{P}, \quad (45)$$

349 has been used. The pressure-axial stress 2D-vector $\mathbf{P} = (P, \sigma_{zz})$ is transformed
 350 into a linear combination of those in 2D-vector \mathcal{P} from (45). The Laplace
 351 transform of (44) then leads to

$$352 \quad (s^2 - \mathbf{C}_{\mathcal{P}}^2 \partial_Z^2) \tilde{\mathcal{P}} = \mathbf{0}, \quad (46)$$

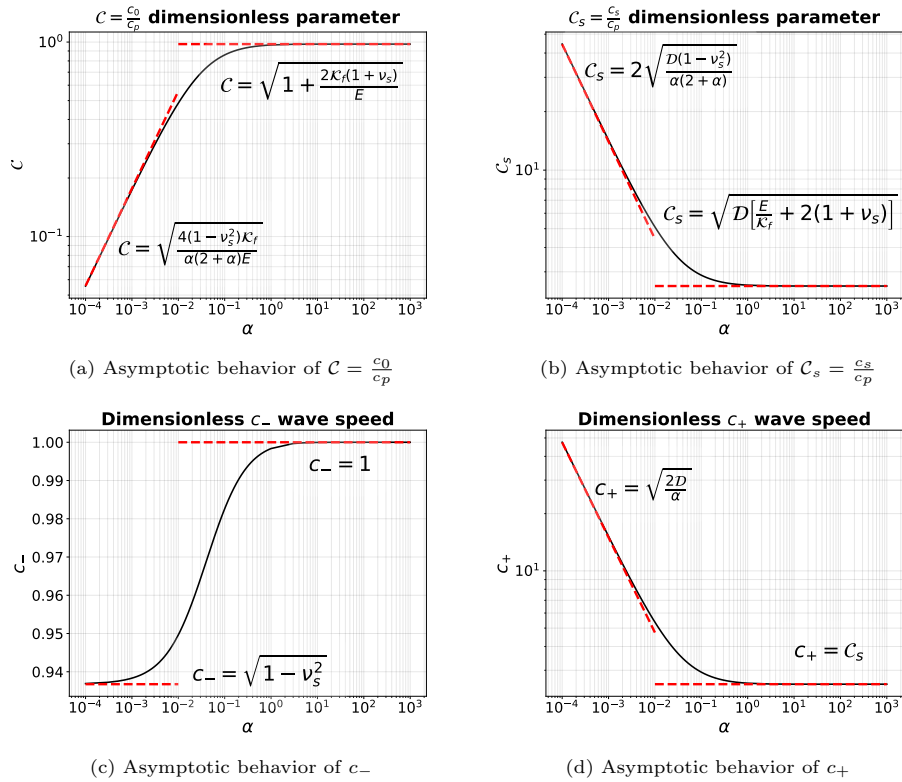


Figure 5: (taken from [65]) Wall thickness α -dependence ($\alpha = \frac{e}{R_0}$) of dimensionless characteristic FSI wave speeds for $\nu_s = 0.35$ and $\mathcal{D} = 0.1122$, $\frac{E}{\mathcal{K}_f} = 54$. The red dashed lines indicate various analytical asymptotic behaviors.

353 $\mathcal{C}_{\mathcal{P}}^2$ being diagonal given (45). A solution can be found for the spatial ODE
 354 system leading to

$$355 \quad \tilde{\mathcal{P}}(Z, s) = \mathbf{E}(Z, s)\tilde{\mathcal{P}}^{\mathcal{D}}(s) + \mathbf{F}(Z, s)\tilde{\mathcal{P}}^{\mathcal{N}}(s), \quad (47)$$

356 with 2×2 diagonal matrices

$$357 \quad \mathbf{E}(Z, s) = \begin{pmatrix} \cos\left(\frac{is}{c_-}Z\right) & 0 \\ 0 & \cos\left(\frac{is}{c_+}Z\right) \end{pmatrix}, \quad \mathbf{F}(Z, s) = \begin{pmatrix} \sin\left(\frac{is}{c_-}Z\right) & 0 \\ 0 & \sin\left(\frac{is}{c_+}Z\right) \end{pmatrix}, \quad (48)$$

and $\tilde{\mathcal{P}}^{\mathcal{D}}(s), \tilde{\mathcal{P}}^{\mathcal{N}}(s)$ 2D-vectors yet to be found. $\tilde{\mathcal{P}}^{\mathcal{D}}(s)/\tilde{\mathcal{P}}^{\mathcal{N}}(s)$ provide the Dirichlet/Neumann mode-dependent amplitude of $\tilde{\mathcal{P}}(Z, s)$ respectively associated with the condition imposed at location $Z = 0$ because $\mathbf{E}(0, s) = \mathbf{I}$ and $\mathbf{F}(0, s) = \mathbf{0}$. A general set of boundary conditions is stated as

$$\begin{pmatrix} \mathcal{N} & \mathcal{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{Q} & \mathcal{R} \end{pmatrix}_{(4 \times 8)} \begin{pmatrix} \mathcal{P}(0) \\ \partial_Z \mathcal{P}(0) \\ \mathcal{P}(1) \\ \partial_Z \mathcal{P}(1) \end{pmatrix}_{(4 \times 1)} = \mathbf{0}, \quad (49)$$

358 with $\mathcal{N}, \mathcal{M}, \mathcal{Q}, \mathcal{R}$ 2×2 matrices associated with the Dirichlet/Neumann couplings
 359 at both ends. It is worth mentioning that condition 49 needs to be enlarged in the
 360 case where the applied boundary conditions depend on the velocity. Then, a four-
 361 dimensional wave vector including the fluid velocity and solid longitudinal acceleration
 362 needs to be considered as performed in [50]. Introducing notation $\mathcal{C}_{\mathcal{P}}^{-1} = \begin{pmatrix} c_-^{-1} & 0 \\ 0 & c_+^{-1} \end{pmatrix}$,
 363 from definition (48) one gets

$$364 \quad \partial_Z \mathbf{E}(Z, s) = -is\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{F}(Z, s), \quad \partial_Z \mathbf{F}(Z, s) = is\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{E}(Z, s) \quad (50)$$

365 Combining the expression of (47), (48) (50) with the Laplace transform of the bound-
 366 ary condition system (49) (into which the Laplace transform of the Dirac distribution
 367 $\delta(\tau)$ equals one, i.e $\tilde{\delta} = 1$), one finds

$$368 \quad \begin{pmatrix} \tilde{\mathcal{P}}^{\mathcal{D}} \\ \tilde{\mathcal{P}}^{\mathcal{N}} \end{pmatrix}(s) = \mathcal{B}^{-1}(s)\mathcal{S}, \quad (51)$$

369 with \mathcal{S} being a constant 4-vector depending on the precise set of boundary conditions
 370 (explicit examples are given in [53]) and

$$371 \quad \mathcal{B} = \begin{pmatrix} \mathcal{N} & is\mathcal{M}\mathcal{C}_{\mathcal{P}}^{-1} \\ \mathcal{Q}\mathbf{E}(1) - is\mathcal{R}\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{F}(1) & \mathcal{Q}\mathbf{F}(1) + is\mathcal{R}\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{E}(1) \end{pmatrix}. \quad (52)$$

372 The inverse of (52) is then needed to find $\tilde{\mathcal{P}}$ from solving (51). One can note that the
373 general solution for arbitrary closure law can easily be deduced from solution (51) by
374 multiplying the constant source term \mathcal{S} by the Laplace transform of the closure law
375 (which depends on s). Alternatively, in time-domain, the general closure law solution
376 is found from a convolution product with the impulse response solution as detailed in
377 [50]. The impulse response is thus a generic solution. Following notations of [53] by
378 introducing the adjugate matrix of \mathcal{B} , namely $\text{adj}[\mathcal{B}]$ one can write

$$379 \quad \mathcal{B}^{-1}(s) = \frac{\text{adj}[\mathcal{B}(s)]}{\det \mathcal{B}(s)}, \quad (53)$$

380 Let us furthermore introduce the two matrices

$$381 \quad \mathbf{IO} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and, } \mathbf{OI} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (54)$$

382 vector $\tilde{\mathcal{P}}(s, Z)$ can then be found using (47), (51) and (53) to reach

$$383 \quad \tilde{\mathcal{P}}(s, Z) = [\mathbf{E}(Z, s)\mathbf{IO} + \mathbf{F}(Z, s)\mathbf{OI}] \frac{(\text{adj}[\mathcal{B}])}{\det \mathcal{B}(s)} \mathcal{S}, \quad (55)$$

384 (55) is the formal solution for the 2D-vector $\tilde{\mathcal{P}}(s, Z)$ in the frequency domain. For
385 specific sets of boundary conditions, this formal solution can be further developed into
386 explicit analytical expressions. In [53] the complete analytical solutions of three fam-
387 ilies of boundary conditions are detailed, thus not repeated here. Nevertheless, there
388 is one salient and generic feature of this solution which is of distinct importance : it
389 diverges for specific values of s called poles. It can be shown from applying inverse
390 Laplace transform and Cauchy theorem [53] that these poles provide the specific nat-
391 ural resonant frequencies of the wave system or, equivalently the specific oscillating
392 modes of the time-domain solution. The ensemble of these discrete resonant frequen-
393 cies is the solution's spectrum. As discussed in [53], from solution (55) one can find
394 that this divergent condition at poles is given by condition

$$395 \quad \mathcal{S}_{\mathcal{P}} = \{s \in \mathbb{C} \mid \det \mathcal{B}(s) = 0\}. \quad (56)$$

396 Condition (56) can be translated into a transcendental equation, the roots of which
397 are denoted s_k . One can find in [53] three configurations where the transcendental
398 equation is explicitly given and for which, in each case, the root s_k is purely imaginary,
399 i.e.

$$400 \quad s_k = i\lambda_k \quad , \text{ with, } \lambda_k \in \mathbb{R}. \quad (57)$$

Curiously, it is only recently that a formal one-to-one mapping between TMM Laplace domain solutions and time domain ones has been clarified [50, 53]. One important consequence of this one-to-one mapping between Laplace domain and time-domain solutions is the appearance of discrete spectrum of λ_k , leading to $\sin(\lambda_k t)$ time oscillation modes. If no friction model is taken into account in these FSI function coupling solutions these modes are not damped. Including friction models produces damping with a resulting time decay specific to each mode as discussed in 2.3. Let us now add some aspects concerning time-domain solutions. Following notations of [50] the mode decomposition of time-domain solutions is performed over the eigen-function based $\Phi_k(Z)$ of the heterogeneous operator \mathcal{H} based on $\mathcal{C}_{\mathcal{P}}^2$ defined as

$$\mathcal{H}\Phi_k(Z) \equiv \mathcal{C}_{\mathcal{P}}^2 \cdot \partial_Z^2 \Phi_k(Z) = -\lambda_k^2 \Phi_k(Z), \quad (58)$$

The eigenfunctions have been given in [50] as $\Phi_k = \phi_k / \|\phi_k\|$

$$\phi_k(Z) = \begin{pmatrix} \cos\left(\frac{\lambda_k Z}{c_-}\right) + \tan\left(\frac{\lambda_k}{c_-}\right) \sin\left(\frac{\lambda_k Z}{c_-}\right) \\ -\frac{c_+}{\beta c_-} \left(\cos\left(\frac{\lambda_k Z}{c_+}\right) + \tan\left(\frac{\lambda_k}{c_+}\right) \sin\left(\frac{\lambda_k Z}{c_+}\right) \right) \end{pmatrix}, \quad (59)$$

with β defined as

$$\beta = \frac{c_+ c_-^2 - 1}{c_- c_+^2 - 1}. \quad (60)$$

Furthermore one has to define a general scalar product,

$$\forall \Psi, \Psi' \in L^2(\mathbb{R}) \times L^2(\mathbb{R}), \quad \langle \Psi', \Psi \rangle = \sum_{j=1}^2 \eta_j \int_0^1 \Psi'_j(Z) \Psi_j(Z) dZ, \quad (61)$$

with $j = 1, 2$ referring to the j^{th} components of vector $\boldsymbol{\eta} \equiv (\eta_1, \eta_2) \in \mathbb{R}^2$, a real vector which is adapted to each specific problem, so that the operator 58 associated with boundary conditions 49 is self-adjoint. More precisely, invoking the definition of \mathcal{H} in (58), the search for self-adjoint condition for operator \mathcal{H} , equipped with scalar product (61), performing a double integration by parts leads to

$$\begin{aligned} \langle \mathcal{H}\Psi, \Psi' \rangle &= \langle \mathcal{C}_{\mathcal{P}}^2 \cdot \partial_Z^2 \Psi, \Psi' \rangle = \langle \Psi, \mathcal{C}_{\mathcal{P}}^2 \cdot \partial_Z^2 \Psi' \rangle + \\ &\quad \sum_{j=1}^2 \eta_j c_j^2 \left(\left[\partial_Z \Psi_j(Z) \Psi'_j(Z) - \Psi_j(Z) \partial_Z \Psi'_j(Z) \right]_0^1 \right), \end{aligned} \quad (62)$$

where c_j^2 are the j^{th} diagonal terms of $\mathcal{C}_{\mathcal{P}}^2$. From (62) self-adjoint property $\langle \mathcal{H}\Psi, \Psi' \rangle = \langle \Psi, \mathcal{H}\Psi' \rangle$, is thus obtained from condition

$$\sum_{j=1}^2 \eta_j c_j^2 \left[\partial_Z \Psi_j(Z) \Psi'_j(Z) - \Psi_j(Z) \partial_Z \Psi'_j(Z) \right]_0^1 = 0. \quad (63)$$

Eigenfunctions $\Phi_k(Z)$ are chosen of norm unity, i.e $\langle \Phi_k, \Phi_k \rangle = 1$ [50]. Operator \mathcal{H} self-adjointness is a prerequisite to get a discrete spectrum composed of distinct eigenvalues as well as an orthogonal base decomposition. Using a proper choice of scalar product weight $\boldsymbol{\eta} \equiv (\eta_1, \eta_2)$ satisfying 63 permit to write the general time-domain solutions for the FSI problem to be decomposed into some homogeneous part and some particular solution taking care of non-homogeneous boundary conditions at the edges

$$\mathcal{P}(Z, \tau) = \sum_{S_p} a_k(\tau) \Phi_k(Z) + \mathcal{P}_p(Z, \tau), \quad (64)$$

where $a_k(\tau)$ are the mode amplitudes and \mathcal{P}_p is a particular solution lying in the kernel of \mathcal{H} , i.e

$$\mathcal{H}\mathcal{P}_p = \mathbf{0}, \quad (65)$$

so that, it can be decomposed as a linear and constant field

$$\mathcal{P}_p(Z, \tau) = Z\mathcal{P}_p^1(\tau) + \mathcal{P}_p^0(\tau), \quad (66)$$

where \mathcal{P}_p^1 and \mathcal{P}_p^2 are two time functions used to map boundary conditions as well as initial conditions as further detailed in [50]. Let us investigate further the homogeneous part of the time-domain solution in order to find how the spectrum solution condition appears in this case, and how does it compares to the one obtained in frequency domain. Transposing the TMM method in time domain one can propagate the boundary value from $Z = 0$ to a solution of (58) at coordinate Z from choosing

$$\Phi_k(Z) = \mathbf{E}(Z, i\lambda_k)\Phi_k(0) - \frac{1}{\lambda_k}\mathcal{C}_{\mathcal{P}}\mathbf{F}(Z, i\lambda_k)\partial_Z\Phi_k(0) \quad (67)$$

which is built, from (48) definitions, so that when using (50) and again, $\mathbf{E}(0, i\lambda_k) = \mathbf{I}$, $\mathbf{F}(0, i\lambda_k) = \mathbf{0}$ one gets self-consistent $\Phi_k(0) = \Phi_k(0)$ both sides in (67). Similarly from derivating (67) using (50) again one finds

$$\partial_Z\Phi_k(Z) = \lambda_k\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{F}(Z, i\lambda_k)\Phi_k(0) + \mathbf{E}(Z, i\lambda_k)\partial_Z\Phi_k(0), \quad (68)$$

so that once again $\partial_Z\Phi_k(0) = \partial_Z\Phi_k(0)$ both sides in (68). Hence, using (67) and (68) at $Z = 1$, it is possible to express boundary condition (49) as

$$\left(\begin{array}{cc} \mathcal{N} & \mathcal{M} \\ \mathcal{Q}\mathbf{E}(1, i\lambda_k) - \lambda_k\mathcal{R}\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{F}(1, i\lambda_k) & -\frac{1}{\lambda_k}\mathcal{C}_{\mathcal{P}}[\mathcal{Q}\mathbf{F}(1, i\lambda_k) - \lambda_k\mathcal{R}\mathcal{C}_{\mathcal{P}}^{-1}\mathbf{E}(1, i\lambda_k)] \end{array} \right) \begin{pmatrix} \Phi_k(0) \\ \partial_Z\Phi_k(0) \end{pmatrix} = 0 \quad (69)$$

401 A non-trivial solution to (69) boundary condition is only possible if the matrix has a
402 non-empty kernel, i.e if its determinant equals zero. Comparing (69) with 52 shows

403 that these two matrices have a determinant which is proportional, so that the zero
 404 determinant condition leads to the same spectrum. Hence the discrete spectrum solu-
 405 tion is identical either found from frequency domain or time domain. Furthermore, in
 406 time-domain, the homogeneous solution value and derivative at one node of the single
 407 pipe pertain to the null-space of a given matrix. We will find in the section 4.4.2 that
 408 the spectrum of vibrating modes into a network share a similar property, but for a
 409 distinct matrix.

410 2.2.2. Junction coupling within complex domains

411 Within pipe systems and networks, even in the case of linear boundary conditions,
 412 most of the literature analyze junction coupling using numerical methods. This is
 413 mostly because in this more complex context, analytical solutions are most often not
 414 possible to find. Nevertheless section 4.4 covers recent advances whereby some alge-
 415 braic analytical solutions are discussed in networks. Since dealing with an hyperbolic
 416 problem, classical numerical methods have been adapted and developed to solve water-
 417 hammer wave propagation using Method of Characteristics (MOC) [66, 67, 68, 69, 70],
 418 Finite Element Method (FEM) [71, 72, 73, 74, 75, 76, 77], Finite-Volume method (FV)
 419 [78, 79, 80, 81, 82, 83] or coupled MOC–FEM [84]. The respective pros and cons of
 420 these various methods have been discussed in several specific reviews [27, 15, 85], so
 421 that a similar meticulous discussion will not be repeated here. When FSI’s ef-
 422 fects are ignored considering the pressure transient dynamics in hydraulic systems
 423 [13], MOC is one of the most popular methods. Rather than considering a pressure
 424 wave second order propagation operator, it is more precise and easier to consider
 425 the first order coupled hyperbolic problem associated with the pressure and velocity
 426 (82). Since for water-hammer wave propagation within pipes having elastic properties
 427 (such as metallic materials), the wave velocity is homogeneous along each pipe, MOC
 428 provides an easy to implement, very weakly dissipative integration method. MOC
 429 nevertheless has two intrinsic drawback/weakness. First, a Courant–Friedrichs–Lewy
 430 (CFL) condition (obviously not specific to MOC’s method) $c_{p_{ij}} \Delta t_{ij} / \Delta Z_{ij} < 1$ has to
 431 be fulfilled along each pipe ij , connecting node i to node j whereby the wave velocity
 432 is $c_{p_{ij}}$ prescribing a constraint between time-step Δt_{ij} and spatial discretization ΔZ_{ij} .
 433 Since the CFL condition has to be fulfilled in each pipe, the most restrictive time step
 434 enslaves all others so that the time step is $\Delta t = \min_{ij} \Delta t_{ij}$. The CFL condition also
 435 prescribes the spatial discretization $\Delta Z_{ij} = c_{p_{ij}} \Delta t$ in each pipe. This constraint con-

436 siderably enlarges the number of unknowns of the problem as well as imposes a small
437 constant time-step (without the possible use of adaptative time-stepping). The sec-
438 ond drawback is that even if MOC is well adapted to non-dispersive wave propagation,
439 i.e propagation with constant wave speed, this framework is limited and not always
440 relevant. For example, in the case of water-hammer within visco-elastic pipes, the
441 wave-speed is not constant over time. In this case, MOC method is not well adapted
442 even though some scheme and approximations have been developed to extent its ap-
443 plication in this context [27, 15, 85]. On the contrary, FEM and FV methods permit
444 the use of adaptative time stepping and/or implicit time integration schemes debili-
445 tating the numerical cost of the CFL constraint. Both FEM and FV can be applied
446 with stabilizing hyperbolic schemes (the most popular being the Godunov scheme for
447 FV [78, 79, 86], and possibly the Streamline Upwind Petrov–Galerkin SUPG for FEM
448 [87]). One distinct advantage of FE over FV is its ability to deal with FSI from solving
449 the coupled solid elasto-dynamic problem. Furthermore, considering FEM method in
450 more than one dimension in space (e.g non axi-symmetrical breathing varying along θ)
451 also permit to handle much more general deformations (e.g yaw, torsion, etc..) than
452 pipe breathing modes mostly considered in this review.

453 *2.3. Friction coupling with shear stress dissipation modeling*

454 As recently discussed in an authorized survey by A. E. Vardy [88], friction is
455 one mechanism of water-hammer’s damping having attracted much attention, by the
456 author himself but also many others as will be discussed just below. Many modeling ef-
457 forts have indeed been dedicated to model the experimental damping of water-hammer
458 waves, which turns-out to be a subtle, difficult, but also central issue in the topic. Let
459 us start briefly from mentioning why this issue definitely owns a practical interest.
460 From neglecting wave dissipation and dispersion, surge analysis is a current engineering
461 computation which permits to localize the most dangerous spots, i.e locations where
462 the water-hammer pressure can exceed security prescriptions into a given installation,
463 resulting from a given operation/incident within the network (e.g. valve closing, hy-
464 draulic motor tripping, check valve failure, pipe breakage, etc...). Obviously this
465 analysis can be overprotective, and possibly alarming. Hence more accurate predic-
466 tions for water-hammer events is of interest for lowering the cost of water-hammer
467 protection equipment and security design of a given installation. This is where friction
468 modeling kicks-in. As for any modeling, two strategies can be pursued : on the one

469 hand, a practical one, dealing with developing specific, dedicated and accurate models
470 able to describe observations. On the other hand, a more generic one trying to de-
471 cipher which friction mechanisms are involved, and how they could be modeled in a
472 generic way. Before entering into the details of friction model, we would first like to
473 mention (for honesty and perhaps clarity) that this review is more oriented toward the
474 second modeling effort. One motivation behind this tropism is the ability to quantify
475 the respective contributions involving various superposed effects, so as to be predictive
476 in different situations. Nevertheless, albeit many research efforts, this issue has not
477 reached this level of maturity, as now detailed.

478 Some wall-shear-stress models emerged from the hydraulic analysis of viscous flow
479 in pipes, ignoring (FSI) effects. In many cases, the acoustic hypothesis is used for the
480 fluid and the Poisson coupling is discarded, [89, 90, 91, 13, 92].

481

Nevertheless before discussing the various dissipation models proposed in the lit-
erature, let us step out to properly derive how the wall shear-stress happens to be the
key ingredient in this issue. As previously mentioned in 2.1 the momentum balance
second equation of Tijsseling's four-equation FSI model (34) can be deduced using
Reynolds transport theorem. Considering a flexible tube, i.e a tube whose radius and
section varies in time, where a velocity field \mathbf{u}^* flows, momentum balance in the fluid
(without body force) reads

$$\rho_f^* \frac{d\mathbf{v}^*}{dt^*} = \nabla \cdot \boldsymbol{\sigma}_f^*, \quad (70)$$

where the fluid stress-tensor of a Newtonian fluid is

$$\boldsymbol{\sigma}_f^* = -P^* \mathbf{I} + \mu_f^* (\nabla \mathbf{v}^* + \nabla \mathbf{v}^{*T}) = \boldsymbol{\sigma}_f^* = -P^* \mathbf{I} + 2\mu_f^* \mathbf{e}_f^*, \quad (71)$$

where the deviatoric part of $\boldsymbol{\sigma}_f^*$ is related to the symmetric velocity gradient $\mathbf{e}_f^* = 1/2(\nabla \mathbf{v}^* + \nabla \mathbf{v}^{*T})$ defined in (71). Let us first consider (70)'s r.h.s. Integrating (70) into the infinitesimal volume $\Omega(t^*) = S^*(z^*, t^*) \times \epsilon_z$, based upon the product between the section S^* with the infinitesimal thickness ϵ_z along z direction, whilst using the divergence theorem leads to

$$\int_{\Omega(t^*)} \nabla \cdot \boldsymbol{\sigma}_f^* = \int_{S^*(z^*+\epsilon_z, t)} \boldsymbol{\sigma}_f^* \cdot \mathbf{n} ds + \int_{S^*(z^*, t)} \boldsymbol{\sigma}_f^* \cdot \mathbf{n} dS + \epsilon_z \int_{C^*} \boldsymbol{\sigma}_f^* \cdot \mathbf{n} dC^* \quad (72)$$

482 where \mathcal{C} is again the contour of surface $S^*(z^*, t)$. (72) gives

$$\begin{aligned} \int_{\Omega(t^*)} \nabla \cdot \boldsymbol{\sigma}_f^* &= -S^* (\overline{P}(z^* + \epsilon_z) - \overline{P}(z^*)) \mathbf{e}_z + \epsilon_z \int_{\mathcal{C}^*} \boldsymbol{\tau}_w^* d\mathcal{C}^* \\ &+ 2\mu_f \int_{S^*(z^* + \epsilon_z, t) \cup S^*(z^*, t)} \mathbf{e}_f^* \cdot \mathbf{e}_z, \end{aligned} \quad (73)$$

where \overline{P} denotes the surface average pressure, i.e $\overline{P} = \int_{S^*} P ds / S^*$. Let us now take care of the integral of (70)'s l.h.s. For this we use the Reynolds transport theorem (29) for vector field $\boldsymbol{\Psi} = \rho_f^* \mathbf{v}^*$ which leads to

$$\frac{d}{dt^*} \int_{\Omega(t^*)} \rho_f^* \mathbf{v}^* d\Omega = \int_{\Omega(t^*)} \left(\frac{\partial(\rho_f^* \mathbf{v}^*)}{\partial t^*} + \nabla \cdot (\rho_f^* \mathbf{v}^* \mathbf{v}^*) \right) d\Omega \quad (74)$$

Developing the r.h.s integrand term of (74) leads to

$$\frac{\partial(\rho_f^* \mathbf{v}^*)}{\partial t^*} + \nabla \cdot (\rho_f^* \mathbf{v}^* \mathbf{v}^*) = \mathbf{v}^* \left(\frac{\partial \rho_f^*}{\partial t^*} + \nabla \cdot (\rho_f^* \mathbf{v}^*) \right) + \rho_f^* \left(\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^* \nabla \cdot \mathbf{v}^* \right) \quad (75)$$

Since the local version of the integrated mass balance (4)'s reads

$$\partial_{t^*} \rho_f^* + \nabla \cdot (\rho_f^* \mathbf{v}^*) = 0, \quad (76)$$

this permits to simplify the r.h.s of (75), so that using (76) in (75) and (74) then leads to the equality

$$\frac{d}{dt^*} \int_{\Omega(t^*)} \rho_f^* \mathbf{v}^* d\Omega = \int_{\Omega(t^*)} \rho_f^* \frac{d\mathbf{v}^*}{dt^*} d\Omega \quad (77)$$

Using now the domain $\Omega(t^*) = S^*(z^*, t^*) \times \epsilon_z$, integrating (70)'s l.h.s using (77) and (73) whilst dividing both sides by ϵ_z , projecting the vectorial equality along \mathbf{e}_z and taking the limit $\epsilon_z \rightarrow 0$ leads to

$$\frac{d}{dt^*} \int_{S^*} \rho_f^* w^* ds = -S^* \frac{\partial \overline{P}}{\partial z^*} + \int_{S^*} \mu_f^* \frac{\partial^2 w^*}{\partial z^{*2}} ds + \int_{\mathcal{C}^*} \boldsymbol{\tau}_w^* d\mathcal{C}^*. \quad (78)$$

We now consider axi-symmetric wave perturbations (pipe breathing perturbations) for which $\boldsymbol{\tau}_w^*$ is uniform along \mathcal{C}^* . Furthermore, a core acoustic wave velocity for which the longitudinal velocity field w^* is uniform and denoted W^* is also considered. On (78)'s l.h.s, there is small contribution of boundary layers where w^* tends to zero near the boundary, but these regions are small. More precisely denoting δR_0^* the boundary layer thickness, as $\delta \equiv \sqrt{1/\epsilon Re_p}$ with $\delta \ll 1$, $\epsilon = R_0^*/L \ll 1$ and $Re_p = R_0^* c_p^*/\nu \gg 1$. The correction from these regions to (78)'s l.h.s are $O(\delta)$. Similarly, on (78)'s r.h.s, a core acoustic perturbation having constant longitudinal velocity W^* is such

that $\partial^2 W^*/\partial z^{*2} = 0$, so that a non-zero contribution of the viscous dissipation term $\partial^2 w^*/\partial z^{*2}$ is found only within the boundary layer. This leads to

$$\frac{d}{dt^*}(S^* \rho_f^* W^*)(1 + O(\delta)) = -S^* \frac{\partial \bar{P}}{\partial z^*} + 2\pi R^* \tau_w^*(1 + O(\delta)), \quad (79)$$

Furthermore, from integrated mass-balance (4) one gets

$$\frac{d}{dt^*}(\rho_f^* S^*) + \rho_f^* S^* \partial_{z^*} W^* = 0, \quad (80)$$

so that, finally, using (80) in (79) leads to

$$\rho_f^* \partial_{t^*} W^*(1 + O(\delta)) = -\frac{\partial \bar{P}}{\partial z^*} + \frac{2}{R^*} \tau_w^*(1 + O(\delta)), \quad (81)$$

Since $\delta \ll 1$, $O(\delta)$ corrections are generally neglected so that (81) is identical with the second line of (33). (81)'s r.h.s thus involve a source term proportional to wall shear stress and responsible for the wave damping. It is directly related to the viscous dissipation arising within the liquid boundary layer. Hydraulic water-hammer modeling has mainly been focused on dissipative viscous losses, whilst disregarding (FSI) effects. More precisely, this means that most of the hydraulic literature ignores the stress-acceleration equations (34) and only consider (33) discarding Poisson's coupling, i.e in the $\nu_s \rightarrow 0$ limit. In this limit, the set of four-(FSI) equations introduced in (33)-(34), simplifies to two coupled first order hyperbolic equations for the fluid pressure P and longitudinal velocity W^* [93, 94]

$$\left[\partial_{t^*} + \begin{pmatrix} 0 & \rho_f^* c_p^{*2} \\ \frac{1}{\rho_f^*} & 0 \end{pmatrix} \partial_{z^*} \right] \begin{pmatrix} P^* \\ W^* \end{pmatrix} = -\frac{2\tau_w^*}{\rho_f^* R_0^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (82)$$

The coupled first order hyperbolic problem (82) can be re-casted into a second order wave equation, in dimensionless form, using $z^* = ZL$ as quoted in [13, 95]

$$[\partial_t^2 - \partial_Z^2] P = 2\delta \frac{\partial \tau_w}{\partial Z}. \quad (83)$$

483 When viscosity effects are taken into account, one needs a wall shear-stress model.
484 This issue turns out to be central, for a correct evaluation of the wave damping pre-
485 diction and this is why it has attracted so many studies in the literature. But, as
486 previously stated, it is a subtle issue, for which there is still surprisingly remaining
487 open questions at the present state of the art, as will be discussed at the end of this
488 section. Furthermore, the dimensionless pressure wave (83) shows that the dissipation

489 rate is slow since the r.h.s is small, i.e $O(\delta)$. This specific asymptotic property has
 490 nevertheless been used only recently, and will thus be discussed in section 4.1.

A first intuitive empirical and popular wall shear-stress model is built from the extension of the steady-state Darcy-Weisbach friction law (Cf. [19, 96, 13]) to hydraulic transient. The use of the redesigned quasi-steady wall shear rate τ_{qst}

$$\tau_{qst}^*(z^*, t^*) = \frac{\rho_f^* f_{DW} W^*(z^*, t^*) |W^*(z^*, t^*)|}{8}, \quad (84)$$

491 through the dimensionless Darcy-Weisbach coefficient f_{DW} , is nevertheless question-
 492 able for transient investigations. Poor experimental agreement is sometimes found
 493 using this dissipation model [97, 98, 99]. Furthermore, even if the extension of steady
 494 friction to quasi-steady one seems a plausible model at first sight, one should bear in
 495 mind that (84) has not been established within any rigorous derivation framework.
 496 Also, this wall-shear stress model considers that the rapid wave shear is coupled with
 497 the steady-state velocity. How such coupling between steady and transient happens
 498 for the shear stress is elusive. We will consider this issue in a much deeper way in
 499 section (4.2), so as to address the fluid mechanics behind a possible coupling between
 500 the steady-flow and the unsteady one, still not well understood at the present state of
 501 the art.

This quasi-steady model has then been enriched to improve the experimental atten-
 uation's predictability. Several classes of model, [16] have been established: (i) instan-
 taneous material acceleration-based (IMAB) models and, (ii) weighting function-based
 (WFB). Although conceptually different, these two approaches seek to account for the
 same physical observation, that the near-wall dynamics does not instantaneously re-
 spond to the core velocity variations. The energetic dissipation emerges from the time
 response delay between the central part of the flow and its boundary layer. Finally,
 these models are based on a decomposition of the total wall shear rate into a quasi-
 steady component, via the use of the Darcy-Weisbach model (84), and a transient one
 τ_{tr}^* , [13]

$$\tau_w^* = \tau_{qst}^* + \tau_{tr}^*. \quad (85)$$

502 The τ_{tr}^* component is then expected to fill the gaps between the model predictions
 503 and the experimental observations, when only a quasi-steady wall shear stress model
 504 is considered.

Instantaneous material acceleration-based (IMAB) model. IMAB relies on semi-empirical observations and assumes a linear variation of the transient wall shear stress τ_{tr}^* , with respect to the mean flow acceleration $\partial_{t^*} W^*$. These models arise from the experimental work of [100]. The author analyzed the turbulence structure in a pressurized flow with or without orifices. [100] highlighted the time delay between the response of boundary lines with respect to the mean flow variations. He then proposes the following transient wall shear stress model

$$\tau_{tr}^* = \frac{\rho_f k_3 R_0^*}{2} \partial_{t^*} W^*, \quad (86)$$

where k_3 account for the boundary response deviation. When re-injecting the above transient wall shear stress expression into the hyperbolic constitutive equation set (82), it turns out that no energetic damping arises, [101]. Indeed, the time-derivatives of both (82) and (86) can be factorized so that no source term remains in the hyperbolic system r.h.s. However, the structure of the hyperbolic system, i.e. its eigenvalues and eigenvectors, are modified by k_3 . The wave speed is thus modified by a $1/\sqrt{1+k_3}$ prefactor. To account for energetic losses, the (IMAB) model was then later completed by [102, 98]. The authors added a convective term to the transient shear-stress (86) and proposed that

$$\tau_{tr}^*(x, t) = \frac{\rho_f^* k_3 R_0^*}{2} (\partial_{t^*} W^* + c_p^* \cdot \text{sgn}(W) |\partial_{z^*} W^*|), \quad (87)$$

where $\text{sgn}(W)$ stands for the accelerating or decelerating transient flow phase. This inertial contribution, similar to the Navier's inertial terms, creates a source term in the hyperbolic system (82). Consequently, the (IMAB) model intends to model both the attenuation and the phase shift of overpressure waves. Some authors, [103, 104, 105], also developed a slightly distinct model by using a second semi-empirical parameter k_3'

$$\tau_{tr}(x, t)^* = \frac{\rho_f^* R_0^*}{2} \left(k_3 \partial_{t^*} W^* + k_3' c_p^* \cdot \text{sgn}(W^*) |\partial_{z^*} W^*| \right) \quad (88)$$

505 and then distinguished the phenomenon of wave speed deviation from the damping
 506 one, [101]. When the single coefficient model is used in (87), an approximation arising
 507 from theoretical arguments is used to estimate the Reynolds number dependence of
 508 k_3 , [106]

$$k_3 = \frac{\sqrt{0.00476}}{2}, \text{ if } Re \leq 2000, \quad (89)$$

$$k_3 = \frac{1}{2} \sqrt{\frac{7.41}{Re^{\log\left(\frac{14.3}{Re^{0.05}}\right)}}}, \text{ else,} \quad (90)$$

509 where $Re = W_0 R_0 / \nu_f$ is the Reynolds number, W_0 is a reference axial fluid velocity
510 magnitude and ν_f is the fluid kinematic viscosity. The review of [13] provides an
511 excellent state-of-the-art of (IMAB) models. In addition to the use of semi-empirical
512 models, some analytical developments relying on a weighting function-based (WFB)
513 approach, have also been carried out.

Weighting function-based (WFB) model. WFB seeks for an analytical description of the viscous shear energetic losses. One of the first noteworthy contributions has been provided by [107]. In the low-Mach number acoustic framework, i.e. neglecting Navier's inertial terms and decomposing the fluid variables into steady and perturbed components, the authors performed a Laplace domain analysis of the fluid mass and momentum conservation equations. They found a radial- Bessel-dependent solution for the axial fluid velocity $W^*(r)$. The transient wall shear stress was then also derived, directly from a direct computation of the shear

$$\tau_{tr}^* = \rho_f^* \nu_f^* \partial_r^* W^* \Big|_{r=R_0}. \quad (91)$$

This theoretical approach is also consistent with [7] for the maximum overpressure prediction. Furthermore, [13] highlighted the relevance of a previously defined dimensionless parameter associated with the pressure waves damping, which is now known as the water-hammer small parameter δ

$$\delta^2 = \frac{\nu_f^* L^*}{c_p^* R_0^{*2}} = \frac{\nu_f^*}{R_0^{*2}} \cdot \frac{L^*}{c_p^*} \equiv \frac{t_c^*}{t_{vis}^*} \equiv \frac{1}{\epsilon Re_p}, \quad (92)$$

where $t_{vis}^* = R_0^{*2} / \nu_f^*$ is the viscous diffusion time-scale within the boundary layer, and $t_c^* = L^* / c_p^*$ is the advective time-scale of the wave, $\epsilon = R_0^* / L^*$ is the inner pipe's radius to its length ratio and $Re_p = R_0^* c_p^* / \nu_f^*$ is the pulse re-scaled Reynolds number. This small parameter is the cornerstone of the asymptotic analysis of water hammer as will be discussed in section 4.1. Approximatively at the same time as Holmboe, Zielke [108, 109] delivered a famous analysis of fully developed laminar boundary layer and derived a diffusion equation of the axial fluid velocity, forced by the longitudinal pressure gradient

$$\left[\partial_t^* - \nu_f^* \frac{\partial_r^*}{r^*} (r^* \partial_r^*) \right] W^* = -\frac{1}{\rho_f^*} \partial_{z^*} P^*. \quad (93)$$

Solving (93) in the Laplace domain, and then performing an inverse Laplace transform using Cauchy's residue theorem leads to a time-convoluted form of the wall shear stress

$$\tau_w^* = \underbrace{\frac{4\rho_f^*\nu_f^*}{R_0^*}W^*}_{\tau_{qst}^*} + \underbrace{\frac{2\rho_f^*\nu_f^*}{R_0^*}\int_0^{t^*}\Theta_{Zielke}(t^*-\tau^*)\partial_\tau^*Wd\tau^*}_{\tau_{tr}^*}, \quad (94)$$

514 where $\Theta_{Zielke}(t)$ is the convolution kernel. For practical use of this kernel Zielke has
 515 provided some approximate fitting of the form

$$\Theta_{Zielke}(t^*) = \sum_{i=0}^5 m_i \left(\frac{t^*}{t_{vis}^*}\right)^{\frac{i-1}{2}}, \quad \text{if } \frac{t}{t_{vis}} \leq 0.02, \quad (95)$$

$$\Theta_{Zielke}(t^*) = \sum_{i=0}^4 e^{-n_i \frac{t^*}{t_{vis}^*}}, \quad \text{if } \frac{t^*}{t_{vis}^*} > 0.02, \quad (96)$$

where n_i, m_i parameters can be found in [109]. The presence of the τ_{qst} term in (94) arises from the fact that W. Zielke did not decompose his fluid velocity and pressure fields into steady and perturbed components as it is classically done in the low Mach number acoustic framework. This point is highlighted here and will be discussed further in section 4.1. It is interesting to point out that the W. Zielke's kernel is convoluted with the fluid mean acceleration. It then reflects that the underlying physical phenomenon governing the energetic damping in the (WFB) models and in the (IMAB) ones is the same (Cf. (86)), and relies on the non-instantaneous response of the boundary layer with respect to the core acceleration. Whilst the (IMAB) assumes a direct linear relation between the wall shear stress and the mean acceleration using k_3 , the (WFB) embeds all the historic mean flow variations through a time convolution with $\Theta_{Zielke}(t)$. The scaling of the W. Zielke convolution kernel is obtained regarding both the first term of (95) and by setting up the characteristic advective time scale $\tau_c = \frac{L^*}{c_p^*}$, thus leading to

$$\Theta_{Zielke} \equiv O\left(\frac{1}{\delta}\right), \quad (97)$$

with δ as introduced in (92). The convolution kernel scaling thereby merges with the conclusion of [107]. Finally, the author confronted his theory to the experimental data of [107] and a very close agreement was found as revealed in Figure 6. [110] extended W. Zielke's work into an asymptotic analysis which supposes a near wall inner concentrated laminar boundary layer, or a skin friction model. The flow in the core area, bulk or outer region, is then considered as inviscid. The Figure 7a provides a schematic representation of the [110]'s asymptotic model. During a transient event, the authors supposed that: "*the fluid remains divided into a turbulent core and laminary boundary layer and that, the boundary-layer thickness remains constant.*" The

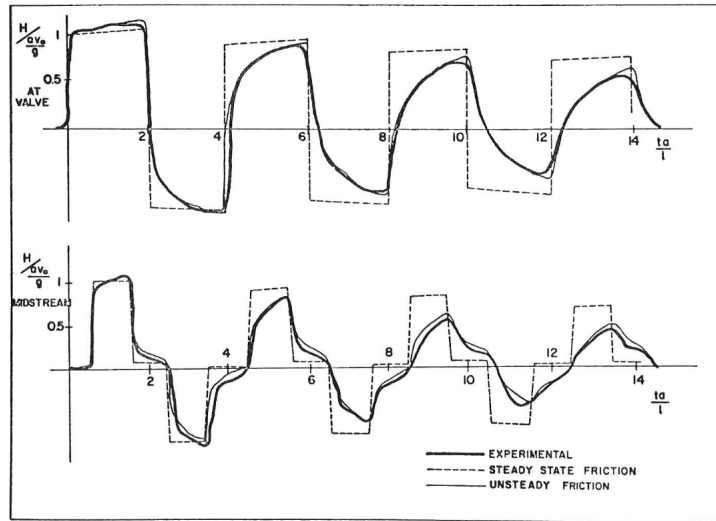
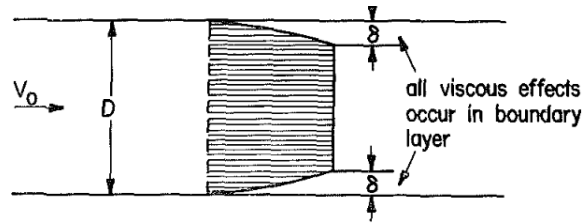
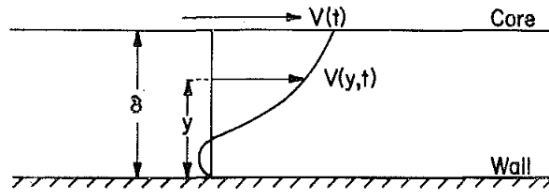


Figure 6: [109]’s theoretical predictions compared to [107]’s experimental data. The following notation are used: H the hydraulic head line, a the pulse wave speed.



(a) Laminar boundary layer concept, [110].



(b) Axial velocity matching at the fluid boundary layer interface, [110].

Figure 7: [110] boundary layer model for pressure waves energetic damping.

boundary layer thickness, and hence its dynamics, are thus governed by the preexisting flow regime. The dimensionless steady boundary layer thickness δ_{st} , follows from the

equilibrium of steady state viscous terms with the initial the pressure gradient

$$\delta_{st} = \frac{4}{f_{DW} Re}. \quad (98)$$

Two approaches are then proposed to evaluate the attenuation of pressure waves. The first one relies on solving the inner region problem associated with the diffusion equation upon the axial velocity in the boundary layer. A velocity matching at the boundary layer interface is then performed to ensure kinematic continuity conditions (Cf. Fig.7b). The second method involves energy balance from taking into account energy losses via the quadratic integral of the wall shear-stress over the boundary layer thickness. [97] extended a similar asymptotic approach from designing a two-dimensional model to describe the viscous losses in both the laminar and turbulent regions. By considering the pipe as an in-extensible solid, i.e. without radial dilatation, they decomposed the fluid into a succession of interconnected concentric ring of small thicknesses. Each layer of fluid is coupled to others via momentum transfers, radial kinematic and shear stress continuity, as depicted in Figure 8. Ensuring mass and

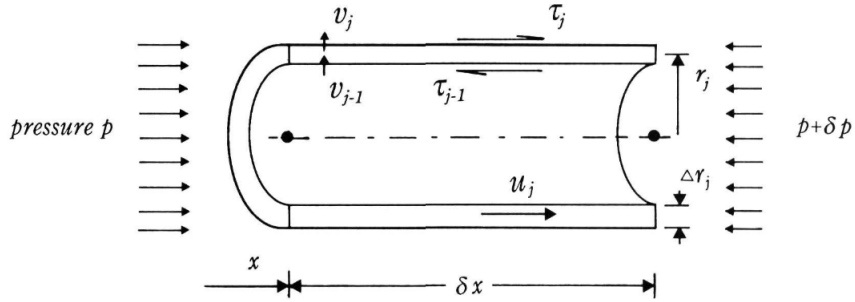


Figure 8: Momentum balance on a cylinder element, [97]

momentum conservation within each rings, [97] derive a coupled hyperbolic system forced by the radial transfer. For turbulent flow regimes, a five-region model has been adopted to model the shear rate. The authors found: "*pleasantly surprising that Zielke's expression is so successful even through his assumed (laminar) initial velocity profile differs markedly from reality*". [113] extended the laminar framework of [110] by taking into account the Reynolds-dependence of the flow in its convolution kernel. The flow is again divided into two regions: (i) the acoustic outer (*bulk*) region where the velocity field is radially uniform (ii) the inner boundary layer (*annulus*) where viscous effects are concentrated. The dimensionless boundary layer thickness is once again set

up to match with the preexisting flow conditions. Inspired by the experimental results of [114], [110] highlights the effects of the initial Reynolds number upon the convolution kernel structure as illustrated in Figure 9. In the boundary layer, a diffusion equation is

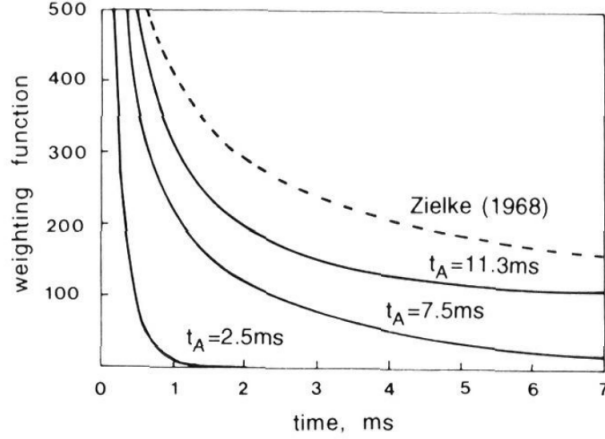


Figure 9: [113] discussion on the convolution kernel Reynolds-dependence according to the experimental work of [114].

once again derived in [113], which merges with [109, 110] analysis. The improvement and elegance of [113]’s model lies in the inner-outer *asymptotic matching* handling. The authors finally yielded the derivation of a Reynolds-dependent transient wall shear stress kernel expression (via the δ_{st} parameter)

$$\Theta(t)_{\text{Vardy et al. (1993)}} \approx \frac{1}{\delta_{st}} \sum_{k \in \mathbb{N}^*} e^{-[\frac{k\pi}{\delta_{st}}]^2 \frac{t^*}{v_{vis}^*}} \equiv O\left(\frac{1}{\delta_{st}}\right). \quad (99)$$

The experimental contributions of [115, 116] on the turbulent kinematic viscosity distribution in pipes, permitted [106] to extend [113] to account for higher Reynolds numbers ($Re \gg 10^5$). This new model, valid in a smooth pipe, is based on an idealized radial distribution of the turbulent kinematic viscosity in the boundary-layer as shown in Figure 10a. In this work, a core viscosity ν_c to wall viscosity ν_w ratio is introduced and used to characterize the turbulent kinematic distribution

$$\frac{\nu_c}{\nu_w} \equiv \sigma_{\text{Vardy}} \approx 0.173 (f_{DW} Re)^{1.12}, \quad (100)$$

whilst a new dimensionless boundary layer thickness is set up to scale in

$$\frac{b}{R_0} = \delta_{st} \frac{u_c}{U} \frac{\sigma_{\text{Vardy}} - 1}{\ln(\sigma_{\text{Vardy et al.}})}, \quad (101)$$

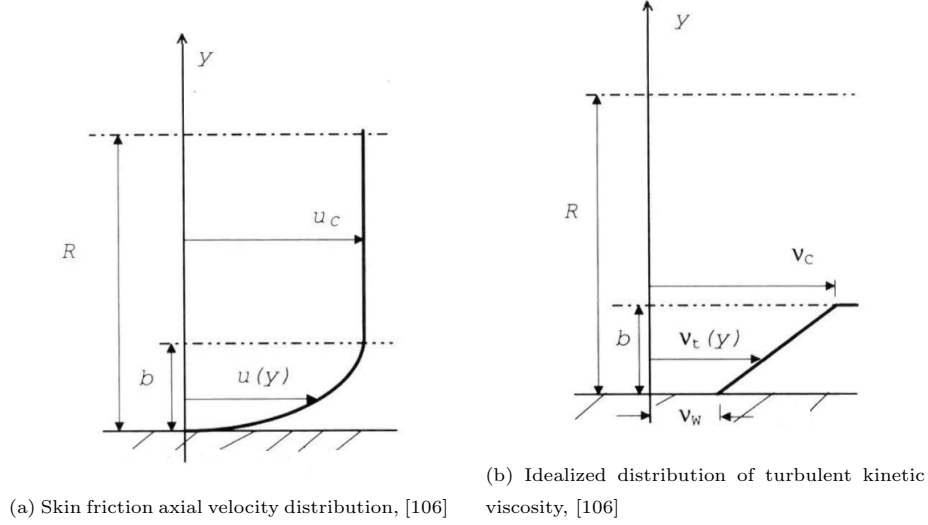


Figure 10: Idealized velocity and viscosity distribution for the skin friction model of [106].

where u_c is the uniform core velocity, U the mean flow velocity per section and b is the dimensional boundary layer thickness. In the limit σ_{Vardy} tends to unity, the dimensionless boundary layer thickness of [113], presented in (98), is recovered. A similar asymptotic analysis as in [113] is carried out in [106] yielding to the derivation of a modified convolution kernel

$$\Theta(t)_{\text{Vardy et al. (1995)}} = \frac{A^* e^{-B^* \frac{t^*}{t_{vis}^*}}}{\sqrt{\frac{t^*}{t_{vis}^*}}}, \quad A^* = \frac{1}{2\sqrt{\pi}}, \quad B^* = 0.135 Re^{\log_{10}\left(\frac{14.3}{Re^{0.05}}\right)}. \quad (102)$$

For $t \equiv O\left(\frac{L}{c_p}\right)$, one finds

$$\Theta(t)_{\text{Vardy et al. (1995)}} \equiv O\left(\frac{1}{\delta}\right). \quad (103)$$

Under the hypothesis of constant transient acceleration, i.e. if $\partial_t W$ constant in (94), [106] derived a straightforward relation between the semi-empirical deviation constant k_3 , and their inertial shear coefficient B^*

$$k_3 \approx \frac{1}{2\sqrt{B^*}}. \quad (104)$$

This relation provides a plausible justification for the derivation of (89)-(90). [106] point-out the Reynolds-dependence of parameter k_3 . Additional discussions concerning the relations between [106]'s model and the (IMAB) ones are developed within

[102]. Finally, it is important to mention that many expressions have been proposed to model A^* and B^* from model (102) derived from several approximations. For example, [117] extend their previous model by modifying both (i) their idealized turbulent viscosity distribution (until then considered as infinite in the acoustic core) and, (ii) considering the boundary-layer thickness as independent from Reynolds. [117] reached a new convolution kernel via a modification of the B^* coefficient

$$B^* = \frac{Re^{\log_{10}\left(\frac{15.29}{Re^{0.0567}}\right)}}{12.86}, \quad (105)$$

which appear seemingly valid over a wider range of Reynolds numbers, i.e $Re \in [2 \cdot 10^3, 10^8]$. Ref. [118] gives a final answer in order to take pipe's roughness into account. It is noteworthy discussing [117, 119, 118]'s assumptions. The authors indeed supposed that the idealize turbulent eddy viscosity profile does not instantaneously respond to the mean flow variations. Quoting the authors: "*The change in the effective viscosity occurs during the period when the shape of the velocity profile is changing, not during the earlier period when the velocity amplitude increases uniformly. That is, there is a phase lag between the step change in mean velocity and the resulting change in the effective viscosity.*" This assumption is known in this literature as the "*frozen viscosity*" model and has been later-on analyzed by [120, 118, 13]. [120, 119, 118, 13] confirmed the relevance of the "*frozen viscosity*" approach as long as the shear pulse diffusion through the viscous sub-layer time scale, i.e. $\tau_{diff,sublayer} \equiv \frac{\sqrt{2}R_0}{u_*}$ with $u_*^2 = \frac{f_{DW}W_0^2}{8}$ the friction velocity, is smaller than the advective wave time scale, i.e. $\tau_{adv} \equiv \frac{L}{c_p^*}$. A validity condition then follows

$$\frac{4R_0c_p}{\sqrt{f_{DW}LW_0}} \gg 1, \quad (106)$$

or otherwise, invoking the definition of δ and δ_{st} in (92) and (98), respectively

$$\delta^2 \ll \sqrt{f_{DW}}\delta_{st}, \quad (107)$$

516 It obviously follows from the condition (107), that it should not be expected close
 517 agreement between the [117, 119] model predictions and the experimental data, for
 518 observation times upper than $\tau_{diff,sublayer}$. To test these models, comparisons between
 519 theoretical/semi-empirical predictions and experimental data were carried out in [98].
 520 The authors found excellent agreement for all the models herein presented. Other
 521 experimental validations were carried-out in [122, 99]. [122, 99] performed a series of
 522 experiments in an elastic copper pipe of length $L = 98m$, inner radius $R_0 = 8 \cdot 10^{-3}m$

Friction Model Category	Model Name	Equation
Steady and quasisteady friction	Darcy Weisbach	$f_s = \frac{8fL}{\pi^2 g D^5} Q^2$
Instantaneous mean flow velocity	Hino	$f_u = 0.188 \left(V \sqrt{\frac{4L}{v\pi a}} \right)^{-\frac{1}{2.85}}$
Instantaneous mean flow velocity and local acceleration	Daily	$\frac{f_u}{f_s} = 1 + c_2 \frac{2D}{f_s V^2} \frac{dV}{dt}$
Instantaneous mean flow velocity, local acceleration, and convective acceleration	Brunone	$f_u = \frac{kD}{V V } \left(\frac{\partial V}{\partial t} - a \frac{\partial V}{\partial x} \right)$
Instantaneous mean flow velocity, local acceleration, and convective acceleration	Vitkovský	$f_u = \frac{kD}{V V } \left(\frac{\partial V}{\partial t} - a \cdot \text{sign}(V) \left \frac{\partial V}{\partial x} \right \right)$
Instantaneous mean flow velocity and past velocity changes weights	Convolutional model	$f_u = \frac{16\mu}{\rho D^2 A} \left(\frac{\partial Q}{\partial t} * W(\tau) \right)$
	Vardy and Brown weighting function	$W(\tau) = \frac{A^\times e^{-B^\times \tau}}{\sqrt{\tau}}$

Figure 11: Friction coefficient f_u , i.e. $\tau_{tr} \equiv \frac{\rho_f f_u W|W|}{4}$ for the characterization of transient overpressure waves damping in pipes. Original chart of [121].

523 and wall thickness $e = 10^{-3}m$ for a wide Reynolds number range, $R_e \in [1100, 15800]$.
524 Despite their conclusions merge those of [98], they highlighted that (WFB) models:
525 ”have to be singled out. These models predict almost superbly the wave front shape and
526 preserve the frequency. However, it is symptomatic that for higher Reynolds number
527 (over approx 10^5) the damping effect observed in the calculated courses is greater than
528 in the experimental ones”. An observation also shared recently by [123, 124]. An
529 overview of all damping models can be gathered from the work of [121] from which
530 Figure 11 has been taken.

531 Beside pipe wall’s boundary-layer related damping models, some analysis suggest that
532 acoustic impedance discontinuities might also affect the phase as well as the damping
533 of water-hammer waves [70, 111]. How to handle and model the wave interaction
534 nearby discontinuities is still under debate [70, 111] and many possible improvements
535 are suggested from a huge acoustic literature that this review does not cover. However,
536 a recent asymptotic analysis [112] in complex geometries might help improving long-
537 wavelength approximation models when impedance discontinuities have to be taken
538 into account.

539 Finally, it is worth mentioning that despite of all these modeling efforts, water-
540 hammer friction models, are yet only partially successful when confronted with exper-
541 imental observations. One possible explanation — maybe an optative one— is that

542 some issues might have been missed. As mentioned earlier, some researches evoked
543 the possible influence of turbulence [106, 117] of the "steady" flow to influence the
544 water-hammer wave friction propagating through it. Also, at the roots of the quasi-
545 steady Darcy-Weisbach approximation (84) is a coupling between the water-hammer
546 wave and the steady flow underneath. In both cases, the friction model depends on
547 the coupling arising between the steady base flow and water-hammer wave, but, to
548 our knowledge, the mechanical origin of this coupling has not been clearly elucidated
549 yet. A possible path toward clarifying this issue will be described in section 4.2 where
550 a recent asymptotic study is discussed.

551 **3. Poisson coupling in more complex contexts**

552 *3.1. More complex vibrating degree of freedom*

553 I should be pointed out that, depending on the system's Degree Of Freedom
554 (DOF), several additional vibrations must be taken into account (e.g. torsion, bending
555 etc.). The recent reviews of [15, 17] provide insights into this modeling when all pipe's
556 DOF are considered, leading to an increasingly larger number of equations (e.g eight
557 DOF leads to sixteen coupled equations [17]). Obviously, in this context of extended
558 DOF analysis, only very simple pipe's configurations have been considered without
559 taking into account friction coupling.

560 *3.2. Visco-elastic FSI effects*

Water hammer pressure waves propagating into liquid filled pipes having visco-
elastic solid walls experience a strong attenuation both in hydraulic contexts [88]
as well as bio-mechanical ones [125, 126, 127, 128, 18]. By the last quarter of the
twentieth century, Rieutord et al. [129] demonstrated that visco-elastic stress-strain
response of a pipe wall has a strong influence on the water-hammer waves propaga-
tion. A few years later some experiments from the same team [130] complemented
with a 1D theoretical model [131] confirmed their first observations supported by a
theoretical understanding of it. As opposed to the case of a purely elastic solid, when
considering a visco-elastic wall, the water-hammer wave velocity becomes dispersive
[125, 40, 132], i e, the wave velocity depends on the considered frequency. Secondly,
the wave amplitude is exponentially damped because the wave velocity acquires an
imaginary component coming from visco-elastic dissipation [133, 31, 104, 134]. Both

damping and dispersivity are related to the creep-functions J^* defined as the strain to stress ratio. More precisely considering the normal stress σ_{rr}^* and the longitudinal deformation ξ^* ,

$$J^* = \frac{\epsilon_{rr}^*}{\sigma_{rr}^*} \equiv \frac{\partial_r \xi^*}{\sigma_{rr}^*} \equiv \frac{1}{E^*}, \quad (108)$$

where the strain tensor ϵ^* and displacement gradients are related by

$$\epsilon^* = \frac{1}{2} \left(\nabla^* \xi^* + \nabla^{*T} \xi^* \right). \quad (109)$$

561 A similar exponential damping is also present in water-hammer wave propagation
562 within purely elastic pipes as discussed in 2.3. Nevertheless it results from a very
563 distinct mechanism associated with the viscous dissipation within boundary layers
564 [95, 135]. In many cases however, the visco-elastic damping dominates over the vis-
565 cous one. This visco-elastic damping is filtering high-frequency oscillating elastic
566 modes. Since water-hammer wave propagation has been found material dependent,
567 the pioneering studies of [129, 130, 131] inspired many others, following the similar
568 footsteps, combining experimental measurements with modeling associated with solid
569 creep-functions displaying Kelvin-Voigt behavior (Cf [136, 137, 31, 104, 138] among
570 many others). The applicative interest and the relevance of the topic motivated many
571 further studies whereby one could enrich the Kelvin-Voigt model [139, 140] to better
572 fit with observations. Alternatively, some authors also included both solid visco-elastic
573 damping and fluid one, through time-convolution, shear-stress models [141, 142, 143].
574 Because the modelling relies on many parameters, combined with time-convolution
575 many approaches are possible to match experiments raising a number of questions
576 including wave-speed calibration in visco-elastic pipes [138, 144]. The influence of the
577 visco-elastic stress response has been more recently considered in a Fluid-Structure-
578 Interaction (FSI) context as more extensively discussed in the recent review of [15].
579 In this context Kelvin-Voigt solid responses of the creep function have also been used
580 in FSI four-equations models [145, 146, 147] in order to improve the relevance of the
581 modeling. Furthermore, for improving data fitting, a series of Kelvin-Voigt units are
582 often considered [145, 148, 146, 147]. However, in these previous modeling efforts, the
583 creep-function parameters are calibrated [149] not only to describe the visco-elastic
584 properties of the solid but also the considered pipe configuration associated with a
585 specific length, thickness, diameter and boundary conditions. Let us now first discuss
586 in more details visco-elastic models without FSI effects.

587 *3.2.1. Two-equations water hammer models within visco-elastic pipes*

588 When considering an acoustic fluid region whilst disregarding the influence of vis-
 589 cous boundary layer, the dimensionless low-Mach, long-wavelength mass conservation
 590 and momentum balance in the core fluid region (the outer region of the asymptotic
 591 framework) leads to the following two-equation hyperbolic problem [149, 43, 135, 65]

$$\partial_\tau P + \mathcal{C}^2 \partial_Z W = -2\alpha \mathcal{C}^2 \partial_\tau \xi \Big|_{R=1}, \quad (110)$$

$$\partial_\tau W = -\partial_Z P. \quad (111)$$

592 In (110)-(111), as used in (38) a Joukowski's dimensionless pressure is chosen, i.e
 593 $P^* = (\rho_f c_p^* W^*) P$, built upon reference steady fluid velocity W^* , also used for dimen-
 594 sionless velocity W defined as $w^* = W^* W$. As in section 2.2.1 dimensionless time
 595 is again based upon wave traveling reference time, i.e $t^* = (L/c_p^*) \tau$, associated with
 596 the pulsed wave-speed c_p^* and longitudinal reference length L^* , also used for dimen-
 597 sionless longitudinal length $Z^* = L^* Z$. Finally the dimensionless longitudinal solid
 598 displacement $\xi^* = \xi_0^* \xi$ is related to a reference length $\xi_0^* = R_0^* \alpha W^* / c_p^*$ the origin of
 599 which comes from kinematic boundary conditions [135]. From the r.h.s of (110) and
 600 the long-wavelength approximation, continuity relations of the radial velocity at the
 601 wall $\partial_\tau \xi \Big|_{R=1}$ is automatically satisfied. The dimensional version of (110)'s r.h.s was
 602 derived from Reynolds transport theorem using (30) in section 2.1 as resulting from a
 603 kinematic driven FSI coupling. Writing (110)-(111) in Fourier space leads to

$$i\omega \tilde{P} + \mathcal{C}^2 \partial_Z \tilde{W} = -2i\alpha \omega \mathcal{C}^2 \tilde{\xi} \Big|_{R=1}, \quad (112)$$

$$i\omega \tilde{W} = -\partial_Z \tilde{P}. \quad (113)$$

These equations are complemented with Fluid/Solid interface boundary conditions.
 To express them, one first needs to define the linear constitutive relation between the
 solid stress tensor $\boldsymbol{\sigma}^*$ and strain tensor $\boldsymbol{\epsilon}^*$. In frequency domain and dimensional form,
 these constitutive visco-elastic solid equations read

$$\tilde{\boldsymbol{\sigma}}^* = \tilde{\lambda}_s^*(\omega^*) \text{Tr}(\tilde{\boldsymbol{\epsilon}}^*) \mathbf{I} + 2\tilde{\mu}_s^*(\omega^*) \tilde{\boldsymbol{\epsilon}}^* \quad , \quad \text{with,} \quad \tilde{\boldsymbol{\epsilon}}^* = \frac{1}{2} \left(\nabla^* \tilde{\boldsymbol{\xi}}^* + \nabla^{*\text{T}} \tilde{\boldsymbol{\xi}}^* \right). \quad (114)$$

604 where $\text{Tr}(\tilde{\boldsymbol{\epsilon}}^*) = \nabla^* \cdot \tilde{\boldsymbol{\xi}}^*$ is the trace of tensor $\tilde{\boldsymbol{\epsilon}}^*$ which also equals the divergence
 605 of the displacement vector. $\tilde{\lambda}_s^*(\omega^*)$ and $\tilde{\mu}_s^*(\omega^*)$ are the generalized Lamé coefficients,
 606 dependent on pulsation ω^* . These coefficients are usually found experimentally from
 607 using Dynamic Mechanical Analysis (DMA) measurements). Since dimensionless for-
 608 mulations are better for comparing various models in the same reference framework,

609 the Joukowski pressure is used as a reference pressure, for dimensionless stress ten-
 610 sor $\boldsymbol{\sigma}^* = \rho_f c_p^* W^* \boldsymbol{\sigma}$. Each Lamé coefficient is made dimensionless using the Young
 611 modulus, $(\lambda_e^*, \mu_e^*) = E^*(\lambda_e, \mu_e)$, so that in dimensionless form, frequency domain and
 612 cylindrical coordinates the visco-elastic equation (114) reads

$$\tilde{\sigma}_{rr} = \alpha \left[\frac{2}{\tilde{C}_{\mu_s}(\omega)} + \frac{1}{\tilde{C}_{\lambda_s}(\omega)} \right] \partial_R \tilde{\xi} + \frac{\alpha}{\tilde{C}_{\lambda_s}(\omega)} \left[\partial_Z \tilde{\zeta} + \frac{\tilde{\xi}}{R} \right], \quad (115)$$

$$\tilde{\sigma}_{\theta\theta} = \alpha \left[\frac{2}{\tilde{C}_{\mu_s}(\omega)} + \frac{1}{\tilde{C}_{\lambda_s}(\omega)} \right] \frac{\tilde{\xi}}{R} + \frac{\alpha}{\tilde{C}_{\lambda_s}(\omega)} \left[\partial_Z \tilde{\zeta} + \partial_R \tilde{\xi} \right], \quad (116)$$

$$\tilde{\sigma}_{zz} = \alpha \left[\frac{2}{\tilde{C}_{\mu_s}(\omega)} + \frac{1}{\tilde{C}_{\lambda_s}(\omega)} \right] \partial_Z \tilde{\zeta} + \frac{\alpha}{\tilde{C}_{\lambda_s}(\omega)} \frac{\partial_R}{R} (R \tilde{\xi}), \quad (117)$$

$$\epsilon^2 \frac{\tilde{C}_{\mu_s}(\omega)}{\alpha} \tilde{\sigma}_{rz} = \partial_R \tilde{\zeta} + \epsilon^2 \partial_Z \tilde{\xi}, \quad (118)$$

Ignoring external constraints applied in the solid radial direction (supposing a zero external normal stress) whilst using $\tilde{\sigma}_{rr}$ defined in (115), the continuity of the normal and tangential stress as well as axial velocity read (Cf [65] for more details)

$$\tilde{\sigma}_{rr} = -\tilde{P} \quad , \text{ at } R = 1 \qquad \tilde{\sigma}_{rr} = 0 \quad , \text{ at } R = 1 + \alpha \quad (119)$$

$$\tilde{\sigma}_{rz} = 0 \quad , \text{ at } R = 1 \qquad \tilde{\sigma}_{rz} = 0 \quad , \text{ at } R = 1 + \alpha \quad (120)$$

Note that, for dimensionless radial distance r , since the dimensionless thickness of the pipe is α , the outer wall is reached as $R = 1 + \alpha$. Kinematic condition between the solid and the fluid at $R = 1$ in Fourier space read

$$\tilde{w} = \alpha i \omega \tilde{\zeta} \Big|_{R=1} \qquad \tilde{u} = \alpha i \omega \tilde{\xi} \Big|_{R=1} \quad (121)$$

It is noteworthy to mention that, the dimensionless form of the two-equation model proposed by Covas et al. [31] reads

$$i \omega \tilde{P} + \partial_Z \tilde{W} = - \frac{2D}{\alpha C_s^2} i \omega \tilde{I}_{Cov} \tilde{P}, \quad (122)$$

$$i \omega \tilde{W} = -\partial_Z \tilde{P}, \quad (123)$$

$$(124)$$

613 having the very same terms as (112)-(113) but for introducing the creep function \tilde{I}_{Cov}
 614 in the r.h.s term of (122). Using the relation between normal stress and pressure
 615 obtained from normal-stress continuity boundary condition (119), since the normal
 616 displacement $\tilde{\xi} \Big|_{R=1}$ in (112) is linearly related with the radial velocity of (121) which
 617 is also linearly related to the normal stress, and thus to the wall pressure, the r.h.s

618 term of (122) can be interpreted as resulting from some FSI effect. This is why the
 619 full FSI problem is now considered.

620 3.2.2. Four-equations water hammer models within visco-elastic pipes

The dimensionless form of the four-equation model proposed by Keramat et al. [145] is (the dimensionless formulation of this model is derived in [135]’s Appendix)

$$i\omega\tilde{W} = -\partial_Z\tilde{P}, \quad (125)$$

$$i\omega\tilde{P} + \partial_Z\tilde{W} - 2i\omega\alpha\nu_e\partial_Z\tilde{\zeta} = -\frac{2\mathcal{D}(1-\nu_e^2)}{\alpha C_s^2}i\omega\tilde{\mathcal{I}}_{Ker}\tilde{P}, \quad (126)$$

$$i\omega\tilde{\sigma}_{zz} - i\omega\frac{\nu_e}{\alpha}\tilde{P} - i\omega\frac{\alpha C_s^2}{\mathcal{D}}\partial_Z\tilde{\zeta} = -i\omega\tilde{\mathcal{I}}_{ker}\tilde{\sigma}_{zz} + \frac{\nu_e}{\alpha}i\omega\tilde{\mathcal{I}}_{Ker}\tilde{P}, \quad (127)$$

$$\frac{\alpha}{\mathcal{D}}\omega^2\tilde{\zeta} + \partial_Z\tilde{\sigma}_{zz} = 0, \quad (128)$$

where $\tilde{\mathcal{I}}_{Ker}$ is the Fourier transform of kernels proposed in Keramat et al. [145]). The Covas et al. [31]’s model can be derived in the $\nu_e \rightarrow 0$ limit of the Keramat et al. [145]’s one resulting in decoupling fluid axial dynamics to the solid’s one. Nevertheless, it is important to stress that, in this limit the right-hand-side of (122) is non-zero, resulting from kinematic continuity condition which produces a FSI term into (112) already found in Skalak’s model (24), as previously mentioned. Both [31] and [145] then consider N_{kv} Kelvin-Voigt elements to build their convolution kernel interpreted as a creeping law, each having its own exponential times-decay τ_k , amplitudes J_k , to model their convolution kernels, [148]

$$\left(\tilde{\mathcal{I}}_{Cov}, \tilde{\mathcal{I}}_{Ker}\right) = \sum_{k=1}^{N_{kv}} \frac{E_e J_k}{1 + i\omega \frac{c_p^e \tau_k}{L}}. \quad (129)$$

621 3.2.3. Generalized 3D visco-elastic rheology

Various Kelvin-Voigt models have been previously considered in the literature [150, 149, 139, 140, 141, 146]. Nevertheless among those almost every model has used a 1D scalar relationship between the stress and the strain. However, general visco-elastic Kelvin-Voigt models can be formulated in 3D as provided by [151] for the stress-strain relation

$$\boldsymbol{\sigma}^* = \lambda_e^* (1 + \tau_\lambda \partial_{t^*}) (\nabla^* \cdot \boldsymbol{\xi}) \mathbf{I} + \mu_e^* (1 + \tau_\mu \partial_{t^*}) \left(\nabla^* \boldsymbol{\xi}^* + \nabla^{*T} \boldsymbol{\xi}^{*} \right), \quad (130)$$

having four parameters : two elastic Lamé coefficients λ_e^*, μ_e^* and two visco-elastic times τ_λ, τ_μ . A more general 3D visco-elastic rheology can be formulated [135]

$$a(1 + \tau_r \partial_{t^*}) \boldsymbol{\sigma}^* = \lambda_e^* (1 + \tau_\lambda \partial_{t^*}) (\nabla^* \cdot \boldsymbol{\xi}^*) \mathbf{I} + \mu_e^* (1 + \tau_\mu \partial_{t^*}) (\nabla^* \boldsymbol{\xi}^* + \nabla^{*T} \boldsymbol{\xi}^*), \quad (131)$$

where now, six constitutive parameters $a, \tau_r, \tau_\lambda, \tau_\mu, \lambda_e^*, \mu_e^*$ are considered, with again (λ_e^*, μ_e^*) the elastic Lamé coefficients and $(\tau_r, \tau_\lambda, \tau_\mu)$ visco-elastic characteristic times, all independent of frequency ω^* . (131) general visco-elastic rheology encapsulates all previous models used in the context of water-hammer analysis [126, 134, 152, 153, 154, 155, 156], as detailed in table 1. Among those parameters a might be chosen equal to one if the elastic part of the visco-elastic response match with the purely elastic one, since $1/a$ essentially appears as a visco-elastic rescaling of elastic parameters λ_e^* and μ_e^* . Now, from the Fourier transform of (131) and identification with (114) the

	a	τ_r	λ_e^*	τ_λ	μ_e^*	τ_μ
Carcione et al. [152]			✓	✓	✓	✓
Eringen, Canic et al.[153, 126]			✓	✓	✓	✓
Kisilova et al. [134]		✓			✓	✓
Bland [154]			✓		✓	
Ieşan [155]			✓	✓	✓	✓
Sharma et al. [156]			✓	✓	✓	✓

Table 1: Comparative table of 3D rheological parameters taken from literature.

generalized Lamé coefficients can be deduced for this rheology

$$\tilde{\lambda}_s^*(\omega^*) = \lambda_e^* \frac{1 + i\omega^* \tau_\lambda}{a(1 + i\omega^* \tau_r)}, \quad \text{and} \quad \tilde{\mu}_s^*(\omega^*) = \mu_e^* \frac{1 + i\omega^* \tau_\mu}{a(1 + i\omega^* \tau_r)}. \quad (132)$$

Generalized Poisson and Young modulus can also be found from these rheological parameters

$$\tilde{\nu}_s^*(\omega^*) = \nu_e \frac{1 + i\omega^* \tau_\lambda}{1 + i\omega^* \tau_\nu}, \quad \text{and} \quad \tilde{E}_s^*(\omega^*) = \frac{E_e(1 + i\omega^* \tau_\mu)(1 + i\omega^* \tau_E)}{a(1 + i\omega^* \tau_r)(1 + i\omega^* \tau_\nu)}, \quad (133)$$

where the above introduced times-scale τ_ν and τ_E are given by

$$\tau_\nu = \frac{\lambda_e \tau_\lambda + \mu_e \tau_\mu}{\lambda_e + \mu_e}, \quad \text{and} \quad \tau_E = \frac{3\lambda_e \tau_\lambda + 2\mu_e \tau_\mu}{3\lambda_e + 2\mu_e}. \quad (134)$$

In the following the dimensionless generalized Lamé coefficients are defined,

$$\tilde{\lambda}_s^*(\omega^*) = \tilde{\lambda}(\omega^*)\lambda_e^* \quad , \quad \text{and} \quad \tilde{\mu}_s^*(\omega^*) = \tilde{\mu}_s(\omega^*)\mu_e^* \quad , \quad (135)$$

as well as the dimensionless generalized Poisson and Young modulus

$$\tilde{\nu}_s^*(\omega^*) = \tilde{\nu}_s(\omega^*)\nu_e \quad , \quad \text{and} \quad \tilde{E}_s^*(\omega^*) = E_e\tilde{E}_s(\omega^*) \quad . \quad (136)$$

Also, the Fourier transform of the dimensionless creep function $\tilde{J}_s(\omega)$ (108) reads from (136)

$$\tilde{J}_s(\omega) = \frac{1}{\tilde{E}_s} = a \frac{(1 + i\omega\tau_r \frac{c_p}{L})(1 + i\omega\tau_\nu \frac{c_p}{L})}{(1 + i\omega\tau_\mu \frac{c_p}{L})(1 + i\omega\tau_E \frac{c_p}{L})} \quad . \quad (137)$$

622 where dimensionless frequency ω is related to the dimensional one $\omega^* = \omega c_p/L$ using
623 the advective time-scale L/c_p .

624 3.3. FSI Rheology-based four-equations dimensionless visco-elastic water-hammer

625 In [135], a rheology-based four-equations FSI model has been derived from the
626 general visco-elastic rheology (131) hereby presented in dimensionless form

$$i\omega\tilde{P} + \partial_Z\tilde{W} - 2i\omega\alpha\nu_e\partial_Z\tilde{\zeta} = -i\omega\chi_e\tilde{\mathcal{I}}_P^F\tilde{P} + \frac{2\mathcal{D}\nu_e}{\mathcal{C}_s^2}i\omega\tilde{\mathcal{I}}_\sigma^F\tilde{\sigma}_{zz} \quad , \quad (138)$$

$$i\omega\tilde{W} = -\partial_Z\tilde{P} \quad (139)$$

$$i\omega\tilde{\sigma}_{zz} - \frac{2\nu_e}{\alpha(2+\alpha)}i\omega\tilde{P} - i\omega\frac{\alpha\mathcal{C}_s^2}{\mathcal{D}}\partial_Z\tilde{\zeta} = -i\omega\tilde{\mathcal{I}}_\sigma^S\tilde{\sigma}_{zz} + \frac{2\nu_e}{\alpha(2+\alpha)}i\omega\tilde{\mathcal{I}}_P^S\tilde{P} \quad , \quad (140)$$

$$\frac{\alpha}{\mathcal{D}}\omega^2\tilde{\zeta} + \partial_Z\tilde{\sigma}_{zz} = 0 \quad , \quad (141)$$

627 where the hereby introduced visco-elastic extra terms $\tilde{\mathcal{I}}_P^F$, $\tilde{\mathcal{I}}_\sigma^F$, $\tilde{\mathcal{I}}_P^S$ and $\tilde{\mathcal{I}}_\sigma^S$ read

$$\tilde{\mathcal{I}}_P^F = \frac{1}{\tilde{E}_s(\omega)} \frac{1 - \nu_e^2\tilde{\nu}_s(\omega) + \frac{\alpha(2+\alpha)}{2}(1 + \nu_e\tilde{\nu}_s(\omega))}{1 - \nu_e^2 + \frac{\alpha(2+\alpha)}{2}(1 + \nu_e)} - 1 \quad , \quad (142)$$

$$\tilde{\mathcal{I}}_\sigma^F = \frac{\tilde{\nu}_s(\omega) - 1}{\tilde{E}_s(\omega)} \quad , \quad (143)$$

$$\tilde{\mathcal{I}}_P^S = -\left(1 - \frac{\tilde{\nu}_s(\omega)}{\tilde{E}(\omega)}\right) \quad , \quad (144)$$

$$\tilde{\mathcal{I}}_\sigma^S = \frac{1 - \tilde{E}_s(\omega)}{\tilde{E}_s(\omega)} \quad . \quad (145)$$

It is important to stress the similarity between (138)-(141)'s model and Keramat et al. [31, 145]'s ones. Considering [145]'s hypothesis that the generalized Young modulus ν_s^* equals the elastic one ν_e , i.e. $\tilde{\nu}_s = 1$, with small dimensionless tube's thickness i.e. $\alpha \ll 1$, visco-elastic kernels (142)-(145) simplify to

$$\tilde{\mathcal{I}}_P^F = \tilde{\mathcal{I}}_P^S = \tilde{\mathcal{I}}_\sigma^S = \tilde{J}_s(\omega) - 1 \quad , \quad \text{and} \quad \tilde{\mathcal{I}}_\sigma^F = 0 \quad . \quad (146)$$

628 As a matter of fact within $\tilde{\nu}_s = 1$ ($\tilde{\nu}_s^* = \nu_e$) and $\alpha \rightarrow 0$ hypothesis, one finds that the
629 r.h.s of (139) displays a convolution product with the pressure only, as (126) when
630 $\mathcal{I}_\sigma^F = 0$ as provided by (146). Furthermore, the kernel associated with $\tilde{\mathcal{I}}_P^F$, $\tilde{\mathcal{I}}_P^S$ and
631 $\tilde{\mathcal{I}}_\sigma^S$ is the same, so that both r.h.s terms of (140) share the same kernel respectively
632 applied to the pressure and the axial stress. The very same feature is satisfied by the
633 r.h.s of (127). Hence, the visco-elastic rheological based model (138)-(141) is similar
634 with Keramat et al. (2011) (125)-(128) when using the $\tilde{\nu}_s = 1$ and $\alpha \rightarrow 0$ i.e, in the
635 limit of thin-wall and without visco-elastic contribution to the Poisson coupling, the
636 [135]'s model directly matches with [31, 145] provided by the explicit rheology-based
637 creep-function family (137). In the more general case $\tilde{\nu}_s \neq 1$, [135] gives explicit
638 derivations of visco-elastic extra-terms versus rheological parameters reading

$$\tilde{\mathcal{I}}_P^F = a \frac{\left(1 + i\omega \frac{\tau_r}{t_c}\right) \left(1 + i\omega \frac{\tau_\nu}{t_c}\right) \nu_e^2 \frac{1+i\omega \frac{\tau_\lambda}{t_c}}{1+i\omega \frac{\tau_\mu}{t_c}} - 1 - \frac{\alpha(2+\alpha)}{2} \left(1 + \nu_e \frac{1+i\omega \frac{\tau_\lambda}{t_c}}{1+i\omega \frac{\tau_\nu}{t_c}}\right)}{\left(1 + i\omega \frac{\tau_\mu}{t_c}\right) \left(1 + i\omega \frac{\tau_E}{t_c}\right) \nu_e^2 - 1 - \frac{\alpha(2+\alpha)}{2} (1 + \nu_e)} \quad (147)$$

$$\tilde{\mathcal{I}}_\sigma^F = a \frac{\left(1 + i\omega \frac{\tau_r}{t_c}\right) \left(1 + i\omega \frac{\tau_\nu}{t_c}\right)}{\left(1 + i\omega \frac{\tau_\mu}{t_c}\right) \left(1 + i\omega \frac{\tau_E}{t_c}\right)} \left(\frac{1 + i\omega \frac{\tau_\lambda}{t_c}}{1 + i\omega \frac{\tau_\nu}{t_c}} - 1\right), \quad (148)$$

$$\tilde{\mathcal{I}}_P^S = - \left(1 - a \frac{\left(1 + i\omega \frac{\tau_r}{t_c}\right) \left(1 + i\omega \frac{\tau_\lambda}{t_c}\right)}{\left(1 + i\omega \frac{\tau_\nu}{t_c}\right) \left(1 + i\omega \frac{\tau_E}{t_c}\right)}\right), \quad (149)$$

$$\tilde{\mathcal{I}}_\sigma^S = - \left(1 - a \frac{\left(1 + i\omega \frac{\tau_r}{t_c}\right) \left(1 + i\omega \frac{\tau_\nu}{t_c}\right)}{\left(1 + i\omega \frac{\tau_\mu}{t_c}\right) \left(1 + i\omega \frac{\tau_E}{t_c}\right)}\right). \quad (150)$$

where, again the convective time-scale is defined as $t_c = L/c_p$. As previously considered in (39) the wave system resulting from the visco-elastic FSI four equations hyperbolic problem (138)-(141) can thus be recast into two coupled waves for the two-component pressure/stress vector $\tilde{\mathbf{P}} \equiv [\tilde{P}, \tilde{\sigma}_{zz}]$ following [65]

$$\omega^2 \tilde{\mathbf{P}} + [\tilde{c}_p^v]^2 \begin{pmatrix} 1 & 2\nu_e \mathcal{D} \frac{1+\tilde{\mathcal{I}}_\sigma^S + \tilde{\mathcal{I}}_\sigma^F}{1+\tilde{\mathcal{I}}_\sigma^S} \\ \frac{2\nu_e}{\alpha(2+\alpha)} \frac{1+\tilde{\mathcal{I}}_P^S}{1+\tilde{\mathcal{I}}_\sigma^S} & \mathcal{C}_s e^2 \frac{1+\chi_e \tilde{\mathcal{I}}_P^F}{1+\tilde{\mathcal{I}}_\sigma^S} + \frac{4\nu_e^2 \mathcal{D}}{\alpha(2+\alpha)} \frac{1+\tilde{\mathcal{I}}_P^S}{1+\tilde{\mathcal{I}}_\sigma^S} \end{pmatrix} \partial_Z^2 \tilde{\mathbf{P}} = \mathbf{0}. \quad (151)$$

The characteristic equation associated with this propagating operator is

$$\begin{aligned} \left(\frac{\tilde{c}_\pm}{\tilde{c}_p^v}\right)^4 - \left(\frac{\tilde{c}_\pm}{\tilde{c}_p^v}\right)^2 \left[1 + \mathcal{C}_s e^2 \frac{1 + \chi_e \tilde{\mathcal{I}}_P^F}{1 + \tilde{\mathcal{I}}_\sigma^S} + \frac{4\nu_e^2 \mathcal{D}}{\alpha(2+\alpha)} \frac{1 + \tilde{\mathcal{I}}_P^S}{1 + \tilde{\mathcal{I}}_\sigma^S}\right] + \\ \mathcal{C}_s e^2 \frac{1 + \chi_e \tilde{\mathcal{I}}_P^F}{1 + \tilde{\mathcal{I}}_\sigma^S} + \frac{4\nu_e^2 \mathcal{D}}{\alpha(2+\alpha)} \frac{1 + \tilde{\mathcal{I}}_P^S}{1 + \tilde{\mathcal{I}}_\sigma^S} \left(1 - \frac{1 + \tilde{\mathcal{I}}_\sigma^S + \tilde{\mathcal{I}}_\sigma^F}{1 + \tilde{\mathcal{I}}_\sigma^S}\right) = 0, \quad (152) \end{aligned}$$

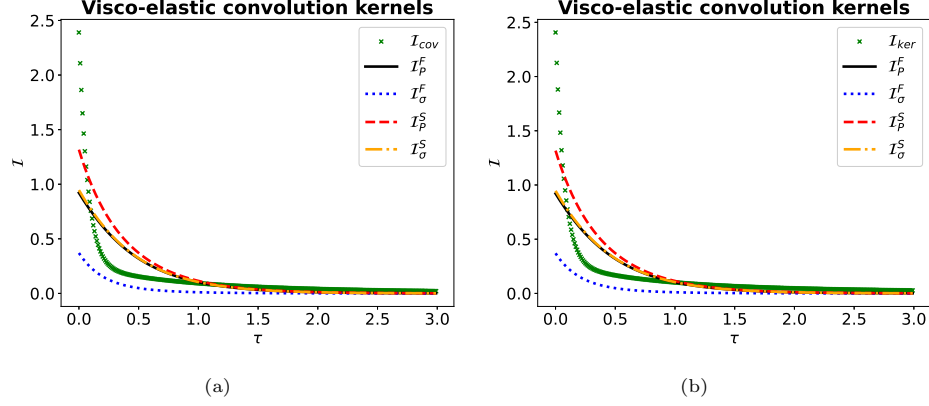


Figure 12: Comparison to (a) Covas et al. [31]'s and (b) Keramat et al. [145]'s visco-elastic convolution kernels for the experimental data of Covas et al. [137, 31]. Dimensionless time $\tau = t^* c_{p,T}^* / L^*$ has been used.

where the effective frequency-dependent corrective visco-elastic phase velocity $\tilde{c}_p^v(\omega)$ found in [135] is

$$\tilde{c}_p^v(\omega) = \frac{1}{\sqrt{1 + \chi_e \tilde{\mathcal{I}}_P^F - \frac{4\mathcal{D}\nu_e^2}{\alpha(2+\alpha)c_s^2} \tilde{\mathcal{I}}_\sigma^F \frac{1+\tilde{\mathcal{I}}_P^S}{1+\tilde{\mathcal{I}}_\sigma^S}}}. \quad (153)$$

639 The root of (152) can be found, generalizing (42) to a general visco-elastic rheology

$$\tilde{c}_\pm^2 = [\tilde{c}_p^v]^2 \frac{\hat{c} \pm \sqrt{\hat{c}^2 - 4 \left[C_s^e \frac{1+\chi_e \tilde{\mathcal{I}}_P^F}{1+\tilde{\mathcal{I}}_\sigma^S} + \frac{4\nu_e^2 \mathcal{D}}{\alpha(2+\alpha)} \frac{1+\tilde{\mathcal{I}}_P^S}{1+\tilde{\mathcal{I}}_\sigma^S} \left(1 - \frac{1+\tilde{\mathcal{I}}_\sigma^S + \tilde{\mathcal{I}}_\sigma^F}{1+\tilde{\mathcal{I}}_\sigma^S} \right) \right]}}{2}, \quad (154)$$

$$\hat{c} = 1 + C_s^e \frac{1 + \chi_e \tilde{\mathcal{I}}_P^F}{1 + \tilde{\mathcal{I}}_\sigma^S} + \frac{4\nu_e^2 \mathcal{D}}{\alpha(2+\alpha)} \frac{1 + \tilde{\mathcal{I}}_P^S}{1 + \tilde{\mathcal{I}}_\sigma^S} \equiv \frac{\tilde{c}_+^2 + \tilde{c}_-^2}{[\tilde{c}_p^v]^2}. \quad (155)$$

640 The various visco-elastic convolution kernels are compared in Figure 12 for the exper-
641 imental data of [31] presented in Table 2. Figure 13 shows the least-squares difference
642 fitting of the pressure solution (151) obtained in [65]. Even though in each case a
643 fitting of the pressure signal is performed, it is nice to observe that the various models
644 [31], [145] or [65] do not perform alike, in this fitting. This is expected since both the
645 model and the fitting parameters differ. From the rheological fitting on the pressure
646 signal, the visco-elastic kernels can be compared. Even though these kernel families
647 present similar exponential decay types, $\tilde{\mathcal{I}}_{Cov}$ and $\tilde{\mathcal{I}}_{Ker}$ display faster attenuation than
648 the various kernels $\mathcal{I}_P^F, \mathcal{I}_\sigma^F, \mathcal{I}_P^S, \mathcal{I}_\sigma^S$ of [65] as can be observed in Fig 14c. Also, both
649 \mathcal{I}_P^F and \mathcal{I}_σ^S are very similar for the obtained visco-elastic parameters. Concerning the

650 velocity dispersivity prediction provided in Figure 14, one should note that every
 651 model displays a similar trend for the norm of complex velocities: it varies from a
 652 minimum value at $\omega = 0$ within a narrow low-frequency region (associated with a long
 653 time behaviour) so as to reach a constant plateau for large $|\omega|$ values.

654 Hence, at short-time/large $|\omega|$ most dispersivity of the wave velocity is lost and
 655 the visco-elastic response is very much like the elastic one, [157, 158]. This allows to
 656 define a 'dispersive' frequency gap band depicted within vertical orange dotted lines for
 657 which visco-elastic effects are important. The 'dispersive' frequency gap band $\Delta\omega_v$ is
 658 more precisely defined as the 95% difference velocity region from the asymptotic high-
 659 frequency regime, as exemplified in the inset of figure 14a. The larger this dispersive
 660 gap-band, and the deeper the $\omega = 0$ velocity, the larger visco-elastic effects are. One
 661 can observe in figure 14 that the dispersive gap is wider for Covas et al. [31] and
 662 Keramat et al. [145] models than for the hereby model (in black) for parameters
 663 obtained from the same data set [137, 31].

664 4. Recent developments

665 4.1. Two-time scale asymptotic analysis of water-hammer without FSI

In this section more recent advances obtained from complementing the spatial asymptotic analysis discussed in section 2.3 with a two-time scale asymptotic approach are now presented. Indeed, not only the water-hammer small parameter δ (92) is useful to define the dimensionless size of the wave perturbation boundary layer nearby the solid wall. It is also relevant to define a long-time scale associated with the momentum relaxation within it. More precisely, as provided in (92) since δ is related to the ratio between advection time to viscous relaxation time $\delta^2 = t_c/t_{vis}$, and since, as found by Zielke (95), Vardy (102) or others, the velocity time variations within the boundary layer are controlled by t_{vis} , one realizes that since momentum relaxation within the boundary layer varies as $t/t_{vis} = \delta^2 t/t_c = \delta^2 \tau$ when chossing a dimensionless time $\tau = t/t_c$. In other words, the time response of the wave perturbation within the boundary layer is slow compared to the wave advection time. Approximated solutions for water-hammer waves based upon this multi-time scale nature of the water-hammer wave have been developed in several contributions [159, 160, 161, 162, 95, 163]. In the following we will nevertheless first focus on the contribution of Mei & Jing [95] which investigates how a two-time scale perturbation analysis permits to predict the

exponential decay of the wave envelope. Introducing the slow time-scale $T = \delta\tau$ permits to decompose time variations as

$$\partial_{t^*} = \frac{c_p}{L}(\partial_\tau + \delta\partial_T) \quad (156)$$

666 Either non-FSI or FSI effects can be considered within this multi-time scale approx-
 667 imation approach. Since, the latter requires much involved developments, we first
 668 consider the non-FSI problem analyzed in [95] for a single pipe.

669 An asymptotic solution to the slowly damped pressure wave (83) is set up from
 670 evaluating the leading order wall-shear stress associated with damped propagation
 671 from solving the undamped problem first. The dimensionless pressure P , velocity W
 672 and wall shear-stress $\tau_w \equiv \tau_f|_{R=1}$ are thus asymptotically expanded as

$$P = P^0 + \delta P^1 + \dots, \quad W = W^0 + \delta W^1 + \dots, \quad \tau_w = \tau_w^0 + \delta\tau_w^1 + \dots \quad (157)$$

Using (157) and (156) in (83) provide the leading order undamped problem

$$[\partial_\tau^2 - \partial_Z^2] P^0 = 0 \quad (158)$$

673 which is solved together with mixed Dirichlet-Neumann boundary conditions

$$P^0(Z=0) = 0, \quad \frac{\partial P^0}{\partial Z}(Z=1) = D(t) \quad (159)$$

where $D(t)$ is a closure law to be specified (and chosen as a triangular shape in [95]),
 but we will simplify their result to either a general closure $D(t)$ or a Dirac distribution
 $D(t) \equiv \delta(t)$. In Laplace domain, the solution (158) and (159) is

$$P^0(Z, s) = \frac{\sinh(sZ)}{s \cosh s} D(s) \quad (160)$$

The time-domain solution can either be found from inverse Laplace transform, or direct
 solution. The time-domain solution for the leading order pressure is reported in [95]
 for a Dirac distribution closure $D(t) \equiv \delta(t)$, $D(s) = 1$ and reads

$$P^0(Z, t) = 2 \sum_{k \in \mathbb{N}} (-1)^k \frac{\sin(\lambda_k Z)}{\lambda_k} \sin(\lambda_k \tau) \quad \& \quad \lambda_k = \pi \left(\frac{1}{2} + k \right). \quad (161)$$

674 From (161), one realizes that boundary conditions (159) are satisfied. The same re-
 675 sult can also be obtained from (158) decomposing the pressure into an homogeneous
 676 component having homogeneous boundary conditions, as well as a particular solution
 677 dealing with the non-homogeneous r.h.s of (159) with $D(t) \equiv \delta(t)$

$$P^0(Z, t) = P_p^0(Z, t) + P_h^0(Z, t) \quad \text{with,} \quad P_p^0(Z, t) = Z\delta(t), \quad (162)$$

In the following, it will be useful to define the spatial orthogonal basis such that

$$\phi_{\lambda_k}(Z) = \sqrt{2} \operatorname{sgn}(\lambda_k) \sin(\lambda_k Z) \quad \& \quad \int_0^1 \phi_{\lambda_k} \phi_{\lambda_{k'}} dZ = \langle \phi_{\lambda_k}, \phi_{\lambda_{k'}} \rangle = \delta_{\lambda_k, \lambda_{k'}} + \delta_{\lambda_k, -\lambda_{k'}}. \quad (163)$$

where $\delta_{\lambda_k, \lambda_{k'}}$ is the Kronecker symbol associated with the equality of λ_k and $\lambda_{k'}$. Given $\lambda_k = \pi(\frac{1}{2} + k)$ for $k \in \mathbb{Z}$, the resulting $\lambda_k \in \mathbb{R}$ can indeed lead to $\pm\lambda_k$ as possible eigenvalues, and since $\phi_{\lambda_k} = \phi_{-\lambda_k}$ there are indeed two possible non-zero contributions to the base decomposition, i.e $\pm\lambda_k$. Since the Laplace transform of the particular solution defined in (162) is $\tilde{P}_h^0(Z, s) = Z$, it decomposes into the ϕ_{λ_k} base as follows

$$\tilde{P}_p = \frac{1}{2} \sum_{\lambda_k \in \mathbb{R}} \langle Z, \phi_{\lambda_k} \rangle \phi_{\lambda_k}, \quad \& \quad \langle Z, \phi_{\lambda_k} \rangle = \frac{(-1)^k}{\sqrt{2} \lambda_k^2}. \quad (164)$$

Decomposing the Laplace transform of the homogeneous part as

$$\tilde{P}_h^0(Z, s) = \sum_{\lambda_k \in \mathbb{R}} \tilde{a}^0(s) \phi_{\lambda_k}, \quad (165)$$

whilst using (158), (164) and (165) leads to

$$\sum_{\lambda_k \in \mathbb{R}} (s^2 + \lambda_k^2) \tilde{a}^0(s) \phi_{\lambda_k} = -s^2 \tilde{P}_p = \frac{-s^2}{2} \sum_{\lambda_k \in \mathbb{R}} \langle Z, \phi_{\lambda_k} \rangle \phi_{\lambda_k}, \quad (166)$$

678 the projection of which over ϕ_{λ_k} , leads to

$$\tilde{a}_{\lambda_k}^0(s) + \tilde{a}_{-\lambda_k}^0(s) = -\frac{s^2}{s^2 + \lambda_k^2} \langle Z, \phi_{\lambda_k} \rangle = \left(\frac{\lambda_k^2}{s^2 + \lambda_k^2} - 1 \right) \langle Z, \phi_{\lambda_k} \rangle \quad (167)$$

$$\tilde{a}_{\lambda_k}^0(s) + \tilde{a}_{-\lambda_k}^0(s) = \left[\frac{\lambda_k}{2i} \left(\frac{1}{s - i\lambda_k} - \frac{1}{s + i\lambda_k} \right) - 1 \right] \langle Z, \phi_{\lambda_k} \rangle \quad (168)$$

so that, given the condition $\tilde{a}_{\lambda_k}^{0*} = \tilde{a}_{-\lambda_k}^0$ one finds

$$\tilde{a}_{\lambda_k}^0 = -\frac{1}{2} \left[\frac{\pm i \lambda_k}{s \mp \lambda_k} + 1 \right] \langle Z, \phi_{\lambda_k} \rangle. \quad (169)$$

Using the Laplace transform of sinus function $\mathcal{L}(\sin(\lambda_k \tau)) = \lambda_k / (s^2 + \lambda_k^2)$, (161) can be rewritten using (164) as

$$\tilde{P} = \sum_{k \in \mathbb{N}} \frac{\lambda_k^2}{s^2 + \lambda_k^2} \phi_{\lambda_k} \langle Z, \phi_{\lambda_k} \rangle = \sum_{k \in \mathbb{N}} \left[1 - \frac{s^2}{s^2 + \lambda_k^2} \right] \phi_{\lambda_k} \langle Z, \phi_{\lambda_k} \rangle \quad (170)$$

which can now easily be identified with

$$\tilde{P}_h^0(Z, s) + \tilde{P}_p^0(Z, s) = \sum_{\lambda_k \in \mathbb{R}} \tilde{a}_{\lambda_k}^0(s) \phi_{\lambda_k} + \frac{1}{2} \sum_{\lambda_k \in \mathbb{R}} \langle Z, \phi_{\lambda_k} \rangle \phi_{\lambda_k} = \sum_{k \in \mathbb{N}} \left[1 - \frac{s^2}{s^2 + \lambda_k^2} \right] \phi_{\lambda_k} \langle Z, \phi_{\lambda_k} \rangle. \quad (171)$$

Momentum balance into the core acoustic domain for leading order reads

$$\frac{\partial W^0}{\partial \tau} = -\partial_Z P^0. \quad (172)$$

679 From which one finds,

$$W^0(Z, t) = 2 \sum_{k \in \mathbb{N}} (-1)^k \frac{\cos(\lambda_k Z)}{\lambda_k} \cos(\lambda_k \tau) \quad (173)$$

The wall shear-stress is furthermore obtained in [95] from solving the boundary-layer problem. In the boundary-layer, i.e, the inner domain of the asymptotic analysis, one needs to solve the dimensionless momentum conservation (93), which includes the viscous contribution related to dimensionless pipe radius $R = R^*/R_0^*$, i.e

$$\frac{\partial W^0}{\partial \tau} = -\partial_Z P^0 + \frac{\delta^2}{R} \partial_R (R \partial_R W^0). \quad (174)$$

Defining the inner boundary-layer transverse coordinate,

$$y = (1 - r)/\delta, \quad (175)$$

as well as inner boundary-layer pressure p and longitudinal velocity w , (172) leads to a boundary-layer momentum diffusion dynamics according to

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial y^2} \right) w^0(y, \tau) = -\partial_Z p^0 \quad (176)$$

(176) is a diffusion problem driven by the longitudinal pressure gradient associated with no-slip boundary condition

$$w^0(y, t)|_{y=0} = 0. \quad (177)$$

The Laplace transform of (176) reads

$$\left(s - \frac{\partial^2}{\partial y^2} \right) \tilde{w}^0 = -\partial_Z \tilde{p}^0. \quad (178)$$

The solution for \tilde{w} with boundary condition (177) is

$$\tilde{w}^0 = -\frac{1}{s} \left(1 - e^{-\sqrt{s}y} \right) \partial_Z \tilde{p}^0. \quad (179)$$

From, (179), one can evaluate the shear-rate

$$\frac{\partial \tilde{w}^0}{\partial y} = -\frac{\sqrt{s}}{s} e^{-\sqrt{s}y} \partial_Z \tilde{p}^0, \quad (180)$$

which, evaluated at $y = 0$, gives the wall shear-stress

$$\tilde{\tau}_w^0 = \frac{\partial \tilde{w}^0}{\partial y} \Big|_{y=0} = -\frac{\sqrt{s}}{s} \partial_Z \tilde{p}^0 = -\frac{\sqrt{s}}{s} \partial_Z \tilde{P}^0, \quad (181)$$

since the outer pressure P^0 and the inner one p^0 match together, and since neither depends on the transverse coordinate, i.e $P^0 = p^0$. From (181), the r.h.s of (83) can be evaluated, feeding the first order pressure field

$$[s^2 - \partial_Z^2] \tilde{P}^1 = -2 \frac{\sqrt{s}}{s} \partial_Z^2 \tilde{P}^0(s, Z) = -2 \frac{\sqrt{s}}{s} \partial_Z^2 \tilde{P}_h^0(s, Z) \quad (182)$$

680 since, $\partial_Z^2 \tilde{P}_p^0(s, Z) = 0$. In [95], P^1 solution in time-domain is explicitly solved from the
 681 inverse Laplace transform performing some heavy computations. Among the obtained
 682 complicated expressions, a specific one is linearly diverging with time. The so-called
 683 "secularity" condition is then introduced so as to zeros these diverging terms. In
 684 fact, this secularity condition happens when "resonant modes" are triggered by the
 685 slow-time perturbation. The resulting secularity condition then provides the envelope
 686 attenuation of the leading order, which is the main useful result of the two-time scale
 687 asymptotic analysis.

Realizing that the precise solution of P^1 is not needed in order to evaluate the secular condition [65] gives a short-cut to avoid cumbersome developments, and provides it from realizing that secularity is related to double poles in the inverse Laplace transform of \tilde{P}^1 . This comes from the following result : given any analytical function $\phi(s)$ in the complex plane, the inverse Laplace transform of the double poles through Cauchy's residue theorem effectively leads to a linear divergence in time

$$\mathcal{L}^{-1} \left(\frac{\phi(s)}{(s \pm i\lambda_k)^2} \right) (\tau) = \lim_{s \rightarrow \pm i\lambda_k} \partial_s [\phi(s) e^{s\tau}] = [\phi(\pm i\lambda_k)\tau + \partial_s \phi(\pm i\lambda_k)] e^{\pm i\lambda_k \tau}. \quad (183)$$

Here we adapt the approach of [65] for the simpler case treated in [95] so as to better emphasize the origin of resonance modes. At order one, the solution for \tilde{P}^1 only has an homogeneous part for the particular one is zero since homogeneous boundary conditions apply at this order. Hence, $\tilde{P}^1(Z, s)$ can be decomposed into the orthogonal base

$$\tilde{P}^1 = \sum_{k \in \mathbb{Z}} a_1^k(\tau) \phi_{\lambda_k}(Z) \quad (184)$$

Using this decomposition in (182) whilst using the leading-order pressure solution (161) leads to

$$\sum_{\lambda_k \in \mathbb{R}} (s^2 + \lambda_k^2) \tilde{a}_{\lambda_k}^1(s) \phi_{\lambda_k}(Z) = -\frac{\sqrt{s}}{s} 2\sqrt{2} \sum_{k \in \mathbb{N}} (-1)^k \frac{s^2}{s^2 + \lambda_k^2} \phi_{\lambda_k}(Z) \quad (185)$$

where the Laplace transform of (158) have led to $\partial_Z^2 \tilde{P}^0(s, Z) = \partial_Z^2 \tilde{P}_h^0(s, Z) = s^2 \tilde{P}_h^0(s, Z)$, and where $\mathcal{L}(\sin(\lambda_k \tau)) = \lambda_k / (s^2 + \lambda_k^2)$ has been used. When projecting (254) on ϕ_{λ_k}

modes one finds in (254)'s r.h.s a term proportional to $1/(s^2 + \lambda_k^2)^2$ i.e proportional to $1/(s + i\lambda_k)^2(s - i\lambda_k)^2$ having two double poles in $s = \pm i\lambda_k$. From (183) one realizes that these double poles lead to a linear divergence with time, which is inconsistent with the fact that each perturbative $a_1^k(\tau)$ term has to keep being $O(1)$ so as the order-one correction δP^1 is also kept $O(\delta)$. To remove this divergent term, the leading-order solution has to be enriched with some slow-time scale envelope, so that P_h^0 (165) is re-written

$$P_h^0(Z, s, T) = \sum_{k \in \mathbb{Z}} A_{\lambda_k}^0(T) \tilde{a}_{\lambda_k}^0(s) \phi_{\lambda_k} \quad (186)$$

with $T = \delta\tau$ and the normalization condition $A_{\lambda_k}^0(T) = 1$ for each $k \in \mathbb{Z}$. Using (186) and (156) in (83) using (181) then leads to

$$[\partial_\tau^2 - \partial_Z^2] P^1 + 2\partial_T \partial_\tau P^0 = -2 \frac{\sqrt{s}}{s} \partial_Z^2 \tilde{P}^0 \quad (187)$$

In Laplace domain, (187) becomes

$$[s^2 - \partial_Z^2] \tilde{P}^1 = -2\partial_T s \tilde{P}_h^0 - 2 \frac{\sqrt{s}}{s} s^2 \tilde{P}_h^0 \quad (188)$$

so that using decomposition (186) and (184), in (188) one gets

$$\sum_{k \in \mathbb{Z}} (s^2 + \lambda_k^2) \tilde{a}_{\lambda_k}^1(s) \phi_{\lambda_k}(Z) = \sum_{k \in \mathbb{Z}} 2s [-\partial_T A_{\lambda_k}^0 - \sqrt{s} A_{\lambda_k}^0] \tilde{a}_{\lambda_k}^0(s) \phi_{\lambda_k}. \quad (189)$$

688 Projecting (189) over ϕ_{λ_k} whilst using (169) leads to

$$\tilde{a}_{\lambda_k}^1(s) + \tilde{a}_{-\lambda_k}^1(s) = \frac{s}{\lambda_k^2 + s^2} \sum_{\pm} 2[-\partial_T A_{\pm\lambda_k}^0 - \sqrt{s} A_{\pm\lambda_k}^0] a_{\pm\lambda_k}^0 \quad (190)$$

$$= \frac{s \langle Z, \phi_{\lambda_k} \rangle}{(\lambda_k^2 + s^2)} \sum_{\pm} [\partial_T A_{\pm\lambda_k}^0 + \sqrt{s} A_{\pm\lambda_k}^0] \left[\frac{\pm i \lambda_k}{s \mp \lambda_k} + 1 \right], \quad (191)$$

so that the double poles are canceled if the two secularity conditions

$$\lim_{s \rightarrow \pm i \lambda_k} [-\partial_T A_{\pm\lambda_k}^0(T) - \sqrt{s} A_{\pm\lambda_k}^0(T)] = 0, \quad (192)$$

are met. Hence, slow time-scale amplitude dependence has to be chosen so as to cancel this term, so that

$$A_{\pm\lambda_k}^0(T) = e^{-\sqrt{\pm i \lambda_k T}} = e^{-\frac{1 \pm i}{2} \sqrt{|\lambda_k| T}} \quad (193)$$

Then, using the slow amplitude solution (193) in (186) whilst using (169) and inverse Laplace transform leads to the final damped solution

$$P(Z, t, T) = 2 \sum_{k \in \mathbb{N}} (-1)^k \frac{\sin(\lambda_k Z)}{\lambda_k} e^{-\sqrt{\frac{\lambda_k}{2}} T} \sin \left(\lambda_k \tau - \sqrt{\frac{\lambda_k}{2}} T \right), \quad \& \quad \lambda_k = \pi \left(\frac{1}{2} + k \right), \quad (194)$$

689 with again, $T = \delta\tau$. This solution has been compared with several experimental
690 recordings in [65], and will be illustrated in section 4.3. We will discuss in section
691 4.3 how most of the foot-steps detailed in this section can be extended when FSI
692 effects are taken into account. Nevertheless before entering into this matter, let us
693 now discuss how a systematic asymptotic derivation of water-hammer four-equation
694 FSI models can be handled so as to better understand what are the constraints on the
695 dimensionless parameter for water-hammer formulation (33)-(34) to hold.

696 4.2. Asymptotic analysis of water-hammer with FSI

697 Most of water-hammer studies are given, performed and analyzed with the implicit
698 hypothesis or possibly the explicit measure of small Mach number. Nevertheless, how
699 precisely small the Mach should be for water hammer wave modeling (33)-(34) to
700 be valid has remained mostly uninformed until recently. More generally, a systematic
701 derivation of water-hammer waves from constitutive equations (Solid Lamé's equations
702 coupled with compressible Navier-Stokes ones) has only been recently achieved in [65].
703 This section discusses some aspects of this derivation which sheds new light into the
704 mechanism underpinning the wave propagation model, and noteworthy the possible
705 couplings between the wave and the underlying flow's steady-state.

The outer/inner fluid pressure P_f^*/p_f^* , axial velocity W_f^*/w_f^* , and radial velocity
 U_f^*/u_f^* , are split into steady, denoted with subscript st , and unsteady components
(without subscript) following classical acoustic approach, ([164])

$$P_f^* = P^*(r, z, t) + P_{st}^*(r, z), \quad p_f^* = p^*(r, z, t) + p_{st}^*(r, z), \quad (195)$$

$$W_f^* = W^*(r, z, t) + W_{st}(r, z), \quad w_f^* = w^*(r, z, t) + w_{st}^*(r, z), \quad (196)$$

$$U_f^* = U^*(r, z, t), \quad u_f^* = u^*(r, z, t). \quad (197)$$

706 As the steady-state is assumed unidirectional, the outer/inner radial velocity compo-
707 nents U_f^*/u_f^* , are unsteady. Finally, the fluid stress tensors are $\sigma_f^* = \sigma_{st}^* + \sigma^*$

$$\sigma_{st}^* = (-P_{st}^* + \lambda_f \partial_z W_{st}^*) \mathbf{I} + \mu_f \begin{pmatrix} 0 & \cdots & \partial_r W_{st}^* \\ \cdots & 0 & \cdots \\ \partial_r W_{st}^* & \cdots & 0 \end{pmatrix}, \quad (198)$$

$$\sigma^* = \left(-p^* + \lambda_f \left[\frac{\partial_r}{r} (ru^*) + \partial_z w^* \right] \right) \mathbf{I} + 2\mu_f \begin{pmatrix} \partial_r u^* & \cdots & \frac{\partial_r w^* + \partial_z u^*}{2} \\ \cdots & \frac{u^*}{r} & \cdots \\ \frac{\partial_r w^* + \partial_z u^*}{2} & \cdots & \partial_z w^* \end{pmatrix} \quad (199)$$

where the volume viscosity λ_f has been used. In the following the dimensionless ratio Γ of volume viscosity to dynamic viscosity will also be used :

$$\Gamma = \frac{\lambda_f}{\mu_f} \quad (200)$$

708 *4.2.1. Dimensionless numbers setting*

Two dimensionless numbers (similar to Cauchy's number) are introduced

$$\mathcal{C}_G = \frac{\rho_f^* c_p^{*2}}{\mathcal{G}^*} \equiv \frac{2\rho_f^* c_p^{*2}(1 + \nu_s)}{E^*}, \quad \& \quad \mathcal{C}_{\lambda_s} = \frac{\rho_f^* c_p^{*2}}{\lambda_s^*} \equiv \frac{\mathcal{C}_G(1 - 2\nu_s)}{2\nu_s}, \quad (201)$$

where

$$\mathcal{G}^* = \frac{E^*}{2(1 + \nu_s)}, \quad \& \quad \lambda_s^* = \frac{\nu_s E^*}{(1 + \nu_s)(1 - 2\nu_s)}, \quad (202)$$

are the solid shear modulus and the second Lamé-Clapeyron coefficient, respectively. The overpressure wave velocity $c_{p,T}^*$, given in (35), is thus a corrective formulation of c_0^* due to the tube elastic constraints. By introducing parameter

$$\chi = \frac{2\mathcal{K}_f^*}{\alpha E^* \mathcal{C}^2} \left(\frac{2(1 - \nu_s^2)}{2 + \alpha} + \alpha(1 + \nu_s) \right) \equiv \frac{2\nu_s \mathcal{C}_{\lambda_s} + (1 + \alpha)^2 \mathcal{C}_G}{\alpha(2 + \alpha)}, \quad (203)$$

the dimensionless version of (35) is defined as the dimensionless wave speed $c_p = c_{p,T}^*/c_0^*$ and fulfills

$$c_p^2 = \frac{1}{1 + \chi \mathcal{C}^2}, \quad \text{with} \quad \mathcal{C} = \frac{c_0^*}{c_{p,T}^*} \quad (204)$$

where $1 + \chi \mathcal{C}^2$ is a corrective fluid pulse-wave speed factor. Regarding c_p definition in (204), it should be noted that $\mathcal{C}^2 > 1$ implies $c_p < 1$ ([6, 10, 19, 165]). Dimensionless parameters associated with boundary layer thickness δ (92), aspect ratio $\epsilon = R_0^*/L^*$, Reynolds number Re , pulsed Reynolds number Re_p and Mach number \mathcal{M} are related as

$$Re_p = \frac{c_p R_0}{\nu_f} \gg 1, \quad Re = \frac{W_0 R_0}{\nu_f} = \mathcal{M} Re_p, \quad (205)$$

$$\delta^2 = \frac{\nu_f L}{c_p R_0^2} = \frac{1}{\epsilon Re_p} \ll 1, \quad \mathcal{M} = \frac{W_0}{c_p} \ll 1. \quad (206)$$

Low-Mach number i.e. $\mathcal{M} \ll 1$ and long-wavelength, i.e. $\epsilon \ll 1$ ([28, 32, 166, 43]), are generally considered for water-hammer validity [13, 167]. But, much

more precisely, they have been asymptotically established in [65] within the following condition

$$\delta^2 \gg \mathcal{M} > \frac{\mathcal{M}}{\mathcal{C}^2}, \quad \delta \gg \epsilon^2, \quad \delta \gg \alpha\mathcal{M}, \quad 1 \gg \epsilon \gg \alpha\mathcal{M}. \quad (207)$$

709 We now provide some supplementary details of this asymptotic analysis, es-
 710 pecially those related to $O(\delta)$ corrections which were considered in the previ-
 711 ous section but also other possible corrections/contributions such as $O(\mathcal{M}/\delta)$,
 712 $O(\mathcal{M}\epsilon Re)$ and $O(\frac{\mathcal{M}\epsilon Re}{\delta})$.

713 4.2.2. Navier–Stokes equations

Taking W_{st} as the reference dimensionless velocity for the steady- longitudinal flow, and as previously W_0^* for the disturbance, and since long-wavelength assumption implies that the radial fluid velocity component is ϵ smaller than the axial one, dimensionless pressure and velocity fulfill

$$P_{st}^* = \rho_{f_0} W_0^{*2} P_{st}(R, Z), \quad W_{st}^* = W_0^* W_{st}(R, Z), \quad (208)$$

$$P^* = \rho_{f_0} c_p W_0^* P(R, Z, \tau), \quad p^* = \rho_{f_0} c_p W_0^* p(y, Z, \tau), \quad (209)$$

$$W^* = W_0^* W(R, Z, \tau), \quad w^* = W_0^* w(y, Z, \tau), \quad (210)$$

$$U^* = \epsilon W_0 U(R, Z, \tau), \quad u^* = \epsilon W_0 u(y, Z, \tau), \quad (211)$$

where capital letters have been used to define the outer variables, and lower case has been chosen for inner variables. Note that the perturbed velocity and the steady state both scale as W_0^* since the perturbation is supposed to arise from an order-one fraction of the steady state velocity. From (208)-(211) it is also useful to define dimensionless stress and shear as

$$\tau_f^* = -\frac{\rho_f^* \nu_f^* W_0^*}{\delta R_0^*} \tau_f(y, Z, \tau), \quad \tau_f = \partial_y w(y, Z, \tau), \quad (212)$$

$$\sigma^* = \rho_{f_0}^* c_p^* W_0^* \sigma, \quad \sigma_{st}^* = \rho_{f_0}^* W_0^{*2} \sigma_{st}, \quad (213)$$

714 with,

$$\sigma_{st} = \left(-P_{st} + \Gamma \frac{(\epsilon\delta)^2}{\mathcal{M}} \partial_Z W_{st} \right) \mathbf{I} + \frac{\epsilon\delta}{\mathcal{M}} \begin{pmatrix} 0 & \cdots & -\partial_y W_{st} \\ \cdots & 0 & \cdots \\ -\partial_y W_{st} & \cdots & \epsilon\delta \partial_Z W_{st} \end{pmatrix}, \quad (214)$$

$$\boldsymbol{\sigma} = \left(-p + \Gamma (\epsilon\delta)^2 \left(-\frac{\partial_y [(1-\delta y)u]}{\delta(1-\delta y)} + \partial_Z w \right) \right) \mathbf{I} + \epsilon\delta \begin{pmatrix} -2\epsilon\partial_y u & \cdots & -\partial_y w + \epsilon^2\partial_Z u \\ \cdots & 2\epsilon\delta\frac{u}{1-\delta y} & \cdots \\ -\partial_y w + \epsilon^2\partial_Z u & \cdots & 2\epsilon\delta\partial_Z w \end{pmatrix}. \quad (215)$$

Integrating the fluid isentropic compression law (6) the fluid density is subjected to pressure variations following

$$\rho_f^*(r, z, t) = \rho_{f_0}^* e^{\frac{P_f^*(r, z, t)}{\mathcal{K}_f}} = \rho_{f_0}^* e^{\frac{P^*(r, z, t) + P_{st}^*(r, z)}{\mathcal{K}_f}}, \quad (216)$$

so that by introducing the dimensionless density $\rho_f = \rho_f^*/\rho_{f_0}^*$ whilst using scalings (208)–(211) yields

$$[1, \nabla, \partial_\tau] \rho_f = e^{\frac{\mathcal{M}}{\mathcal{C}^2}(P + \mathcal{M}P_{st})} \left[1, \frac{\mathcal{M}}{\mathcal{C}^2} \nabla (P + \mathcal{M}P_{st}), \frac{\mathcal{M}}{\mathcal{C}^2} \partial_\tau P \right], \quad (217)$$

with ∇ the dimensionless nabla operator, \mathcal{C}^2 define in (204) and $\mathcal{M}/\mathcal{C}^2 \ll 1$. Obviously, in the inner region (217) holds from replacing P by the inner pressure p . The Navier–Stokes equations, which follow from fluid mass and momentum conservations, are

$$(\partial_t + W_f^* \partial_z + U^* \partial_r) \rho_f^* + \rho_f^* \left(\partial_z W_f^* + \frac{1}{r} \partial_r (r \partial_r U^*) \right) = 0, \quad (218)$$

$$\begin{aligned} \rho_f^* (\partial_t + W_f^* \partial_z + U^* \partial_r) W_f^* &= -\partial_z P_f^* \\ &+ \rho_{f_0} \nu_f \left((1 + \Gamma) \partial_z \left[\partial_z W_f^* + \frac{\partial_r}{r} (r U^*) \right] + \left(\frac{\partial_r}{r} (r \partial_r) + \partial_z^2 \right) W_f^* \right), \end{aligned} \quad (219)$$

$$\begin{aligned} \rho_f^* (\partial_t + W_f^* \partial_z + U^* \partial_r) U^* &= -\partial_r P_f^* \\ &+ \rho_{f_0} \nu_f \left((1 + \Gamma) \partial_r \left[\partial_z W_f^* + \frac{\partial_r}{r} (r U^*) \right] + \left(\frac{\partial_r}{r} (r \partial_r) - \frac{1}{r^2} + \partial_z^2 \right) U^* \right), \end{aligned} \quad (220)$$

716 where Γ definition (200) has been used. They are now decomposed into steady-state
717 and perturbations.

718 4.2.3. Dimensionless steady-state fluid equations

At steady-state, the fluid unsteady components vanish, so that the dimensionless steady version of (218)–(220) reads

$$\left(\frac{\mathcal{M}}{\mathcal{C}} \right)^2 W_{st} \partial_z P_{st} + \partial_z W_{st} = 0, \quad (221)$$

$$\mathcal{M}e^{(\frac{\mathcal{M}}{\epsilon})^2 P_{st}} W_{st} \partial_Z W_{st} = -\mathcal{M} \partial_Z P_{st} + (\epsilon \delta)^2 (2 + \Gamma) \partial_Z^2 W_{st} + \delta^2 \frac{\partial_R}{R} (R \partial_R) W_{st}, \quad (222)$$

$$\frac{\mathcal{M}}{\epsilon^2} \partial_R P_{st} = (1 + \Gamma) \delta^2 \partial_R \partial_Z W_{st}, \quad (223)$$

From (221), the leading-order dimensionless steady solution (W_{st}^0, P_{st}^0) fulfills

$$\partial_Z W_{st}^0 = 0. \quad (224)$$

On the other hand, $\epsilon^2 \delta^2 / \mathcal{M} = \epsilon / Re \ll 1$ follows from the definition of δ, ϵ and \mathcal{M} in (205)–(206) so that using (224) in (223) leads to find that the steady-state leading-order pressure field is uniform per section

$$\partial_R P_{st}^0 = 0. \quad (225)$$

Finally, the steady-state leading-order axial mass conservation equation (222) results in equalizing a R-dependent function to a Z-dependent one

$$\mathcal{M} \partial_Z P_{st}^0 = \delta^2 \frac{\partial_R}{R} (R \partial_R W_{st}^0), \quad (226)$$

719 yielding the steady-state velocity W_{st}

$$W_{st}^0 = \frac{\mathcal{M}}{4\delta^2} \partial_Z P_{st}^0 [R^2 - 1] = \frac{\mathcal{M}}{4\delta} \partial_Z P_{st}^0 [\delta y - 2] y, \quad (227)$$

720 since $R = 1 - \delta y$. Note that, in the core region e.g. $R = 0$, $W_{st}^0 \sim \mathcal{M}/\delta^2 = \epsilon Re$
 721 which results from the steady-state balance between the inertial pressure with viscous
 722 dissipation. Furthermore, from (227), one can evaluate the steady-state leading order
 723 shear stress

$$\partial_y w_{st}^0 \equiv \tau_{st}^0 = \frac{\mathcal{M}}{4\delta} \partial_Z P_{st}^0 [\delta y - 2] + \frac{\mathcal{M}}{4} \partial_Z P_{st}^0 y, \quad (228)$$

724 which is thus found $\tau_{st}^0 \sim O(\mathcal{M}/\delta)$. From (228) one can now compare the relative
 725 contribution of the steady to transient friction in (85). Since $\tau_{tr} = \tau_f = \partial_y w(y, Z, \tau) \sim$
 726 $O(1)$ in (212) it is easy to find that the relative steady-state shear stress contribution
 727 compared to unsteady one is $O(\mathcal{M}/\delta)$. It first means that decomposition (85) is
 728 meaningful only if \mathcal{M}/δ is not too small, but also that if $\mathcal{M}/\delta \sim \delta$, then, there is a
 729 need to take into account the steady-state shear, when considering viscous dissipation
 730 for the unsteady contributions arising at $O(\delta)$. In section 4.2.5 it is shown that such

731 condition, i.e $\mathcal{M}/\delta \sim \delta$ also necessitates to take care of perturbed shear-stress radial
732 convection, a task not yet achieved in the literature. Last but not least, as done in
733 the next section, since the transient can always be written as the superposition of a
734 perturbation over a pre-existing steady-state both in the fluid and the solid, every
735 aspect of the steady-state including normal stress and shear-stress at the liquid/solid
736 interface can be subtracted from the analysis. In other words, decomposition (85)
737 cannot involve the steady shear τ_{st} for transient analysis, and this is why a quasi-
738 steady τ_{qst} is used. However, even if a Darcy-Weisbach quasi-steady shear (84) is
739 sometimes successfully used in engineering applications, to the best of our knowledge,
740 it is still lacking for theoretical basis. As a possible path for future investigation we will
741 however provide the conditions for steady-state and transient to be coupled together
742 from inertial non-linearities in the following sections.

743 4.2.4. Dimensionless unsteady bulk fluid equations

Subtracting the steady-state relations (221)–(223), from (218)–(220) whilst using scalings (208)–(211), the outer region dimensionless mass and momentum conservation equations read

$$\partial_\tau P + \mathcal{M}([W\partial_Z + U\partial_R](P + \mathcal{M}P_{st}) + W_{st}\partial_Z P) + \mathcal{C}^2 \left[\partial_Z W + \frac{1}{R}\partial_R(RU) \right] = 0, \quad (229)$$

$$e^{\frac{\mathcal{M}}{c^2}(P + \mathcal{M}P_{st})} (\partial_\tau W + \mathcal{M}([W\partial_Z + U\partial_R](W + W_{st}) + W_{st}\partial_Z W)) = -\partial_Z P + (\epsilon\delta)^2(1 + \Gamma)\partial_Z \left[\partial_Z W + \frac{1}{R}\partial_R(RU) \right] + \delta^2 \left(\frac{\partial_R}{R} R\partial_R + \epsilon^2 \partial_Z^2 \right) W, \quad (230)$$

$$e^{\frac{\mathcal{M}}{c^2}(P + \mathcal{M}P_{st})} (\partial_\tau + \mathcal{M}[(W + W_{st})\partial_Z + U\partial_R])U = -\frac{1}{\epsilon^2}\partial_R P + \delta^2(1 + \Gamma)\partial_R \left[\partial_Z W + \frac{1}{R}\partial_R(RU) \right] + \delta^2 \left(\frac{\partial_R}{R} (R\partial_R) - \frac{1}{R^2} + \epsilon^2 \partial_Z^2 \right) U, \quad (231)$$

744 The steady-state contributions into the unsteady fluid equations (229)–(231) appear
745 to be driven by the Mach number, i.e are $O(\mathcal{M})$. More precisely, from (227) one
746 finds that the outer steady-state contributions in (229)–(231) are $O(\mathcal{M}\epsilon Re)$, thus not
747 significant as far as $\delta \gg \mathcal{M}(\epsilon Re)$. Following the asymptotic framework of [65], since
748 $\delta^2 \gg \mathcal{M}$, no steady-state contributions arise into the outer region (core region of the
749 pipe). What happens in the unsteady boundary-layer is now discussed.

750 4.2.5. Dimensionless unsteady fluid boundary-layer equations

In the inner viscous zone, using rescaled coordinate y defined in (175) dimensionless Navier–Stokes equations are

$$\partial_\tau p + \mathcal{M} \left(\left[w \partial_Z - \frac{u}{\delta} \partial_y \right] (p + \mathcal{M} P_{st}) + w_{st} \partial_Z p \right) + \mathcal{C}^2 \left[\partial_Z w - \frac{1}{\delta} \frac{1}{1 - \delta y} \partial_y ((1 - \delta y)u) \right] = 0, \quad (232)$$

$$\begin{aligned} e^{\frac{\mathcal{M}}{\mathcal{C}^2} (p + \mathcal{M} P_{st})} \left(\partial_\tau w + \mathcal{M} \left(\left[w \partial_Z - \frac{u}{\delta} \partial_y \right] (w + w_{st}) + w_{st} \partial_Z w \right) \right) \\ = -\partial_Z p + (\epsilon \delta)^2 (1 + \Gamma) \partial_Z \left[\partial_Z w - \frac{1}{\delta} \frac{1}{1 - \delta y} \partial_y ((1 - \delta y)u) \right] \\ + \left(\frac{\partial_y}{1 - \delta y} ((1 - \delta y) \partial_y) + (\epsilon \delta)^2 \partial_Z^2 \right) w, \quad (233) \end{aligned}$$

$$\begin{aligned} e^{\frac{\mathcal{M}}{\mathcal{C}^2} (p + \mathcal{M} P_{st})} \left(\partial_\tau + \mathcal{M} \left[(w + w_{st}) \partial_Z - \frac{u}{\delta} \partial_y \right] \right) u = \frac{1}{\delta \epsilon^2} \partial_y p \\ - (1 + \Gamma) \partial_y \left[\delta \partial_Z w - \frac{1}{1 - \delta y} \partial_y ((1 - \delta y)u) \right] \\ + \left(\frac{\partial_y}{1 - \delta y} ((1 - \delta y) \partial_y) - \frac{\delta^2}{(1 - \delta y)^2} + (\epsilon \delta)^2 \partial_Z^2 \right) u. \quad (234) \end{aligned}$$

(233) can be expanded and simplified to

$$\begin{aligned} e^{\frac{\mathcal{M}}{\mathcal{C}^2} (p + \mathcal{M} P_{st})} \left(\partial_\tau w + \mathcal{M} \left((w + w_{st}) \partial_Z w - \frac{u}{\delta} (\tau_f + \partial_y w_{st}) + w \partial_Z w_{st} \right) \right) \\ = -\partial_Z p + (\epsilon \delta)^2 (1 + \Gamma) \partial_Z \left[\partial_Z w - \frac{1}{\delta} \left(\partial_y u - \delta \frac{u}{1 - \delta y} \right) \right] \\ + \partial_y^2 w - \delta \frac{\tau_f}{1 - \delta y} + (\epsilon \delta)^2 \partial_Z^2 w, \quad (235) \end{aligned}$$

751 where dimensionless shear-rate $\tau_f = \partial_y w$ defined in (212) has been used. From (235),
752 one finds that various possible dissipation term can arise

- 753 • $O\left(\frac{\mathcal{M}}{\mathcal{C}^2}\right)$ and $O\left(\left(\frac{\mathcal{M}}{\mathcal{C}}\right)^2\right)$ fluid density compressibility effects (217);
- 754 • $O(\epsilon^2 \delta)$ radial flow compressibility effects within the inner region;
- 755 • $O(\epsilon^2 \delta^2)$ and $O(\epsilon^2 \delta)$ axial diffusion and radial flow compressibility;
- 756 • $O(\mathcal{M})$ and $O(\mathcal{M} \epsilon Re)$, axial inertial corrections;
- 757 • $O\left(\frac{\mathcal{M}}{\delta}\right)$ and $O\left(\frac{\mathcal{M} \epsilon Re}{\delta}\right)$ radial inertial transport of viscous shear;
- 758 • $O(\delta)$ radial diffusion transport of viscous shear.

759

760 This very last effect is the only one been analyzed by [65] reaching to the "classi-
761 cal" four equation FSI (33)-(34) model (more precisely a perturbative version of it, as
762 further detailed in 4.2.6). Furthermore, since the leading-order steady dimensionless
763 inner velocity field (227) is $w_{st} \sim O(\mathcal{M}/\delta^2) \equiv O(\epsilon Re)$ the steady-state/perturbation
764 couplings terms in (235) are either $O(\mathcal{M}\epsilon Re)$ or $O(\mathcal{M}\epsilon Re/\delta)$. As already mentioned
765 in (207), [65] have considered the asymptotic setting (207) for which all these contri-
766 butions are not significant, so that transient and steady-state are decoupled. Never-
767 theless, (235) shows that there is a possible non-trivial one-way coupling between the
768 steady-state and the perturbation when the steady-state inertia rises, i.e for $\epsilon Re > 1$,
769 when $\mathcal{M}\epsilon Re/\delta \sim \delta$, i.e $\delta^2 \sim \mathcal{M}\epsilon Re$ (keeping with the largest perturbative term only).
770 This asymptotic regime has not yet been analyzed in the literature but it might result
771 in a deeper understanding of why, as inertia increases, there is a need for changing the
772 friction model depending on the steady flow inertia as already mentioned in section
773 2.3.

774 4.2.6. Dimensionless boundary condition and FSI perturbative two-wave formu- 775 lation

776 From (198)-(199) and (195) it has been shown in [65] that dimensionless stress
777 boundary conditions at the solid-fluid interface settled at $R = 1 + O(\alpha\mathcal{M})$ and at the
778 solid external boundary at $R = 1 + \alpha + O(\alpha\mathcal{M})$ read

$$\sigma_{rr}|_{R=1+O(\alpha\mathcal{M})} = -p|_{y=O(\frac{\alpha\mathcal{M}}{\delta})} + O(\alpha\epsilon\mathcal{M}), \quad \& \quad \sigma_{rr}|_{R=1+\alpha+O(\alpha\mathcal{M})} = O(\alpha\epsilon\mathcal{M}), \quad (236)$$

$$\sigma_{rz}|_{R=1+O(\alpha\mathcal{M})} = -\delta\tau_w|_{y=O(\frac{\alpha\mathcal{M}}{\delta})} + O(\alpha\epsilon\mathcal{M}), \quad \& \quad \sigma_{rz}|_{R=1+\alpha+O(\alpha\mathcal{M})} = O(\alpha\epsilon\mathcal{M}), \quad (237)$$

779 It is important to note that the transient shear-stress coupling between the solid
780 and the fluid found in (237) appears as an $O(\delta)$ correction. At leading order, the
781 solid response is independent from the fluid dissipation. This very small detail is the
782 corner-stone of the perturbative approach of [65], since it implies that FSI can only
783 occur perturbatively and should be derived as such. Nevertheless, including $O(\delta)$ then
784 also necessitates, for asymptotic consistency, to take care of small time-scale variations
785 from (156) also providing $O(\delta)$ corrections.

Using fluid pressure/longitudinal stress vector \mathbf{P} previously defined in (40), a per-
turbative expansion of \mathbf{P} is considered as

$$\mathbf{P}^0 + \delta\mathbf{P}^1 = \begin{pmatrix} P^0 + \delta P^1 \\ \sigma_{zz}^0 + \delta\sigma_{zz}^1 \end{pmatrix}. \quad (238)$$

Inlet/outlet boundary conditions are also needed for the wave problem to be closed. For a single pipe, having dimensionless length unity, they read as follows (Cf [65] for more information)

$$\partial_Z P^0|_{Z=1} = \delta(\tau) \quad , \text{ and, } \quad \partial_Z P^1|_{Z=1} = 0, \quad (239)$$

where $\delta(\tau)$ is the Dirac distribution. Upstream, the homogeneous Dirichlet condition applied on the pressure trivially leads to

$$P^0|_{Z=0} = P^1|_{Z=0} = 0. \quad (240)$$

In the solid, the upstream and downstream conditions are

$$\partial_Z \sigma_{zz}^0|_{Z=0\&1} = 0 \quad , \text{ and, } \quad \partial_Z \sigma_{zz}^1|_{Z=0\&1} = -\frac{2}{\alpha(2+\alpha)} \tau_w^0|_{Z=0\&1}. \quad (241)$$

[65] establishes a perturbative coupled waves equation reading

$$(\partial_\tau^2 - \mathbf{C}_{\mathbf{P}}^2 \partial_Z^2) [\mathbf{P}^0 + \delta \mathbf{P}^1] = -2\delta \left[\partial_T \partial_\tau \mathbf{P}^0 - \partial_Z \tau_w^0 \left(\frac{1 + \frac{2\nu_s \mathcal{D}}{\alpha(2+\alpha)}}{\frac{1}{\alpha(2+\alpha)}} \left[2\nu_s \mathcal{D} + \mathcal{C}_s^2 + \frac{4\nu_s^2 \mathcal{D}}{\alpha(2+\alpha)} \right] \right) \right], \quad (242)$$

786 where matrix $\mathbf{C}_{\mathbf{P}}^2$ defined in (40) has been used. (242) is the dimensionless
 787 perturbative version of (33)-(34), and the strict equivalence between the wave
 788 operator has been shown in [65]. Note that the limit $\nu_s \rightarrow 0$, (242) degenerates
 789 ates to the non-FSI dissipative pressure wave (83). The leading-order of (242)
 790 displays a parabolic form without dissipation associated with a fast time-scale
 791 wave propagation already analyzed in section 2.2.1, as opposed to the additional
 792 slow-time scale damping that arises when $O(\delta)$ corrections are considered. This
 793 short-time behavior appears because the dissipation in the fluid boundary layer
 794 does not have time to develop, and the coupled system remains purely conservative.
 795 We now present how this perturbative FSI couple wave can be analyzed
 796 within a two-time scale asymptotic analysis similar to section 4.1.

797 4.3. Two-time scale asymptotic analysis of water-hammer with FSI

The Laplace transform of the transient wall shear-stress has been derived in [65] and reads

$$\partial_Z \tilde{\tau}_w^0 = -s\sqrt{s} \left(\left[1 - (1 - 2\nu_s) \frac{2\nu_s \mathcal{D}}{\alpha \mathcal{C}_s^2 (2 + \alpha)} \right] \tilde{P}^0 + (1 - 2\nu_s) \frac{\mathcal{D}}{\mathcal{C}_s^2} \tilde{\sigma}_{zz}^0 \right). \quad (243)$$

798 The axial gradient of the fluid wall shear stress is thus a linear combination of
 799 \tilde{P}^0 and σ_{zz}^0 . The Laplace transform of (242) then reads

$$(s^2 - \mathbf{C}_P^2 \partial_Z^2) \tilde{\mathbf{P}}^0 = \mathbf{0}, \quad (244)$$

$$(s^2 - \mathbf{C}_P^2 \partial_Z^2) \tilde{\mathbf{P}}^1 = -2s [\partial_T + \sqrt{s} \mathbf{E}] \tilde{\mathbf{P}}^0, \quad (245)$$

where matrix \mathbf{E} has been introduced

$$\mathbf{E} = \frac{1}{2\nu_s} \begin{pmatrix} \left(1 - (1 - 2\nu_s) \frac{c^2 - 1}{c_s^2}\right) \left(1 + \frac{2\nu_s \mathcal{D}}{\alpha(2 + \alpha)}\right) & \frac{2\nu_s \mathcal{D}(1 - 2\nu_s)}{c_s^2} \left(1 + \frac{2\nu_s \mathcal{D}}{\alpha(2 + \alpha)}\right) \\ \left(1 - (1 - 2\nu_s) \frac{c^2 - 1}{c_s^2}\right) \frac{c^2 - (1 - 2\nu_s)}{\alpha(2 + \alpha)} & \frac{2\nu_s \mathcal{D}(1 - 2\nu_s)}{c_s^2} \frac{c^2 - (1 - 2\nu_s)}{\alpha(2 + \alpha)} \end{pmatrix}. \quad (246)$$

800 (244) and (245) are the FSI generalization of (158) and (188). 2×2 matrix \mathbf{E}
 801 (246) then permits to find the damping rate from following the very same foot-
 802 steps as in section 4.1. Using previously defined base change (45) associated with
 803 diagonal base $\mathcal{P} = \mathbf{\Pi}^{-1} \mathbf{P}$, and decomposing \mathcal{P} over eigenfunctions (58)-(59),
 804 then leads to

$$\tilde{\mathcal{P}}^0(Z, s, T) = \sum_{\lambda_k \in \mathbb{R}} \tilde{a}_{\lambda_k}^0(s) A_{\lambda_k}(T) \Phi_{\lambda_k}(Z) + \tilde{\mathcal{P}}_p^0(Z), \quad (247)$$

$$\tilde{\mathcal{P}}^1(Z, s) = \sum_{\lambda_k \in \mathbb{R}} \tilde{a}_{\lambda_k}^1(s) \Phi_{\lambda_k}(Z) + \tilde{\mathcal{P}}_p^1(Z, s), \quad (248)$$

805 with particular solution associated with non-homogeneous boundary conditions
 806 (239)-(240) and (241)

$$\tilde{\mathcal{P}}_p^0(Z) = \frac{Z}{\det(\mathbf{\Pi})} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (249)$$

$$\tilde{\mathcal{P}}_p^1(Z, s) = \frac{1}{\alpha(2 + \alpha)\sqrt{s}} \left(\frac{Z^2}{1 - \frac{c_+}{c - \beta}} \begin{pmatrix} 1 \\ \frac{-c_+}{c - \beta} \end{pmatrix} - \frac{4\nu_s \mathcal{D} \left(\frac{Z^2}{2} - Z\right)}{c_-^2 - 1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{c - \beta}{c_+} \end{pmatrix} \right) \partial_Z \tilde{\mathcal{P}}^0(Z, s).$$

The Laplace transform of (244) leads to

$$\sum_{\lambda_k \in \mathbb{R}} (s^2 - \mathcal{H}) \tilde{a}_{\lambda_k}^0(s) \Phi_{\lambda_k}(Z) = -s^2 \tilde{\mathcal{P}}_p^0(Z), \quad (251)$$

where operator \mathcal{H} has been defined in (58). Using (244), (247) and the orthog-
 onality of the eigenfunction basis one gets

$$\tilde{a}_{\lambda_k}^0(s) + \tilde{a}_{-\lambda_k}^0(s) = \left[\frac{\lambda_k}{2i} \left(\frac{1}{s - i\lambda_k} - \frac{1}{s + i\lambda_k} \right) - 1 \right] \langle \tilde{\mathcal{P}}_p^0(Z), \Phi_{\lambda_k}(Z) \rangle. \quad (252)$$

leading to

$$\tilde{a}_{\pm\lambda_k}^0(s) = -\frac{1}{2} \left[\frac{\pm i\lambda_k}{s \mp i\lambda_k} + 1 \right] \langle \tilde{\mathcal{P}}_p^0(Z), \Phi_{\lambda_k}(Z) \rangle. \quad (253)$$

a result which generalizes (169). At next order, the Laplace transform of (245) is

$$\begin{aligned} \sum_{\lambda_k \in \mathbb{R}} (s^2 + \lambda_k^2) \tilde{a}_{\lambda_k}^1(s) \Phi_{\lambda_k}(Z) &= -(s^2 - \mathcal{H}) \tilde{\mathcal{P}}_p^1(Z, s) - 2s\sqrt{s}\mathcal{E}\tilde{\mathcal{P}}_p^0(Z, s) \\ + s \sum_{\lambda_k \in \mathbb{R}} [\partial_T + \sqrt{s}\mathcal{E}] A_{\lambda_k}(T) &\left[\frac{i\lambda_k}{s - i\lambda_k} + 1 \right] \langle \tilde{\mathcal{P}}_p^0(Z, s), \Phi_{\lambda_k}(Z) \rangle \Phi_{\lambda_k}(Z). \end{aligned} \quad (254)$$

where $\mathcal{E} = \mathbf{\Pi} \cdot \mathbf{E} \cdot \mathbf{\Pi}^{-1}$ is the base-transform of matrix \mathbf{E} with, again the base change matrix $\mathbf{\Pi}$ defined in (45). As previously detailed in section 4.1, the secularity condition is found from suppressing resonant modes of $\tilde{a}_{\lambda_k}^1(s)$ solutions. Defining the 2-component vector function

$$\mathcal{J}_{\lambda_k}(Z, s) = \frac{\lambda_k}{\alpha(2 + \alpha) \left(1 - \frac{c_- \beta}{c_+}\right)} \begin{pmatrix} \left(s^2 \left(\frac{Z^2}{2} - Z\right) - c_-^2\right) \frac{\tan\left(\frac{\lambda_k}{c_-}\right)}{c_-} \\ - \left(s^2 \left(\frac{Z^2}{2} - Z\right) - c_+^2\right) \frac{\tan\left(\frac{\lambda_k}{c_+}\right)}{c_+} \end{pmatrix}, \quad (255)$$

[65] found that the FSI secularity condition reads

$$\lim_{s \rightarrow \pm i\lambda_k} \left(\partial_T + \sqrt{s} \langle \mathcal{E} \Phi_{\lambda_k}(Z) - \frac{\mathcal{J}_{\lambda_k}(Z, s)}{s^2}, \Phi_{\lambda_k}(Z) \rangle \right) A_{\pm\lambda_k}(T) = 0. \quad (256)$$

807 This leads to the FSI slow-time attenuation amplitude

$$A_{\lambda_k}(T) = e^{-\sqrt{i\lambda_k} \frac{T}{\mathcal{T}_{\lambda_k}}}, \quad (257)$$

$$\mathcal{T}_{\lambda_k}^{-1} = \langle \mathcal{E} \Phi_{\lambda_k}(Z) + \frac{\mathcal{J}_{\lambda_k}(Z, s = i\lambda_k)}{\lambda_k^2}, \Phi_{\lambda_k}(Z) \rangle, \quad (258)$$

808 More explicit (and rather cumbersome) computation of \mathcal{T}_{λ_k} are detailed in [65]'s
 809 Appendix. [65] found that the limit $\nu_s \rightarrow 0$ of (258) tends to (193). This FSI
 810 theoretical prediction has been compared to experimental results as well as
 811 non-FSI attenuation in [65]. More precisely, the pressure signature is compared
 812 at two distinct locations in Figure 15 from ([99])'s data set. The special case
 813 $\nu_s \rightarrow 0$ or that of ([95]) is again depicted. Each analytical solutions exhibits
 814 excellent agreement for both amplitude and phase for every considered pipe's

Table 2: Physical and geometrical properties for the analysis of the reservoir pipe anchored valve system. (*) refers to unavailable data in the original article. They were estimated in [65] based up available properties of pure copper tube and water.

Article	Density ($kg \cdot m^{-3}$)	Elasticity ($10^9 Pa$)	ν_f ($m^2 \cdot s^{-1}$)	ν_s	Geometry (m)
([107])	$\rho_{f_0}^* = 998.3$ $\rho_{s_0}^* = 8935.0$	$\mathcal{K}_f^* = 2.1$ $E^* = 127.0$	$3.967 \cdot 10^{-5}$	0.34^*	$R = 0.0127$ $e = 0.001651$ $L = 36.088$
([99])	$\rho_{f_0} = 1000.0$ $\rho_{s_0} = 8890.0$	$\mathcal{K}_f = 2.1$ $E = 120.0$	$9.493 \cdot 10.0^{-7.0}$	0.35	$R = 0.008$ $e = 0.001$ $L = 98.11$
([98])	$\rho_{f_0}^* = 1000.0$ $\rho_{s_0}^* = 8960.0$	$\mathcal{K}_f^* = 2.1$ $E^* = 130.0$	$1.182 \cdot 10^{-6.0}$	0.3^*	$R = 0.01105$ $e = 0.00163$ $L = 37.23$
([168])	$\rho_{f_0}^* = 1000.0$ $\rho_{s_0}^* = 8960.0$	$\mathcal{K}_f^* = 2.1$ $E^* = 130.0$	$10^{-6.0}$	0.3^*	$R = 0.01$ $e = 0.001$ $L = 15.22$

815 locations with experimental observations. No parameter fit is used. The vari-
816 ety of observed patterns of the pressure signal depicted in Figure 15 and the
817 surprisingly precise predictions provided by the theory results from the com-
818 plex mode decomposition $\Phi_{\lambda_k}(Z)$, each with its own phase. In Figure (15a), a
819 deeper analysis of the pressure signature reveals that ([95])’s theory leads to a
820 better agreement with experimental data in the early times, i.e. $\tau \ll O(1/\delta)$.
821 At longer times, both models correctly describe the attenuation, ([95])’s theory
822 under-attenuating, whilst the hereby developed one slightly over-attenuating.
823 In Figure (15b) however, the present analysis shows excellent agreement with
824 experimental data at long time, ([95])’s theory again under-attenuating. It is
825 worth noting that these differences are minor in both configurations as the (FSI)
826 coupling has little influences in this experimental data set.
827 To deepen the analysis of the new prediction for (FSI) damping, Figures (16a)–

828 (16d) then focus on the damping envelope of the first exponential mode. A
 829 comparison with four sets of experiments is provided. For each experiment, the
 830 pressures of the envelope peaks are extracted, non-dimensionalized, and com-
 831 pared with the theoretical damping trend. Figures (16a)–(16d) reveal a very
 832 good agreement between the predictions and experiments for laminar and tran-
 833 sitional Reynolds numbers. As the first mode damping is dominant over others
 834 at long time, the match between predictions and observations becomes better
 835 with time, as expected.

836 Last, but not least, the asymptotic analysis of the FSI water-hammer wave
 837 [65] has permitted to derive the FSI transient wall shear-stress which indeed
 838 differs from the non-FSI one (181). In frequency domain, both transient shear-
 839 rate and wall-shear stress have been found equal to

$$\tilde{\tau}_f^0 = \partial_y \tilde{w}^0 = -\sqrt{s} \left[\frac{1}{s} \partial_Z \tilde{P}^0 + s\alpha \tilde{\zeta}^0 \right] e^{-\sqrt{s}y}, \quad (259)$$

$$\tilde{\tau}_w^0 = -\sqrt{s} \left[\frac{1}{s} \partial_Z \tilde{P}^0 + s\alpha \tilde{\zeta}^0 \right]. \quad (260)$$

840 where $\tilde{\zeta}^0$ is the leading order longitudinal solid displacement at the fluid-solid in-
 841 terface $R = 1$ whose relation with $\mathbf{P}^0 = \begin{pmatrix} P^0 \\ \sigma_{zz}^0 \end{pmatrix}$ components (and their Laplace
 842 transform) are

$$\frac{\alpha \mathcal{C}_s^2}{\mathcal{D}} \partial_Z \tilde{\zeta}^0 = \tilde{\sigma}_{zz}^0 - \frac{2\nu_s}{\alpha(2 + \alpha)} \tilde{p}^0 \quad (261)$$

Comparing (260) with (181) reveals that a supplementary FSI term associated
 with the solid longitudinal displacement should be included in the wall shear
 stress. [65] also found that re-writing (260) in time-domain in a form similar
 with Zielke’s one in [109] leads to

$$\tau_w^0 = \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{\partial_{\tau'} [W^0 - \alpha \partial_{\tau'} \zeta^0]}{\sqrt{\tau - \tau'}} d\tau'. \quad (262)$$

843 (262) shows that the supplementary FSI term leads to consider the *relative* ac-
 844 celeration of the fluid to that of the pipe’s wall, rather than the fluid acceleration
 845 only (chosen by Zielke in [109]) as previously suggested in [44]. This result with
 846 simple physical interpretation however necessitates to solve for the FSI problem

847 in order to find the wall longitudinal acceleration. This has been solved in [65]
 848 where the relative FSI contribution to the wall shear-stress has been analyzed
 849 as illustrated in figure (19) and (20). Figure (19) displays the time variations of
 850 the transient wall shear-stress, showing huge peaks synchronized with the wave
 851 front's passages. When comparing the FSI and non-FSI results for two distinct
 852 density ratio \mathcal{D} , one finds that the FSI relative contribution increases as \mathcal{D} rises
 853 to $O(1)$ (more precisely [65] have analyzed the case $\mathcal{D} = 1$). Figure 20 analyzed
 854 further the relative FSI contribution to the wall shear-stress. It shows that this
 855 relative contribution can become as large as 50% for a relative wall thickness of
 856 10% and Poisson modulus $\nu_s = 0.3$. These results suggest that for the in-vivo
 857 biomedical context of blood pressure pulse wave propagation within vascular
 858 network, the FSI contribution to transient wall-shear stress cannot be ignored.

859 4.4. Theoretical analysis of junction coupling within complex networks

860 This section covers how wave propagation within networks can be formulated
 861 within a pipe network described by a graph. Most of the content of this section
 862 discusses the success toward an "algebraic" (a wording to be clarified later on
 863 in section 4.4.2) formulation of water-hammer wave propagation within graphs.

864 Section 4.4.1 discuss the frequency domain approach whereas section 4.4.2
 865 reviews time-domain one. In the following we denote usual graphs $G(\mathcal{V}, \mathcal{E})$
 866 having vertex set \mathcal{V} and edge one \mathcal{E} the cardinal of which are respectively denoted
 867 V and E . The graph adjacency matrix is denoted \mathbf{A} . In section 4.4.2 the concept
 868 of metric graph denoted $\mathcal{G}(\mathcal{V}, \mathcal{E})$ will also be introduced.

869 4.4.1. Discrete graph-wave solutions for wave propagation within networks

One of the first generalizations of the TM method to networks has first been
 performed into simple networks such as those analyzed by S. Kim 21. The
 looped network along with its spectrum, i.e. linear network admittance system
 for [169, 170]'s analysis is depicted in Figure 21. [171] develop a frequency-
 domain network admittance formulation (from Laplace transform) to solve the
 first order coupled hyperbolic pressure/velocity problem (82) (expressed as a

pressure/flux in [171]) within general water distribution networks. A series impedance approach leads to a graph-node spatial discretization of the one-dimensional gradients, and also to some analytical node-to-node relation along each pipe-line similar to TMM's solutions (47)-(48) (except for being related to pressure/flux vector rather than pressure/stress since (47)-(48) are related to FSI's TMM). More explicitly following [172, 173], the Laplace transform of (82) for each network link e_{ij} joining nodes i and j , parametrized by dimensionless coordinate Z , leads to a parametric linear relation between the pressure p_{ij} , the flux q_{ij} and their gradients

$$Z_{ij}(Z, s)q_{ij}(Z) = -\partial_Z p_{ij}(Z) \quad , \quad Y_{ij}(Z, s)p_{ij}(Z) = -\partial_Z q_{ij}(Z), \quad (263)$$

defining the shunt admittance per unit-length Y_{ij} related to the compressibility in the flow produced by pressure variations, and Z_{ij} the impedance per unit length. These quantities permit to evaluate the "propagation operator" $\Gamma_{ij} = \sqrt{Y_{ij}Z_{ij}}$ [172, 173]. [171] has found the frequency-domain graph admittance matrix \mathfrak{Y} related to the adjacency matrix \mathbf{A} as

$$\mathfrak{Y}_{ij} = -\delta_{ij} \left(\sum_{m \in \mathfrak{N}(i)} A_{im} \frac{1}{Z_{ij}(s) \tanh(\Gamma_{ij}(s))} \right) + A_{ij} \frac{1}{Z_{ij}(s) \sinh(\Gamma_{ij}(s))}. \quad (264)$$

Introducing the pressure field at each node of the network as vector ϕ whose components are ϕ_j for j in the vertex set of the graph, i.e $j \in [1, V]$, Matrix \mathfrak{Y} relates each pressure/flux at network's nodes as

$$\mathfrak{Y}(s)\phi = \mathbf{Q}. \quad (265)$$

870 In [171] (265) is used to build a frequency domain network transfer matrix
871 for transient computation inside network, when including boundary conditions.
872 [174] consider the extension of [171]'s framework, supplementing the "passive"
873 graph nodes with "compound nodes" for which a dynamical behavior with non-
874 linear (more precisely quadratic non-linearities) relations between input/output
875 is specified. These "compound nodes" are modeling complex hydraulic compo-
876 nents —such as junctions, air vessels, valves— in water distribution networks.

877 There are taken care-off in [174] from an expansion of the graph nodes with sup-
878plementary nodes associated with the new degree of freedom of each "compound
879nodes". When linearizing the "compound nodes" [174] provides an extension of
880[171] from generalizing the frequency-domain graph admittance matrix \mathfrak{Y} (264).
881[174] also provides direct comparison between time-domain numerical computa-
882tion of the complete non-linear transient problem solved with the MOC method
883and the frequency-domain graph admittance matrix method. The numerical
884results of [174] illustrate that the higher the pressure, the higher the error as-
885sociated with non-linearities, as expected. They also suggest that, for realistic
886pressure range of water-hammer within water distribution networks, neglecting
887the compound nodes's non-linearities does not produce a large error. This obser-
888vation should obviously be strengthened in order to more precisely know under
889which precise conditions these non-linearities have to be taken into account.

890 [176, 175] have developed a numerical Inverse Laplace Transform (NILT) pro-
891cedure allowing a time-domain numerical evaluation of the previously described
892frequency-domain network admittance formulation for networks, as illustrated
893in Figure 22. Figure 22 shows that the low-frequency components of the pres-
894sure signal can be well approximated by the (NILT) method. [177] recently used
895the NILT method [176, 175] to investigate the transient behavior of an Y-pipe
896experimental setup. In the following section, we now describe how time-domain
897graph's solutions can also be developed.

898 4.4.2. *Spectral quantum graph method for solving wave propagation within net-* 899 *works*

Quantum graphs is the name for operators equipping a metric graph, i.e
a graph whose one-dimensional edges have a physical length. This theoretical
concept has been used to study vibrations and spectrum structure of metric
graphs [178], quantum chaos e.g [179, 180], wave scattering [181, 182], and more
recently wave propagation in networks [183, 184, 185]. Even if the applica-
tion of this method for water-hammer propagation modeling is still very recent,
since it is a promising one (because it opens future theoretical advances to be

made into wave's propagation within complex networks, where most previous investigations could only address them from direct numerical simulation, —e.g with MOC method—), it is hereby discussed in more details following the time-domain approach proposed in [185]. The interest of quantum graph method over traditional numerical modeling already discussed in section 2.2.2 is that the wave solution is searched as a decomposition onto a continuous base defined over the entire metric graph. Each element of the base is denoted Ψ_λ . It generalizes into a network the eigenfunction of the Laplacian operator previously considered in (58). On the contrary to traditional approaches this base, does not need to be discretized along each edge e_{ij} joining vertex i and j of the graph since it displays an analytical shape along those, i.e it is a spectral base. Such spectral discretization along each network's edge can save a huge numerical cost for large networks as discussed in [184]. The approach is nevertheless yet restricted to address linear wave operators solutions. Furthermore on the contrary to traditional numerical discrete approaches, a quantum graph formulation permits to establish general results, as the ones found in [185], that numerical computations can only illustrate. Before describing more theoretically this concept, let-us first illustrate it on a simple 3-star graph depicted in figure 23. Figure 24 shows the first two eigenmodes Ψ_{λ_1} and Ψ_{λ_2} onto the 3-star metric graph of figure 23. One can observe in figure 24 that each eigenmode is continuous along each internal node of the graph (i.e in this example, only node O), and also that the sum of the derivatives on each node is zero, which is precisely why these modes are called Kirchhoff eigenmodes. Let-us now provide more information about eigenmode Ψ_λ , denoting $\Psi_{ij}(x)$ its component along each edge e_{ij} of length ℓ_{ij} joining vertex i and j of the graph. Each $\Psi_{ij}(x)$ fulfills to be an eigenfunction of the one dimensional Laplacian

$$-\frac{d^2}{dx^2}\Psi_{ij}(x) = \lambda^2\Psi_{ij}(x). \quad (266)$$

Furthermore, inside any vertex, there is a Dirichlet continuity at vertex i and j

$$\Psi_{ij}(0) = \phi_i \quad \Psi_{ij}(\ell_{ij}) = \phi_j. \quad (267)$$

Denoting A_{ij} the components of the graph adjacency matrix \mathbf{A} , boundary conditions at each vertex i reads

$$-\sum_{j<i} A_{ij} \frac{d}{dx} \Psi_{ij}(\ell_{ij}) + \sum_{j>i} A_{ij} \frac{d}{dx} \Psi_{ij}(0) = h_i \phi_i. \quad (268)$$

Condition (268) is called a Kirchhoff-Robin condition. For $h_i = 0$, it degenerates into a Kirchhoff condition. In the limit $h_i \rightarrow \infty$, it provides a Dirichlet condition at vertex i where it imposes $\phi_i = 0$. Given continuity conditions (267), the spectral base $\Psi_{ij}(x)$ is

$$\Psi_{ij}(x) = \frac{A_{i,j}}{\sin(\lambda \ell_{ij})} (\phi_i \sin(\lambda(\ell_{ij} - x)) + \phi_j \sin(\lambda x)). \quad (269)$$

The restriction of Ψ_λ over the vertex set \mathcal{V} i.e $\Psi_\lambda|_{\mathcal{V}}$ defines a V -component vector denoted ϕ_λ having ϕ_j amplitudes with $j \in [1, V]$ where V is the cardinal of \mathcal{V} . Furthermore parameter λ not yet specified, needs to be set so that the operator boundary conditions (268) at each vertex are satisfied. It can be shown (Cf [186, 187, 188, 189, 185] for more information) that specific values of λ called the quantum graph operator spectrum permits these conditions to arise. Defining secular matrix \mathfrak{A} as

$$\mathfrak{A}_{ij}(\lambda, \mathbf{h}) = -\delta_{ij} \left(\sum_{m \in \mathfrak{N}(i)} A_{im} \cot(\lambda \ell_{im}) + \frac{h_i}{\lambda} \right) + A_{ij} \frac{1}{\sin(\lambda \ell_{ij})}. \quad (270)$$

where $\mathfrak{N}(i) = \{m \mid A_{im} \neq 0\}$ being the neighbors of vertex i associated with non-zero components of adjacency matrix \mathbf{A} . The analogy between the secular time-domain matrix (270) and the frequency domain one (264) should be pointed out. Matrix \mathfrak{A} is built so that (268) is equivalent to

$$\sum_{j=1}^V \mathfrak{A}_{ij} \phi_j = \mathfrak{A} \phi_\lambda = 0, \quad (271)$$

where, again, ϕ_λ is the V -component vector of amplitudes ϕ_j . (271) can have a non-trivial solution different from zero when the so-called secular condition is met

$$\det \mathfrak{A}(\lambda, \mathbf{h}) = 0. \quad (272)$$

900 (272) provides an algebraic formulation for the wave's spectrum. It is very
 901 nice to observe that this condition generalizes to any metric graph the one
 902 used for a single pipe for FSI TMM solutions (56) or the time-domain ones
 903 (69). There is indeed a deep mathematical reason for this, since the associate
 904 operator having linear boundary conditions, they can be expressed as resulting
 905 from the action of a matrix acting on the field value (and possibly its gradient).
 906 It results in the common feature that the discrete possible frequencies called the
 907 spectrum is found from an algebraic zero determinant condition. (272) provides
 908 the spectrum of (infinite) discrete set of values of positive ordered λ such that
 909 $0 \leq \lambda_1 < \lambda_2 \dots$. Let-us illustrate more precisely this concept on the 3-star
 910 graph illustrated in figure 23 whose adjacency matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (273)$$

911 Using definition (270) the secular matrix reads in this case

$$\mathfrak{A}(\lambda, \mathbf{h}) = \begin{bmatrix} -\cot(\ell_{21}\lambda) & \frac{1}{\sin(\ell_{21}\lambda)} & 0 & 0 \\ \frac{1}{\sin(\ell_{21}\lambda)} & -\frac{h_m}{\lambda} - \cot(\ell_{21}\lambda) - \cot(\ell_{32}\lambda) - \cot(\ell_{24}\lambda) & \frac{1}{\sin(\ell_{32}\lambda)} & \frac{1}{\sin(\ell_{24}\lambda)} \\ 0 & \frac{1}{\sin(\ell_{32}\lambda)} & -\cot(\ell_{32}\lambda) & 0 \\ 0 & \frac{1}{\sin(\ell_{24}\lambda)} & 0 & -\cot(\ell_{24}\lambda) \end{bmatrix} \quad (274)$$

whose determinant is

$$\det \mathfrak{A}(\lambda, \mathbf{h}) = \left[\left(\left(\frac{h_m}{\lambda} \cot(\ell_{24}\lambda) - 1 \right) \cot(\ell_{23}\lambda) - \cot(\ell_{24}\lambda) \right) \cot(\ell_{21}\lambda) - \cot(\ell_{23}\lambda) \cot(\ell_{24}\lambda) \right]. \quad (275)$$

Numerically solving for $\det \mathfrak{A}(\lambda_n, \mathbf{0}) = 0$ gives the eigenvalues λ_n of the Kirchhoff
 modes. This spectrum is complemented with the vertex vector ϕ_{λ_n} of quantum
 graph modes n (269) obtained from computing the one dimensional null-space
 of matrix $\mathfrak{A}(\lambda_n, \mathbf{0})$ (here a 4-vector) which then sets Ψ_{λ_n} . [185] provides tables
 with λ_n and ϕ_{λ_n} values of the first twenty Kirchhoff modes. Figure 24 illustrates
 the first two Kirchhoff mode Ψ_{λ_1} and Ψ_{λ_2} , showing as expected an increasing

number of minimum and maximum as λ_n increases. The spectral base of the quantum graph then permits to find the time-domain pressure-wave solution as decomposed into it. In [185] the two-time scale analysis has furthermore been used to analyze the damped wave pressure problem (83) within the entire network. Following the very same steps as in section 4.1, the pressure solution is decomposed into

$$\mathbf{P}_{\mathcal{G}} = \mathbf{P}_{\mathcal{G}}^0 + \delta\mathbf{P}_{\mathcal{G}}^1 + \dots \quad (276)$$

912 In [185] each term of the expansion (276) is then decomposed into an homoge-
913 neous and a particular solution similarly as (162) and (186)

$$\begin{aligned} \mathbf{P}_{\mathcal{G}}^0 &= \sum_{n \in \mathbb{N}} (A_n(T)a_n e^{i\lambda_n \tau} + A_n(T)^* a_n^* e^{-i\lambda_n \tau}) \Psi_{\lambda_n} + \mathbf{P}_p \\ \mathbf{P}_{\mathcal{G}}^0 &= \sum_{n \in \mathbb{N}} (A_n(T)a_n e^{i\lambda_n \tau} + cc) \Psi_{\lambda_n} + \mathbf{P}_p, \end{aligned} \quad (277)$$

where cc stands for complex-conjugate, and where Ψ_{λ_n} is the Kirchoff quantum-graph base for which the vertex boundary condition (268) is a Kirchoff one, i.e $h_i = 0$ for every vertex $i \in V$. The Kirchoff spectrum given by condition (272) results from the intrinsic graph structure of the network. Furthermore, the particular solution \mathbf{P}_p has been computed in [185] for a given time-domain triggering event arising at a given origin vertex v_O of the graph where a closure law $P_O(t)$ is imposed so that there is locally a non-zero flow acceleration (typically related to a valve opening or closure) leading to

$$\sum_{e_k} \frac{\partial \mathbf{P}_{\mathcal{G}}^0}{\partial x}(\mathbf{x}_{v_O}, \tau) = P_O(\tau), \quad (278)$$

[185] found that the particular solution \mathbf{P}_p can also be decomposed into a distinct quantum graph base being a Kirchoff-Robin-Fourier one denoted $\Psi_{\lambda_m}^p$ that will not be detailed here. At first order one gets (83)

$$\left(\frac{\partial^2}{\partial \tau} - \frac{\partial^2}{\partial x^2} \right) p_k^1 = 2 \left(\frac{\partial \tau_w^0}{\partial x} - \frac{\partial}{\partial T} \frac{\partial}{\partial \tau} p_k^0 \right), \quad (279)$$

where within each edge, the pressure leading order and first order are respectively denoted p_k^0 and p_k^1 for $k \in E$. Since, the first order problem is then

decomposed into the Laplacian eigenfunctions

$$\mathbf{P}_{\mathcal{G}}^1 = \sum_{n \in \mathbb{N}} [a_n^1(\tau) + cc] \Psi_{\lambda_n} + \sum_{m \in \mathbb{N}} [a_m^{1p}(\tau) + cc] \Psi_{\lambda_m}^p. \quad (280)$$

914 From (277), one can evaluate the right-hand-side of (279) in the Laplace domain
 915 which reads, using the Laplace transform of the wall-shear-stress (243)

$$2 \left(\frac{\partial \tilde{\tau}_w^0}{\partial x} - \frac{\partial}{\partial T} s \tilde{\mathbf{P}}_{\mathcal{G}}^0 \right) = 2 \left(\sum_{n \in \mathbb{N}} \left[\frac{a_n^0}{s - i\lambda_n} \left(A_n^0 \frac{\sqrt{s}}{s} \lambda_n^2 - s \frac{\partial A_n^0}{\partial T} \right) + cc \right] \Psi_{\lambda_n} \right. \\ \left. + \sum_{m \in \mathbb{N}} \left[\frac{a_m^{0p}}{s - i\lambda_m^p} \left(A_m^0 \frac{\sqrt{s}}{s} \lambda_m^{p2} - s \frac{\partial A_m^{0p}}{\partial T} \right) + cc \right] \Psi_{\lambda_m^p} \right) \quad (281)$$

916 Writing (279) in Laplace domain whilst using (281), the first-order problem
 917 written on \mathcal{G} reads

$$(s^2 - \Delta_{\mathcal{G}}) \mathbf{P}_{\mathcal{G}}^1 = 2 \left(\sum_{n \in \mathbb{N}} \left[\frac{a_n^0}{s - i\lambda_n} \left(A_n^0 \frac{\sqrt{s}}{s} \lambda_n^2 - s \frac{\partial A_n^0}{\partial T} \right) + cc \right] \Psi_{\lambda_n} + \right. \\ \left. \sum_{m \in \mathbb{N}} \left[\frac{a_m^{0p}}{s - i\lambda_m^p} \left(A_m^{0p} \frac{\sqrt{s}}{s} \lambda_m^{p2} - s \frac{\partial A_m^{0p}}{\partial T} \right) + cc \right] \Psi_{\lambda_m^p} \right). \quad (282)$$

918 Projecting (282) over Ψ_{λ_n} permits to find the secularity solution to cancel the
 919 resonant double poles, [185] found

$$\lim_{s \rightarrow i\lambda_n} \left(A_n \frac{\sqrt{s}}{s} \lambda_n^2 - s \frac{\partial A_n}{\partial T} \right) = 0, \\ \lim_{s \rightarrow -i\lambda_n} \left(A_n^* \frac{\sqrt{s}}{s} \lambda_n^2 - s \frac{\partial A_n^*}{\partial T} \right) = 0, \quad (283)$$

leading to the same consistent secularity solution

$$A_n(T) = e^{-\sqrt{-i\lambda_n}T} = e^{-(\frac{1-i}{\sqrt{2}})\sqrt{\lambda_n}T}, \quad (284)$$

920 Now, using (284) in (277) leads to the damped leading-order solution for the
 921 wave solution

$$\mathbf{P}_{\mathcal{G}}^0 = \sum_{n \in \mathbb{N}} e^{-\sqrt{\frac{\lambda_n}{2}}T} \left(a_n^0 e^{i\sqrt{\frac{\lambda_n}{2}}T} e^{i\lambda_n t} + cc \right) \Psi_{\lambda_n} \\ + \sum_{m \in \mathbb{N}} e^{-\sqrt{\frac{\lambda_m^p}{2}}T} \left(a_m^{p0} e^{i\sqrt{\frac{\lambda_m^p}{2}}T} e^{i\lambda_m^p t} + cc \right) \Psi_{\lambda_m^p}. \quad (285)$$

922 It is noteworthy to mention that (285) generalizes for a network the damping
 923 analysis described in section 4.1 and 4.3 for a single pipe. Nevertheless, one

924 deep relation between the presented results is that the homogeneous spectrum
925 results from the applied boundary conditions.

926 **5. Conclusion and perspectives**

927 This review dedicated to the mechanics underpinning water-hammer waves
928 has provided both a short historical survey as well as the author's state-of-the-
929 art of the current understanding of this complex topic. Since water-hammer
930 involves both wave propagation within the solid and the fluid, it does involve
931 FSI at the liquid-solid interface and necessitates to take into account both the
932 influence of non-stationary boundary layers and inlet/outlet FSI boundary con-
933 ditions.

934 The first two sections of this review have emphasized and summarized the
935 crucial role of three couplings (Poisson, junction, friction) as already stressed
936 in previous reviews [12, 13, 15, 16, 18]. This review has not only covered the
937 established sets of equations for water-hammer modeling, their possible deriva-
938 tion from first principles, but also, their possible theoretical solutions. This
939 viewpoint is perhaps the most original stance since previous ones most often
940 discussed water-hammer's numerical computation.

941 Furthermore recent developments related to the influence of visco-elastic
942 properties of the pipe's wall, asymptotic analysis of water-hammer, as well as
943 the theoretical analysis of wave propagation within networks have also been
944 discussed. Concerning the visco-elastic solid, this review has emphasized that
945 even though Kelvin-Voigt models for the creep-functions are widespread in the
946 literature (and indubitably operational) it is interesting to consider rheological
947 visco-elastic models for providing an intrinsic description which does not depend
948 on the specific water-hammer test (since, as opposed to Kelvin-Voigt models,
949 visco-elastic models do not depend on the water-hammer test detail, i.e closure
950 law, pipe thickness, length, diameter, etc...).

951 In the future the rheological visco-elastic parameters might possibly be esti-
952 mated from Dynamical Mechanical Analysis (DMA) independently of the water

953 hammer test, so that one could try to model visco-elastic water-hammer without
954 parameter fitting (as opposed to the traditional Kelvin-Voigt model's approach).
955 This is indeed an interesting perspective to mention.

956 Concerning the asymptotic analysis of water-hammer this review has tried to
957 emphasize the relationship between previous contributions in the literature ad-
958 dressing traditional boundary layer asymptotic matching, two-time scale anal-
959 ysis, or both. Recent advances have permitted the asymptotic perturbative
960 derivation of the four equation FSI model. They have also permitted to bring
961 to the fore that some distinct asymptotic regimes could happen, as emphasized
962 in this review. In the future, some further developments could maybe permit
963 a deeper understanding of the couplings between the steady-state flow and the
964 water-hammer wave, perhaps leading to the derivation of a quasi-steady shear
965 stress model alternative to the quasi-steady Darcy-Weisbach model. The use
966 of asymptotic analysis might also be useful in the future to approximate reflec-
967 tion/transmission coefficients resulting from the presence of singularities (e.g
968 sudden diameter changes, blockages, etc.) within networks in order to provide
969 improved long-wavelength approximation models for water-hammer waves.

970 Last but not least, this review has covered the theoretical analysis of water-
971 hammer propagation within networks. This topic is stimulating since a modern
972 viewpoint recently brought new concepts and results in this interesting area.
973 Among those, the notion that the wave spectrum within networks can be found
974 from an algebraic condition related to the secular matrix (which is related to the
975 graph adjacency matrix) has been emphasized. There is much to develop in this
976 area, in order to progress along water-hammer modeling in complex networks.
977 We believe this research topic will grow in the future. It might also impact the
978 related growing topic of transient-based defect detection methods.

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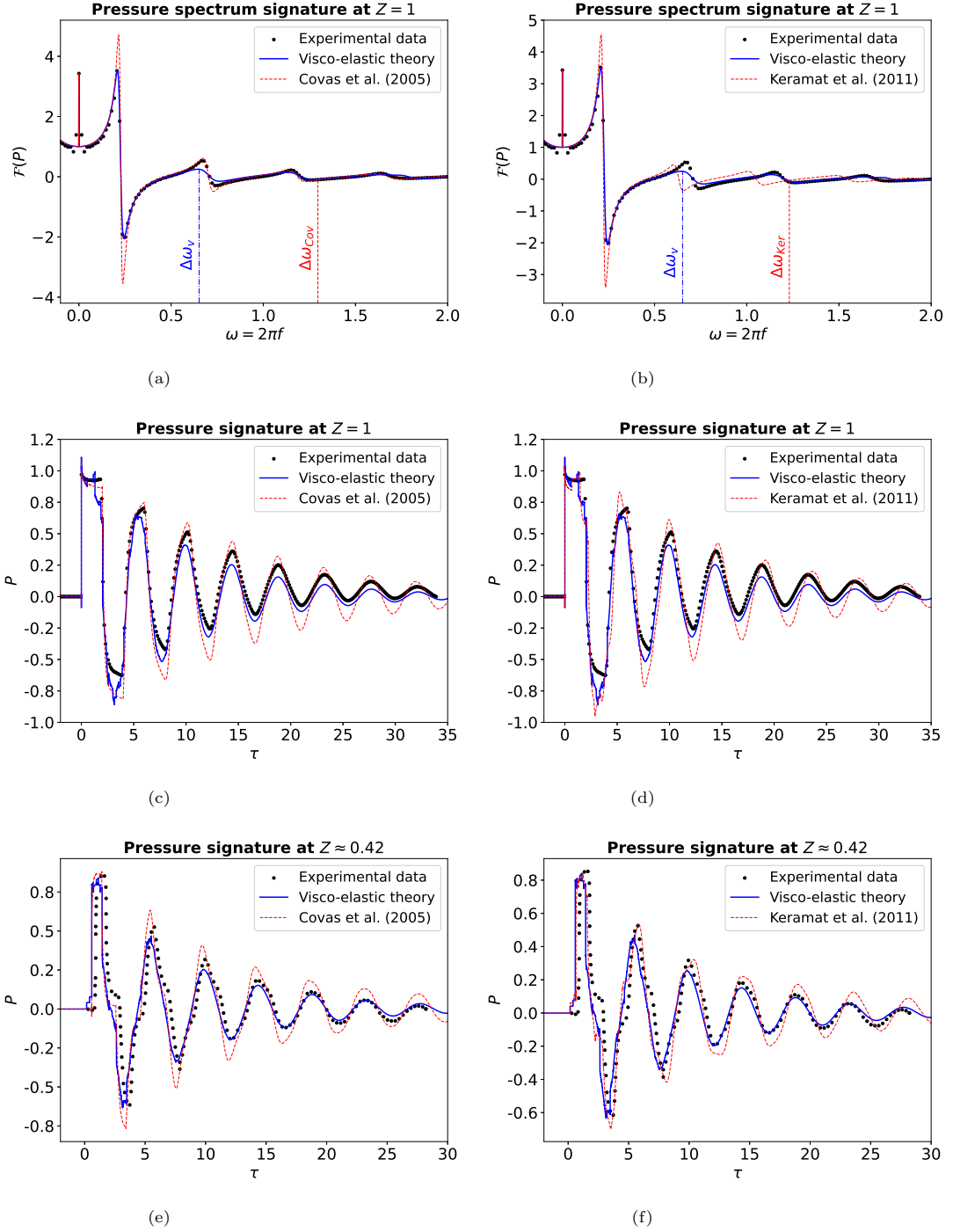
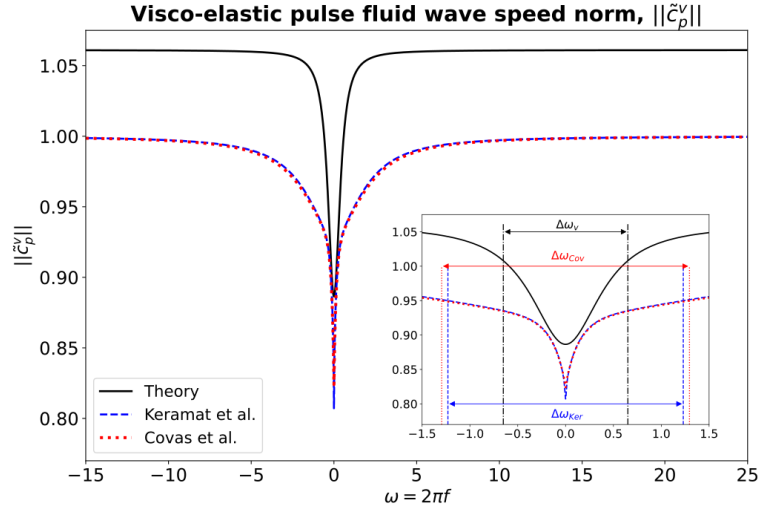
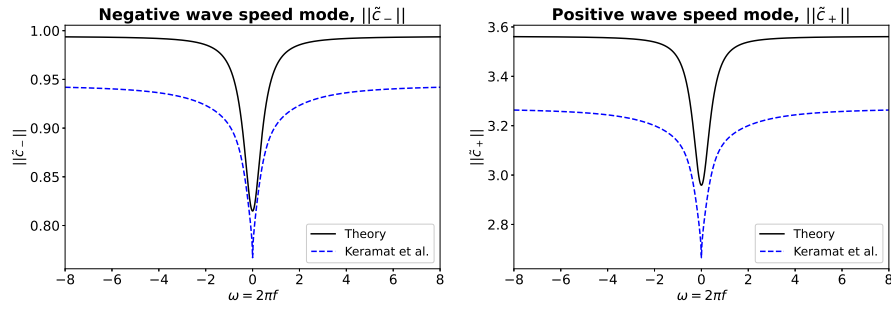


Figure 13: Comparison between the rheology-based model of [65] (referred to as "Visco-elastic theory") and [31] and [145] ones for the pressure signal at various location for the reservoir-pipe-anchored valve configuration. Dimensionless pulsation $\omega = \omega^* L^* / c_{p,T}^*$ using (35) have been used in (a) and (b), dimensionless time $\tau = t^* c_{p,T}^* / L^*$ in (c), (d), (e) and (f).



(a)



(b)

(c)

Figure 14: Comparison of frequency domain dependence of dimensionless visco-elastic velocities obtained from fitting parameters to the experimental data of [137, 31]. (a) $\tilde{c}_p^v(\omega)$ (153), (b) \tilde{c}_+ (154), (c) \tilde{c}_- (154). Dimensionless pulsation $\omega = \omega^* L^* / c_{p,T}^*$ have been used.

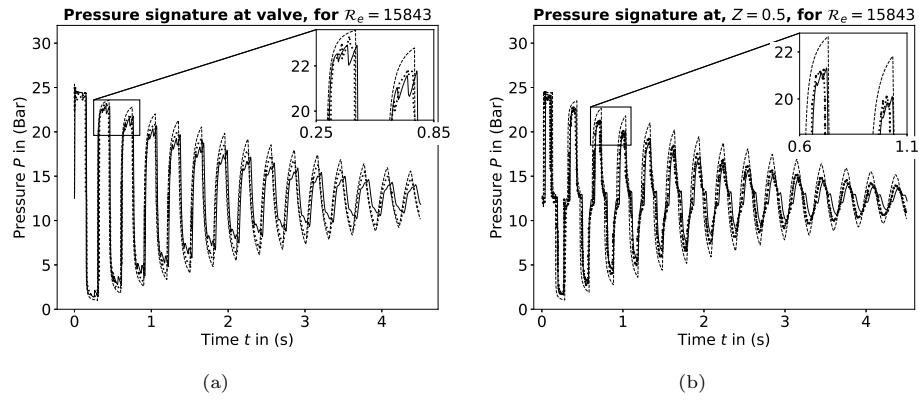
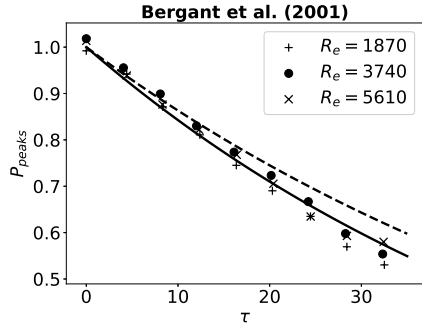
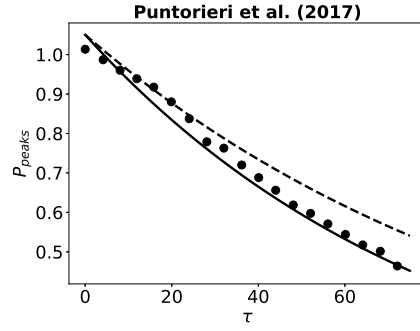


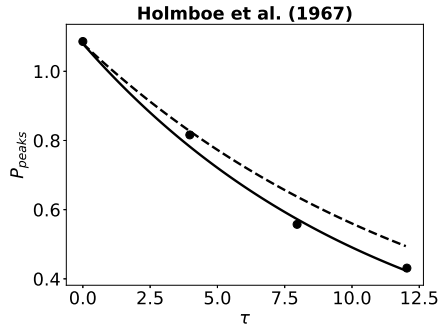
Figure 15: Pressure signature compared with experimental data of ([99]). Experimental data are depicted with black dotted lines while theoretical results from [65] are depicted with continuous lines. ([95])'s solution (no-FSI) is provided with dashed line. Dimensionless numbers are $\mathcal{M} = 7.2 \cdot 10^{-4}$, $\epsilon = 8.2 \cdot 10^{-5}$, $\alpha = 0.125$, $\delta = 3.3 \cdot 10^{-2}$ and $\mathcal{D} = 0.11$.



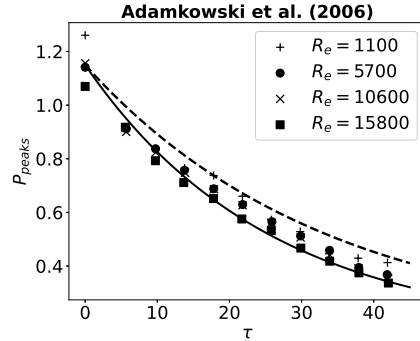
(a) $\mathcal{M} = [7.65^{(+)}, 15.3^{(\bullet)}, 23^{(\times)}] \cdot 10^{-5}$, $\epsilon = 3 \cdot 10^{-4}$, $\alpha = 0.146$, $\delta = 1.7 \cdot 10^{-2}$



(b) $\mathcal{M} = 1.53 \cdot 10^{-4}$, $\epsilon = 6.57 \cdot 10^{-4}$, $\alpha = 0.1$, $\delta = 1.1 \cdot 10^{-2}$



(c) $\mathcal{M} = 1.88 \cdot 10^{-4}$, $\epsilon = 3.5 \cdot 10^{-4}$, $\alpha = 0.13$, $\delta = 8.2 \cdot 10^{-2}$



(d) $\mathcal{M} = [5.05^{(+)}, 26.0^{(\bullet)}, 48.3^{(\times)}, 72.0^{(\blacksquare)}] \cdot 10^{-5}$, $\epsilon = 8.2 \cdot 10^{-5}$, $\alpha = 0.125$, $\delta = 3.3 \cdot 10^{-2}$

Figure 16: First exponential damping mode comparison with experimental dimensionless pressure peaks, P_{peaks} . The present theory is depicted with a continuous line while ([95])'s theory ($\nu_s \rightarrow 0$) is depicted by a dashed line.

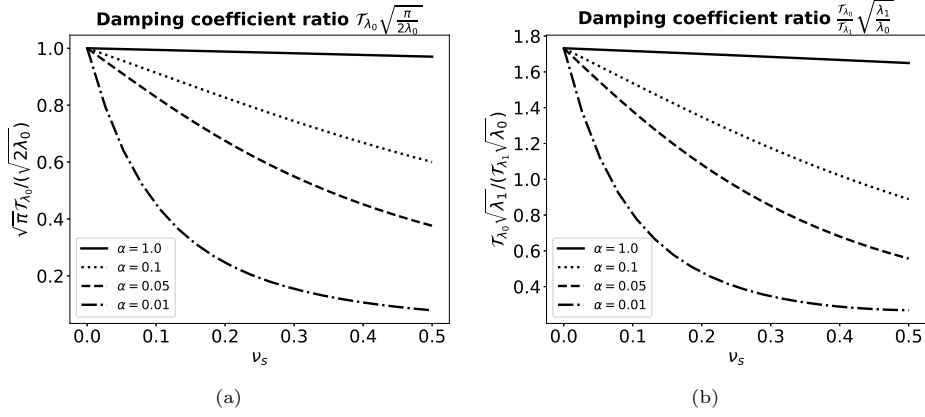


Figure 17: Exponential damping coefficient analysis: (a) Analysis of $\mathcal{T}_0 \sqrt{\frac{\pi}{2\lambda_0}}$, (b) Analysis of $\frac{\mathcal{T}_0}{\mathcal{T}_1} \sqrt{\frac{\lambda_1}{\lambda_0}}$. Iso- α lines are represented based on ([99])'s data from Table 2.

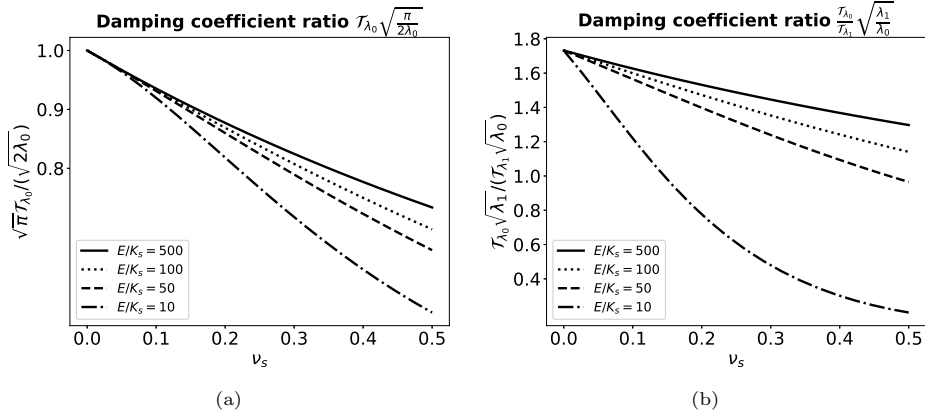


Figure 18: Exponential damping coefficient analysis: (a) Analysis of $\mathcal{T}_0 \sqrt{\frac{\pi}{2\lambda_0}}$, (b) Analysis of $\frac{\mathcal{T}_0}{\mathcal{T}_1} \sqrt{\frac{\lambda_1}{\lambda_0}}$. Iso- E lines are represented based on ([99])'s data from Table 2.

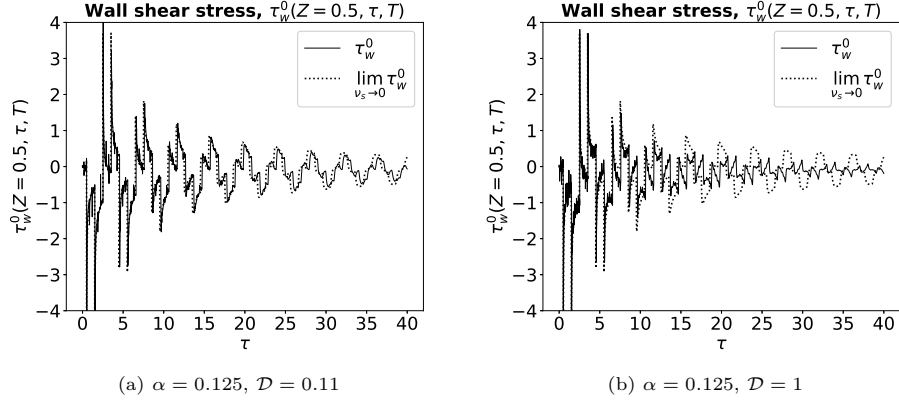


Figure 19: Dimensionless fluid wall shear stress τ_w^0 at middle's pipe location. ([99])'s data from Table 2 are used with (a) $\mathcal{D} = 0.11$, (b) $\mathcal{D} = 1$.

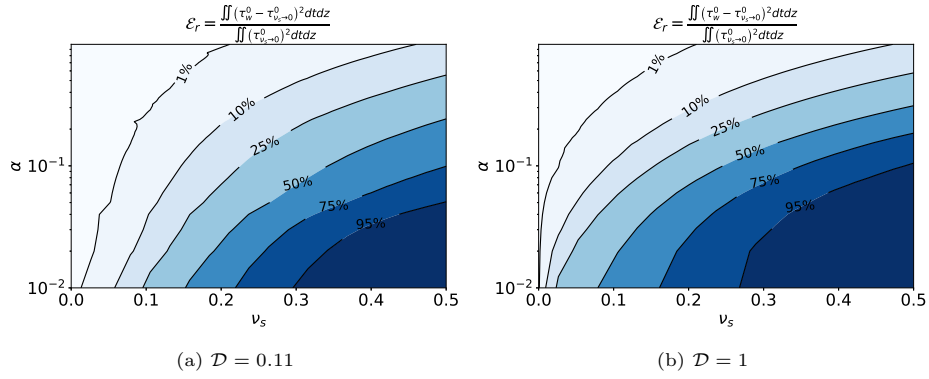
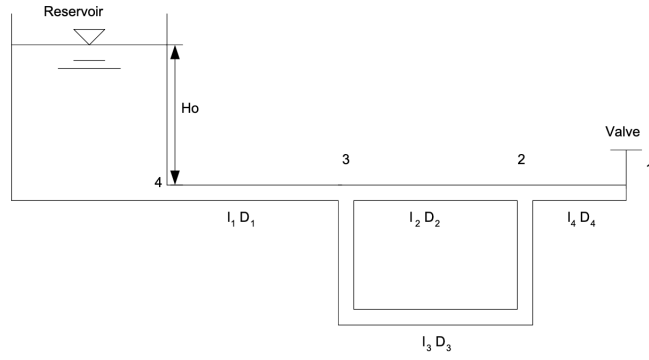


Figure 20: Relative difference \mathcal{E}_r between (FSI) and no-(FSI) ($\nu_s \rightarrow 0$) fluid wall shear rate. ([99])'s data from Table 2 are used with (a) density ratio $\mathcal{D} = 0.11$, (b) $\mathcal{D} = 1$.



$$\begin{bmatrix}
 Z_{c2} \sinh \gamma_2 l_2 & 0 & 0 & 0 & 1 & 0 \\
 \cosh \gamma_2 l_2 & 0 & -1 & 0 & 0 & 0 \\
 0 & Z_{c3} \sinh \gamma_3 l_3 & 0 & 0 & 1 & 0 \\
 0 & \cosh \gamma_3 l_3 & 0 & -1 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & -1
 \end{bmatrix}
 \cdot
 \begin{bmatrix}
 Q_{312}/Q_{311} \\
 Q_{313}/Q_{311} \\
 Q_{212}/Q_{311} \\
 Q_{213}/Q_{311} \\
 H_2/Q_{311} \\
 Q_{214}/Q_{311}
 \end{bmatrix}
 =
 \begin{bmatrix}
 Z_3 \cosh \gamma_2 l_2 \\
 Z_3 \sinh \gamma_2 l_2 / Z_{c2} \\
 Z_3 \cosh \gamma_3 l_3 \\
 Z_3 \sinh \gamma_3 l_3 / Z_{c3} \\
 1 \\
 0
 \end{bmatrix}$$

Figure 21: Hydraulic looped network and its associated network admittance linear system studied by S. Kim [170].

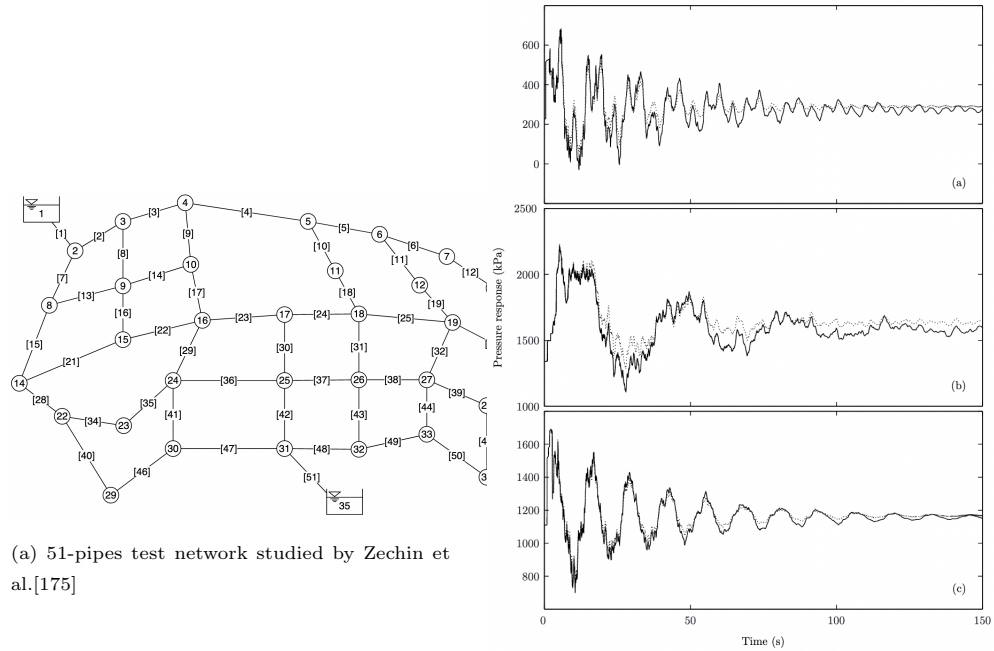


Figure 22: Investigation of the pressure dynamic within a complex network using the numerical inverse Laplace transform (NILT) procedure from Zechin et al. [176, 175]

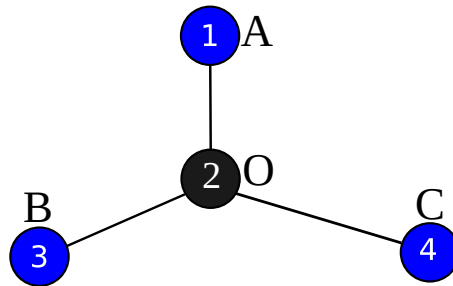
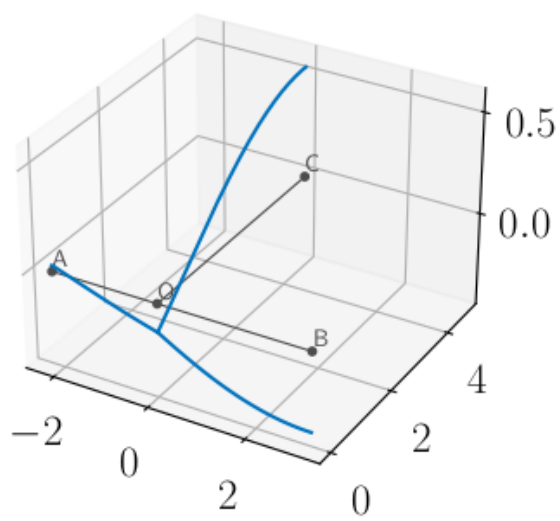
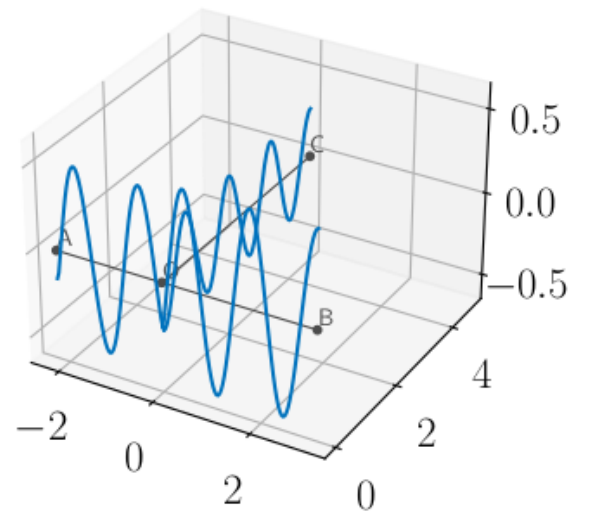


Figure 23: 3-star metric graph having length $\ell_{21} = 2.211$, $\ell_{23} = 3.111$ and $\ell_{24} = 4.711$.



(a) Kirchhoff mode Ψ_{λ_1} .



(b) Kirchhoff mode Ψ_{λ_2} .

Figure 24: First two Kirchhoff eigenmode of the 3-star graph 23.