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PROBABILISTIC WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION ON THE $2d$ SPHERE I: POSITIVE REGULARITIES

NICOLAS BURQ, NICOLAS CAMPS, CHENMIN SUN, AND NIKOLAY TZVETKOV

ABSTRACT. We establish the probabilistic well-posedness of the nonlinear Schrödinger equation on the $2d$ sphere \mathbb{S}^2 . The initial data are distributed according to Gaussian measures with typical regularity $H^s(\mathbb{S}^2)$, for $s > 0$. This level of regularity goes significantly beyond existing deterministic results, in a regime where the flow map cannot be extended uniformly continuously.

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1. INTRODUCTION

1.1. **Context.** The study of the nonlinear Schrödinger equation (NLS) on general Riemannian manifolds was initiated in [7] as a sequence of works by Bourgain [1] on the flat torus. It was shown in [7] that in the cases of Riemannian surfaces the Cauchy problem for the cubic NLS is locally well-posed in the Sobolev space H^s , $s > \frac{1}{2}$. This result is of interest because the classical methods, based only on Sobolev embeddings, give the well-posedness under the much stronger restriction $s > 1$ (one half of the dimension). In the case of the sphere \mathbb{S}^2 the restriction $s > \frac{1}{2}$ was relaxed to $s > \frac{1}{4}$ in [8]. As shown in [6] the $s > \frac{1}{4}$ restriction is in a sense optimal because this is the limit of the semi-linear well-posedness methods. The goal of this paper is to show that, in the case of the $2d$ sphere \mathbb{S}^2 , one can go beyond the $s > \frac{1}{4}$ threshold by randomizing the initial data in a Sobolev space of low regularity. This is in the spirit of the program initiated in [11], which aims to study dispersive

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PDE with random data (not necessarily related to an invariant measure) beyond the deterministic thresholds, where some instabilities are known to occur.

1.2. Setup and main results. Let \mathbb{S}^2 be the unit sphere in \mathbb{R}^3 , endowed with the canonical metric, where we normalize the measure on \mathbb{S}^2 to adopt the convention $\int_{\mathbb{S}^2} 1 = 1$. The cubic NLS, posed on \mathbb{S}^2 , we consider in this article is¹

$$i\partial_t u - (-\Delta + 1)u = (|u|^2 - 2\|u\|_{L^2}^2)u, \quad (1.1)$$

where $u: \mathbb{R}_t \times \mathbb{S}^2 \rightarrow \mathbb{C}$ and Δ is the Laplace-Beltrami operator on \mathbb{S}^2 . It is well-known that $-\Delta + 1$ is a self adjoint operator on $L^2(\mathbb{S}^2)$ with domain the Sobolev space $H^2(\mathbb{S}^2)$ and that its spectrum is discrete with eigenvalues λ_n^2 given by

$$\lambda_n^2 = n^2 + n + 1, \quad n = 0, 1, 2, \dots$$

Moreover, λ_n^2 has multiplicity $2n + 1$ and the corresponding eigenspace consists of the spherical harmonics of degree n (i.e. the restriction of the harmonic polynomials of degree n on \mathbb{R}^3 to \mathbb{S}^2). We denote by E_n the eigenspace of $-\Delta + 1$ associated with the eigenvalue λ_n^2 .

If u is a solution of (1.1) then $v = e^{it-2it\|u\|_{L^2}^2}u$ is a solution of

$$i\partial_t v + \Delta v = |v|^2 v \quad (1.2)$$

which is the traditionally studied NLS. For a sake of convenience, in this article we will restrict to (1.1) but all results we obtain can be easily rephrased in the context of (1.2) thanks to the straightforward link between the two equations.

We denote by $\langle f|g \rangle$ the scalar product in $L^2(\mathbb{S}^2)$. For $n \geq 0$, we denote by π_n the orthogonal projection of $L^2(\mathbb{S}^2)$ on E_n . More precisely, if $(\mathbf{b}_{n,k})_{|k| \leq n}$ is an orthonormal basis of E_n then

$$\pi_n(f) = \sum_{|k| \leq n} \langle f | \mathbf{b}_{n,k} \rangle \mathbf{b}_{n,k}, \quad \forall f \in L^2(\mathbb{S}^2).$$

The L^p mapping properties of π_n and the precise concentration properties of its kernel will play an important role in the analysis below. Since π_n is invariant under the action of the rotation group $SO(3)$, the kernel of π_n is invariant under the rotation, hence $\sum_{|k| \leq n} |\mathbf{b}_{n,k}(x)|^2$ is independent of $x \in \mathbb{S}^2$ (see e.g. [9]). Therefore

$$\sum_{|k| \leq n} |\mathbf{b}_{n,k}(x)|^2 = \sum_{|k| \leq n} \|\mathbf{b}_{n,k}\|_{L^2(\mathbb{S}^2)}^2 = 2n + 1. \quad (1.3)$$

If f is decomposed as $f = \sum_{n \geq 0} \pi_n(f)$ then the Sobolev norm $H^s(\mathbb{S}^2)$ of f is equivalent to

$$\sum_{n \geq 0} \langle \lambda_n \rangle^{2s} \|\pi_n f\|_{L^2}^2, \quad (1.4)$$

¹In this work, the Wick-ordered nonlinearity is only favorable when estimating the regularity for purely random objects.

where by definition for $x \in \mathbb{R}$, $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$. We next define the random initial data we will consider. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of i.i.d. complex standard Gaussian random variables $(g_{n,k}(\omega))_{n \in \mathbb{N}, |k| \leq n}$ on it, i.e.

$$g_{n,k}(\omega) = \frac{\operatorname{Re}(g_{n,k})(\omega) + i \operatorname{Im}(g_{n,k})(\omega)}{\sqrt{2}},$$

where $\operatorname{Re}(g_{n,k}), \operatorname{Im}(g_{n,k})$ are independent real-valued normal random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let again $(\mathbf{b}_{n,k})_{|k| \leq n}$ be an orthonormal basis of E_n . For $\alpha \in \mathbb{R}$, we set

$$\phi_\alpha(x, \omega) = \sum_{n \geq 0} \frac{1}{\lambda_n^\alpha} \sum_{|k| \leq n} g_{n,k}(\omega) \mathbf{b}_{n,k}(x). \quad (1.5)$$

Thanks to (1.3) and (1.4), we can easily compute the typical Sobolev regularity of $\phi_\alpha(x, \omega)$. Namely

$$\mathbb{E} \|\phi_\alpha(\cdot, \omega)\|_{H^s}^2 = \sum_{n \geq 0} \langle \lambda_n \rangle^{2s-2\alpha} (2n+1)$$

which is finite if and only if $s < \alpha - 1$. Therefore $\phi_\alpha(\cdot, \omega)$ belongs almost surely to the Sobolev space $H^s(\mathbb{S}^2)$ for $s < \alpha - 1$. One may also show (see e.g. [11, Lemma B.1]) that $\phi_\alpha(\cdot, \omega)$ does almost surely not belong to $H^{\alpha-1}(\mathbb{S}^2)$. We also observe that thanks to the invariance of the gaussians under rotations, we have that the law of $\omega \mapsto \phi_\alpha(\cdot, \omega)$ (as a random variable on $H^s(\mathbb{S}^2)$ for some $s < \alpha - 1$) is independent of the choice of the basis $(\mathbf{b}_{n,k})_{|k| \leq n}$, and will be denoted by μ_α .

For $N \geq 0$, we define the projectors Π_N, Π_N^\perp by

$$\Pi_N = \sum_{n \leq N} \pi_n, \quad \Pi_N^\perp := \operatorname{Id} - \Pi_N.$$

We denote by v_N the solution of (1.1) with truncated initial data $v_N|_{t=0} = \Pi_N \phi_\alpha$:

$$i\partial_t v_N - (-\Delta + 1)v_N = |v_N|^2 v_N, \quad v_N|_{t=0} = \Pi_N \phi_\alpha(\cdot, \omega). \quad (1.6)$$

The initial data in (1.6) is smooth and therefore, thanks to [7], v_N is a (unique) globally defined smooth function on $\mathbb{R} \times \mathbb{S}^2$. We now can formulate our main result concerning the probabilistic well-posedness of (1.1) with data (1.5).

Theorem 1.1. *Let $\alpha > 1$. There is a set Σ of full probability such that for every $\omega \in \Sigma$ there exists $T_\omega > 0$ such that the sequence of smooth solutions $(v_N)_{N \geq 1}$ in (1.6) with the initial data $\Pi_N \phi_\alpha$, converges in $L^\infty([-T_\omega, T_\omega]; H^s(\mathbb{S}^2))$, $s < \alpha - 1$ to a limit which satisfies the cubic NLS in a distributional sense on $(-T_\omega, T_\omega) \times \mathbb{S}^2$.*

We give a more precise version in Theorem 1.2. For $\alpha > \frac{5}{4}$, the statement of Theorem 1.1 is implied by the local well-posedness of (1.1) in $H^s(\mathbb{S}^2)$, $s > \frac{1}{4}$, obtained in [8]. For $\alpha \in (1, \frac{5}{4}]$, Theorem 1.1 requires a proof going much beyond the analysis [8]. In particular, it requires a proof of the result of [8] which is less dependent on the bilinear Strichartz estimates for the free evolution and which is sensitive to the smoothing effect coming from the modulations in the Bourgain spaces associated

with the Schrödinger equation on the sphere. These smoothing effects are not exploited in [8] while they play a key role in the probabilistic well-posedness of (1.1) when estimating the regularity of the stochastic objects involved in the construction of the solutions.

The proof of Theorem 1.1 is based on a probabilistic resolution scheme originally developed by Bourgain in [2]. This scheme was further refined in recent years in [3, 13, 20, 15, 16, 17, 22, 24]. In the present work we will strongly benefit from these developments, especially [3, 15]. The severe difficulty in the analysis on \mathbb{S}^2 , revealed by [4], is the divergence of the first nonlinear Picard iteration with data ϕ_α in $H^{\alpha-1}$ while in the case of the flat tori the first nonlinear Picard iteration enjoys almost $\frac{1}{2}$ derivative smoothing. This derivative smoothing is crucially exploited by Bourgain in [2] and the subsequent papers on flat tori.

We point out that in our analysis of NLS with data (1.5) we are strongly inspired by [9]. This allows us to use Weyl asymptotics ideas in the context of Khinchin and Wiener chaos type estimates. In contrast with the case of flat tori, a significant part of the analysis on \mathbb{S}^2 is performed in the physical space, requiring in particular the introduction of vector valued Fourier-Lebesgue spaces. We also point out that even in the local in time analysis we use propagation of measure global arguments on the high dimensional space E_n , $n \gg 1$. Arguments of this spirit are not presented in the analysis on flat tori.

The globalization of the solutions obtained in Theorem 1.1 is a challenging open problem. It would also be interesting to decide whether Theorem 1.1 holds for more general randomized functions, in the spirit of [11, 12].

In the forthcoming work [5], we will consider the case $\alpha = 1$. In the case $\alpha = 1$ the statement of Theorem 1.1 should be properly modified because the sequence $(v_N)_{N \geq 1}$ is not expected to converge. We will however show that, after introducing some new ideas, it converges after a suitable renormalization. Moreover, thanks to invariant measure considerations, we will show that the limit solutions are globally defined.

1.3. Structure of the nonlinearity. Denote the wick cubic power

$$\mathcal{N}(u) := |u|^2 u - 2\|u\|_{L^2}^2 u.$$

By abusing the notation, we denote $\mathcal{N}(\cdot, \cdot, \cdot)$ the canonical trilinear form such that $\mathcal{N}(u, u, u) = \mathcal{N}(u)$.

We isolate the *singular trilinear form*:

$$\mathcal{N}_{(0,1)}(f, g, h) := \sum_{n, n_2, n_3} \pi_n(\pi_n f \cdot (\pi_{n_2} g \diamond \pi_{n_3} h)), \quad (1.7)$$

where

$$(g \diamond h)(x) := \bar{g}(x)h(x) - \langle h|g \rangle \quad (1.8)$$

is the wick-product of functions g and h . We mean $|g|^{\diamond 2} = |g|^2 - \|g\|_{L^2}^2$.

It was shown in [4] that, on \mathbb{S}^2 , contrarily to the case of \mathbb{T}^2 , Bourgain's re-centering ansatz

$$u(t) = e^{it\Delta}\phi^\omega + r(t)$$

fails to solve NLS on \mathbb{S}^2 , as $r(t)$ cannot be viewed as a remainder living in more regular spaces. Roughly speaking, the Duhamel's integration of the high-low-low type resonant interaction

$$\mathcal{N}_{(0,1)}(u_{\text{high}}, u_{\text{low}}, u_{\text{low}}) \quad (1.9)$$

does not gain any regularity. This compromise the standard semi-linear iteration scheme based on the Picard's iterations. Note that in the flat case, the zero momentum condition (a property inherited from the structure of the plane waves) and the Wick ordering remove these interactions from the nonlinearity.

Inspired by [15], our resolution is based on the random averaging operator ansatz.

1.4. Random averaging operator (RAO) ansatz. We now prepare and state in Theorem 1.2 the more precise statement of Theorem 1.1, in which we describe the precise structure of the limit solutions. For a dyadic integer $N \in 2^{\mathbb{N}}$, denote by

$$\mathbf{P}_N = \Pi_N - \Pi_{\frac{N}{2}} \quad (1.10)$$

the sharp dyadic projection. Let u_N be the (smooth) solution of the Wick-ordered cubic Schrödinger equation:

$$i\partial_t u_N + (\Delta - 1)u_N = \mathcal{N}(u_N), \quad u_N|_{t=0} = \Pi_N \phi^\omega, \quad (1.11)$$

Let ψ_N be the solution of the linear equation

$$(i\partial_t + \Delta - 1)\psi_N = 2\Pi_N \mathcal{N}_{(0,1)}(\psi_N, u_{N/2}, u_{N/2}), \quad \psi_N|_{t=0} = \phi_N^\omega := \mathbf{P}_N \phi^\omega. \quad (1.12)$$

For each N , with the knowledge of $u_{N/2}$, the properties of ψ_N are encoded in the random averaging operator (RAO) denoted $\mathcal{H}_n^N(t)$ on E_n and implicitly defined by

$$f_n \mapsto \mathcal{H}_n^N(t)(f_n) := e^{-it\lambda_n^2} f_n - 2i \int_0^t e^{-i(t-t')\lambda_n^2} \pi_n(\mathcal{H}_n^N(t')(f_n) \cdot |u_{N/2}(t')|^{\diamond 2}) dt'. \quad (1.13)$$

Given an orthonormal base $(\mathbf{b}_{n,k})_{|k| \leq n}$ of E_n , we can also define the operator $\mathcal{H}_n^N(t)$ by its matrix elements

$$H_{n;\ell,k}^N(t) := \langle \mathcal{H}_n^N(t)(\mathbf{b}_{n,k}) | \mathbf{b}_{n,\ell} \rangle = \langle \mathbf{b}_{n,k}^N(t) | \mathbf{b}_{n,\ell} \rangle, \quad (1.14)$$

where $\mathbf{b}_{n,k}^N(t, x)$ is the solution of

$$(i\partial_t - \lambda_n^2)\mathbf{b}_{n,k}^N(t) := 2\pi_n(\mathbf{b}_{n,k}^N(t) \cdot |u_{N/2}(t)|^{\diamond 2}), \quad \mathbf{b}_{n,k}^N|_{t=0} = \mathbf{b}_{n,k}. \quad (1.15)$$

In other words,

$$\mathcal{H}_n^N(t)(\mathbf{b}_{n,k}) = \mathbf{b}_{n,k}^N(t) = \sum_{|\ell| \leq n} H_{n;\ell,k}^N(t) \mathbf{b}_{n,\ell}.$$

Note that $\mathcal{H}_n^N(t)$ does not depend on the basis of E_n , and it turns out that for all $t \in \mathbb{R}$ the operator $\mathcal{H}_n^N(t)$ is an L^2 -isometry of E_n .

We can now express the *colored* Gaussian variables ψ_N , referred to as *terms of type (C)*, as follows:

$$\psi_N(t, x) = \sum_{\frac{N}{2} < n \leq N} \pi_n \psi_N(t, x) = \sum_{\frac{N}{2} < n \leq N} \frac{1}{\lambda_n^{\alpha-\frac{1}{2}}} e_n^N(t, x), \quad (1.16)$$

where, given \mathcal{H}_n^N the random averaging operator (RAO) defined in (1.13), we let

$$e_n^N(t, x) := \mathcal{H}_n^N(t)(e_n^\omega). \quad (1.17)$$

With these notations, for $x \in \mathbb{S}^2$,

$$e_n^N(t, x) = \frac{1}{\sqrt{2n+1}} \sum_{|k| \leq n} g_{n,k}(\omega) \mathbf{b}_{n,k}(t, x), \quad e_n^N(0, x) = e_n^\omega(x).$$

Heuristically, the colored object ψ_N captures the singular interaction $\mathcal{N}_s(\cdot, \cdot, \cdot)$, and it can be viewed as a correction of the Bourgain's re-centering ansatz:

$$\psi_N(t) = e^{it\Delta} \phi_N^\omega + e^{it\Delta} \Theta_N(t)(\phi_N^\omega) + \text{remainder}, \quad (1.18)$$

where $\Theta_N(t)$ is the linear operator (with the knowledge of $u_{\frac{N}{2}}$) defined by

$$\phi \mapsto -i\Pi_N \int_0^t e^{i(t-t')\Delta} \mathcal{N}_{(0,1)}(\mathbf{P}_N \phi, u_{\frac{N}{2}}, u_{\frac{N}{2}})(t') dt'.$$

To describe the smooth solution u_N we decompose

$$u_N = \sum_{M \leq N} v_M, \quad v_M := u_M - u_{M/2},$$

and we impose

$$v_M := \psi_M + w_M. \quad (1.19)$$

The remainders w_M , referred to as *terms of type (D)*, will be estimated in a functional space that can be embedded into $C_{\text{loc}}(\mathbb{R}_t, H^{\frac{1}{2}+}(\mathbb{S}^2))$. Since

$$\Pi_N \mathcal{N}_{(0,1)}(\psi_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}) = \mathcal{N}_{(0,1)}(\psi_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}),$$

the equation for w_N reads:

$$\begin{aligned} (i\partial_t + \Delta - 1)w_N &= \mathcal{N}(u_{\frac{N}{2}} + v_N) - \mathcal{N}(u_{\frac{N}{2}}) - 2\Pi_N \mathcal{N}_{(0,1)}(\psi_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}) \\ &= 2(\mathcal{N}(v_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}) - \mathcal{N}_{(0,1)}(\psi_N, u_{\frac{N}{2}}, u_{\frac{N}{2}})) + \mathcal{N}(u_{\frac{N}{2}}, v_N, u_{\frac{N}{2}}) \\ &\quad + 2\mathcal{N}(v_N, v_N, u_{\frac{N}{2}}) + \mathcal{N}(v_N, u_{\frac{N}{2}}, v_N) + \mathcal{N}(v_N). \end{aligned} \quad (1.20)$$

We see from the equation above that the ansatz (1.19) removes the most singular high-low-low type interaction from the equation solved by the remainder w_N .

After solving (1.20), the full ansatz can be written as

$$u_N = \psi_{\leq N} + w_{\leq N}, \quad \psi_{\leq N} := \sum_{L \leq N} \psi_L, \quad w_{\leq N} = \sum_{L \leq N} w_L. \quad (1.21)$$

Formally taking $N \rightarrow \infty$, the local structure of (1.1) is given by

$$u = \psi_{\leq \infty} + w_{\leq \infty}.$$

We are now ready to formulate the more precise statement of the local well-posedness Theorem 1.1.

Theorem 1.2. *Let $\alpha > 1$. There exist absolute constants $C_1 > c_1 > 0$ and $\delta_0 > 0$, such that the following statements hold. For $R \geq 1$, let $\tau_R := R^{-C_1}$. There exists a measurable set $\Sigma_R \subset H^{\alpha-1-}(\mathbb{S}^2)$ with*

$$\mu_\alpha(\Sigma_R^c) < C_1 e^{-c_1 R^{\delta_0}},$$

such that for all $\phi \in \Sigma_R$, the sequence of smooth solutions u_N of (1.11) with initial data $\Pi_N \phi$ converges to u in $C([- \tau_R, \tau_R]; H^{\alpha-1-}(\mathbb{S}^2))$.

Moreover, the limit u is solution to the Wick-ordered cubic Schrödinger equation and can be written

$$u(t) = \psi_{\leq \infty}(t) + w_{\leq \infty}(t),$$

where, for some $s_0 > \frac{1}{2}$,

$$\psi_{\leq \infty} \in C([- \tau_R, \tau_R]; \mathcal{C}^{0-}(\mathbb{S}^2)) \quad , \quad w_{\leq \infty} \in C([- \tau_R, \tau_R]; H^{s_0}(\mathbb{S}^2)).$$

1.5. Main ideas of the proof. With the objects introduced earlier we can now explain in more details the ingredients of our proofs.

Having identified the singular interactions in [4] we now have to construct an adapted refined ansatz in order to show the strong convergence of the regularized solutions $(u_N)_N$, and to prove the decomposition claimed in Theorem 1.2. As mentioned earlier the overall strategy is inspired from both [3] and [15].

– *Non-perturbative resolution scheme.* For the derivative nonlinear wave equation considered in [3], Picard’s iterations are not regularizing and this compromises the standard semi-linear approach of [2, 11]. Instead, [3] developed a probabilistic quasi-linear (or non-perturbative) resolution scheme. It has been known since [6] that solving deterministically NLS on \mathbb{S}^2 when $s \leq \frac{1}{4}$ would require a quasi-linear ansatz too. We proved in [4] that it is also the case when the initial data are randomly distributed, thus confirming that a resolution scheme like in [3] was needed to go beyond the Cauchy theory achieved in [8] twenty years ago.

– *Ansatz.* To address the Gibbs measure problem with general power type nonlinearities for NLS on \mathbb{T}^2 , [15] identified the structure of the probabilistic objects in the resolution scheme of [3]. This formulation, particularly the random averaging operators (RAO) as expressed in (1.21) in our context, provides the foundations for our ansatz. Leveraging our understanding of the singular interactions [4], we arrive at the precise description (1.21) of the limit solutions. A notable aspect of our work is that the RAO are not regularizing: in the limit $N \rightarrow \infty$,

$$\psi_N(t) - e^{it\Delta} \phi_N^\omega$$

does not exhibit smoother properties than the initial data. This is in contrast with the previous works [15, 16, 17] and [22, 24]. In particular, the random tensors theory introduced in [16] to capture the smoothing effect in the Picard iterations, does not seem to improve the (RAO) method for NLS on \mathbb{S}^2 .

– *Random objects of type (C)*. These terms, which are not smoother but display a probabilistic structure, encode the singular interactions that contribute to the main part of the effective nonlinear dynamics. In our context, they are a superposition of time-dependent eigenfunctions $e_n^N(t)$ obtained by applying the operator $\mathcal{H}_n^N(t)$ (unitary on E_n) to the initial Gaussian spherical harmonics e_n^ω , as expressed in (1.16).

In the present work, the rigorous construction of the random averaging operators necessitates the functional framework established in Section 4 and the multilinear estimates from Section 6.

Another new aspect of our analysis is the use of the law invariance stated in Lemma 3.7. This property comes from the unitary structure of the RAO [17] and the probabilistic independence between the RAO and the high-frequency component of the initial data, first observed in [3].

We use the law-invariance to prove pointwise-in-time estimates for resonant interactions as well as $L^\infty(\mathbb{S}^2)$ -bounds for time-dependent eigenfunctions $e_n^N(t)$. However, when we use Fourier-restriction norms to exploit the time-oscillations, we have to consider averaged in time quantities for which we cannot use the law-invariance. Instead, we prove large deviation bounds in Section 7 that allow to capture the $L^\infty(\mathbb{S}^2)$ -bounds in the Fourier-Lebesgue norm of $e_n^N(t)$.

– *Remainder terms of type (D)*. These smoother terms denoted w_N solve the equation (1.20). We prove semi-classical energy estimates stated in (5.14), which imply the convergence of $(w_N)_N$ at regularity $H^{s_0}(\mathbb{S}^2)$ for some $s_0 > \frac{1}{2}$. In particular, we recover the gain of almost $\frac{1}{2}$ derivatives obtained by Bourgain on \mathbb{T}^2 .

Since some parts of the trilinear estimates could not be performed using general linear or bilinear Strichartz estimates, we have to carefully analyse the nonlinear interactions in Fourier-restriction spaces in order to exploit the time-oscillations. This necessitated the development of a more flexible analysis going much beyond the methods used in [8].

In Section 8 we derive the trilinear probabilistic estimates for the (C)(C)(C) terms, and the deterministic estimates for the interactions with at least one term of type (D) in Section 9, respectively. For the (C)(C)(C) interactions we also cover the case $\alpha = 1$ (corresponding to the Gaussian free field on \mathbb{S}^2), which explains the length of Section 8.

Finally, we note that our proof of Theorem 1.2 can be strengthened to get a stability result in the spirit of [15, Proposition 5.5], which is useful in order to iterate in time the local well-posedness result.

1.6. Remark on a completely resonant system. For the following completely resonant system

$$i\partial_t u + \Delta u = \sum_{n,m} \pi_n(\pi_n u |\pi_m u|^2), \quad (t, x) \in \mathbb{R} \times \mathbb{S}^2, \quad (1.22)$$

one easily verifies that for every $s \in \mathbb{R}$, the H^s -norm of the smooth solution $u(t)$ to (1.22) is conserved. Such conservation laws lead to invariant measures of the formal

form

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\|u\|_{H^\alpha(\mathbb{S}^2)}^2\right) du.$$

As a by-product of our analysis for proving Theorem 1.2 (as well as the stability statement), together with Bourgain's invariant measure argument, we have the following almost sure global well-posedness result for the completely resonant system of the sphere, at positive regularity:

Theorem 1.3. *Let $\alpha > 1$. There is a set Σ of full probability such that for every $T > 0$, $\omega \in \Sigma$, the sequence of smooth solution $(u_N)_{N \geq 1}$ of (1.22) with initial data $\Pi_N \phi_\alpha^\omega$ converges in $L^\infty([-T, T]; H^\beta(\mathbb{S}^2))$, $\beta < \alpha - 1$ to a limit which satisfies (1.22) in a distributional sense on $[-T, T] \times \mathbb{S}^2$.*

We stress out that for $\alpha < \frac{5}{4}$, we do not know how to prove Theorem 1.3 without using the machinery elaborated in the proof of Theorem 1.2.

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2. DICTIONARY OF GLOBAL NOTATIONS

We summarize generic notations and some notations for important objects.

- *Estimates:* We will adopt the generic PDE notations $f \lesssim g$, $f \gtrsim g$, and $f \sim g$ if $f \lesssim g$ and $f \gtrsim g$. If we want to emphasize the dependence of a parameter A in the implicit constants, we will write as $f \lesssim_A g$. These notations will be often used in the proof, but we will only use C, C_A .
- *Probability:* Fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{B}_{\leq N}$ denotes the σ -algebra generated by Gaussians $(g_{n,k})_{|k| \leq n, n \leq N}$, and \mathcal{B}_N denotes the σ -algebra generated by Gaussians $(g_{n,k})_{|k| \leq n, N < n \leq 2N}$. Given a sub- σ algebra \mathcal{B} of \mathcal{F} and $p \geq 1$, denote by

$$\|X(\omega)\|_{L^p_\omega|\mathcal{B}} := \left(\mathbb{E}[|X(\omega)|^p|\mathcal{B}]\right)^{\frac{1}{p}}$$

the conditional L^p -moment of a random variable $X(\omega)$ with respect to \mathcal{B} .

- *Operators:* For Banach spaces \mathcal{X}, \mathcal{Y} , we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from \mathcal{X} to \mathcal{Y} . If $\mathcal{X} = \mathcal{Y}$, we denote it simply by $\mathcal{L}(\mathcal{X})$.
- *Dyadic frequencies:* Capital numbers N, M, L, N_j, \dots always refer to dyadic integers, when they appear as subscript/superscript of functions/operators as well as the summation. Given a set of dyadic numbers N_1, N_2, N_3, \dots , we denote by $N_{(1)}, N_{(2)}, N_{(3)}$ the non-increasing ordering among them. In particular, $N_{(1)} \geq N_{(2)} \geq N_{(3)}$.

- *Projections:* For $n \in \mathbb{N}$, $\pi_n : L^2(\mathbb{S}^2) \rightarrow E_n$ is the orthonormal projection on the eigenspace E_n of the eigenvalue $\lambda_n^2 = n^2 + n + 1$. The space E_n is identified with the L^2 topology. Given $f \in E_n$ and $1 \leq r \leq \infty$ we denote by $\|f\|_{L^r(E_n)} := \|\pi_n f\|_{L^r(\mathbb{S}^2)}$. For dyadic intergers N , Π_N, \mathbf{P}_N defined in (1.10) are the (sharp) Littlewood-Paley projectors.
- *Important objects:* u_N is the solution of the truncated NLS equation (1.11) with initial data $\Pi_N \phi^\omega$, and ψ_N is the solution of (1.12) with initial data $\mathbf{P}_N \phi^\omega$. $\mathcal{H}_n^N(t)$ is the random averaging operator (RAO).
- *Time cut-off functions:* $\chi, \chi_1, \varphi, \eta \dots$. For $T > 0$, we denote by $\chi_T(\cdot) := \chi(T^{-1}\cdot)$.
- For Lebesgue spaces L^q , we will use the conjugate exponent $q' = \frac{q}{q-1}$ (with the convention that $q' = 1$ if $q = \infty$ and $q' = \infty$ if $q = 1$).
- *Generic small parameters:* We reserve a very small parameter $\sigma \in (0, 2^{-100})$ throughout this article and define several related large/small parameters $q_\sigma, \gamma_\sigma, \delta_\sigma, \theta_\sigma$ around some specific numbers such that:

$$\frac{1}{q_\sigma} = \sigma, \quad 1 - \gamma_\sigma = \sigma - \sigma^{10}, \quad 1 - \gamma_{1,\sigma} = \sigma - \sigma^{15} \quad (2.1)$$

$$\delta_\sigma = \sigma^{20}, \quad s_\sigma = \frac{1}{2} + 100\sigma, \quad \theta_\sigma = \sigma^5. \quad (2.2)$$

Note that $1 < \gamma_{1,\sigma} q'_\sigma < \gamma_\sigma q'_\sigma < 1$ and the hierarchy of the smallness is needed to estimate deterministic objects into the Fourier-restriction type spaces, for technical reason. In summary, we have the following hierarchy of parameters:

$$0 < \delta \ll \gamma_1 - \frac{1}{q'} \ll \gamma - \frac{1}{q'} \ll \theta \ll \frac{1}{q} \ll s - \frac{1}{2} \ll 1.$$

- *Special symbols:*

(a) For simplicity, later we will write

$$n \approx L \quad \iff \quad \frac{L}{2} < n \leq L. \quad (2.3)$$

(b) The wick square $|f|^{\circ 2}$ is defined in (1.8) and the exotic products $f \circledast g$ and $f \circledcirc g$ are defined in (5.16)

Notation for the pairing and non-pairing products: Note that

$$\mathcal{N}(u, u, u) = \mathcal{N}^{(1)}(u, u, u) + \mathcal{N}^{(2)}(u, u, u) + \mathcal{N}^{(3)}(u, u, u), \quad (2.4)$$

where

$$\begin{aligned}\mathcal{N}^{(1)}(f_1, f_2, f_3) &= \sum_{\substack{n_1, n_2, n_3 \\ n_2 \neq n_1, n_2 \neq n_3}} \pi_{n_1} f_1 \overline{\pi_{n_2} f_2} \pi_{n_3} f_3, \\ \mathcal{N}^{(2)}(f_1, f_2, f_3) &= \sum_{\substack{n_1, n_2 \\ n_1 \neq n_2}} [\pi_{n_2} f_3 \overline{\pi_{n_2} f_2} - \langle \pi_{n_2} f_3 | \pi_{n_2} f_2 \rangle] \pi_{n_1} f_1 \\ &\quad + \sum_{\substack{n_2, n_3 \\ n_2 \neq n_3}} [\pi_{n_2} f_1 \overline{\pi_{n_2} f_2} - \langle \pi_{n_2} f_1 | \pi_{n_2} f_2 \rangle] \pi_{n_3} f_3, \\ \mathcal{N}^{(3)}(f_1, f_2, f_3) &= \sum_n [\pi_n f_1 \overline{\pi_n f_2} \pi_n f_3 - \langle \pi_n f_1 | \pi_n f_2 \rangle \pi_n f_3 - \langle \pi_n f_3 | \pi_n f_2 \rangle \pi_n f_1].\end{aligned}$$

As the wick product is not always needed, we denote by

$$\mathcal{N}^{(0)}(f_1, f_2, f_3) := \sum_{n_0, n_1, n_2, n_3} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3).$$

We will add $(,)$ to present paired, and $[,]$ to present non-paired constraints that appear as subscripts of $\mathcal{N}^{(j)}(f_1, f_2, f_3)$, $j = 0, 1, 2, 3$. For example, for $j, k \in \{0, 1, 2, 3\}$ such that $(-1)^j + (-1)^k = 0$,

$$\mathcal{N}_{(j,k)}^{(0)}(f_1, f_2, f_3) = \sum_{n_0, n_1, n_2, n_3} \mathbf{1}_{n_j = n_k} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3), \quad (2.5)$$

If several pairing conditions are satisfied simultaneously, we write them as accumulated subscripts. For example,

$$\mathcal{N}_{(0,1)(2,3)}^{(0)}(f_1, f_2, f_3) = \sum_{n_0, n_1, n_2, n_3} \mathbf{1}_{n_0 = n_1} \mathbf{1}_{n_2 = n_3} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3).$$

For non-pairing constraints, we express the condition $n_j \neq n_k$ as subscript $[j, k]$:

$$\mathcal{N}_{[j,k]}^{(0)}(f_1, f_2, f_3) = \sum_{n_0, n_1, n_2, n_3} \mathbf{1}_{n_j \neq n_k} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3). \quad (2.6)$$

When there are several indices in the bracket, we mean that non-pairing conditions are satisfied for each pair (j, k) such that $(-1)^j + (-1)^k = 0$. For example, the completely non-paired product can be represented by the subscript $[0, 1, 2, 3]$:

$$\mathcal{N}_{[0,1,2,3]}^{(0)}(f_1, f_2, f_3) := \sum_{\substack{n_0, n_1, n_2, n_3 \\ \text{distinct}}} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3).$$

Similarly, if a pairing condition (j, k) and a non-pairing condition $[l, i]$ simultaneously appears, we will use the accumulate subscript $(j, k)[l, i]$:

$$\mathcal{N}_{(j,k)[l,i]}^{(0)}(f_1, f_2, f_3) = \sum_{\substack{n_0, n_1, n_2, n_3 \\ n_j = n_k, n_l \neq n_i}} \pi_{n_0} (\pi_{n_1} f_1 \pi_{n_2} \overline{f_2} \pi_{n_3} f_3).$$

3. EIGENFUNCTION ESTIMATES AND PROBABILISTIC SOBOLEV EMBEDDING

3.1. Eigenfunction estimates on \mathbb{S}^2 . The next two propositions are true on any Riemannian compact surface.

Proposition 3.1 (Eigenfunction estimate, [23]).

$$\|\pi_n f\|_{L^p(\mathbb{S}^2)} \leq C \begin{cases} n^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}, & 2 \leq p \leq 6 \\ n^{\frac{1}{2}-\frac{2}{p}}, & 6 \leq p \leq \infty. \end{cases}$$

Proposition 3.2 (Bilinear eigenfunction estimate, [8]). *Assume that $n_1 \geq n_2$, then for any $f, g \in L^2(\mathbb{S}^2)$,*

$$\|\pi_{n_1} f \cdot \pi_{n_2} g\|_{L^2(\mathbb{S}^2)} \leq C n_2^{\frac{1}{4}} \|f\|_{L^2(\mathbb{S}^2)} \|g\|_{L^2(\mathbb{S}^2)}.$$

We will need a probabilistic estimate for the random spherical harmonics for which the proof is standard:

Lemma 3.3. *There exist $C > 0$ such that for all $2 \leq q < \infty$, $n \in \mathbb{N}$ and $x \in \mathbb{S}^2$,*

$$\|e_n^\omega\|_{L^q_\omega} \lesssim \sqrt{r}.$$

Moreover, there exists $C_0 > c_0 > 0$, such that for every $R \geq 1$,

$$\mathbb{P}[\|e_n^\omega\|_{L^q(\mathbb{S}^2)} > R] \leq C_0 e^{-c_0 R^2}.$$

Proof. The first part of Lemma 3.3 is a consequence of Khintchine's inequality and the exact Weyl's law (1.3). The second claim can be viewed as a rough version of the probabilistic Sobolev embedding from the work [10]. \square

3.2. Elementary properties of the random averaging operators. We first establish a simple but crucial property is that $\mathcal{H}_n^N(t)$ is unitary on E_n :

Lemma 3.4. *For fixed $t \in [0, T]$, the operator $\mathcal{H}_n^N(t)$ is unitary on E_n . In particular, the matrix element $(H_{n;k,k'}^N(t))$ expressed in a given orthonormal basis $(\mathbf{b}_{n,k})_{|k| \leq n}$ verifies*

$$\sum_{|k| \leq n} H_{n;k,\ell}^N(t) \overline{H_{n;k,\ell'}^N(t)} = \delta_{\ell\ell'}, \quad \sum_{|\ell| \leq n} H_{n;k,\ell}^N(t) \overline{H_{n;k',\ell}^N(t)} = \delta_{kk'}, \quad (3.1)$$

for all $t \in \mathbb{R}$.

This is the consequence of Lemma B.1 and the fact that the potential operator

$$f_n \mapsto \pi_n(f_n \cdot |u_{\frac{N}{2}}(t)|^{\circ 2})$$

is self-adjoint. This is a consequence of Lemma B.1. This type of unitary property for random averaging operators was first observed in [17]. As in [17], we will use it to achieve some cancellation for Wick's cubic power of colored Gaussians. In addition, we use this to prove the law of the colored terms is invariant (see Proposition 3.6). This will be useful to handle some resonant interactions that are removed by the wick-ordering when NLS is posed on \mathbb{T}^d .

We have the following key ingredient is the following key cancellation from Wick-ordering (this is a global aspect of the RAO theory, first appeared in [17]).

Lemma 3.5.

$$\begin{aligned}
 |e_n^N(t, x)|^2 - \|e_n^N(t, x)\|_{L_x^2}^2 &= \sum_{\substack{|k|, |k'|, |\ell|, |\ell'| \leq n \\ \ell \neq \ell'}} H_{n;k, \ell}^N(t) \overline{H}_{n;k', \ell'}^N(t) \frac{g_{n, \ell} \overline{g}_{n, \ell'}}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) \\
 &+ \sum_{|k|, |k'|, |\ell| \leq n} H_{n;k, \ell}^N(t) \overline{H}_{n;k', \ell}^N(t) \frac{|g_{n, \ell}|^2 - 1}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) \\
 &- \sum_{|\ell| \leq n} \frac{|g_{n, \ell}|^2 - 1}{2n+1}. \quad (3.2)
 \end{aligned}$$

Proof. To see the formula above, we expand directly $|e_n^N(t, x)|^2 - \|e_n^N(t, x)\|_{L_x^2}^2$ as

$$\begin{aligned}
 &\sum_{k, k', \ell, \ell'} H_{n;k, \ell}^N \overline{H}_{n;k', \ell'}^N \frac{g_{n, \ell} \overline{g}_{n, \ell'}}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) - \sum_{k, \ell, \ell'} H_{n;k, \ell}^N \overline{H}_{n;k, \ell'}^N \frac{g_{n, \ell} \overline{g}_{n, \ell'}}{2n+1} \\
 &= \sum_{\substack{k, k', \ell, \ell' \\ \ell \neq \ell'}} H_{n;k, \ell}^N \overline{H}_{n;k', \ell'}^N \frac{g_{n, \ell} \overline{g}_{n, \ell'}}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) \\
 &+ \sum_{k, k', \ell} H_{n;k, \ell}^N \overline{H}_{n;k', \ell}^N \frac{|g_{n, \ell}|^2}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) - \sum_{\ell} \frac{|g_{n, \ell}|^2}{2n+1},
 \end{aligned}$$

where to the last step, we used the unitary property (Lemma 3.4). Using again Lemma 3.4, the second line on the right hand side can be computed as

$$\sum_{k, k', \ell} H_{n;k, \ell}^N \overline{H}_{n;k', \ell}^N \frac{|g_{n, \ell}|^2 - 1}{2n+1} \mathbf{b}_{n, k}(x) \overline{\mathbf{b}}_{n, k'}(x) + \sum_{\ell} \frac{|\mathbf{b}_{n, k}(x)|^2}{2n+1} - \sum_{\ell} \frac{|g_{n, \ell}|^2}{2n+1}.$$

By Weyl, we have

$$\sum_{\ell} \frac{|\mathbf{b}_{n, k}(x)|^2}{2n+1} = \sum_{\ell} \frac{1}{2n+1}.$$

Regrouping terms, the proof is complete. \square

Besides, we develop a new property for colored-Gaussians $e_n^N(t) = \mathcal{H}_n^N(t)(e_n^\omega)$ on E_n , which is helpful in the estimate for the resonant interactions, for which we cannot use time-modulation.

Proposition 3.6. *For fixed $n > N/2$ and $t \in \mathbb{R}$, the random field $e_n^N(t)$ has the same law as e_n^ω :*

$$\mathcal{L}(e_n^N(t)) = \mathcal{L}(e_n^\omega) = \mathcal{N}_{E_n}(0; \text{Id}).$$

We need the following Lemma from elementary probability theory:

Lemma 3.7. *Let $X \sim \mathcal{N}_{\mathbb{R}^n}(0; \text{Id})$ be a random Gaussian vector on a probability space $(\Omega; \mathcal{A}, \mathbb{P})$. Assume that $\mathcal{G} \subseteq \mathcal{F}$ is a sub σ -algebra, and that X is independent of \mathcal{G} . Let $A \in \mathcal{O}(n)$ be a random orthogonal matrix that is \mathcal{G} -measurable. Then*

$$\mathcal{L}(AX) = \mathcal{L}(X).$$

Proof. We compute the characteristic function of AX : given $\xi \in \mathbb{R}^n$,

$$\varphi_{AX}(\xi) = \mathbb{E}[e^{i\langle AX, \xi \rangle}] = \mathbb{E}[e^{i\langle X, A^*\xi \rangle}] = \mathbb{E}[\mathbb{E}[e^{i\langle X, A^*\xi \rangle} \mid \mathcal{G}]].$$

Since $A^*\xi$ is \mathcal{G} -measurable and that X is independent of \mathcal{G} , we deduce that

$$\mathbb{E}[e^{i\langle X, A^*\xi \rangle} \mid \mathcal{G}] = \varphi_X(A^*\xi) = e^{-\frac{1}{2}\langle A^*\xi, A^*\xi \rangle} = e^{-\frac{1}{2}\langle AA^*\xi, \xi \rangle} = e^{-\frac{1}{2}|\xi|^2} = \varphi_X(\xi).$$

• Hence,

$$\varphi_{AX}(\xi) = \varphi_X(\xi).$$

Since this holds for every $\xi \in \mathbb{R}^n$, we conclude that $\mathcal{L}(AX) = \mathcal{L}(X) = \mathcal{N}_{\mathbb{R}^n}(0, \text{Id})$. \square

Proof of Proposition 3.6. For fixed $n > N/2$, the colored field $e_n^N = \mathcal{H}_n^N(t)(e_n^\omega)$ is the image of the normal Gaussian variable e_n^ω on E_n under the rotation $\mathcal{H}_n^N(t)$. Note that $\mathcal{H}_n^N(t)$ depends on the σ -algebra that is $\mathcal{F}_{N/2}$ -measurable, hence independent of e_n^ω since $n \leq N/2$. It follows from Lemma 3.7 that for every $t \in [0, T]$,

$$\mathcal{L}(e_n^N(t)) = \mathcal{L}(e_n^\omega). \quad (3.3)$$

This completes the proof of Corollary 3.6. \square

4. FUNCTIONAL SPACES FOR FUNCTIONS AND OPERATORS

4.1. Fourier-Lebesgue restriction spaces for functions. Recall that $E_n = \pi_n L^2(\mathbb{S}^2)$ is a closed subspace of $L^2(\mathbb{S}^2)$ of dimension $2n + 1$, endowed with the $L^2(\mathbb{S}^2)$ norm. For a space-time Schwartz distribution $F \in \mathcal{S}'(\mathbb{R} \times \mathbb{S}^2)$, $\pi_n F$ can be viewed as a Schwartz distribution with values in E_n , namely $\pi_n F \in \mathcal{S}'(\mathbb{R}; E_n)$. We denote by $\widehat{F}(\tau, \cdot)$ its time Fourier transform. By decomposing into eigenspaces E_n , we have

$$\widehat{F}(\tau, x) = \sum_{n=1}^{\infty} \widehat{\pi_n F}(\tau, x).$$

For $p, q, r \in [1, \infty]$, consider the restriction norm:

$$\|F\|_{X_{p,q,r}^{s,\gamma}} := \|\langle n \rangle^s \langle \tau + \lambda_n^2 \rangle^\gamma \widehat{\pi_n F}(\tau, \cdot)\|_{l_n^p L_\tau^q L_x^r} = \|\langle n \rangle^s \langle \kappa \rangle^\gamma \pi_n \widehat{e^{-it\Delta} F}(\kappa, \cdot)\|_{l_n^p L_\kappa^q L_x^r}.$$

It will be more convenient to introduce the notation of twisted time-Fourier transform. For a function $F_n \in \mathcal{S}'(\mathbb{R}; E_n)$, we denote by

$$\widetilde{F}_n(\kappa, x) := \widehat{F}_n(\kappa - \lambda_n^2, x) = \widehat{e^{-it\Delta} F_n}(\kappa, x).$$

For a general distribution $F \in \mathcal{S}'(\mathbb{R} \times \mathbb{S}^2)$, we denote by

$$\widetilde{F}(\kappa, \cdot) := \widehat{e^{-it\Delta} F}(\kappa, \cdot) = \sum_{n=1}^{\infty} \widetilde{\pi_n F}(\kappa, \cdot).$$

Though E_n is finite-dimensional space, different norms are not asymptotically equivalent as $n \rightarrow \infty$. By abusing the notation, we denote by $L^r(E_n)$ the space E_n endowed with the $L^r(\mathbb{S}^2)$ norm, with the convention $L^2(E_n) = E_n$.

Set

$$\|F_n\|_{X_{q,r}^{s,\gamma}(E_n)} := \|\langle \kappa \rangle^\gamma \pi_n \widehat{e^{-it\Delta} F_n}(\kappa, x)\|_{L_\kappa^q L_x^r} = \|\langle \kappa \rangle^\gamma \widetilde{\pi_n F_n}(\kappa, x)\|_{L_\kappa^q L_x^r},$$

then

$$\|F\|_{X_{p,q,r}^{s,\gamma}} = \|\pi_n F\|_{L_n^p X_{q,r}^{s,\gamma}(E_n)}.$$

Note that $X_{2,2,2}^{0,b}$ is just the usual $X^{0,b}$ space in the literature. One keeps in mind that in our trilinear estimates, we will put remainders in $X_{2,2,2}^{0,\frac{1}{2}+}$ and terms like linear evolutions in $X_{\infty,\infty,1}^{1+}$ essentially. Given an interval $I \subset \mathbb{R}$, we define the norm $X_{p,q,r}^{s,b}(I)$ by

$$\|F\|_{X_{p,q,r}^{s,b}(I)} = \inf\{\|F_1\|_{X_{p,q,r}^{s,b}} : F_1|_I = F\}.$$

Let $\chi \in C_c^\infty((-1, 1))$ and $\chi_T(t) := \chi(T^{-1}t)$ be a time cutoff function, we have the following localization property:

Proposition 4.1. *Let $1 \leq p, r \leq \infty$, and $1 \leq q < \infty$, $0 < \gamma \leq \gamma_1 < 1$. For any $u \in X_{p,q,r}^{\gamma_1}$ such that $u(t=0, \cdot) = 0$, we have*

$$\|\chi_T(t)u\|_{X_{p,q,r}^{s,\gamma}} \lesssim T^{\gamma_1-\gamma} \|u\|_{X_{p,q,r}^{s,\gamma_1}}$$

for all $0 < T \leq 1$.

We provide the proof in Appendix (see Lemma A.4). We have the embedding property:

Lemma 4.2. *Let $1 < q < \infty$, $1 \leq r \leq \infty$ and $\gamma \in (\frac{1}{q}, 1)$. Then $X_{q,r}^{0,\gamma}(E_n) \hookrightarrow C^\alpha(\mathbb{R}; L^r(E_n))$ for any $\alpha \in (0, \gamma - \frac{1}{q}]$.*

The proof is a direct consequence of Lemma A.5. Denote by

$$\mathcal{I}F(t) := \int_0^t e^{i(t-t')\Delta} F(t') dt'$$

the Duhamel operator, we have the following linear inhomogeneous estimate:

Proposition 4.3. *For any $s \in \mathbb{R}$, $q \in (1, \infty)$, $\frac{1}{q'} < \gamma \leq 1$ and any compactly supported smooth time cutoff $\chi(t)$, we have for any $0 < \theta < 1 - \gamma$,*

$$\|\chi_T(t)\mathcal{I}\pi_n F\|_{X_{q,r}^{s,\gamma}(E_n)} \leq C_{\eta,\gamma} T^\theta \|\pi_n F\|_{X_{q,r}^{s,\gamma-1+\theta}(E_n)},$$

$$\|\chi_T(t)\mathcal{I}\pi_n F\|_{X_{p,q,r}^{s,\gamma}} \leq C_{\eta,\gamma} T^\theta \|F\|_{X_{p,q,r}^{s,\gamma-1+\theta}},$$

where $\chi_T(t) = \chi(T^{-1}t)$.

The proof is a direct consequence of the abstract Lemma A.8 and Lemma A.4. We end this Section by recalling the semi-classical Strichartz estimate from Burq–Gérard–Tzvetkov and its formulation in Fourier restriction space. We stress out that this estimate is true on any compact surface.

Proposition 4.4 (Semi-classical Strichartz estimate, [7]). *For all $2 \leq p, q \leq +\infty$ with $q \neq +\infty$ admissible*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$

and for all compact interval $I \subset \mathbb{R}$, there exists $C(I, p, q) > 0$ such that for all $u \in H^{\frac{1}{p}}(\mathbb{S}^2)$,

$$\|e^{it\Delta} \phi\|_{L^p L^q(I \times \mathbb{S}^2)} \leq C(I, p, q) \|\phi\|_{H^{\frac{1}{p}}(\mathbb{S}^2)}.$$

Moreover, if χ is a test function on \mathbb{R} , then for all $b > 1$ and $u \in X^{\frac{1}{p}, b}$,

$$\|\chi(t)u\|_{L^p L^q(\mathbb{R} \times \mathbb{S}^2)} \leq C(\eta, p, q) \|u\|_{X^{\frac{1}{p}, b}}.$$

The bound in Fourier restriction spaces follows from the transference principle. Semi-classical Strichartz estimates are useful in the deterministic analysis of Section 9.

4.2. Fourier restriction norms for operators. For the eigenspace E_n and a time-dependent linear operator $\mathcal{T}_n(t) : E_n \rightarrow E_n$. Denote by $(T_{n;k,k'}(t))_{|k|,|k'| \leq n}$ the matrix element of \mathcal{T}_n under a given orthonormal basis $(\mathbf{b}_{n,k})_{|k| \leq n}$ of E_n , i.e.

$$\mathcal{T}_n(t)(\mathbf{b}_{n,k}) = \sum_{|\ell| \leq n} T_{n;\ell,k}(t) \mathbf{b}_{n,\ell}.$$

Denote by

$$\widehat{\mathcal{T}}_n(\tau) := \int_{\mathbb{R}} \mathcal{T}_n(t) e^{-it\tau} dt$$

the time Fourier transform of the linear operator $\mathcal{H}_n^{2N,T}(t)$, where the integral is in the sense of Bochner, and we set

$$\widetilde{\mathcal{T}}_n(\kappa) := e^{it\lambda_n^2} \widehat{\mathcal{T}}_n(t)(\kappa)$$

the twisted time-Fourier transform of the operator $\mathcal{T}_n(t)$.

We will use the norm $S_n^{q,\gamma}$ for any linear operator $\mathcal{T}_n(t), \mathcal{L}_n(t) \in \mathcal{S}'(\mathbb{R}; \mathcal{L}(E_n))$:

$$\|\mathcal{T}_n\|_{S_n^{q,\gamma}} := \|\langle \tau + \lambda_n^2 \rangle^\gamma \widehat{\mathcal{T}}_n(\tau)\|_{E_n \rightarrow L_\tau^q E_n}, \quad (4.1)$$

$$\|\mathcal{L}_n\|_{S_n^{q,\gamma,*}} := \|\langle \tau - \lambda_n^2 \rangle^\gamma \widehat{\mathcal{L}}_n(\tau)\|_{E_n \rightarrow L_\tau^q E_n} \quad (4.2)$$

In terms of twisted Fourier transform,

$$\|\mathcal{T}_n\|_{S_n^{q,\gamma}} = \|\langle \kappa \rangle^\gamma \mathcal{F}_t(e^{i\lambda_n^2 t} \mathcal{T}_n(t))(\kappa)\|_{E_n \rightarrow L_\kappa^q E_n}, \quad (4.3)$$

$$\|\mathcal{L}_n\|_{S_n^{q,\gamma,*}} = \|\langle \kappa \rangle^\gamma \mathcal{F}_t(e^{-i\lambda_n^2 t} \mathcal{L}_n(t))(-\kappa)\|_{E_n \rightarrow L_\kappa^q E_n}. \quad (4.4)$$

In terms of matrix element,

$$\|\mathcal{T}_n\|_{S_n^{q,\gamma}} = \|\langle \kappa \rangle^\gamma \widehat{T}_{n,\ell,k}(\kappa - \lambda_n^2)\|_{l_\kappa^2 \rightarrow L_\kappa^q l_\ell^2}, \quad \|\mathcal{L}_n\|_{S_n^{q,\gamma,*}} = \|\langle \kappa \rangle^\gamma \widehat{L}_{n,\ell,k}(\kappa + \lambda_n^2)\|_{l_\kappa^2 \rightarrow L_\kappa^q l_\ell^2}.$$

Given $\mathcal{H}_n(t) \in \mathcal{S}'(\mathbb{R}; \mathcal{L}(E_n))$, in application, we will control $S_n^{q,\gamma}$ -norm for $\mathcal{H}_n(t)$ while $S_n^{q,\gamma,*}$ -norm for its adjoint $(\mathcal{H}_n(t))^*$. Note that

$$\widetilde{\mathcal{H}}_n(\kappa) = \mathcal{F}_t(e^{i\lambda_n^2 t} \mathcal{H}_n(t))(\kappa) = \int_{\mathbb{R}} e^{i\lambda_n^2 t - it\kappa} \mathcal{H}_n(t) dt.$$

For fixed $\kappa \in \mathbb{R}$, as an operator acting on E_n , the adjoint of $\widetilde{\mathcal{H}}_n(\lambda)$ is given by

$$(\widetilde{\mathcal{H}}_n(\kappa))^* = \mathcal{F}_t(e^{-i\lambda_n^2 t} (\mathcal{H}_n(t))^*)(-\kappa) = \int_{\mathbb{R}} e^{-i\lambda_n^2 t + it\kappa} (\mathcal{H}_n(t))^* dt.$$

The matrix elements of $\widetilde{\mathcal{H}}_n(\kappa)$ is $\widetilde{H}_{n;\ell,k}(\kappa) = \widehat{H}_{n;\ell,k}(\kappa - \lambda_n^2)$, and the matrix element of $(\widetilde{\mathcal{H}}_n(\kappa))^*$ is

$$(\widetilde{\mathcal{H}}_n(\kappa))^*_{\ell,k} = \widehat{H}_{n;k,\ell}(\lambda_n^2 - \kappa) = \overline{\widehat{H}_{n;k,\ell}(\kappa - \lambda_n^2)} = \overline{\widetilde{H}_{n;k,\ell}(\kappa)}.$$

By definition, we have clearly that

$$\|\mathcal{H}_n\|_{S_n^{q,\gamma}} = \|\langle \kappa \rangle^\gamma \widetilde{\mathcal{H}}_n(\kappa)\|_{E_n \rightarrow L_\kappa^q E_n} = \|\mathcal{H}_n\|_{E_n \rightarrow X_{q,2}^{0,\gamma}(E_n)} = \|\langle \kappa \rangle^\gamma \widetilde{H}_{n;\ell,k}(\kappa)\|_{l_k^2 \rightarrow L_\kappa^q l_\ell^2}.$$

Since the matrix element

$$\mathcal{F}_t(e^{-it\lambda_n^2}(\mathcal{H}_n(t))^*)_{\ell,k}(-\kappa) = \widehat{H}_{n;k,l}(-\kappa + \lambda_n^2) = \overline{\widehat{H}_{n;k,\ell}(\kappa - \lambda_n^2)},$$

we have

$$\begin{aligned} \|\mathcal{H}_n^*\|_{S_n^{q,\gamma,*}} &= \|\langle \kappa \rangle^\gamma \mathcal{F}_t(e^{-it\lambda_n^2}(\mathcal{H}_n(t))^*)(-\kappa)\|_{E_n \rightarrow L_\kappa^q E_n} \\ &= \|\langle \kappa \rangle^\gamma \widetilde{H}_{n;\ell,k}(\kappa)\|_{l_\ell^2 \rightarrow L_\kappa^q l_k^2}. \end{aligned} \quad (4.5)$$

The $S_n^{q,\gamma}$ norm allows us to control the $X_{q,\infty}^{0,\gamma}(E_n)$ norm of $e_n^{N,\dagger}$, while the $S_n^{q,\gamma,*}$ norm will be used in the probabilistic trilinear estimate for purely colored-Gaussian terms (see Section 8). In particular, we need the following type of pointwise bound:

Lemma 4.5. *Assume that $\mathcal{T}_n(t) \in S_n^{q,\gamma,*}$, then for any $\mathbf{f}_n \in E_n$,*

$$\sup_{x \in \mathbb{S}^2} \|\langle \kappa \rangle^\gamma \widetilde{\mathcal{T}}_n(t)(\mathbf{f}_n)(\kappa, x)\|_{L_\kappa^q} \leq \sqrt{2n+1} \|\mathcal{T}_n\|_{S_n^{q,\gamma,*}} \|\mathbf{f}_n\|_{E_n}.$$

Proof. For example, for $\mathbf{f}_n = \sum_{|k| \leq n} a_k \mathbf{b}_{n,k} \in E_n$ written in the bases $(\mathbf{b}_{n,k})$, we have

$$\mathcal{T}_n(t)(\mathbf{f}_n)(x) = \sum_{|\ell|, |k| \leq n} T_{n;\ell,k}(t) \cdot a_k \mathbf{b}_{n,\ell}(x).$$

Hence by Cauchy-Schwarz, for fixed $x \in \mathbb{S}^2$,

$$\begin{aligned} \|\langle \kappa \rangle^\gamma \widetilde{\mathcal{T}}_n(t)(\mathbf{f}_n)(\kappa, x)\|_{L_\kappa^q} &= \left\| \sum_{|\ell|, |k| \leq n} \langle \kappa \rangle^\gamma \cdot \widetilde{T}_{n;\ell,k}(\kappa) \cdot \overline{a_k} \overline{\mathbf{b}_{n,\ell}}(x) \right\|_{L_\kappa^q} \\ &\leq \|a_k\|_{l_k^2} \left\| \left(\sum_{|k| \leq n} \left| \sum_{|\ell| \leq n} \langle \kappa \rangle^\gamma \widetilde{T}_{n;\ell,k}(\kappa) \mathbf{b}_{n,\ell}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\kappa^q} \\ &= \|\mathbf{f}_n\|_{E_n} \left\| \left(\sum_{|k| \leq n} \left| \sum_{|\ell| \leq n} \langle \kappa \rangle^\gamma \widetilde{T}_{n;\ell,k}(\kappa) \mathbf{b}_{n,\ell}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\kappa^q}. \end{aligned}$$

We conclude by the following pointwise bound:

$$\left\| \left(\sum_{|k| \leq n} \left| \sum_{|\ell| \leq n} \langle \kappa \rangle^\gamma \widetilde{T}_{n;\ell,k}(\kappa) \mathbf{b}_{n,\ell}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\kappa^q} \leq \sqrt{2n+1} \|\langle \kappa \rangle^\gamma (\widetilde{\mathcal{T}}_n(\kappa))^*\|_{E_n \rightarrow L_\kappa^q E_n}. \quad (4.6)$$

Indeed, by Weyl (1.3), the left hand side can be bounded by

$$\|\langle \kappa \rangle^\gamma \widetilde{T}_{n;\ell,k}(\kappa)\|_{l_\ell^2 \rightarrow L_\kappa^q l_k^2} \left(\sum_{|\ell| \leq n} |\overline{\mathbf{b}_{n,\ell}}(x)|^2 \right)^{\frac{1}{2}} = \sqrt{2n+1} \|\langle \kappa \rangle^\gamma (\widetilde{\mathcal{T}}_n(\kappa))^*\|_{E_n \rightarrow L_\kappa^q E_n},$$

This completes the proof. \square

5. KEY INDUCTION STEPS

In this section, we present the precise statement of Theorem 1.2 and the induction steps. For the reason of clarity, we will only prove the convergence for dyadic sequences $(u_N)_{N \in 2^{\mathbb{N}}}$. Slight modification leads to the convergence of the full sequence, see [15] or [24] for details.

5.1. Equations for the RAO and remainders. The Cauchy problem (1.15) can be reformulated as a linear equation for $\mathcal{H}_n^N(t)$:

$$\mathcal{H}_n^N(t) = e^{-it\lambda_n^2} \text{Id}_{E_n} + \frac{2}{i} \int_0^t e^{-i(t-t')\lambda_n^2} \pi_n(\mathcal{H}_n^N(t')(\cdot) \cdot |u_{N/2}(t')|^{\diamond 2}) dt'. \quad (5.1)$$

To solve the fixed-point problem for the norm $S_n^{q,\gamma}$, we need also to estimate the adjoint of the twisted-Fourier transform of $\mathcal{H}_n^N(t)$. Define the potential operator \mathcal{P}_n^N

$$\mathcal{P}_n^N(t) : F(t) \mapsto \pi_n(\pi_n F(t) \cdot |u_N(t)|^{\diamond 2}), \quad (5.2)$$

The Duhamel formula for the adjoint operator reads:

$$(\mathcal{H}_n^N(t))^* = e^{i\lambda_n^2 t} \text{Id}_{E_n} - \frac{2}{i} \int_0^t e^{i\lambda_n^2 t'} \mathcal{P}_n^N(t') \circ \mathcal{H}_n^N(t') \circ (\mathcal{H}_n^N(t))^* dt'. \quad (5.3)$$

To see this, using the unitary property of $\mathcal{H}_n^N(t)$, we write

$$\begin{aligned} (\mathcal{H}_n^N(t))^* &= e^{i\lambda_n^2 t} \text{Id}_{E_n} + (\mathcal{H}_n^N(t))^* - e^{i\lambda_n^2 t} \mathcal{H}_n^N(t) (\mathcal{H}_n^N(t))^* \\ &= e^{i\lambda_n^2 t} \text{Id}_{E_n} + (\mathcal{H}_n^N(t))^* - \left(\text{Id}_{E_n} + \frac{2}{i} \int_0^t e^{it'\lambda_n^2} \mathcal{P}_n^N(t') \circ \mathcal{H}_n^N(t') dt' \right) (\mathcal{H}_n^N(t))^* \\ &= e^{i\lambda_n^2 t} \text{Id}_{E_n} - \frac{2}{i} \int_0^t e^{i\lambda_n^2 t'} \mathcal{P}_n^N(t') \circ \mathcal{H}_n^N(t') \circ (\mathcal{H}_n^N(t))^* dt'. \end{aligned}$$

From (1.20), the equation verified by w_N reads

$$\begin{aligned} w_N(t) &= 2\mathcal{I}\mathcal{N}_{(0,1)}(w_N, u_{N/2}, u_{N/2}) + 2\mathcal{I}\mathcal{N}_{[0,1]}(\psi_N + w_N, u_{N/2}, u_{N/2}) \\ &\quad + \mathcal{I}\mathcal{N}(u_{N/2}, \psi_N + w_N, u_{N/2}) + 2\mathcal{I}\mathcal{N}(\psi_N + w_N, \psi_N + w_N, u_{N/2}) \\ &\quad + \mathcal{I}\mathcal{N}(\psi_N + w_N, u_{N/2}, \psi_N + w_N) + \mathcal{I}\mathcal{N}(\psi_N + w_N), \end{aligned} \quad (5.4)$$

where we recall that

$$\mathcal{N}_{[0,1]}(f, g, h) := \sum_{\substack{n, n_1, n_2, n_3 \\ n \neq n_1}} \pi_n(\pi_{n_1} f \cdot \pi_{n_2} \bar{g} \diamond \pi_{n_3} h). \quad (5.5)$$

5.2. The extensions and local well-posedness statements. In practice, we have to restrict the random averaging operators and remainders on some short time interval of size $O(T)$ and extend these restrictions that are defined for all time. As our analysis involves the adjoint of RAO, the extension procedure becomes slightly more delicate than in [15]. These extensions will be defined inductively as follows:

A. Initialization:

Given the small number $T \in (0, \frac{1}{2})$ and the bump function $\chi \in C_c^\infty((-1, 1))$ such that $\chi(t) \equiv 1$ for $|t| \leq \frac{1}{2}$. First for the given initial frequency $N_0 = 2$, we define $u_{N_0}^\dagger(t) = \chi_T(t)u_{N_0}(t)$, where u_{N_0} is the solution of (1.11) with $N = 2$. Define for $N_0/2 < n \leq N_0$,

$$\mathcal{H}_n^{N_0, \dagger}(t) = \chi(t)\mathcal{H}_n^{N_0}(t), \quad \mathcal{G}_n^{N_0, \dagger}(t) := \chi(t)(\mathcal{H}_n^{N_0}(t))^*,$$

and for $L \leq N_0$, we set simply

$$\psi_L^\dagger := u_L^\dagger - u_{\frac{L}{2}}^\dagger, \quad w_L^\dagger \equiv 0.$$

B. Heredity: Assuming that for some $N \geq N_0$, the objects $\mathcal{H}_m^{M, \dagger}(t), \mathcal{G}_m^{M, \dagger}(t), w_M^\dagger(t)$ and $\psi_M^\dagger(t)$ are defined for $N_0 \leq M \leq N$, compactly supported in $(-1, 1)$ and they coincide with respectively $\mathcal{H}_m^M(t), (\mathcal{H}_m^M(t))^*, w_M(t), \psi_M(t)$ on $[-\frac{T}{2}, \frac{T}{2}]$.

Recall the notation $n \approx 2N$ for $N < n \leq 2N$.

Step B1: The potential operator: For $n \approx 2N$, we define the extension of the linear operator $\mathcal{P}_n^{2N, \dagger}(t)$ via

$$\mathcal{P}_n^{2N, \dagger}(t)(F(t)) := \pi_n(\pi_n F(t) \cdot |u_N^\dagger(t)|^{\otimes 2}). \quad (5.6)$$

We have $\mathcal{P}_n^{2N, \dagger}(t) = \mathcal{P}_n^{2N}(t)$ for $|t| \leq \frac{T}{2}$, and one verifies that for every $t \in \mathbb{R}$, $\mathcal{P}_n^{2N, \dagger}(t)$ is a self-adjoint operator on E_n .

Step 2: Random averaging operators and colored gaussians: For $n \approx 2N$, we define $\mathcal{H}_n^{2N, \dagger}$ by the equation (provided that the solution exists and is unique)

$$\mathcal{H}_n^{2N, \dagger}(t) = \chi(t)e^{-it\lambda_n^2} \text{Id}_{E_n} + \frac{2}{i}\chi_{2T}(t) \int_0^t e^{-i\lambda_n^2(t-t')} \mathcal{P}_n^{2N, \dagger}(t') \circ \mathcal{H}_n^{2N, \dagger}(t') dt'. \quad (5.7)$$

Note that for all $n \approx 2N$ and $t \in \mathbb{R}$, the operator $\mathcal{H}_n^{2N, \dagger}$ is $\mathcal{B}_{\leq N}$ -measurable. Moreover, when $|t| \leq \frac{T}{2}$, we have

$$\mathcal{H}_n^{2N, \dagger}(t) = \mathcal{H}_n^{2N}(t).$$

For the extension of the adjoint operator, instead of taking directly the adjoint, we introduce the operator $\mathcal{G}_n^{2N, \dagger}(t)$ as the solution of the linear equation

$$\mathcal{G}_n^{2N, \dagger}(t) = \chi(t)e^{i\lambda_n^2 t} \text{Id}_{E_n} - \frac{2}{i}\chi_{2T}(t) \int_0^t e^{i\lambda_n^2 t'} \mathcal{P}_n^{2N, \dagger}(t') \circ \mathcal{H}_n^{2N, \dagger}(t') \circ \mathcal{G}_n^{2N, \dagger}(t) dt'. \quad (5.8)$$

We observe that for $|t| \leq \frac{T}{2}$,

$$\mathcal{G}_n^{2N, \dagger}(t) = (\mathcal{H}_n^{2N}(t))^*,$$

but $\mathcal{H}_n^{2N, \dagger}(t), \mathcal{G}_n^{2N, \dagger}(t)$ differ from $\mathcal{H}_n^{2N}(t), (\mathcal{H}_n^{2N}(t))^*$ on $[-T, T]$. Nevertheless we deduce from Lemma B.2 that $\mathcal{H}_n^{2N, \dagger}(t)$ is still unitary² on $[-T, T]$, and that

$$\mathcal{G}_n^{2N, \dagger}(t) = (\mathcal{H}_n^{2N, \dagger}(t))^*.$$

²Here we only need the fact that $\mathcal{P}_n^{2N, \dagger}(t)$ is self-adjoint for all $t \in \mathbb{R}$.

We split the perturbative components of $\mathcal{H}_n^{N,\dagger}(t)$, $\mathcal{G}_n^{N,\dagger}(t)$ into

$$\mathbf{h}_n^{2N,\dagger}(t) := \mathcal{H}_n^{N,\dagger}(t) - \chi(t)e^{-i\lambda_n^2 t} \text{Id}_{E_n}, \quad \mathbf{g}_n^{2N,\dagger}(t) := \mathcal{G}_n^{2N,\dagger}(t) - \chi(t)e^{i\lambda_n^2 t} \text{Id}_{E_n}. \quad (5.9)$$

For $|t| \leq T$,

$$\mathbf{g}_n^{2N,\dagger}(t) = (\mathbf{h}_n^{2N,\dagger}(t))^*$$

and the operators $\mathbf{h}_n^{2N,\dagger}(t)$, $\mathbf{g}_n^{2N,\dagger}(t)$ are compactly supported on $|t| \leq 2T$. Next, we define

$$e_n^{2N,\dagger}(t) := \mathcal{H}_n^{N,\dagger}(t)(e_n^\omega), \quad \psi_{2N}^\dagger(t) := \chi_T(t) \sum_{n \approx 2N} \frac{e_n^{2N,\dagger}(t)}{\lambda_n^{\alpha-\frac{1}{2}}}. \quad (5.10)$$

where

$$\mathbf{f}_n^{2N,\dagger}(t) = \chi_T(t) \mathbf{h}_n^{2N,\dagger}(t)(e_n^\omega), \quad \mathbf{f}_{2N}^\dagger(t) := \sum_{N < n \leq 2N} \frac{\mathbf{f}_n^{2N,\dagger}(t)}{\lambda_n^{\alpha-\frac{1}{2}}}.$$

From the notion above, we have for $n \approx 2N$,

$$\pi_n \psi_{2N}^\dagger(t) = \frac{\chi_T(t) e^{-it\lambda_n^2} e_n^\omega + \mathbf{f}_n^{2N,\dagger}(t)}{\lambda_n^{\alpha-\frac{1}{2}}}. \quad (5.11)$$

Note that $\mathbf{f}_{2N}^\dagger(t)$, $\psi_{2N}^\dagger(t)$ are compactly supported on $|t| \leq T$ that coincides with $\psi_{2N}(t) - e^{it\Delta} \mathbf{P}_N \phi$, $\psi_{2N}(t)$ on $|t| \leq \frac{T}{2}$, with respectively. Note that $\psi_{2N}^\dagger(t)$ solves the equation

$$(i\partial_t + \Delta - 1)\psi_{2N}^\dagger = 2\Pi_{2N} \mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, u_N^\dagger, u_N^\dagger)$$

only on $[-T/2, T/2]$.

The advantage of defining $\psi_{2N}^\dagger(t)$ only on $[-T, T]$ is that $\pi_n \psi_{2N}^\dagger(t) = \chi_T(t) e_n^{2N,\dagger}(t)$ and we could apply the invariance property (which is a global information) for $e_n^{2N,\dagger}$ when solving the operator equation (5.7) and the equation (5.12) for the remainder. The motivation of separating the operator $\mathbf{h}_n^{2N,\dagger}(t)$ from $\mathcal{H}_n^{2N,\dagger}(t)$ is that, when global information of the colored Gaussian $\pi_n \psi_{2N}^\dagger(t)$ is not needed, we decompose it as the sum of

$$\lambda_n^{-\frac{1}{2}} \chi_T(t) \mathcal{T}_n^{2N,\dagger}(t)(e_n^\omega), \quad \mathcal{T}_n^{2N,\dagger} \in \{\chi(t) e^{-it\lambda_n^2} \text{Id}_{E_n}, \varphi_T(t) \mathbf{h}_n^{2N,\dagger}(t)\},$$

where $\varphi_T(t) \chi_T(t) \mathbf{h}_n^{2N,\dagger}(t) = \chi_T(t) \mathbf{h}_n^{2N,\dagger}(t)$ and $\text{supp}(\varphi_T) \subset [-T, T]$. In practice, the output of our estimate for operators $\mathcal{T}_n^{2N}(t)$ will only depend on $\|\mathcal{T}_n^{2N}\|_{S_n^{q,\gamma}}$ and $\|(\mathcal{T}_n^{2N})^*\|_{S_n^{q,\gamma,*}}$ norms.

We provide some large deviation bounds on the stochastic objects in Section 7.

Step 3: Remainder: Finally we define w_{2N}^\dagger by solving the integral equation³

$$\begin{aligned} w_{2N}^\dagger(t) = & 2\chi_T(t) [\mathcal{I}\mathcal{N}_{(0,1)}(w_{2N}^\dagger, u_N^\dagger, u_N^\dagger) + \mathcal{I}\mathcal{N}_{[0,1]}(\psi_{2N}^\dagger + w_{2N}^\dagger, u_N^\dagger, u_N^\dagger)] \\ & + \chi_T(t) [\mathcal{I}\mathcal{N}(u_N^\dagger, \psi_{2N}^\dagger + w_{2N}^\dagger, u_N^\dagger) + 2\mathcal{I}\mathcal{N}(\psi_{2N}^\dagger + w_{2N}^\dagger, \psi_{2N}^\dagger + w_{2N}^\dagger, u_N^\dagger)] \\ & + \chi_T(t) [\mathcal{I}\mathcal{N}(\psi_{2N}^\dagger + w_{2N}^\dagger, u_N^\dagger, \psi_{2N}^\dagger + w_{2N}^\dagger) + \mathcal{N}(\psi_{2N}^\dagger + w_{2N}^\dagger)]. \end{aligned} \quad (5.12)$$

Note that (5.12) coincides with (1.20) only on the interval $[-T/2, T/2]$.

³Here we truncate the Duhamel formula by $\chi_T(t)$ instead of $\chi_{2T}(t)$. The reason is that on $\text{supp}(\chi_T)$, the extended operators $(\mathcal{H}_n^{2N,\dagger}, \mathcal{G}_n^{2N,\dagger})$ form an unitary pair.

At the end of this step, we define

$$u_{2N}^\dagger(t) := u_N^\dagger(t) + \psi_{2N}^\dagger(t) + w_{2N}^\dagger(t).$$

Denote by

$$\psi_{\leq 2N}^\dagger := \sum_{M \leq 2N} \psi_M^\dagger, \quad w_{\leq 2N}^\dagger := \sum_{M \leq 2N} w_M^\dagger.$$

From the iterative scheme of construction, we summarize that:

- We denote the extension of u_N, ψ_N, w_N , and $\mathcal{H}_n^N(t), (\mathcal{H}_n^N(t))^*$ by

$$u_N^\dagger, \psi_N^\dagger, w_N^\dagger, \mathcal{H}_n^{N,\dagger}(t), \mathcal{G}_n^{N,\dagger}(t)$$

- The functions $u_N^\dagger, \psi_N^\dagger, w_N^\dagger$ are supported in $(-T, T)$, while $\mathcal{H}_n^{2N,\dagger}(t), \mathcal{G}_n^{2N}(t)$ are supported on $(-2T, 2T)$. They coincide with the corresponding objects without \dagger on $[-\frac{T}{2}, \frac{T}{2}]$.
- The operators $(\mathcal{H}_n^{2N,\dagger}(t), \mathcal{G}_n^{2N,\dagger}(t))$ form a unitary pair when $|t| \leq T$.
- The random operators $\mathcal{H}_n^{N,\dagger}(t), \mathcal{G}_n^{N,\dagger}(t)$ are $\mathcal{B}_{\leq \frac{N}{2}}$ measurable, keeping the initial law invariant for $|t| \leq T$ (see Lemma 7.1).
- The random functions $w_{\leq N}^\dagger, u_N^\dagger, \psi_{\leq N}^\dagger$ are $\mathcal{B}_{\leq N}$ measurable.

Next we define an a priori parameter-dependent local bounds statement:

Definition 5.1 (Loc(N)). *Let $N \geq 2$ be a dyadic number, $0 < T \ll 1$, $R \geq 1$ and $0 < \sigma \ll 1$, $(q, \delta, \gamma, \gamma_1, s, \theta) = (q_\sigma, \gamma_\sigma, \gamma_{1,\sigma}, \delta_\sigma, s_\sigma, \theta_\sigma)$ the σ -dependent parameters in (2.1). The statement Loc(N) holds when for all $2 \leq M \leq N$ the following properties are satisfied: If $N = 2$, then with $T = \frac{1}{2}$,*

$$\|\psi_{\leq 2}^\dagger\| = \|u_2^\dagger\|_{L_t^\infty H_x^{10}} \leq R,$$

and if $N > 2$, then $T = R^{-\frac{10}{\theta}}$,

- For any $\frac{M}{2} < n \leq M$,

$$\|\mathbf{h}_n^{M,\dagger}\|_{S_n^{q,\gamma}} + \|\mathbf{g}_n^{M,\dagger}\|_{S_n^{q,\gamma,*}} \leq R^{-1},$$

and

$$\|\mathcal{H}_n^{M,\dagger}\|_{S_n^{q,\gamma}} + \|\mathcal{G}_n^{M,\dagger}\|_{S_n^{q,\gamma,*}} \leq R.$$

- The functions ψ_M^\dagger verify the bounds (see (5.16) for the definition of \ominus):

$$\|\psi_N^\dagger\|_{X_{q,\gamma,\infty}^0} \leq RT^{-\gamma+\frac{1}{q}} N^{-(\alpha-\frac{1}{2})+\frac{1}{q}+2\delta}, \quad \|\psi_N^\dagger \overline{\psi_N^\dagger}\|_{L_t^q L_x^\infty} \leq CR^2 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta}. \quad (5.13)$$

- The functions w_M^\dagger verify the bounds: for all $M \leq N$ and $L \geq 2M$,

$$\|w_M^\dagger\|_{X^{0,b}} \leq M^{-s} R^{-1}, \quad \|\Pi_L^\dagger w_M^\dagger\|_{X^{0,b}} \leq \left(\frac{M}{L}\right)^{10} M^{-s} R^{-1}. \quad (5.14)$$

Remark 5.2. Note that when $\text{Loc}(N)$ holds with parameters (T, R, σ) , then we also have that for all $2 \leq M \leq N$,

$$\|(\chi_T \mathcal{H}_n^{M,\dagger})^*\|_{S_n^{q,\gamma,*}} = \|\chi_T \mathcal{G}_n^{M,\dagger}\|_{S_n^{q,\gamma,*}} \lesssim T^{\frac{1}{q}-\gamma} \|\mathcal{G}_n^{M,\dagger}\|_{S_n^{q,\gamma,*}} \lesssim T^{\frac{1}{q}-\gamma} R. \quad (5.15)$$

We used Proposition A.1 to obtain the first inequality.

We now state the key induction Proposition.

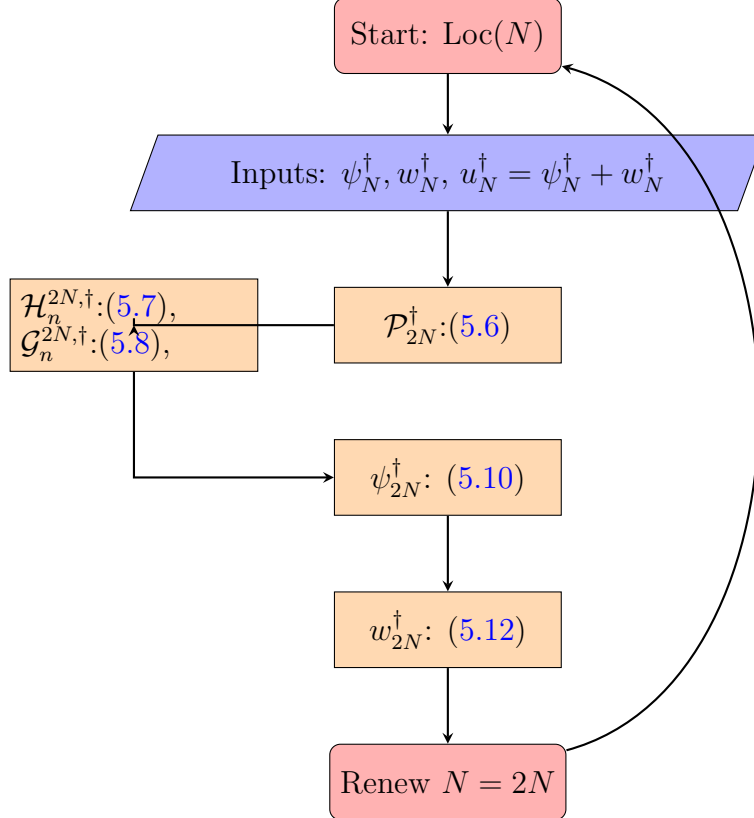
Lemma 5.3 (Key induction Proposition). *Assume that $\alpha > 1$. There exist a sufficiently small $\sigma \in (0, 1)$ and accordingly the parameters $(q, \delta, \gamma, s, \theta) = (q_\sigma, \gamma_\sigma, \delta_\sigma, s_\sigma, \theta_\sigma)$, constants $C_0 > 1 > c_0 > 0$, depending on parameters $q, \delta, \gamma, s, \theta, \alpha$, such that for any $R \geq 1$ and $T = R^{-\frac{10}{\theta}}$, the following statement holds:*

If $\text{Loc}(N)$ holds for any $\omega \in \Omega_N$ where Ω_N is a $\mathcal{B}_{\leq N}$ measurable set, then $\text{Loc}(2N)$ holds for $\omega \in \Omega_{2N}$, where Ω_{2N} is another measurable set such that

$$\mathbb{P}(\Omega_N \setminus \Omega_{2N}) < C_0 e^{-N^{c_0} R^\delta}.$$

Note that the parameters $(\sigma, q, \delta, \gamma, s, \theta, C_0, c_0)$ are indeed absolute constants, and the key induction property does not require any constraint on R which measures the large deviation of the desired events. Consequently, we have:

Corollary 5.4. *Let $(\sigma, q, \delta, \gamma, s, \theta, C_0, c_0)$ be the same set of parameter in Proposition 5.3. Then for any sufficiently large $R \geq 1$, $\text{Loc}(N)$ holds for all $N \in 2^{\mathbb{N}}$, outside a set of probability measure $C_0 e^{-cR^{\delta_0}}$, where $C_0, c, \delta_0 > 0$ are absolute constants depending only on the single parameter σ .*



Note: Though $T = R^{-\frac{10}{\theta}}$, we write down the T -dependence in the bound for ψ_M^\dagger for the reason of clarity.

5.3. Key estimates for RAOs. The proof of Proposition 5.3 relies on a set of multi-linear estimates for functions and operators, both of deterministic type and stochastic type.

Define

$$f^{\ominus}g := \sum_{n_2 \neq n_3} \pi_{n_2} f \cdot \pi_{n_3} g, \quad f^{\ominus}g := \sum_n (\pi_n f \cdot \pi_n g - \langle \pi_n f | \pi_n g \rangle). \quad (5.16)$$

Hence

$$\psi_{\leq N}^\dagger \ominus \overline{\psi_{\leq N}^\dagger} = \sum_{1 \leq L \leq N} \psi_L^\dagger \ominus \overline{\psi_L^\dagger}, \quad (5.17)$$

with

$$\begin{aligned} \psi_L^\dagger \ominus \overline{\psi_L^\dagger}(t, x) &:= \sum_{\frac{L}{2} \leq n < L} (|\pi_n \psi_L^\dagger(t, x)|^2 - \|\pi_n \psi_L^\dagger(t, \cdot)\|_{L^2}^2) \\ &= \chi_T(t)^2 \sum_{\frac{L}{2} \leq n < L} \lambda_n^{-2(\alpha - \frac{1}{2})} (|e_n^{L, \dagger}(t, x)|^2 - \|e_n^{L, \dagger}(t)\|_{L^2}^2). \end{aligned} \quad (5.18)$$

Remark 5.5. The contribution $\psi_L^\dagger \ominus \overline{\psi_L^\dagger}$ will be roughly estimated in $L_{t,x}^\infty$ spaces (see Lemma 7.3) where we will make use of the fact that $\pi_n \psi_L^\dagger = \lambda_n^{-(\alpha - \frac{1}{2})} \chi_T(t) \mathcal{H}_n^{L, \dagger}(t)(e_n^\omega)$ and $\mathcal{H}_n^{L, \dagger}(t)$ is unitary on $\text{supp } \chi_T(\cdot)$. This global information for RAO motivates the definition 5.10 for $\psi_N^\dagger(t)$.

Recall that in the RAO ansatz,

$$u_N^\dagger(t) = \psi_{\leq N}^\dagger(t) + w_{\leq N}^\dagger(t), \quad w_{\leq N}^\dagger(t) \in X^{s_0, b}, \quad \psi_{\leq N}^\dagger \in X_{q, q, \infty}^{\alpha - \frac{1}{2} - \frac{1}{q} - 3\delta, \gamma_1}$$

for $s_0 = s_\sigma - \sigma$, $\delta = \delta_\sigma$ and $\gamma_1 = \gamma_{1, \sigma} \in (\frac{1}{q'}, 1)$. To estimate the norm of $\mathcal{P}_n^{2N, \dagger}(t)$, we expand $|u_N^\dagger(t)|^{\circ 2}$, the wick square of u_N^\dagger by

$$\begin{aligned} |u_N^\dagger(t)|^2 - \|u_N^\dagger(t)\|_{L_x^2}^2 &= |\psi_{\leq N}^\dagger(t)|^{\circ 2} + |w_{\leq N}^\dagger(t)|^{\circ 2} \\ &\quad + 2\text{Re}(\psi_{\leq N}^\dagger(t) \overline{w_{\leq N}^\dagger(t)}) - 2\text{Re} \left\langle \psi_{\leq N}^\dagger(t) | w_{\leq N}^\dagger(t) \right\rangle, \end{aligned} \quad (5.19)$$

where

$$|\psi_{\leq N}^\dagger(t)|^{\circ 2} := |\psi_{\leq N}^\dagger(t)|^2 - \|\psi_{\leq N}^\dagger(t)\|_{L_x^2}^2, \quad |w_{\leq N}^\dagger(t)|^{\circ 2} = |w_{\leq N}^\dagger(t)|^2 - \|w_{\leq N}^\dagger(t)\|_{L^2}^2.$$

We further split

$$|\psi_{\leq N}^\dagger(t)|^{\circ 2} = \overline{\psi_{\leq N}^\dagger(t)} \ominus \psi_{\leq N}^\dagger(t) + \overline{\psi_{\leq N}^\dagger(t)} \ominus \psi_{\leq N}^\dagger(t). \quad (5.20)$$

We note that

$$\Pi_N \psi_{\leq N}^\dagger = \psi_{\leq N}^\dagger, \quad \Pi_N w_{\leq N}^\dagger \neq w_{\leq N}^\dagger.$$

Let us start with the estimates for $S_n^{q, \gamma}$ norm of the RAOs $\mathbf{h}_n^{2N, \dagger}$ and $\mathbf{g}_n^{2N, \dagger}$ in the fixed-point problems:

Proposition 5.6. *There exists a sufficiently small $\sigma \in (0, 1)$, such that if the statement $\text{Loc}(N)$ holds with respect to arguments (T, R, σ) , then for any $N < n \leq 2N$, $\mathcal{T}_n \in S_n^{q,\gamma}$,*

$$\left\| \chi_{2T}(t) \int_0^t e^{-i\lambda_n^2(t-t')} \mathcal{P}_n^{2N,\dagger}(t') \circ \mathcal{T}_n(t') dt' \right\|_{S_n^{q,\gamma}} \leq CT^\theta \|\mathcal{T}_n\|_{S_n^{q,\gamma}} \cdot T^{-2\gamma + \frac{2}{q}} R^2.$$

Proof. Thanks to Proposition 4.3, it suffices to show that for any $F_n \in X_{q,2}^{0,\gamma}(E_n)$,

$$\begin{aligned} & \|\mathcal{P}_n^{2N,\dagger}(t)(F_n(t))\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \\ & \lesssim \|F_n\|_{X_{q,2}^{0,\gamma}(E_n)} (\|\bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger\|_{L_t^{q'} L_x^\infty} + \|\psi_{\leq N}^\dagger\|_{X_{q,q,\infty}^{\beta,\gamma}}^2 + \|w_{\leq N}^\dagger\|_{X^{s,b}}^2), \end{aligned} \quad (5.21)$$

where $\beta = \alpha - \frac{1}{2} - \frac{1}{q} - 3\delta$.

We decompose $\mathcal{P}_n^{2N}(t)(F_n)$ into I + II + III, where

$$\text{I} := e^{-i\lambda_n^2 t} \pi_n(e^{i\lambda_n^2 t} \pi_n F_n(t) \bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger),$$

$$\text{II} := e^{-i\lambda_n^2 t} \pi_n(e^{i\lambda_n^2 t} \pi_n F_n(t) \text{Re}(\psi_{\leq N}^\dagger(t) \bar{w}_{\leq N}^\dagger(t))) - \text{Re} \left\langle \psi_{\leq N}^\dagger(t) | w_{\leq N}^\dagger(t) \right\rangle \cdot \pi_n F_n(t),$$

$$\text{III} := e^{-i\lambda_n^2 t} \pi_n(e^{i\lambda_n^2 t} \pi_n F_n(t) |w_{\leq N}^\dagger(t)|^2) - \|w_{\leq N}^\dagger(t)\|_{L^2}^2 \cdot \pi_n F_n(t),$$

$$\text{IV} := e^{-i\lambda_n^2 t} \pi_n(e^{i\lambda_n^2 t} \pi_n F_n(t) \bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger).$$

By Lemma A.6 (since E_n is a Hilbert space),

$$\begin{aligned} \|\text{I}\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} & \leq \|\text{I}\|_{X_{q,2}^{0,0}(E_n)} = \|\mathcal{F}_t(\pi_n(e^{i\lambda_n^2 t} \pi_n F_n(t) \bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger))(\lambda)\|_{L_\lambda^q E_n} \\ & \lesssim \|e^{i\lambda_n^2 t} \pi_n(\pi_n F_n(t) \bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger)\|_{L_t^{q'} E_n} \\ & \lesssim \|F_n\|_{L_t^\infty L_x^2} \|\bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger\|_{L_t^{q'} L_x^\infty} \\ & \lesssim \|F_n\|_{X_{q,2}^{0,\gamma}(E_n)} \|\bar{\psi}_{\leq N}^\dagger \ominus \psi_{\leq N}^\dagger\|_{L_t^{q'} L_x^\infty}, \end{aligned}$$

where to the last step, we used Lemma 4.2, thanks to the fact that $\gamma \in (\frac{1}{q}, 1)$.

The estimate of the first contribution in II is a direct consequence of Proposition 6.1. Moreover, for fixed t , using the embedding property $X^{s,b}, \hookrightarrow C_t H_x^s$ and $X_{q,\infty}^{0,\gamma}(E_n) \hookrightarrow C_t L^\infty(E_n)$, we have

$$\begin{aligned} \left| \left\langle \psi_{\leq N}^\dagger(t) | w_{\leq N}^\dagger(t) \right\rangle \right| & \leq \sum_{n \leq N} \|\pi_n \psi_{\leq N}^\dagger\|_{L_t^\infty L_x^\infty(E_n)} \|\pi_n w_{\leq N}^\dagger\|_{L_x^\infty L_x^2} \\ & \leq \sum_{n \leq N} n^{-(s+\beta)} \|n^\beta \pi_n \psi_{\leq N}^\dagger\|_{X_{q,\infty}^{0,\gamma}(E_n)} \|n^s \pi_n w_{\leq N}^\dagger\|_{X^{0,b}} \\ & \lesssim \|\psi_{\leq N}^\dagger\|_{X_{q,q,\infty}^{\beta,\gamma}} \|w_{\leq N}^\dagger\|_{X^{s,b}}. \end{aligned}$$

Hence for the second term of II, its $X_{q,2}^{0,\gamma-1+\theta}(E_n)$ norm can be bounded by its $L_{t,x}^2$ norm, leading to the bound

$$\|F_n\|_{X_{q,2}^{0,\gamma}(E_n)} \|\psi_{\leq N}^\dagger\|_{X_{q,q,\infty}^{\beta,\gamma}} \|w_{\leq N}^\dagger\|_{X^{s,b}}.$$

The estimate of the first term in III is implied by Proposition 6.2, while the estimate for the second term in III is straightforward. Finally, the estimate for IV is implied by Proposition 6.3. We conclude the proof of (5.21). This completes the proof of Proposition 5.6. \square

A similar proposition claims that the Duhamel operator above can be extended as a bounded operator on $X_{q,2}^{q,\gamma}(E_n)$. This will allow us to control the equation for the operator $\mathcal{G}_n^{2N,\dagger}$.

Proposition 5.7. *There exists a sufficiently small $\sigma \in (0, 1)$, such that if the statement Loc(N) holds with respect to arguments (T, R, σ) , then for any $N < n \leq 2N$, $\mathcal{T}_n, \mathcal{S}_n \in S_n^{q,\gamma}$,*

$$\begin{aligned} \left\| \chi_{2T}(t) \left(\int_0^t e^{i\lambda_n^2 t'} \mathcal{P}_n^{2N,\dagger}(t') \circ \mathcal{T}_n(t') dt' \right) \circ \mathcal{S}_n(t) \right\|_{S_n^{q,\gamma,*}} \\ \leq CT^{\theta-\gamma+\frac{1}{q'}} \|\mathcal{T}_n\|_{S_n^{q,\gamma}} \|\mathcal{S}_n\|_{S_n^{q,\gamma,*}} \cdot T^{-2\gamma+\frac{2}{q'}} R^2. \end{aligned}$$

Proof. By definition (4.4), we need to show that for any $\mathbf{f}_n \in E_n$,

$$\begin{aligned} \left\| \chi_{2T}(t) \left(\int_0^t \mathcal{P}_n^{2N,\dagger}(t') \circ e^{i\lambda_n^2 t'} \mathcal{T}_n(t') dt' \right) \circ e^{-i\lambda_n^2 t} \mathcal{S}_n(t)(\mathbf{f}_n) \right\|_{\mathcal{FL}_q^\gamma(E_n)} \\ \lesssim T^{\theta-\gamma+\frac{1}{q'}} \|\mathcal{T}_n\|_{S_n^{q,\gamma,*}} \|e^{-i\lambda_n^2 t} \mathcal{S}_n(t)(\mathbf{f}_n)\|_{\mathcal{FL}_q^\gamma(E_n)} (T^{-2\gamma+\frac{2}{q'}} R^2 + \|\overline{\psi_{\leq N}^\dagger} \ominus \psi_{\leq N}^\dagger\|_{L_t^q L_x^\infty}), \end{aligned}$$

where we refer to (A.1) the abstract notation of Fourier-Lebesgue spaces. Set

$$\mathbf{A}(t') = e^{i\lambda_n^2 t'} \mathcal{P}_n^{2N,\dagger}(t') \circ \mathcal{T}_n(t'), \quad \mathbf{F}(t) = e^{-i\lambda_n^2 t} \mathcal{S}_n(t)(\mathbf{f}_n).$$

By Lemma A.9,

$$\left\| \chi_{2T}(t) \int_0^t \mathbf{A}(t')(\mathbf{F}(t)) dt' \right\|_{\mathcal{FL}_q^\gamma(E_n)} \lesssim T^{\theta-\gamma+\frac{1}{q'}} \|\mathbf{A}(t)\|_{E_n \rightarrow \mathcal{FL}_q^{\gamma-1+\theta}(E_n)} \|\mathbf{F}(t)\|_{\mathcal{FL}_q^\gamma(E_n)}.$$

As a consequence of (5.21), we have

$$\begin{aligned} \|\mathbf{A}(t)\|_{E_n \rightarrow \mathcal{FL}_q^{\gamma-1+\theta}(E_n)} &\lesssim \|\mathcal{T}_n\|_{S_n^{q,\gamma}} (\|\overline{\psi_{\leq N}^\dagger} \ominus \psi_{\leq N}^\dagger\|_{L_t^q L_x^\infty} + \|\psi_{\leq N}^\dagger\|_{X_{q,q,\infty}^{\beta,\gamma}}^2 + \|w_{\leq N}^\dagger\|_{X_{s,b}}^2) \\ &\lesssim \|\mathcal{T}_n\|_{S_n^{q,\gamma}} (T^{-2\gamma+\frac{2}{q'}} R^2 + \|\overline{\psi_{\leq N}^\dagger} \ominus \psi_{\leq N}^\dagger\|_{L_t^q L_x^\infty}). \end{aligned}$$

This completes the proof of Proposition 5.7. \square

5.4. Key trilinear estimates. Next we state multi-linear estimates for remainders. We separate them into different classes.

Definition 5.8. *For a space-time function⁴ v_N , we say that it is of type (C), if*

$$v_N = \chi_T(t) \sum_{N < n \leq 2N} \frac{\mathcal{T}_n^N(t)(e_n^\omega)}{\lambda_n^{\alpha-\frac{1}{2}}} =: \chi_T(t) \mathcal{T}^N(t)(\phi_N),$$

where $\mathcal{T}_n^N(t) \in S_n^{q,\gamma}$ is $\mathcal{B}_{\leq N/2}$ -measurable,

$$\|\mathcal{T}_n^N(t)\|_{S_n^{q,\gamma}} \leq R.$$

⁴Here the notation v_N should not be confused with the $u_N - u_{N/2}$ in the previous section.

and $\mathcal{T}^N := \sum_{N < n \leq 2N} \mathcal{T}_n^N$ satisfies

$$\|\mathcal{T}^N(t)(\phi_N)\|_{X_{q,\tilde{q},\infty}^{0,\gamma}} \leq N^{-\frac{1}{2} + \frac{1}{q} + \frac{\delta}{2}} R.$$

It is of type (D), if $v_N = z_N \in X^{0,b}$ and

$$\|z_N\|_{X^{0,b}} \leq N^{-s} R^{-1}.$$

A typical type (C) term is ψ_N^\dagger for which the corresponding operators are $\mathcal{H}_n^{N,\dagger}$. We enlarge the definition of type (C) term to include RAO $\mathbf{h}_n^{N,\dagger}$ and $\chi(t)e^{-it\lambda_n^2} \text{Id}_{E_n}$ appearing in the decomposition of $\mathcal{H}_n^{N,\dagger}$. A typical type (D) term is the remainder w_N^\dagger , solution to (5.12). However, we introduce the notation to distinguish w_N^\dagger . In practice, knowing $\text{Loc}(N)$, we will solve the fixed-point problem for w_{2N}^\dagger in the ball $\{w : \|w\|_{X^{0,b}} \leq (2N)^{-s}\}$. For this reason, it is more convenient to establish estimates for general type (D) terms.

We begin with some deterministic estimates that allows to control the resonant high-low-low component:

Proposition 5.9. *There exists a sufficiently small $\sigma \in (0, 1)$, such that if the statement $\text{Loc}(N)$ holds with arguments (T, R, σ) , then for any $N < n \leq 2N$,*

$$\|\chi_T(t) \mathcal{I} \mathcal{P}_n^{2N,\dagger}\|_{\mathcal{L}(X^{0,b})} \leq CT^{\theta-2(\gamma-\frac{1}{q'})} R^2.$$

Consequently, for any $z_{2N} \in X^{0,b}$,

$$\|\chi_T(t) \mathcal{I} \mathcal{N}_{(0,1)}(z_{2N}, u_N^\dagger, u_N^\dagger)\|_{X^{0,b}} \leq CT^{\theta-2(\gamma-\frac{1}{q'})} \|z_{2N}\|_{X^{0,b}} R^2.$$

Proposition 5.9 allows to control the first term of (5.12), and the proof follows in the same way as for Proposition 5.6, hence we omit it.

Let us analyze the remaining type of nonlinearities in the equation of w_{2N}^\dagger (with $N_{(1)} = 2N$):

- (C)(C)(C) type $\mathcal{N}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)$:
 - If $N_1 = 2N, N_2, N_3 \leq N$, only $\mathcal{N}_{[0,1]}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)$ are presented.
 - Otherwise $N_2 = 2N$ or $N_{(1)} = N_{(2)} = 2N$, all possible combinations.
- (C)(D)(D) types $\mathcal{N}(\psi_{N_1}^\dagger, z_{N_2}^\dagger, z_{N_3}^\dagger), \mathcal{N}(z_{N_1}^\dagger, \psi_{N_2}^\dagger, z_{N_3}^\dagger)$ and $\mathcal{N}(z_{N_1}^\dagger, z_{N_2}^\dagger, \psi_{N_3}^\dagger)$:
 - If $N_1 = 2N, N_2, N_3 \leq N$, only $\mathcal{N}_{[0,1]}(\psi_{N_1}^\dagger, z_{N_2}^\dagger, z_{N_3}^\dagger)$
 - Otherwise $N_2 = 2N$ or $N_{(1)} = N_{(2)} = 2N$, all possible combinations.
- (C)(C)(D) types $\mathcal{N}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, z_{N_3}^\dagger), \mathcal{N}(\psi_{N_1}^\dagger, z_{N_2}^\dagger, \psi_{N_3}^\dagger)$ and $\mathcal{N}(z_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)$:
 - If $N_1 = 2N, N_2, N_3 \leq N$, only $\mathcal{N}_{[0,1]}(z_{N_1}^\dagger, z_{N_2}^\dagger, z_{N_3}^\dagger)$.
 - Otherwise $N_2 = 2N, N_1, N_3 \leq N$ or $N_{(1)} = N_{(2)} = 2N$, all possible combinations.
- (D)(D)(D) types $\mathcal{N}(z_{N_1}^\dagger, z_{N_2}^\dagger, z_{N_3}^\dagger)$: all possible combinations.

We collect the deterministic trilinear estimates in the following proposition which treated (D)(D)(C), (D)(C)(C) and (D)(D)(D) terms together. Note that we do not cover the case of the Gibbs measure ($\alpha = 1$) here, and refined probabilistic estimates together with a modified resolution scheme are needed. These will be developed in our forthcoming work [5].

Proposition 5.10. *Let $\alpha > 1$ and $0 < \epsilon_0 < \min(\frac{1}{2}(s - \frac{1}{2}), \alpha - 1)$. We suppose that $\text{Loc}(N)$ is true with parameters (T, R, σ) and that the terms ψ_{2N}^\dagger also satisfy the bound (5.13). Let N_1, N_2, N_3 be dyadic integers such that $2N = N_{(1)} \geq N_{(2)} \geq N_{(3)}$, and $v_{N_j}^\dagger$ is either $\psi_{N_j}^\dagger$ (of type (C)) or $z_{N_j}^\dagger$ (of type (D)). We have:*

(1) *If there is at least one term z_{2N}^\dagger then*

$$\|\chi_T \mathcal{I}\mathcal{N}(z_{2N}^\dagger, \overline{v_{N_2}^\dagger}, v_{N_3}^\dagger)\|_{X^{0,b}} + \|\chi_T \mathcal{I}\mathcal{N}(v_{N_1}^\dagger, \overline{z_{2N}^\dagger}, v_{N_3}^\dagger)\|_{X^{0,b}} \leq CT^{\frac{\theta}{2}} R \|z_{2N}^\dagger\|_{X^{0,b}} N_{(2)}^{-\epsilon_0}.$$

(2) *If there is at least one term ψ_{2N}^\dagger and no term z_{2N}^\dagger . Then,*

$$\|\chi_T \mathcal{I}\mathcal{N}(v_{N_1}^\dagger, \psi_{2N}^\dagger, v_{N_3}^\dagger)\|_{X^{0,b}} + \|\chi_T \mathcal{I}\mathcal{N}_{[0,1]}(\psi_{2N}^\dagger, v_{N_2}, \psi_{2N}^\dagger)\|_{X^{0,b}} \leq CT^{\frac{\theta}{2}} R N_{(1)}^{-s} N_{(2)}^{-\epsilon_0}.$$

(3) *If $N_{(2)} = N_{(1)} = 2N$ and at least one of $v_{N_2}^\dagger, v_{N_3}^\dagger$ is of type (D), then*

$$\|\chi_T \mathcal{I}\mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger)\|_{X^{0,b}} \leq CT^{\frac{\theta}{2}} R N_{(1)}^{-s} N_{(2)}^{-\epsilon_0}.$$

We prove Proposition 5.10 in Section 9. Let us now state the trilinear probabilistic estimates. In contrast to the above deterministic estimate, we cover the Gibbs case corresponding to $\alpha = 1$.

Proposition 5.11 ((C)(C)(C) type interactions). *Let $\alpha \geq 1$. Suppose that $\text{Loc}(N)$ holds with parameters (T, R, σ) , for some sufficiently small $\sigma \in (0, 1)$. Let N_1, N_2, N_3 be dyadic integers such that $2N = N_{(1)} \geq N_{(2)} \geq N_{(3)}$. Let $\psi_{N_j}^\dagger$ be of type (C). Then there exist $\epsilon_0 > 0$, independent of α , and a $\mathcal{B}_{\leq 2N}$ -measurable set Ξ with $\mathbb{P}(\Omega \setminus \Xi) < C_0 e^{-(\log N)^{\epsilon_0} R^{\delta_0}}$, such that the following estimates hold on Ξ :*

- *If $N_1 = 2N$ and $N_2, N_3 \leq N$, then*

$$\|\chi_T(t) \mathcal{I}\mathcal{N}_{[0,1]}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{X^{0,b}} \leq T^{\frac{\theta}{2}} R^4 N_{(1)}^{-s-\epsilon_0}.$$

- *If $N_2 = 2N$ or $N_{(1)} = N_{(3)} = 2N$, then*

$$\|\chi_T(t) \mathcal{I}\mathcal{N}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{X^{0,b}} \leq T^{\frac{\theta}{2}} R^4 N_{(1)}^{-s-\epsilon_0}.$$

5.5. Proof of Proposition 5.3 by assuming multi-linear estimates. In this subsection, we show that Proposition 5.6, Proposition 5.7, Proposition 5.9, Proposition 5.10 and Proposition 5.11 imply the key-induction Lemma 5.3.

Let $R \gg 1$ be a fixed parameter. Assuming $\text{Loc}(N)$ (see Definition 5.1) holds for all $\omega \in \Omega_N$ which is $\mathcal{B}_{\leq N}$ -measurable.

•Step 1 : Fixed-point for the RAOs

We solve the fixed-point problem (5.7), written in term of $\mathbf{h}_n^{2N,\dagger}$:

$$\mathbf{h}_n^{2N,\dagger}(t) = -2i\chi_{2T}(t) \int_0^t e^{-i\lambda_n^2(t-t')} \mathcal{P}_n^{2N,\dagger}(t') \circ \mathcal{H}_n^{2N,\dagger}(t') dt' \quad (5.22)$$

Thanks to Proposition 5.6, the $S_n^{q,\gamma}$ norm of the right hand side is bounded by

$$CT^{\theta-2\gamma+\frac{2}{q}} R^2 \|\mathcal{H}_n^{2N,\dagger}(t)\|_{S_n^{q,\gamma}} \leq CT^{\frac{\theta}{2}} R^2 (1 + \|\mathbf{h}_n^{2N,\dagger}\|_{S_n^{q,\gamma}}),$$

where we used the fact that $\theta - 2\gamma + \frac{2}{q'} > \frac{\theta}{2}$, thanks to the choices of parameters. For $T \sim R^{-\frac{10}{\theta}}$, the fixed-point problem (5.22) has a unique solution in the ball of $S_n^{q,\gamma}$ with radius $R > 0$.

Similarly, for the fixed-point problem (5.8) written in term of $\mathbf{g}_n^{2N,\dagger}$:

$$\mathbf{g}_n^{2N,\dagger}(t) = 2i\chi_{2T}(t) \left(\int_0^t e^{i\lambda_n^2 t'} \mathcal{P}_n^{2N,\dagger}(t') \circ \mathcal{H}_n^{2N,\dagger}(t') dt' \right) \circ \mathcal{G}_n^{2N,\dagger}(t), \quad (5.23)$$

by Proposition 5.7, the $S_n^{q,\gamma,*}$ -norm of the right hand side is bounded by

$$CT^{\frac{\theta}{2}} R^2 (1 + \|\mathbf{g}_n^{2N,\dagger}\|_{S_n^{q,\gamma,*}}),$$

provided that $\theta - 3\gamma + \frac{3}{q'} > \frac{\theta}{2}$, which allows to find the unique fixed-point in the ball of $S_n^{q,\gamma,*}$ with radius $R > 0$. Consequently, for any $N < n \leq 2N$, we obtain that

$$\|\mathbf{h}_n^{2N,\dagger}\|_{S_n^{q,\gamma}} + \|\mathbf{g}_n^{2N,\dagger}\|_{S_n^{q,\gamma,*}} \leq R^{-1},$$

and

$$\|\mathcal{H}_n^{2N,\dagger}\|_{S_n^{q,\gamma}} + \|\mathcal{G}_n^{2N,\dagger}\|_{S_n^{q,\gamma,*}} \leq R.$$

From Lemma 7.2 and Lemma 7.3, by deleting a subset Ξ_N from Ω_N of probability smaller than $C_0 e^{-c_0(2NR)^{\delta_0}}$, we have

$$\|\psi_{2N}^\dagger\|_{X_{q,q,\infty}^{0,\gamma}} \leq T^{-\gamma + \frac{1}{q'}} R N^{-(\alpha - \frac{1}{2}) + \frac{1}{q} + 2\delta}, \quad \|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}\|_{L_t^q L_x^\infty} \leq R^2 (2N)^{-2(\alpha - \frac{1}{2}) + \frac{1}{2} + \frac{1}{q} + 2\delta}.$$

• **Step 2: Fixed-point problem for the remainder w_{2N}^\dagger**

Now we solve the fixed-point problem (5.12) for w_{2N}^\dagger in the closed set

$$\mathbf{B}_{2N} := \{z : \|z\|_{X^{0,b}} \leq (2N)^{-s} R^{-1}, \sup_{L \geq 4N} \|\Pi_L^\perp z\|_{X^{0,b}} \leq (L/2N)^{-10} (2N)^{-s} R^{-1}\}$$

Below we only show that with the choice $T \sim R^{-\frac{10}{\theta}}$, the mapping

$$\begin{aligned} \Phi(z_{2N}) := & 2\chi_T(t) [\mathcal{I}\mathcal{N}_{(0,1)}(z_{2N}, u_N^\dagger, u_N^\dagger) + 2\mathcal{I}\mathcal{N}_{[0,1]}(\psi_{2N}^\dagger + z_{2N}, u_N^\dagger, u_N^\dagger)] \\ & + 2\chi_T(t) [\mathcal{I}\mathcal{N}(u_N^\dagger, \psi_{2N}^\dagger + z_{2N}, u_N^\dagger) + \mathcal{I}\mathcal{N}(\psi_{2N}^\dagger + z_{2N}, \psi_{2N}^\dagger + z_{2N}, u_N^\dagger)] \\ & + \chi_T(t) [\mathcal{I}\mathcal{N}(\psi_{2N}^\dagger + z_{2N}, u_N^\dagger, \psi_{2N}^\dagger + z_{2N}) + \mathcal{I}\mathcal{N}(\psi_{2N}^\dagger + z_{2N})]. \end{aligned} \quad (5.24)$$

maps \mathbf{B}_{2N} to itself, since the contraction property can be proved in a similar way upon elementary algebraic manipulation.

First, from Proposition 5.9, for any type (D) term $z_{2N} \in \mathbf{B}_{2N}$, we have

$$\|\chi_T(t) \mathcal{I}\mathcal{N}_s(z_{2N}, u_N^\dagger, u_N^\dagger)\|_{X^{s_1,b}} \leq CT^{\theta - 2\gamma + \frac{2}{q'}} R^2 \|z_{2N}\|_{X^{0,b}} \leq CT^{\frac{\theta}{2}} R (2N)^{-s}, \quad (5.25)$$

thanks to the condition of parameters $\theta - 2\gamma + \frac{2}{q'} > \frac{\theta}{2}$.

For other contributions on the right hand side of (5.12) (replacing w_{2N}^\dagger by z_{2N}), we decompose $u_N^\dagger = \sum_{L \leq N} (\psi_L^\dagger + w_L^\dagger)$ and estimate each dyadic pieces. For (C)(C)(C)

type interactions are linear combination of the following terms

$$\begin{aligned} & \sum_{N_2, N_3 \leq N} \chi_T(t) \mathcal{I}\mathcal{N}_{[0,1]}(\psi_{2N}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger), \quad \sum_{N_1, N_3 \leq N} \chi_T(t) \Pi_{2N} \mathcal{I}\mathcal{N}(\psi_{N_1}^\dagger, \psi_{2N}^\dagger, \psi_{N_3}^\dagger) \\ & \sum_{N_2 \leq 2N} \chi_T(t) \mathcal{I}\mathcal{N}(\psi_{2N}^\dagger, \psi_{N_2}^\dagger, \psi_{2N}^\dagger), \quad \sum_{N_3 \leq N} \chi_T(t) \mathcal{I}\mathcal{N}(\psi_{2N}^\dagger, \psi_{2N}^\dagger, \psi_{N_3}^\dagger). \end{aligned}$$

According to Proposition 5.11, the $X^{0,b}$ -norm of these terms can be bounded by

$$CT^{\frac{\theta}{2}} R^4 (2N)^{-s+\epsilon_0} (\log N)^3,$$

outside a subset of probability smaller than $C_0 e^{-(\log N)^{c_0} R^{c_0}}$, where the $\log N$ factor comes from the dyadic summation.

For other terms ((C)(C)(D), (C)(D)(D) and (D)(D)(D) interactions), we apply Proposition 5.10 to control the $X^{0,b}$ -norm of the sum of these dyadic pieces by

$$\sum_{N_{(3)} \leq N_{(2)} \leq 2N} CT^{\frac{\theta}{2}} R^2 (2N)^{-s} N_{(2)}^{-\epsilon_0(\alpha-1)} \leq CCT^{\frac{\theta}{2}} R^2 (2N)^{-s}.$$

For $T = R^{-\frac{10}{\theta}}$ and sufficiently large $R > 1$, we deduce that

$$CT^{\frac{\theta}{2}} R^2 (2N)^{-s} + CT^{\frac{\theta}{2}} R^3 (2N)^{-s+\epsilon_0} (\log N)^3 + CT^{\frac{\theta}{2}} R^2 (2N)^{-s} \leq (2N)^{-s} R^{-1}.$$

To conclude, it remains to estimate $\|\Pi_L^\perp \Phi(z_{2N})\|_{X^{0,b}}$ for all $L \geq 4N$. The proof is exactly the same as in the previous paragraphs. Here we sketch the main point to gain of the extra factor $(L/2N)^{-10}$. Indeed, since ψ_M^\dagger is exactly supported on frequencies $m \in (M/2, M]$, so for (C)(C)(C) terms, when $L \geq 2^4 N$, say, $\Pi_L^\perp \mathcal{N}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger) = 0$. For other terms $\Pi_L^\perp \mathcal{N}(v_{N_1}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger)$ where there are at least one $v_{N_j}^\dagger$ is of type (D), we decompose each type (D) term as $\Pi_{L/4} z_{N_j}^\dagger + \Pi_{L/4}^\perp v_{N_j}^\dagger$. As $L/4 \geq 4N$, then for non-zero contributions of $\Pi_L^\perp \mathcal{N}(\dots)$, there is at least one $\Pi_{L/4}^\perp z_{N_j}^\dagger$, which will contribute a factor $C(L/4N)^{-10} \cdot (2N)^{-s} R^{-1}$. With these additional arguments, we conclude the proof of the key-induction Lemma 5.3.

6. SOME DETERMINISTIC ESTIMATES FOR OPERATORS

Let $n \in \mathbb{N}$, recall the notation

$$\mathcal{N}_{(0,1)}(f_1, f_2, f_3) := \sum_{n, n_2, n_3} \pi_n(\pi_n f_1 \pi_{n_2} \bar{f}_2 \pi_{n_3} f_3) \quad (6.1)$$

and a slightly different variant (in order to deal with the product $f_2 \otimes f_3$)

$$\mathcal{N}_{(0,1)[2,3]}(f_1, f_2, f_3) := \sum_{\substack{n, n_2, n_3 \\ n_2 \neq n_3}} \pi_n(\pi_n f_1 \pi_{n_2} \bar{f}_2 \pi_{n_3} f_3). \quad (6.2)$$

We provide in this section some deterministic estimates that are needed for the key estimates for RAOs. Our estimates hold under weaker assumptions for the regularity of the Fourier-Lebesgue norms. So they will be useful to treat the Gibbs-measure

regularity data, i.e. $\alpha = 1$. To state the following propositions, we assume that parameters $q, \gamma, \gamma_0, \theta, \delta$ satisfies the following constraint

$$1 - \gamma = \frac{1}{q} - \frac{1}{q^{10}}, \quad \gamma_0 = \frac{1}{q} + \frac{1}{q^{10}}, \quad \delta < \frac{1}{q^{20}}, \quad \theta = \frac{1}{q^5}. \quad (6.3)$$

Proposition 6.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}; \mathbb{R})$. There exist $q \in [2, \infty)$ large enough, and $\gamma, \gamma_0, \delta, \theta$ obeying (6.3), such that for any $s > \frac{1}{4} + \frac{2}{q}, \frac{1}{2} < b < 1$, and for any $F_1 \in X_{q,2,2}^{0,\gamma}, F_2 \in X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}, F_3 \in X^{s,b}$, we have*

$$\begin{aligned} \|\varphi(t)\pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} &\lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}} \|F_3\|_{X^{s,b}}, \\ \|\varphi(t)\pi_n \mathcal{N}_{(0,1)}(F_1, F_3, F_2)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} &\lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}} \|F_3\|_{X^{s,b}}, \end{aligned}$$

where the implicit bound is independent of n and $0 < T < 1$.

Similarly, we have:

Proposition 6.2. *Let $\varphi \in \mathcal{S}(\mathbb{R}; \mathbb{R})$. There exist $q \in [2, \infty)$ large enough, and $\gamma, \gamma_0, \delta, \theta$ obeying (6.3), such that for any $s > \frac{1}{4} + \frac{2}{q}, \frac{1}{2} < b < 1$, and for any $F_1 \in X_{q,2,2}^{0,\gamma}, F_2, F_3 \in X^{s,b}$,*

$$\|\varphi(t)\pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X^{s,b}} \|F_3\|_{X^{s,b}},$$

where the implicit bound is independent of n and $0 < T < 1$.

For non-pairing part, we have similar estimate:

Proposition 6.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}; \mathbb{R})$. There exist $q \in [2, \infty)$ large enough, and $\gamma, \gamma_0, \delta, \theta$ obeying (6.3), such that for any $s > \frac{1}{4} + \frac{2}{q}, \frac{1}{2} < b < 1$, and for any $F_1 \in X_{q,2,2}^{0,\gamma}, F_2, F_3 \in X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}$,*

$$\|\varphi(t)\mathcal{N}_{(0,1)[2,3]}(F_1, F_2, F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}} \|F_3\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}},$$

where the implicit bound is independent of n and $0 < T < 1$.

Before proving the above propositions, we state an elementary Lemma useful in the analysis.

Lemma 6.4. *Assume that*

$$\alpha_2, \alpha_3 \in (0, 1), \quad \alpha_2 + \alpha_3 > \frac{1}{2},$$

then for any $\tilde{\kappa} \in \mathbb{R}, 1 \leq r \leq 2$

$$\sum_{n_3 \geq 1} \lambda_{n_3}^{-2\alpha_3} \left(\sum_{n_2 \geq 1} \lambda_{n_2}^{-r\alpha_2} (\tilde{\kappa} + \lambda_{n_2}^2 - \lambda_{n_3}^2)^{-10r} \right)^{\frac{2}{r}} \lesssim 1,$$

where the implicit bound is independent of $\tilde{\kappa} \in \mathbb{R}$.

Proof. By making change of variable $m = |\lambda_{n_3}^2 - \lambda_{n_2}^2|$, we have

$$\begin{aligned} \sum_{n_2:1 \leq n_2 \leq n_3} \lambda_{n_2}^{-r\alpha_2} \langle \tilde{\kappa} + \lambda_{n_2}^2 - \lambda_{n_3}^2 \rangle^{-10r} &= \sum_{0 \leq m \leq \lambda_{n_3}^2 - 1} \langle \tilde{\kappa} - m \rangle^{-10r} (\lambda_{n_3}^2 - m)^{-\frac{r\alpha_2}{2}} \\ &\lesssim \langle \tilde{\kappa} - \lambda_{n_3}^2 \rangle^{-\frac{r\alpha_2}{2}}, \end{aligned}$$

and

$$\begin{aligned} \sum_{n_2:n_2 > n_3} \lambda_{n_2}^{-r\alpha_2} \langle \tilde{\kappa} + \lambda_{n_2}^2 - \lambda_{n_3}^2 \rangle &= \sum_{m \geq 0} \langle \tilde{\kappa} + m \rangle^{-10r} \langle m + \lambda_{n_3}^2 \rangle^{-\frac{r\alpha_2}{2}} \\ &\lesssim \langle \tilde{\kappa} - \lambda_{n_3}^2 \rangle^{-\frac{r\alpha_2}{2}}. \end{aligned}$$

Therefore, the desired sum can be estimated by

$$\sum_{n_3 \geq 1} \lambda_{n_3}^{-2\alpha_3} \langle \tilde{\kappa} - \lambda_{n_3}^2 \rangle^{-\alpha_2} \lesssim 1,$$

thanks to the fact that $2\alpha_2 + 2\alpha_3 > 1$. \square

6.1. Multilinear estimates for $\mathcal{N}_{(0,1)}$ and $\mathcal{N}_{(0,1)[2,3]}$. The proof will follow from an interpolation argument. First we prove some general estimates which leave some flexibility of indices:

Lemma 6.5. *Assume that $q_1, q_2 \in [2, \infty)$ and $\gamma_j \in (\frac{1}{q_j'}, 1)$. For any $\gamma_0 \in (\frac{1}{q_1}, 1)$ $\delta > 0, b > \frac{1}{2}$, then uniformly in $n \in \mathbb{N}$,*

$$\begin{aligned} \|\varphi(t)\pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q_1,2}^{0,-\gamma_0}(E_n)} \\ \lesssim_{q_j, \gamma_j, \delta} \|\pi_n F_1\|_{X_{q_1,2}^{0,\gamma_1}(E_n)} \|F_2\|_{X_{q_2,q_2,\infty}^{\frac{1}{2} - \frac{2}{q_2} - 3\delta, \gamma_2}} \|F_3\|_{X_{2,2,2}^{\frac{1}{4} + \frac{2}{q_2} + 4\delta, b}}, \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} \|\varphi(t)\pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q_1,2}^{0,0}(E_n)} \\ \lesssim_{q_j, \gamma_j} \|\pi_n F_1\|_{X_{q_1,2}^{0,\gamma_1}(E_n)} \|F_2\|_{X_{q_2,q_2,\infty}^{1,\gamma_2}} \|F_3\|_{X_{2,2,2}^{2,b}}. \end{aligned} \quad (6.5)$$

Proof. First we prove (6.4). By duality, it suffices to show that for any $G_n \in X_{q_1,2}^{0,\gamma_0}(E_n)$ with $\|G_n\|_{X_{q_1,2}^{0,\gamma_0}(E_n)} \leq 1$, there holds

$$\begin{aligned} \left| \int_{\mathbb{R}} \int_{\mathbb{S}^2} \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3) \cdot \varphi(t) \overline{G_n}(t, x) \right| \\ \lesssim \|\pi_n F_1\|_{X_{q_1,2}^{0,\gamma_1}(E_n)} \|F_2\|_{X_{q_2,q_2,\infty}^{\beta_2, \gamma_2}} \|F_3\|_{X^{\beta_3, b}}, \end{aligned} \quad (6.6)$$

where

$$\beta_2 = \frac{1}{2} - \frac{2}{q_2} - 3\delta, \quad \beta_3 = \frac{1}{4} + \frac{2}{q_2} + 4\delta$$

throughout the proof. By Plancherel in time, we have

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{S}^2} \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3) \cdot \varphi(t) \overline{G}_n(t, x) dx dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{S}^2} \sum_{n_2, n_3} \pi_n F_1(t) \pi_{n_2} \overline{F}_2(t) \pi_{n_3} F_3(t) \varphi(t) \overline{G}_n(t) dt dx \\
&= \int_{\mathbb{R}^4} d\vec{\tau} \int_{\mathbb{S}^2} \sum_{n_2, n_3} \pi_n \widehat{F}_1(\tau_1, x) \pi_{n_2} \overline{\widehat{F}_2}(\tau_2, x) \pi_{n_3} \widehat{F}_3(\tau_3, x) \overline{\widehat{G}_n}(\tau_0, x) \overline{\widehat{\varphi}}(\tau_1 - \tau_2 + \tau_3 - \tau_0) dx \\
&= \int_{\mathbb{R}^4} d\vec{\kappa} \int_{\mathbb{S}^2} \sum_{n_2, n_3} \pi_n \widetilde{F}_1(\kappa_1, x) \pi_{n_2} \overline{\widetilde{F}_2}(\kappa_2, x) \pi_{n_3} \widetilde{F}_3(\kappa_3, x) \overline{\widetilde{G}_n}(\kappa_0, x) \overline{\widetilde{\varphi}}(\vec{\kappa} - \Omega(\vec{n})) dx,
\end{aligned}$$

where $\kappa_j = \tau_j + \lambda_{n_j}^2$, $d\vec{\tau} = d\tau_0 d\tau_1 d\tau_2 d\tau_3$, $d\vec{\kappa} = d\kappa_1 d\kappa_2 d\kappa_3 d\kappa_0$, $\vec{\kappa} = \kappa_1 - \kappa_2 + \kappa_3 - \kappa_0$, and $\Omega(\vec{n}) = -\lambda_{n_2}^2 + \lambda_{n_3}^2$. Set

$$\begin{aligned}
\mathbf{a}_n^{(1)}(\kappa_1) &= \|\langle \kappa_1 \rangle^{\gamma_1} \pi_n \widetilde{F}_1(\kappa_1, \cdot)\|_{L_x^2}, & \mathbf{a}_{n_2}^{(2)}(\kappa_2) &= n_2^{\beta_2} \|\langle \kappa \rangle^{\gamma_2} \pi_{n_2} \overline{\widetilde{F}_2}(\kappa_2, \cdot)\|_{L_x^\infty}, \\
\mathbf{a}_{n_3}^{(3)}(\kappa_3) &= n_3^{\beta_3} \|\langle \kappa_3 \rangle^b \pi_{n_3} \widetilde{F}_3(\kappa_3, \cdot)\|_{L_x^2}, & \mathbf{a}_n^{(0)}(\kappa_0) &= \|\langle \kappa_0 \rangle^{\gamma_0} \overline{\widetilde{G}_n}(\kappa_0, \cdot)\|_{L_x^2}.
\end{aligned}$$

Applying the bilinear eigenfunction estimate from Proposition 3.2, we obtain

$$\begin{aligned}
& \int_{\mathbb{S}^2} \pi_n \widetilde{F}_1(\kappa_1, x) \pi_{n_2} \overline{\widetilde{F}_2}(\kappa_2, x) \pi_{n_3} \widetilde{F}_3(\kappa_3, x) \overline{\widetilde{G}_n}(\kappa_0, x) dx \\
& \lesssim n_3^{\frac{1}{4}} n_2^{-\beta_2} n_3^{-\beta_3} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3)}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^{\gamma_1} \langle \kappa_2 \rangle^{\gamma_2} \langle \kappa_3 \rangle^b}.
\end{aligned}$$

It remains to control the expression

$$\int_{\mathbb{R}^4} d\vec{\kappa} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1)}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^{\gamma_1} \langle \kappa_2 \rangle^{\gamma_2} \langle \kappa_3 \rangle^b} \sum_{n_2, n_3} \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3) |\overline{\widetilde{\varphi}}(\vec{\kappa} - \Omega(\vec{n}))| n_2^{-\beta_2} n_3^{-\beta_3 + \frac{1}{4}}. \quad (6.7)$$

By Hölder, the sum in the integrand can be bounded by

$$\|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^{q_2}} \|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{\ell_{n_3}^2} \|\overline{\widetilde{\varphi}}(\vec{\kappa} - \Omega(\vec{n})) n_2^{-\beta_2} n_3^{-\beta_3 + \frac{1}{4}}\|_{\ell_{n_3}^2 \ell_{n_2}^{q_2'}} \lesssim_{\delta, q_2} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^{q_2}} \|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{\ell_{n_3}^2},$$

where to the last step we applied Lemma 6.4 with parameters $r = q_2' \in [1, 2]$, $\alpha_2 = \beta_2 \in (0, 1)$ and $\alpha_3 = \beta_3 - \frac{1}{4} \in (0, 1)$, satisfying $\alpha_2 + \alpha_3 = \beta_2 + \beta_3 - \frac{1}{4} > \frac{1}{2}$. Then, by Hölder and the fact that $\gamma_0 q > 1$, $\gamma_j q_j' > 1$, $j = 1, 2$, we obtain that

$$(6.7) \lesssim \|\mathbf{a}_n^{(0)}(\kappa_0)\|_{L_{\kappa_0}^{q_0'}} \|\mathbf{a}_n^{(1)}(\kappa_1)\|_{L_{\kappa_1}^{q_1}} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{L_{\kappa_2}^{q_2} \ell_{n_2}^{q_2}} \|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{L_{\kappa_3}^2 \ell_{n_3}^2}.$$

This proves (6.4).

Next, to prove (6.5), by Corollary A.7,

$$\|\varphi(t) \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q_1, 2}^{0,0}(E_n)} \lesssim \|\varphi(t) \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{L_t^{q_1'} L_x^2}.$$

The right hand side can be controlled simply by

$$\|\pi_n F_1\|_{L_t^\infty L_x^2} \|F_2\|_{L_t^\infty L_x^\infty} \|F_3\|_{L_t^\infty L_x^\infty}.$$

By Sobolev embedding and the embedding property claimed in Lemma 4.2,

$$\|\pi_n F_1\|_{L_t^\infty L_x^2} \lesssim \|\pi_n F_1\|_{X_{q_1,2}^{0,\gamma_1}(E_n)}, \quad \|F_3\|_{L_t^\infty L_x^\infty} \lesssim \|F_3\|_{L_t^\infty H_x^2} \lesssim \|F_3\|_{X^{2,b}}.$$

Moreover,

$$\|F_2\|_{L_x^\infty} = \left\| \sum_{n_2} \pi_{n_2} F_2 \right\|_{L_x^\infty} \leq \left(\sum_{n_2} n_2^{q_2} \|\pi_{n_2} F_2\|_{L_x^\infty}^{q_2} \right)^{\frac{1}{q_2}} \left(\sum_{n_2} \frac{1}{n_2^{q_2'}} \right)^{\frac{1}{q_2'}} \lesssim_{q_2} \left\| n_2 \pi_{n_2} F_2 \right\|_{\ell_{n_2}^{q_2} L_x^\infty}.$$

Hence

$$\|F_2\|_{L_t^\infty L_x^\infty} \lesssim \|n_2 \pi_{n_2} F_2\|_{L_t^\infty \ell_{n_2}^{q_2} L_x^\infty} \lesssim \|n_2 \pi_{n_2} F_2\|_{\ell_{n_2}^{q_2} L_t^\infty L_x^\infty} \lesssim \|F_2\|_{X_{q_2,2}^{1,\gamma_2,\infty}},$$

where we used Lemma 4.2 to the last inequality. The proof of Lemma 6.5 is now complete. \square

Similarly, we have:

Lemma 6.6. *Assume that $q \in [2, \infty)$ and $\gamma \in (\frac{1}{q}, 1)$. For any $\gamma_0 \in (\frac{1}{q}, 1)$ $\delta > 0$, $b > \frac{1}{2}$, then*

$$\|\varphi(t) \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q,2}^{0,-\gamma_0}(E_n)} \lesssim_{q,\gamma,\delta} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X^{\frac{1}{4}+\frac{2}{q}+4\delta,b}} \|F_3\|_{X^{\frac{1}{4}+\frac{2}{q}+4\delta,b}}, \quad (6.8)$$

and

$$\|\varphi(t) \pi_n \mathcal{N}_{(0,1)}(F_1, F_2, F_3)\|_{X_{q,2}^{0,0}(E_n)} \lesssim_{q_j,\gamma_j} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X^{2,b}} \|F_3\|_{X^{2,b}}. \quad (6.9)$$

Proof. The proof follows from very similar argument as in the proof of Lemma 6.5. Below we only indicate the different places.

For the proof of (6.8), we modify the function $\mathbf{a}_{n_2}^{(2)}(\kappa_2)$ as

$$\mathbf{a}_{n_2}^{(2)}(\kappa_2) = n_2^{\beta_2} \|\langle \kappa \rangle^b \pi_{n_2} \tilde{F}_2(\kappa, \cdot)\|_{L_x^2}, \quad \text{where } \beta_2 = \frac{1}{4} + \frac{2}{q} + 4\delta.$$

Note that $\beta_3 = \frac{1}{4} + \frac{2}{q} + 4\delta$ remains unchanged. Using the bilinear eigenfunction estimate (Proposition 3.2), the expression to be estimated becomes

$$\int_{\mathbb{R}^4} d\vec{\kappa} \sum_{n_2, n_3} (n_2 n_3)^{\frac{1}{4}} n_2^{-\beta_2} n_3^{-\beta_3} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3)}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^b \langle \kappa_3 \rangle^b} |\widehat{\varphi}(\vec{\kappa} - \Omega(\vec{n}))|.$$

Here to estimate the sum over n_2, n_3 , we use Schur's test. More precisely, we observe that

$$\sup_{n_2} \sum_{n_3} |\widehat{\varphi}(\vec{\kappa} - \Omega(\vec{n}))| + \sup_{n_3} \sum_{n_2} |\widehat{\varphi}(\vec{\kappa} - \Omega(\vec{n}))| \lesssim 1.$$

Thanks to the fact that $\beta_2 = \beta_3 > \frac{1}{4}$, we obtain the bound

$$\sum_{n_2, n_3} (n_2 n_3)^{\frac{1}{4}-\beta_2} \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3) |\widehat{\varphi}(\vec{\kappa} - \Omega(\vec{n}))| \lesssim \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^2} \|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{\ell_{n_3}^2}.$$

The rest argument is to use Hölder to control the integral over κ_j , as in the previous case.

To prove (6.9), the analysis is exactly the same as before, but we simply control $\|F_2\|_{L_t^\infty L_x^\infty}$ by $\|F_2\|_{X^{2,b}}$ using the Sobolev embedding. This proves Lemma 6.6. \square

Finally we establish an estimate to deal with the product $F_2 \otimes F_3$ when F_2, F_3 are both in the Fourier-Lebesgue space $X_{q,q,\infty}^{\frac{1}{2}-\frac{2}{q}-3\delta,\gamma}$.

Lemma 6.7. *Assume that $q \in [2, \infty)$ and $\gamma \in (\frac{1}{q}, 1)$. For any $\gamma_0 \in (\frac{1}{q}, 1)$ $\delta > 0$, $b > \frac{1}{2}$,*

$$\|\varphi(t)\pi_n \mathcal{N}_{(0,1)[2,3]}(F_1, F_2, F_3)\|_{X_{q,2}^{0,-\gamma_0}(E_n)} \lesssim_{q,\gamma,\delta} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{2}{q}-3\delta,\gamma}} \|F_3\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{2}{q}-3\delta,\gamma}}, \quad (6.10)$$

$$\|\varphi(t)\pi_n \mathcal{N}_{(0,1)[2,3]}(F_1, F_2, F_3)\|_{X_{q,2}^{0,0}(E_n)} \lesssim_{q_j,\gamma_j} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|F_2\|_{X_{q,q,\infty}^{1,\gamma}} \|F_3\|_{X_{q,q,\infty}^{1,\gamma}}. \quad (6.11)$$

Proof. The proof is also very similar as the proof of Lemma 6.5, so we only indicate the different places. For (6.10), we adjust the function

$$\mathbf{a}_{n_3}^{(3)}(\kappa_3) := n_3^{\beta_3} \|\langle \kappa_3 \rangle^\gamma \pi_{n_3} \widetilde{F}_3(\kappa_3, \cdot)\|_{L_x^\infty}, \quad \beta_3 = \frac{1}{2} - \frac{2}{q} - 3\delta,$$

and $\beta_2 = \frac{1}{2} - \frac{2}{q} - 3\delta$ remains unchanged. By Cauchy-Schwarz,

$$\begin{aligned} & \int_{\mathbb{S}^2} \pi_n \widetilde{F}_1(\kappa_1, x) \pi_{n_2} \widetilde{F}_2(\kappa_2, x) \pi_{n_3} \widetilde{F}_3(\kappa_3, x) \widetilde{G}_n(\kappa_0, x) dx \\ & \lesssim (n_2 n_3)^{-\beta} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3)}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^\gamma}, \end{aligned}$$

and we need to control

$$\begin{aligned} & \int_{\mathbb{R}^4} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) d\vec{\kappa}}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^\gamma} \sum_{\substack{n_2, n_3 \\ n_2 \neq n_3}} \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3) |\widetilde{\varphi}(\vec{\kappa} - \Omega(\vec{n}))| (n_2 n_3)^{-\beta_3} \\ & \leq \int_{\mathbb{R}^4} \frac{\mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1)}{\langle \kappa_0 \rangle^{\gamma_0} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^\gamma} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^q} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^q} d\vec{\kappa} \\ & \quad \times \sum_{\substack{n_2, n_3 \leq n \\ n_2 \neq n_3}} |\widetilde{\varphi}(\vec{\kappa} - \Omega(\vec{n}))| (n_2 n_3)^{-\beta}, \end{aligned}$$

where we used $\ell^q \hookrightarrow \ell^\infty$. By the divisor bound (here we make use of the fact $n_2 \neq n_3$):

$$\sup_l \#\{n_2 \neq n_3 : n_j \sim N_j, \lambda_{n_2}^2 - \lambda_{n_3}^2 = l\} \lesssim_\epsilon (N_2 N_3)^\epsilon,$$

the sum in the second line is uniformly bounded. By Hölder, we obtain (6.10).

Next, to prove (6.5), by Lemma A.7,

$$\begin{aligned}
& \|\varphi(t)\pi_n\mathcal{N}_{(0,1)[2,3]}(F_1, F_2, F_3)\|_{X_{q,2}^{0,0}(E_n)} \\
& \lesssim \|\varphi(t)\pi_n\mathcal{N}_{(0,1)[2,3]}(F_1, F_2, F_3)\|_{L_t^{q'}L_x^2} \\
& \lesssim \left\| \sum_{n_2, n_3} \|\pi_{n_1}F_1(t)\|_{L_x^2} \|\pi_{n_2}F_2(t)\|_{L_x^\infty} \|\pi_{n_3}F_3(t)\|_{L_x^\infty} \right\|_{L_t^\infty} \\
& \leq \|\pi_n F_1(t)\|_{L_t^\infty L_x^2} \left\| \|n_2 \pi_{n_2} F_2(t)\|_{\ell_{n_2}^q L_x^\infty} \|n_3 \pi_{n_3} F_3(t)\|_{\ell_{n_3}^q L_x^\infty} \right\|_{L_t^\infty},
\end{aligned}$$

where to the last step, we used the Hölder inequality for the sum and the fact that

$$\|n_2^{-1}\|_{\ell_{n_2}^{q'}} < \infty, \|n_3^{-1}\|_{\ell_{n_3}^{q'}} < \infty.$$

Finally, by Minkowski and the embedding property $X_{q,\infty}^{0,\gamma}(E_n) \hookrightarrow L_t^\infty L_x^\infty(E_n)$ (Lemma 4.2), we conclude the proof. \square

6.2. Proof of main propositions. Now we provide the proof of main propositions in this section by an interpolation argument.

Proof of Proposition 6.1, 6.2 and 6.3. We take a q large enough, then from the choice of parameters (6.3), we must have

$$\begin{aligned}
& \frac{1-\gamma-\theta}{\gamma_0} \cdot \left(s - \frac{1}{4} - \frac{2}{q} - 4\delta\right) > (2-s) \cdot \frac{\gamma_0 - (1-\gamma-\theta)}{\gamma_0}, \\
& \frac{1}{q} \cdot \frac{1-\gamma-\theta}{\gamma_0} > \left(\frac{1}{2} + \frac{1}{q} + 3\delta\right) \cdot \frac{\gamma_0 - (1-\gamma-\theta)}{\gamma_0}.
\end{aligned} \tag{6.12}$$

Indeed, the left hand sides of inequalities above are at least of order $O(q^{-1})$ while the right hand sides are of order $O(q^{-4})$.

To prove the desired inequality in Proposition 6.1, we will apply (6.4), (6.5) to frequency-localized F_2, F_3 and then interpolate. More precisely, we decompose

$$F_j = \sum_{N_j} \mathbf{P}_{N_j} F_j, \quad j = 2, 3.$$

For fixed N_2, N_3 , applying (6.4) with $q_1 = q_2 = q$,

$$\begin{aligned}
& \|\varphi(t)\pi_n\mathcal{N}_{(0,1)}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,-\gamma_0}(E_n)} \\
& \lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|\mathbf{P}_{N_2}F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{2}{q}-3\delta,\gamma}} \|\mathbf{P}_{N_3}F_3\|_{X^{\frac{1}{4}+\frac{2}{q}+4\delta,b}} \\
& \lesssim N_2^{-\frac{1}{q}} N_3^{-\left(s-\frac{1}{4}-\frac{2}{q}-4\delta\right)} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|\mathbf{P}_{N_2}F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}} \|\mathbf{P}_{N_3}F_3\|_{X^{s,b}}.
\end{aligned}$$

From (6.5),

$$\begin{aligned}
& \|\varphi(t)\pi_n\mathcal{N}_{(0,1)}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,0}(E_n)} \\
& \lesssim \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|\mathbf{P}_{N_2}F_2\|_{X_{q,q,\infty}^{1,\gamma}} \|\mathbf{P}_{N_3}F_3\|_{X^{2,b}} \\
& \lesssim N_2^{\frac{1}{2}+\frac{1}{q}+3\delta} N_3^{2-s} \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)} \|\mathbf{P}_{N_2}F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}} \|\mathbf{P}_{N_3}F_3\|_{X^{s,b}}.
\end{aligned}$$

Interpolate the above inequalities (since $0 < 1 - \gamma - \theta < \gamma_0$), we have

$$\begin{aligned} & \|\varphi(t)\pi_n\mathcal{N}_{(0,1)}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \\ & \leq \|\varphi(t)\mathcal{N}_{n,\text{res}}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,-\gamma_0}(E_n)}^{\frac{1-\gamma-\theta}{\gamma_0}} \\ & \quad \|\varphi(t)\mathcal{N}_{n,\text{res}}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,0}(E_n)}^{1-\frac{1-\gamma-\theta}{\gamma_0}}. \end{aligned}$$

Thanks to (6.12), we gain negative powers in N_2, N_3 for the right hand side, which is summable. This proves the first estimate of Proposition 6.1. The second one is similar and we omit the detail.

For Proposition 6.2 and Proposition 6.3, we only need to adjust the argument above. By Lemma 6.6 and interpolation,

$$\begin{aligned} & \|\varphi(t)\pi_n\mathcal{N}_{(0,1)}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \\ & \lesssim (N_2N_3)^{-\left(s-\frac{1}{4}-\frac{2}{q}-4\delta\right)\frac{1-\gamma-\theta}{\gamma_0}}(N_2N_3)^{(2-s)\frac{\gamma_0-(1-\gamma-\theta)}{\gamma_0}} \\ & \quad \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)}\|\mathbf{P}_{N_2}F_2\|_{X^{s,b}}\|\mathbf{P}_{N_3}F_3\|_{X^{s,b}}, \end{aligned}$$

where the powers of N_2, N_3 are negative. The same argument by Lemma 6.3, we have

$$\begin{aligned} & \|\varphi(t)\pi_n\mathcal{N}_{(0,1)[2,3]}(F_1, \mathbf{P}_{N_2}F_2, \mathbf{P}_{N_3}F_3)\|_{X_{q,2}^{0,\gamma-1+\theta}(E_n)} \\ & \lesssim (N_2N_3)^{-\frac{1}{q}\cdot\frac{1-\gamma-\theta}{\gamma_0}}(N_2N_3)^{\left(\frac{1}{2}+\frac{1}{q}+3\delta\right)\frac{\gamma_0-(1-\gamma-\theta)}{\gamma_0}} \\ & \quad \|\pi_n F_1\|_{X_{q,2}^{0,\gamma}(E_n)}\|\mathbf{P}_{N_2}F_2\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}}\|\mathbf{P}_{N_3}F_3\|_{X_{q,q,\infty}^{\frac{1}{2}-\frac{1}{q}-3\delta,\gamma}}. \end{aligned}$$

These bounds are conclusive, thanks to (6.12). \square

7. LARGE DEVIATION ESTIMATES FOR LINEAR RANDOM OBJECT

In this section we prove the bounds claimed in $\text{Loc}(N)$ (Definition 5.1) at step $2N$ for the random objects. An important ingredient is the equivalence in law of product of colored random spherical harmonics.

7.1. Equivalence in law. Denote by $\mathcal{B}_{\leq N}$ the σ -algebra generated by gaussians of frequency $\leq N$. Recall that for $n \in \mathbb{N}$, E_n is the eigenspace with respect to the eigenvalue λ_n^2 , and E_n is isometric to \mathbb{C}^{2n+1} , equipped with the L^2 norm. Recall that for $n \approx N$ and $|t| \leq T$, the colored term is

$$e_n^{N,\dagger}(t) = \mathcal{H}_n^{N,\dagger}(t)(e_n^\omega),$$

where $\mathcal{H}_n^{N,\dagger}(t)$ is a unitary operator on E_n for $|t| \leq T$ and is $\mathcal{B}_{\leq N/2}$ measurable.

Lemma 7.1 (law-equivalence). *Let $l \in \mathbb{N}^*$ and $n_1, \dots, n_l \in (N/2, N]$ (not necessarily distinct). Let F be a bounded Borel measurable functional on $\mathbb{C} \times E_{n_1} \times \dots \times E_{n_l}$. Let Y be $\mathcal{B}_{\leq N/2}$ measurable. Then for all $|t| \leq T$,*

$$\mathbb{E}\left[F(Y, \mathcal{H}_{n_1}^{N,\dagger}(t)(e_{n_1}^\omega), \dots, \mathcal{H}_{n_l}^{N,\dagger}(t)(e_{n_l}^\omega)) \mid \mathcal{B}_{\leq \frac{N}{2}}\right] = \mathbb{E}\left[F(Y, e_{n_1}^\omega, \dots, e_{n_l}^\omega) \mid \mathcal{B}_{\leq \frac{N}{2}}\right], \text{ a.s.}$$

In particular, if Y is independent of \mathcal{B}_N and F is of the product form $F(Y, \cdot) = Y \cdot G(\cdot)$, then

$$\mathbb{E}[YG(\mathcal{H}_{n_1}^{N,\dagger}(t)(e_{n_1}^\omega), \dots, \mathcal{H}_{n_l}^{N,\dagger}(t)(e_{n_l}^\omega)) | \mathcal{B}_{\leq \frac{N}{2}}] = \mathbb{E}[Y] \cdot \mathbb{E}[G(e_{n_1}^\omega, \dots, e_{n_l}^\omega)], \text{ a.s.}$$

Proof. We identify the eigenspace E_n as \mathbb{C}^{2n+1} endowed with the $L^2(E_n)$, which is isometric to the canonical norm of \mathbb{C}^{2n+1} . Let $\mathbf{U}(2n+1)$ denote the unitary group on $E_n = \mathbb{C}^{2n+1}$. Up to modifying the functional F , we can assume that n_1, n_2, \dots, n_l are distinct. There holds

$$X^\omega := (e_{n_1}^\omega, \dots, e_{n_l}^\omega), \quad \mathcal{H}(X) := (\mathcal{H}_{n_1}^N(e_{n_1}^\omega), \dots, \mathcal{H}_{n_l}^N(e_{n_l}^\omega)).$$

Since $\mathcal{B}_{\leq N/2}$ is generated by $Z := (e_n^\omega)_{n \leq N/2}$, then almost surely, $Y = \varphi(Z)$ for some Borel measurable function φ on the product space $\prod_{n \leq N/2} E_n$. Similarly, since $\mathcal{H}_{n_j}^N$ are $\mathcal{B}_{\leq N/2}$ measurable there exist some $\mathbf{U}(2n+1)$ -valued Borel functions U_{n_j} such that $\mathcal{H}_{n_j}^N = U_{n_j}(Z)$.

It suffices to show that for any bounded Borel function G ,

$$\mathbb{E}[F(Y, \mathcal{H}(X))G(Y)] = \mathbb{E}[F(Y, X)G(Y)], \quad \text{a.s.} \quad (7.1)$$

Denote by $\mu_Z(\cdot)$ the distribution of $Z = (e_n^\omega)_{n \leq N/2}$ on $\prod_{n \leq N/2} \mathbb{C}^{2n+1}$ and $\mu_X(\cdot)$ the distribution of X on $\prod_{j=1}^l \mathbb{C}^{2n_j+1}$. By independence between X, Z , the joint distribution of (Z, X) is $\mu_Z \otimes \mu_X$. Thus

$$\begin{aligned} & \mathbb{E}[F(Y, \mathcal{H}(X))G(Y)] \\ &= \int_{\prod_{n \leq N/2} \mathbb{C}^{2n+1}} G(\mathbf{z}) d\mu_Y(\mathbf{z}) \int_{\prod_{j=1}^l \mathbb{C}^{2n_j+1}} F(\varphi(\mathbf{z}), U_{n_1}(\mathbf{z})x_{n_1}, \dots, U_{n_l}(\mathbf{z})x_{n_l}) d\mu_X(\mathbf{x}), \end{aligned}$$

It suffices to show that for almost every \mathbf{z} ,

$$\begin{aligned} & \int_{\prod_{j=1}^l \mathbb{C}^{2n_j+1}} F(\varphi(\mathbf{z}), U_{n_1}(\mathbf{z})x_{n_1}, \dots, U_{n_l}(\mathbf{z})x_{n_l}) d\mu_X(\mathbf{x}) \\ &= \int_{\prod_{j=1}^l \mathbb{C}^{2n_j+1}} F(\varphi(\mathbf{z}), x_{n_1}, \dots, x_{n_l}) d\mu_X(\mathbf{x}). \end{aligned}$$

This amounts to show that for fixed \mathbf{z} , the random variables

$$U(X) =: (U_{n_1}(\mathbf{z})e_{n_1}, \dots, U_{n_l}(\mathbf{z})e_{n_l}) \quad \text{and} \quad X = (e_{n_1}, \dots, e_{n_l})$$

have the same law. This follows from the fact that

$$\mu_X = \otimes_{j=1}^l \mathcal{N}_{\mathbb{C}^{2n_j+1}}(0, 1),$$

and from Lemma 3.7 (the unitary group leaves the complex normal distribution invariant). This yields (7.1) and completes the proof of Lemma 7.1. \square

7.2. Fourier-Lebesgue norm of the colored term. In the first Lemma, we obtain deduce some bounds on the Fourier-Lebesgue norms of colored terms from the random information encoded in the random averaging operators (RAOs). Recall the definitions (5.8), (5.9) and (5.10) of $\mathcal{G}_n^{2N,\dagger}$, $\mathbf{h}_n^{2N,\dagger}$, $\mathbf{g}_n^{2N,\dagger}$ and $\mathbf{f}_n^{2N,\dagger}$.

Lemma 7.2 (Fourier-Lebesgue bounds). *Assume that Ξ is a $\mathcal{B}_{\leq N}$ -measurable set such that on Ξ ,*

$$\|\mathbf{g}_n^{2N,\dagger}(t)\|_{S_n^{q,\gamma,*}} \leq R^{-1}, \quad \|\mathcal{G}_n^{2N,\dagger}(t)\|_{S_n^{q,\gamma,*}} \leq R.$$

Then there exists a $\mathcal{B}_{\leq 2N}$ -measurable set Ξ' such that $\mathbb{P}(\Omega \setminus \Xi') < C_0 e^{-c_0(NR)^{\delta_0}}$, such that on $\Xi \setminus \Xi'$,

$$\|\mathbf{f}_n^{2N,\dagger}\|_{X_{q,\infty}^{0,\gamma}(E_n)} \leq N^{\frac{1}{q}+\delta} R, \quad \|\psi_{2N}^\dagger\|_{X_{q,\infty}^{0,\gamma}} \leq T^{-\gamma+\frac{1}{q}} RN^{-(\alpha-\frac{1}{2})+\frac{1}{q}+2\delta},$$

where $C_0, c_0, \delta_0 > 0$ are parameters depending only on q, δ, γ .

Proof. Thanks to Lemma 3.3 and Sobolev embedding, we have

$$\|e^{-it\lambda_n^2} e_n^\omega\|_{X_{q,q,\infty}^0} \lesssim_q RN^{\frac{1}{q}+2\delta}$$

outside a set of probability smaller than $\mathcal{O}(e^{-c_0 N^\theta R^2})$. Then, according to the decomposition (5.11), it remains to prove the bound for $\mathbf{f}_n^{2N,\dagger}$.

To simplify the notation, we drop the subindex n and superindex N in the proof for $\mathbf{f}_n^{2N,\dagger}(t)$, $\mathbf{h}_{n;\ell,k}^{2N,\dagger}(t)$, where given an orthonormal basis $(\mathbf{b}_{n,k})_{|k|\leq n}$ of E_n , we denote by $\mathbf{h}_{n;\ell,k}^{2N,\dagger}(t)$ the matrix-element of $\mathbf{h}_n^{2N,\dagger}(t)$.

Let us first make two observations. Thanks to the eigenfunction estimate, by losing $N^{\frac{1}{q}}$, it suffices to estimate the $X_{q,q}^{0,\gamma}(E_n)$ -norm of \mathbf{f}^\dagger and to prove the bound

$$\|\mathbf{f}^\dagger\|_{X_{q,q}^{0,\gamma}(E_n)} \leq N^\delta R$$

outside of the set with probability smaller than $\mathcal{O}(e^{-c_0(NR)^\theta})$. Moreover, since \mathbf{f}^\dagger is compactly supported on $[-T, T]$, we may replace the operator \mathbf{h}^\dagger (which is supported on $[-2T, 2T]$) by $\eta(t/T)\mathbf{h}^\dagger$ for some $\eta \in C_c^\infty((-1, 1))$ such that $\eta(t/T)\mathbf{f}^\dagger = \mathbf{f}^\dagger$. Since $(\mathbf{h}^\dagger)^* = \mathbf{g}^\dagger$ on $|t| \leq T$, we have $(\eta(t/T)\mathbf{f}^\dagger)^* = \eta(t/T)\mathbf{g}^\dagger$. This simple observation allows us to replace the operator norm of the adjoint $(\mathbf{h}^\dagger)^*$ by the operator norm \mathbf{g}^\dagger in the arguments below.

From the discussion above, we implicitly insert $\eta(t/T)$ in the operators $\mathbf{h}^\dagger(t)$, $\mathbf{g}^\dagger(t)$ and obtain

$$\langle \lambda \rangle^\gamma \tilde{\mathbf{f}}^\dagger(\lambda, x) = \sum_{|k|, |\ell| \leq n} \langle \lambda \rangle^\gamma \tilde{\mathbf{h}}_{\ell,k}^\dagger(\lambda) \frac{g_{n,k}(\omega)}{\sqrt{\lambda_n}} \mathbf{b}_{n,\ell}(x).$$

By Chebyshev and Minkowski, for $q < \infty$ and $p \geq q$,

$$\begin{aligned} \mathbb{P}[\|\mathbf{1}_\Xi \cdot \langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_{\lambda, x}^q} > N^\delta R] &\leq \frac{1}{(N^\delta R)^p} \|\mathbf{1}_\Xi \cdot \langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_\omega^p L_x^q L_\lambda^q}^p \\ &= \frac{1}{(N^\delta R)^p} \left\| \mathbf{1}_\Xi \cdot \left\| \langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_{\lambda, x}^q} \right\|_{L_\omega^p \mathcal{B}_{\leq N}} \right\|_{L_\omega^p}^p \\ &\leq \frac{1}{(N^\delta R)^p} \left\| \mathbf{1}_\Xi \cdot \left\| \langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_\omega^p \mathcal{B}_{\leq N}} \right\|_{L_{\lambda, x}^q} \right\|_{L_\omega^p}^p \end{aligned}$$

Recall that the matrices $\widetilde{\mathbf{h}}_{\ell, k}^\dagger(\lambda)$ are independent of the Borel σ -algebra generated by Gaussians \mathcal{B}_{2N} . Therefore, for fixed λ, x the conditional Wiener-chaos estimate yields

$$\|\mathbf{1}_\Xi \cdot \langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_\omega^p \mathcal{B}_{\leq N}} \leq Cp^{\frac{1}{2}} \left\| \mathbf{1}_\Xi \cdot \sum_{|k|, |\ell| \leq n} \langle \lambda \rangle^\gamma \widetilde{\mathbf{h}}_{\ell, k}^\dagger(\lambda) \frac{g_{n, k}(\omega)}{\sqrt{\lambda_n}} \mathbf{b}_{n, \ell}(x) \right\|_{L_\omega^p \mathcal{B}_{\leq N}}.$$

Taking L_λ^q of the inequality above, we have for fixed x ,

$$\|\langle \lambda \rangle^\gamma \widetilde{\mathbf{f}}^\dagger(\lambda, x)\|_{L_\lambda^q L_\omega^p \mathcal{B}_{\leq N}} \leq \frac{Cp^{\frac{1}{2}}}{n^{\frac{1}{2}}} \left\| \left(\sum_{|k| \leq n} \left| \sum_{|\ell| \leq n} \langle \lambda \rangle^\gamma \widetilde{\mathbf{h}}_{\ell, k}^\dagger(\lambda) \mathbf{b}_{n, \ell}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\lambda^q}. \quad (7.2)$$

Applying (4.6) to $\mathcal{T}_n = \mathbf{h}^\dagger$, we obtain the bound

$$\left(\sum_{|k| \leq n} \left| \sum_{|\ell| \leq n} \langle \lambda \rangle^\gamma \widetilde{\mathbf{h}}_{\ell, k}^\dagger(\lambda) \mathbf{b}_{n, \ell}(x) \right|^2 \right)^{\frac{1}{2}} \leq Cn^{\frac{1}{2}} \|\langle \lambda \rangle^\gamma (\widetilde{\mathbf{h}}^\dagger(\lambda))^*\|_{E_n \rightarrow L_\lambda^q E_n}.$$

Plugging into (7.2) and using the fact that $(\mathbf{h}^\dagger(t))^* = \eta(t/T) \mathbf{g}^\dagger(t)$, we finally obtain the bound

$$\begin{aligned} (7.2) &\leq Cp^{\frac{1}{2}} \|\langle \lambda \rangle^\gamma (\widetilde{\mathbf{h}}^\dagger(\lambda))^*\|_{E_n \rightarrow L_\lambda^q E_n} \\ &= Cp^{\frac{1}{2}} \|\langle \lambda \rangle^\gamma \mathcal{F}_t(e^{-it\lambda_n^2} \mathbf{g}^\dagger(t))(-\lambda)\|_{E_n \rightarrow L_\lambda^q E_n} = Cp^{\frac{1}{2}} \|\eta\left(\frac{t}{T}\right) \mathbf{g}^\dagger(t)\|_{S_n^{q, \gamma, *}}. \end{aligned}$$

Since $\mathbf{g}^\dagger(0) = 0$, by Corollary A.4, the right hand side is bounded by

$$Cp^{\frac{1}{2}} \|\mathbf{g}^\dagger\|_{S_n^{q, \gamma, *}} \leq Cp^{\frac{1}{2}} R^{-1}$$

on Ξ . Taking the L_x^q on and optimizing the choice of p , we obtain the desired estimate. \square

7.3. Pointwise bound on the colored terms. In this subsection, we prove large deviation estimates for norms defined in the physical space. We use the equivalence in law of Lemma 3.7 to reduce the analysis to time $t = 0$, when the random objects are much simpler.

Lemma 7.3 (Pointwise bounds). *Suppose that on a $\mathcal{B}_{\leq N}$ -measurable set Ξ , the colored Gaussian ψ_{2N}^\dagger is well defined and $\text{Loc}(N)$ is true with parameters (T, R, σ) with σ sufficiently small, and $(q, \gamma, \delta) = (q_\sigma, \gamma_\sigma, \delta_\sigma)$. There exists a $\mathcal{B}_{\leq 2N}$ -measurable set Ξ' with $\mathbb{P}(\Omega \setminus \Xi') < C_0 e^{-c_0(NR)^{\delta_0}}$ such that on $\Xi \setminus \Xi'$,*

$$\|\psi_{2N}^\dagger\|_{L_t^q L_x^\infty} \leq RN^{-(\alpha-1)+\delta}, \quad \|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}\|_{L_t^q L_x^\infty} \leq R^2 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\delta}, \quad (7.3)$$

where the constants $C_0, c_0, \delta_0 > 0$ only depend on q, δ, γ .

Proof. As in the proof of Lemma 7.2 (and using a Sobolev embedding) we reduce the proof to a moment bounds: there exists $C > 0$ such that for all $r \geq q$,

$$\|\psi_{2N}^\dagger\|_{L_\omega L_{t,x}^q} \leq \sqrt{r} C N^{-(\alpha-1)}, \quad (7.4)$$

$$\|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}\|_{L_\omega L_{t,x}^q} \leq r C N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}}. \quad (7.5)$$

We first prove (7.4). Recall the decomposition

$$\psi_{2N}^\dagger(t, x) = \sum_{n \approx 2N} \pi_n \psi_{2N}^\dagger(t) = \sum_{n \approx 2N} \lambda_n^{-(\alpha-\frac{1}{2})} \chi_T(t) e_n^{2N, \dagger}(t).$$

We have from the equivalence in law of Lemma 7.1 that for fixed t, x and $n \approx 2N$,

$$\|e_n^{2N, \dagger}(t, x)\|_{L_\omega} = \|\|e_n^{2N, \dagger}(t, x)\|_{L_\omega | \mathcal{B}_{\leq N}}\|_{L_\omega} = \|e_n^\omega(x)\|_{L_\omega} \lesssim \sqrt{r},$$

where we used Lemma 3.3. Then (7.4) follows from the Minkowski's inequality and the triangle inequality in the sum over $n \approx 2N$.

Let us now turn to the proof of (7.5). According to equivalence of laws, for fixed t, x ,

$$\begin{aligned} \|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}(t, x)\|_{L_\omega} &= \chi_T(t)^2 \|\| \sum_{n \approx 2N} \lambda_n^{-2(\alpha-\frac{1}{2})} (|e_n^{2N, \dagger}(t, x)|^2 - \|e_n^{2N, \dagger}\|_{L^2(\mathbb{S}^2)}^2) \|_{L_\omega | \mathcal{B}_{\leq N}}\|_{L_\omega} \\ &= \chi_T(t)^2 \|\| \sum_{n \approx 2N} \lambda_n^{-2(\alpha-\frac{1}{2})} (|e_n^\omega(x)|^2 - \|e_n^\omega\|_{L^2(\mathbb{S}^2)}^2) \|_{L_\omega}, \end{aligned}$$

Then, for $n \approx 2N$ we fix an orthonormal basis $(\mathbf{b}_{n,k})_{|k| \leq n}$ of E_n and we decompose into two parts:

$$\begin{aligned} |e_n^\omega(x)|^2 - \|e_n^\omega\|_{L^2(\mathbb{S}^2)}^2 &= \frac{1}{(2n+1)} \sum_{k \neq k'} g_{n,k}(\omega) \bar{g}_{n,k'}(\omega) \mathbf{b}_{n,k}(x) \bar{\mathbf{b}}_{n,k'}(x) \\ &\quad + \frac{1}{(2n+1)} \sum_k |g_{n,k}(\omega)|^2 (|\mathbf{b}_{n,k}(x)|^2 - 1). \end{aligned}$$

According to this decomposition, we write

$$\sum_{n \approx L} \lambda_n^{-2(\alpha-\frac{1}{2})} (|e_n^\omega(x)|^2 - \|e_n^\omega\|_{L^2}^2) = \text{I}(x) + \text{II}(x).$$

Note that we have no dependence on t , thanks to the invariance of the law. Note also that on \mathbb{T}^d there is no term $\text{II}(x)$ since the amplitude of a plane wave is always one.

•**Term I**(x): We deduce from hypercontractivity that

$$\begin{aligned} \mathbb{E}[|I(x)|^r]^{\frac{1}{r}} &= \mathbb{E}\left[\left|\sum_{n \approx 2N} \lambda_n^{-2(\alpha-\frac{1}{2})} \sum_{\substack{k, k' \\ k \neq k'}} g_{n,k}(\omega) \bar{g}_{n,k'}(\omega) \frac{\mathbf{b}_{n,k}(x) \bar{\mathbf{b}}_{n,k'}(x)}{2n+1}\right|^r\right]^{\frac{1}{r}} \\ &\lesssim r \mathbb{E}\left[\left|\sum_{n \approx 2N} \lambda_n^{-2(\alpha-\frac{1}{2})} \sum_{\substack{k, k' \\ k \neq k'}} g_{n,k}(\omega) \bar{g}_{n,k'}(\omega) \frac{\mathbf{b}_{n,k}(x) \bar{\mathbf{b}}_{n,k'}(x)}{2n+1}\right|^2\right]^{\frac{1}{2}} \\ &\lesssim r \left(\sum_{n \approx 2N} \lambda_n^{-4(\alpha-\frac{1}{2})} \sum_{\substack{k, k' \\ k \neq k'}} \frac{|\mathbf{b}_{n,k}(x)|^2 |\mathbf{b}_{n,k'}(x)|^2}{(2n+1)^2}\right)^{\frac{1}{2}}, \end{aligned}$$

where we used the independence on the last line. We conclude from Weyl's law (1.3) that

$$\mathbb{E}[|I(x)|^r]^{\frac{1}{r}} \leq r \left(\sum_{n \approx L} \lambda_n^{-4(\alpha-\frac{1}{2})} \left(\sum_{|k| \leq n} \frac{|\mathbf{b}_{n,k}(x)|^2}{2n+1}\right)^2\right)^{\frac{1}{2}} \lesssim r N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}}.$$

Note that the above bound does not depend on x .

•**Term II**(x): For the second term, we also use hypercontractivity to obtain

$$\mathbb{E}[|II(x)|^r]^{\frac{1}{r}} \leq r \mathbb{E}\left[\left(\sum_{n \approx 2N} \lambda_n^{-2(\alpha-\frac{1}{2})} \sum_k |g_{n,k}(\omega)|^2 \frac{|\mathbf{b}_{n,k}(x)|^2 - 1}{2n+1}\right)^2\right]^{\frac{1}{2}}.$$

We observe from Weyl's law (1.3) that for every n and $x \in \mathbb{S}^2$,

$$\sum_k |g_{n,k}(\omega)|^2 (|\mathbf{b}_{n,k}(x)|^2 - 1) = \sum_k (|g_{n,k}(\omega)|^2 - 1) \frac{|\mathbf{b}_{n,k}(x)|^2 - 1}{2n+1}.$$

This allows us to use random oscillations and to deduce from the independence that

$$\begin{aligned} \mathbb{E}[|II(x)|^r]^{\frac{1}{r}} &\leq r \left(\sum_{n \approx 2N} \lambda_n^{-4(\alpha-\frac{1}{2})} \sum_{|k| \leq n} \mathbb{E}[(|g_{n,k}(\omega)|^2 - 1)^2] \left(\frac{|\mathbf{b}_{n,k}(x)|^2 - 1}{2n+1}\right)^2\right)^{\frac{1}{2}} \\ &\leq r \left(\sum_{n \approx 2N} \lambda_n^{-4(\alpha-\frac{1}{2})} \sum_{|k| \leq n} \left(\frac{|\mathbf{b}_{n,k}(x)|^2 - 1}{2n+1}\right)^2\right)^{\frac{1}{2}} \\ &\leq r \left(\sum_{n \approx 2N} \lambda_n^{-4(\alpha-\frac{1}{2})} \left(\sum_{|k| \leq n} \left|\frac{|\mathbf{b}_{n,k}(x)|^2 - 1}{2n+1}\right|\right)^2\right)^{\frac{1}{2}}, \end{aligned}$$

where we used $\ell^1 \hookrightarrow \ell^2$. Applying the Weyl's law (1.3) we conclude that

$$\mathbb{E}[|II(x)|^r]^{\frac{1}{r}} \lesssim r \left(\sum_{n \approx 2N} \lambda_n^{-4(\alpha-\frac{1}{2})}\right)^{\frac{1}{2}} \lesssim r N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}}.$$

This gives (7.5) and completes the proof of Lemma 7.3. \square

Summarizing Lemma 7.2 and Lemma 7.3, we obtain the bounds on the stochastic objects claimed in $\text{Loc}(N)$ (Definition 5.1) at step $2N$.

The rest of the paper is devoted to prove the bounds (5.14) on the remainder w_{2N}^\dagger , which is solution to the equation (5.12). We may implicitly suppose that the random objects of generation $2N$ satisfy the bound (5.13) in $\text{Loc}(2N)$.

8. LARGE DEVIATION ESTIMATES FOR THE (C)(C)(C) TERMS

Given N_1, N_2, N_3 and $\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger$ of type (C)(C)(C) we derive some moments estimates, from which we deduce the large deviation bounds claimed in Proposition 5.11. Note that the estimates obtained in this Section are effective when $\alpha = 1$, on the support of the Gibbs measure.

In what follows, denote by $N_{(1)} \geq N_{(2)} \geq N_{(3)}$ the decreasing rearrangement of N_1, N_2, N_3 .

Proposition 8.1. *Suppose that $\text{Loc}(N)$ holds for parameters (T, R, σ) , with σ sufficiently small. Let Ξ'_{2N} be a \mathcal{B}_{2N} -measurable set as in Lemma 7.2 and Lemma 7.3. Then, there exists $C > 0$ such that for all $\psi_{N_j}^\dagger$ of type (C), for all $r \geq q$ and*

$$b_1 > \frac{1}{2}, \quad s_1 < \alpha - \frac{1}{2} + \frac{1}{q} + 4\delta,$$

we have

- For all N_1, N_2, N_3 with $N_{(1)} = 2N$ and $N_{(1)} = N_{(2)}$ or $N_{(1)} = N_2$,

$$\|\chi_{\mathcal{N}_{(0,1)}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{L_{\omega}(\Omega_N \cap \Xi'_{2N}; X^{0, -b_1})} \lesssim_{\alpha, s} (r^{\frac{1}{2}} R T^{-\gamma + \frac{1}{q'}})^3 N^{-s_1}.$$

- For all N_1, N_2, N_3 with $N_1 = 2N$,

$$\|\chi_{\mathcal{N}_{[0,1]}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{L_{\omega}(\Omega_N \cap \Xi'_{2N}; X^{0, -b_1})} \lesssim_{\alpha, s} (r^{\frac{1}{2}} R T^{-\gamma + \frac{1}{q'}})^3 N^{-s_1}.$$

We stress out that the constraint $N_{(1)} = N_{(2)}$ or $N_{(1)} = N_2$ when $\mathcal{N} = \mathcal{N}_{(0,1)}$ is necessary in order to have the nonlinear smoothing effect, as proved in [4]. This constraint is imposed by our ansatz (1.12).

Proof: Proposition 8.1 implies 5.11. Using that

$$\|\chi_T \mathcal{I}(\chi_{\mathcal{N}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger))\|_{X^{0, b}} \lesssim T^{1-b-b'} \|\chi_{\mathcal{N}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{X^{0, -b'}},$$

for $b + b' < 1$, with $0 \leq b' < \frac{1}{2}$. Interpolate with the trivial estimate ($N_{(1)} = 2N$)

$$\|\chi_{\mathcal{N}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{L_{\omega}(\Omega_N \cap \Xi'_{2N}; X^{0, 0})} \lesssim_{\alpha} N^{10} R^3,$$

we have

$$\|\chi_{\mathcal{N}}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger)\|_{L_{\omega}(\Omega_N \cap \Xi'_{2N}; X^{0, -b'})} \lesssim r^{\frac{3b'}{2b_1}} R^3 T^{(-\gamma + \frac{1}{q'}) \frac{3b'}{b_1}} N^{-\frac{b'}{b_1} s_1 + 10(1 - \frac{b'}{b_1})}.$$

Obviously, for $b', b_1, b' < \frac{1}{2} < b_1$, close enough to $\frac{1}{2}$, we have $\frac{b'}{b_1} s_1 - 10(1 - \frac{b'}{b_1}) > s + \sigma$. Then for $\gamma, \frac{1}{q'}$ close enough to 1 such that $1 - b - b' > \gamma - \frac{1}{q'}$, we complete the proof of Proposition 5.11 by the standard argument using Chebyshev. We remark that the growth R^4 in the statement of Proposition 5.11 is responsible for the decay

$C_0 \exp(-(\log N)^{c_0} R^{c_0})$ for the exceptional set, since the growth in the moment bound is R^3 . \square

Let us start with a general probabilistic multilinear estimate used in many places of the proof.

Lemma 8.2 (*k*-linear probabilistic estimates at generation $2N$). *Fix $N \geq 1$ such that $\text{Loc}(N)$ holds for parameters (R, T, σ) , and let Ξ'_{2N} be as in Lemma 7.2 and Lemma 7.3. For all $k \geq 1$, for every $\mathcal{B}_{\leq N}$ -measurable function $f \in L_x^\infty L_\omega^r \ell_{n_1, \dots, n_k} L_{\kappa_1, \dots, \kappa_k}^q$ and every x in \mathbb{S}^2 ,*

$$\begin{aligned} & \left\| \int_{\mathbb{R}^k} d\kappa_1 \cdots d\kappa_k \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} \left(\prod_{i=1}^k \pi_{n_i} \widetilde{\psi_{2N}^{\dagger \iota_i}}(\kappa_i, x) \right) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x) \right\|_{L_\omega^r(\Xi'_{2N})} \\ & \lesssim_k \left(r^{\frac{1}{2}} T^{-\gamma + \frac{1}{q}} R N^{-(\alpha - \frac{1}{2})k} \right) \left\| \left(\prod_{i=1}^k \langle \kappa_i \rangle^{-\gamma} \right) f_{n_1, \dots, n_k} \right\|_{L_\omega^r(\Xi'_{2N}; \ell_{n_1, \dots, n_k}^2 L_{\kappa_1, \dots, \kappa_k}^q)}, \end{aligned} \quad (8.1)$$

where ι_i stands for the possible conjugaison bar, which is not important here.

This type of estimate allows one to control the time-modulated multilinear interactions between k -colored Gaussian variables from the same generation $2N$, possibly interacting with a random function constructed at a former generation, which is therefore $\mathcal{B}_{\leq N}$ -measurable.

Remark 8.3. *Lemma 8.2 will be repeatedly applied in several parts of the multilinear probabilistic estimates, with $k \leq 3$. Hence, we neglect the dependence on k of the constants.*

Proof. It turns out that the conjugaison bar plays no role we suppose that $\iota_i = 1$ for all $i \in \{1, \dots, k\}$. Recall that for $n \approx 2N$,

$$\pi_n \psi_{2N}^{\dagger}(t, x) = \chi_T(t) \lambda_n^{-(\alpha - \frac{1}{2})} e_n^{2N, \dagger}(t, x) = \frac{\lambda_n^{-(\alpha - \frac{1}{2})}}{(2n + 1)^{\frac{1}{2}}} \sum_{|k|, |\ell| \leq n} \chi_T(t) H_{n; k, \ell}^{2N, \dagger}(t) g_{n, \ell}(\omega) \mathbf{b}_{n, k}(x).$$

We also stress out that $\mathcal{H}^{2N, \dagger}$ is $\mathcal{B}_{\leq N}$ -measurable, and is independent of the Gaussian variables $(g_{n, \ell})_{|\ell| \leq n}$ when $n \approx 2N$. Hence, we deduce from the conditional Wiener chaos estimate and from the non-pairing condition that almost surely in ω ,

$$\begin{aligned} \text{l.h.s (8.1)} &= \left\| \int_{\mathbb{R}^k} \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} \prod_{i=1}^k \pi_{n_i} \widetilde{\psi_{2N}^{\dagger}}(\kappa_i, x) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x) d\kappa_1 \cdots d\kappa_k \right\|_{L_\omega^r L_\omega^r | \mathcal{B}_{\leq N}} \\ &\lesssim r^{\frac{k}{2}} \left\| \int_{\mathbb{R}^k} \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} \prod_{i=1}^k \pi_{n_i} \widetilde{\psi_{2N}^{\dagger}}(\kappa_i, x) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x) d\kappa_1 \cdots d\kappa_k \right\|_{L_\omega^r L_\omega^2 | \mathcal{B}_{\leq N}}. \end{aligned}$$

Expanding the colored Gaussian variables and using independence along with the non-pairing condition, we conclude that almost surely in ω ,

$$\begin{aligned} & \left\| \int_{\mathbb{R}^k} d\kappa_1 \cdots d\kappa_k \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} \prod_{i=1}^k \widetilde{\pi_{n_i} \psi_{2N}^\dagger}(\kappa_i, x) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \right\|_{L_\omega^2 \mathcal{B}_{\leq N}}^2 \\ &= \int_{\mathbb{R}^{2k}} d\kappa_1 \cdots d\kappa_k d\kappa'_1 \cdots d\kappa'_k \sum_{\sigma \in \mathfrak{S}_k} \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \overline{f_{n_{\sigma(1)}, \dots, n_{\sigma(k)}}(\kappa'_1, \dots, \kappa'_k, x, \omega)} \\ &\times \prod_{i=1}^k \lambda_{n_i}^{-2(\alpha - \frac{1}{2})} \sum_{|\ell_i| \leq n_i} \left(\sum_{|k_i| \leq n_i} (\chi_T \widetilde{H_{n_i; k_i, \ell_i}^{2N, \dagger}})(\kappa_i) \frac{\mathbf{b}_{n_i, k_i}(x)}{(2n_i + 1)^{\frac{1}{2}}} \right) \overline{\left(\sum_{|k'_i| \leq n_i} (\chi_T \widetilde{H_{n_i; k'_i, \ell_i}^{2N, \dagger}})(\kappa'_i) \frac{\mathbf{b}_{n_i, k'_i}(x)}{(2n_i + 1)^{\frac{1}{2}}} \right)}, \end{aligned}$$

where \mathfrak{S}_k is the permutation group of $\{1, 2, \dots, k\}$.

We do Cauchy-Schwarz in the sums over ℓ_i to see that the right-hand-side is controlled by

$$\begin{aligned} & \lesssim \int_{\mathbb{R}^{2k}} d\kappa_1 \cdots d\kappa_k d\kappa'_1 \cdots d\kappa'_k \\ & \sum_{\sigma \in \mathfrak{S}_k} \sum_{n_1, \dots, n_k} |f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \overline{f_{n_{\sigma(1)}, \dots, n_{\sigma(k)}}(\kappa'_1, \dots, \kappa'_k, x, \omega)}| \\ & \times \prod_{i=1}^k \lambda_{n_i}^{-2(\alpha - \frac{1}{2})} \left(\sum_{|\ell_i| \leq n_i} \left| \sum_{|k_i| \leq n_i} (\chi_T \widetilde{H_{n_i; k_i, \ell_i}^{2N, \dagger}})(\kappa_i) \frac{\mathbf{b}_{n_i, k_i}(x)}{(2n_i + 1)^{\frac{1}{2}}} \right|^2 \right)^{\frac{1}{2}} \\ & \times \left(\sum_{|\ell_i| \leq n_i} \left| \sum_{|k_i| \leq n_i} (\chi_T \widetilde{H_{n_i; k_i, \ell_i}^{2N, \dagger}})(\kappa'_i) \frac{\mathbf{b}_{n_i, k_i}(x)}{(2n_i + 1)^{\frac{1}{2}}} \right|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and we conclude from Hölder's inequality in the κ_i 's variables that

$$\begin{aligned} & \lesssim \sum_{n_1, \dots, n_k} \left\| \left(\prod_{i=1}^k \langle \kappa_i \rangle^{-\gamma} \lambda_{n_i}^{-(\alpha - \frac{1}{2})} \right) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \right\|_{L_{\kappa_1, \dots, \kappa_k}^{q'}}^2 \\ & \prod_{i=1}^k \left\| \langle \kappa_i \rangle^\gamma \left(\sum_{|\ell_i| \leq n_i} \left| \sum_{|k_i| \leq n_i} (\chi_T \widetilde{H_{n_i; k_i, \ell_i}^{2N, \dagger}})(\kappa_i) \frac{\mathbf{b}_{n_i, k_i}(x)}{(2n_i + 1)^{\frac{1}{2}}} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_{\kappa_i}^q}^2. \end{aligned}$$

The result follows from (4.6) and from the induction assumption $\text{Loc}(N)$, together with the definition of the norm (4.5) and the operator bounds at generation $2N$ and (5.15), the above quantity can be estimated by

$$\begin{aligned} & \sum_{n_1, \dots, n_k} \left\| \left(\prod_{i=1}^k \langle \kappa_i \rangle^{-\gamma} \lambda_{n_i}^{-(\alpha - \frac{1}{2})} \right) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \right\|_{L_{\kappa_1, \dots, \kappa_k}^{q'}}^2 \left\| (\chi_T \mathcal{H}_n^{2N, \dagger})^* \right\|_{\ell_n^\infty S_n^{q, \gamma, *}}^{2k} \\ & \lesssim_k (T^{-\gamma + \frac{1}{q'}} R N^{-(\alpha - \frac{1}{2})})^{2k} \sum_{n_1, \dots, n_k} \left\| \left(\prod_{i=1}^k \langle \kappa_i \rangle^{-\gamma} \right) f_{n_1, \dots, n_k}(\kappa_1, \dots, \kappa_k, x, \omega) \right\|_{L_{\kappa_1, \dots, \kappa_k}^{q'}}^2. \end{aligned}$$

We conclude by taking the L^r -norm in ω . \square

We are now ready to prove Proposition 8.1.

Proof of Proposition 8.1. The proof of Proposition 8.1 boils down to show that there exists $\delta > 0$ such that for all $r \geq 2$,

$$\begin{aligned} & \left\| \langle \kappa \rangle^{-b_1} \int_{\mathbb{R}^3} d\vec{\kappa} \sum_{(n_1, n_2, n_3) \in \Gamma(n)} \widehat{\chi}(\vec{\kappa} - \Omega(\vec{n})) \right. \\ & \left. \pi_n(\widetilde{\pi_{n_1} \psi_{N_1}^\dagger(\kappa_1)} \widetilde{\pi_{n_2} \psi_{N_2}^\dagger(\kappa_2)} \diamond \pi_{n_3} \widetilde{\psi_{N_3}^\dagger(\kappa_3)}) \right\|_{L_\omega^r(\Xi'_{2N}; \ell_n^2 L_{\kappa, x}^2)} \lesssim (r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q'}})^3 N^{-s_1} \end{aligned} \quad (8.2)$$

where we recall the notations

$$\vec{n} = (n_1, n_2, n_3, n), \quad \vec{\kappa} := \kappa_1 - \kappa_2 + \kappa_3 - \kappa, \quad d\vec{\kappa} = d\kappa_1 d\kappa_2 d\kappa_3, \quad \Omega(\vec{n}) := \lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_n^2.$$

Given n , the set $\Gamma(n)$ represents a certain constraint to be specified. We defined the Wick product denoted by \diamond in (1.8).

Remark 8.4. *Before starting the proof we stress out that, by considerations on the degree of the spherical harmonics, only the terms with $n \lesssim N_{(1)}$ contribute to the sum over n .*

To prove (8.2) and to control the moments of order r , we use different features of the colored Gaussian random variables, depending on the pairing configuration $\Gamma(n)$ and on the relative size of N_1, N_2, N_3 and $2N$.

Case 1: Pairing $n_1 = n$. In this case $\psi_{N_2}^\dagger$ and $\psi_{N_3}^\dagger$ play a symmetric role, so we may suppose that $N_2 \geq N_3$.

Only the terms with $N_{(1)} = N_{(2)} = 2N$ or $N_{(1)} = N_2 = 2N$ contribute. Indeed, the other frequency interactions are absorbed in the ansatz (1.12) when $\mathcal{N} = \mathcal{N}_{(0,1)}$. We have

$$\text{l.h.s (8.2)} = \left\| \langle \kappa \rangle^{-b_1} \int_{\mathbb{R}^3} \widehat{\chi}(\vec{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2)) \mathcal{N}_{(0,1)}(\psi_{N_1}^\dagger, \psi_{N_2}^\dagger, \psi_{N_3}^\dagger) d\vec{\kappa} \right\|_{L_\omega(\Omega_N \cap \Xi'_{2N}; \ell_n^2 L_\kappa^2 L_x^2)}.$$

• **Case 1.1: Partial pairing:** $\mathcal{N}_{(0,1)[2,3]}$. In this case we have $N_1 \leq 2N$, $\max(N_2, N_3) = 2N$. Since the resonant function is $\Omega(\vec{n}) = \lambda_{n_2}^2 - \lambda_{n_3}^2$, with $n_2 \neq n_3$, we leverage these time-oscillations to sum over n_2 and n_3 together without losing powers in N_2 or N_3 .

First, observe that we save a square root in the sum over n thanks to the orthogonal projectors π_n . Then, we get rid of the terms $\pi_n \widetilde{\psi_{N_1}^\dagger(\kappa_1)}$ by using deterministic bounds:

$$\begin{aligned} \text{l.h.s (8.2)} &= \left\| \chi_{\mathcal{N}_{(0,1)[2,3]}}(\psi_{N_1}^\dagger, \psi_{2N}^\dagger, \psi_{N_3}^\dagger) \right\|_{L_\omega(\Xi'_{2N}, X^{0, -b_1})} \\ &\lesssim \left\| \langle \kappa \rangle^{-b_1} \sum_{n_2 \neq n_3} \int_{\mathbb{R}^3} \widehat{\chi}(\vec{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2)) \pi_n \widetilde{\psi_{N_1}^\dagger(\kappa_1, x)} \widetilde{\pi_{n_2} \psi_{N_2}^\dagger(\kappa_2, x)} \widetilde{\pi_{n_3} \psi_{N_3}^\dagger(\kappa_3, x)} d\vec{\kappa} \right\|_{L_\omega L_\kappa^2 L_x^2 \ell_n^2} \\ &\lesssim \left\| \langle \kappa_1 \rangle^\gamma \pi_n \widetilde{\psi_{N_1}^\dagger} \right\|_{L_\omega^\infty(\Omega_N \cap \Xi'_{2N}; \ell_n^q L_{\kappa_1}^q L_x^\infty)} \left\| \langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_n(\kappa, \kappa_1, x) \right\|_{L_\omega L_{\kappa, x}^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'}}, \end{aligned}$$

with

$$f_n(\kappa_1, \kappa, x) := \int_{\mathbb{R}^2} \sum_{n_2 \neq n_3} \widehat{\chi}(\tilde{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2)) \overline{\pi_{n_2} \psi_{N_2}^\dagger(\kappa_2, x)} \widetilde{\pi_{n_3} \psi_{N_3}^\dagger(\kappa_3, x)} d\kappa_2 d\kappa_3.$$

We obtain

$$\text{l.h.s (8.2)} \lesssim \|\psi_{N_1}^\dagger\|_{L^\infty(\Omega_N \cap \Xi'_{2N}; X_{q, \gamma}^{0, \infty})}, \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_n(\kappa, \kappa_1, x)\|_{L_\omega^r L_{\kappa, x}^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'}}.$$

Using the definition of the set Ξ'_{2N} we deduce that

$$\text{l.h.s (8.2)} \lesssim RT^{-\gamma + \frac{1}{q'}} N_1^{-(\alpha - \frac{1}{2}) + \frac{1}{q} + 2\delta} \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_n(\kappa, \kappa_1, x)\|_{L_\omega^r(\Xi'_{2N}; L_{\kappa, x}^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'})}.$$

We are now in a good position to apply Lemma 8.2. If $N_2 = N_3$ we can directly apply Lemma 8.2 with $k = 2$, and we end up with the bound

$$(r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q'}})^3 N_1^{-(\alpha - \frac{1}{2}) + \frac{1}{q} + 2\delta} N_2^{-2(\alpha - \frac{1}{2})} \leq (r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q'}})^3 N_{(1)}^{-(\alpha - \frac{1}{2}) + \frac{1}{q} + 2\delta} N_{(2)}^{-(\alpha - \frac{1}{2})},$$

which is conclusive. Otherwise $N_2 > N_3$ and we apply the Lemma twice. In this case, set

$$f_n(\kappa, \kappa_1, x) =: \int_{\mathbb{R}} \sum_{n_2 \approx N_2} \overline{\pi_{n_2} \psi_{N_2}^\dagger(\kappa_2)} f_{n, n_2}(\kappa, \kappa_1, \kappa_2, x) d\kappa_2,$$

with

$$f_{n, n_2}(\kappa, \kappa_1, \kappa_2, x) := \int_{\mathbb{R}} \sum_{n_3 \approx N_3} \widehat{\chi}(\tilde{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2)) \overline{\pi_{n_3} \psi_{N_3}^\dagger(\kappa_3)} d\kappa_3.$$

To control f_n , we use Lemma 8.2 applied at generation N_2 with f_{n, n_2} and $k = 1$:

$$\begin{aligned} \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_n(\kappa, \kappa_1, x)\|_{L_\omega^r L_{\kappa, x}^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'}} &\leq \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_n(\kappa, \kappa_1, x)\|_{L_\omega^r L_{\kappa, x}^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'} L_\omega^1 \mathcal{B}_{\leq N_3}} \\ &\lesssim r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q'}} N_2^{-(\alpha - \frac{1}{2})} \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} \langle \kappa_2 \rangle^{-\gamma} f_{n, n_2}\|_{L_\omega^r L_\kappa^2 L_x^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1}^{q'} \ell_{n_2}^2 L_{\kappa_2}^{q'}}. \end{aligned}$$

Subsequently, we write

$$f_{n, n_2}(\kappa, \kappa_1, \kappa_2) = \int_{\mathbb{R}} \sum_{n_3 \approx N_3} \overline{\pi_{n_3} \psi_{N_3}^\dagger(\kappa_3)} f_{n, n_2, n_3}(\kappa, \kappa_1, \kappa_2, \kappa_3, x) d\kappa_3,$$

with

$$f_{n, n_2, n_3}(\kappa, \kappa_1, \kappa_2, \kappa_3) = \widehat{\chi}(\tilde{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2)).$$

We apply Lemma 8.2 at generation N_3 with f_{n, n_2, n_3} and $k = 1$ and Minkowski ($2 > q'$) to conclude that

$$\begin{aligned} \text{l.h.s (8.2)} &\lesssim r(RT^{-\gamma + \frac{1}{q'}})^3 N_1^{-(\alpha - \frac{1}{2}) + \frac{1}{q} + 2\delta} (N_2 N_3)^{-(\alpha - \frac{1}{2})} \\ &\quad \|\mathbf{1}_{n_2 \neq n_3} \langle \kappa \rangle^{-b_1} (\langle \kappa_1 \rangle \langle \kappa_2 \rangle \langle \kappa_3 \rangle)^{-\gamma} \widehat{\chi}(\tilde{\kappa} - (\lambda_{n_2}^2 - \lambda_{n_3}^2))\|_{L_\kappa^2 \ell_n^{\frac{2q}{q-2}} L_{\kappa_1, \kappa_2, \kappa_3}^{q'} \ell_{n_2, n_3}^2}. \end{aligned}$$

Since $n_2 \neq n_3$ in the sums, we can conclude from the divisor bound

$$\sup_l \#\{n_2 \neq n_3 : n_j \sim N_j, \lambda_{n_2}^2 - \lambda_{n_3}^2 = l\} \lesssim_\epsilon (N_2 N_3)^\epsilon, \quad (8.3)$$

that we can sum over n_2 and n_3 together. We obtain that for all $\epsilon > 0$,

$$\text{l.h.s (8.2)} \lesssim_{\epsilon} r(RT^{-\gamma+\frac{1}{q'}})^3 N_1^{-(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta} (N_2 N_3)^{-(\alpha-\frac{1}{2})+\epsilon},$$

where we used that the sum over n runs over $\mathcal{O}(N_1)$ -terms (because we have the condition $n = n_1$). This is conclusive at least when $\alpha \geq 1$. In the case where $N_{(1)} = N_{(2)} = 2N$, we get the bound

$$r(RT^{-\gamma+\frac{1}{q'}})^3 N_{(1)}^{-(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta} (N_{(2)})^{-(\alpha-\frac{1}{2})+\epsilon} \lesssim r(RT^{-\gamma+\frac{1}{q'}})^3 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta+\epsilon}.$$

In the cases where $N_2 = 2N$, we get the bound

$$r(RT^{-\gamma+\frac{1}{q'}})^3 N_{(2)}^{-(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta} N_{(1)}^{-(\alpha-\frac{1}{2})+\epsilon} \lesssim r(RT^{-\gamma+\frac{1}{q'}})^3 N^{-(\alpha-\frac{1}{2})+\frac{1}{q}+2\delta+\epsilon}.$$

We conclude in both cases.

• **Case 1.2 Full pairing:** $\mathcal{N}_{(0,1)(2,3)}$ In this resonant case we need to exploit (at least when $n_1 \neq n_2$) the Wick renormalization, especially Lemma 7.3. Moreover, we cannot expect to gain from the modulation (since there is no time oscillation), so we do not have to consider Fourier transform in time.

For the interaction $\mathcal{N}_{(0,1)}$ we supposed $N_{(2)} = 2N$ or $N_2 = 2N$. Since here we have $N_2 = N_3$, and hence $N_2 \geq N_{(2)}$, we necessarily have $N_2 = 2N$. Therefore, we have to exploit the Wick ordering and the saving in N_2 from Lemma 7.3. We have

$$\begin{aligned} \text{l.h.s (8.2)} &= \|\chi \sum_{n \approx N_1} \pi_n (\pi_n \psi_{N_1}^\dagger (\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}))\|_{L_\omega(\Omega_N \cap \Xi'_{2N}; X^{0,-b_1})} \\ &\lesssim \|\chi \sum_{n \approx N_1} \pi_n (\pi_n \psi_{N_1}^\dagger (\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}))\|_{L_\omega(\Omega_N \cap \Xi'_{2N}; X^{0,0})} \\ &\lesssim \|\chi \pi_n \psi_{N_1}^\dagger (\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger})\|_{L_\omega(\Omega_N \cap \Xi'_{2N}; \ell_n^2 L_t^2 L_x^2)}. \end{aligned}$$

Using the assumption, we have that for all $\omega \in \Xi'_{2N}$

$$\|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}\|_{L_t^q L_x^\infty} \leq CR^2 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta},$$

Therefore,

$$\begin{aligned} \text{l.h.s (8.2)} &\lesssim \left\| \|\chi \pi_n \psi_{N_1}^\dagger\|_{L_t^{\frac{2q}{q-2}} \ell_n^2 L_x^2} \|\psi_{2N}^\dagger \ominus \overline{\psi_{2N}^\dagger}\|_{L_t^q L_x^\infty} \right\|_{L_\omega(\Omega_N \cap \Xi'_{2N})} \\ &\lesssim \|\pi_n \psi_{N_1}^\dagger\|_{L^r(\Omega_N; L_t^{\frac{2q}{q-2}} \ell_n^2 L_x^2)} R^2 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta}. \end{aligned}$$

Using that $\pi_n \psi_{N_1}^\dagger(t, x) = \lambda_n^{-(\alpha-\frac{1}{2})} \mathcal{H}_n^{N_1, \dagger}(t)(e_n^\omega)(x)$, and that $\mathcal{H}_n^{N_1, \dagger}(t)$ is unitary, we obtain that for all t and $n \approx N_1$

$$\|\pi_n \psi_{N_1}^\dagger(t)\|_{L^r(\Omega_N; \ell_n^2 L_x^2)} \lesssim \|e_n^\omega\|_{L_\omega(\Omega_N L_x^2)} \left(\sum_{n \approx N_1} \lambda_n^{-2(\alpha-\frac{1}{2})} \right)^{\frac{1}{2}} \lesssim RN_1^{1-\alpha}. \quad (8.4)$$

Hence, by Minkowski (when $r \geq \frac{2q}{q-2}$),

$$\text{l.h.s (8.2)} \lesssim R^3 N_1^{1-\alpha} N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta} \lesssim R^3 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+2\delta}.$$

This is conclusive when $\alpha \geq 1$.

Case 2: No pairing, n_1, n_2, n_3, n are distinct. In this paragraph we have $N_{(1)} = 2N$, and without loss of generality we suppose $N_1 = N_{(1)} \geq N_2 \geq N_3$ (since the three terms are symmetric in this configuration).

We only present the situation when we have three different generations, namely $2N = N_1 > N_2 > N_3$, in which case we apply Lemma 8.2 with $k = 1$ three times. Otherwise we would proceed similarly: if $N_1 = N_2 = 2N$, say, we apply Lemma 8.2 at generation $2N$ with $k = 2$ first and then at generation N_3 with $k = 1$.

In Case 2, when $2N = N_1 > N_2 > N_3$, we have

$$\text{l.h.s (8.2)} = \|\langle \kappa \rangle^{-b_1} \int_{\mathbb{R}} \pi_{n_1} \widetilde{\psi_{2N}^\dagger}(\kappa_1, x) f_{n_1}(\kappa_1, n, \kappa, x) d\kappa_1\|_{L_\omega^r \ell_n^2 L_\kappa^2 L_x^2},$$

where, given $n_1 \approx 2N$,

$$f_{n_1}(\kappa_1, n, \kappa, x) := \sum_{\substack{n_2 \approx N_2, n_3 \approx N_3 \\ n_2 \neq n_3}} \int_{\mathbb{R}^2} \widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n})) \pi_{n_2} \widetilde{\psi_{N_2}^\dagger}(\kappa_2, x) \pi_{n_3} \widetilde{\psi_{N_3}^\dagger}(\kappa_3, x) d\kappa_2 d\kappa_3,$$

is a $\mathcal{B}_{\leq N}$ -measurable function. We first apply the conditional Minkowski's inequality and Lemma 8.2 at generation N_1 with f_{n_1} and $k = 1$ to see that

$$\begin{aligned} \text{l.h.s (8.1)} &\lesssim \|\langle \kappa \rangle^{-b_1} \int_{\mathbb{R}} \pi_{n_1} \widetilde{\psi_{N_1}^\dagger}(\kappa_1, x) f_{n_1}(\kappa_1, n, \kappa, x) d\kappa_1\|_{L_\omega^r \ell_n^2 L_\kappa^2 L_x^2 L_\omega^r | \mathcal{B}_{\leq N}} \\ &\lesssim r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q}} N^{-(\alpha - \frac{1}{2})} \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_{n_1}\|_{L_\omega^r \ell_n^2 L_\kappa^2 L_x^2 L_{n_1}^{q'} L_{\kappa_1}^{q'}} \\ &\leq r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q}} N^{-(\alpha - \frac{1}{2})} \|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_{n_1}\|_{L_\omega^r \ell_n^2 L_\kappa^2 L_x^2 L_{n_1}^{q'} L_{\kappa_1}^{q'} L_\omega^r | \mathcal{B}_{\leq N_2}}, \end{aligned}$$

where we have used the Minkowski in the last step.

Subsequently, we write

$$f_{n_1}(\kappa_1, \kappa, n, x) := \int_{\mathbb{R}} \sum_{n_2 \approx N_2} \pi_{n_2} \widetilde{\psi_{N_2}^\dagger}(\kappa_2, x) f_{n_1, n_2}(\kappa_1, \kappa_2, n, \kappa, x) d\kappa_2,$$

with

$$f_{n_1, n_2}(\kappa_1, \kappa_2, n, \kappa, x) := \sum_{n_3 \approx N_3} \int_{\mathbb{R}} \pi_{n_3} \widetilde{\psi_{N_3}^\dagger}(\kappa_3, x) \widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n})) d\kappa_3.$$

We apply Lemma 8.2 at generation N_2 with f_{n_1, n_2} (which is a $\mathcal{B}_{\leq N_3}$ -measurable function, with $N_3 \leq \frac{N_2}{2}$) and $k = 1$ together with the Minkowski to deduce that

$$\begin{aligned} &\|\langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_{n_1}\|_{L_\omega^r(\Omega_N; \ell_n^2, L_\kappa^2 L_x^2 L_{\kappa_1}^{q'} L_\omega^r | \mathcal{B}_{\leq N_2})} \\ &\lesssim r^{\frac{1}{2}} RT^{-\gamma + \frac{1}{q}} N_2^{-(\alpha - \frac{1}{2})} \left\| \|\langle \kappa \rangle^{-b_1} \prod_{i=1}^2 \langle \kappa_i \rangle^{-\gamma} f_{n_1, n_2}\|_{L_x^2 L_{\kappa_1, \kappa_2}^{q'} L_\kappa^2 \ell_{n, n_1, n_2}^2} \right\|_{L_\omega^r(\Omega_N \cap \Xi'_{2N})}. \end{aligned}$$

Hence,

$$\text{l.h.s (8.2)} \lesssim r (RT^{-\gamma + \frac{1}{q}})^2 (NN_2)^{-(\alpha - \frac{1}{2})} \|\langle \kappa \rangle^{-b_1} \prod_{i=1}^2 \langle \kappa_{n_i} \rangle^{-\gamma} f_{n_1, n_2}\|_{L_\omega^r(\Omega_N; L_x^2 L_{\kappa_1, \kappa_2}^{q'} L_\kappa^2 L_{n, n_1, n_2}^2)}.$$

Then, we apply Lemma 8.2 at generation N_3 with f_{n_1, n_2, n_3} and $k = 1$, where

$$f_{n_1, n_2, n_3}(\kappa_1, \kappa_2, \kappa_3, n, \kappa) := \widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n})) = \widehat{\chi}(\tilde{\kappa} - (\lambda_n^2 - \lambda_{n_1}^2 + \lambda_{n_2}^2 - \lambda_{n_3}^2)).$$

We obtain

$$\text{l.h.s (8.2)} \lesssim r^{\frac{3}{2}} (RT^{\gamma - \frac{1}{q'}})^3 (NN_2N_3)^{-(\alpha - \frac{1}{2})} \|\langle \kappa \rangle^{-b_1} \prod_{i=1}^3 \langle \kappa_i \rangle^{-\gamma} \widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n}))\|_{L_{\kappa}^2 L_{\kappa_1, \kappa_2, \kappa_3}^{q'} \ell_{n, n_1, n_2, n_3}^2}.$$

Then, we can use the divisor bound as in (8.3) together with Remark 8.4 sum over n_1 and n together for fixed $\vec{\kappa}$ and n_2, n_3 : for all $\epsilon > 0$,

$$\begin{aligned} \text{l.h.s (8.2)} &\lesssim (r^{\frac{1}{2}} RT^{\gamma - \frac{1}{q'}})^3 (N_1 N_2 N_3)^{-(\alpha - \frac{1}{2})} (N_2 N_3)^{\frac{1}{2}} \\ &\quad \left\| \langle \kappa \rangle^{-b_1} \prod_{i=1}^3 \langle \kappa_i \rangle^{-\gamma} \kappa^{-b_1} \widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n})) \right\|_{L_{\kappa, \kappa_1, \kappa_2, \kappa_3}^{\infty} \ell_{n_2, n_3}^{\infty} \ell_{n, n_1}^2} \left\| \right\|_{L_{\kappa}^2 L_{\kappa_1, \kappa_2, \kappa_3}^q} \\ &\lesssim_{\epsilon} r^{\frac{3}{2}} (RT^{\gamma - \frac{1}{q'}})^3 N^{-(\alpha - \frac{1}{2}) + \epsilon} (N_2 N_3)^{1 - \alpha}, \end{aligned}$$

which is conclusive at least when $\alpha \geq 1$.

Case 3: Partial pairing with wick-ordering: $\mathcal{N}_{[0,1](23)}$. We distinguish between two cases, whether $N_1 \leq N_2 = N_3 = 2N$, or $N_2 = N_3 \leq N$ and $N_1 = 2N$.

• **Case 3.1:** $N_1 \leq N_2 = 2N$. In this case we only exploit the gain from the Wick ordering. For this reason, we do not need to use the Fourier transform in time. In case 3.1 we have

$$\begin{aligned} \text{l.h.s (8.2)} &= \|\chi \sum_n \sum_{\substack{n_1 \approx N_1 \\ n_1 \neq n}} \pi_n(\pi_{n_1} \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; X^{0, -b_1})} \\ &\lesssim \|\chi \sum_n \sum_{\substack{n_1 \approx N_1 \\ n_1 \neq n}} \pi_n(\pi_{n_1} \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; X^{0, 0})} \end{aligned}$$

Observe that

$$\begin{aligned} &\|\chi \sum_n \sum_{\substack{n_1 \approx N_1 \\ n_1 \neq n}} \pi_n(\pi_{n_1} \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; L_{t,x}^2)} \\ &\leq \|\chi \sum_n \sum_{n_1 \approx N_1} \pi_n(\pi_{n_1} \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; L_{t,x}^2)} \\ &\quad + \|\chi \sum_n \pi_n(\pi_n \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; L_{t,x}^2)}. \end{aligned}$$

We have already controlled the second term in Case 1.2 (in the present case when $N_2 = 2N$), and we deduce from this bound that

$$\text{l.h.s (8.2)} \lesssim \|\chi \sum_n \sum_{n_1 \approx N_1} \pi_n(\pi_{n_1} \psi_{N_1}^{\dagger}(\psi_{2N}^{\dagger} \ominus \psi_{2N}^{\dagger}))\|_{L_{\omega}^r(\Omega_N \cap \Xi'_{2N}; L_{t,x}^2)} + R^3 N^{-2(\alpha - \frac{1}{2}) + \frac{1}{2} + \frac{1}{q} + 2\delta}.$$

It suffices therefore to handle the first term, where we do not assume the non-pairing condition. By duality, we have that for fixed $\omega \in \Omega_N \cap \Xi'_{2N}$,

$$\begin{aligned} & \left\| \chi \sum_n \sum_{n_1 \approx N_1} \pi_n(\pi_{n_1} \psi_{N_1}^\dagger(\psi_{2N}^\dagger)) \right\|_{L^r_\omega(\Omega_N \cap \Xi'_{2N}; L^2_{t,x})} \\ & \leq \left\| \sup_{\|v\|_{L^2_{t,x}} \leq 1} \left| \int_{\mathbb{R} \times \mathbb{S}^2} \chi(t) \psi_{N_1}^\dagger(t, x) (\psi_{2N}^\dagger \ominus \psi_{2N}^\dagger)(t, x) \bar{v}(t, x) \right| dt dx \right\|_{L^r_\omega(\Omega_N \cap \Xi'_{2N})} \\ & \lesssim \left\| \sup_{\|v\|_{L^2_{t,x}} \leq 1} \|\psi_{N_1}^\dagger\|_{L^r_\omega L^{\frac{2q}{q-2}}_t L^2_x} \|\psi_{2N}^\dagger \ominus \psi_{2N}^\dagger\|_{L^q_t L^\infty_x} \|v\|_{L^2_{t,x}} \right\|_{L^r_\omega(\Omega_N \cap \Xi'_{2N})}. \end{aligned}$$

We have from orthogonality and form (8.4) that

$$\|\psi_{N_1}^\dagger\|_{L^r_\omega L^{\frac{2q}{q-2}}_t L^2_x} \lesssim r^{\frac{1}{2}} N_1^{1-\alpha}.$$

According to the assumption $\omega \in \Xi'_{2N}$,

$$\|\psi_{2N}^\dagger \ominus \psi_{2N}^\dagger\|_{L^q_t L^\infty_x} \lesssim R^2 N^{-2(\alpha - \frac{1}{2}) + \frac{1}{2} + \frac{1}{q} + 2\delta}$$

We conclude that in Case 3.1 we have the same estimate as in Case 1.2:

$$\text{l.h.s (8.2)} \lesssim R^2 N^{-2(\alpha - \frac{1}{2}) + \frac{1}{2} + \frac{1}{q} + 2\delta},$$

for all $\alpha \geq 1$.

• **Case 3.2:** $N_2 \leq N$, $N_1 = 2N$ In this case, it suffices to gain over N_1 by using the time-modulation, and the non-pairing condition $n \neq n_1$.

We do not exploit the saving from the Wick-ordering (for instance, when $N_2 \ll N_1$ this saving is of no use). In Case 3.2, we have

$$\begin{aligned} \text{l.h.s (8.2)} \lesssim & \left\| \langle \kappa \rangle^{-b_1} \sum_{n_1 \approx 2N} \int_{\mathbb{R}^3} \widehat{\chi}(\tilde{\kappa} - (\lambda_n^2 - \lambda_{n_1}^2)) \pi_n \left(\pi_{n_1} \widetilde{\psi_{N_1}^\dagger}(\kappa_1, x) \right. \right. \\ & \left. \left. \widetilde{\psi_{N_2}^\dagger}(\kappa_2, x) \ominus \widetilde{\psi_{N_2}^\dagger}(\kappa_3, x) \right) d\vec{\kappa} \right\|_{L^r_\omega \ell_n^2 L^2_\kappa L^2_x}. \end{aligned}$$

Since $N_2 \leq N$, the random function $\psi_{N_2}^\dagger$ is $\mathcal{B}_{\leq N}$ -measurable, we can readily apply Lemma 8.2 at generation $2N$ with $k = 1$, and

$$f_{n_1}(\kappa, \kappa_1, x, n) := \int_{\mathbb{R}^2} \widehat{\chi}(\tilde{\kappa} - (\lambda_n^2 - \lambda_{n_1}^2)) \widetilde{\psi_{N_2}^\dagger}(\kappa_2, x) \ominus \widetilde{\psi_{N_2}^\dagger}(\kappa_3, x) d\kappa_2 d\kappa_3.$$

We obtain

$$\text{l.h.s (8.2)} \lesssim r^{\frac{1}{2}} R T^{-\gamma + \frac{1}{q'}} N^{-(\alpha - \frac{1}{2})} \left\| \langle \kappa \rangle^{-b_1} \langle \kappa_1 \rangle^{-\gamma} f_{n_1}(\kappa, \kappa_1, x, n) \right\|_{L^r_\omega \ell_n^2 L^2_\kappa L^2_x \ell_{n_1}^2 L^{q'}_{\kappa_1}}.$$

It follows from the Minkowski's inequality ($q' < 2$) and the Hölder's inequality that

$$\begin{aligned} \text{l.h.s (8.2)} \lesssim & r^{\frac{1}{2}} R T^{-\gamma + \frac{1}{q'}} N^{-(\alpha - \frac{1}{2})} \\ & \sum_{n_2 \approx N_2} \left\| \langle \kappa \rangle^{-b_1} (\langle \kappa_1 \rangle \langle \kappa_2 \rangle \langle \kappa_3 \rangle)^{-\gamma} \widehat{\chi}(\tilde{\kappa} - (\lambda_n^2 - \lambda_{n_1}^2)) \right\|_{L^2_\kappa L^{q'}_{\kappa_1, \kappa_2, \kappa_3} \ell_{n, n_1}^2} \|\psi_{N_2}^\dagger\|_{X_{q, \infty}^{0, \gamma}}^2. \end{aligned}$$

Using the bound (5.13) for $\psi_{N_2}^\dagger$, which holds on Ω_N , we obtain that for all $\omega \in \Omega_N \cap \Xi'_{2N}$,

$$\begin{aligned} \text{l.h.s (8.2)} &\lesssim r^{\frac{1}{2}} (RT^{-\gamma+\frac{1}{q'}})^3 N^{-(\alpha-\frac{1}{2})+\epsilon} N_2^{1-2(\alpha-\frac{1}{2})+\frac{2}{q}+4\delta} \\ &\quad \|\langle \kappa \rangle^{-b_1} (\langle \kappa_1 \rangle \langle \kappa_2 \rangle \langle \kappa_3 \rangle)^{-\gamma} \widehat{\chi}(\tilde{\kappa} - (\lambda_n^2 - \lambda_{n_1}^2))\|_{L_\kappa^2 L_{\kappa_1, \kappa_2, \kappa_3}^{q'} \ell_{n, n_1}^2}. \end{aligned}$$

We conclude from the divisor bound and Remark 8.4, which allows to sum $n, n_1 \lesssim N$ together, that for all $\epsilon > 0$,

$$\text{l.h.s (8.2)} \lesssim_\epsilon r^{\frac{1}{2}} (RT^{-\gamma+\frac{1}{q'}})^3 N^{-(\alpha-\frac{1}{2})+\epsilon} N_2^{1-2(\alpha-\frac{1}{2})+\frac{2}{q}+4\delta}.$$

This completes the proof of Proposition 8.1 when $\alpha \geq 1$. \square

9. DETERMINISTIC TRILINEAR ESTIMATES

In this section, we prove Proposition 5.10, hence finish the proof of our main result. We organize the terms to be estimated into following groups: Below, $v_{N_j}^\dagger \in \{\psi_{N_j}^\dagger, z_{N_j}^\dagger\}$:

- Group I: There is at least one type (D) term z_M^\dagger with $M \geq \frac{1}{100}N$.
- Group II: The type (C) term ψ_{2N}^\dagger appears on the first (or the third) position and are not in Group I. This group contains terms of the form

$$\mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, \psi_{2N}^\dagger, z_{N_3}^\dagger), \mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, z_{N_2}^\dagger, \psi_{2N}^\dagger), \mathcal{N}_{[0,1]}(\psi_{2N}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger),$$

where at least one of $v_{N_2}^\dagger, v_{N_3}^\dagger$ is of type (D). Note that we removed the singular interactions $\mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, v_{N_2}, v_{N_3})$ with $N_2, N_3 \leq N$ thanks to the ansatz.

- Group III: The type (C) term ψ_{2N}^\dagger appears on the second position and are not in Group I or Group II. This group contains terms of the form

$$\mathcal{N}(v_{N_1}^\dagger, \psi_{2N}^\dagger, v_{N_3}^\dagger),$$

$N_1, N_3 \leq N$, $v_{N_i}^\dagger$ is of type (C) or (D) with at least one of type (D).

Before estimating the interactions in each group, we prove an elementary Lemma useful to control terms of type (D). This Lemma exploit the fact that type (D) terms are not any function in $X^{s,b}$. Indeed, they satisfy a semiclassical energy estimate and are essentially oscillating at frequency N .

Lemma 9.1. *If Loc(N) is true then for all $M \leq N$ and for z_M^\dagger of type (D) and $\epsilon_0 > 0$,*

$$\|z_M^\dagger\|_{X^{s-\epsilon_0,b}} \lesssim R^{-1} M^{-\epsilon_0}.$$

Proof. A Littlewood–Paley decomposition gives

$$\begin{aligned} \|z_M^\dagger\|_{X^{s-\epsilon_0,b}} &\leq \sum_K K^{s-\epsilon_0} \|P_K z_M^\dagger\|_{X^{0,b}} \lesssim \sum_{K \leq M} K^{s-\epsilon_0} \|z_M^\dagger\|_{X^{0,b}} + R^{-1} \sum_{K \geq 2M} \left(\frac{M}{K}\right)^{10-s} K^{-\epsilon_0} \\ &\lesssim R^{-1} M^{-\epsilon_0}. \end{aligned}$$

This proves Lemma 9.1. \square

We now handle terms in Group I (which contains terms of type (D) with constructed at a generation close to N). The goal is to place this high-frequency term in $X^{0,b}$ (to spare derivatives). The proof involves only the semiclassical Strichartz estimate recalled in Proposition 4.4.

Lemma 9.2. *Suppose that $0 < \epsilon_0 < \frac{1}{2}(s - \frac{1}{2})$. There exists $C > 0$ such that for all function $v_{N_2}^\dagger, v_{N_3}^\dagger$ of type (C) or (D), and $b_1 > \frac{1}{2}$,*

$$\|\chi \mathcal{N}(z_{N_1}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger)\|_{X^{0,-b_1}} \leq C \|z_{N_1}^\dagger\|_{X^{0,b}} \prod_{i \in \{2,3\}} \min(\|v_{N_i}^\dagger\|_{X^{s-2\epsilon_0,b}}, \|v_{N_i}^\dagger\|_{L_t^q L_x^\infty}).$$

Similarly, for all functions $v_{N_1}^\dagger, v_{N_3}^\dagger$ of type (C) or (D),

$$\|\chi \mathcal{N}(v_{N_1}^\dagger, z_{N_2}^\dagger, v_{N_3}^\dagger)\|_{X^{0,-b_1}} \leq C \|z_{N_2}^\dagger\|_{X^{0,b}} \prod_{i \in \{1,3\}} \min(\|v_{N_i}^\dagger\|_{X^{s-2\epsilon_0,b}}, \|v_{N_i}^\dagger\|_{L_t^q L_x^\infty}).$$

Proof. By duality it suffices to show that for all $v \in X^{0,b_1}$ such that $\|v\|_{X^{0,b_1}} \leq 1$, there holds

$$\begin{aligned} & \left| \int_{\mathbb{R}} \int_{\mathbb{S}^2} \chi(t) \mathcal{N}(z_{N_1}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger)(t, x) \overline{v(t, x)} dt dx \right| \\ & \leq C \|z_{N_1}^\dagger\|_{X^{0,b}} \|v\|_{X^{0,b_1}} \prod_{i \in \{2,3\}} \min(\|v_{N_i}^\dagger\|_{X^{s-2\epsilon_0,b}}, \|v_{N_i}^\dagger\|_{L_t^q L_x^\infty}). \end{aligned}$$

Since the Wick renormalization plays no role in the analysis below, without loss of generality, we will replace the nonlinearity $\mathcal{N}(f_1, f_2, f_3)$ by the usual multiplication $f_1 \bar{f}_2 f_3$. By Hölder's inequality,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \int_{\mathbb{S}^2} \chi(t) \mathcal{N}(z_{N_1}^\dagger, v_{N_2}^\dagger, v_{N_3}^\dagger)(t, x) \overline{v(t, x)} dt dx \right| \\ & \leq \|z_{N_1}^\dagger\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2} \|v_{N_2}^\dagger\|_{L_t^{p_2} L_x^\infty} \|v_{N_3}^\dagger\|_{L_t^{p_3} L_x^\infty} \|\chi\|_{L_t^{1-\frac{1}{p_2}-\frac{1}{p_3}}}, \end{aligned}$$

where $p_2, p_3 \in (2, +\infty)$, to be chosen later, satisfy $\frac{1}{p_2} + \frac{1}{p_3} < 1$. By Lemma 4.2 we obtain

$$\|z_{N_1}^\dagger\|_{L_t^\infty L_x^2} \lesssim \|z_{N_1}^\dagger\|_{X^{0,b}}, \quad \|v\|_{L_t^\infty L_x^2} \lesssim \|v\|_{X^{0,b_1}}.$$

As for the terms v_{N_2} and v_{N_3} , of type (C) or (D), we proceed as follows. When v_{N_i} is of type (C), we take $p_i = q$ (we specified the parameters in (2.1)). When v_{N_i} is of type (D), we let (p_i, q_i) be an admissible pair and we apply the Sobolev embedding and the semi-classical Strichartz estimate from [7] recalled in Proposition 4.4:

$$\|v_{N_i}^\dagger\|_{L_t^{p_i} L_x^\infty} \lesssim \|v_{N_i}^\dagger\|_{X^{\frac{2}{q_i} + \frac{1}{p_i}, b}} \lesssim \|v_{N_i}^\dagger\|_{X^{1-\frac{1}{p_i}, b}},$$

where we used the admissibility condition $\frac{2}{p_i} + \frac{2}{q_i} = 1$. The desired bound for terms of type (D) follows by choosing $p_i > 2$ sufficiently close to 2, such that $1 - \frac{1}{p_i} < s - 2\epsilon_0$. This is possible thanks to the assumption $\epsilon_0 < \frac{1}{2}(s - \frac{1}{2})$. \square

To handle terms in Group II, we distinguish a special contribution where our analysis relies on the Wick ordering:

Lemma 9.3. *For all $v_{N_3}^\dagger$ of type (D), we have*

$$\|\chi\mathcal{N}_{(1,2)}(\psi_{2N}^\dagger, \psi_{2N}^\dagger, v_{N_3}^\dagger)\|_{X^{0,-b_1}} \lesssim R^2 N^{-2(\alpha-\frac{1}{2})+\frac{1}{2}+\frac{1}{q}+\delta} \|v_{N_3}^\dagger\|_{X^{0,b}}.$$

Proof. The proof is in the physical space, in the spirit of the proof of Lemma 9.2. Here, we place $\psi_{2N}^\dagger \ominus \psi_{2N}^\dagger$ in $L_t^q L_x^\infty$ and apply Lemma 7.3. \square

It remains to handle the other interactions in Group II. In the next Lemma we control the remaining interactions not covered by Lemma 9.2 and Lemma 9.3 (even if there might be some overlap in some cases).

Lemma 9.4 (Remaining interactions with ψ_{2N}^\dagger in first position). *Suppose that for $i \in \{2, 3\}$, $N_i \leq 2N$ and v_{N_i} is of type (C) or (D) with at least one of type (D). For all $\epsilon > 0$ and $\epsilon_0 > 0$ with $\epsilon_0 < \min(s - \frac{1}{2}, \alpha - 1)$, we have*

$$\|\chi\mathcal{N}_{[0,1][1,2]}(\psi_{2N}^\dagger, v_{N_2}, v_{N_3})\|_{X^{0,-b_1}} \lesssim_{\epsilon_0} RN^{-(\alpha-\frac{1}{2})+\epsilon} (N_2 N_3)^{-\epsilon_0}. \quad (9.1)$$

Moreover, if $N_3 \leq 2N$,

$$\|\chi\mathcal{N}_{(01)}(\psi_{2N}^\dagger, \psi_{2N}^\dagger, z_{N_3}^\dagger)\|_{X^{0,-b_1}} \lesssim_{\epsilon_0} RN^{-(\alpha-\frac{1}{2})-(\alpha-1)} N_3^{-(s-\frac{1}{2})+\epsilon_0}. \quad (9.2)$$

If $N_2 \leq 2N$,

$$\|\chi\mathcal{N}(\psi_{2N}^\dagger, z_{N_2}^\dagger, \psi_{2N}^\dagger)\|_{X^{0,-b_1}} \lesssim_{\epsilon_0} RN^{-(\alpha-\frac{1}{2})-(\alpha-1)} N_2^{-(s-\frac{1}{2})+\epsilon_0}. \quad (9.3)$$

Proof. Thanks to the high frequency assumption in $\text{Loc}(N)$ we can suppose that the terms of type (D) are truncated at frequency $4N$. Consequently the output frequency is less than $100N$. As in the proof of Lemma 9.2, it suffices to replace the nonlinearity $\mathcal{N}...$ by the usual multiplication, subjecting to the specific constraint.

We first prove (9.1).

• **Case 1:** (C)(C)(D). By duality and Cauchy–Schwarz in $x \in \mathbb{S}^2$,

$$\begin{aligned} & \|\chi\mathcal{N}_{[0,1][1,2]}(\psi_{2N}^\dagger, \psi_{N_2}^\dagger, \Pi_{2N_3} z_{N_3}^\dagger)\|_{X^{0,-b_1}} \\ & \lesssim \sum_{\substack{n_i \lesssim N \\ n_0 \neq n_1, n_1 \neq n_2}} (\lambda_{n_1} \lambda_{n_2})^{-(\alpha-\frac{1}{2})} \int_{\mathbb{R}^4} \frac{|\widehat{\chi}(\tilde{\kappa} - \Omega(\vec{n}))|}{\langle \kappa_0 \rangle^{b_1} \langle \kappa_1 \rangle^b \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^b} \prod_{i=0}^3 \mathbf{a}_{n_i}^{(i)}(\kappa_i) d\vec{\kappa}. \end{aligned} \quad (9.4)$$

Denote by

$$\begin{aligned} \mathbf{a}_{n_0}^{(0)}(\kappa_0) & := \langle \kappa_0 \rangle^{b_1} \|\pi_{n_0} \widehat{v}_{n_0}(\kappa_0 - \lambda_{n_0}^2)\|_{L_x^2}, & \mathbf{a}_{n_1}^{(1)}(\kappa_1) & := \langle \kappa_1 \rangle^\gamma \|\widehat{e}_{n_1}^{2N, \dagger}(\kappa_1 - \lambda_{n_1}^2)\|_{L_x^\infty}, \\ \mathbf{a}_{n_2}^{(2)}(\kappa_2) & := \langle \kappa_2 \rangle^\gamma \|\widehat{e}_{n_2}^{N_2, \dagger}(\kappa_2 - \lambda_{n_2}^2)\|_{L_x^\infty}, & \mathbf{a}_{n_3}^{(3)}(\kappa_3) & := \langle \kappa_3 \rangle^b \|\pi_{n_3} \widehat{z}_{N_3}^\dagger(\kappa_3 - \lambda_{n_3}^2)\|_{L_x^2}, \end{aligned}$$

where v is a function in X^{0,b_1} with $\|v\|_{X^{0,b_1}} \leq 1$. Under our assumptions, there holds

$$\|\mathbf{a}_{n_0}^{(0)}(\kappa_0)\|_{\ell_{n_0}^2 L_{\kappa_0}^2} = \|v\|_{X^{0,b_1}} \leq 1, \quad \|\mathbf{a}_{n_i}^{(i)}(\kappa_i)\|_{L_{\kappa_i}^q \ell_{n_i}^\infty} \lesssim R \quad \text{for } i \in \{1, 2\},$$

and for all $\epsilon > 0$, according to Lemma 9.1,

$$\|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{L_{\kappa_3}^2 \ell_{n_3}^1} \lesssim_\epsilon \|n_3^{\frac{1}{2}+\epsilon} \mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{L_{\kappa_3}^2 \ell_{n_3}^2} \lesssim_\epsilon \|z_{N_3}^\dagger\|_{X^{\frac{1}{2}+\epsilon, b}} \lesssim_\epsilon N_3^{-(s-\frac{1}{2})+\epsilon} R^{-1}. \quad (9.5)$$

For fixed n_2, n_3 and $\vec{\kappa}$,

$$\sum_{\substack{n_0, n_1 \lesssim N \\ n_0 \neq n_1}} |\widehat{\chi}(\vec{\kappa} - \Omega(\vec{n}))| \mathbf{a}_{n_1}^{(1)}(\kappa_1) \lambda_{n_1}^{-(\alpha-\frac{1}{2})} \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})+\epsilon} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty}.$$

Indeed, for fixed n_2 and n_3 , we have from the small divisor bound that

$$\begin{aligned} & \sum_{\substack{n_0 \approx 2N, n_1 \lesssim N \\ n_0 \neq n_1}} |\widehat{\chi}(\vec{\kappa} - \Omega(\vec{n}))| \mathbf{a}_{n_1}^{(1)}(\kappa_1) \lambda_{n_1}^{-(\alpha-\frac{1}{2})} \\ & \lesssim N^{-(\alpha-\frac{1}{2})} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty} \sum_m \sum_{\substack{n_0, n_1 \lesssim N \\ n_0 \neq n_1}} \mathbf{1}_{\lambda_{n_1}^2 - \lambda_{n_0}^2 = m} |\widehat{\chi}(\vec{\kappa} - \lambda_{n_3}^2 + \lambda_{n_2}^2 - m)| \\ & \lesssim N^{-(\alpha-\frac{1}{2})} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty} \sup_m \#\{n_0 \neq n_1 \lesssim N, \lambda_{n_1}^2 - \lambda_{n_0}^2 = m\} \\ & \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})+\epsilon} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty}. \end{aligned}$$

We deduce from the Schur's test that for fixed n_3 and $\vec{\kappa}$,

$$\begin{aligned} & \sum_{\substack{n_0, n_1, n_2 \lesssim N \\ n_0 \neq n_1, n_2 \neq n_1}} (\lambda_{n_1} \lambda_{n_2})^{-(\alpha-\frac{1}{2})} |\widehat{\chi}(\vec{\kappa} - \Omega(\vec{n}))| \mathbf{a}_{n_0}^{(0)}(\kappa_0) \mathbf{a}_{n_1}^{(1)}(\kappa_1) \mathbf{a}_{n_2}^{(2)}(\kappa_2) \\ & \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})+\epsilon} N_2^{\frac{1}{2}} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty} \|\mathbf{a}_{n_0}^{(0)}(\kappa_0)\|_{\ell_{n_0}^2} \|\lambda_{n_2}^{-(\alpha-\frac{1}{2})} \mathbf{a}_{n_2}^{(0)}(\kappa_2)\|_{\ell_{n_2}^2} \\ & \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})+\epsilon} N_2^{-(\alpha-1)} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{\ell_{n_1}^\infty} \|\mathbf{a}_{n_0}^{(0)}(\kappa_0)\|_{\ell_{n_0}^2} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^\infty}. \end{aligned}$$

Summing over n_3 in (9.4) and integrating over $\vec{\kappa}$, we conclude from Hölder's inequality that

$$\begin{aligned} |(9.4)| & \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})-\epsilon} N_2^{-(\alpha-1)} \|\mathbf{a}_{n_0}^{(0)}(\kappa_0)\|_{L_{\kappa_0}^2 \ell_{n_0}^2} \|\mathbf{a}_{n_1}^{(1)}(\kappa_1)\|_{L_{\kappa_1}^q \ell_{n_1}^\infty} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{L_{\kappa_2}^q \ell_{n_2}^\infty} \\ & \lesssim_\epsilon N^{-(\alpha-\frac{1}{2}-\epsilon)} N_2^{-(\alpha-1)} N_3^{-(s-\frac{1}{2})+\epsilon} R, \end{aligned}$$

which is conclusive.

• **Case 2:** (C)(D)(C). This case is analog to the previous one but inverting the role of n_2 and n_3 , by first fixing n_2 while summing over n_0, n_1, n_3 . When we apply the Schur's test, we apply the divisor bound

$$\sup_m \#\{n_1, n_3 \lesssim N \mid \lambda_{n_1}^2 + \lambda_{n_3}^2 = m\} \lesssim_\epsilon N^\epsilon.$$

• **Case 3:** (C)(D)(D). We do the same analysis as in Case 1, but replacing $\|\lambda_{n_2}^{-(\alpha-\frac{1}{2})} \mathbf{a}_{n_2}^{(0)}(\kappa_2)\|_{\ell_{n_2}^2}$ by

$$\|\langle \kappa_2 \rangle^b \pi_{n_2} \widehat{z_{N_2}^\dagger}(\kappa_2 - \lambda_{n_2}^2)\|_{L_{\vec{\kappa}}^2 \ell_x^2 L_x^\infty} \lesssim_\epsilon \|z_{N_2}^\dagger\|_{X^{\frac{1}{2}+\epsilon, b}} \lesssim N_2^{-(s-\frac{1}{2})+\epsilon} R^{-1},$$

where we used the Sobolev embedding and applied Lemma 9.1 to obtain the last bound.

We now turn to the proof of (9.2). In this case, we have the analog of (9.4):

$$\begin{aligned} & \|\chi \mathcal{N}_{(0,1)}(\psi_{2N}^\dagger, \psi_{2N}^\dagger, \Pi_{2N_3} z_{N_3}^\dagger)\|_{X^{0,-b_1}} \\ & \lesssim \sum_{\substack{n, n_2, n_3 \\ n, n_2 \approx N}} (\lambda_n \lambda_{n_2})^{-(\alpha-\frac{1}{2})} \int_{\mathbb{R}^4} \frac{|\widehat{\chi}(\vec{\kappa} - \Omega(\vec{n}))|}{\langle \kappa_0 \rangle^{b_1} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^b} \mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) \mathbf{a}_{n_2}^{(2)}(\kappa_2) \mathbf{a}_{n_3}^{(3)}(\kappa_3) d\vec{\kappa}, \end{aligned} \quad (9.6)$$

where the $\mathbf{a}^{(i)}$ for $i \in \{0, 1, 2, 3\}$ are as in (9.4). First, we fix n, n_3 and $\vec{\kappa}$ to sum over n_2 :

$$\sum_{n_2 \approx N} \lambda_{n_2}^{-(\alpha-\frac{1}{2})} \mathbf{a}_{n_2}^{(2)}(\kappa_2) |\widehat{\chi}(\vec{\kappa} - (\lambda_{n_1}^2 - \lambda_{n_2}^2 + \lambda_{n_3}^2 - \lambda_{n_0}^2))| \lesssim N^{-(\alpha-\frac{1}{2})} \|\mathbf{a}_{n_2}^{(2)}(\kappa_2)\|_{\ell_{n_2}^\infty}.$$

By Cauchy-Schwarz, summing over n gives

$$\sum_{n \approx N} \lambda_n^{-(\alpha-\frac{1}{2})} \mathbf{a}_n^{(0)}(\kappa_0) \mathbf{a}_n^{(1)}(\kappa_1) \lesssim N^{-(\alpha-1)} \|\mathbf{a}_n^{(0)}\|_{\ell_n^2} \|\mathbf{a}_n^{(1)}\|_{\ell_n^\infty}.$$

We conclude by integrating over $\vec{\kappa}$ using Hölder, and summing over n_3 using (9.5) to finally obtain (9.2). Since the proof of (9.3) is similar, we omit it. This finishes the proof of Lemma 9.4 \square

We now handle the interactions when ψ_{2N}^\dagger is at the second position and the other terms are of generation $\leq N$.

Lemma 9.5 (Remaining interactions with ψ_{2N}^\dagger in second position). *Suppose that for $i \in \{1, 3\}$, $N_i \leq N$ and v_{N_i} is of type (C) or (D) with at least one of type (D). Moreover, for type (D) terms we suppose that $N_i \leq \frac{1}{50}N$. For all $b_1 > \frac{1}{2}$, $\epsilon > 0$ and $\epsilon_0 > 0$ with $\epsilon_0 < \min(s - \frac{1}{2}, \alpha - 1)$, we have*

$$\|\chi \mathcal{N}(v_{N_1}, \psi_{2N}^\dagger, v_{N_3})\|_{X^{0,-b_1}} \lesssim_\epsilon N^{-(\alpha-\frac{1}{2})+\epsilon} (N_1 N_3)^{-\epsilon_0}. \quad (9.7)$$

Proof. The proof is similar to the proof of Lemma 9.4. We reduce to the case when the terms of type are truncated at frequency $4N$ and the output frequency is less than $100N$. As in the proof of Lemma 9.2, it suffices to replace the nonlinearity $\mathcal{N}...$ by the usual multiplication, subjecting to the specific constraint.

• **Case 1:** (C)(C)(D). By duality and Cauchy-Schwarz in $x \in \mathbb{S}^2$,

$$\begin{aligned} & \|\chi \mathcal{N}(\psi_{N_1}^\dagger, \psi_{2N}^\dagger, \Pi_{2N_3} z_{N_3}^\dagger)\|_{X^{0,-b_1}} \\ & \lesssim \sum_{\substack{n_0 \lesssim N, n_1 \approx N_1, n_2 \approx 2N, n_3 \lesssim N_3}} (\lambda_{n_1} \lambda_{n_2})^{-(\alpha-\frac{1}{2})} \int_{\mathbb{R}^4} \frac{|\widehat{\chi}(\vec{\kappa} - \Omega(\vec{n}))|}{\langle \kappa_0 \rangle^{b_1} \langle \kappa_1 \rangle^\gamma \langle \kappa_2 \rangle^\gamma \langle \kappa_3 \rangle^b} \prod_{i=0}^3 \mathbf{a}_{n_i}^{(i)}(\kappa_i) d\vec{\kappa}, \end{aligned} \quad (9.8)$$

where

$$\begin{aligned} \mathbf{a}_{n_0}^{(0)}(\kappa_0) & := \langle \kappa_0 \rangle^{b_1} \|\pi_{n_0} \widehat{v}_{n_0}(\kappa_0 - \lambda_{n_0}^2)\|_{L_x^2}, & \mathbf{a}_{n_1}^{(1)}(\kappa_1) & := \langle \kappa_1 \rangle^\gamma \|e_{n_1}^{\widehat{N_1, \dagger}}(\kappa_1 - \lambda_{n_1}^2)\|_{L_x^\infty}, \\ \mathbf{a}_{n_2}^{(2)}(\kappa_2) & := \langle \kappa_2 \rangle^\gamma \|e_{n_2}^{\widehat{2N, \dagger}}(\kappa_2 - \lambda_{n_2}^2)\|_{L_x^\infty}, & \mathbf{a}_{n_3}^{(3)}(\kappa_3) & := \langle \kappa_3 \rangle^b \|\pi_{n_3} z_{N_3}^\dagger(\kappa_3 - \lambda_{n_3}^2)\|_{L_x^2}, \end{aligned}$$

where v is a function in X^{0,b_1} with $\|v\|_{X^{0,b_1}} \leq 1$. Under our assumptions, we have that for all $\epsilon > 0$

$$\begin{aligned} \|\mathbf{a}_{n_0}^{(0)}(\kappa_0)\|_{\ell_{n_0}^2 L_{\kappa_0}^2} = \|v\|_{X^{0,b_1}} \leq 1, \quad \|\mathbf{a}_{n_i}^{(i)}(\kappa_i)\|_{L_{\kappa_i}^q \ell_{n_i}^\infty} \lesssim R \quad \text{for } i \in \{1, 2\}, \\ \|\mathbf{a}_{n_3}^{(3)}(\kappa_3)\|_{L_{\kappa_3}^2 \ell_{n_3}^1} \lesssim_\epsilon N_3^{-(s-\frac{1}{2})+\epsilon} R^{-1}, \end{aligned}$$

where the last estimate was proved in (9.5). We now fix n_3 and $\vec{\kappa}$ and we sum over n_0, n_1, n_2 in the same spirit as in the proof of Lemma 9.4, but in this case we will sum over n_0, n_1 using Schur's test and avoid to sum over n_2 by using the modulation. The assumption $N_1 \leq 2N$ ensures the non-pairing condition $n_1 \neq n_2$. Moreover, we use the divisor bound

$$\sup_m \#\{n_0, n_2 \lesssim N \mid \lambda_{n_0}^2 + \lambda_{n_2}^2 = m\} \lesssim_\epsilon N^\epsilon.$$

• **Case 2:** (D)(C)(D). In this case we assume that $N_1 \leq \frac{1}{50}N$ to ensure the non-pairing condition $n_1 \neq n_2$ (otherwise this case falls into the scope of Group I and can be bounded by Lemma 9.2). The rest of the analysis is as in the previous cases and we do not give further details. \square

We are now ready to prove the trilinear deterministic estimates claimed in Proposition 5.10.

Proof of Proposition 5.10. Cases when there is a term z_{2N}^\dagger follows from Lemma 9.2, Lemma 9.1 and an interpolation argument (using [14, Proposition 1.2.1] for instance, in order to have a crude estimate for the $X^{0,0}$ -norm but losing small powers of N_2 only).

The other cases are source terms, so we can afford to lose a small power of N in the interpolation. These terms are covered by Lemma 9.3, 9.4 and 9.5 (always in the case $\alpha > 1$), where we collected before the statement of Proposition 5.9 the different configurations which contribute to the equation (5.12) satisfied by w_{2N}^\dagger . \square

APPENDIX A. BANACH-VALUED FOURIER-LEBESGUE FUNCTION SPACES

Let \mathcal{X} be a Banach space and $\mathbf{F} : \mathbb{R} \rightarrow \mathcal{X}$ a Schwartz function such that

$$\|\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} := \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L^q(\mathbb{R}_\tau; \mathcal{X})} < \infty. \quad (\text{A.1})$$

The space $\mathcal{FL}_q^\gamma(\mathcal{X})$ is the closure of \mathcal{X} -valued Schwartz functions $\mathcal{S}(\mathbb{R}; \mathcal{X})$ with respect to the topology defined via the norm above. We will state and prove estimates for a general Banach space \mathcal{X} in this appendix, but one should keep in mind that in our application, $\mathcal{X} = L^p(E_n)$ for some $1 \leq p \leq \infty$ which is a finite (but large as $n \rightarrow \infty$) dimensional space. Indeed, this section concerns only *estimates* for regular objects in general functional spaces.

A.1. Time localization properties.

Proposition A.1. *Let \mathcal{X} be a Banach space, $1 < q < \infty$, $\gamma \in \mathbb{R}$, and $\varphi \in \mathcal{S}(\mathbb{R})$.*

(1) *For any $\mathbf{F} \in \mathcal{FL}_q^\gamma$,*

$$\|\varphi(t)\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim \|\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})}.$$

(2) *Denote by $\varphi_T(t) = \varphi(T^{-1}t)$ for $0 < T \leq 1$. For $0 \leq \gamma \leq \gamma_1 < 1$ and $0 < T \leq 1$, we have*

$$\|\varphi_T(t)\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma_1}(\mathcal{X})}, \text{ if } \gamma_1 < \frac{1}{q'}$$

and

$$\|\varphi_T(t)\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \log\left(1 + \frac{1}{T}\right) \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma_1}(\mathcal{X})}, \text{ if } \gamma_1 = \frac{1}{q'},$$

and

$$\|\varphi_T(t)\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\frac{1}{q'} - \gamma} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma_1}(\mathcal{X})}, \text{ if } \gamma_1 > \frac{1}{q'},$$

(3) *If in addition, $\mathbf{F}(0) = 0$, for any $0 < \gamma \leq \gamma_1 < 1$ such that $\gamma_1 > \frac{1}{q'}$, and $0 < T \leq 1$, we have*

$$\|\varphi_T(t)\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma_1}(\mathcal{X})}.$$

Remark A.2. *It will be clear from the proof that all implicit bounds in Proposition A.1 can be chosen to depend on*

$$\|\varphi\|_{W_t^{10,1}(\mathbb{R})} := \sum_{k=0}^{10} \|\varphi^{(k)}\|_{L^1(\mathbb{R})}.$$

Proof. (1) By Fourier inversion formula,

$$\varphi(t)\mathbf{F}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\eta)\mathbf{F}(t)e^{it\eta} d\eta.$$

Since $\langle \tau \rangle^\gamma \lesssim \langle \eta \rangle^{|\gamma|} \langle \tau - \eta \rangle^\gamma$, we deduce that

$$\|\varphi(t)\mathbf{F}(t)\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim \int_{\mathbb{R}} |\widehat{\varphi}(\eta)| \langle \eta \rangle^{|\gamma|} \|\langle \tau - \eta \rangle^\gamma \widehat{\mathbf{F}}(\tau - \eta)\|_{L_\tau^q(\mathcal{X})} d\eta \leq \|\varphi\|_{\mathcal{FL}_1^{|\gamma|}(\mathbb{R})} \|\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})}.$$

Since $\varphi \in \mathcal{S}(\mathbb{R})$, we obtain the first inequality.

(2) Now let us prove the second inequality with time cutoff $\varphi_T(t)$. First we consider the case $0 \leq \gamma \leq \gamma_1 < 1$.

We split $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where

$$\widehat{\mathbf{F}}_1(\tau) = \mathbf{1}_{|\tau| \geq \frac{1}{T}} \widehat{\mathbf{F}}(\tau), \quad \widehat{\mathbf{F}}_2(\tau) = \mathbf{1}_{|\tau| < \frac{1}{T}} \widehat{\mathbf{F}}(\tau),$$

where $\widehat{\mathbf{F}}(\tau, \cdot)$ is the time-Fourier transform of \mathbf{F} . It suffices to show that

$$(a) \quad \|\langle \tau \rangle^\gamma (\widehat{\varphi}_T * \widehat{\mathbf{F}}_1)(\tau)\|_{L_\tau^q(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \|\langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta)\|_{L_\eta^q(\mathcal{X})}; \quad (\text{A.2})$$

$$(b) \quad \|\langle \tau \rangle^\gamma (\widehat{\varphi}_T * \widehat{\mathbf{F}}_2)(\tau)\|_{L_\tau^q(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \|\langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta)\|_{L_\eta^q(\mathcal{X})}. \quad (\text{A.3})$$

To prove (a), we note that

$$\begin{aligned} \|\langle \tau \rangle^\gamma (\widehat{\varphi}_T * \widehat{\mathbf{F}}_1)(\tau)\|_{\mathcal{X}} &= \left\| \int_{\mathbb{R}} K_T^+(\tau, \eta) \langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \\ &\leq \int_{\mathbb{R}} |K_T^+(\tau, \eta)| \langle \eta \rangle^{\gamma_1} \|\widehat{\mathbf{F}}(\eta)\|_{\mathcal{X}} d\eta, \end{aligned}$$

where

$$K_T^+(\lambda, \eta) = \widehat{\varphi}(T(\lambda - \eta)) \frac{T \langle \lambda \rangle^\gamma}{\langle \eta \rangle^{\gamma_1}} \mathbf{1}_{|\eta| \geq \frac{1}{T}}.$$

By Schur's test, it suffices to show that

$$\sup_{\lambda \in \mathbb{R}} \int_{\mathbb{R}} |K_T^+(\lambda, \eta)| d\eta \leq C_1 T^{\gamma_1 - \gamma}, \quad \sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}} |K_T^+(\lambda, \eta)| d\lambda \leq C_2 T^{\gamma_1 - \gamma}$$

with C_1, C_2 independent of T . One can check these two inequalities by direct computation, here we explain it in an informal way. Since $\widehat{\varphi}$ is a Schwartz function, $|\lambda - \eta|$ is essentially bounded by $O(\frac{1}{T})$. Due to the fact that $|\eta| \geq \frac{1}{T}$, λ is essentially constraint in the region $|\lambda| \lesssim \frac{1}{T}$. Since the length of the integration is of size $\frac{1}{T}$, we deduce that two integrations are bounded by $O(1)T^{\gamma_1 - \gamma}$. We note that the (a) holds for $0 \leq \gamma \leq \gamma_1 < 1$.

Next we prove (b). By splitting $\widehat{\varphi}_T(\lambda - \eta) = [\widehat{\varphi}_T(\lambda - \eta) - \widehat{\varphi}_T(\lambda)] + \widehat{\varphi}_T(\lambda)$, we have

$$\begin{aligned} \|(\widehat{\varphi}_T * \widehat{\mathbf{F}}_2)(\lambda)\|_{\mathcal{X}} &= \left\| \int_{|\eta| < \frac{1}{T}} \widehat{\varphi}_T(\lambda - \eta) \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \\ &\leq \left\| \int_{|\eta| < \frac{1}{T}} [\widehat{\varphi}_T(\lambda - \eta) - \widehat{\varphi}_T(\lambda)] \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} + |\widehat{\varphi}_T(\lambda)| \left\| \int_{|\eta| < \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}}. \end{aligned} \quad (\text{A.4})$$

Since $\widehat{\varphi}_T(\lambda) = T \widehat{\varphi}(T\lambda)$, we have

$$[\widehat{\varphi}_T(\lambda - \eta) - \widehat{\varphi}_T(\lambda)] \mathbf{1}_{|\eta| < 1} = T \cdot O(|T\eta| \langle T\lambda \rangle^{-100}).$$

By Hölder, we deduce that

$$\begin{aligned} &\left| \int_{|\eta| < \frac{1}{T}} \|\widehat{\mathbf{F}}(\eta)\|_{\mathcal{X}} [\widehat{\varphi}_T(\lambda - \eta) - \widehat{\varphi}_T(\lambda)] d\eta \right| \\ &\leq T^2 \langle T\lambda \rangle^{-100} \|\langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta)\|_{L_q^q(\mathcal{X})} \left\| \frac{|\eta|}{\langle \eta \rangle^{\gamma_1}} \right\|_{L_{q'}(\eta \leq \frac{1}{T})} \\ &\lesssim T^{\gamma_1 + 1 - \frac{1}{q'}} \langle T\lambda \rangle^{-100} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})}. \end{aligned}$$

Multiplying by $\langle \lambda \rangle^\gamma$ and taking the L_λ^q norm of the above quantity, we obtain the bound

$$\left\| \langle \lambda \rangle^\gamma \int_{|\eta| < \frac{1}{T}} [\widehat{\varphi}_T(\lambda - \eta) - \widehat{\varphi}_T(\lambda)] \widehat{\mathbf{F}}(\eta) d\eta \right\|_{L_\lambda^q(\mathcal{X})} \lesssim T^{\gamma_1 - \gamma} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})}.$$

We note that there is no constraint about $0 \leq \gamma \leq \gamma_1 < 1$ for this contribution.

Finally,

$$\left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\| \left\| \int_{|\eta| < \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \Big\|_{L_\lambda^q} \leq \left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\|_{L_\lambda^q} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})} \left\| \frac{1}{\langle \eta \rangle^{\gamma_1}} \mathbf{1}_{|\eta| < \frac{1}{T}} \right\|_{L_\eta^{q'}}. \quad (\text{A.5})$$

Since for $0 < T \leq 1$, $\langle \lambda \rangle^\gamma \leq T^{-\gamma} \langle T\lambda \rangle^\gamma$, we have $\left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\|_{L_\lambda^q} \lesssim T^{\frac{1}{q'} - \gamma}$. If $0 \leq \gamma \leq \gamma_1 < \frac{1}{q}$, $\left\| \langle \eta \rangle^{-\gamma_1} \mathbf{1}_{|\eta| < T^{-1}} \right\|_{L_\eta^{q'}} \lesssim T^{\gamma_1 - \frac{1}{q'}}$, we obtain that

$$\left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\| \left\| \int_{|\eta| < \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \Big\|_{L_\lambda^q} \lesssim T^{\gamma_1 - \gamma} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})}.$$

When $\gamma_1 = \frac{1}{q}$, we get an extra factor $\log\left(1 + \frac{1}{T}\right)$. When $\gamma_1 > \frac{1}{q}$ and $0 \leq \gamma \leq \gamma_1$, we only get the bound $T^{\frac{1}{q'} - \gamma} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})}$.

(3) If in addition $\mathbf{F}(0) = 0$, we are able to improve the estimate (b) of (A.2). This is an analogue of Proposition 2.7 of [15].

From the argument in (2), we realize that it suffices to improve the second term on the right hand side of (A.4):

$$\left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\| \left\| \int_{|\eta| < \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \Big\|_{L_\lambda^q} \lesssim T^{\gamma_1 - \gamma} \|\mathbf{F}\|_{\mathcal{F}L_q^{\gamma_1}(\mathcal{X})} \quad (\text{A.6})$$

in the range $\gamma_1 > \frac{1}{q}$. The point is to exploit the cancellation from the condition $\mathbf{F}(0) = 0$. By the Fourier inversion formula, we have

$$\int_{\mathbb{R}} \widehat{\mathbf{F}}(\eta) d\eta = 0 = \int_{|\eta| < \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta + \int_{|\eta| \geq \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta. \quad (\text{A.7})$$

Therefore, the left hand side of (A.6) is equal to

$$\left\| \langle \lambda \rangle^\gamma T \widehat{\varphi}(T\lambda) \right\| \left\| \int_{|\eta| \geq \frac{1}{T}} \widehat{\mathbf{F}}(\eta) d\eta \right\|_{\mathcal{X}} \Big\|_{L_\lambda^q}.$$

By Hölder's inequality and the fact that $\gamma_1 > \frac{1}{q}$,

$$\int_{|\eta| \geq \frac{1}{T}} \|\widehat{\mathbf{F}}(\eta)\|_{\mathcal{X}} d\eta \leq \|\langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta)\|_{L_\eta^q(\mathcal{X})} \cdot T^{\gamma_1 - \frac{1}{q}},$$

hence (using $\langle \lambda \rangle \leq T^{-1} \langle T\lambda \rangle$ if $0 < T \leq 1$)

$$\left\| \langle \lambda \rangle^\gamma T \widehat{\varphi}(T\lambda) \right\| \left\| \int_{|\eta| \geq \frac{1}{T}} \|\widehat{\mathbf{F}}(\eta)\|_{\mathcal{X}} d\eta \right\|_{L_\lambda^q} \lesssim T^{\gamma_1 - \gamma} \|\langle \eta \rangle^{\gamma_1} \widehat{\mathbf{F}}(\eta)\|_{L_\eta^q(\mathcal{X})}.$$

This completes the proof of Proposition A.1. \square

Remark A.3. Note that on the right hand side of (A.5), when $\gamma_1 > \frac{1}{q}$, the function $\langle \eta \rangle^{-\gamma_1}$ belongs to $L^{q'}$, and we do not gain positive power in T when integrating the region $|\eta| \leq \frac{1}{T}$ to compensate the power $T^{\frac{1}{q'} - \gamma}$ from $\left\| \langle \lambda \rangle^\gamma \widehat{\varphi}_T(\lambda) \right\|_{L_\lambda^q}$. By applying vanishing property, finally we are allowed to integrate $\langle \eta \rangle^{-\gamma_1}$ from the region $|\eta| > \frac{1}{T}$, and this explains the gain.

Corollary A.4. *Let $1 \leq p, r \leq \infty$, and $1 \leq q < \infty$, $\frac{1}{q'} < \gamma \leq \gamma_1 < 1$. For any $u \in X_{p,q,r}^{\gamma_1}$ such that $u(t=0, \cdot) = 0$, we have*

$$\|\chi_T(t)u\|_{X_{p,q,r}^{s,\gamma}} \lesssim T^{\gamma_1-\gamma} \|u\|_{X_{p,q,r}^{s,\gamma_1}}$$

for all $0 < T < 1$.

Proof. It suffices to prove that for $u = \pi_n u \in E_n$ such that $u|_{t=0} = 0$, the following inequality:

$$\|\chi_T(t)u\|_{X_{q,r}^{0,\gamma}(E_n)} \lesssim T^{\gamma_1-\gamma} \|u\|_{X_{q,r}^{0,\gamma_1}(E_n)}. \quad (\text{A.8})$$

Note that with $v_n(t) = e^{i\lambda_n^2 t} u(t)$, this is equivalent to

$$\|\chi_T(t)v_n(t)\|_{\mathcal{F}L_q^\gamma(L^r(E_n))} \lesssim T^{\gamma_1-\gamma} \|v_n\|_{\mathcal{F}L_q^{\gamma_1}(L^r(E_n))},$$

the consequence of (3) of Proposition A.1. \square

A.2. Embedding properties.

Lemma A.5. *Let \mathcal{X} be a Banach space, $1 < q < \infty$ and $\gamma \in (\frac{1}{q'}, 1)$. Then for any $\mathbf{F} \in C_c^\infty(\mathbb{R}; \mathcal{X})$, $\mathbf{F} \in C^\alpha(\mathbb{R}; \mathcal{X})$ with $0 < \alpha \leq \gamma - \frac{1}{q'}$. More precisely,*

$$\|\mathbf{F}\|_{C^\alpha(\mathbb{R}; \mathcal{X})} \lesssim \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{\mathcal{F}L_q^\gamma(\mathcal{X})},$$

where the implicit constant is uniform.

Proof. Pick $t, t' \in \mathbb{R}$, $0 < |t - t'| < 1$. By the inverse Fourier transform for vector-valued function $f \in C_c^\infty(\mathbb{R}; \mathcal{X})$,

$$\begin{aligned} \mathbf{F}(t) - \mathbf{F}(t') &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\mathbf{F}}(\tau) e^{it'\tau} (e^{i(t-t')\tau} - 1) d\tau \\ &= \frac{1}{2\pi} \left(\int_{|\tau| \leq \frac{1}{|t-t'|}} + \int_{|\tau| > \frac{1}{|t-t'|}} \right) \langle \tau \rangle^{-\gamma} \cdot \langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau) e^{it'\tau} (e^{i(t-t')\tau} - 1) d\tau. \end{aligned}$$

For the contribution $|\tau| > \frac{1}{|t-t'|}$, by Hölder,

$$\begin{aligned} \left\| \int_{|\tau| > \frac{1}{|t-t'|}} \langle \tau \rangle^{-\gamma} \cdot \langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau) e^{it'\tau} (e^{i(t-t')\tau} - 1) d\tau \right\|_{\mathcal{X}} &\leq 2 \|\langle \tau \rangle^{-\gamma} \mathbf{1}_{|\tau| |t-t'| > 1}\|_{L_{\tau'}^{q'}} \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L_{\tau'}^q \mathcal{X}} \\ &\lesssim |t - t'|^{\gamma - \frac{1}{q'}} \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L^q(\mathbb{R}; \mathcal{X})}. \end{aligned}$$

For the contribution $|\tau| \leq \frac{1}{|t-t'|}$, from $|e^{i(t-t')\tau} - 1| \leq |t - t'| |\tau|$, we have

$$\begin{aligned} \left\| \int_{|\tau| \leq \frac{1}{|t-t'|}} \langle \tau \rangle^{-\gamma} \cdot \langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau) e^{it'\tau} (e^{i(t-t')\tau} - 1) d\tau \right\|_{\mathcal{X}} &\leq |t - t'| \|\langle \tau \rangle^{1-\gamma} \mathbf{1}_{|\tau| |t-t'| \leq 1}\|_{L_{\tau'}^{q'}} \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L_{\tau'}^q \mathcal{X}} \\ &\lesssim |t - t'|^{\gamma - \frac{1}{q'}} \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L^q(\mathbb{R}; \mathcal{X})}. \end{aligned}$$

The boundedness of $\|\mathbf{F}(t)\|_{\mathcal{X}}$ follows similarly, and we omit the detail. The proof of Lemma A.5 is complete. \square

Lemma A.6 (Vector-valued inequality: Theorem 4.5.1 of [19]). *Let \mathcal{H} be a separable Hilbert space and $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a linear operator that is bounded from $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for some $1 < p, r < \infty$. Then T can be extended as a bounded linear operator from $L^p(\mathbb{R}^d; \mathcal{H})$ to $L^r(\mathbb{R}^d; \mathcal{H})$, denoted by \mathbf{T} , such that*

$$\|\mathbf{T}(\mathbf{F})\|_{L^r(\mathbb{R}^d; \mathcal{H})} \leq C_{p,r} \|T\|_{L^p(\mathbb{R}) \rightarrow L^r(\mathbb{R})} \|\mathbf{F}\|_{L^p(\mathbb{R}^d; \mathcal{H})}.$$

In particular, the Fourier transform extends to the space $L^2(\mathbb{R}; \mathcal{H})$.

Proof. The proof can be found in the book [19]. Here we provide the proof for the case $r \geq p$ (which is enough for our need) by using the probabilistic method. Let $(\mathbf{h}_j)_{j \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} . Then any $\mathbf{F} \in \mathcal{S}(\mathbb{R}; \mathcal{H})$ can be represented as $\mathbf{F} = \sum_{j \geq 1} F_j \mathbf{h}_j$, where $F_j \in \mathcal{S}(\mathbb{R}^d)$. It suffices to show that for any finite number m ,

$$\left\| \left(\sum_{j=1}^m |TF_j|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbb{R})} \leq C_{p,r} \|T\|_{L^p \rightarrow L^r} \left\| \left(\sum_{j=1}^m |F_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \quad (\text{A.9})$$

Once this inequality is proved, a density argument allows us to conclude. Consider a sequence of i.i.d. random variables $(\epsilon_j^\omega)_{j \in \mathbb{N}}$, such that

$$\mathbb{P}[\epsilon_j^\omega = +1] = \mathbb{P}[\epsilon_j^\omega = -1] = \frac{1}{2}.$$

Consider the random function $F^\omega := \sum_{j=1}^m \epsilon_j^\omega F_j$. For each fixed event ω , from the boundedness of T , we have

$$\|TF^\omega\|_{L^r(\mathbb{R}^d)} \leq \|T\|_{L^p \rightarrow L^r} \|F^\omega\|_{L^p(\mathbb{R}^d)}.$$

Since $r \geq p$, by taking the L_ω^r norm on both sides and using Minkowski, we have

$$\|TF^\omega\|_{L^r(\mathbb{R}^d; L_\omega^r)} \leq \|T\|_{L^p \rightarrow L^r} \|F^\omega\|_{L^p(\mathbb{R}^d; L_\omega^r)}. \quad (\text{A.10})$$

By the Khintchine inequality, for fixed $x \in \mathbb{R}^d$,

$$\left\| \sum_{j=1}^m \epsilon_j^\omega TF_j(x) \right\|_{L_\omega^r} \sim_r \left(\sum_{j=1}^m |F_j(x)|^2 \right)^{\frac{1}{2}}, \quad \left\| \sum_{j=1}^m \epsilon_j^\omega TF_j(x) \right\|_{L_\omega^r} \sim_r \left(\sum_{j=1}^m |F_j(x)|^2 \right)^{\frac{1}{2}}.$$

Plugging into (A.10), we obtain (A.9). \square

Corollary A.7 (Hausdorff-Young). *Let \mathcal{H} be a Hilbert space and $\mathbf{F} \in L^{p'}(\mathbb{R}; \mathcal{H})$ for some $p \geq 2$. Then $\mathbf{F} \in \mathcal{F}L_p^0(\mathcal{H})$ and*

$$\|\mathbf{F}\|_{\mathcal{F}L_p^0(\mathcal{H})} \lesssim_p \|\mathbf{F}\|_{L^{p'}(\mathbb{R}; \mathcal{H})}.$$

Proof. First, if $\mathbf{F} \in L^1(\mathbb{R}; \mathcal{H})$, by the triangle inequality, we get

$$\|\widehat{\mathbf{F}}(\tau)\|_{\mathcal{H}} \leq \int_{\mathbb{R}} \|\mathbf{F}(t)\|_{\mathcal{H}} dt = \|\mathbf{F}\|_{L^1(\mathbb{R}; \mathcal{H})}.$$

If $2 \leq p < \infty$, by Hausdorff-Young and Lemma A.6, the Fourier transform \mathcal{F} extends to $\mathbf{F} \in L^{p'}(\mathbb{R}; \mathcal{H})$, and the desired inequality holds. \square

A.3. Inhomogeneous linear estimates.

Lemma A.8. *Let \mathcal{X} be a Banach space and $\mathbf{F} : \mathbb{R} \rightarrow \mathcal{X}$ a Schwartz function such that*

$$\|\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} := \|\langle \tau \rangle^\gamma \widehat{\mathbf{F}}(\tau)\|_{L^q(\mathbb{R}; \mathcal{X})} < \infty.$$

Then for any $\chi \in C_c^\infty(\mathbb{R})$, $\chi_T(t) := \chi(T^{-1}t)$, $0 < T \leq 1$, $\frac{1}{q'} < \gamma \leq 1$,

$$\|\chi_T(t)\mathcal{I}\mathbf{F}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})},$$

where the implicit constant does not depend on $T > 0$.

Proof. We follow the argument in [18]. Write

$$\begin{aligned} \chi_T(t)\mathcal{I}\mathbf{F}(t) &= \frac{\chi_T(t)}{2\pi} \int_0^t dt' \int_{\mathbb{R}} \widehat{\mathbf{F}}(\tau) e^{it'\tau} d\tau = \frac{\chi_T(t)}{2\pi} \int_{\mathbb{R}} \widehat{\mathbf{F}}(\tau) \frac{e^{it\tau} - 1}{i\tau} d\tau \\ &= \frac{\chi_T(t)}{2\pi} \underbrace{\sum_{k \geq 0} \frac{t^k}{k!} \int_{T|\tau| \leq 1} (i\tau)^{k-1} \widehat{\mathbf{F}}(\tau) d\tau}_{\text{I}} - \underbrace{\frac{\chi_T(t)}{2\pi} \int_{T|\tau| > 1} (i\tau)^{-1} \widehat{\mathbf{F}}(\tau) d\tau}_{\text{II}} \\ &\quad + \underbrace{\frac{\chi_T(t)}{2\pi} \int_{T|\tau| > 1} e^{it\tau} (i\tau)^{-1} \widehat{\mathbf{F}}(\tau) d\tau}_{\text{III}}. \end{aligned}$$

Since

$$\begin{aligned} \int_{T|\tau| \leq 1} |(i\tau)^{k-1}| \|\widehat{\mathbf{F}}(\tau)\|_{\mathcal{X}} d\tau &\leq T^{-(k-1)} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})} \left(\int_{|\tau| \leq 1} \langle \tau \rangle^{(1-\gamma)q'} d\tau \right)^{\frac{1}{q'}} \\ &\leq CT^{-(k-\gamma+\frac{1}{q'})} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}, \text{ provided that } \gamma \leq 1, \end{aligned}$$

and

$$\begin{aligned} \int_{T|\tau| > 1} |\tau|^{-1} \|\widehat{\mathbf{F}}(\tau)\|_{\mathcal{X}} d\tau &\leq \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})} \left(\int_{T|\tau| > 1} \langle \tau \rangle^{-q'\gamma} d\tau \right)^{\frac{1}{q'}} \\ &\leq CT^{\gamma-\frac{1}{q'}} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}, \text{ provided that } \gamma > \frac{1}{q'}. \end{aligned}$$

the integrals in I, II are fixed elements of \mathcal{X} . Then

$$\begin{aligned} \|\text{I}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} &\leq C \sum_{k \geq 0} \frac{\|\chi_T(t)t^k\|_{\mathcal{FL}_q^\gamma(\mathbb{R})}}{k!} \left\| \int_{T|\tau| \leq 1} (i\tau)^{k-1} \widehat{\mathbf{F}}(\tau) d\tau \right\|_{\mathcal{X}} \\ &\leq C \sum_{k \geq 0} \frac{k^2 T^{k-\gamma+\frac{1}{q'}}}{k!} \cdot T^{-(k-\gamma+\frac{1}{q'})} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})} \leq C \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}, \end{aligned}$$

where we use the fact that

$$\|\chi_T(t)(t/T)^k\|_{\mathcal{FL}_q^\gamma(\mathbb{R})} \lesssim T^{\frac{1}{q'}-\gamma} \|\chi(t)t^k\|_{\mathcal{FL}_q^\gamma(\mathbb{R})} \lesssim T^{\frac{1}{q'}-\gamma} \|\chi(t)t^k\|_{W_t^{2,q'}} \lesssim k^2 T^{\frac{1}{q'}-\gamma}.$$

and

$$\|\text{II}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \leq C \|\chi_T(t)\|_{\mathcal{FL}_q^\gamma(\mathbb{R})} T^{\gamma-\frac{1}{q'}} \|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}.$$

For III, set

$$J_T(t) := \frac{1}{2\pi} \int_{T|\tau|>1} \frac{\widehat{\mathbf{F}}(\tau)}{i\tau} e^{it\tau} d\tau,$$

then by Hölder, we have

$$\|J_T\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \leq C\|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}, \quad \|J_T\|_{\mathcal{FL}_q^0(\mathcal{X})} \leq CT^\gamma\|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}.$$

From $\langle\eta\rangle^\gamma \leq C\langle\eta - \tau\rangle^\gamma + C\langle\tau\rangle^\gamma$ for any $\tau \in \mathbb{R}$, we deduce that

$$\begin{aligned} \|\text{III}\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} &\leq C\|(\langle\cdot\rangle^\gamma \widehat{\chi}_T) * \widehat{J}_T(\eta)\|_{L_\eta^q \mathcal{X}} + C\|\widehat{\chi}_T * (\langle\cdot\rangle^\gamma \widehat{J}_T)(\eta)\|_{L_\eta^q \mathcal{X}} \\ &\leq C\|\langle\cdot\rangle^\gamma \widehat{\chi}_T\|_{L^1} \|J_T\|_{\mathcal{FL}_q^0(\mathcal{X})} + C\|\widehat{\chi}_T\|_{L^1} \|J_T\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \\ &\leq C\|\mathbf{F}\|_{\mathcal{FL}_q^{\gamma-1}(\mathcal{X})}. \end{aligned}$$

where to the second inequality, we used Young's convolution inequality, and

$$\|\langle\cdot\rangle^\gamma \widehat{\chi}_T\|_{L^1} \lesssim T^{-\gamma}.$$

The proof of Lemma A.8 is now complete. \square

Lemma A.9. *Let $\chi \in C_c^\infty(\mathbb{R})$, $0 < T < 1$, $1 < q < \infty$, $\theta \geq 0$, $\gamma \in (\frac{1}{q'}, 1)$. Then for any $\mathbf{A}(t) \in \mathcal{S}(\mathbb{R}; \mathcal{L}(\mathcal{X}))$, the operator $\chi_T(t) \int_0^t \mathbf{A}(t') dt' \in \mathcal{L}(\mathcal{FL}_q^\gamma(\mathcal{X}))$. Moreover, for any $\mathbf{F}(t) \in \mathcal{FL}_q^\gamma(\mathcal{X})$,*

$$\left\| \chi_T(t) \left(\int_0^t \mathbf{A}(t') dt' \right) (\mathbf{F}(t)) \right\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\theta - \gamma + \frac{1}{q'}} \|\mathbf{A}\|_{\mathcal{L}(\mathcal{X}, \mathcal{FL}_q^{\gamma-1+\theta}(\mathcal{X}))} \|\mathbf{F}(t)\|_{\mathcal{FL}_q^\gamma(\mathcal{X})}.$$

Remark A.10. *The reason for which we make the assumption $\mathbf{A}(t) \in \mathcal{S}(\mathbb{R}; \mathcal{L}(\mathcal{X}))$ is to avoid the issue of justifying interchange of orders of integral as well as the time Fourier transform. This regularity assumption is always satisfied in our application, as every object is smooth (in time and in space).*

Remark A.11. *Comparing to Lemma A.8, we have an extra loss $T^{-\gamma + \frac{1}{q'}}$. This is basically due to the fact that $\|\chi_T(t)\mathbf{F}(t)\|_{\mathcal{FL}_q^\gamma(\mathcal{X})} \lesssim T^{\frac{1}{q'} - \gamma} \|\mathbf{F}(t)\|_{\mathcal{FL}_q^\gamma(\mathcal{X})}$ since $\gamma > \frac{1}{q'}$.*

Proof. The proof is similar to the proof of Lemma A.8. Without loss of generality, we assume that $\mathbf{F}(t) \in \mathcal{S}(\mathbb{R}; \mathcal{X})$ so that all manipulation can be justified. The function $\mathbf{A}(t')\mathbf{F}(t)$ is well-defined with its time Fourier transform (with respect to t' variable) is $\widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t)$. Moreover, the Fourier transform in t -variable is $\widehat{\mathbf{A}}(\tau_1)\widehat{\mathbf{F}}(\tau)$.

We write $\chi_T(t) \int_0^t \mathbf{A}(t')\mathbf{F}(t) dt'$ as

$$\begin{aligned} &\frac{\chi_T(t)}{2\pi} \int_0^t dt' \int_{\mathbb{R}} \widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t) e^{it_1\tau_1} d\tau_1 = \frac{\chi_T(t)}{2\pi} \int_{\mathbb{R}} \frac{e^{it\tau_1} - 1}{i\tau_1} \widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t) d\tau_1 \\ &= \underbrace{\frac{\chi_T(t)}{2\pi} \int_{T|\tau_1| \leq 1} \frac{\widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t)}{i\tau_1} \sum_{k=1}^{\infty} \frac{(i\tau_1 t)^k}{k!} d\tau_1}_{\text{I}} - \underbrace{\frac{\chi_T(t)}{2\pi} \int_{T|\tau_1| > 1} \widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t) \frac{1}{i\tau_1} d\tau_1}_{\text{II}} \\ &+ \underbrace{\frac{\chi_T(t)}{2\pi} \int_{T|\tau_1| > 1} \widehat{\mathbf{A}}(\tau_1)\mathbf{F}(t) \frac{e^{it\tau_1}}{i\tau_1} d\tau_1}_{\text{III}}. \end{aligned}$$

•**Estimate for I:** Note that

$$\begin{aligned} & \int_{T|\tau_1| \leq 1} |(i\tau_1)^{k-1}| \|\langle \tau \rangle^\gamma \widehat{\mathbf{A}}(\tau_1) \mathcal{F}_t(\chi_T(t)t^k \mathbf{F})(\tau)\|_{\mathcal{X}} d\tau_1 \\ & \leq T^{-k+1} \|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \mathcal{F}_t(\chi_T(t)t^k \mathbf{F})(\tau)\|_{L^q_{\tau_1}(\mathcal{X})} \left(\int_{T|\tau_1| \leq 1} \langle \tau_1 \rangle^{(1-\gamma-\theta)q'} d\tau_1 \right)^{\frac{1}{q'}} \\ & \lesssim T^{\gamma-\frac{1}{q'}+\theta} \|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \mathcal{F}_t(\chi_T(t))(t/T)^k \mathbf{F})(\tau)\|_{L^q_{\tau_1}(\mathcal{X})}. \end{aligned}$$

Set $\mathbf{G}_k(t) = \chi_T(t)(t/T)^k \mathbf{F}(t)$. From Remark A.2 and Proposition A.1,

$$\|\mathbf{G}_k\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})} \lesssim T^{\frac{1}{q'}-\gamma} \|\chi(t)t^k\|_{W_t^{10,1}} \lesssim k^{10} T^{\frac{1}{q'}-\gamma} \|\mathbf{F}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})},$$

where the implicit constant is independent of k . Therefore,

$$\begin{aligned} \|\mathbf{I}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})} & \lesssim T^{\gamma-\frac{1}{q'}+\theta} \sum_{k=1}^{\infty} \frac{\|\|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \widehat{\mathbf{G}}_k(\tau)\|_{\mathcal{X}}\|_{L^q_{\tau, \tau_1}}}{k!} \\ & = T^{\gamma-\frac{1}{q'}+\theta} \sum_{k=1}^{\infty} \frac{\|\|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \widehat{\mathbf{G}}_k(\tau)\|_{L^q_{\tau_1}(\mathcal{X})}\|_{L^q_{\tau}}}{k!} \\ & \leq T^{\gamma-\frac{1}{q'}+\theta} \sum_{k=1}^{\infty} \frac{\|\|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \|\langle \tau \rangle^\gamma \widehat{\mathbf{G}}_k(\tau)\|_{\mathcal{X}}\|_{L^q_{\tau}}}{k!} \\ & \lesssim T^\theta \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \sum_{k=1}^{\infty} \frac{k^{10} \|\mathbf{F}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})}}{k!} \\ & \lesssim T^\theta \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \|\mathbf{F}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})}, \end{aligned}$$

as desired.

•**Estimate for II:**

$$\begin{aligned} & \int_{T|\tau_1| > 1} \frac{\|\langle \tau \rangle^\gamma \mathbf{A}(\tau_1) \widehat{\chi_T \mathbf{F}}(\tau)\|_{\mathcal{X}}}{|\tau_1|} d\tau_1 \\ & \leq \|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \widehat{\chi_T \mathbf{F}}(\tau)\|_{L^q_{\tau_1}(\mathcal{X})} \left(\int_{T|\tau_1| > 1} \langle \tau_1 \rangle^{-q'(\gamma+\theta)} d\tau_1 \right)^{\frac{1}{q'}} \\ & \lesssim T^{\gamma-\frac{1}{q'}+\theta} \|\langle \tau \rangle^\gamma \langle \tau_1 \rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1) \widehat{\chi_T \mathbf{F}}(\tau)\|_{L^q_{\tau_1}(\mathcal{X})}, \end{aligned}$$

provided that $\gamma > \frac{1}{q'}$. The right hand side can be majorized by

$$T^{\gamma-\frac{1}{q'}+\theta} \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \|\langle \tau \rangle^\gamma \widehat{\chi_T \mathbf{F}}(\tau)\|_{\mathcal{X}}.$$

Taking the L^q_{τ} -norm, we obtain the bound

$$T^{\gamma-\frac{1}{q'}+\theta} \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \|\chi_T \mathbf{F}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})} \lesssim T^\theta \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}^{\gamma-1+\theta}} \|\mathbf{F}\|_{\mathcal{F}L^q_{\tau}(\mathcal{X})},$$

where we have used (2) of Proposition A.1.

•**Estimate of III** : Finally, for III, we denote by

$$\mathbf{J}(\tau) = \frac{1}{2\pi} \int_{T|\tau_1|>1} \frac{\widehat{\mathbf{A}}(\tau_1)\widehat{\chi_T\mathbf{F}}(\tau - \tau_1)}{i\tau_1} d\tau_1$$

and we need to control $\|\langle\tau\rangle^\gamma \mathbf{J}(\tau)\|_{L^q_\tau \mathcal{X}}$. First,

$$\langle\tau\rangle^\gamma \|\mathbf{J}(\tau)\|_{\mathcal{X}} \leq \int_{T|\tau_1|>1} \langle\tau\rangle^\gamma \langle\tau_1\rangle^{-1} \|\widehat{\mathbf{A}}(\tau_1)\widehat{\chi_T\mathbf{F}}(\tau - \tau_1)\|_{\mathcal{X}} d\tau_1.$$

Next, we observe that if $T|\tau_1| > 1$ and $|\tau| \lesssim |\tau_1|$, $\langle\tau\rangle^\gamma \langle\tau_1\rangle^{-1} \lesssim T^\theta \langle\tau_1\rangle^{\gamma-1+\theta}$. If $|\tau| \gg |\tau_1|$, we have $|\tau| \lesssim |\tau - \tau_1|$, hence $\langle\tau\rangle^\gamma \langle\tau_1\rangle^{-1} \sim T^\theta \langle\tau - \tau_1\rangle^\gamma \langle\tau_1\rangle^{\gamma-1+\theta} \langle\tau_1\rangle^{-\gamma}$. Thus

$$\begin{aligned} \langle\tau\rangle^\gamma \|\mathbf{J}(\tau)\|_{\mathcal{X}} &\lesssim T^\theta \int_{T|\tau_1|>1} \|\langle\tau_1\rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau_1)\widehat{\chi_T\mathbf{F}}(\tau - \tau_1)\|_{\mathcal{X}} d\tau_1 \\ &+ T^\theta \int_{T|\tau_1|>1} \|\langle\tau_1\rangle^{\gamma-1+\theta} \langle\tau - \tau_1\rangle^\gamma \widehat{\mathbf{A}}(\tau_1)\widehat{\chi_T\mathbf{F}}(\tau - \tau_1)\|_{\mathcal{X}} \cdot \langle\tau_1\rangle^{-\gamma} d\tau_1. \end{aligned} \quad (\text{A.11})$$

By changing variables, the first term on the right hand side can be written as

$$T^\theta \int_{T|\tau - \tau_1|>1} \|\langle\tau - \tau_1\rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau - \tau_1)\widehat{\chi_T\mathbf{F}}(\tau_1)\|_{\mathcal{X}} d\tau_1.$$

By Minkowski, its L^q_τ -norm can be bounded by

$$\begin{aligned} &T^\theta \int_{\mathbb{R}} \|\langle\tau - \tau_1\rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau - \tau_1)\widehat{\chi_T\mathbf{F}}(\tau_1)\|_{L^q_\tau \mathcal{X}} d\tau_1 \\ &= T^\theta \int_{\mathbb{R}} \|\langle\tau\rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau)\langle\tau_1\rangle^\gamma \widehat{\chi_T\mathbf{F}}(\tau_1)\|_{L^q_\tau \mathcal{X}} \langle\tau_1\rangle^{-\gamma} d\tau_1 \\ &\leq T^\theta \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}{}^{\gamma-1+\theta}(\mathcal{X})} \|\chi_T \mathbf{F}\|_{\mathcal{F}L^q_\tau(\mathcal{X})} \|\langle\tau_1\rangle^{-\gamma}\|_{L^{q'}_{\tau_1}} \\ &\lesssim T^{\theta + \frac{1}{q'} - \gamma} \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau}{}^{\gamma-1+\theta}(\mathcal{X})} \|\mathbf{F}\|_{\mathcal{F}L^q_\tau(\mathcal{X})}, \end{aligned}$$

where to the last step, we have used Hölder and (2) in Proposition A.1.

Similarly, to control the L^q_τ -norm of the second term on the right hand side of (A.11), we apply Hölder and Minkowski to get

$$\begin{aligned} &T^\theta \|\|\langle\tau - \tau_1\rangle^{\gamma-1+\theta} \langle\tau_1\rangle^\gamma \widehat{\mathbf{A}}(\tau - \tau_1)\widehat{\chi_T\mathbf{F}}(\tau_1)\|_{L^q_{\tau_1} \mathcal{X}} \|\langle\tau - \tau_1\rangle^{-\gamma} \mathbf{1}_{T|\tau - \tau_1|>1}\|_{L^{q'}_{\tau_1}}\|_{L^q_\tau} \\ &\lesssim T^{\theta + \gamma - \frac{1}{q'}} \|\|\langle\tau - \tau_1\rangle^{\gamma-1+\theta} \widehat{\mathbf{A}}(\tau - \tau_1)\langle\tau_1\rangle^\gamma \widehat{\chi_T\mathbf{F}}(\tau_1)\|_{L^q_{\tau_1} \mathcal{X}}\|_{L^q_{\tau_1}} \\ &\lesssim T^{\theta + \gamma - \frac{1}{q'}} \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau_1}{}^{\gamma-1+\theta}(\mathcal{X})} \|\chi_T \mathbf{F}\|_{\mathcal{F}L^q_{\tau_1}(\mathcal{X})} \\ &\lesssim T^\theta \|\mathbf{A}\|_{\mathcal{X} \rightarrow \mathcal{F}L^q_{\tau_1}{}^{\gamma-1+\theta}(\mathcal{X})} \|\mathbf{F}\|_{\mathcal{F}L^q_{\tau_1}(\mathcal{X})}, \end{aligned}$$

where we used (2) in Proposition A.1 in the last step. This completes the proof of Proposition A.9. \square

APPENDIX B. SOME ELEMENTARY ESTIMATES

Lemma B.1. *Let $\mathcal{A}(t)$ be a operator-valued function that is self-adjoint on a Hilbert space \mathcal{X} for every $t \in \mathbb{R}$. Let $\mathcal{H}(t)$ be the solution operator of the well-posed Cauchy problem*

$$i\partial_t \mathcal{H}(t) = \mathcal{A}(t)\mathcal{H}(t), \quad \mathcal{H}(0) = \text{Id}.$$

Then $\mathcal{H}(t)$ is unitary.

Proof. Note that $\mathcal{H}(t)$ is a linear operator for any $t \in \mathbb{R}$. Since $\mathcal{H}(0) = \text{Id}$, it suffices to show that $\mathcal{H}(t)$ preserves the norm of the Hilbert-space. For any $f \in \mathcal{X}$, consider

$$\frac{d}{dt} \|\mathcal{H}(t)f\|_{\mathcal{X}}^2 = 2\text{Re}(\partial_t \mathcal{H}(t)f, \mathcal{H}(t)f)_{\mathcal{X}} = 2\text{Im}(\mathcal{A}(t)\mathcal{H}(t)f, \mathcal{H}(t)f)_{\mathcal{X}} = 0,$$

since $(\mathcal{A}(t))^* = \mathcal{A}(t)$ for all $t \in \mathbb{R}$. Therefore, $\|\mathcal{H}(t)f\|_{\mathcal{X}} = \|\mathcal{H}(0)f\|_{\mathcal{X}} = \|f\|_{\mathcal{X}}$, for any $f \in \mathcal{X}$. This leads to $\mathcal{H}(t)\mathcal{H}(t)^* = \text{Id}$. \square

Lemma B.2. *Let $\mathcal{A}(t)$ be in Lemma B.1. Let $0 < T < 1$. Assume that $\chi \in C_c^\infty(\mathbb{R})$ is a bump function that equals to 1 on $|t| \leq \frac{1}{2}$ and vanishes for $|t| \geq 1$. For the unique solutions $\mathcal{H}^\dagger(t), \mathcal{G}^\dagger(t)$ of equations*

$$\begin{aligned} \mathcal{H}^\dagger(t) &= \chi(t)\text{Id} - i\chi\left(\frac{t}{2T}\right) \int_0^t \mathcal{A}(t')\mathcal{H}^\dagger(t')dt', \\ \mathcal{G}^\dagger(t) &= \chi(t)\text{Id} + i\chi\left(\frac{t}{2T}\right) \int_0^t \mathcal{A}(t')\mathcal{H}^\dagger(t')\Gamma^\dagger(t)dt'. \end{aligned}$$

Then for all $|t| \leq T$, $\mathcal{H}^\dagger(t)$ is unitary and $\Gamma^\dagger(t) = (\mathcal{H}^\dagger(t))^$.*

Proof. Note that $\mathcal{H}^\dagger(t)$ solves the equation

$$\begin{cases} i\partial_t \mathcal{H}^\dagger(t) = \chi_{2T}(t)\mathcal{A}(t)\mathcal{H}^\dagger(t) + \chi'(t) - \frac{i}{2T}\chi'\left(\frac{t}{2T}\right) \int_0^t \mathcal{A}(t')\mathcal{H}^\dagger(t')dt', \\ \mathcal{H}^\dagger(t) = \text{Id}. \end{cases}$$

We observe that for $|t| \leq T$, $\mathcal{H}^\dagger(t)$ solves the same equation as $\mathcal{H}(t)$ with the same initial data. By uniqueness of the linear equation. By the well-posedness of the Cauchy problem, we have $\mathcal{H}^\dagger(t) = \mathcal{H}(t)$ for $|t| \leq T$. For the $\mathcal{G}^\dagger(t)$ part, as it solves the same integral equation

$$\mathcal{G}^\dagger(t) = \text{Id} + i \int_0^t \mathcal{A}(t')\mathcal{H}(t')\mathcal{G}^\dagger(t)dt'$$

as $(\mathcal{H}(t))^*$ for $|t| \leq T$, by uniqueness, we have $\mathcal{G}(t)^\dagger = (\mathcal{H}(t))^*$. \square

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