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Lower bounds on fibered Yang-Mills functionals: generic nefness and semistability of direct images

Siarhei Finski

Abstract. The main goal of this paper is to generalize a part of the relationship between mean curvature and Harder-Narasimhan filtrations of holomorphic vector bundles to arbitrary polarized fibrations. More precisely, for a polarized family of complex projective manifolds, we establish lower bounds on a fibered version of Yang-Mills functionals in terms of the Harder-Narasimhan slopes of direct image sheaves associated with high tensor powers of the polarization. We discuss the optimality of these lower bounds and, as an application, provide an analytic characterisation of a fibered version of generic nefness. As another application, we refine the existent obstructions for finding metrics with constant horizontal mean curvature. The study of the semiclassical limit of Hermitian Yang-Mills functionals lies at the heart of our approach.

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1 Introduction

Consider a holomorphic submersion $\pi : X \rightarrow B$ between compact complex manifolds X and B of dimensions $n + m$ and m respectively, $n, m \in \mathbb{N}$. Let L be a holomorphic line bundle over X , which is relatively ample with respect to π . We fix a *Gauduchon Hermitian form* ω_B on B , i.e. a positive $(1, 1)$ -form, such that $\partial\bar{\partial}\omega_B^{m-1} = 0$, see [37]. The main goal of this paper is to study the relationship between the so-called horizontal mean curvature of the fibration, which is a certain differential-geometric invariant of the family defined using ω_B , and Harder-Narasimhan ω_B -slopes of direct images $E_k := R^0\pi_*L^k$, which are algebraic invariants.

More precisely, consider a Hermitian metric h^L on L , which is positive along the fibers of π . We denote by $\omega(h^L) := \frac{\sqrt{-1}}{2\pi}R^L$ the first Chern form of (L, h^L) , where R^L is the curvature of the Chern connection. When h^L is clear from the context, we omit it from the above notation.

As ω is positive along the fibers, it provides a (smooth) decomposition of the tangent space TX of X into the vertical component $T^V X$, corresponding to the tangent space of the fibers, and the horizontal component $T^H X$, corresponding to the orthogonal complement of $T^V X$ with respect to ω . The form ω then decomposes as $\omega = \omega_V + \omega_H$, $\omega_V \in \mathcal{C}^\infty(X, \wedge^{1,1}T^V X)$, $\omega_H \in \mathcal{C}^\infty(X, \wedge^{1,1}T^H X)$. Upon the natural identification of $T^H X$ with π^*TB , we may view ω_H as an element from $\mathcal{C}^\infty(X, \wedge^{1,1}\pi^*T^*B)$. The triple $(\pi, \omega, T^H X)$ then defines a *Kähler family* in the sense of [6, Definition 1.4]. We define the *horizontal mean curvature*, $\wedge_{\omega_B}\omega_H(h^L) \in \mathcal{C}^\infty(X)$, as

$$\wedge_{\omega_B}\omega_H(h^L) := \frac{\omega_H(h^L) \wedge \omega_B^{m-1}}{\omega_B^m}. \quad (1.1)$$

We say that h^L is *fibered Einstein* if $\wedge_{\omega_B} \omega_H(h^L)$ is a constant. By decomposition into horizontal and vertical components, it is easy to see that this condition is equivalent to

$$\omega(h^L)^{n+1} \wedge \pi^* \omega_B^{m-1} = c \cdot \omega(h^L)^n \wedge \pi^* \omega_B^m, \quad (1.2)$$

where c is a constant. By integrating (1.2), we see that the constant c is independent of h^L , since ω_B is Gauduchon, see Section 4 for details. Remark the similarity of (1.2) with the J-equation if $m = 1$, and with optimal symplectic connection equation of Dervan-Sektnan, see [26, Proposition 2.7], if the fibers are Fano. For families of manifolds given by projectivizations of vector bundles, the condition (1.2) was introduced by Kobayashi [46], who called such metrics *Finsler-Einstein* metrics; then Feng-Liu-Wan [33] generalized it for general Kähler submersions, and called such metrics *geodesic Einstein* metrics. If instead of a closed manifold B , one considers manifolds with boundary, the equation (1.2) was studied extensively in the past: if B is a 1-dimensional annuli and $c = 0$, this is a geodesic equation in Mabuchi space [57], see Semmes [67] or Donaldson [30], if B is a bounded smooth strongly pseudoconvex domain in \mathbb{C}^n , $c = 0$ and ω_B is the standard Kähler form, (1.2) was called Wess-Zumino-Witten equation in [30] due to its connection with [73].

Remark that in the important case when $X := \mathbb{P}(E^*)$ for some holomorphic vector bundle E over B , $L := \mathcal{O}(1)$, and h^L is induced by a Hermitian metric h^E on E , (L, h^L) is fibered Einstein if and only if (E, h^E) is Hermite-Einstein, cf. Remark 3.4 for details.

The first main observation of this paper is that the correspondence between fibered Einstein and Hermite Einstein equations is much tighter. Indeed, as we shall see, relying on the work of Ma-Zhang [55], [56], cf. Theorem 2.2, the fibered Einstein equation for L is *mutatis mutandis* the semi-classical limit (i.e. $k \rightarrow \infty$) of the Hermite-Einstein equation for $E_k := R^0 \pi_* L^k$. Let us now explain the first manifestation of this correspondence.

Recall that a *slope* (or ω_B -slope) of a coherent sheaf \mathcal{E} over B is defined as $\mu(\mathcal{E}) := \deg(\mathcal{E})/\text{rk}(\mathcal{E})$, where the degree, $\deg(\mathcal{E})$, is given by $\deg(\mathcal{E}) := \int_B [c_1(\det \mathcal{E})] \cdot [\omega_B^{m-1}]$; here and after the intersection product is for Bott-Chern and Aeppli cohomology classes, ω_B^{m-1} represents an Aeppli cohomology class since ω_B is Gauduchon, see Section 4 for details, and $\det \mathcal{E}$ is the Knudsen-Mumford determinant of \mathcal{E} , see [45]. A torsion-free coherent sheaf \mathcal{E} is called *semistable* or ω_B -*semistable* if for every coherent subsheaf \mathcal{F} of \mathcal{E} , verifying $\text{rk}(\mathcal{F}) > 0$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. When $\dim B = 1$, these notions clearly do not depend on ω_B .

Theorem 1.1. *Assume that L admits an approximate fibered Einstein metric, i.e. there is $c \in \mathbb{R}$, such that for any $\epsilon > 0$, there is a relatively positive metric h_ϵ^L on L , verifying the following bound*

$$\left| \wedge_{\omega_B} \omega_H(h_\epsilon^L) - c \right| < \epsilon. \quad (1.3)$$

Then the vector bundles $E_k := R^0 \pi_ L^k$ are asymptotically semistable, i.e. for any quotient sheaves \mathcal{Q}_k of E_k , $\text{rk}(\mathcal{Q}_k) > 0$, and any $\epsilon > 0$ for k big enough, we have $\mu(\mathcal{Q}_k) \geq (1 - \epsilon) \cdot \mu(E_k)$.*

Remark 1.2. It is likely that there is an even closer relationship between fibered Einstein metrics on L and Hermite Einstein metrics on E_k , paralleling the known correspondence between constant scalar curvature and balanced metrics, see Donaldson [31].

When Theorem 1.1 is applied to $\pi : \mathbb{P}(E^*) \rightarrow B$ for some vector bundle E over B , and $L := \mathcal{O}(1)$, due to a precise relation between the slopes of $\text{Sym}^k E = R^0 \pi_* L^k$, $k \in \mathbb{N}$, and E , see [17, §3.2], we recover the well-known fact, cf. [47, Theorem 6.10.13], that if E admits approximate Hermite-Einstein metrics, then E is semistable.

The asymptotic semistability condition from Theorem 1.1 seems rather difficult to verify at first sight. We will now discuss some numerical obstructions for it. More precisely, for an irreducible complex analytic subspace $Y \subset X$ of dimension $k + m$, $k \geq 0$, such that the restriction of π to Y , $\pi|_Y : Y \rightarrow B$, is a surjection, we define the ω_B -slope, $\mu(Y)$, as

$$\mu(Y) = \frac{1}{k+1} \cdot \frac{\int_Y [c_1(L)^{k+1}] \cdot [\omega_B^{m-1}]}{\int_Y [c_1(L)^k] \cdot [\omega_B^m]}. \quad (1.4)$$

By Serre vanishing theorem, for $k \in \mathbb{N}^*$ big enough, the sheaf $\mathcal{Q}_k := R^0 \pi|_{Y,*} L|_Y^k$ can be realized as a quotient of E_k through the restriction map, see proof of Proposition 4.7 for details. By the asymptotic version of Riemann-Roch-Grothendieck theorem, see Theorem 4.1, which we establish in our singular setting, we can calculate the asymptotics of the slopes of \mathcal{Q}_k and E_k , as $k \rightarrow \infty$. By comparing the asymptotics of these slopes, we obtain in Section 4 the following result.

Theorem 1.3. *If the vector bundles E_k are asymptotically semistable, then X is numerically semistable, i.e. for any Y as above, we have $\mu(Y) \geq \mu(X)$. Moreover, if $\dim B = 1$, then asymptotic semistability of E_k is equivalent to numerical semistability of X .*

Remark 1.4. A combination of Theorems 1.1 and 1.3 shows that existence of approximate fibered Einstein metrics on L implies $\mu(Y) \geq \mu(X)$ for Y above. Feng-Liu-Wan [33, Theorem 2.2], cf. also Wan-Wang [72], established this by different means under an assumption, requiring among others that the projection of the singular locus of Y to B has codimension at least 2.

As we explain later, Theorem 1.1 is a direct consequence of a more refined result concerning lower bounds on fibered Yang-Mills functionals. More precisely, for any relatively positive metric h^L on L and any $c \in \mathbb{R}$, $p \in [1, +\infty[$, we define the fibered Yang-Mills functional as

$$\begin{aligned} \text{FYM}_{p,c}(\pi, h^L) &:= \int_X |\wedge_{\omega_B} \omega_H(x) - c|^p \omega^n \wedge \pi^* \omega_B^m(x), \\ \text{FYM}_{+\infty,c}(\pi, h^L) &:= \sup_{x \in X} |\wedge_{\omega_B} \omega_H(x) - c|. \end{aligned} \quad (1.5)$$

We will now show that asymptotic Harder-Narasimhan slopes of E_k , as $k \rightarrow \infty$, yield lower bounds for this functionals. To readers familiar with Hermitian Yang-Mills theory, see Atiyah-Bott [2, Proposition 8.20], Donaldson [28, Proposition 5] and Daskalopoulos-Wentworth [20, §2.3, 2.4], this will sound very natural. Indeed, again from the work of Ma-Zhang [55], [56], one can view the horizontal mean curvature of L as the semi-classical limit (i.e. $k \rightarrow \infty$) of the mean curvature of E_k . From this, we establish the lower bounds on the fibered Yang-Mills functionals through the limits of the lower bounds on the Hermitian Yang-Mills functionals of E_k .

To explain this in details, recall first that any torsion-free coherent sheaf \mathcal{E} on $(B, [\omega_B])$ admits a unique filtration by subsheaves \mathcal{E}_i , $i = 1, \dots, s$, also called *Harder-Narasimhan filtration*:

$$\mathcal{E} = \mathcal{E}_s \supset \mathcal{E}_{s-1} \supset \dots \supset \mathcal{E}_1 \supset \mathcal{E}_0 = \{0\}, \quad (1.6)$$

such that for any $1 \leq i \leq s-1$, the quotient sheaf $\mathcal{E}_i/\mathcal{E}_{i-1}$ is the maximal semistable (torsion-free coherent) subsheaf of $\mathcal{E}/\mathcal{E}_{i-1}$, i.e. for any subsheaf \mathcal{F} of a (torsion-free coherent) sheaf $\mathcal{E}/\mathcal{E}_{i-1}$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ and $\text{rk}(\mathcal{F}) \leq \text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ if $\mu(\mathcal{F}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. For the proof of this result in the setting of Gauduchon form ω_B , see either Bruasse [14] or Greb-Kebekus-Peternell [38, Corollary 2.27]. We define the *Harder-Narasimhan slopes*, $\mu_1, \dots, \mu_{\text{rk}(\mathcal{E})}$

of \mathcal{E} , so that $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ appears among $\mu_1, \dots, \mu_{\text{rk}(\mathcal{E})}$ exactly $\text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ times, and the sequence $\mu_1, \dots, \mu_{\text{rk}(\mathcal{E})}$ is non-increasing. We let $\mu_{\min} := \mu_{\text{rk}(\mathcal{E})}$, $\mu_{\max} := \mu_1$.

We denote $N_k := \text{rk}(E_k)$, and let $\mu_1^k, \dots, \mu_{N_k}^k$ be the Harder-Narasimhan slopes of E_k , and $\mu_{\min}^k, \mu_{\max}^k$ be the minimal and the maximal slopes. Define the probability measure η_k^{HN} on \mathbb{R} as

$$\eta_k^{HN} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta \left[\frac{\mu_i^k}{k} \right], \quad (1.7)$$

where $\delta[x]$ is the Dirac mass at $x \in \mathbb{R}$. Our lower bounds for the fibered Yang-Mills functionals will build upon the following result.

Theorem 1.5. *The sequence of measures η_k^{HN} converges weakly, as $k \rightarrow \infty$, to a probability measure η^{HN} on \mathbb{R} , and the limits below exist and relate with η^{HN} as follows*

$$\eta_{\min}^{HN} := \lim_{k \rightarrow \infty} \frac{\mu_{\min}^k}{k} \leq \text{ess inf } \eta^{HN}, \quad \eta_{\max}^{HN} := \lim_{k \rightarrow \infty} \frac{\mu_{\max}^k}{k} = \text{ess sup } \eta^{HN}. \quad (1.8)$$

Remark 1.6. The proof of Theorem 1.5 follows the arguments from Chen [16] and [36, Theorem 1.1], establishing Theorem 1.5 in the projective setting for flat maps $\pi : X \rightarrow B$, for $\dim B = 1$ and $\dim B \geq 1$ respectively. The only difference is that due to the lack of algebraicity, the proofs from [16] and [36] of the linear upper bound on μ_{\max}^k in $k \in \mathbb{N}^*$, crucial for Theorem 1.5, do not work. Here this bound is obtained by a differential-geometric argument, see Proposition 2.3.

We are finally ready to state our lower bounds for the fibered Yang-Mills functionals.

Theorem 1.7. *For any $c \in \mathbb{R}$, $p \in [1, +\infty[$, and any relatively positive metric h^L on L , we have*

$$\begin{aligned} \text{FYM}_{p,c}(\pi, h^L) &\geq \int_{\mathbb{R}} |x - c|^p d\eta^{HN}(x) \cdot \int_X [\omega^n] \cdot \pi^*[\omega_B^m], \\ \text{FYM}_{+\infty,c}(\pi, h^L) &\geq \max \left\{ |\eta_{\min}^{HN} - c|, |\eta_{\max}^{HN} - c| \right\}. \end{aligned} \quad (1.9)$$

Remark 1.8. a) Clearly, the second bound is equivalent to the following one

$$\inf_{x \in X} \wedge_{\omega_B} \omega_H(x) \leq \eta_{\min}^{HN}, \quad \sup_{x \in X} \wedge_{\omega_B} \omega_H(x) \geq \eta_{\max}^{HN}, \quad (1.10)$$

which clarifies that Theorem 1.7 refines Theorem 1.1, see Proposition 4.5 for details.

b) We stress out that while the left hand-side of (1.9) depends on h^L , the right-hand side doesn't.

Remark that similar lower bounds in the context of constant scalar curvature metrics were obtained by Donaldson [32] for Calabi functional. Here, as in [32], we expect the bounds from Theorem 1.7 to be tight. In other words, it seems likely that the following conjecture holds.

Conjecture. *In the notations of Theorem 1.7, for any $p \in [1, +\infty[$, $c \in \mathbb{R}$, we have*

$$\begin{aligned} \inf_{h^L} \text{FYM}_{p,c}(\pi, h^L) &= \int_{\mathbb{R}} |x - c|^p d\eta^{HN}(x) \cdot \int_X [\omega^n] \cdot \pi^*[\omega_B^m], \\ \inf_{h^L} \text{FYM}_{+\infty,c}(\pi, h^L) &= \max \left\{ |\eta_{\min}^{HN} - c|, |\eta_{\max}^{HN} - c| \right\}, \end{aligned} \quad (1.11)$$

where the infimum is taken among all relatively positive metrics h^L on L .

Remark that if the above Conjecture holds, it will provide a geometric interpretation of η^{HN} , as Radon measures of compact support on \mathbb{R} are determined by their shifted absolute moments. Also, by Proposition 4.5, it will imply the inverse implication to Theorem 1.1.

For $\pi : \mathbb{P}(E^*) \rightarrow B$, $L = \mathcal{O}(1)$, where E is some holomorphic vector bundle over a complex compact manifold B , one can show that Conjecture holds by the existence of the approximate critical hermitian structures on vector bundles, cf. Daskalopoulos-Wentworth [20, Definition 3.9], and the calculation of the asymptotic slopes of $\text{Sym}^k E = R^0 \pi_* L^k$, due to Chen [17, Theorem 1.2]. By (1.10), the following result is another partial justification of the Conjecture.

Theorem 1.9. *The following identity holds $\sup_{h^L} \inf_{x \in X} \wedge_{\omega_B} \omega_H(x) = \eta_{\min}^{HN}$, where the supremum is taken among all relatively positive metrics h^L on L .*

We assume now that B (and, hence, X) is projective. Recall that a line bundle L on X is called *nef* if for any irreducible curve C in X , $\int_C c_1(L) \geq 0$. It is well-known that this condition is equivalent to the existence of metrics with an arbitrary small negative part of the curvature, see [22, Proposition 4.2] for a precise statement. One of the fundamental concerns in complex geometry is the study of variations of this statement, giving a “dictionary” between algebraic and analytic definitions of various positivity conditions. As examples, we have the numerical characterization of the Kähler cone due to Demailly-Păun [25], the characterization of pseudoeffectivity in terms of pairing with movable curves due to Boucksom-Demailly-Păun-Peternell [13], or Andreotti-Grauert theorem [1] and its partial converses, cf. Demailly [23] or Yang [77]. Further down this line, let’s explain an application of Theorem 1.9 towards generic fibered nefness.

We fix a very ample integral multipolarization $[\omega_{B,1}], \dots, [\omega_{B,m-1}]$ on B , which is a collection of very ample classes from $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$. We say that a \mathbb{Q} -line bundle L on X is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -*generically fibered nef with respect to π* if there is $l_0 \in \mathbb{N}^*$, such that for any regular curve $C = C(l) \subset B$, $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$, $l_i \geq l_0$, $i = 1, \dots, m-1$, given by a complete intersection of *generic* divisors from classes $l_1[\omega_{B,1}], \dots, l_{m-1}[\omega_{B,m-1}]$, the restriction of $c_1(L)$ to $\pi^{-1}(C)$ is nef. When π is the projectivization of a vector bundle, equivalent definition was given by Miyaoka [59], see also Peternell [60]. The general case was introduced in [36]. We say L is *stably* $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -*generically fibered nef with respect to π* if for some (or any) ample line bundle L_0 on X , for any $\epsilon > 0$, $\epsilon \in \mathbb{Q}$, the \mathbb{Q} -line bundle $L \otimes L_0^\epsilon$ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -*generically fibered nef with respect to π* . Recall that L is called *relatively nef with respect to π* if its restriction to every fiber is nef. As we explain in Section 3, from the previously obtained algebraic description of η_{\min}^{HN} from [36, Corollary 1.4], Theorem 1.9 can be used to prove the following result.

Theorem 1.10. *Consider a holomorphic submersion $\pi : X \rightarrow B$ between projective manifolds B, X . A relatively nef line bundle L on X is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef if and only if for any (or some) Kähler forms $\omega_{B,1}, \dots, \omega_{B,m-1}$ on B in $[\omega_{B,1}], \dots, [\omega_{B,m-1}]$, and any (or some) Kähler form ω_X on X , for any $\epsilon > 0$, there is a Hermitian metric h_ϵ^L on L , such that*

$$\omega(h_\epsilon^L) \wedge \pi^* \omega_{B,1} \wedge \dots \wedge \pi^* \omega_{B,m-1} \geq -\epsilon \cdot \omega_X^m, \quad (1.12)$$

where by this we mean that the volume forms obtained by the restriction to every m -dimensional complex hyperplane of each side of (1.12) compares as required in (1.12), cf. [24, (III.1.6)].

Remark 1.11. Curiously, even though the forms $\omega_{B,1}, \dots, \omega_{B,m-1}$ are Kähler, if these forms are different, then in the proof of Theorem 1.10, we need to apply Theorem 1.9 for a non-Kähler Gauduchon form ω_B , constructed from $\omega_{B,1}, \dots, \omega_{B,m-1}$. This was our main motivation to write this article in the current generality. However, Theorem 1.10 is new even if the forms are equal.

In conclusion, it seems for us that a proof of the Conjecture should rely either on the techniques of geometric flows or continuity method as in Donaldson [28] and Uhlenbeck-Yau [71]. In this vein, the recent apriori bounds on for Monge-Ampère and Hessian equations established by Guo-Phong-Tong [40] and Guo-Phong-Tong-Wang [41] will likely play an important role. We also mention that due to the appearance of the numerical obstructions from Theorem 1.3 and the resemblance of our equation with J-equation, it is very likely that the recent result of Chen [15] and Datar-Pingali [21] on numerical criterion for solutions of J-equation will be useful as well.

Remark that for Hermitian Yang-Mills functional, the analogous conjecture holds due to results of Atiyah-Bott [2], Daskalopoulos-Wentworth [20], Sibley [68], Jacob [44] and Li-Zhang-Zhang [49], cf. Theorem 3.5 for a precise statement. Recall also that Sjöström Dyrefelt established optimal lower bounds for Donaldson's J-functional in [69]. This should be related with the above Conjecture, as in the context of constant scalar curvature metrics, it is known that finding metrics with approximately constant scalar curvature is related with lower bounds on Mabuchi K-energy, see Bando [3] or Phong-Song-Sturm-Weinkove [62]. Finally, we mention recent works of Xia [75], Hisamoto [43] and Dervan-Székelyhidi [27], cf. also Collins-Hisamoto-Takahashi [19], proving versions of the above Conjecture in the context of constant scalar curvature metrics.

In a different direction, when B is a bounded smooth strongly pseudoconvex domain in \mathbb{C}^n , $c = 0$ and ω_B is the standard Kähler form on $B \subset \mathbb{C}^n$, Donaldson [29] and Coifman-Semmes [18] established that Dirichlet problem associated with the Hermite-Einstein equation has solutions for any vector bundle over B (in particular for E_k , $k \in \mathbb{N}$). Wu in [74] showed that Dirichlet problem associated with (1.2) has always weak solutions, and these solutions can be obtained as the semiclassical limit of the solutions of the Hermite-Einstein equations on E_k , cf. also [63], [65], [70] for earlier results in this direction. In other words, a phenomenon similar to Theorem 1.7 is present: there is a relation between Hermite-Einstein equation and Wess-Zumino-Witten equation. The major difference between these developments and our paper is that in our boundaryless setting, neither Hermite-Einstein equations nor fibered Einstein equations have solutions in general, and the methods of [29], [18], [74] do not apply.

This article will be organized as follows. In Section 2, we will establish Theorems 1.5 and 1.7. We discuss how horizontal curvature behaves with respect to a restriction to a subfamily in Section 3, and using this, we establish Theorems 1.9, 1.10. Finally, in Section 4, we establish a numerical obstruction for asymptotic semistability of direct images from Theorem 1.3.

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2 Fibered Yang-Mills functionals through the semiclassical limit

The main goal of this section is to prove Theorems 1.5 and 1.7. The theory of Toeplitz operators and Hermitian Yang-Mills theory, which we recall below, will be particularly useful for that.

We begin by recalling some facts about Toeplitz operators. Let Y be a complex projective manifold of dimension n with an ample line bundle L . We fix a positive Hermitian metric h^L on L . We denote by ω its first Chern form, $c_1(L, h^L)$. For smooth sections f, f' of L^k , $k \in \mathbb{N}$, over Y , we define the L^2 -scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_{h^L}$ induced by h^L as follows

$$\langle f, f' \rangle_{L^2(Y)} := \int_Y \langle f(x), f'(x) \rangle_{h^L} \cdot \omega^n(x). \quad (2.1)$$

Recall that the *Bergman projector* B_k is given by the orthogonal projection (with respect to the scalar product (2.1)) from the space of L^2 -sections of L^k to $H^0(Y, L^k)$. For any bounded function f on Y , we then define the Toeplitz operator, $T_k(f) : H^0(Y, L^k) \rightarrow H^0(Y, L^k)$, as follows

$$T_k(f)(s) = B_k(f \cdot s), \quad s \in H^0(Y, L^k). \quad (2.2)$$

Proposition 2.1. For any bounded function $f : Y \rightarrow \mathbb{R}$, the following inequalities hold

$$\inf f \cdot \text{Id}_{H^0(Y, L^k)} \leq T_k(f) \leq \sup f \cdot \text{Id}_{H^0(Y, L^k)}, \quad (2.3)$$

where by $A \leq B$ we mean that the difference $B - A$ is positive definite. Moreover, if f is smooth, then for any continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\text{Tr}[\phi(T_k(f))]}{\dim H^0(Y, L^k)} &= \frac{\int_{x \in Y} \phi(f(x)) \omega^n(x)}{\int_Y [\omega^n]}, \\ \lim_{k \rightarrow \infty} \|\phi(T_k(f))\| &= \max \{ |\sup \phi(f)|, |\inf \phi(f)| \}, \end{aligned} \quad (2.4)$$

where $\|\cdot\|$ is the operators norm.

Proof. The statement (2.3) follows from the trivial fact that if f is a positive function, then the operator $T_k(f)$ is positive-definite. The statement (2.4) is a restatement of the weak convergence of spectral measures of Toeplitz operators due to Boutet de Monvel-Guillemin [48, Theorem 13.13]. For an alternative proof through Bergman kernel expansion, see Ma-Marinescu [50, Theorem 7.4.1], Barron-Ma-Marinescu-Pinsonnault [4, Theorem 3.8] or [34, Appendix A]. See also Ma-Marinescu [52], [53], [54], [51] for generalizations and more refined results. \square

Now, the reason why Toeplitz operators are relevant to this paper is because they appear as the principal term in the asymptotic expansion of the curvature of L^2 -metrics on direct images of a polarized fibrations. More precisely, consider a proper holomorphic submersion $\pi : X \rightarrow B$ between complex manifolds X and B of dimensions $n + m$ and m respectively, $n, m \in \mathbb{N}$. Let L be a holomorphic line bundle over X , which is relatively ample with respect to π . Endow L with a relatively positive Hermitian metric h^L . Let $k \in \mathbb{N}$ be big enough so that $E_k := R^0 \pi_* L^k$ is locally free. The L^2 -product (2.1) then defines a smooth Hermitian metric h^{E_k} on E_k . We denote by $R^{E_k} \in \mathcal{C}^\infty(B, \wedge^2 T^* B \otimes \text{End}(E_k))$ the curvature of its Chern connection.

Theorem 2.2 (Ma-Zhang [56, Theorem 0.4]). *There are $C > 0$, $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$,*

$$\left\| \frac{\sqrt{-1}}{2\pi} R^{E_k} - k \cdot T_k(\omega_H) \right\| \leq C, \quad (2.5)$$

where $\|\cdot\|$ is the operator norm, and we naturally extended the definition of Toeplitz operators from functions to bounded sections of $\pi^* \wedge^2 T^* B$.

As we shall explain below, Theorem 2.2 is the crucial ingredient connecting fibered Yang-Mills functionals with Hermitian Yang-Mills functionals. But before this, let us mention another application of Theorem 2.2 to the study of Harder-Narasimhan slopes of direct images.

We fix now a *Gauduchon Hermitian form* ω_B on B . As before Theorem 1.5, we denote by μ_{\max}^k the maximal Harder-Narasimhan ω_B -slope of E_k .

Proposition 2.3. There is $C > 0$, such that $\mu_{\max}^k \leq Ck$ for any $k \in \mathbb{N}^*$.

Proof. For $p = 1, \dots, \text{rk}(E_k)$, we denote by $R^{\wedge^p E_k}$ the curvature of the Chern connection on $\wedge^p E_k$, induced by the metric $h^{\wedge^p E_k}$ induced by h^{E_k} . By Theorem 2.2, we conclude that there is $C > 0$, such that for any $k \in \mathbb{N}^*$, we have

$$\frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R^{\wedge^p E_k} \leq Cpk \cdot \text{Id}_{\wedge^p E_k}. \quad (2.6)$$

Let L be a line subbundle of $\wedge^p E_k$. We denote by h^L the Hermitian metric on L , induced by the metric h^{E_k} . By (2.6) and the well-known principle that curvature decreases in holomorphic sub-bundles, cf. [24, (V.14.6)], we deduce

$$\wedge_{\omega_B} c_1(L, h^L) \leq Cpk. \quad (2.7)$$

However, it is classical, cf. [47, Proofs of Lemma 5.7.16 and Theorem 5.8.3], that we have

$$\mu_{\max}^k \leq \max_{p=1, \dots, \text{rk}(E_k)} \sup_{L \subset \wedge^p E_k} \frac{1}{p} \int_B c_1(L, h^L) \wedge \omega_B^{m-1}, \quad (2.8)$$

where the second supremum is taken over line subbundles L . We conclude by (2.7) and (2.8). \square

Proof of Theorem 1.5. Taking into account the linear bound from Proposition 2.3, the proof of Theorem 1.5 is the same as in Chen [16] and [36, Theorem 1.1]. Let us briefly recall the main steps for completeness. We introduce the (non-increasing) filtrations $\mathcal{F}_k(\lambda)$, $\lambda \in \mathbb{R}$, of E_k by coherent (torsion-free) subsheaves (defined over B), so that $\mathcal{F}_k(\lambda)$ is the maximal subsheaf of E_k such that all of its Harder-Narasimhan slopes are bigger than λ . The filtration \mathcal{F}_k is just a “renaming” of the Harder-Narasimhan filtration of E_k . Now, for any $b \in B$, we denote by \mathcal{F}_b the filtration induced by $\mathcal{F}_k(\lambda)$ on $R(X_b, L_b) = \bigoplus_{k=0}^{\infty} H^0(X_b, L_b^k)$ of the fiber $X_b = \pi^{-1}(b)$, $L_b = L|_{X_b}$, $b \in B$. It was established in [16] for $\dim B = 1$ and in [36, Proposition 2.5] for any projective B , that for generic $b \in B$, the above filtration is submultiplicative, i.e. for any $t, s \in \mathbb{R}$, $k, l \in \mathbb{N}$, we have

$$\mathcal{F}_b^t H^0(X_b, L_b^k) \cdot \mathcal{F}_b^s H^0(X_b, L_b^l) \subset \mathcal{F}_b^{t+s} H^0(X_b, L_b^{k+l}). \quad (2.9)$$

Remark, however, that the projectivity assumption was never used in [36, Proposition 2.5], and so submultiplicativity holds for general complex manifolds B . Theorem 1.5 is then a formal consequence of the submultiplicativity and Proposition 2.3, saying that the above filtration is bounded in the terminology of [12]. For the (different) proofs of this last result, see [16, Théorème 3.4.3], [12, Theorem A] and [35, Theorem 1.9]. \square

Let us now recall some crucial facts from Hermitian Yang-Mills theory, following the pioneering work of Atiyah-Bott [2] and later developments by Donaldson [28], Daskalopoulos-Wentworth [20] and others. We fix a compact complex manifold B of dimension m with a Gauduchon Hermitian form ω_B on B . Let E be a holomorphic vector bundle of rank r over B . For a

Hermitian metric h^E on E , we denote by R^E its curvature. For any $p \in [1, +\infty[$, $c \in \mathbb{R}$, we define the *Hermitian Yang-Mills functional* as

$$\begin{aligned} \text{HYM}_{p,c}(E, h^E) &:= \int_B \text{Tr} \left| \frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R_x^E - c \cdot \text{Id}_E \right|^p \omega_B^m(x), \\ \text{HYM}_{+\infty,c}(E, h^E) &:= \sup_{x \in X} \left\| \frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R_x^E - c \cdot \text{Id}_E \right\|, \end{aligned} \quad (2.10)$$

where $\|\cdot\|$ means the operator norm.

As in (1.7), we denote the Harder-Narasimhan ω_B -slopes of E by μ_1, \dots, μ_r . Let $\mu_{\min} := \mu_r$, $\mu_{\max} := \mu_1$, be the minimal and the maximal slopes. We define the probability measure

$$\mu_E := \frac{1}{r} \sum_{i=1}^r \delta[\mu_i]. \quad (2.11)$$

The following result lies at the heart of this paper.

Theorem 2.4. *For any $c \in \mathbb{R}$, $p \in [1, +\infty[$, and any Hermitian metric h^E on E , we have*

$$\begin{aligned} \text{HYM}_{p,c}(E, h^E) &\geq r \cdot \int_{\mathbb{R}} |x - c|^p d\mu_E(x) \cdot \int_B [\omega_B^m], \\ \text{HYM}_{+\infty,c}(E, h^E) &\geq \max \left\{ |\mu_{\min} - c|, |\mu_{\max} - c| \right\}. \end{aligned} \quad (2.12)$$

For the proof of Theorem 2.4, remark first that for $p = +\infty$, it follows directly from Theorem 2.4 for $p \in [1, +\infty[$ by taking p -th root and then a limit $p \rightarrow +\infty$. For the proof of Theorem 2.4 for $p \in [1, +\infty[$, see Atiyah-Bott [2, Proposition 8.20] if $\dim B = 1$, Daskalopoulos-Wentworth [20, Lemma 2.17, Corollary 2.22, Proposition 2.25] if B is Kähler of any dimension (even though the article [20] is written for surfaces, cf. Sibley [68, §3.2]), and Li-Zhang-Zhang [49, Theorem 1.6 and Corollary 4.4] for the non-Kähler case of any dimension. It is remarkable that the bounds from Theorem 2.4 are actually tight, see Theorem 3.5 for a precise statement.

Proof of Theorem 1.7. Conserving the notations introduced in (1.7), define the probability measure, $\eta_{k,0}^{HN}$, $k \in \mathbb{N}$, on \mathbb{R} as

$$\eta_{k,0}^{HN} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta[\mu_i^k], \quad (2.13)$$

We apply Theorem 2.4 for (E_k, h^{E_k}) , for any $c \in \mathbb{R}$, $p \in [1, +\infty[$, $k \in \mathbb{N}$, to get

$$\begin{aligned} \text{HYM}_{p,ck}(E_k, h^{E_k}) &\geq N_k \cdot \int_{\mathbb{R}} |x - ck|^p d\eta_{k,0}^{HN}(x) \cdot \int_B [\omega_B^m], \\ \text{HYM}_{+\infty,ck}(E_k, h^{E_k}) &\geq \max \left\{ |\mu_{\min}^k - ck|, |\mu_{\max}^k - ck| \right\}. \end{aligned} \quad (2.14)$$

Directly from Theorem 2.2 and Proposition 2.1, for any $p \in [1, +\infty[$, $c \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\text{HYM}_{p,ck}(E_k, h^{E_k})}{k^p \cdot N_k} &= \frac{\text{FYM}_{p,c}(\pi, h^L)}{\int_{X_b} [\omega^n]}, \\ \lim_{k \rightarrow \infty} \frac{\text{HYM}_{+\infty,ck}(E_k, h^{E_k})}{k} &= \text{FYM}_{+\infty,c}(\pi, h^L), \end{aligned} \quad (2.15)$$

where $b \in B$ is an arbitrary point, and X_b is the fiber of π at b .

We now divide both sides of the first inequality of (2.14) by $k^p \cdot N_k$, take the limit $k \rightarrow \infty$, and apply Theorem 1.5 and (2.15) to deduce

$$\text{FYM}_{p,c}(\pi, h^L) \geq \int_{\mathbb{R}} |x - c|^p d\eta^{HN}(x) \cdot \int_{X_b} [\omega^n] \cdot \int_B [\omega_B^m]. \quad (2.16)$$

This establishes Theorem 1.7 for $p \in [1, +\infty[$, as $\int_{X_b} [\omega^n] \cdot \int_B [\omega_B^m] = \int_X [\omega^n] \cdot \pi^* [\omega_B^m]$. To get Theorem 1.7 for $p = +\infty$, we divide both sides of the second inequality of (2.14) by k , take limit $k \rightarrow \infty$, and apply Theorem 1.5 and (2.15). \square

3 Horizontal curvature on subfamilies and generic fibered nefness

The main goal of this section is to establish Theorems 1.9, 1.10. For this, we construct a sequence of metrics on the polarizing line bundle from a sequence of Hermitian metrics on direct images. To show that horizontal mean curvature behaves well under this procedure, we rely on a fibered analogue of the principle that ‘‘a curvature of a vector bundle increases under taking quotients’’.

More precisely, consider a holomorphic submersion $\pi : X \rightarrow B$ between compact complex manifolds X and B . We denote $m := \dim B$. Consider an embedding $\iota : Y \hookrightarrow X$ of a smooth complex manifold Y , such that restriction of π , $\pi|_Y : Y \rightarrow B$, is a submersion. We fix a $(1, 1)$ -form ω_X on X , which is positive along the fibers of π , and denote $\omega_Y := \iota^* \omega_X$. We fix a Hermitian $(1, 1)$ -form ω_B on B , and denote by $\wedge_{\omega_B} \omega_{Y,H} \in \mathcal{C}^\infty(Y)$, $\wedge_{\omega_B} \omega_{X,H} \in \mathcal{C}^\infty(X)$, the horizontal mean curvatures of ω_Y and ω_X respectively.

Lemma 3.1. For any $y \in Y$, we have $\wedge_{\omega_B} \omega_{Y,H}(y) \geq \wedge_{\omega_B} \omega_{X,H}(\iota(y))$.

Remark 3.2. An equivalent result was established in [33, (2.2)] by a slightly different method.

Proof. Let us fix $y \in Y$, and denote by $b := \pi|_Y(y)$, $x := \iota(y)$, and by e_1, \dots, e_m an orthonormal basis of $T_b^{1,0} B$ with respect to ω_B . We denote by $e_1^X, \dots, e_m^X \in T_x^{1,0} X$ the horizontal lifts of e_1, \dots, e_m , defined with respect to ω_X , i.e. $d\pi(e_i^X) = e_i$, $i = 1, \dots, m$ and e_1^X, \dots, e_m^X are orthogonal (with respect to ω_X) to the tangent space of the fibers, $T^V X$, of π . Similarly, we denote by $e_1^Y, \dots, e_m^Y \in T_y^{1,0} Y$ the horizontal lifts of e_1, \dots, e_m , defined with respect to ω_Y . Clearly, using implicitly the embedding of $T_y Y$ in $T_x X$ through ι , we can write $e_i^Y = e_i^X + v_i$, where $v_i \in T_x^V X$. But then, since e_i^X and v_i are orthogonal with respect to ω_X , and ω_X is positive in the vertical directions, we obtain $\sqrt{-1} \omega_Y(e_i^Y, \bar{e}_i^Y) = \sqrt{-1} \omega_X(e_i^X, \bar{e}_i^X) + \sqrt{-1} \omega_X(v_i, \bar{v}_i) \geq \sqrt{-1} \omega_X(e_i^X, \bar{e}_i^X)$. By taking a sum of the above inequality over all $i = 1, \dots, m$, we establish the needed inequality. \square

Another ingredient we need is the calculation of the horizontal mean curvature for projectivizations of vector bundles. More precisely, let (F, h^F) be a Hermitian vector bundle over B of rank r . Let $\mathcal{O}(1)$ be the hyperplane bundle over $\mathbb{P}(F^*)$, $\pi : \mathbb{P}(F^*) \rightarrow B$. We endow $\mathcal{O}(1)$ with the metric $h^{\mathcal{O}(1)}$ induced by h^F . We denote by R^F the curvature of the Chern connection on (F, h^F) , by ω the first Chern class of $(\mathcal{O}(1), h^{\mathcal{O}(1)})$, and by ω_H its horizontal component.

Lemma 3.3. In the above notations, for any $x \in X$, we have

$$\inf_{y \in \mathbb{P}(F_x^*)} \wedge_{\omega_B} \omega_H(y) = \inf_{f \in F_x, \|f\|_{h^F} = 1} \left\langle \frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R^F f, f \right\rangle_{h^F}. \quad (3.1)$$

Proof. We fix some local coordinates $z := (z_1, \dots, z_n)$ on B , centered at $x \in B$, and a local normal frame f_1, \dots, f_r of F at x , defined in a neighborhood U of x . By a *normal frame* we mean the one satisfying $\langle f_i, f_j \rangle_{h^F} = \delta_{ij} - \sum_{\lambda\mu} d_{\lambda\mu ij} z_\lambda \bar{z}_\mu + O(|z|^3)$ for some constants $d_{\lambda\mu ij}$. We denote by f_1^*, \dots, f_r^* the dual frame of F^* . The above data defines a trivialization of $U \times \mathbb{P}(\mathbb{C}^r) \rightarrow \mathbb{P}(F^*)$ near $\pi^{-1}(x)$ as follows. For $a := (a_1, \dots, a_r)$, where $a_i \in \mathbb{C}$, $1 \leq i \leq r$, and not all a_i are equal to zero, the trivialization is given by the map $(z, [a]) \rightarrow [\sum_{i=1}^r a_i f_i^*(z)] \in \mathbb{P}(F^*)$. Now we take $a_1 = 1$ and denote $b_i := a_i$, $2 \leq i \leq r$, $b := (b_i)$. Then (z, b) gives a chart for $\mathbb{P}(F^*)$. The well-known formula, cf. [24, Formula (V.15.15)], shows that at the point $(x, [f_1^*]) \in \mathbb{P}(F^*)$, the curvature, $R^{\mathcal{O}(1)}$, of the hyperplane bundle $(\mathcal{O}(1), h^{\mathcal{O}(1)})$, can be expressed as follows

$$R_{(x, [f_1^*])}^{\mathcal{O}(1)} = \sum_{2 \leq j \leq r} db_j \wedge d\bar{b}_j + \langle R^F f_1, f_1 \rangle_{h^F}. \quad (3.2)$$

In particular, we see that the vertical part of the form $\omega = c_1(\mathcal{O}(1), h^{\mathcal{O}(1)})$ is the Fubini-Study form induced by h^F , and the horizontal part of ω , ω_H , evaluated at $(x, [f_1^*]) \in \mathbb{P}(E^*)$, coincides with $\frac{\sqrt{-1}}{2\pi} \langle R^F f_1, f_1 \rangle_{h^F}$. The result follows directly from this. \square

Remark 3.4. From the proof of the above lemma, we see that $\wedge_{\omega_B} \omega_H$ is constant if and only if $\wedge_{\omega_B} R^F$ is the identity endomorphism up to a constant, which means that ω is fibered Einstein if and only if (F, h^F) is Hermite Einstein.

Finally, the last ingredient we need is about the optimality of the bounds on the Hermitian Yang-Mills functional from Theorem 2.4.

Theorem 3.5. *In the notations Theorem 2.4, for any $c \in \mathbb{R}$, $p \in [1, +\infty[$, we have*

$$\begin{aligned} \inf_{h^E} \text{HYM}_{p,c}(E, h^E) &= r \cdot \int_{\mathbb{R}} |x - c|^p d\mu_E(x) \cdot \int_B [\omega_B^m], \\ \inf_{h^E} \text{HYM}_{+\infty,c}(E, h^E) &= \max \left\{ |\mu_{\min} - c|, |\mu_{\max} - c| \right\}. \end{aligned} \quad (3.3)$$

where the infimum is taken over all Hermitian metrics h^E on E .

For the proof of Theorem 3.5 for $p \in [1, +\infty[$, see [2, Proposition 8.20] if $\dim B = 1$. For higher dimensions, this is a direct consequence of the existence of L^p -approximate critical hermitian structure on E , see [20, Definition 3.9] for the definition, and [20, Theorem 3.11] and Sibley [68, Theorem 1.3] for the proofs if B is Kähler of dimension 2 and any dimension respectively. See also Jacob [44, Theorems 2, 3] for an alternative proof in the Kähler setting. In the non-Kähler setting and for $p = +\infty$, the result was established in [49, Theorems 1.2, 1.7].

In what follows, we will only need the following result, which is a direct consequence of Theorem 3.5 for $p = +\infty$.

Corollary 3.6 ([49, Theorem 1.3]). For any $\epsilon > 0$, there is a Hermitian metric h_ϵ^E on E , such that the associated curvature, R_ϵ^E , for any $b \in B$, $e \in E_b$, verifies $\frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R_\epsilon^E \geq (\mu_{\min} - \epsilon) \cdot \text{Id}_E$.

Proof of Theorem 1.9. First of all, for a given $\epsilon > 0$, let $k \in \mathbb{N}$ be such that $\frac{\mu_{\min}^k}{k} > \eta_{\min}^{HN} - \frac{\epsilon}{2}$. Using Corollary 3.6, we construct a metric h_ϵ^k on E_k such that for the associated curvature, $R_\epsilon^{E_k}$, we have

$$\frac{\sqrt{-1}}{2\pi} \wedge_{\omega_B} R_\epsilon^{E_k} \geq \left(\mu_{\min}^k - \frac{\epsilon}{2} \right) \cdot \text{Id}_{E_k}. \quad (3.4)$$

We denote by ω_k the $(1, 1)$ -form on $\mathbb{P}(E_k^*)$, given by the first Chern class of the curvature of the hyperplane line bundle induced by the metric h_k^ϵ . We denote by $\omega_{H,k}$ the horizontal part of this curvature. From Lemma 3.3, the choice of $k \in \mathbb{N}$ and (3.4), we deduce

$$\inf_{x \in \mathbb{P}(E_k^*)} \wedge_{\omega_B} \omega_{H,k}(x) \geq k \cdot (\eta_{\min}^{HN} - \epsilon). \quad (3.5)$$

We will now assume that k was chosen big enough so that L^k is relatively ample. Consider now the Kodaira embedding $\iota_k : X \hookrightarrow \mathbb{P}(E_k^*)$. It is well-known that there is a canonical isomorphism between $\iota_k^* \mathcal{O}(1)$ and L^k . We denote by h_ϵ^L the metric induced on L by the pull-back; then $\omega(h_\epsilon^L) = \frac{1}{k} \iota_k^* \omega_{H,k}$. By Lemma 3.1 and (3.5), we conclude that $\inf_{x \in X} \wedge_{\omega_B} \omega_H(h_\epsilon^L) \geq \eta_{\min}^{HN} - \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, we deduce $\sup_{h^L} \inf_{x \in X} \wedge_{\omega_B} \omega_H(h^L) \geq \eta_{\min}^{HN}$. In combination with the upper bound from (1.10), this finishes the proof. \square

Now, let us establish Theorem 1.10. As in the statement of Theorem 1.10, we fix any Kähler forms $\omega_{B,1}, \dots, \omega_{B,m-1}, \omega_X$. Then the form $\omega_{B,1} \wedge \dots \wedge \omega_{B,m-1}$ is positive in the sense of [24, (III.1.1)]. By Michelsohn [58, (4.8)], there is a Hermitian form ω_B on B , verifying

$$\omega_B^{m-1} = \omega_{B,1} \wedge \dots \wedge \omega_{B,m-1}. \quad (3.6)$$

Remark that ω_B is automatically Gauduchon, but not necessarily Kähler. Since the form ω_B is Gauduchon, it makes sense to define the ω_B -degree and study the Harder-Narasimhan ω_B -slopes, as we did before Theorem 1.1. Remark that due to a relation (3.6), this ω_B -degree coincides with the degree associated with a multipolarization $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$, as defined in [36, before (1.1)]. Below, the invariant η_{\min}^{HN} and other quantities are calculated with respect to ω_B . The following result, alongside with Theorem 1.9, lie at the core of the proof of Theorem 1.10.

Proposition 3.7 ([36, Proposition 5.2]). A relatively ample line bundle L over X is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to π if and only if $\eta_{\min}^{HN} \geq 0$.

Proof of Theorem 1.10. Let us first establish Theorem 1.10 under an additional assumption that L is relatively ample. We assume first that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to π . By Theorem 1.9 and Proposition 3.7, we establish that for any $\epsilon > 0$, there is a relatively positive Hermitian metric h_ϵ^L on L , such that $\wedge_{\omega_B} \omega_H(h_\epsilon^L) > -\epsilon$. From the definition of \wedge_{ω_B} and the trivial fact that there is $C > 0$, such that $\omega_B^m < C\omega_X^m$, we establish

$$\omega_H(h_\epsilon^L) \wedge \pi^* \omega_{B,1} \wedge \dots \wedge \pi^* \omega_{B,m-1} \geq -C\epsilon \cdot \omega_X^m. \quad (3.7)$$

But the form $\omega(h_\epsilon^L)$ is relatively positive, so (3.7) implies (1.12) for $\epsilon := C\epsilon$, which finishes the proof of one direction of Theorem 1.10 under an additional assumption that L is relatively ample.

To prove the opposite direction under the same additional assumption that L is relatively ample, assume that we have a sequence of metrics h_ϵ^L , verifying (1.12). We will now show that one can cook up a sequence of relatively positive metrics, $h_{\epsilon,0}^L$, verifying similar bounds. Indeed, let us fix an arbitrary relatively positive metric h_0^L on L . By (1.12), it is easy to see that there is $c > 0$, such that for any $\epsilon > 0$, over the fibers, the following inequality is satisfied $c_1(L, h_\epsilon^L) \geq -\epsilon c \cdot c_1(L, h_0^L)$. Then an easy calculation shows that the sequence of metrics $h_{\epsilon,0}^L := (h_\epsilon^L)^{1-2c\epsilon} \cdot (h_0^L)^{2c\epsilon}$ is positive along the fibers and verifies the inequality (1.12) with $C\epsilon$ in place of ϵ , for some $C > 0$. Then as in (3.7), there is $C > 0$, such that for any $\epsilon > 0$, we have $\wedge_{\omega_B} \omega_H(h_{\epsilon,0}^L) > -C\epsilon$. By Theorem 1.9,

we then conclude that $\eta_{\min}^{HN} \geq 0$, which implies that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef by Proposition 3.7.

We now only assume that L is relatively nef. We assume first that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to π . Now, for any $\delta \in \mathbb{Q}$, $\delta > 0$, consider the \mathbb{Q} -line bundle $L_\delta := L \otimes L_0^\delta$, where L_0 is some ample line bundle on X . Clearly, L_δ is relatively ample, and it is also stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to π . The already established relatively ample case of Theorem 1.10 says that L_δ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef if and only if for any $\epsilon > 0$, there is a Hermitian metric $h_\epsilon^{L_\delta}$ on L_δ , such that the analogue of (1.12) holds. Let h_0^L be now an arbitrary positive metric on L_0 . It is easy to see that if δ and ϵ are sufficiently small, then the metric h_ϵ^L on L , which is constructed as the only metric verifying $h_\epsilon^{L_\delta} = h_\epsilon^L \cdot (h_0^L)^\delta$, will satisfy the analogue of (1.12) (for $C\epsilon$ instead of ϵ for some $C > 0$). This shows one direction of Theorem 1.10.

Inversely, if for any $\epsilon > 0$ there is a metric h_ϵ^L as in (1.12), then the metrics $h_\epsilon^{L_\delta}$, defined by the above formula, will also satisfy a similar inequality. Hence, by the already established case of Theorem 1.10, L_δ is then stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef for any $\delta \in \mathbb{Q}$, $\delta > 0$. In particular, for any $\delta \in \mathbb{Q}$, $\delta > 0$, the line bundle $L_{2\delta} = L_\delta \otimes L_0^\delta$ is $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef for any $\delta \in \mathbb{Q}$, $\delta > 0$, which means that L is stably $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef, as $L_{2\delta} = L \otimes L_0^{2\delta}$. This finishes the proof. \square

4 Asymptotic Riemann-Roch-Grothendieck and semistability

The main goal of this section is to prove a numerical obstruction for asymptotic semistability of direct images from Theorem 1.3. This will be based on an asymptotic version of Riemann-Roch-Grothendieck theorem, which we establish here in the singular setting.

To begin, let us recall some basic facts about Bott-Chern and Aeppli cohomologies. Let Y be a compact complex manifold of dimension n . We denote by $\Omega^{(p,q)}(Y)$ the vector space of (p, q) -differential forms on Y , $p, q \in \mathbb{N}$, and define $\partial : \Omega^{(p,q)}(Y) \rightarrow \Omega^{(p+1,q)}(Y)$, $\bar{\partial} : \Omega^{(p,q)}(Y) \rightarrow \Omega^{(p,q+1)}(Y)$, as usual. Recall that Bott-Chern cohomology, $H_{BC}^{p,q}(Y)$, is defined as

$$H_{BC}^{p,q}(Y) := \frac{(\ker \partial : \Omega^{(p,q)}(Y) \rightarrow \Omega^{(p+1,q)}(Y)) \cap (\ker \bar{\partial} : \Omega^{(p,q)}(Y) \rightarrow \Omega^{(p,q+1)}(Y))}{\text{im } \partial \bar{\partial} : \Omega^{(p-1,q-1)}(Y) \rightarrow \Omega^{(p,q)}(Y)}. \quad (4.1)$$

Recall that Aeppli cohomology, $H_A^{p,q}(Y)$, is defined as

$$H_A^{p,q}(Y) := \frac{\ker \partial \bar{\partial} : \Omega^{(p,q)}(Y) \rightarrow \Omega^{(p+1,q+1)}(Y)}{(\text{im } \partial : \Omega^{(p-1,q)}(Y) \rightarrow \Omega^{(p,q)}(Y)) + (\text{im } \bar{\partial} : \Omega^{(p,q-1)}(Y) \rightarrow \Omega^{(p,q)}(Y))}. \quad (4.2)$$

It is standard that for compact Kähler manifolds, the two cohomologies coincide. For $p, q = 0, \dots, n$, we have the natural (non-degenerate, see [66, §2.c]) pairing

$$\wedge : H_{BC}^{p,q}(Y) \times H_A^{n-p,n-q}(Y) \rightarrow \mathbb{C}, \quad (4.3)$$

given by the wedge product and integration. If $p : Y \rightarrow B$ is a holomorphic map between compact complex manifolds, then for $s := \dim Y - \dim B$, we have a natural map $p_* : H_{BC}^{p,q}(Y) \rightarrow H_{BC}^{p-s,q-s}(B)$, defined by the pairing (4.3) and the pull-back p^* .

The Bott-Chern cohomology can be generally defined for arbitrary complex analytic spaces Y in the sense of [24, Definition II.5.2], where one considers differential forms on Y obtained by

pullbacks of smooth differential forms through local embeddings of the space into complex vector spaces. Due to a theorem of Lelong, cf. [24, Theorem III.2.7], the intersection pairing (4.3) can still be defined in this setting, and so the slope (1.4) is well-defined. Also, using the non-degeneracy of the wedge product (4.3) for compact manifolds, we can extend the definition of the pushforward for maps between a compact complex analytic spaces Y and a compact manifold B .

Now, let E be a holomorphic vector bundle over Y . Using Chern-Weil theory, one can construct for any Hermitian metric h^E on E a corresponding Chern character form, $\text{ch}(E, h^E)$, which is a d -closed form in $\bigoplus_{p=0}^{+\infty} \Omega^{(p,p)}(Y)$. Bott-Chern in [11] showed that the resulting class in Bott-Chern cohomology doesn't depend on the choice of the metric. This gives a definition of the Chern character, $\text{ch}(E)$ of a vector bundle E with values in Bott-Chern cohomology.

Following earlier work by Qiang [64], Bismut-Shu-Wei in [7] generalized the definition of the Chern character with values in Bott-Chern cohomology for any coherent sheaf \mathcal{E} on Y . If \mathcal{E} has a finite locally free projective resolution (which is always the case if Y is projective), this construction corresponds to the one given by the alternating sum of Chern characters of the resolution. By [7, §8.6], the (k, k) -component of the Chern character, $\text{ch}_k(\mathcal{E})$, we have

$$\text{ch}_0(\mathcal{E}) = \text{rk}(\mathcal{E}), \quad \text{ch}_1(\mathcal{E}) = c_1(\det \mathcal{E}), \quad (4.4)$$

where $\det \mathcal{E}$ is the Knudsen-Mumford determinant [45]. Without entering into details of the construction, we mention that the absence of finite locally free projective resolutions of coherent sheaves for general complex manifolds is circumvented in [7] by the use of so-called antiholomorphic superconnections, introduced by Block [8], see [7, Theorem 6.7].

Now, recall that for any proper holomorphic map $p : Y \rightarrow B$, and any coherent sheaf \mathcal{E} , Grauert theorem tells that the direct image sheaves $R^q p_* \mathcal{E}$, $q \in \mathbb{N}$, are coherent. The main result of this section goes as follows.

Theorem 4.1. *Let Y be an irreducible compact complex analytic space, L an arbitrary line bundle on Y and \mathcal{E} a coherent sheaf on Y . Let $p : Y \rightarrow B$ be a holomorphic map to a compact complex manifold B . Then for any $r \in \mathbb{N}$, and $s := \dim Y - \dim B + r$, in the Bott-Chern cohomology*

$$\lim_{k \rightarrow \infty} \frac{1}{k^s} \sum_{t=0}^{\dim Y} (-1)^t \text{ch}_r(R^t p_*(\mathcal{E} \otimes L^k)) = \frac{\text{rk}(\mathcal{E})}{s!} \cdot p_*(c_1(L)^s). \quad (4.5)$$

Remark 4.2. Despite a huge amount of literature, we were not able to find the proof of this result under the stated hypotheses (even for projective Y, B). For flat maps p and relatively ample L , this result can be alternatively established using Knudsen-Mumford expansions, see [45] and Phong-Ross-Sturm [61, Theorem 3]. Note, however, that flatness doesn't pass through subfamilies, and there is no flatness assumption in Theorem 1.3.

The proof of this result relies on the recent result of Bismut-Shu-Wei [7] establishing the Riemann-Roch-Grothendieck theorem in Bott-Chern cohomology for arbitrary holomorphic maps between (*smooth!*) complex manifolds, relying on the Hypoelliptic Laplacian of Bismut [5], and generalizing previous results as [10]. More precisely, the main result of [7] says the following.

Theorem 4.3. *Let $p : Y \rightarrow B$ be a holomorphic map between compact complex manifolds. Then for any coherent sheaf \mathcal{E} on Y , the following identity holds in Bott-Chern cohomology*

$$\text{Td}(TB) \cdot \sum_{t=0}^{\dim Y} (-1)^t \text{ch}(R^t p_*(\mathcal{E})) = p_*(\text{Td}(TY) \cdot \text{ch}(\mathcal{E})), \quad (4.6)$$

where $\text{Td}(TB)$, $\text{Td}(TY)$ are the Todd classes.

Proof of Theorem 4.1. Remark first that for smooth manifolds Y , the result follows directly from Theorem 4.3 by (4.4) and the fact that the 0-degree part of the Todd class is identity. If $\dim Y = 0$, then Y is automatically smooth, and, hence, Theorem 4.1 holds as stated.

We will argue by induction on the dimension of Y . For this, we consider a resolution of singularities $f : \hat{Y} \rightarrow Y$ of Y . For any $q = 1, \dots, \dim Y$, we define the sheaves \mathcal{Q}_q on Y as $\mathcal{Q}_q := R^q f_* f^* \mathcal{E}$, where $f^* \mathcal{E} := f^{-1} \mathcal{E} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_{\hat{Y}}$. We define the sheaf \mathcal{Q}_0 on Y by the following short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow R^0 f_* f^* \mathcal{E} \rightarrow \mathcal{Q}_0 \rightarrow 0. \quad (4.7)$$

By Grauert theorem, the sheaves \mathcal{Q}_q , $q = 0, \dots, \dim Y$, are coherent. Since the resolution of singularities is biholomorphic away from a subset of singular points of Y , and over the locally free locus of \mathcal{E} , by projection formula, cf. [42, Exercise II.5.1d)], we have $R^0 f_* f^* \mathcal{E} = \mathcal{E}$, the supports of the sheaves \mathcal{Q}_q , $q = 0, \dots, \dim Y$, are *proper* analytic subsets of Y , which, by irreducibility of Y , have strictly smaller dimension than Y , cf. [24, Proposition II.4.2.6]. By this and the usual *de-visage* techniques, see [39, Théorème 3.1.2], cf. [42, Proposition I.7.4], for any $q = 0, \dots, \dim Y$, there is $r(q) \in \mathbb{N}$, and complex analytic subspaces $\iota_{i,q} : Z_{i,q} \hookrightarrow Y$, with some ideal sheaves $\mathcal{I}_{i,q}$ on $Z_{i,q}$ and a filtration $\mathcal{F}_{i,q}$ of \mathcal{Q}_q , $i = 0, \dots, r(q)$, $\mathcal{F}_{0,q} = \{0\}$, $\mathcal{F}_{r(q),q} = \mathcal{Q}_q$, $\mathcal{F}_{i-1,q} \subset \mathcal{F}_{i,q}$, $i = 1, \dots, r(q)$, such that for any $i = 1, \dots, r(q)$, we have $\mathcal{F}_{i,q}/\mathcal{F}_{i-1,q} = \iota_{i,q,*}(\mathcal{I}_{i,q})$. We denote $p_{i,q} := p \circ \iota_{i,q}$, $i = 1, \dots, r(q)$, $\hat{p} := p \circ f$. We argue that for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{t=0}^{\dim Y} (-1)^t \text{ch}(R^t p_*(\mathcal{E} \otimes L^k)) &= \sum_{t=0}^{\dim Y} (-1)^t \text{ch}(R^t \hat{p}_*(f^* \mathcal{E} \otimes f^* L^k)) \\ &\quad - \sum_{t,u=0}^{\dim Y} (-1)^{t+u} \sum_{i=1}^{r(q)} \text{ch}(R^t (p_{i,u})_*(\mathcal{I}_{i,u} \otimes \iota_{i,u}^* L^k)). \end{aligned} \quad (4.8)$$

Once (4.8) is established, Theorem 4.1 would follow by induction, as the space \hat{Y} is smooth, and so for the first summand on the right-hand side of (4.8), the smooth version of Theorem 4.1 applies, and the second summand doesn't contribute to the asymptotics by induction hypothesis, as all $Z_{i,q}$ have strictly smaller dimensions than Y .

Now, let us establish (4.8). First of all, since in derived category, there is a canonical isomorphism between the functors $R\hat{p}_*$ and $Rp_* Rf_*$, [9, Proposition I.9.15], cf. [7, (3.13)], and the construction of Chern character, defined using derived category of coherent sheaves, factors through the K -theory of the derived category of coherent sheaves [7, Theorem 8.11 and §8.9], we have

$$\sum_{t=0}^{\dim Y} (-1)^t \text{ch}(R^t \hat{p}_*(f^* \mathcal{E} \otimes f^* L^k)) = \sum_{t,u=0}^{\dim Y} (-1)^{t+u} \text{ch}(R^t p_*(R^u f_*(f^* \mathcal{E} \otimes f^* L^k)). \quad (4.9)$$

Now, from the exact sequence (4.7), using again the fact that the construction of the Chern character passes through the formation of K -theory, we obtain

$$\sum_{t=0}^{\dim Y} (-1)^t \text{ch}(R^t p_*(R^0 f_*(f^* \mathcal{E} \otimes f^* L^k))$$

$$= \sum_{t=0}^{\dim Y} (-1)^t \operatorname{ch}(R^t p_*(\mathcal{E} \otimes L^k)) + \sum_{t=0}^{\dim Y} (-1)^t \operatorname{ch}(R^t p_*(\mathcal{Q}_0 \otimes L^k)). \quad (4.10)$$

Similarly, for any $q = 0, \dots, \dim Y$, we obtain

$$\sum_{t=0}^{\dim Y} (-1)^t \operatorname{ch}(R^t p_*(\mathcal{Q}_q \otimes L^k)) = \sum_{t=0}^{\dim Y} (-1)^t \sum_{i=0}^{r(q)} \operatorname{ch}(R^t p_*(\iota_{i,q,*}(\mathcal{J}_{i,q}) \otimes L^k)). \quad (4.11)$$

Remark, however, that since $\iota_{i,q}$ is a closed embedding, $\iota_{i,q,*}$ is an exact functor, so we have $R^v \iota_{i,q,*} = 0$ for $v = 1, \dots, \dim Y$, and $R^0 \iota_{i,q,*} = \iota_{i,q,*}$. Moreover, by the projection formula, cf. [42, Exercise II.5.1d)], we have $R^0 \iota_{i,q,*}(\mathcal{J}_{i,q}) \otimes L^k = R^0 \iota_{i,q,*}(\mathcal{J}_{i,q} \otimes \iota_{i,q}^* L^k)$. In particular, for any $t, q = 0, \dots, \dim Y$, $i = 1, \dots, r(q)$, we can write

$$\operatorname{ch}(R^t p_*(\iota_{i,q,*}(\mathcal{J}_{i,q}) \otimes L^k)) = \sum_{v=0}^{\dim Y} (-1)^v \operatorname{ch}(R^t p_*(R^v \iota_{i,q,*}(\mathcal{J}_{i,q} \otimes \iota_{i,q}^* L^k))). \quad (4.12)$$

But using the same argument as in (4.9), we have

$$\sum_{t,v=0}^{\dim Y} (-1)^{t+v} \operatorname{ch}(R^t p_*(R^v \iota_{i,q,*}(\mathcal{J}_{i,q} \otimes \iota_{i,q}^* L^k))) = \sum_{t=0}^{\dim Y} (-1)^t \operatorname{ch}(R^t (p_{i,q})_*(\mathcal{J}_{i,q} \otimes \iota_{i,q}^* L^k)). \quad (4.13)$$

Now, a combination of (4.9), (4.10), (4.11), (4.12) and (4.13), gives us (4.8). \square

Now, let us finally establish an application of Theorem 4.1 towards the study of Harder-Narasimhan slopes of direct images. We fix an irreducible compact complex analytic space Y of dimension $k + m$, $k \geq 0$, with a surjective holomorphic map $\pi : Y \rightarrow B$. Let ω_B be a Gauduchon Hermitian form on B . Let L be a relatively ample line bundle on Y . Recall that in (1.4), we defined the slope of Y , and before Theorem 1.1, we defined the slopes of coherent sheafs.

Lemma 4.4. The following identity holds $\mu(Y) \cdot \int_B [\omega_B^m] = \lim_{k \rightarrow \infty} \mu(R^0 \pi_* L^k) / k$.

Proof. By Serre vanishing theorem, the higher direct images $R^v \pi_* L^k$, $v \geq 1$, vanish. By (4.4), we conclude that

$$\mu(R^0 \pi_* L^k) = \frac{\int_Y [\operatorname{ch}_1(R^0 \pi_* L^k)] \cdot [\omega_B^{m-1}]}{\operatorname{ch}_0(R^0 \pi_* L^k)}. \quad (4.14)$$

The result now follows directly from Theorem 4.1. \square

Proposition 4.5. The vector bundles E_k are asymptotically semistable if and only if $\eta_{\min}^{HN} = \eta_{\max}^{HN}$. Moreover, if E_k are asymptotically semistable, then for any subsheaves \mathcal{F}_k of E_k , $\operatorname{rk}(\mathcal{F}_k) > 0$, and any $\epsilon > 0$, for k big enough, we have $\mu(\mathcal{F}_k) \leq (1 + \epsilon) \cdot \mu(E_k)$.

Proof. Remark first that the maximal and the minimal slopes satisfy

$$\begin{aligned} \mu_{\max}^k &= \sup \left\{ \mu(\mathcal{F}_k) : \mathcal{F}_k \text{ is a subsheaf of } E_k \right\}, \\ \mu_{\min}^k &= \inf \left\{ \mu(\mathcal{Q}_k) : \mathcal{Q}_k \text{ is a quotient sheaf of } E_k \right\}. \end{aligned} \quad (4.15)$$

Now, by Theorem 1.5, as $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \frac{\mu(E_k)}{k} = \int_{\mathbb{R}} x d\eta^{HN}(x). \quad (4.16)$$

From (4.15) and (4.16), we see that E_k are asymptotically semistable if and only if $\eta_{\min}^{HN} = \text{ess sup } \eta^{HN}$. However, by Theorem 1.5, $\text{ess sup } \eta^{HN}$ coincides with η_{\max}^{HN} , which finishes the proof of the first part of the theorem. The proof of the second statement of Proposition 4.5 follows directly by (4.15) and the first part. \square

Remark 4.6. From [36, Proposition 5.1], cf. also [76], we know that if $\dim B = 1$, then $\text{ess inf } \eta^{HN}$ coincides with η_{\min}^{HN} . The above proof shows that for $\dim B = 1$, the condition on the quotient sheaves is equivalent to asymptotic semistability.

Proposition 4.7. For any complex analytic subspace Y of X as in Theorem 1.3, the following bound holds $\mu(Y) \cdot \int_B [\omega_B^m] \geq \eta_{\min}^{HN}$.

Proof. Consider the following short exact sequence of sheaves associated with Y

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Y \rightarrow 0, \quad (4.17)$$

where \mathcal{I}_Y is the ideal sheaf of Y , consisting of local holomorphic functions on X , vanishing along Y , and \mathcal{O}_Y is the structure sheaf of Y associated with the reduced scheme structure of Y , i.e. defined by (4.17). By considering a long exact sequence of direct images associated with (4.17) and the map π , and using Serre vanishing theorem, we conclude that the restriction map $R^0 \pi_* L^k \rightarrow R^0 \pi|_{Y,*} L|_Y^k$ is surjective. Then, in the notations of Theorem 1.5, we have $\mu_{\min}^k \leq \mu(R^0 \pi|_{Y,*} L|_Y^k)$. We deduce Proposition 4.7 from Lemma 4.4 by dividing by k and passing to the limit $k \rightarrow \infty$. \square

Proof of Theorem 1.3. First of all, by Lemma 4.4, we conclude that

$$\lim_{k \rightarrow \infty} \mu(E_k)/k = \mu(X) \cdot \int_B [\omega_B^m] \geq \eta_{\min}^{HN}. \quad (4.18)$$

Moreover, since $\text{ess sup } \eta^{HN}$ coincides with η_{\max}^{HN} by [36, Theorem 1.1], by (4.16), we conclude that the equality in the above inequality holds if and only if $\eta_{\min}^{HN} = \eta_{\max}^{HN}$, i.e. when E_k is asymptotically semistable by Proposition 4.5. In particular, if E_k is asymptotically semistable, then by $\eta_{\min}^{HN} = \eta_{\max}^{HN}$, Proposition 4.7 and (4.18), we establish the first part of Theorem 1.3.

Let us now establish the second part. By a reformulation of the result of Xu-Zhuang [76, Lemma 2.26 and Proposition 2.28] from [36, (1.5)], we have

$$\eta_{\min}^{HN} = \inf_{C \subset X} \mu(C) \cdot \int_B [\omega_B^m], \quad (4.19)$$

where C runs over all irreducible curves in X , with project surjectively to B . In particular, from (4.19), we conclude that

$$\eta_{\min}^{HN} \geq \inf_{Y \subset X} \mu(Y) \cdot \int_B [\omega_B^m], \quad (4.20)$$

where Y are as in Theorem 1.3. A combination of (4.18) and (4.20) shows that if $\inf_{Y \subset X} \mu(Y) = \mu(X)$, then both (4.18) and (4.20) are actually equalities. By the remark after (4.18), this implies that E_k is asymptotically semistable, which finishes the proof. \square

Remark 4.8. It is interesting to know if the second part of Theorem 1.3 continues to hold for $\dim B > 1$. There are several potential pitfalls for that. First, since B is not necessarily projective, there might be very few analytic subspaces $Y \subset X$, projecting surjectively to B . Second, even for projective B , the main result of [36, Theorem 1.4] implies that the analogue of the bound (4.19) becomes tight if one considers among Y all subcurves in X projecting to generic curves over the base. But it seems that there is much more curves like that than analytic subspaces projecting surjectively to the base.

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