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# On Harder-Narasimhan slopes of direct images

Siarhei Finski

**Abstract.** For a polarized family of complex projective manifolds, we study the asymptotic distribution of Harder-Narasimhan slopes of direct image sheaves associated with high tensor powers of the polarization. We establish a theorem of Mehta-Ramanathan type, showing that this asymptotic distribution can be recovered from the analogous asymptotic distributions associated with base changes of the family over generic curves.

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## 1 Introduction

Consider a surjective flat holomorphic map  $\pi : X \rightarrow B$  between complex projective manifolds  $X$  and  $B$  of dimensions  $n + m$  and  $m$  respectively,  $n, m \in \mathbb{N}$ . Let  $L$  be a holomorphic line bundle over  $X$ , which is relatively ample with respect to  $\pi$ . We fix a *multipolarization*  $[\omega_B]$ , which is a collection of Kähler classes  $[\omega_{B,i}] \in H^{1,1}(B)$ ,  $i = 1, \dots, m - 1$ . The goal of this paper is to study the Harder-Narasimhan  $[\omega_B]$ -slopes of direct image sheaves  $E_k := R^0\pi_*L^k$  of  $L^k$  for big  $k \in \mathbb{N}$ .

More precisely, recall that a *slope* (or  $[\omega_B]$ -slope) of a coherent sheaf  $\mathcal{E}$  over  $B$  is defined as  $\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ , where the degree,  $\deg(\mathcal{E})$ , is given by  $\deg(\mathcal{E}) := \int_B c_1(\det(\mathcal{E})) \cdot [\omega_{B,1}] \cdots [\omega_{B,m-1}]$ , where  $\det \mathcal{E}$  is Knudsen-Mumford determinant of  $\mathcal{E}$ , see [29]. A torsion-free coherent sheaf  $\mathcal{E}$  is called *semistable* (or  $[\omega_B]$ -*semistable*) if for every coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ , verifying  $\mathrm{rk}(\mathcal{F}) > 0$ , we have  $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ . When the multipolarization is *symmetric*, i.e.  $[\omega_{B,1}] = \cdots = [\omega_{B,m-1}]$ , we recover the usual notion of slopes and semistability, as in [30].

Recall that any torsion-free coherent sheaf  $\mathcal{E}$  on  $(B, [\omega_B])$  admits a unique filtration by subsheaves  $\mathcal{E}_i$ ,  $i = 1, \dots, s$ , also called *Harder-Narasimhan filtration*:

$$\mathcal{E} = \mathcal{E}_s \supset \mathcal{E}_{s-1} \supset \cdots \supset \mathcal{E}_1 \supset \mathcal{E}_0 = \{0\}, \quad (1.1)$$

such that for any  $1 \leq i \leq s - 1$ , the quotient sheaf  $\mathcal{E}_i/\mathcal{E}_{i-1}$  is the maximal semistable (torsion-free) subsheaf of  $\mathcal{E}/\mathcal{E}_{i-1}$ , i.e. for any subsheaf  $\mathcal{F}$  of a (torsion-free) sheaf  $\mathcal{E}/\mathcal{E}_{i-1}$ , we have  $\mu(\mathcal{F}) \leq \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$  and  $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$  if  $\mu(\mathcal{F}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ . For the proof, see [28, Theorem 1.6.7] for symmetric multipolarization and [21, Corollary 2.27] for the general case.

We define the *Harder-Narasimhan slopes*,  $\mu_1, \dots, \mu_{\mathrm{rk}(\mathcal{E})}$  of  $\mathcal{E}$ , so that  $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$  appears among  $\mu_1, \dots, \mu_{\mathrm{rk}(\mathcal{E})}$  exactly  $\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$  times, and the sequence  $\mu_1, \dots, \mu_{\mathrm{rk}(\mathcal{E})}$  is non-increasing. We call  $\mu_{\min} := \mu_{\mathrm{rk}(\mathcal{E})}$  and  $\mu_{\max} := \mu_1$ , the minimal and the maximal slopes respectively.

By the flatness of  $\pi$ , for  $k \in \mathbb{N}$  big enough (which we assume implicitly from now on), the coherent sheaves  $E_k$  are locally free, cf. [24, Theorem III.9.9]. We denote  $N_k := \mathrm{rk}(E_k)$ , and

let  $\mu_1^k, \dots, \mu_{N_k}^k$  be the Harder-Narasimhan slopes of  $E_k$ . Let  $\mu_{\min}^k, \mu_{\max}^k$  be the minimal and the maximal slopes. We define the probability measure  $\eta_k^{HN}$  on  $\mathbb{R}$  as follows

$$\eta_k^{HN} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta \left[ \frac{\mu_i^k}{k} \right], \quad (1.2)$$

where  $\delta[x]$  is the Dirac mass at  $x \in \mathbb{R}$ . In Section 2, we establish the following result, serving as a starting point of the current work.

**Theorem 1.1.** *The sequence of measures  $\eta_k^{HN}$  converges weakly, as  $k \rightarrow \infty$ , to a probability measure  $\eta^{HN}$  on  $\mathbb{R}$ , which is absolutely continuous with respect to the Lebesgue measure, except perhaps for a point mass at  $\text{ess sup } \eta^{HN}$ . Also, the limits below exist and relate with  $\eta^{HN}$  as follows*

$$\eta_{\min}^{HN} := \lim_{k \rightarrow \infty} \frac{\mu_{\min}^k}{k} \leq \text{ess inf } \eta^{HN}, \quad \eta_{\max}^{HN} := \lim_{k \rightarrow \infty} \frac{\mu_{\max}^k}{k} = \text{ess sup } \eta^{HN}. \quad (1.3)$$

When  $\dim B = 1$ , the convergence part of Theorem 1.1 was established by Chen [7]. For this, he verified that Harder-Narasimhan filtrations of  $E_k$  are submultiplicative, see (2.3), and  $\mu_{\max}^k$  grows at most linearly in  $k$ . Theorem 1.1 then follows from the general theory of bounded submultiplicative filtrations developed by Chen [7] and Boucksom-Chen [4], cf. Theorem 2.2. Our contribution in Theorem 1.1 consists in extending the results of Chen for  $\dim B > 1$ .

When the base of the family is a curve, the measure  $\eta^{HN}$  is easier to study for various reasons: the subsheaves in the Harder-Narasimhan filtration become locally free, there is a very precise relation between the slope and the number of holomorphic sections, cf. [8, Lemma 2.1], and the notion of slope doesn't depend on the choice of multipolarization.

The main goal of this article is to show that  $\eta^{HN}$ ,  $\eta_{\min}^{HN}$ ,  $\eta_{\max}^{HN}$  can be recovered from the analogous quantities associated with base changes of our family over generic curves in  $B$ .

For this, we need to assume that the multipolarization  $[\omega_B]$  is *integral* and *very ample*, i.e.  $[\omega_{B,i}] \in H^2(X, \mathbb{Z})$  and very ample for any  $i = 1, \dots, m-1$ . Consider a regular curve  $C = C(l) \subset B$  given by Bertini theorem as an intersection of  $m-1$  generic divisors in the classes  $l_i[\omega_{B,i}]$ ,  $i = 1, \dots, m-1$ , where  $l \in \mathbb{N}^{*(m-1)}$ ,  $l = (l_1, \dots, l_{m-1})$ . It is well-known, cf. Corollary 3.8, that for such *generic* curves  $C$ , the Harder-Narasimhan slopes of  $E_k|_C$  do not depend on the choice of  $C$  (for a fixed parameter  $l \in \mathbb{N}^{*(m-1)}$ ). In particular, the sequence of probability measures  $\eta_{k,l}^{HN|C}$ , defined for  $E_k|_C$  analogously to  $\eta_k^{HN}$ , and their weak limits, as  $k \rightarrow \infty$ , which we denote by  $\eta_l^{HN|C}$ , as well as the limiting minimal and maximal slopes, cf. (1.3), which we denote by  $\eta_{\min,l}^{HN|C}$ ,  $\eta_{\max,l}^{HN|C}$ , are well-defined and exist by Theorem 1.1.

For  $a \in \mathbb{R}^*$ , we introduce the homothety map  $H^a : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x/a$ .

**Theorem 1.2.** *The measures  $H_*^{l_1 \cdots l_{m-1}}(\eta_l^{HN|C})$  converge weakly to  $\eta^{HN}$ , as  $l = (l_1, \dots, l_{m-1}) \rightarrow \infty$  (i.e.  $l_i \rightarrow \infty$ ,  $i = 1, \dots, m-1$ ). Moreover,*

$$\lim_{l \rightarrow \infty} \frac{\eta_{\min,l}^{HN|C}}{l_1 \cdots l_{m-1}} = \eta_{\min}^{HN}, \quad \lim_{l \rightarrow \infty} \frac{\eta_{\max,l}^{HN|C}}{l_1 \cdots l_{m-1}} = \eta_{\max}^{HN}. \quad (1.4)$$

*Remark 1.3.* For the rationale behind the normalization by  $l_1 \cdots l_{m-1}$ , see (2.8).

As an application of Theorem 1.2 and a result from Xu-Zhuang [48], we give an algebraic interpretation of  $\eta_{\min}^{HN}$  and  $\eta_{\max}^{HN}$ . More precisely, we say that  $\alpha \in H^{1,1}(X)$  is  $[\omega_B]$ -*generically fibered nef (resp. psef) with respect to  $\pi$*  if there is  $l_0 \in \mathbb{N}^*$ , such that for any regular curve  $C = C(l) \subset B$ ,  $l = (l_1, \dots, l_{m-1})$ ,  $l_i \geq l_0$ , defined as before Theorem 1.2, the restriction of  $\alpha$  to  $\pi^{-1}(C)$  is nef (resp. psef), i.e. for any irreducible curve  $C'$  in  $\pi^{-1}(C)$ , the pairing  $\int_{C'} \alpha$  is non negative (resp. the restriction of  $\alpha$  to  $\pi^{-1}(C)$  contains a positive current). For families given by projectivization of vector bundles, equivalent definition was given by Miyaoka [37], see also Peternell [38]. In analogy with the definition of the nef (resp. psef) threshold, for a class  $\alpha \in H^{1,1}(X)$ , we then define the  $[\omega_B]$ -*generic nef (resp. psef) threshold with respect to  $\pi$*  as the supremum over all  $T \in \mathbb{R}$ , such that  $\alpha - T(\pi^*[\omega_{B,0}]/\int_B[\omega_{B,0}] \cdots [\omega_{B,m-1}])$  is  $[\omega_B]$ -generically fibered nef (resp. psef) with respect to  $\pi$  for some Kähler class  $[\omega_{B,0}] \in H^{1,1}(B, \mathbb{C})$  (it is easy to see that the definition is independent on the choice of  $[\omega_{B,0}]$ , and for relatively ample  $\alpha$ , this supremum is finite).

**Corollary 1.4.** When the multipolarization  $[\omega_B]$  is integral, the quantity  $\eta_{\min}^{HN}$  (resp.  $\eta_{\max}^{HN}$ ) equals to  $[\omega_B]$ -generic nef (resp. psef) threshold of  $c_1(L)$  with respect to  $\pi$ .

As we shall explain in Section 5, if  $\dim B = 1$ , from the characterization of the pseudoeffective cone [5] and [48], we can get the following numerical formulas

$$\eta_{\min}^{HN} = \inf_{C \in \mathcal{C}_{irr}} \frac{\int_C c_1(L)}{\deg(\pi|_C)}, \quad \eta_{\max}^{HN} = \inf_{C \in \mathcal{C}_{mov}} \frac{\int_C c_1(L)}{\deg(\pi|_C)}, \quad (1.5)$$

where  $\mathcal{C}_{irr}$  is the set of irreducible curves  $C \subset X$ , such that  $\pi|_C$  is surjective,  $\mathcal{C}_{mov} \subset \mathcal{C}_{irr}$  is the subset of movable curves, i.e. curves which can be put in a family, which cover  $X$ , and  $\deg(\pi|_C)$  is the topological degree of  $\pi|_C$ . By Corollary 1.4, similar formulas can be established if  $\dim B > 1$ .

To prove Theorem 1.2, we use a theorem of Mehta-Ramanathan [36], showing that semistability behaves well under restriction over generic curves of sufficiently big degrees. Using this theorem, in (4.8), we show that the coupling of  $\eta^{HN}$  with an arbitrary test function on  $\mathbb{R}$  can be calculated as a double limit, where the first limit is taken in  $l$  (corresponding to the degree of the generic curve,  $C$ ) and the second limit is in  $k$  (corresponding to the index of the direct image sheaves,  $E_k$ ). The main content of Theorem 1.2 is that we can interchange the order of these limits. Philosophically, even though we do not literally pursue this direction, this means that there is a certain weak uniformity in the degree of the generic curve required in Mehta-Ramanathan theorem for the vector bundles  $E_k$  for all parameters  $k \in \mathbb{N}$ . In order to show that we can change the order of limits, we rely on the techniques of Shatz [44], to describe how Harder-Narasimhan slopes behave under the variation of the manifold (in our case, a specialization of the generic curve) and on the analysis of Chen [7] on the study of submultiplicative filtrations.

Let us finally mention that recently there's been a surge of interest in the study of asymptotics of Harder-Narasimhan slopes, especially when the base of the family is a curve. Chen studied its applications in Arakelov geometry in [7] and algebraic geometry in [8]. It was applied in the study of the moduli space of K-stable Fano varieties by Codogni-Patakfalvi [11], Xu-Zhuang [48] and more recently by Hattori [25]. Related ideas have been used in diophantine approximations, see Faltings-Wüstholz [15], cf. Grieve [22]. Sektnan-Tipler [42] have studied a related question of semistability of pullbacks along submersions. These developments were part of the motivation for the current work. Another motivation comes from the sequel of this article, cf. [18], where we study the relation between  $\eta^{HN}$  and differential geometry using some results from this paper.

This article is organized as follows. In Section 2, we recall some crucial facts from the theory of submultiplicative filtrations and establish Theorem 1.1. In Section 3, we study how Harder-Narasimhan filtrations behave in families. The results of that section will be particularly important in Section 4, where we establish Theorem 1.2. Finally, in Section 5, we establish Corollary 1.4.

**Notation.** A word *generic* in this article means outside a countable union of proper analytic subspaces in some connected complex manifold. We say that a curve  $C \subset B$  is a complete intersection in an integral multipolarization  $[\omega_B] = ([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ , if it coincides with the intersection of  $m - 1$  divisors in  $B$  in the classes  $[\omega_{B,i}]$ ,  $i = 1, \dots, m - 1$ . For  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , we denote  $l[\omega_B] := (l_1[\omega_{B,1}], \dots, l_{m-1}[\omega_{B,m-1}])$ . By an abuse of notations, a  $m - 1$  tuple of ample line bundles on  $B$  will also be called a multipolarization.

A sequence  $a_n \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , is called *multiplicative* if there is  $b_n \in \mathbb{N}^*$ , such that  $a_{n+1} = a_n \cdot b_n$ . A sequence  $a_n \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , is called *subadditive* (resp. *superadditive*) if for any  $n, m \in \mathbb{N}$ , we have  $a_{n+m} \leq a_n + a_m$  (resp.  $a_{n+m} \geq a_n + a_m$ ). We say that a property holds *eventually* if it holds for sufficiently big parameters.

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## 2 Submultiplicative nature of Harder-Narasimhan filtrations

The main goal of this section is to prove Theorem 1.1. For this, we will first recall some crucial points of the theory of submultiplicative filtrations on section rings.

Recall that a *non-increasing*  $\mathbb{R}$ -filtration  $\mathcal{F}$  of a vector space  $V$  is a map from  $\mathbb{R}$  to vector subspaces of  $V$ ,  $t \mapsto \mathcal{F}^t V$ , verifying  $\mathcal{F}^t V \subset \mathcal{F}^s V$  for  $t > s$ , and such that  $\mathcal{F}^t V = V$  for sufficiently small  $t$  and  $\mathcal{F}^t V = \{0\}$  for sufficiently big  $t$ . We say that  $\mathcal{F}$  is *graded* if it respects the grading of  $V$ . It is *left-continuous* if for any  $t \in \mathbb{R}$ , there is  $\epsilon_0 > 0$ , such that  $\mathcal{F}^t V = \mathcal{F}^{t-\epsilon} V$  for any  $0 \leq \epsilon \leq \epsilon_0$ . All filtrations in this article are assumed to be non-increasing left-continuous and graded if applicable. We associate with any filtration the probability measure,  $\eta(\mathcal{F}, V)$ , on  $\mathbb{R}$ , which we also call the *jumping measure*, defined as follows

$$\eta(\mathcal{F}, V) := \frac{1}{\dim V} \sum_{j=1}^{\dim V} \delta[\eta(\mathcal{F}, j)], \quad (2.1)$$

where  $\eta(\mathcal{F}, j)$  are the *jumping numbers*, defined as follows  $\eta(\mathcal{F}, j) := \sup\{t \in \mathbb{R} : \dim \mathcal{F}^t V \geq j\}$ . Alternatively, we can define this measure as  $\eta(\mathcal{F}, V) = -\frac{1}{\dim V} \frac{d}{d\lambda} \dim \mathcal{F}^\lambda V$ .

Now, for an ample line bundle  $L$  over a complex projective manifold  $Y$ , define the *section ring*

$$R(Y, L) := \bigoplus_{k=1}^{\infty} H^0(Y, L^k). \quad (2.2)$$

A graded filtration  $\mathcal{F}$  on  $R(Y, L)$  is called *submultiplicative* if for any  $t, s \in \mathbb{R}$ ,  $k, l \in \mathbb{N}$ , we have

$$\mathcal{F}^t H^0(Y, L^k) \cdot \mathcal{F}^s H^0(Y, L^l) \subset \mathcal{F}^{t+s} H^0(Y, L^{k+l}). \quad (2.3)$$

We say that  $\mathcal{F}$  is *bounded* if there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ ,  $\mathcal{F}^{Ck} H^0(Y, L^k) = \{0\}$ . Remark that it is an immediate consequence of the submultiplicativity and the fact that  $R(Y, L)$  is a finitely generated ring that there is  $C > 0$ , such that  $\mathcal{F}^{-Ck} H^0(Y, L^k) = H^0(Y, L^k)$ .

We let  $N_k = \dim H^0(Y, L^k)$ , and denote by  $\eta_{\mathcal{F}}(1, k) \geq \dots \geq \eta_{\mathcal{F}}(N_k, k)$  the *jumping numbers* of the filtration  $\mathcal{F}$  on  $H^0(X, L^k)$ . We denote by  $\eta_{\mathcal{F}, \max}(k) := \eta_{\mathcal{F}}(1, k)$  the maximal and by  $\eta_{\mathcal{F}, \min}(k) := \eta_{\mathcal{F}}(N_k, k)$  the minimal jumping numbers. The following result is well-known, cf. [7, Propositions 3.2.4, 3.2.6].

**Proposition 2.1.** For any submultiplicative filtration  $\mathcal{F}$  on  $R(Y, L)$ , the sequence  $\eta_{\mathcal{F}, \max}(k)$  (resp.  $\eta_{\mathcal{F}, \min}(k)$ ),  $k \in \mathbb{N}$ , is (resp. eventually) superadditive. In particular, the following limits exist  $\lim_{k \rightarrow \infty} \eta_{\mathcal{F}, \max}(k)/k$ ,  $\lim_{k \rightarrow \infty} \eta_{\mathcal{F}, \min}(k)/k$ , in  $\mathbb{R} \cup \{+\infty\}$ , and they are finite if  $\mathcal{F}$  is bounded.

Define the sequence of *jumping measures*  $\eta_{\mathcal{F}, k}$ ,  $k \in \mathbb{N}^*$ , of  $\mathcal{F}$  on  $\mathbb{R}$  as follows

$$\eta_{\mathcal{F}, k} := \frac{1}{N_k} \sum_{j=1}^{N_k} \delta \left[ \frac{\eta_{\mathcal{F}}(j, k)}{k} \right] = H_*^k \left( \eta(\mathcal{F}, H^0(X, L^k)) \right), \quad (2.4)$$

where  $H^k$  is the homothety map, defined before Theorem 1.2.

**Theorem 2.2** (Chen [7], Boucksom-Chen [4]). *For any bounded submultiplicative filtration  $\mathcal{F}$  on  $R(Y, L)$ , the jumping measures  $\eta_{\mathcal{F}, k}$ ,  $k \in \mathbb{N}^*$ , converge weakly, as  $k \rightarrow \infty$ , to a probability measure  $\eta_{\mathcal{F}}$ . Moreover, the following holds*

$$\lim_{k \rightarrow \infty} \frac{\eta_{\mathcal{F}, \min}(k)}{k} \leq \text{ess inf } \eta_{\mathcal{F}}, \quad \lim_{k \rightarrow \infty} \frac{\eta_{\mathcal{F}, \max}(k)}{k} = \text{ess sup } \eta_{\mathcal{F}}. \quad (2.5)$$

Also, the measure  $\eta_{\mathcal{F}}$  is absolutely continuous with respect to the Lebesgue measure, except perhaps for a point mass at  $\text{ess sup } \eta_{\mathcal{F}}$ .

*Remark 2.3.* Boucksom-Chen in [4] gave an explicit expression of  $\eta_{\mathcal{F}}$  in terms of the Okounkov body associated with  $\mathcal{F}$ . Witt Nyström [47] gave an expression of  $\eta_{\mathcal{F}}$  in terms the geodesic ray in the space of positive metrics on  $L$  associated with the filtration, when the filtration is given by the weight of a  $\mathbb{C}^*$ -action; Hisamoto [26] later generalized it for any finitely generated filtrations; then, the author in [16] further generalized it to all bounded submultiplicative filtrations by giving a different proof, based purely on complex pluripotential theory and semiclassical input from Tian [45] and Phong-Sturm [39]. This was further generalized by the author in [17] in the relative setting, following Chen-Maclean [9], cf. also Boucksom-Jonsson [6], and relying on the results of Zelditch [49], Chen-Sun [10], Berndtsson [2], Dai-Liu-Ma [12], Ma-Marinescu [32], [33] and [16].

Now, for any  $\lambda \in \mathbb{R}$ , we consider the subalgebra  $R^{(\lambda)}(Y, L) := \bigoplus_{k=0}^{\infty} \mathcal{F}^{\lambda k} H^0(Y, L^k)$  of  $R(Y, L)$ .

**Theorem 2.4** (Boucksom-Chen [4]). *For any  $\lambda \in \mathbb{R}$ , the following limit exists*

$$\text{vol}(R^{(\lambda)}(Y, L)) := \lim_{k \rightarrow \infty} \frac{\dim \mathcal{F}^{\lambda k} H^0(Y, L^k)}{N_k} = \int_{\lambda}^{+\infty} d\eta_{\mathcal{F}}. \quad (2.6)$$

Moreover, the function  $\lambda \mapsto \text{vol}(R^{(\lambda)}(Y, L))^{1/\dim Y}$  is concave on  $]-\infty, \text{ess sup } \eta_{\mathcal{F}}[$ .

Now, we specialize this general theory to the case of Harder-Narasimhan filtrations. We use the notations from Introduction. We introduce the (non-increasing) filtrations  $\mathcal{F}_k(\lambda)$ ,  $\lambda \in \mathbb{R}$ , of  $E_k$  by coherent (torsion-free) subsheaves (defined over  $B$ ) so that  $\mathcal{F}_k(\lambda)$  is the maximal subsheaf of  $E_k$  such that all of its Harder-Narasimhan slopes are bigger than  $\lambda$ . Clearly, the filtration  $\mathcal{F}_k$  is just a “renaming” of the Harder-Narasimhan filtration of  $E_k$ .

The main idea behind the above “renaming” of Harder-Narasimhan filtration is that the filtration then becomes submultiplicative. More precisely, for any  $b \in B$ , we denote by  $\mathcal{F}_b$  the filtration induced by  $\mathcal{F}_k(\lambda)$  on  $R(X_b, L_b) = \bigoplus_{k=0}^{\infty} H^0(X_b, L_b^k)$ , for  $X_b = \pi^{-1}(b)$ ,  $L_b = L|_{X_b}$ ,  $b \in B$ .

**Proposition 2.5.** The filtration  $\mathcal{F}_b$  is submultiplicative for generic  $b \in B$ .

When  $\dim B = 1$ , Proposition 2.5 is due to Chen [7, §4.3.2], and the statement holds non-generically. The proof of the general case remains verbatim modulo some technical modifications which enter into play due to the fact that Harder-Narasimhan filtrations are filtrations by subsheaves (and not by vector bundles) if  $\dim B > 1$ . More precisely, we need the following basic results.

**Lemma 2.6.** For any  $\lambda > 0$ , we have  $\mu_{\max}^{HN}(E_k/\mathcal{F}_k(\lambda)) < \lambda$ .

*Proof.* It follows directly from the definition of  $\mathcal{F}_k(\lambda)$ .  $\square$

For the following result, we recall that the determinant line of a coherent sheaf is canonically trivial if the support of the sheaf has codimension at least 2, cf. [30, Proposition 5.6.14]. In particular, Harder-Narasimhan filtrations can be defined for coherent sheaves, which are torsion-free in codimension 2, cf. [28, Theorem 1.6.7]. In this case, the filtration will be given by coherent subsheaves, which are torsion-free in codimension 2, and their subsequent quotients will also be torsion-free in codimension 2. In particular, the notions of maximal and minimal slopes are well-defined. We borrow for those notions the previously-defined notations of the maximal and the minimal Harder-Narasimhan slopes.

**Lemma 2.7** ([28, Proposition 1.2.7 and Theorem 1.6.6]). Let  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  be a map between coherent sheaves  $\mathcal{E}, \mathcal{F}$ , such that both  $\mathcal{E}$  and  $\mathcal{F}$  are torsion-free in codimension 2. Assume that  $\mu_{\max}^{HN}(\mathcal{F}) < \mu_{\min}^{HN}(\mathcal{E})$ . Then  $\phi = 0$  outside an analytic subset of codimension at least 2.

*Remark 2.8.* In [28], authors treat symmetric multipolarization, but the generalization is direct.

The reason why it is necessary to consider sheaves which are not torsion-free in the context of this article is that in general a tensor product of two torsion-free coherent sheaves is not torsion-free in general, cf. [41, p. 261]. However, the following result is true.

**Lemma 2.9.** Let  $\mathcal{E}_1, \mathcal{E}_2$  be two torsion-free coherent sheaves, then their tensor product is torsion-free in codimension 2. Similarly, for a torsion-free coherent sheaf  $\mathcal{E}$ , the sheaves  $\text{Sym}^k \mathcal{E}$ ,  $k \in \mathbb{N}^*$ , are torsion-free in codimension 2.

*Proof.* The statement follows directly from the fact that torsion-free coherent sheaves are locally-free in codimension 2, cf. [30, Theorem 5.5.8, Corollary 5.5.15]  $\square$

*Proof of Proposition 2.5.* It suffices to verify that the natural map  $\mathcal{F}_k(\lambda) \otimes \mathcal{F}_l(\mu) \rightarrow E_k/\mathcal{F}_{k+l}(\lambda + \mu)$  is zero outside a subset of codimension at least 2, for any  $\lambda, \mu \in \mathbb{R}$ ,  $k, l \in \mathbb{N}$ . But this is a direct consequence of Lemmas 2.6, 2.7, 2.9, the fact that a tensor product of two semistable sheaves is semistable (in the category of sheaves which are torsion free in codimension 2), and the slope behaves additively with respect to tensor products. The last two results were established by Maruyama [34] in the case of symmetric mutipolarization and by Greb-Kebekus-Peternell [21, Theorem 4.2] in general.  $\square$

The next result proves the boundness of submultiplicative filtrations. In the context of filtrations associated with test configurations, the importance of such bounds in analytic approach to the study of submultiplicative filtrations has been underlined by Phong-Sturm in [40, Lemma 4].

**Proposition 2.10.** The filtration  $\mathcal{F}_b$  is bounded for any  $b \in B$ .

When  $m := \dim B = 1$ , this was established by Chen [7, Proposition 4.3.5] by relying on the very explicit relation between the degree and the number of holomorphic sections of a stable vector bundle (which doesn't extend to higher dimensions), cf. also [48, Lemma-Definition 2.26]. We will show below that the statement for projective  $B$  of any dimension reduces to the one considered by Chen. For this, the following result will be crucial.

**Lemma 2.11.** Assume that the multipolarization  $[\omega_B]$  is integral and very ample. Let  $\mathcal{E}$  be a torsion-free coherent sheaf over  $B$ . Then for generic regular curves  $C \subset B$ , which are complete intersections in  $l[\omega_B]$ ,  $l \in \mathbb{N}^{*(m-1)}$ , the restriction,  $\mathcal{E}|_C$ , of  $\mathcal{E}$  to  $C$  is torsion-free, and we have

$$\begin{aligned} \mu_{[\omega_B], \max}(\mathcal{E}) &\leq l_1 \cdots l_{m-1} \cdot \mu_{\max}(\mathcal{E}|_C), \\ \mu_{[\omega_B], \min}(\mathcal{E}) &\geq l_1 \cdots l_{m-1} \cdot \mu_{\min}(\mathcal{E}|_C). \end{aligned} \quad (2.7)$$

*Proof.* First, the statement about the torsion-free restriction is well-known, cf. [23, Théorème 12.1.1] or [28, Lemma 3.1.1]. Now, as  $C$  is a complete intersection in  $l[\omega_B]$ , we have

$$\mu(\mathcal{E}|_C) = l_1 \cdots l_{m-1} \cdot \mu_{[\omega_B]}(\mathcal{E}). \quad (2.8)$$

Remark also that the maximal and the minimal slopes admit the following useful characterization

$$\begin{aligned} \mu_{[\omega_B], \max}(\mathcal{E}) &= \max \left\{ \mu_{[\omega_B]}(\mathcal{F}) : \mathcal{F} \text{ is a subsheaf of } \mathcal{E} \right\}, \\ \mu_{[\omega_B], \min}(\mathcal{E}) &= \min \left\{ \mu_{[\omega_B]}(\mathcal{Q}) : \mathcal{Q} \text{ is a quotient sheaf of } \mathcal{E} \right\}. \end{aligned} \quad (2.9)$$

The result now follows from (2.8) and (2.9) applied for  $\mathcal{E}$  and  $\mathcal{E}|_C$ .  $\square$

*Proof of Proposition 2.10.* First of all, since  $B$  is projective, any multipolarization can be bounded from above by an integral multipolarization. Remark that the slopes will only increase by this procedure, so it is sufficient to assume that the multipolarization  $[\omega_B]$  is integral and very ample.

Now, by Lemma 2.11, the maximal Harder-Narasimhan  $[\omega_B]$ -slope of  $E_k$  is bounded from above by the maximal Harder-Narasimhan slope of  $E_k|_C$ , where  $C$  is a regular complete intersection curve in  $[\omega_B]$ . But since the maximal slopes of  $E_k|_C$  grow at most linearly in  $k$  by the already mentioned result of Chen, we deduce the same for the maximal slopes of  $E_k$ .  $\square$

*Proof of Theorem 1.1.* It follows directly from Theorem 2.2 and Propositions 2.5, 2.10.  $\square$

### 3 Relative slopes on degenerating families of varieties

The main goal of this section is to describe how Harder-Narasimhan filtrations behave in families. More precisely, we first recall that in any flat family, Harder-Narasimhan slopes of the generic fiber are well-defined. Following Shatz [44], we then make a relation between the Harder-Narasimhan slopes of the generic fiber and the Harder-Narasimhan slopes of the specialization of the generic fiber. As an application of this, we show that the sequence of measures composed of Harder-Narasimhan slopes of the restrictions of a given torsion-free sheaf to generic curves of a given degree satisfy certain subadditivity properties in terms of degrees of curves.

More precisely, our setting will be as follows. Let  $p : B \rightarrow S$  be a proper surjective flat holomorphic map of relative dimension  $m$  between complex connected manifolds  $B$  and  $S$ , and let  $F = (F_1, \dots, F_{m-1})$  be a relative multipolarization, which means that  $F_i$  are relatively ample line



bundles with respect to  $p$  on  $B$ ,  $i = 1, \dots, m-1$ . We assume for simplicity that  $p$  has a smooth fiber, which implies that the set of smooth fibers is an open dense subset of  $S$ , see [23, Théorème 12.2.4]. Let  $\mathcal{E}$  be a flat family of torsion-free coherent sheaves on the fibers of  $p$ , cf. [28, Definition 2.1.1]. The set of fibers of  $p$  such that the restriction of  $\mathcal{E}$  to them is torsion-free is open in  $S$  by [23, Théorème 12.1.1]. In particular, the Harder-Narasimhan filtrations (with respect to the multipolarization  $c_1(F)|_{B_s} := (c_1(F_1)|_{B_s}, \dots, c_1(F_{m-1})|_{B_s})$ ) of the torsion-free coherent sheaf  $\mathcal{E}|_{B_s}$  are well-defined for generic  $s \in S$ .

**Proposition 3.1.** The  $c_1(F)|_{B_s}$ -slopes of the Harder-Narasimhan filtrations of  $\mathcal{E}|_{B_s}$  are constant for generic  $s \in S$ .

The following two results will make the proof of Proposition 3.1.

**Lemma 3.2** ([20, Example 20.3.3]). The  $c_1(F)|_{X_s}$ -slopes of  $\mathcal{E}|_{B_s}$  are constant for  $s \in S$ .

**Theorem 3.3.** *There is a complex manifold  $T$  with a birational morphism  $f : T \rightarrow S$ , such that for  $p_0 : B \times_S T \rightarrow B$ , there is a filtration of  $p_0^*\mathcal{E}$  by subsheaves:*

$$p_0^*\mathcal{E} = HN(\mathcal{E})_s \supset HN(\mathcal{E})_{s-1} \supset \dots \supset HN(\mathcal{E})_1 \supset HN(\mathcal{E})_0 = \{0\}, \quad (3.1)$$

such that the factors  $HN(\mathcal{E})_i/HN(\mathcal{E})_{i-1}$  are flat for  $i = 1, \dots, s$ , and there is a dense open subset  $U \subset T$ , such that the restriction of the above filtration to a fiber at  $t \in U$  corresponds to the Harder-Narasimhan filtration of the fiber.

*Remark 3.4.* The filtration above is called *relative Harder-Narasimhan filtration*. The proof of the above statement for symmetric multipolarizations can be found in [28, Theorem 2.3.2], and in the general case in [46, Corollary 6.6].

*Proof of Proposition 3.1.* It follows directly from Theorem 3.3 and Lemma 3.2, applied to all subsheaves of the relative Harder-Narasimhan filtration.  $\square$

We will now apply this general theory to study Harder-Narasimhan filtrations of the restrictions of a given sheaf to generic curves. We fix a complex projective manifold  $B$  of dimension  $m$  with very ample line bundles  $F_1, \dots, F_{m-1}$ . For  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , we consider the space  $S_l := \mathbb{P}(H^0(B, F_1^{l_1})) \times \dots \times \mathbb{P}(H^0(B, F_{m-1}^{l_{m-1}}))$ , parametrizing  $\dim B - 1$  hyperplanes in  $H^0(B, F_1^{l_1})^*, \dots, H^0(B, F_{m-1}^{l_{m-1}})^*$ . We define the *incidence variety*  $\mathcal{C}_l$  as follows

$$\mathcal{C}_l := \left\{ (b \in B, s \in S_l) : b \text{ lies in the intersection of hyperplanes parametrized by } s \right\}, \quad (3.2)$$

where we implicitly identified  $B$  with its images in  $\mathbb{P}(H^0(B, F_i^{l_i})^*)$ ,  $i = 1, \dots, m-1$ , through the Kodaira map. We then have the natural maps  $p : \mathcal{C}_l \rightarrow S_l$  and  $p_0 : \mathcal{C}_l \rightarrow B$ . It is classical, cf. [28, §3.1], that  $p_0$  is a locally trivial bundle with fibers given by the product of projective spaces. In particular,  $\mathcal{C}_l$  is a smooth manifold. The fibers of the projection  $p$  are given by the intersection of the  $\dim B - 1$  divisors given by the zero-locus of sections parametrized by a point in  $S_l$ . Hence, these fibers are generically curves. More precisely, the following result holds true.

**Proposition 3.5.** The set  $U \subset S_l$  (resp.  $V \subset U$ ) such that the fibers of  $p$  at  $U$  have dimension 1 (resp. of regular values of  $p$ ) is a non-empty open subset in Zariski topology, i.e. its complement is an analytic subset.

*Proof.* The statement for  $V$  is the Bertini theorem, cf. [24, Theorem II.8.18]. Let us now establish the statement about  $U$ . Define the function  $n : S_l \rightarrow \mathbb{N}$  as

$$n(s) = \text{maximal dimension of an irreducible component of } p^{-1}(s). \quad (3.3)$$

Since  $p^{-1}(s)$  is given by the intersection of  $\dim B - 1$  divisors, by Serre's inequality on height, cf. [13, Exercice II.11.11], the dimension of every irreducible component of  $p^{-1}(s)$ ,  $s \in S_l$ , is at least 1. In particular, we have  $U := \{s \in S_l : n(s) < 2\}$ . But  $C_l$  is projective and, hence,  $p$  is proper. In particular, the function  $n$  is upper semicontinuous, cf. [23, Théorème 13.1.5]. Hence, the set  $\{s \in S_l : n(s) < 2\}$  is open, which finishes the proof.  $\square$

**Proposition 3.6.** The restriction of  $p$  to  $p^{-1}(U)$  is flat.

*Remark 3.7.* The same statement appeared in [35, Proposition 1.5] with a different proof.

*Proof.* Since both  $U$  and  $p^{-1}(U)$  are smooth manifolds, and the dimension of the fibers of  $p$  over  $p^{-1}(U)$  equals to 1 by the definition of the set  $U$ , and  $\dim C_l - \dim S_l = 1$  by a simple dimension count, the result follows from the miracle flatness theorem, cf. [24, Exercise III.10.9].  $\square$

**Corollary 3.8.** Assume  $[\omega_B]$  is very ample. For any torsion-free coherent sheaf  $\mathcal{E}$  over  $B$ , the Harder-Narasimhan slopes of the restriction,  $\mathcal{E}|_C$ , (locally free by Lemma 2.11) to a generic regular curve  $C \subset B$ , which is a complete intersection in  $[\omega_B]$ , do not depend on the choice of  $C$ .

*Proof.* It is a direct consequence of Propositions 3.1, 3.5, 3.6.  $\square$

In the second part of this section, we discuss the relation between the slopes of generic and special fibers of the family. For this, we need to introduce a certain order on filtrations.

Let  $\mathcal{E}$  be a torsion-free coherent sheaf of rank  $r$  over a complex Kähler manifold  $B$  with a fixed multipolarization  $[\omega_B]$ . We fix an arbitrary filtration of  $\mathcal{E}$  by coherent subsheaves  $\mathcal{E} = \mathcal{F}_s \supset \mathcal{F}_{s-1} \supset \dots \supset \mathcal{F}_1 \supset \mathcal{F}_0 = \{0\}$ , and define the *polytope of the filtration* as the graph of a partially linear function, defined on  $[0, r]$ , as a linear interpolation of nodes  $(\text{rk}(\mathcal{F}_i), \deg(\mathcal{F}_i))$ ,  $i = 0, \dots, s$ , where the degree is calculated with respect to  $[\omega_B]$ . A polytope is then a certain piecewise linear path, connecting  $(0, 0)$  with  $(r, \deg(\mathcal{E}))$ . We say that a polytope *dominates* another polytope if it lies above it.

The following alternative description of this order will be particularly useful in what follows. We define the slopes of the filtration as a non-increasing sequence of numbers  $\mu_1^{\mathcal{F}}, \dots, \mu_r^{\mathcal{F}}$ , such that  $\mu(\mathcal{F}_r/\mathcal{F}_{r-1})$  appears exactly  $\text{rk}(\mathcal{F}_r/\mathcal{F}_{r-1})$  times. We fix two filtrations of  $\mathcal{E}$  by subsheaves, and denote their slopes by  $\mu_1^{\mathcal{F},j}, \dots, \mu_r^{\mathcal{F},j}$ ,  $j = 1, 2$ . Then by [1, (12.1)], the first filtration dominates the second one if and only if

$$\mu_1^{\mathcal{F},1} + \dots + \mu_k^{\mathcal{F},1} \geq \mu_1^{\mathcal{F},2} + \dots + \mu_k^{\mathcal{F},2}, \quad (3.4)$$

for any  $k = 1, \dots, r - 1$ . Remark that we always have  $\mu_1^{\mathcal{F},1} + \dots + \mu_r^{\mathcal{F},1} = \text{rk}(\mathcal{E}) \cdot \mu(\mathcal{E}) = \mu_1^{\mathcal{F},2} + \dots + \mu_r^{\mathcal{F},2}$ . Alternatively, we define the probability measures  $\mu^{\mathcal{F},j}$ ,  $j = 1, 2$ , as in (1.2):

$$\mu^{\mathcal{F},j} := \frac{1}{r} \sum_{i=1}^r \delta[\mu_i^{\mathcal{F},j}]. \quad (3.5)$$

Then by [27], cf. Atiyah-Bott [1, (12.2)], the above partial order is equivalent to the following one: for any convex function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , we have

$$\int \phi \mu^{\mathcal{F},1} \geq \int \phi \mu^{\mathcal{F},2}. \quad (3.6)$$

More generally, for arbitrary probability measures  $\mu_1, \mu_2$  on  $\mathbb{R}$ , we say that  $\mu_1$  dominates  $\mu_2$  if the analogue of (3.6) holds.

The relevance of this partial order to Harder-Narasimhan filtrations comes from the following extremal characterization of them.

**Theorem 3.9** (Satz [44, Theorem 2 and Remark on p.174]). *Harder-Narasimhan filtration of a torsion-free coherent sheaf dominates all the other filtrations by coherent subsheaves.*

*Remark 3.10.* The article [44] is written for symmetric multipolarization, but it readily extends to the general case, as it only relies on Lemma 2.7 and the existence of relative Harder-Narasimhan filtrations, which holds for non-symmetric multipolarization, cf. Theorem 3.3.

As we shall see below, the above partial order is also useful in the study of how Harder-Narasimhan filtrations behave under specialization. For this, we assume from now on that our family of manifolds,  $p : B \rightarrow S$ , is defined over a unit disc, i.e.  $S = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . We then decompose the central fiber,  $B_0$ , as  $\sum n_j \cdot B_0^j$ , where  $B_0^j$ ,  $j = 1, \dots, t$ , are irreducible components, which we assume to be smooth for simplicity, and  $n_j$  are the multiplicities. We define the probability measure  $\mu_0^{HN}$  on  $\mathbb{R}$  as follows

$$\mu_0^{HN} = \frac{1}{r} \sum_{i=1}^r \delta \left[ \sum_{j=1}^t n_j \cdot \mu_i^{0,j} \right], \quad (3.7)$$

where  $\mu_i^{0,j}$ ,  $j = 1, \dots, t$ ,  $i = 1, \dots, r$ , are the slopes of the Harder-Narasimhan filtrations of  $\mathcal{E}|_{B_0^j}$  (which we assume to be torsion-free) with respect to  $c_1(F)|_{B_0^j}$ . We denote by  $\mu_*^{HN}$  the probability measure composed of the slopes of the Harder-Narasimhan filtrations of the restriction of  $\mathcal{E}$  to generic fibers (this doesn't depend on the choice of generic fiber by Proposition 3.1). In other words, by Theorem 3.3, we have

$$\mu_*^{HN} = \frac{1}{r} \sum_{i=1}^r \delta[\mu_i^*], \quad (3.8)$$

where  $\mu_i^*$  are the slopes of the restriction of the relative Harder-Narasimhan filtration  $HN(\mathcal{E})_i$ ,  $i = 0, \dots, s$ , from Theorem 3.3 to fibers.

**Theorem 3.11.** *The probability measure  $\mu_0^{HN}$  dominates  $\mu_*^{HN}$ .*

*Remark 3.12.* When  $B = B_0 \times \mathbb{D}$ , and  $p : B \rightarrow \mathbb{D}$  is the natural projection, the result is due to Satz [44, Theorem 3]. When  $p$  is of relative dimension 1, see [35, Proposition 4.3] for another related result. Our proof of the general case is very much inspired by [44].

In the proof of Theorem 3.11, we will use the following simple result.

**Lemma 3.13.** Let  $\mu_i^j = \frac{1}{N} \sum_{r=1}^N \delta[a_{i,r}^j]$ ,  $i, j = 1, 2$ , be probability measures on  $\mathbb{R}$  for some  $a_{i,r}^j \in \mathbb{R}$ , ordered in such a way that  $a_{i,r}^j$  are non-increasing in  $r$ , and such that  $\sum a_{i,r}^1 = \sum a_{i,r}^2$ . Assume that the measure  $\mu_1^1$  dominates  $\mu_2^2$  for  $i = 1, 2$ . Then for the measures  $\mu^j := \frac{1}{N} \sum_{r=1}^N \delta[a_{1,r}^j + a_{2,r}^j]$ ,  $\mu^1$  dominates  $\mu^2$ .

*Proof.* It follows directly from the characterization (3.4).  $\square$

*Proof of Theorem 3.11.* By the flatness of the family and the factors  $HN(\mathcal{E})_i/HN(\mathcal{E})_{i-1}$ , cf. [20, Example 20.3.3], we conclude that

$$\mu_i^* = \sum_{j=1}^t n_j \cdot \mu\left((HN(\mathcal{E})_i/HN(\mathcal{E})_{i-1})|_{B_0^j}\right). \quad (3.9)$$

Now, using the notations introduced in (3.7), for any  $j = 1, \dots, t$ , we define the measures

$$\begin{aligned} \mu_{0,j}^{HN} &:= \frac{1}{r} \sum_{i=1}^r \delta[\mu_i^{0,j}], \\ \mu_{*,j}^{HN} &:= \frac{1}{r} \sum_{i=1}^r \delta\left[\mu\left((HN(\mathcal{E})_i/HN(\mathcal{E})_{i-1})|_{B_0^j}\right)\right]. \end{aligned} \quad (3.10)$$

By Theorem 3.9, we conclude that the measures  $\mu_{0,j}^{HN}$  dominate  $\mu_{*,j}^{HN}$ . We deduce Theorem 3.11 from this, Lemma 3.13 and (3.7), (3.8), (3.9), (3.10).  $\square$

We will now apply this general theory to the study of measures associated with restrictions of torsion-free coherent sheaves. For this, we fix a torsion-free coherent sheaf  $\mathcal{E}$  of generic rank  $r$  over  $B$ . Assume that the multipolarization  $[\omega_B]$  is very ample. For any  $l \in \mathbb{N}^{*(m-1)}$ , we denote by  $\mu_i^{HN}(\mathcal{E}|_C, l)$ ,  $i = 1, \dots, r$ , the Harder-Narasimhan slopes of the restriction of  $\mathcal{E}$  to a generic curve  $C$ , given by a complete intersection in  $l[\omega_B]$ , cf. Corollary 3.8. We denote by  $\mu_{\min}^{HN}(\mathcal{E}|_C, l)$  and  $\mu_{\max}^{HN}(\mathcal{E}|_C, l)$  the minimal and maximal slopes respectively. We define the sequence of probability measures,  $\eta_l$ ,  $l \in \mathbb{N}^{*(m-1)}$ , on  $\mathbb{R}$  as follows

$$\eta_l := \frac{1}{r} \sum_{i=1}^r \delta\left[\frac{\mu_i^{HN}(\mathcal{E}|_C, l)}{l_1 \cdots l_{m-1}}\right]. \quad (3.11)$$

We can now state the main result of this section.

**Theorem 3.14.** *For any convex function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , the sequence of real numbers  $a_l := l_1 \cdots l_{m-1} \cdot \int \phi d\eta_l$ ,  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , is coordinatewise subadditive, i.e. for any  $l' \in \mathbb{N}^{*(m-2)}$ ,  $l_1^1, l_1^2 \in \mathbb{N}^*$ , we have  $a_{l^0} \leq a_{l^1} + a_{l^2}$ , for  $l^0 := (l_1^1 + l_1^2, l')$ ,  $l^1 := (l_1^1, l')$ ,  $l^2 := (l_1^2, l)$ , and similarly for other coordinates.*

We will now establish Theorem 3.14. All the curves considered below will naturally be given as fibers of  $p$ , which would parametrize them by points in  $S_l$ , see (3.2). The following result will be crucial for the proof of Theorem 3.14.

**Proposition 3.15.** *For any  $l' = (l_2, \dots, l_{m-1}) \in \mathbb{N}^{*(m-2)}$ ,  $l_1^1, l_1^2 \in \mathbb{N}^*$ , we let  $l^1 := (l_1^1, l')$ ,  $l^2 := (l_1^2, l')$ , and fix some curves  $C_1, C_2 \subset B$ , given by a complete intersection in  $l^1[\omega_B]$ ,  $l^2[\omega_B]$ , as a zero set of holomorphic sections  $s_1^1 \in H^0(B, F_1^{l_1^1})$ ,  $s_i \in H^0(B, F_i^{l_i})$ ,  $i = 2, \dots, m-1$ , and  $s_1^2 \in H^0(B, F_1^{l_1^2})$ ,  $s_i$ ,  $i = 2, \dots, m-1$ , respectively. Then there is a smooth manifold  $\mathcal{C}$  and a flat proper map  $p : \mathcal{C} \rightarrow \mathbb{D}$ , such that the central fiber of  $p$  is given by  $C_1 + C_2$ , and the general fiber is a complete intersection in  $l^0[\omega_B]$ , where  $l^0 := (l_1^1 + l_1^2, l')$ . Moreover, for any countable union of analytic subsets in  $S_{l^0}$ , we can choose  $p$  as a base change along a holomorphic disc in  $S_{l^0}$ , which doesn't lie within any of the analytic subsets.*

*Proof.* Remark first that there is a canonical embedding  $\mathbb{P}(H^0(B, F^{l_1})) \times \mathbb{P}(H^0(B, F^{l_2})) \hookrightarrow \mathbb{P}(H^0(B, F^{l_1+l_2}))$ , given by a composition of the Segre map and the multiplication in the section ring of  $B$ . The needed family can then be constructed by the base change of the family  $\mathcal{C}_{l_0}$  along a holomorphic disc in  $S_{l_0}$ , which is not contained in the countable union of analytic subsets from the statement, and which passes through the image of the point representing  $(C_1, C_2)$  in  $(S_{l_1}, S_{l_2})$ , viewed as a point in  $S_{l_0}$  through the above embedding.

The flatness of the resulting map is a consequence of Proposition 3.6 and the fact that flatness is preserved by base changes, see [24, Proposition 9.2]. The identification of the central fiber follows from the nature of the Segre map: if the holomorphic sections  $s_1^1, s_2, \dots, s_{m-1}$  (resp.  $s_1^2, s_2, \dots, s_{m-1}$ ) vanish along  $C_1$  (resp.  $C_2$ ) with multiplicity 1, then the holomorphic sections  $s_1^1 \cdot s_1^2, s_2, \dots, s_{m-1}$  vanish along  $C_1 + C_2$  with multiplicity 1.  $\square$

We will also need the following technical lemma.

**Lemma 3.16.** Let  $B_1, B_2, B$  be compact complex manifolds, and  $A_1 \subset B_1 \times B, A_2 \subset B_2 \times B$ , be proper analytic subsets. Then there are  $x_1 \in B_1, x_2 \in B_2, y \in B$ , such that  $(x_1, y) \notin A_1$  and  $(x_2, y) \notin A_2$ .

*Proof.* Let  $A$  be a proper analytic subspace in  $C \times B$ , where  $C$  is a compact complex manifold. We denote by  $D \subset B$ , the locus of  $x \in B$ , such that  $A$  doesn't contain  $C \times \{x\}$ . By using the properness of the projection map  $p : A \rightarrow B$  and properness of  $A$ , we argue similarly to (3.3), that  $D$  is open in  $B$  (in Zariski topology). It is also non-empty since  $A$  is a proper subset of  $C \times D$ . By applying this for  $A_1$  and  $A_2$ , we see that the locus of  $y \in B$ , such that  $A_i$  doesn't contain  $B_i \times \{x\}, i = 1, 2$ , is non-empty. It suffices then to pick any  $y \in B$  from this locus, and choose  $x_1, x_2$  arbitrarily away from  $A_1, A_2$ .  $\square$

*Proof of Theorem 3.14.* From Corollary 3.8, the slopes of the Harder-Narasimhan filtrations of  $\mathcal{E}|_C$  are constant for curves  $C \subset B$  given by a complete intersection in  $l^i[\omega_B], i = 0, 1, 2$ , located away from a proper analytic subset,  $A(l^i)$ , in  $S_{l^i}$ . Using Lemma 3.16, we fix curves  $C_i \in S_{l^i} \setminus A(l^i)$ , given by complete intersection in  $l^i[\omega_B], i = 1, 2$ , as required in Proposition 3.15. By Proposition 3.15,  $C_1 + C_2$  can be put as a central fiber in a flat family of curves, obtained as a base change along a holomorphic disc from  $S_{l_0}$ , not contained in  $A(l^0)$ . Since the generic curve from this family doesn't lie in  $A(l^0)$ , the Harder-Narasimhan slopes of the restriction of  $\mathcal{E}$  to a generic curve in this family will then coincide with the Harder-Narasimhan slopes of the restriction of  $\mathcal{E}$  to generic curves given by complete intersection in  $l^0[\omega_B]$ . Directly from this, Theorem 3.11 and (3.6),

$$\sum_{i=1}^r \phi(\mu_i^{HN}(\mathcal{E}|_C, l^1) + \mu_i^{HN}(\mathcal{E}|_C, l^2)) \geq \sum_{i=1}^r \phi(\mu_i^{HN}(\mathcal{E}|_C, l^0)). \quad (3.12)$$

By making a homothety and using the convexity of  $\phi$ , we finish the proof.  $\square$

As a byproduct of Theorem 3.14 and (3.4), we deduce the following.

**Proposition 3.17.** For any  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , we have

$$\begin{aligned} \mu_{\min}^{HN}(E_k|_C, l) &\geq l_1 \cdots l_{m-1} \cdot \mu_{\min}^{HN}(\mathcal{E}|_C, (1, \dots, 1)), \\ \mu_{\max}^{HN}(E_k|_C, l) &\leq l_1 \cdots l_{m-1} \cdot \mu_{\max}^{HN}(\mathcal{E}|_C, (1, \dots, 1)). \end{aligned} \quad (3.13)$$

## 4 Generic base changes and asymptotic distribution of slopes

The main goal of this section is to show that the asymptotic distribution of Harder-Narasimhan slopes of direct images can be calculated through generic base changes, i.e. to establish Theorem 1.2. The following famous result will be at the heart of our approach.

**Theorem 4.1.** *For any semistable torsion-free coherent sheaf  $\mathcal{E}$  over a complex projective manifold  $B$  with an integral multipolarization  $[\omega_B]$ , there is  $l_0 \in \mathbb{N}$ , such that for any generic curve  $C \subset B$ , which is a complete intersection in  $l[\omega_B]$ ,  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ ,  $l_i \geq l_0$ ,  $i = 1, \dots, m-1$ , the restriction of  $\mathcal{E}$  to  $C$  is semistable.*

*Remark 4.2.* When the multipolarization is symmetric, the theorem is due to Mehta-Ramanathan [35, Theorem 6.1], cf. also [28, Theorem 7.11] and [36]. The proof for the general multipolarization remains verbatim.

In particular, from Theorem 4.1 and (2.8), we conclude that for any torsion-free coherent sheaf  $\mathcal{E}$  over  $B$ , the restriction of Harder-Narasimhan filtration of  $\mathcal{E}$  over generic curves  $C \subset B$  as in Theorem 4.1 coincides with the Harder-Narasimhan filtration of  $\mathcal{E}|_C$ .

The main difficulty in the proof of Theorem 1.2 is due to the fact that the value  $l_0$  from Theorem 4.1 depends heavily on  $\mathcal{E}$ . Despite many works on the effective version of Mehta-Ramanathan theorem, see for example Flenner [19] or Bogomolov [3], cf. [28, §7], it seems that all known effective bounds on  $l_0$  depend on the rank and the discriminant of  $\mathcal{E}$ . However, in the notations from Introduction, when  $k \rightarrow \infty$ , both rank and discriminant of  $E_k$  tend to infinity, and so it seems implausible that the value  $l_0$  can be chosen uniformly for all  $k \in \mathbb{N}$ . Remark also that there are some results that relate the Harder-Narasimhan slopes of a sheaf with the Harder-Narasimhan slopes of its restriction to a curve associated with any fixed value  $l_0$ , see for example Grauert-Mulich theorem, cf. [28, §3], or Langer [31, Theorem 3.1]. To the best of authors knowledge, none of these implies Theorem 1.2 directly. Our approach for Theorem 1.2 is then, by necessity, a different one, and it relies on subadditive properties enjoyed by the measures  $\eta_l^{HN|C}$  and  $\eta^{HN}$ , coming from the submultiplicative nature of Harder-Narasimhan filtrations and the results of Section 3.

We will first establish the easier part of Theorem 1.2 about the minimal slopes. For this, we conserve the notations from Introduction and (3.11). Directly from Theorem 4.1, the remark below it and Lemma 2.11, we deduce the following result.

**Lemma 4.3.** The following convergence takes place

$$\lim_{l \rightarrow \infty} \frac{\mu_{\min}^{HN}(E_k|_C, l)}{l_1 \cdots l_{m-1}} = \mu_{\min}^{HN}(E_k), \quad (4.1)$$

where  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ . Moreover, the above limit becomes stationary for  $l$  big enough (i.e. for big enough  $l_0 \in \mathbb{N}$ , and  $l_i \geq l_0$ ,  $i = 1, \dots, m-1$ ), and we always have

$$\frac{\mu_{\min}^{HN}(E_k|_C, l)}{l_1 \cdots l_{m-1}} \leq \mu_{\min}^{HN}(E_k). \quad (4.2)$$

*Remark 4.4.* The same proof shows that the analogous statements hold (with inverse sign in (4.2)) for the maximal slope.

We will also use the following statement, the proof of which is left to the reader.

**Lemma 4.5.** Let  $c_{k,l} \in \mathbb{R}$ ,  $k \in \mathbb{N}$ ,  $l \in \mathbb{N}^{*(m-1)}$ , be such that  $c_{k,l}$  increases in  $k \in \mathbb{N}$ , and the limit  $\lim_{l \rightarrow \infty} c_{k,l}$  exists and satisfies  $c_{k,l} \leq \lim_{l \rightarrow \infty} c_{k,l}$ . Then  $\lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} c_{k,l} = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} c_{k,l}$ .

*Proof of the first convergence of (1.4).* By (4.1) and the definitions of  $\eta_{\min,l}^{HN|C}$  and  $\eta_{\min}^{HN}$ , we see that the first convergence of (1.4) is equivalent to the following statement

$$\lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{\mu_{\min}^{HN}(E_k|_C, l)}{kl_1 \cdots l_{m-1}} = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \frac{\mu_{\min}^{HN}(E_k|_C, l)}{kl_1 \cdots l_{m-1}}, \quad (4.3)$$

where  $k$  goes to infinity along any subsequence which we assume from now on to be multiplicative (as  $k = 2^r$ ,  $r \in \mathbb{N}$ ). Remark, however, that by Proposition 2.1, the sequence  $\mu_{\min}^{HN}(E_k|_C, l)/k$  increases in  $k$ , when  $k$  runs over a multiplicative subsequence. By this and Lemma 4.3, the assumptions of Lemma 4.5 are satisfied for the restriction of the sequence  $c_{k,l} := \mu_{\min}^{HN}(E_k|_C, l)/(kl_1 \cdots l_{m-1})$  to multiplicative subsequences in  $k$ , which finishes the proof.  $\square$

*Remark 4.6.* Curiously, the analogous proof for the maximal slope doesn't work due to Remark 4.4 and the fact that the analogue of Lemma 4.5 with inverse inequality sign doesn't hold.

We will now concentrate on the proof of the rest of Theorem 1.2. For this, we define the sequence of probability measures,  $\eta_{k,l}$ ,  $k \in \mathbb{N}^*$ ,  $l \in \mathbb{N}^{*(m-1)}$ , on  $\mathbb{R}$  as follows

$$\eta_{k,l} := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta \left[ \frac{\mu_i^{HN}(E_k|_C, l)}{kl_1 \cdots l_{m-1}} \right], \quad (4.4)$$

where  $N_k = \text{rk}(E_k)$ , and we used the notations from (3.11).

The key in the proof of Theorem 1.2 lies in the following result, the proof of which we defer to the end of this section.

**Theorem 4.7.** For any convex non-increasing Lipschitz function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , and any  $k \in \mathbb{N}^*$ ,  $l \in \mathbb{N}^{*(m-1)}$ , we have the following decomposition

$$k \int \phi d\eta_{k,l} = a_{k,l} + kb_{k,l}, \quad (4.5)$$

where  $b_{k,l} \rightarrow 0$ , as  $k \rightarrow \infty$ , uniformly on  $l \in \mathbb{N}^{*(m-1)}$ , and  $a_{k,l}$  is subadditive in  $k \in \mathbb{N}$  over multiplicative subsequences.

We will also use the following statement, the proof of which repeats the proof of classical Fekete's Subadditive Lemma, and is left to the interested reader.

**Lemma 4.8.** Let  $c_{k,l} \in \mathbb{R}$ ,  $k \in \mathbb{N}^*$ ,  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , be such that  $c_{k,l}$  is coordinatewise subadditive in  $l \in \mathbb{N}$  in the sense of Theorem 3.14, and it admits the following decomposition  $c_{k,l} = a_{k,l} + kb_{k,l}$ , where  $b_{k,l} \rightarrow 0$ , as  $k \rightarrow \infty$ , uniformly on  $l \in \mathbb{N}^{*(m-1)}$ , and  $a_{k,l}$  is subadditive in  $k \in \mathbb{N}$ . Then  $\lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{c_{k,l}}{kl_1 \cdots l_{m-1}} = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \frac{c_{k,l}}{kl_1 \cdots l_{m-1}}$ .

*Proof of Theorem 1.2.* We will first establish the weak convergence of measures. The statement about the convergence of maximal slopes will follow from the convergence of measures.

Clearly, to verify weak convergence, it suffices to verify that for any smooth test function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  of compact support, as  $l \rightarrow \infty$  over multiplicative subsequences, we have

$$\lim_{l \rightarrow \infty} \int \phi dH_*^{l^{m-1}}(\eta_l^{HN|C}) = \int \phi d\eta^{HN}. \quad (4.6)$$

Remark now that any smooth function of compact support can be represented as a difference of two convex non-increasing Lipschitz functions (to see this, it suffices to decompose  $\phi = (\phi + A \min(x - a, 0)^2) - A \min(x - a, 0)^2$  for carefully chosen  $a, A$ ), and, hence, it suffices to establish (4.6) for such functions. We, hence, assume from now on that  $\phi$  is convex, non-increasing and Lipschitz. By the definition of  $\eta_l^{HN|C}$ , we have

$$\lim_{l \rightarrow \infty} \int \phi dH_*^{l^{m-1}}(\eta_l^{HN|C}) = \lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \int \phi d\eta_{k,l}. \quad (4.7)$$

From Theorem 4.1 and the remark after it, we also have

$$\int \phi d\eta^{HN} = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \int \phi d\eta_{k,l}. \quad (4.8)$$

By Theorem 3.14, the sequence  $(k, l) \mapsto kl_1 \cdots l_{m-1} \int \phi d\eta_{k,l}$ ,  $k \in \mathbb{N}^*$ ,  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , is coordinatewise subadditive in  $l$ . We establish (4.6) by Theorem 4.7, Lemma 4.8 and (4.7), (4.8).

We will now establish the second convergence of (1.4). First of all, from Lemma 2.11, we see

$$\liminf_{l \rightarrow \infty} \frac{\eta_{\max,l}^{HN|C}}{l_1 \cdots l_{m-1}} \geq \eta_{\max}^{HN}. \quad (4.9)$$

Now, by Theorem 2.4, the cumulative function  $F_{HN}(\lambda) := \int_{\lambda}^{\infty} d\eta^{HN}$  (resp.  $F_{HN,l}(\lambda) := \int_{\lambda}^{\infty} dH_*^{l^{m-1}}(\eta_l^{HN|C})$ ) is such that  $F_{HN}^{1/n}$  (resp.  $F_{HN,l}^{1/n}$ ) is concave on  $] - \infty, \text{ess sup } \eta^{HN}[$ . In particular, the function  $F_{HN}$  is continuous on  $\mathbb{R} \setminus \text{ess sup } \eta^{HN}$ . Recall that weak convergence of measures implies pointwise convergence of the cumulative functions over the continuity set. In particular,  $F_{HN,l}(\lambda)$  converge pointwise, as  $l \rightarrow \infty$ , to  $F_{HN}(\lambda)$  away from  $\text{ess sup } \eta^{HN}$ .

Assume now, for the sake of contradiction, that we have a strict inequality

$$\limsup_{l \rightarrow \infty} \frac{\eta_{\max,l}^{HN|C}}{l_1 \cdots l_{m-1}} > \eta_{\max}^{HN}. \quad (4.10)$$

Upon restricting to a subsequence in  $l$ , we may assume that we have  $\liminf$  instead of  $\limsup$  in (4.10). Let  $\epsilon > 0$  be such that the inequality (4.10) is still satisfied if we replace  $\eta_{\max}^{HN}$  by  $\eta_{\max}^{HN} + 4\epsilon$ . We assume that  $l$  is big enough so that  $\eta_{\max}^{HN} + 2\epsilon < \eta_{\max,l}^{HN|C} / (l_1 \cdots l_{m-1})$ . Remark that by Theorem 2.2,  $\eta_{\max}^{HN}$  coincides with the essential supremum of  $\eta^{HN}$ , hence  $F_{HN}(\eta_{\max}^{HN} - \epsilon)$  is strictly positive. By the concavity of  $F_{HN,l}^{1/n}$  on the interval  $] - \infty, \eta_{\max}^{HN} + 2\epsilon[ \subset ] - \infty, \eta_{\max,l}^{HN|C} / (l_1 \cdots l_{m-1})[$ , the fact, following from the first part of the proof, that  $F_{HN,l}(\eta_{\max}^{HN} - \epsilon)$  converges, as  $l \rightarrow \infty$ , to  $F_{HN}(\eta_{\max}^{HN} - \epsilon)$ , which is strictly positive, and the fact that  $F_{HN,l}(\eta_{\max}^{HN} + \epsilon)$  is non-negative, we conclude that there is a constant  $\delta > 0$ , such that  $F_{HN,l}(\lambda) > \delta$  for any  $\lambda \in ] - \infty, \eta_{\max}^{HN} + \epsilon[$ . This clearly contradicts the fact that  $F_{HN,l}(\eta_{\max}^{HN} + \epsilon)$  converges to  $F_{HN}(\eta_{\max}^{HN} + \epsilon) = 0$ . Hence, the initial assumption (4.10) was false, which finishes the proof.  $\square$

We will now prove Theorem 4.7, and for this, the following result will be crucial.

**Proposition 4.9** (Chen [7, Proposition 3.3.3]). For any convex non-increasing function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and any submultiplicative filtration  $\mathcal{F}$  on the ring  $\text{Sym} V := \bigoplus_{k=0}^{\infty} \text{Sym}^k V$  for some vector space  $V$ , the sequence  $k \mapsto k \int \phi \cdot dH_*^k(\eta(\mathcal{F}, \text{Sym}^k V))$  is subadditive.



Remark that if the fibers of  $p$  were projective spaces, then Theorem 4.7 would already follow from Proposition 4.9, as the section ring of the projective space polarized by a hyperplane bundle is isomorphic to the symmetric tensor algebra. Proposition 4.9, however, seemingly doesn't hold if the ring  $\text{Sym}V$  is replaced by a section ring of an arbitrary complex projective manifold. To circumvent this problem, we need a more refined argument, and our solution, which is inspired by [7, §3.4.2], goes through the use of Noether normalization lemma.

More precisely, Noether normalization lemma, cf. [14, Theorem 13.3], says that for any projective manifold  $Y$  of dimension  $n$ , and an ample line bundle  $L$  over  $Y$ , there is  $d \in \mathbb{N}^*$ , and  $s_1, \dots, s_{n+1} \in H^0(Y, L^d)$ , which are algebraically independent in the section ring,  $R(Y, L)$ , and such that the subring,  $R_{(d)}(Y, L)$ , of  $R(Y, L)$ , defined as  $R_{(d)}(Y, L) = \bigoplus_{k=0}^{\infty} H^0(Y, L^{kd})$ , is a finitely generated module over the (polynomial) subring  $A \subset R_{(d)}(Y, L)$  induced by  $s_1, \dots, s_{n+1}$ , i.e.  $A = \text{Sym}\langle s_1, \dots, s_{n+1} \rangle$ .

Now, let  $\mathcal{F}$  be an arbitrary bounded submultiplicative filtration on  $R(Y, L)$ . The following proposition says that the spectral measure of the filtration  $\mathcal{F}$  on  $R(Y, L)$  can be reconstructed from the spectral measure of the restriction of the filtration  $\mathcal{F}$  to  $A \subset R(Y, L)$ .

**Proposition 4.10** (Chen [7, proof of Theorem 3.4.3]). For any convex non-increasing Lipschitz function  $\phi$  on  $\mathbb{R}$ , there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ , we have

$$\left| \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, A_{kd})) - \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, H^0(Y, L^{kd}))) \right| \leq \frac{C}{k}. \quad (4.11)$$

*Remark 4.11.* A combination of Propositions 4.9 and 4.10, and a decomposition argument from the proof of Theorem 1.2 was used in [7] to prove the weak convergence in Theorem 2.2.

Our proof for Theorem 4.7 will then be a combination of Proposition 4.9 and a uniform version of Proposition 4.10. In order to explain the latter statement and its proof, we first recall the main steps from [7] of the proof of Proposition 4.10.

**Lemma 4.12** (Chen [7, Proposition 1.2.5]). Let  $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0$  be a short exact sequence of vector spaces. We fix a filtration  $\mathcal{F}$  on  $V_1$ , and induce the filtrations on  $V_0, V_2$ , which we denote by an abuse of notation by the same letter. Then the following identity holds

$$\dim V_2 \cdot \eta(\mathcal{F}, V_1) = \dim V_2 \cdot \eta(\mathcal{F}, V_0) + \dim V_2 \cdot \eta(\mathcal{F}, V_2), \quad (4.12)$$

where the jumping measures,  $\eta(\mathcal{F}, V_i)$ ,  $i = 0, 1, 2$ , are defined in (2.1).

We denote by  $\text{Frac}(A)$  the field of fractions of  $A$ . As  $R_{(d)}(Y, L)$  is a finitely generated module over  $A$ , the  $\text{Frac}(A)$ -vector space  $\text{Frac}(A) \otimes_A R_{(d)}(Y, L)$  is finitely generated. We denote by  $x_1, \dots, x_r$ ,  $r \in \mathbb{N}$ , some homogeneous elements of  $R_{(d)}(Y, L)$ , which form a basis of the above vector space. Then the space  $B := x_1 A + \dots + x_r A$ ,  $B = \bigoplus_{k=0}^{\infty} B_{kd}$ , is a free  $A$ -submodule of  $R_{(d)}(Y, L)$ , and the quotient  $R_{(d)}(Y, L)/B$  is a  $A$ -torsion module, since  $\text{Frac}(A) \otimes_A R_{(d)}(Y, L)/B = \{0\}$ . Then, as  $R_{(d)}(Y, L)$  is a finitely generated  $A$ -module, the annihilator,  $\text{Ann}(R_{(d)}(Y, L)/B) \subset A$ , of  $R_{(d)}(Y, L)/B$  is non-empty. Hence, the Krull dimension of the  $A$ -module  $R_{(d)}(Y, L)/B$ , which is defined as the Krull dimension of  $A/\text{Ann}(R_{(d)}(Y, L)/B)$ , is strictly smaller than the Krull dimension of  $A$ . Since Krull dimension of a module coincides with the degree of its Hilbert polynomial, see Serre [43, Proposition 6, p. 276], we conclude that there is a constant such that the dimension of the  $k$ -th graded part,  $H^0(Y, L^{kd})/B_{kd}$ , of  $R_{(d)}(Y, L)/B$  is

smaller than  $Ck^{\dim Y - 1}$ . We conclude by this and Lemma 4.12 that for any bounded function  $\phi$ , there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ , we have

$$\left| \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, B_{kd})) - \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, H^0(Y, L^{kd}))) \right| \leq \frac{C}{k}. \quad (4.13)$$

By (4.13), we see that Proposition 4.10 will follow if we prove the following result.

**Lemma 4.13.** For any homogeneous element  $x \in H^0(Y, L^{d_0})$ ,  $d_0 \in \mathbb{N}^*$ ,  $d|d_0$ , the spectral measures of the restriction of the filtration  $\mathcal{F}$  to the submodules  $xA$  and  $A$  compare as follows. For any convex non-increasing function  $\phi$  on  $\mathbb{R}$ , there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ , we have

$$\left| \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, (xA)_{kd})) - \int \phi \cdot dH_*^{kd}(\eta(\mathcal{F}, A_{kd})) \right| \leq \frac{C}{k}. \quad (4.14)$$

*Proof.* In order to prove this, remark that since the  $\text{Frac}(A)$ -vector space  $\text{Frac}(A) \otimes_A R_{(d)}(Y, L)$  is finitely generated, there is  $N \in \mathbb{N}^*$ ,  $N \leq r + 1$ , and homogeneous elements  $a_0 \in A_{d_1}$ ,  $a_1 \in A_{d_1+d_0}$ ,  $\dots$ ,  $a_N \in A_{d_1+Nd_0}$ ,  $a_N \neq 0$ , for some  $d_1 \in \mathbb{N}^*$ , such that  $d|d_1$  and  $a_0x^N + a_1x^{N-1} + \dots + a_N = 0$ . From [7, (61)], for any non-increasing function  $\psi$  on  $\mathbb{R}$ , we have

$$\int \psi \cdot d\eta(\mathcal{F}, A_{kd}) \geq \int \psi \cdot d\eta(\mathcal{F}, (xA)_{kd+d_0}) \geq \int \psi \cdot d\eta(\mathcal{F}, (a_N A)_{kd+d_1+Nd_0}). \quad (4.15)$$

Now, remark that the Krull dimension of the  $A$ -module  $A/a_N A$ , is strictly smaller than the Krull dimension of  $A$ , as  $a_N \neq 0$ . Hence, as in (4.13), we deduce that for any bounded function  $\psi$  on  $\mathbb{R}$ , there is  $C > 0$ , such that we have

$$\left| \int \psi \cdot d\eta(\mathcal{F}, (a_N A)_{kd}) - \int \psi \cdot d\eta(\mathcal{F}, A_{kd}) \right| \leq \frac{C}{k}. \quad (4.16)$$

Also, for any Lipschitz function  $\psi$  on  $\mathbb{R}$ , and any  $d_3 \in \mathbb{R}$ , by the boundness of the filtration  $\mathcal{F}$ , there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ , we have

$$\left| \int \psi \cdot dH_*^{kd} \eta(\mathcal{F}, A_{kd}) - \int \psi \cdot dH_*^{kd+d_3} \eta(\mathcal{F}, A_{kd}) \right| \leq \frac{C}{k}. \quad (4.17)$$

From (4.15), (4.16), (4.17) and the fact that the limit, as  $k \rightarrow \infty$ , of  $\int \psi \cdot dH_*^{kd} \eta(\mathcal{F}, A_{kd})$  exists, which follows from Proposition 4.9, we deduce (4.14).  $\square$

We will now explain that a part of the above reasoning can be done in a family setting. More precisely, consider a surjective flat holomorphic map  $\pi : X \rightarrow B$  between complex projective manifolds  $X$  and  $B$  of dimensions  $n + m$  and  $m$ . Let  $L$  be a holomorphic line bundle over  $X$ , which is relatively ample with respect to  $\pi$ . We argue that there is a subset  $S \subset B$ , which is a countable union of analytic subsets, and  $d^* \in \mathbb{N}$ , such that the following holds. For any  $b \in B \setminus S$ , there is  $d(b) \in \mathbb{N}^*$ , verifying  $d(b) \leq d^*$ , and  $s_1(b), \dots, s_{n+1}(b) \in H^0(X_b, L|_{X_b}^{d(b)})$ , which are algebraically independent in the section ring,  $R(X_b, L|_{X_b})$ , and such that  $R_{(d(b))}(X_b, L|_{X_b})$  is a finitely generated module over the (polynomial) subring  $A(b) \subset R_{(d(b))}(X_b, L|_{X_b})$  induced by  $s_1(b), \dots, s_{n+1}(b)$ . Moreover, there are  $r^*, d_0^* \in \mathbb{N}$  and  $C > 0$ , such that for any  $b \in B \setminus S$ , there are  $r(b), d_0(b) \in \mathbb{N}^*$ , verifying  $r(b) \leq r^*$ ,  $d_0(b) \leq d_0^*$ , and  $x_1(b), \dots, x_{r(b)}(b) \in H^0(X_b, L|_{X_b}^{d_0(b)})$ ,

such that the  $A(b)$ -module  $B(b) := x_1(b)A(b) + \cdots + x_{r(b)}(b)A(b)$  is a free  $A(b)$ -submodule of  $R_{(d(b))}(X_b, L|_{X_b})$ , such that

$$\dim H^0(X_b, L|_{X_b}^{d(b)k}) - \dim B(b)_{d(b)k} \leq Ck^{n-1}. \quad (4.18)$$

To see this, we first recall that in an arbitrary vector bundle, the rank of a set of holomorphic sections at a given point is a lower semicontinuous function (in Zariski topology). We pick an arbitrary  $b_0 \in S$ , and construct the sections  $s_1(b_0), \dots, s_{n+1}(b_0) \in H^0(X_{b_0}, L|_{X_{b_0}}^{d(b_0)})$  by Noether normalization lemma as we did above. Extend them holomorphically (as sections of  $R^0\pi_*L^{d(b_0)}$ ) to  $s_1(b), \dots, s_{N(b_0)}(b) \in H^0(X_b, L|_{X_b}^{d(b_0)})$  for  $b$  in a small neighborhood  $U$  of  $b_0$ .

We argue that away from a countable union of analytic subsets in  $U$ , the sections  $s_1(b), \dots, s_{n+1}(b) \in H^0(X_b, L|_{X_b}^{d(b_0)})$  are algebraically independent. Indeed, this follows from the above lower semicontinuity property, as for any  $k \in \mathbb{N}^*$ , the set of  $k$ -homogeneous monomials in  $s_1(b), \dots, s_{n+1}(b)$ ,  $b \in U$ , is free away from an analytic subset (which is a proper subset of  $U$ , as it doesn't contain  $b_0$ ).

Now, we fix homogeneous elements  $x_1(b_0), \dots, x_{r(b_0)}(b_0)$ , in  $R_{(d(b_0))}(X_{b_0}, L|_{X_{b_0}})$ , such that the associated  $A(b_0)$ -module  $B(b_0) := x_1(b_0)A(b_0) + \cdots + x_{r(b_0)}(b_0)A(b_0)$  is free, and there is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ , we have (4.18) for  $b := b_0$ . We extend  $x_1(b_0), \dots, x_{r(b_0)}(b_0)$  holomorphically in a neighborhood of  $b_0$ . A similar argument to the one before shows that for any  $b$  in this neighborhood away from a countable union of analytic subsets, the  $A(b)$ -module  $B(b)$ , defined analogously to  $B(b_0)$ , is free. Then, since by the flatness,  $\dim H^0(X_b, L|_{X_b}^{d(b)k})$  is a locally constant function of  $b \in S$ , cf. [24, Theorem III.9.9], we conclude that inequality (4.18) holds for  $b$  in this neighborhood away from a countable union of analytic subsets.

We then repeat this procedure for every point  $b_0 \in B$ , and use the compactness of  $B$ , to extract a finite cover of  $B$  by open subsets. In conclusion, a direct repetition of the argument in the proof of Proposition 4.10 yields the following result.

**Proposition 4.14.** There is a countable union of proper analytic subsets  $S$  in  $B$ , such that the following holds. For any  $D > 0$ , any sequence  $b_l \in B \setminus S$ ,  $l \in \mathbb{N}$ , any family of submultiplicative filtrations  $\mathcal{F}(l)$  on  $R(X_{b_l}, L|_{X_{b_l}})$ , which is uniformly bounded in the sense that the absolute value of the maximal and the minimal jumping numbers on degree  $k$ -part are bounded by  $Dk$  (uniformly on  $l$ ), and any convex Lipschitz function  $\phi$  on  $\mathbb{R}$ , there is  $C > 0$ , such that for any sufficiently divisible  $k \in \mathbb{N}^*$ , and any  $l \in \mathbb{N}^*$ , we have

$$\left| \int \phi \cdot dH_*^k(\eta(\mathcal{F}(l), A_k(b_l))) - \int \phi \cdot dH_*^k(\eta(\mathcal{F}(l), H^0(X_{b_l}, L|_{X_{b_l}}^k))) \right| \leq \frac{C}{k}. \quad (4.19)$$

Finally, to finish the argument, we need the following statement.

**Proposition 4.15.** There is  $C > 0$ , such that for any  $k \in \mathbb{N}^*$ ,  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , for any generic regular curve  $C$ , which is a complete intersection in  $l[\omega_B]$ , we have

$$\mu_{\min}^{HN}(E_k|_C, l) \geq -Ckl_1 \cdots l_{m-1}, \quad \mu_{\max}^{HN}(E_k|_C, l) \leq Ckl_1 \cdots l_{m-1}. \quad (4.20)$$

*Proof.* The result now follows from the linear (in  $k$ ) bounds on  $\mu_{\min}^{HN}(E_k|_C, (1, \dots, 1))$ ,  $\mu_{\max}^{HN}(E_k|_C, (1, \dots, 1))$ , cf. Proposition 2.10, and Proposition 3.17.  $\square$

*Proof of Theorem 4.7.* We denote by  $S$  the subset from Proposition 4.14. For any  $l = (l_1, \dots, l_{m-1}) \in \mathbb{N}^{*(m-1)}$ , consider a generic regular curve  $C = C(l) \subset B$ , which is a complete intersection in  $l[\omega_B]$ , and which is not contained in  $S$ . We consider an arbitrary point  $b_l \in C(l) \setminus S$ , and denote by  $\mathcal{F}(l)$  the filtration on  $R(X_{b_l}, L|_{X_{b_l}})$  induced by the Harder-Narasimhan filtration on  $E_k|_C$  with slopes, divided by  $l_1 \cdots l_{m-1}$ . Then by Proposition 4.15, the family of filtrations  $\mathcal{F}(l)$  is uniformly bounded in the sense of Proposition 4.14. By the results of Section 2, the filtrations  $\mathcal{F}(l)$  are submultiplicative. Then by Propositions 4.10 and 4.14, the decomposition (4.5) holds for the sequence  $a_{k,l} = k \int \phi dH_*^k(\eta(\mathcal{F}(l), A_k(b_l)))$ .  $\square$

## 5 Numerical expressions for asymptotic slopes

The main goal of this section is to prove Corollary 1.4. For this, let us first recall the results of Xu-Zhuang from [48]. We conserve the notations from the introduction, and assume for the moment that  $\dim B = 1$ . For a class  $\alpha \in H^{1,1}(X)$ , we then define the *nef* (resp. *psef*) *threshold of  $\alpha$  with respect to  $\pi$*  as the supremum over all  $T \in \mathbb{R}$ , such that  $\alpha - T(\pi^*[\omega_{B,0}]/\int_B[\omega_{B,0}])$  is nef (resp. psef) for some Kähler class  $[\omega_{B,0}]$  on  $B$ .

**Proposition 5.1.** If  $\dim B = 1$ , then  $\eta_{\min}^{HN}$  (resp.  $\eta_{\max}^{HN}$ ) equals to nef (resp. psef) threshold of  $c_1(L)$  with respect to  $\pi$ . Moreover, we have  $\eta_{\min}^{HN} = \text{ess inf } \eta^{HN}$ .

*Proof.* In [48, Proposition 2.28], authors proved that  $\text{ess sup } \eta^{HN}$  equals to psef threshold of  $c_1(L)$  with respect to  $\pi$ . From this and Theorem 2.2, we deduce the statement of Proposition 5.1 for  $\eta_{\max}^{HN}$ . In the proof of [48, Lemma 2.26], authors established that  $\eta_{\min}^{HN}$  is not smaller than nef threshold of  $c_1(L)$  with respect to  $\pi$ . This finishes the proof of Proposition 5.1, as by Theorem 2.2, we have  $\eta_{\min}^{HN} \leq \text{ess inf } \eta^{HN}$ , and by [48, Proposition 2.28],  $\text{ess inf } \eta^{HN}$  equals to nef threshold of  $c_1(L)$  with respect to  $\pi$ .  $\square$

Let us now deduce formulas (1.5) from Proposition 5.1. By the definition of nefness, we see that the nef threshold of  $c_1(L)$  with respect to  $\pi$  equals

$$\inf_{C \in \mathcal{C}_{irr}} \frac{\int_C c_1(L) \cdot \int_C \pi|_C^*[\omega_{B,0}]}{\int_B[\omega_{B,0}]}, \quad (5.1)$$

where  $\mathcal{C}_{irr}$  was defined before (1.5). This gives us the first formula from (1.5) by the definition of the topological degree. The proof of the second formula is identical, except that it is necessary to use the characterization of the dual to the pseudoeffective cone in terms of movable curves due to Boucksom-Demailly-Paun-Peternell [5, Theorem 2.2].

*Proof of Corollary 1.4.* Directly from Proposition 5.1, we have

$$\eta_{\min,l}^{HN|C} = \sup \left\{ T \in \mathbb{R} : \text{the class } c_1(L)|_C - T \cdot \frac{\pi^*[\omega_{B,0}]|_C}{\int_C[\omega_{B,0}]} \text{ is nef} \right\}, \quad (5.2)$$

where the curves  $C = C(l)$  are as in Theorem 1.2. Similar formulas hold for the maximal asymptotic slope. We deduce Corollary 1.4 by (5.2), the identity  $\int_C[\omega_{B,0}] = l^{m-1} \cdot \int_B[\omega_{B,0}] \cdots [\omega_{B,m-1}]$  and Theorem 1.2.  $\square$

Let us now give another interpretation of Corollary 1.4. We say that a relatively ample  $\mathbb{Q}$ -line bundle  $L$  on  $X$  is *stably*  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -*generically fibered nef with respect to  $\pi$*  if for some (or any) ample line bundle  $L_0$  on  $X$ , for any  $\epsilon > 0$ ,  $\epsilon \in \mathbb{Q}$ , the  $\mathbb{Q}$ -line bundle  $L \otimes L_0^\epsilon$  is  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -*generically fibered nef with respect to  $\pi$* .

**Proposition 5.2.** A relatively ample line bundle  $L$  is stably  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$  if and only if  $\eta_{\min}^{HN} \geq 0$ .

*Proof.* We assume first that  $\eta_{\min}^{HN} \geq 0$ . By Corollary 1.4, for any  $\epsilon > 0$ ,  $\epsilon \in \mathbb{Q}$ , the  $\mathbb{Q}$ -line bundle  $L \otimes \pi^* L_1^\epsilon$  is  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$ . But then for any ample line bundle  $L_0$  on  $X$ , any  $\epsilon > 0$ ,  $\epsilon \in \mathbb{Q}$ , the  $\mathbb{Q}$ -line bundle  $L \otimes L_0^\epsilon$  is  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$ , which means that  $L$  is stably  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$ .

Inversely, assume that  $L$  is stably  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$ . Let  $N \in \mathbb{N}^*$  be such that the line bundle  $L_0 := L \otimes \pi^* L_1^N$  is ample (such  $N$  exists since  $L$  is relatively ample). Then for any  $\epsilon > 0$ ,  $\epsilon \in \mathbb{Q}$ , the  $\mathbb{Q}$ -line bundle  $L \otimes L_0^\epsilon$  is  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nef with respect to  $\pi$ . However, since  $L \otimes L_0^\epsilon = L^{1+\epsilon} \otimes \pi^* L_1^{N\epsilon}$ , we see that the  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generic fibered nef threshold of  $c_1(L)$  is non-negative, which by Corollary 1.4 means that  $\eta_{\min}^{HN} \geq 0$ , and this finishes the proof.  $\square$

*Remark 5.3.* From the proof, we see that in the definition of stably  $([\omega_{B,1}], \dots, [\omega_{B,m-1}])$ -generically fibered nefness with respect to  $\pi$ , instead of ample line bundle  $L_0$  over  $X$ , we can consider the pull-back of an ample line bundle over the base.

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