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# Analysis and stabilization of a model of population dynamics with age structure and diffusion

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## Abstract

We analyze a system modeling the evolution of an age and spatially structured population (of Lotka–McKendrick type). We study it by first writing it in an abstract form using several operators. We show that the semigroup associated with the corresponding system is differentiable. Using this property, we show how to prove the exponential stabilization with a finite-dimensional feedback control. We consider two types of controls: one that acts directly on the main equation of evolution and one that acts on the birth equation. One of the main difficulties in the analysis of this system is that the operators involved in the system can depend on the age variable. We use in particular a parabolic evolution operator associated with the main operator of the system. Our stabilization result shows how to extend the framework associated with parabolic system to the case of differentiable semigroups.

**Keywords:** Population dynamics, Feedback stabilization, Differentiable semigroup  
**2020 Mathematics Subject Classification.** 93D15, 92D25

## 1 Introduction

### 1.1 The model

We consider the control of an age and spatially structured population model. We denote by  $p(t, a, x)$  the distribution density of the population at time  $t \geq 0$ , at age  $a \in [0, a_\infty]$  and at space position  $x \in \Omega$ . We assume here that  $a_\infty \in \mathbb{R}_+^*$  and that  $\Omega$  is a bounded regular domain of  $\mathbb{R}^d$ ,  $d \geq 1$ . The corresponding system writes as follows (see, for instance, [5, 18, 24, 40]):

$$\left\{ \begin{array}{ll} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - Lp + \mu p = 0 & (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \Omega, \\ p(t, a, x) = \mathbf{1}_{(a_1, a_2) \times \Gamma}(a, x) u(t, a, x) & (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\infty} \beta(a, x) p(t, a, x) da & (t, x) \in (0, \infty) \times \Omega, \\ p(0, a, x) = p^0(a, x) & (a, x) \in (0, a_\infty) \times \Omega, \end{array} \right. \quad (1.1)$$

where the operator  $L$  is defined by

$$Lp \stackrel{\text{def}}{=} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( \sigma_{i,j} \frac{\partial p}{\partial x_j} \right) + \sum_{j=1}^d m_j \frac{\partial p}{\partial x_j} + rp. \quad (1.2)$$

In the above system,  $\beta$  and  $\mu$  are respectively the birth rate and the death rate. From the model, they are assumed to be nonnegative. We have denoted by  $p^0$  the initial population distribution. The above system is controlled on its boundary by a control  $u = u(t, a, x)$  localized in age and space. More precisely  $(a_1, a_2)$  is a nonempty interval of  $(0, a_\infty)$  and  $\Gamma$  is a nonempty open set of  $\partial\Omega$ .

Our aim is to analyse the above system and to obtain a feedback stabilization result by using the control  $u$  (see Theorem 1.9 below). We will first write the above system under an abstract form (see (1.15) below) and study the corresponding system. We show in particular that the underlying semigroup is differentiable. Such a property will allow us to prove that our system can be stabilized provided a Fattorini-Hautus condition holds true and with a finite-dimensional control (see Theorem 1.3 below).

One of the main difficulties and differences with respect to other results in the literature is that here we assume that the coefficients in (1.1) ( $\sigma$ ,  $m$ ,  $r$ ,  $\mu$  and  $\beta$ ) can depend on the age variable and this leads us to consider operators that can depend on the age variable. More precisely, we impose the following assumptions on the coefficients in (1.1):

- There exists  $\nu > 0$  such that

$$\sigma \in C^\nu([0, a_\infty]; W^{1,\infty}(\Omega)^{d \times d}), \quad (1.3)$$

$$\partial_a \sigma, \partial_{x_i} \sigma, \partial_{x_i, x_j} \sigma \in C^0([0, a_\infty] \times \bar{\Omega})^{d \times d} \quad (1 \leq i, j \leq d), \quad (1.4)$$

$$\sigma(a, x) \text{ symmetric positive-definite for any } a \in [0, a_\infty], x \in \bar{\Omega}, \quad (1.5)$$

$$m \in C^\nu([0, a_\infty]; W^{1,\infty}(\Omega)^d), \quad \partial_{x_j} m \in C^0([0, a_\infty] \times \bar{\Omega})^d \quad (1 \leq j \leq d), \quad r \in C^\nu([0, a_\infty]; L^\infty(\Omega)), \quad (1.6)$$

$$\mu \in C^\nu([0, a_\infty]; L^\infty(\Omega; \mathbb{R}_+)). \quad (1.7)$$

- There exist

$$b > \max\left(\frac{d}{2}, \frac{3}{4}\right), \quad a_\beta \in (0, a_\infty) \quad (1.8)$$

such that

$$\beta \in L^\infty(0, a_\infty; H^b(\Omega)) \cap W^{1,1}(0, a_\infty, L^\infty(\Omega)), \quad (1.9)$$

$$\beta \equiv 0 \quad \text{in } (0, a_\beta) \times \Omega. \quad (1.10)$$

*Remark 1.1.* Some of the above conditions will be used to show the well-posedness of the system, whereas some of these hypotheses are more related to the control problem and in particular to prove a unique continuation condition. Typically, the Hölder continuous conditions and (1.9) are related to the well-posedness, whereas the other conditions such as (1.4), (1.5) corresponds to the stabilization problem.

Another condition used only for the stabilization property is (1.10). Such a condition is already considered in previous works for the control property of such a model (see, for instance, [25, 26]) and models the fact that the young individuals do not participate in the reproduction.

*Remark 1.2.* The standard hypotheses on the mortality rate  $\mu$  yield that it is singular at  $a = a_\infty$  so that condition (1.7) seems unnatural. Nevertheless, with some particular structure of  $\mu$ , it is possible to remove this singularity and to assume (1.7). Assume (instead of (1.7)) that

$$\mu = \mu_0 + \tilde{\mu}, \quad \mu_0 \in L^1_{\text{loc}}([0, a_\infty]; \mathbb{R}_+), \quad \int_0^{a_\infty} \mu_0(a) da = \infty, \quad \tilde{\mu} \in C^\nu([0, a_\infty]; L^\infty(\Omega; \mathbb{R}_+)).$$

In order to avoid the singular mortality rate, one can consider the following change of variables:

$$\begin{aligned} \tilde{p}(t, a, x) &\stackrel{\text{def}}{=} e^{\int_0^a \mu_0(s) ds} p(t, a, x), & \tilde{p}^0(a, x) &\stackrel{\text{def}}{=} e^{\int_0^a \mu_0(s) ds} p^0(a, x), \\ \tilde{u}(t, a, x) &\stackrel{\text{def}}{=} e^{\int_0^a \mu_0(s) ds} u(t, a, x), & \tilde{\beta}(a, x) &\stackrel{\text{def}}{=} e^{-\int_0^a \mu_0(s) ds} \beta(a, x) \end{aligned}$$

and we can check that the system (1.1) is then transformed into

$$\left\{ \begin{array}{l} \frac{\partial \tilde{p}}{\partial t} + \frac{\partial \tilde{p}}{\partial a} - L\tilde{p} + \tilde{\mu}\tilde{p} = 0 \quad (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \Omega, \\ \tilde{p}(t, a, x) = \mathbf{1}_{(a_1, a_2) \times \Gamma}(a, x) \tilde{u}(t, a, x) \quad (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \partial\Omega, \\ \tilde{p}(t, 0, x) = \int_0^{a_\infty} \tilde{\beta}(a, x) \tilde{p}(t, a, x) da \quad (t, x) \in (0, \infty) \times \Omega, \\ \tilde{p}(0, a, x) = \tilde{p}^0(a, x) \quad (a, x) \in (0, a_\infty) \times \Omega. \end{array} \right. \quad (1.11)$$

In particular, with this transformation, we are reduced to work with a mortality rate  $\tilde{\mu} \in C^\nu([0, a_\infty]; L^\infty(\Omega; \mathbb{R}_+))$ .

In addition to the study of (1.1), we are also interested in the feedback stabilization of the same system with a control acting on the birth equation:

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - Lp + \mu p = 0 \quad (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \Omega, \\ p(t, a, x) = 0 \quad (t, a, x) \in (0, \infty) \times (0, a_\infty) \times \partial\Omega, \\ p(t, 0, x) = \int_0^{a_\infty} \beta(a, x) p(t, a, x) da + \mathbf{1}_\omega(x) u(t, x) \quad (t, x) \in (0, \infty) \times \Omega, \\ p(0, a, x) = p^0(a, x) \quad (a, x) \in (0, a_\infty) \times \Omega, \end{array} \right. \quad (1.12)$$

where the control  $u = u(t, x)$  acts on an arbitrary small nonempty open set  $\omega$  of  $\Omega$ . We refer to [22] for a similar problem. In that case, we still assume (1.5), (1.6), (1.7), (1.10) but we replace (1.3) and (1.4) by the stronger hypothesis:

$$\sigma \in C^2([0, a_\infty] \times \bar{\Omega})^{d \times d}. \quad (1.13)$$

We still assume (1.9) but with (1.8) replaced by

$$b > \frac{d}{2}. \quad (1.14)$$

These modifications are related to the control operator associated with (1.12) and in particular the corresponding unique continuation property.

## 1.2 Abstract system with a structure in age

System (1.1) can be written as an abstract system with a structure in age:

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = Ap + Bu \quad (t, a) \in (0, +\infty) \times (0, a_\infty), \\ p(t, 0) = \int_0^{a_\infty} E(a) p(t, a) da \quad t \in (0, +\infty), \\ p(0, a) = p^0(a) \quad a \in (0, a_\infty). \end{array} \right. \quad (1.15)$$

Our aim is to study this abstract controlled system. Let us define the functional framework and let us give our hypotheses on the above operators. First, we consider Hilbert spaces  $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_1^*, \mathcal{U}$  satisfying

$$\mathcal{H}_1 \hookrightarrow \mathcal{H}, \quad \mathcal{H}_1^* \hookrightarrow \mathcal{H}$$

with dense, continuous and compact embeddings. We also define

$$\mathcal{H}_\alpha \stackrel{\text{def}}{=} \begin{cases} [\mathcal{H}, \mathcal{H}_1]_\alpha & \text{if } \alpha \in [0, 1] \\ [\mathcal{H}, \mathcal{H}_1^*]_{-\alpha} & \text{if } \alpha \in [-1, 0] \end{cases}, \quad \mathcal{H}_\alpha^* \stackrel{\text{def}}{=} \begin{cases} [\mathcal{H}, \mathcal{H}_1^*]_\alpha & \text{if } \alpha \in [0, 1] \\ [\mathcal{H}, \mathcal{H}_1]_{-\alpha} & \text{if } \alpha \in [-1, 0] \end{cases}, \quad (1.16)$$

where  $[\cdot, \cdot]_\alpha$  denotes the complex interpolation and where for a Hilbert space  $\mathcal{V} \hookrightarrow \mathcal{H}$ , with dense and continuous embedding,  $\mathcal{V}'$  stands for the dual space of  $\mathcal{V}$  with respect to the pivot space  $\mathcal{H}$ . Note that (see, for instance, [33, Section 1.9.3, p.59 and Section 1.11.3, p.72]) if  $\alpha_1, \alpha_2 \in [-1, 1]$ , with  $\alpha_1 < \alpha_2$ , then

$$\mathcal{H}_{\alpha_2} \hookrightarrow \mathcal{H}_{\alpha_1}, \quad \mathcal{H}_{\alpha_2}^* \hookrightarrow \mathcal{H}_{\alpha_1}^*,$$

with dense, continuous and compact embeddings.

The hypotheses for the operator  $A$  in (1.15) are

$$A \in C^\nu([0, a_\infty]; \mathcal{L}(\mathcal{H}_1, \mathcal{H})), \quad A^* \in C^\nu([0, a_\infty]; \mathcal{L}(\mathcal{H}_1^*, \mathcal{H})), \quad (1.17)$$

for some  $\nu > 0$ . We have denoted by  $A(a)^*$  the adjoint of  $A(a)$ . From the above hypotheses, we can consider  $A(a)$  and  $A(a)^*$  for  $a \in (0, a_\infty)$  as unbounded operators in  $\mathcal{H}$  with respective domains  $\mathcal{D}(A(a)) = \mathcal{H}_1$  and  $\mathcal{D}(A(a)^*) = \mathcal{H}_1^*$ .

We also assume the existence of  $N > 0$ ,  $\lambda_0 \in \mathbb{R}$ , and  $\vartheta \in (\pi/2, \pi)$  such that

$$\Sigma_{\lambda_0, \vartheta} \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \setminus \{\lambda_0\} : |\arg(\lambda - \lambda_0)| < \vartheta\} \subset \rho(A(a)) \quad (a \in [0, a_\infty]), \quad (1.18)$$

and

$$\forall a \in [0, a_\infty], \quad \forall \lambda \in \Sigma_{\lambda_0, \vartheta}, \quad \left\| (\lambda I - A(a))^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{N}{|\lambda - \lambda_0|}. \quad (1.19)$$

With the above hypotheses, we can show (see Appendix A.2) the existence of a constant  $C > 0$  such that

$$\forall a \in [0, a_\infty], \quad \forall p \in \mathcal{H}_1, \quad C^{-1} \|p\|_{\mathcal{H}_1} \leq \|p\|_{\mathcal{H}} + \|A(a)p\|_{\mathcal{H}} \leq C \|p\|_{\mathcal{H}_1}, \quad (1.20)$$

$$\forall a \in [0, a_\infty], \quad \forall \xi \in \mathcal{H}_1^*, \quad C^{-1} \|\xi\|_{\mathcal{H}_1^*} \leq \|\xi\|_{\mathcal{H}} + \|A(a)^*\xi\|_{\mathcal{H}} \leq C \|\xi\|_{\mathcal{H}_1^*}. \quad (1.21)$$

Note also that with the above hypotheses, we can extend the mappings  $A$  and  $A^*$  by extrapolation as

$$A \in C^\nu([0, a_\infty]; \mathcal{L}(\mathcal{H}, \mathcal{H}_{-1})), \quad A^* \in C^\nu([0, a_\infty]; \mathcal{L}(\mathcal{H}, \mathcal{H}_{-1}^*)). \quad (1.22)$$

The hypotheses for the operator  $B$  and  $E$  are

$$B \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{U}, \mathcal{H}_{-\gamma})), \quad \gamma \in [0, 1), \quad (1.23)$$

$$E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H})). \quad (1.24)$$

We can write (1.15) in a “standard” abstract form

$$\begin{cases} p' = \mathbb{A}p + \mathbb{B}u & t > 0, \\ p(0) = p^0 \end{cases} \quad (1.25)$$

by setting

$$\mathbb{H} \stackrel{\text{def}}{=} L^2(0, a_\infty; \mathcal{H}), \quad (1.26)$$

$$\mathcal{D}(\mathbb{A}) \stackrel{\text{def}}{=} \left\{ p \in L^2(0, a_\infty; \mathcal{H}) \cap H^1(0, a_\infty; \mathcal{H}_{-1}) : -\frac{\partial p}{\partial a} + Ap \in \mathbb{H}, p(0) = \int_0^{a_\infty} E(a)p(a) da \right\}, \quad (1.27)$$

$$\mathbb{A} : \mathcal{D}(\mathbb{A}) \rightarrow \mathbb{H}, \quad \mathbb{A}p \stackrel{\text{def}}{=} -\frac{\partial p}{\partial a} + Ap, \quad (1.28)$$

$$\mathbb{U} = L^2(0, a_\infty; \mathcal{U}), \quad \mathbb{B} : \mathbb{U} \rightarrow L^2(0, a_\infty; \mathcal{H}_{-\gamma}), \quad u \mapsto Bu. \quad (1.29)$$

If we consider (1.12) (the case of the birth control) instead of (1.1), we can also write it in an abstract system with a structure in age:

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = Ap & (t, a) \in (0, +\infty) \times (0, a_\infty), \\ p(t, 0) = \int_0^{a_\infty} E(a)p(t, a) da + Bu & t \in (0, +\infty), \\ p(0, a) = p^0(a) & a \in (0, a_\infty). \end{cases} \quad (1.30)$$

In that case, we assume the same hypotheses as before for  $A$  and  $E$  (that is (1.17)–(1.19) and (1.24)) but we replace the hypothesis (1.23) by

$$B \in \mathcal{L}(\mathcal{U}, \mathcal{H}_{-\gamma}), \quad \gamma \in [0, 1/2). \quad (1.31)$$

We will see in Section 4 how to write the above system in the form (1.25).

### 1.3 Main results

In order to obtain the stabilization of (1.15), we consider the corresponding Fattorini-Hautus test: for any  $\lambda \in \mathbb{C}$  and for any  $\xi \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H})$

$$\begin{cases} \frac{\partial \xi}{\partial a} + A^* \xi + E^* \xi(0) = \lambda \xi & a \in (0, a_\infty), \\ \xi(a_\infty) = 0, \\ B^* \xi = 0 & a \in (0, a_\infty), \end{cases} \implies \xi = 0, \quad (1.32)$$

where we have denoted by  $B(a)^* \in \mathcal{L}(\mathcal{H}_1^*, \mathcal{U})$  and by  $E(a)^* \in \mathcal{L}(\mathcal{H})$  the adjoint operators of  $B(a) \in \mathcal{L}(\mathcal{U}, \mathcal{H}_{-1})$  and of  $E(a) \in \mathcal{L}(\mathcal{H})$ .

**Theorem 1.3.** *Assume that  $A$  satisfies (1.17)–(1.19), that  $B$  satisfies (1.23) and that  $E$  satisfies (1.24). Assume moreover (1.32) and that for some  $\theta \in (0, 1)$ ,*

$$E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\theta)) \cap W^{1,1}(0, a_\infty; \mathcal{L}(\mathcal{H})), \quad E^* \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{\gamma/2}^*)). \quad (1.33)$$

*Then system (1.15) can be stabilized exponentially: for any  $\alpha > 0$ , there exists a feedback operator  $\mathbb{F} \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  with a finite-dimensional range such that*

$$\mathcal{D}(\mathbb{A}_\mathbb{F}) \stackrel{\text{def}}{=} \{f \in \mathbb{H} : (\mathbb{A} + \mathbb{B}\mathbb{F})f \in \mathbb{H}\} \quad \text{and} \quad \mathbb{A}_\mathbb{F} f \stackrel{\text{def}}{=} (\mathbb{A} + \mathbb{B}\mathbb{F})f \quad (1.34)$$

*is the infinitesimal generator of a strongly continuous semigroup satisfying*

$$\|e^{t\mathbb{A}_\mathbb{F}} p^0\|_{\mathbb{H}} \leq C e^{-\alpha t} \|p^0\|_{\mathbb{H}} \quad (t \geq 0)$$

*for any  $p^0 \in \mathbb{H}$ .*

*Remark 1.4.* In the above statement, we have used the extrapolation of  $\mathbb{A}$  (see, for instance, [34, Proposition 2.10.3, p. 61]): we can extend  $\mathbb{A}$  as bounded operator  $\mathbb{H} \rightarrow \mathcal{D}(\mathbb{A}^*)'$ , where here  $'$  corresponds to the duality with respect to the pivot space  $\mathbb{H}$ . In particular,  $\mathbb{A} + \mathbb{B}\mathbb{F} : \mathbb{H} \rightarrow \mathcal{D}(\mathbb{A}^*)'$ .

The result of Theorem 1.3 yields the existence of  $N \in \mathbb{N}^*$ ,  $\xi_1, \dots, \xi_N \in \mathbb{H}$  and  $w_1, \dots, w_N \in \mathbb{U}$  such that if we take

$$u(t, \cdot) = \sum_{i=1}^N (p(t, \cdot), \xi_i)_{\mathbb{H}} w_i,$$

then (1.15) admits a unique weak solution  $p \in C^0([0, \infty); \mathbb{H})$  such that

$$\|p(t, \cdot)\|_{\mathbb{H}} \leq C e^{-\alpha t} \|p^0\|_{\mathbb{H}} \quad (t \geq 0).$$

*Remark 1.5.* In the above result, we only need that (1.32) holds for  $\lambda \in \mathbb{C}$  such that  $\text{Re } \lambda \geq -\alpha$ . Below we give a sufficient condition for (1.32) provided that  $A$ ,  $E$  and  $B$  satisfy stronger hypotheses.

We recall the following standard definition: the system associated with  $(A^*, B^*)$  is approximately observable in the nonempty interval  $(a_1, a_2) \subset (0, a_\infty)$  if for any  $\tilde{\xi} \in L^2(a_1, a_2; \mathcal{H}_1^*) \cap H^1(a_1, a_2; \mathcal{H})$

$$\frac{\partial \tilde{\xi}}{\partial a} + A^* \tilde{\xi} = 0 \quad \text{and} \quad B^* \tilde{\xi} = 0 \quad \text{in } (a_1, a_2) \implies \tilde{\xi} = 0. \quad (1.35)$$

*Remark 1.6.* By a standard duality argument, this property is equivalent to the fact that the system associated with  $(A, B)$  is approximately controllable in  $(a_1, a_2)$ , that is for any  $p^0, p^1 \in \mathcal{H}_{-1/2}$ , and for any  $\varepsilon > 0$ , there exists  $u \in L^2(a_1, a_2; \mathcal{U})$  such that the solution of

$$\frac{\partial p}{\partial a} = Ap + Bu \quad \text{in } (a_1, a_2), \quad p(a_1) = p^0, \quad (1.36)$$

satisfies  $\|p(a_2) - p^1\|_{\mathcal{H}_{-1/2}} \leq \varepsilon$ . Note that the existence and uniqueness of solutions for (1.36) with  $p^0 \in \mathcal{H}_{-1/2}$  and  $u \in L^2(a_1, a_2; \mathcal{U})$  is given by Proposition 2.2 below.

The result below shows how to obtain (1.32) by using the approximate observability of the system associated with  $(A^*, B^*)$ :

**Lemma 1.7.** *Assume that  $E \equiv 0$  in  $(0, a_E)$  for some  $a_E \in (0, a_\infty)$ . Assume also that the system associated with  $(A^*, B^*)$  is approximately observable in any nonempty open interval  $(a_1, a_2)$  with  $a_1 < a_E$ . Then (1.32) holds.*

The proof is postponed in the appendix.

In the case of the birth control, that is for the system (1.30), we need the following Fattorini-Hautus test (instead of (1.32): for any  $\lambda \in \mathbb{C}$ ,

$$\begin{cases} \frac{\partial \xi}{\partial a} + A^* \xi + E^* \xi(0) = \lambda \xi & a \in (0, a_\infty), \\ \xi(a_\infty) = 0, \\ B^* \xi(0) = 0, \end{cases} \implies \xi = 0. \quad (1.37)$$

We then have the following result:

**Theorem 1.8.** *Assume that  $A$  satisfies (1.17)–(1.19), that  $B$  satisfies (1.31) and that  $E$  satisfies (1.24). Assume moreover (1.37) and that for some  $\theta \in (0, 1)$ ,*

$$E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\theta)) \cap W^{1,1}(0, a_\infty; \mathcal{L}(\mathcal{H})). \quad (1.38)$$

*Then system (1.30) can be stabilized exponentially: for any  $\alpha > 0$ , there exists a feedback operator  $\mathbb{F} \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  with a finite-dimensional range such that the operator  $(\mathcal{D}(\mathbb{A}_\mathbb{F}), \mathbb{A}_\mathbb{F})$  defined by (1.34) is the infinitesimal generator of a strongly continuous semigroup satisfying*

$$\|e^{t\mathbb{A}_\mathbb{F}} p^0\|_{\mathbb{H}} \leq C e^{-\alpha t} \|p^0\|_{\mathbb{H}} \quad (t \geq 0)$$

for any  $p^0 \in \mathbb{H}$ .

By applying the above abstract result on the system (1.1) and (1.12), we obtain the following results:

**Theorem 1.9.** *Assume (1.2)–(1.10) and that  $a_1 < a_\beta$ . Then the system (1.1) can be stabilized exponentially: for any  $\alpha > 0$ , there exists a feedback operator  $\mathbb{F} \in \mathcal{L}(L^2(0, a_\infty; L^2(\Omega)), L^2(0, a_\infty; L^2(\Gamma)))$  with a finite-dimensional range such that the system (1.1) with  $u = \mathbb{F}(p)$  admits a unique weak solution  $p \in C^0([0, \infty); L^2(0, a_\infty; L^2(\Omega)))$  for any  $p^0 \in L^2(0, a_\infty; L^2(\Omega))$  and such that*

$$\|p(t, \cdot)\|_{L^2(0, a_\infty; L^2(\Omega))} \leq C e^{-\alpha t} \|p^0\|_{L^2(0, a_\infty; L^2(\Omega))} \quad (t \geq 0).$$

**Theorem 1.10.** *Assume (1.2), (1.5), (1.6), (1.9), (1.10), (1.13) and (1.14). Then the system (1.12) can be stabilized exponentially: for any  $\alpha > 0$ , there exists a feedback operator  $\mathbb{F} \in \mathcal{L}(L^2(0, a_\infty; L^2(\Omega)), L^2(0, a_\infty; L^2(\omega)))$  with a finite-dimensional range such that the system (1.12) with  $u = \mathbb{F}(p)$  admits a unique weak solution  $p \in C^0([0, \infty); L^2(0, a_\infty; L^2(\Omega)))$  for any  $p^0 \in L^2(0, a_\infty; L^2(\Omega))$  and such that*

$$\|p(t, \cdot)\|_{L^2(0, a_\infty; L^2(\Omega))} \leq C e^{-\alpha t} \|p^0\|_{L^2(0, a_\infty; L^2(\Omega))} \quad (t \geq 0).$$

*Remark 1.11.* In the above results, we have obtained the feedback stabilization of a model of population dynamics. With respect to the model, we would like to add the property that  $p$  remains nonnegative. In the general case, such a property is quite complicated to obtain. A first step in that direction would be to first obtain a stabilization result in  $L^\infty$  instead of  $L^2$ . A similar difficulty appears and is solved for the controllability problem, see [25].

There is a lot of results in the literature concerning the analysis of models of population dynamics such as system (1.1). We only refer some of them here: [5, 17, 18, 20, 23, 24, 30, 40]. We would like to emphasize in particular many important works done by Walker (again without being exhaustive): [35, 36, 37, 38, 39], etc. Concerning the null controllability of our system, let us mention several works from the literature: [1], [2], [3], [19], [22], [25], [26], [32], etc. Let us also mention [28] where the authors study the detectability and state estimation of the above system with an approach similar to the one we consider here.

The main novelties brought in by our paper are:

- the analysis of a general abstract system with a structure in age, that is system (1.15), in the case where the operators  $A$ ,  $B$  and  $E$  depend on the age variable;
- the study of the corresponding operator  $\mathbb{A}$  and in particular the differentiability of the corresponding semigroup (Theorem 2.13);
- the feedback stabilization of the system with a finite-dimensional control, both in the case of systems (1.15) and (1.30).

For the last part, we extend the framework developed in the case where the semigroup is analytic. In that case, there are many references: [6], [9], [10], [29], etc. just to mention some of them.

The outline of the paper is as follows: in Section 2, we study the operator  $\mathbb{A}$  defined by (1.27), (1.28). We show in particular that it is the infinitesimal generator of a differentiable semigroup if  $E$  satisfies some additional hypotheses in addition to (1.24) (see Theorem 2.13). Then in Section 3, we consider the adjoint system and give some properties of  $\mathbb{A}^*$  that plays an important role in the stabilization analysis since our control operators are not bounded. Finally in Section 4, we show the stabilization properties and prove in particular the main results. In the appendix, we give the proof of some technical results.

*Notation.* In the whole paper, we use  $C$  as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant  $C$  may change from one appearance to another. We also use the notation  $X \lesssim Y$  if there exists a constant  $C > 0$  such that we have the inequality  $X \leq CY$ . The notation  $X \lesssim_k Y$  stands for  $X \leq CY$ , where  $C$  is a positive constant depending on  $k$ .

## 2 Study of the operator $\mathbb{A}$

In this section, we study the properties of the operator  $\mathbb{A}$  defined by (1.27), (1.28).

### 2.1 The Cauchy problem associated with $A$

Here, we recall some results on the Cauchy problem

$$\frac{\partial p}{\partial a} = Ap + f \quad \text{in } (0, a_\infty), \quad p(0) = p^0, \quad (2.1)$$

with  $A$  satisfying the hypotheses (1.17)–(1.19). In that case, we recall (see, [4, Thm 4.4.1, p.63]) the existence of a parabolic evolution operator  $S$  associated with  $A$ . More precisely, if we write

$$\mathcal{T} \stackrel{\text{def}}{=} \{(a, r) \in [0, a_\infty]^2 : a \leq r\}, \quad \mathcal{T}^* \stackrel{\text{def}}{=} \{(a, r) \in [0, a_\infty]^2 : a < r\}, \quad (2.2)$$



then the application  $S$  satisfies the following properties:

$$S \in C^0(\mathcal{T}; \mathcal{L}_s(\mathcal{H})) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H}_1)) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H})) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H}, \mathcal{H}_1)), \quad (2.3)$$

$$\forall r \in [0, a_\infty), \quad S(\cdot, r) \in C^1((r, a_\infty]; \mathcal{L}(\mathcal{H})), \quad (2.4)$$

$$\forall a \in (0, a_\infty], \quad S(a, \cdot) \in C^1([0, a]; \mathcal{L}_s(\mathcal{H}_1, \mathcal{H})), \quad (2.5)$$

$$S(r, r) = I, \quad S(a_3, a_2)S(a_2, a_1) = S(a_3, a_1) \quad (0 \leq a_1 \leq a_2 \leq a_3 \leq a_\infty) \quad (2.6)$$

$$\forall (a, r) \in \mathcal{T}^*, \quad \frac{\partial}{\partial a} S(a, r) = A(a)S(a, r), \quad (2.7)$$

$$\forall (a, r) \in \mathcal{T}^*, \quad \forall p \in \mathcal{H}_1, \quad \frac{\partial}{\partial r} S(a, r)p = -S(a, r)A(r)p. \quad (2.8)$$

Here  $\mathcal{L}_s$  correspond to the space of bounded linear mapping, endowed with the simple convergence topology. We also have, (see [4, Lemma II.5.1.3, p.69]) the existence of a constant  $M > 0$  such that for any  $\alpha \in [0, 1]$ ,  $\theta \in [0, 1]$ ,  $\alpha + \theta \leq 1$ ,

$$\sup_{(a, r) \in \mathcal{T}^*} (a - r)^\theta \|S(a, r)\|_{\mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_{\alpha+\theta})} \leq M. \quad (2.9)$$

Using this parabolic evolution operator  $S$  and the Duhamel formula, we can write the solution of (2.1) as follows

$$p(a) = S(a, 0)p^0 + \int_0^a S(a, r)f(r) dr \quad (a \in [0, a_\infty]). \quad (2.10)$$

More precisely, we have the following classical result that we recall here:

**Proposition 2.1.** *Assume  $f \in L^2(0, a_\infty; \mathcal{H})$  and  $p^0 \in \mathcal{H}$ . Then  $p$  defined by (2.10) satisfies*

$$p \in H^1(0, a_\infty; \mathcal{H}_{-1}) \cap C^0([0, a_\infty]; \mathcal{H})$$

and verifies the system (2.1), where the first equation holds in  $\mathcal{H}_{-1}$ . If we assume that  $f \in C^0([0, a_\infty]; \mathcal{H})$ , then  $p \in C^1([0, a_\infty]; \mathcal{H}_{-1})$ .

*Proof.* The continuity in  $\mathcal{H}$  can be obtained directly from the properties (2.3) and (2.9) of  $S$ . Let us show that  $p \in H^1(0, a_\infty; \mathcal{H}_{-1})$ . First, combining (1.22) and (2.7), we see that  $a \mapsto S(a, 0)p^0$  can be extended a function in  $C^1([0, a_\infty]; \mathcal{H}_{-1})$  with

$$\frac{d}{da} S(a, 0)p^0 = A(a)S(a, 0)p^0 \quad (a \in [0, a_\infty]).$$

Second, we denote by  $F$  the other part of  $p$ :

$$F(a) \stackrel{\text{def}}{=} \int_0^a S(a, r)f(r) dr.$$

We show below that  $F$  is right-differentiable and left-differentiable. Let  $a \in (0, a_\infty)$  and  $h > 0$  small enough so that  $a + h \in (0, a_\infty)$ . We have

$$\frac{F(a+h) - F(a)}{h} = \frac{1}{h} \int_a^{a+h} S(a+h, r)f(r) dr + \int_0^a \frac{S(a+h, r) - S(a, r)}{h} f(r) dr.$$

From (2.7), we deduce that for  $r \in (0, a)$ , as  $h \rightarrow 0^+$ ,

$$\frac{S(a+h, r) - S(a, r)}{h} f(r) \rightarrow A(a)S(a, r)f(r) \quad \text{in } \mathcal{H}.$$

The relation (2.7) also implies that for  $r \in (0, a)$

$$\frac{S(a+h, r) - S(a, r)}{h} f(r) = \frac{1}{h} \int_a^{a+h} A(r')S(r', r)f(r) dr'$$

and thus, with (1.22) and (2.9),

$$\left\| \frac{S(a+h, r) - S(a, r)}{h} f(r) \right\|_{\mathcal{H}_{-1}} \leq \|A\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}, \mathcal{H}_{-1}))} M \|f(r)\|_{\mathcal{H}}.$$

In particular, using the Lebesgue theorem, as  $h \rightarrow 0^+$ ,

$$\int_0^a \frac{S(a+h, r) - S(a, r)}{h} f(r) dr \rightarrow A(a) \int_0^a S(a, r) f(r) dr \quad \text{in } \mathcal{H}_{-1}.$$

Now, we can write

$$\frac{1}{h} \int_a^{a+h} S(a+h, r) f(r) dr = \int_0^1 S(a+h, a+h\tau) f(a+h\tau) d\tau.$$

If  $f \in C^0([0, a_\infty]; \mathcal{H})$ , then using (2.3) and (2.9), we have for any  $\tau \in [0, 1]$ ,

$$S(a+h, a+h\tau) f(a+h\tau) \rightarrow f(a) \quad \text{and} \quad \|S(a+h, a+h\tau) f(a+h\tau)\|_{\mathcal{H}} \leq M \|f\|_{C^0([0, a_\infty]; \mathcal{H})}.$$

We thus deduce that as  $h \rightarrow 0^+$ ,

$$\frac{1}{h} \int_a^{a+h} S(a+h, r) f(r) dr \rightarrow f(a) \quad \text{in } \mathcal{H}.$$

Assume now that  $h > 0$  is small enough so that  $a-h \in (0, a_\infty)$ . We have

$$\frac{F(a) - F(a-h)}{h} = \frac{1}{h} \int_{a-h}^a S(a, r) f(r) dr + \int_0^{a-h} \frac{S(a, r) - S(a-h, r)}{h} f(r) dr.$$

As above, from (2.7), we deduce that for  $r \in (0, a)$ , as  $h \rightarrow 0^+$ ,

$$1_{[0, a-h]}(r) \frac{S(a, r) - S(a-h, r)}{h} f(r) \rightarrow A(a) S(a, r) f(r) \quad \text{in } \mathcal{H}$$

and with (1.22) and (2.9),

$$\left\| 1_{[0, a-h]}(r) \frac{S(a, r) - S(a-h, r)}{h} f(r) \right\|_{\mathcal{H}_{-1}} \leq \|A\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}, \mathcal{H}_{-1}))} M \|f(r)\|_{\mathcal{H}}.$$

In particular, using the Lebesgue theorem, as  $h \rightarrow 0^+$ ,

$$\int_0^{a-h} \frac{S(a, r) - S(a-h, r)}{h} f(r) dr \rightarrow A(a) \int_0^a S(a, r) f(r) dr \quad \text{in } \mathcal{H}_{-1}.$$

On the other hand,

$$\frac{1}{h} \int_{a-h}^a S(a, r) f(r) dr = \int_0^1 S(a, a-h\tau) f(a-h\tau) d\tau.$$

If  $f \in C^0([0, a_\infty]; \mathcal{H})$ , then using (2.3) and (2.9), we have for any  $\tau \in [0, 1]$ ,

$$S(a, a-h\tau) f(a-h\tau) \rightarrow f(a) \quad \text{and} \quad \|S(a, a-h\tau) f(a-h\tau)\|_{\mathcal{H}} \leq M \|f\|_{C^0([0, a_\infty]; \mathcal{H})}.$$

We thus deduce that as  $h \rightarrow 0^+$ ,

$$\frac{1}{h} \int_{a-h}^a S(a, r) f(r) dr \rightarrow f(a) \quad \text{in } \mathcal{H}.$$

Combining the above (left and right) limits, we have obtained that  $F$  is differentiable in  $\mathcal{H}_{-1}$  and that

$$F'(a) = f(a) + A(a) \int_0^a S(a, r) f(r) dr.$$

This shows that if  $f \in C^0([0, a_\infty]; \mathcal{H})$ , then  $F \in C^1([0, a_\infty]; \mathcal{H})$ , and there exists a constant  $C > 0$  such that

$$\|F'\|_{L^2(0, a_\infty; \mathcal{H}_{-1})} \leq C \|f\|_{L^2(0, a_\infty; \mathcal{H})}.$$

Using the density of  $C^0([0, a_\infty]; \mathcal{H})$  in  $L^2(0, a_\infty; \mathcal{H})$ , we deduce that if  $f \in L^2(0, a_\infty; \mathcal{H})$ ,  $F \in H^1(0, a_\infty; \mathcal{H}_{-1})$  and this concludes the proof.  $\square$

We can also state a result of maximal regularity for (2.1). The following proposition is already known for similar hypotheses, but for sake of completeness we proved it in Appendix A.1.

**Proposition 2.2.** *Let  $\alpha \in [0, 1]$ . For any  $f \in L^2(0, a_\infty; \mathcal{H}_{\alpha-1})$  and for any  $p^0 \in [\mathcal{H}_{\alpha-1}, \mathcal{H}_\alpha]_{1/2}$ , there exists a unique solution*

$$p \in L^2(0, a_\infty; \mathcal{H}_\alpha) \cap C^0\left([0, a_\infty]; [\mathcal{H}_{\alpha-1}, \mathcal{H}_\alpha]_{1/2}\right) \cap H^1(0, a_\infty; \mathcal{H}_{\alpha-1}) \quad (2.11)$$

of the system (2.1). If  $\alpha = 1$ , then  $p$  is given by the Duhamel formula (2.10). Moreover, we have the following estimate:

$$\|p\|_{L^2(0, a_\infty; \mathcal{H}_\alpha) \cap H^1(0, a_\infty; \mathcal{H}_{\alpha-1})} \lesssim_\alpha \|p^0\|_{[\mathcal{H}_{\alpha-1}, \mathcal{H}_\alpha]_{1/2}} + \|f\|_{L^2(0, a_\infty; \mathcal{H}_{\alpha-1})}. \quad (2.12)$$

*Remark 2.3.* In the above statement, the solution  $p$  satisfying (2.11) are solutions of the system (2.1) in the sense that the first equation of (2.1) holds in  $L^2(0, a_\infty; \mathcal{H}_{\alpha-1})$ .

*Remark 2.4.* Note that in the particular cases,  $\alpha = 0$  and  $\alpha = 1$ , we have (see, for instance, [33, p.72])

$$[\mathcal{H}_{-1}, \mathcal{H}]_{1/2} = \mathcal{H}_{-1/2} \quad \text{and} \quad [\mathcal{H}, \mathcal{H}_1]_{1/2} = \mathcal{H}_{1/2}.$$

*Remark 2.5.* Note that Proposition 2.2 is proved in [4, Theorem III.4.10.10, pp.188-189] with the additional assumption that the family  $(A(a))_{a \in [0, a_\infty]}$  has uniformly bounded imaginary powers. We don't need this hypothesis since here we work with the space  $L^2(0, T; \mathcal{H})$  with  $\mathcal{H}$  a Hilbert space whereas the result in [4] is obtained with a space  $L^q(0, T; \mathcal{H})$  with  $q > 1$  and with a general Banach space  $\mathcal{H}$ . See [13, Theorem 4.4 and Remark 3.2 (3)] for more details.

## 2.2 Properties of $\mathbb{A}$

With the results of the previous section, we can now show several results on the operator  $\mathbb{A}$  defined by (1.27), (1.28). First we have the following property of the domain of  $\mathbb{A}$ :

**Proposition 2.6.** *The operator  $\mathbb{A}$  is with compact resolvents. For any  $p \in \mathcal{D}(\mathbb{A})$ , the function  $a \mapsto ap(a)$  belongs to  $L^2(0, a_\infty; \mathcal{H}_1) \cap H^1(0, a_\infty; \mathcal{H})$ . Moreover, if  $E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{1/2}))$ , then we have*

$$\mathcal{D}(\mathbb{A}) = \left\{ p \in L^2(0, a_\infty; \mathcal{H}_1) \cap H^1(0, a_\infty; \mathcal{H}) : p(0) = \int_0^{a_\infty} E(a)p(a) da \right\}.$$

*Proof.* Assume  $p \in \mathcal{D}(\mathbb{A})$  and let us set  $f \stackrel{\text{def}}{=} -\partial_a p + Ap \in \mathbb{H}$ . Using (1.24) and  $p \in L^2(0, a_\infty; \mathcal{H})$ , we first deduce that

$$p(0) = \int_0^{a_\infty} E(a)p(a) da \in \mathcal{H}. \quad (2.13)$$

Then we can write  $p = p_1 + p_2$  with

$$\frac{\partial p_1}{\partial a} = Ap_1 + f \quad \text{in } (0, a_\infty), \quad p_1(0) = 0,$$

and

$$\frac{\partial p_2}{\partial a} = Ap_2 \quad \text{in } (0, a_\infty), \quad p_2(0) = p(0) \in \mathcal{H}.$$

Applying Proposition 2.2 with  $\alpha = 1$  gives  $p_1 \in L^2(0, a_\infty; \mathcal{H}_1) \cap H^1(0, a_\infty; \mathcal{H})$ . Moreover, combining Proposition 2.1 and Proposition 2.2, we deduce that  $p_2 = S(\cdot, 0)p(0)$ . From (2.9) with  $(\alpha, \theta) = (0, 1/4)$  we deduce that  $p_2 \in L^2(0, a_\infty; \mathcal{H}_{1/4})$ . Then we have obtained  $p = p_1 + p_2 \in L^2(0, a_\infty; \mathcal{H}_{1/4})$  so that

$$\mathcal{D}(\mathbb{A}) \hookrightarrow L^2(0, a_\infty; \mathcal{H}_{1/4}) \cap H^1(0, a_\infty; \mathcal{H}_{-1}).$$

Using the Aubin-Lions lemma, we deduce that the embedding  $\mathcal{D}(\mathbb{A}) \hookrightarrow \mathbb{H}$  is compact and this shows that  $\mathbb{A}$  is with compact resolvents.

For the next property, we set  $\tilde{p}(a) \stackrel{\text{def}}{=} ap(a)$  that satisfies

$$\frac{\partial \tilde{p}}{\partial a} = A\tilde{p} + f + p \quad \text{in } (0, a_\infty), \quad \tilde{p}(0) = 0$$

and from Proposition 2.2 with  $\alpha = 1$ , we deduce  $\tilde{p} \in L^2(0, a_\infty; \mathcal{H}_1) \cap H^1(0, a_\infty; \mathcal{H})$ .

Finally, if  $E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{1/2}))$ , with (1.24) and  $[\mathcal{H}, \mathcal{H}_{1/2}]_{1/2} = \mathcal{H}_{1/4}$  (see [33, Theorem 2, p. 66]), we deduce by interpolation that  $E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{1/4}))$ . Now, since we have proved that  $p \in L^2(0, a_\infty; \mathcal{H}_{1/4})$ , we deduce from (2.13) that  $p(0) \in \mathcal{H}_{1/4}$ . Thus, from (2.9) with  $(\alpha, \theta) = (1/4, 1/4)$  we deduce  $p_2 = S(\cdot, 0)p(0) \in L^2(0, a_\infty; \mathcal{H}_{1/2})$ . From (2.13) it follows that  $p(0) \in \mathcal{H}_{1/2}$  and applying Proposition 2.2 with  $\alpha = 1$  gives  $p \in L^2(0, a_\infty; \mathcal{H}_1) \cap H^1(0, a_\infty; \mathcal{H})$ . The second part of the proposition is proved.  $\square$

Let us consider the solution (1.25) for  $u = 0$ :

$$p' = \mathbb{A}p \quad (t > 0), \quad p(0) = p^0, \quad (2.14)$$

or equivalently

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = Ap \quad (t, a) \in (0, +\infty) \times (0, a_\infty), \\ p(t, 0) = \int_0^{a_\infty} E(a)p(t, a)da \quad t \in (0, +\infty), \\ p(0, a) = p^0(a) \quad a \in (0, a_\infty). \end{array} \right. \quad (2.15)$$

We set

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \int_0^{a_\infty} E(a)p(t, a)da. \quad (2.16)$$

Formally, the method of characteristics (see Fig. 1) yields that the solution of (2.15) satisfies

$$\left\{ \begin{array}{l} p(t, a) = S(a, 0)\mathcal{E}(t - a) \quad (t > a), \\ p(t, a) = S(a, a - t)p^0(a - t) \quad (t \leq a). \end{array} \right. \quad (2.17)$$

In particular,  $\mathcal{E}$  defined by (2.16) verifies the integral relation

$$\mathcal{E}(t) = \int_0^t \mathbf{1}_{[0, a_\infty]}(a)E(a)S(a, 0)\mathcal{E}(t - a) da + \mathbf{1}_{[0, a_\infty]}(t) \int_t^{a_\infty} E(a)S(a, a - t)p^0(a - t) da \quad (t \geq 0). \quad (2.18)$$

The above formal computations are justified in Proposition 2.12 below. But first, we need the following result, that is similar to Lemma 2.1 in [37]:

**Lemma 2.7.** *For all  $p^0 \in L^1(0, a_\infty; \mathcal{H})$ , equation (2.18) admits a unique solution  $\mathcal{E}_{p^0}$  in  $L^\infty_{\text{loc}}(0, \infty; \mathcal{H})$ . Moreover, if for some  $\theta \in [0, 1]$ ,*

$$E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\theta)), \quad (2.19)$$

*then there exists a constant  $C = C(\theta, \kappa_\theta) > 0$  such that  $\mathcal{E}_{p^0}$  satisfies*

$$\|\mathcal{E}_{p^0}(t)\|_{\mathcal{H}_\theta} \leq C \left( e^{\kappa_\theta t} + \frac{1}{t^\theta} \right) \|p^0\|_{L^1(0, a_\infty; \mathcal{H})} \quad (t > 0), \quad (2.20)$$

*where  $\kappa_\theta \stackrel{\text{def}}{=} M \|E\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\theta))}$ .*

*Proof.* We consider for  $\lambda > 0$ , the Banach space

$$L_\lambda^\infty(0, \infty; \mathcal{H}) \stackrel{\text{def}}{=} \{f \text{ such that } t \mapsto e^{-\lambda t} f(t) \in L^\infty(0, \infty; \mathcal{H})\},$$

endowed with the norm

$$\|f\|_\lambda \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{t \in (0, \infty)} \|e^{-\lambda t} f(t)\|_{\mathcal{H}}.$$

Then we define the mapping

$$\mathcal{I} : L_\lambda^\infty(0, \infty; \mathcal{H}) \rightarrow L_\lambda^\infty(0, \infty; \mathcal{H}),$$

such that

$$\mathcal{I}(\mathcal{E})(t) \stackrel{\text{def}}{=} \int_0^t \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}(t-a) da + \mathbf{1}_{[0, a_\infty]}(t) \int_t^{a_\infty} E(a) S(a, a-t) p^0(a-t) da.$$

Using (2.9) and (1.24), one can check that  $\mathcal{I}$  is well-defined and that

$$\|\mathcal{I}(\mathcal{E})\|_\lambda \leq \frac{\kappa_0}{\lambda} \|\mathcal{E}\|_\lambda + \kappa_0 \|p^0\|_{L^1(0, a_\infty; \mathcal{H})}.$$

Moreover, if  $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in L_\lambda^\infty(0, \infty; \mathcal{H})$ , then using again (2.9) and (1.24), we can show that

$$\|\mathcal{I}(\mathcal{E}^{(1)}) - \mathcal{I}(\mathcal{E}^{(2)})\|_\lambda \leq \frac{\kappa_0}{\lambda} \|\mathcal{E}^{(1)} - \mathcal{E}^{(2)}\|_\lambda,$$

so that  $\mathcal{I}$  is a strict contraction for  $\lambda > \kappa_0$  and thus admits a unique fixed point in  $L_\lambda^\infty(0, \infty; \mathcal{H})$ . Such a fixed-point is a solution of (2.18) in  $L_{\text{loc}}^\infty(0, \infty; \mathcal{H})$ . To prove the uniqueness, let us consider  $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in L_{\text{loc}}^\infty(0, \infty; \mathcal{H})$ , solutions of (2.18). Then  $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}^{(1)} - \mathcal{E}^{(2)}$  satisfies

$$\mathcal{E}(t) = \int_0^t \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}(t-a) da \quad (t \geq 0)$$

and using the Grönwall lemma with (2.9) and (1.24), we deduce that  $\mathcal{E} \equiv 0$ .

Next, we show that (2.19) yields (2.20): we deduce from (2.18) and (2.9) that

$$\|\mathcal{E}_{p^0}(t)\|_{\mathcal{H}_\theta} \leq \kappa_\theta \int_0^t \|\mathcal{E}_{p^0}(a)\|_{\mathcal{H}_\theta} da + \frac{\kappa_\theta}{t^\theta} \|p^0\|_{L^1(0, a_\infty; \mathcal{H})} \quad (t > 0). \quad (2.21)$$

Then applying the Grönwall lemma, we deduce

$$\|\mathcal{E}_{p^0}(t)\|_{\mathcal{H}_\theta} \leq \kappa_\theta^2 \|p^0\|_{L^1(0, a_\infty; \mathcal{H})} \int_0^t \frac{e^{\kappa_\theta(t-s)}}{s^\theta} ds + \frac{\kappa_\theta}{t^\theta} \|p^0\|_{L^1(0, a_\infty; \mathcal{H})} \quad (t > 0) \quad (2.22)$$

and this yields the result.  $\square$

*Remark 2.8.* Note that the first part of the proof of the above lemma follows from [4, Theorem 3.2.2 and Remark 3.2.3], but we prefer to give here an elementary proof of this result.

**Lemma 2.9.** *Let us define the application  $\mathbb{S}$  for  $t \geq 0$  and  $p^0 \in \mathbb{H}$  by*

$$(\mathbb{S}(t)p^0)(a) \stackrel{\text{def}}{=} \begin{cases} S(a, 0) \mathcal{E}_{p^0}(t-a) & (t > a), \\ S(a, a-t) p^0(a-t) & (t \leq a), \end{cases} \quad (a \in [0, a_\infty]), \quad (2.23)$$

where  $\mathcal{E}_{p^0}$  is the unique solution of (2.18). Then  $(\mathbb{S}(t))_{t \geq 0}$  is a strongly continuous semigroup on  $\mathbb{H}$  and there exists a constant  $C > 0$  such that for any  $t \geq 0$ ,

$$\|\mathbb{S}(t)\|_{\mathcal{L}(\mathbb{H})} \leq C e^{\kappa_0 t}. \quad (2.24)$$

*Proof.* The proof is done in the case where  $A$  is independent of  $a$  in [40, Theorem 4]. Here, for sake of completeness, we write the proof in our case by following the proof of [40].

First, we prove that for  $\tau, t > 0$ ,

$$\mathcal{E}_{p^0}(\tau + t) = \mathcal{E}_{\mathbb{S}(\tau)p^0}(t). \quad (2.25)$$

The above relation and (2.23) imply that if  $t_1, t_2 \geq 0$ , then  $\mathbb{S}(t_1)\mathbb{S}(t_2) = \mathbb{S}(t_1 + t_2)$ .

In order to show (2.25), we fix  $\tau > 0$  and we prove that  $\mathcal{E}_{p^0}(\tau + \cdot)$  is a solution of the integral relation (2.18) satisfied by  $\mathcal{E}_{\mathbb{S}(\tau)p^0}$  (that is with  $p^0$  replaced by  $\mathbb{S}(\tau)p^0$ ). Since this integral equation admits a unique solution in  $L_{\text{loc}}^\infty(0, \infty; \mathcal{H})$  (by using Lemma 2.7), this will show (2.25). Let us write the integral relations (2.18) satisfied by  $\mathcal{E}_{p^0}(\tau + \cdot)$  and  $\mathcal{E}_{\mathbb{S}(\tau)p^0}$ :

$$\begin{aligned} \mathcal{E}_{p^0}(\tau + t) &= \int_0^{\tau+t} \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}_{p^0}(\tau + t - a) da \\ &\quad + \mathbf{1}_{[0, a_\infty]}(\tau + t) \int_{\tau+t}^{a_\infty} E(a) S(a, a - t - \tau) p^0(a - t - \tau) da \quad (t \geq 0) \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} \mathcal{E}_{\mathbb{S}(\tau)p^0}(t) &= \int_0^t \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}_{\mathbb{S}(\tau)p^0}(t - a) da \\ &\quad + \mathbf{1}_{[0, a_\infty]}(t) \int_t^{a_\infty} E(a) S(a, a - t) (\mathbb{S}(\tau)p^0)(a - t) da \quad (t \geq 0). \end{aligned} \quad (2.27)$$

From (2.23), we have

$$(\mathbb{S}(\tau)p^0)(a - t) = \begin{cases} S(a - t, 0) \mathcal{E}_{p^0}(\tau + t - a) & (t + \tau > a), \\ S(a - t, a - t - \tau) p^0(a - t - \tau) & (t + \tau \leq a), \end{cases} \quad (a \in [0, a_\infty]). \quad (2.28)$$

Using the above relation and considering the two cases,  $t + \tau > a_\infty$  and  $t + \tau \leq a_\infty$ , we can check that for  $t \geq 0$ ,

$$\begin{aligned} \mathbf{1}_{[0, a_\infty]}(t) \int_t^{a_\infty} E(a) S(a, a - t) (\mathbb{S}(\tau)p^0)(a - t) da &= \int_t^{t+\tau} \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}_{p^0}(\tau + t - a) da \\ &\quad + \mathbf{1}_{[0, a_\infty]}(\tau + t) \int_{\tau+t}^{a_\infty} E(a) S(a, a - t - \tau) p^0(a - t - \tau) da. \end{aligned} \quad (2.29)$$

In particular, the integral relation (2.26) can be written as follows:

$$\begin{aligned} \mathcal{E}_{p^0}(\tau + t) &= \int_0^t \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}_{p^0}(\tau + t - a) da \\ &\quad + \mathbf{1}_{[0, a_\infty]}(t) \int_t^{a_\infty} E(a) S(a, a - t) (\mathbb{S}(\tau)p^0)(a - t) da \quad (t \geq 0). \end{aligned} \quad (2.30)$$

This shows that that  $\mathcal{E}_{p^0}(\tau + \cdot)$  satisfies the same integral relation (2.27) than  $\mathcal{E}_{\mathbb{S}(\tau)p^0}$  and from Lemma 2.7 this yields (2.25).

The relation (2.24) is a consequence of (2.23), (2.9) and (2.20). Let us prove the strong continuity of the semigroup. From (2.23), we have for  $0 < t < a_\infty$ :

$$\begin{aligned} \|\mathbb{S}(t)p^0 - p^0\|_{\mathbb{H}}^2 &= \int_0^t \|S(a, 0) \mathcal{E}_{p^0}(t - a) - p^0(a)\|_{\mathcal{H}}^2 da + \int_0^{a_\infty - t} \|S(a + t, a) p^0(a) - p^0(a + t)\|_{\mathcal{H}}^2 da \\ &\leq \int_0^t \|S(a, 0) \mathcal{E}_{p^0}(t - a) - p^0(a)\|_{\mathcal{H}}^2 da + 2 \int_0^{a_\infty - t} \|S(a + t, a) p^0(a) - p^0(a)\|_{\mathcal{H}}^2 da \\ &\quad + 2 \int_{\mathbb{R}} \|\tilde{p}^0(a) - \tilde{p}^0(a + t)\|_{\mathcal{H}}^2 da \end{aligned} \quad (2.31)$$

where  $\tilde{p}^0 \in L^2(\mathbb{R}; \mathcal{H})$  denotes the extension of  $p^0$  by 0 outside  $(0, a_\infty)$ .

From (2.9) and (2.20), we have, as  $t \rightarrow 0$ ,

$$\int_0^t \|S(a, 0)\mathcal{E}_{p^0}(t-a) - p^0(a)\|_{\mathcal{H}}^2 da \leq Ct \|p^0\|_{\mathbb{H}} + \int_0^t \|p^0(a)\|_{\mathcal{H}}^2 da \rightarrow 0. \quad (2.32)$$

From (2.3), (2.9) and the dominated convergence theorem, as  $t \rightarrow 0$ ,

$$\int_0^{a_\infty-t} \|S(a+t, a)p^0(a) - p^0(a)\|_{\mathcal{H}}^2 da \rightarrow 0. \quad (2.33)$$

From the continuity of the translation operator in  $L^2(\mathbb{R}; \mathcal{H})$ , we deduce that if  $t \rightarrow 0$

$$\int_{\mathbb{R}} \|\tilde{p}^0(a) - \tilde{p}^0(a+t)\|_{\mathcal{H}}^2 da \rightarrow 0.$$

This concludes the proof of this lemma.  $\square$

**Corollary 2.10.** *Let  $p^0 \in \mathbb{H}$ . Then the unique solution  $\mathcal{E}_{p^0}$  of (2.18) satisfies*

$$\mathcal{E}_{p^0}(t) = \int_0^{a_\infty} E(a) (\mathbb{S}(t)p^0)(a) da \quad (2.34)$$

and belongs to  $C^0([0, +\infty); \mathcal{H})$ .

*Proof.* Combining (2.18) and (2.23), we deduce (2.34). Consequently, for  $t_1 \geq 0, t_2 \geq 0$ ,

$$\|\mathcal{E}_{p^0}(t_2) - \mathcal{E}_{p^0}(t_1)\|_{\mathcal{H}}^2 \leq a_\infty \|E\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}))}^2 \|\mathbb{S}(t_2)p^0 - \mathbb{S}(t_1)p^0\|_{\mathbb{H}}^2.$$

Hence, the result is a consequence of the continuity of  $t \mapsto \mathbb{S}(t)p^0 \in \mathbb{H}$ .  $\square$

By combining (2.23), (2.9) and (2.20), we can check the following result

**Corollary 2.11.** *Assume  $E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{\theta/2}))$ , for some  $\theta \in [0, 1)$ . Then the semigroup  $\mathbb{S}$  defined by (2.23) satisfies*

$$\|\mathbb{S}(t)p^0\|_{L^2(0, a_\infty; \mathcal{H}_\theta)} \leq C \left( \frac{1}{t^\theta} + e^{\kappa_\theta/2t} \right) \|p^0\|_{\mathbb{H}} \quad (p^0 \in \mathbb{H}). \quad (2.35)$$

We can now show the following result:

**Proposition 2.12.** *The infinitesimal generator of the strongly continuous semigroup  $(\mathbb{S}(t))_{t \geq 0}$  is the operator  $\mathbb{A}$  defined by (1.27) and (1.28).*

*Proof.* Let us denote by  $\mathbb{A}_0$  the infinitesimal generator of the strongly continuous semigroup  $(\mathbb{S}(t))_{t \geq 0}$  defined in Lemma 2.9 and let us show that  $\mathbb{A}_0 = \mathbb{A}$ .

Let us consider  $\lambda > \kappa_0$  and  $p \in \mathcal{D}(\mathbb{A}_0)$ . We set  $p^0 \stackrel{\text{def}}{=} \lambda p - \mathbb{A}_0 p \in \mathbb{H}$ . From [12, Proposition 2.3, p.100] and (2.23),

$$p(a) = \int_0^\infty e^{-\lambda t} (\mathbb{S}(t)p^0)(a) dt = e^{-\lambda a} \left[ S(a, 0) \int_0^\infty e^{-\lambda t} \mathcal{E}_{p^0}(t) dt + \int_0^a e^{\lambda t} S(a, t) p^0(t) dt \right]. \quad (2.36)$$

From Proposition 2.1, we deduce  $p \in H^1(0, a_\infty; \mathcal{H}_{-1})$  and

$$\frac{\partial p}{\partial a} = -\lambda p + Ap + p^0, \quad (2.37)$$

so that  $-\partial_a p + Ap \in \mathbb{H}$ . Moreover, we also deduce from (2.36) that

$$p(0) = \int_0^\infty e^{-\lambda t} \mathcal{E}_{p^0}(t) dt.$$

Using (2.18), we deduce

$$p(0) = \int_0^\infty e^{-\lambda t} \int_0^t \mathbf{1}_{[0, a_\infty]}(a) E(a) S(a, 0) \mathcal{E}(t-a) da dt + \int_0^{a_\infty} e^{-\lambda t} \int_t^{a_\infty} E(a) S(a, a-t) p^0(a-t) da dt.$$

Combining the above relation with (2.23) and Fubini's theorem yields

$$p(0) = \int_0^{a_\infty} E(a) \int_0^\infty e^{-\lambda t} (\mathbb{S}(t)p^0)(a) dt da = \int_0^{a_\infty} E(a)p(a) da.$$

We thus deduce that  $p \in \mathcal{D}(\mathbb{A})$  and from (2.37), (1.28), we obtain that  $\mathbb{A}_0 p = \mathbb{A} p$ .

Conversely, assume  $p_1 \in \mathcal{D}(\mathbb{A})$  and  $\lambda > \kappa_0$ . From (2.24) and [27, Theorem 5.3, p.20],  $\lambda I - \mathbb{A}_0$  is invertible and thus there exists a unique  $p_0 \in \mathcal{D}(\mathbb{A}_0)$  such that  $(\lambda I - \mathbb{A}_0)p_0 = (\lambda I - \mathbb{A})p_1 \in \mathbb{H}$ . Using the first part of the proof, this implies that  $p_0 \in \mathcal{D}(\mathbb{A})$  and that  $p_0 - p_1 \in \text{Ker}(\lambda I - \mathbb{A})$ . We now show that  $\lambda I - \mathbb{A}$  is injective, so that  $p_1 \in \mathcal{D}(\mathbb{A}_0)$  and  $\mathbb{A}_0 p_1 = \mathbb{A} p_1$ . Assume  $p \in \text{Ker}(\lambda I - \mathbb{A})$ . Then  $p \in \mathcal{D}(\mathbb{A})$ ,

$$\lambda p + \frac{\partial p}{\partial a} - Ap = 0 \quad \text{and} \quad p(0) = \int_0^{a_\infty} E(a)p(a) da.$$

Since  $p(0) \in \mathcal{H}$ , we deduce from (2.3) and (2.7) that  $a \mapsto S(a, 0)p(0)$  is in  $L^2(0, a_\infty; \mathcal{H}) \cap H^1(0, a_\infty; \mathcal{H}_{-1})$ . Applying the uniqueness result in Proposition 2.2 for  $\alpha = 0$ , we thus deduce

$$p(a) = e^{-\lambda a} S(a, 0)p(0)$$

and, with (2.9) and (1.24),

$$\|p(0)\|_{\mathcal{H}} \leq \int_0^{a_\infty} \kappa_0 e^{-\lambda a} da \|p(0)\|_{\mathcal{H}},$$

where we recall that  $\kappa_0 = M \|E\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}))}$ . Since  $\lambda > \kappa_0$ , we deduce that  $p(0) = 0$  and therefore that  $p = 0$ .  $\square$

We now obtain the differentiability of the semigroup generated by  $\mathbb{A}$ :

**Theorem 2.13.** *Assume that for some  $\theta \in (0, 1)$ ,*

$$E \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\theta)) \cap W^{1,1}(0, a_\infty; \mathcal{L}(\mathcal{H})). \quad (2.38)$$

*Then the semigroup generated by  $\mathbb{A}$  is differentiable on  $\mathbb{H}$  for  $t > 2a_\infty$ .*

*Proof.* Assume  $p^0 \in \mathcal{D}(\mathbb{A})$ . Then,  $t \mapsto \mathbb{S}(t)p^0$  is in  $C^1(\mathbb{R}_+; \mathbb{H})$  and from (2.34), we deduce that  $\mathcal{E}_{p^0} \in C^1(\mathbb{R}_+; \mathcal{H})$ . Moreover, the relation (2.18) yields

$$\mathcal{E}_{p^0}(t) = \int_0^{a_\infty} E(a) S(a, 0) \mathcal{E}_{p^0}(t-a) da \quad (t > a_\infty) \quad (2.39)$$

and thus

$$\mathcal{E}'_{p^0}(t) = \int_0^{a_\infty} E(a) S(a, 0) \mathcal{E}'_{p^0}(t-a) da \quad (t > a_\infty). \quad (2.40)$$

Assume  $t > a_\infty$ . Using (2.9) and (2.20), there exists a constant  $C > 0$  such that

$$\|E'(a) S(a, 0) \mathcal{E}_{p^0}(t-a)\|_{\mathcal{H}} \leq C e^{\kappa_0 t} \|E'(a)\|_{\mathcal{L}(\mathcal{H})} \|p^0\|_{L^1(0, a_\infty; \mathcal{H})} \quad (2.41)$$



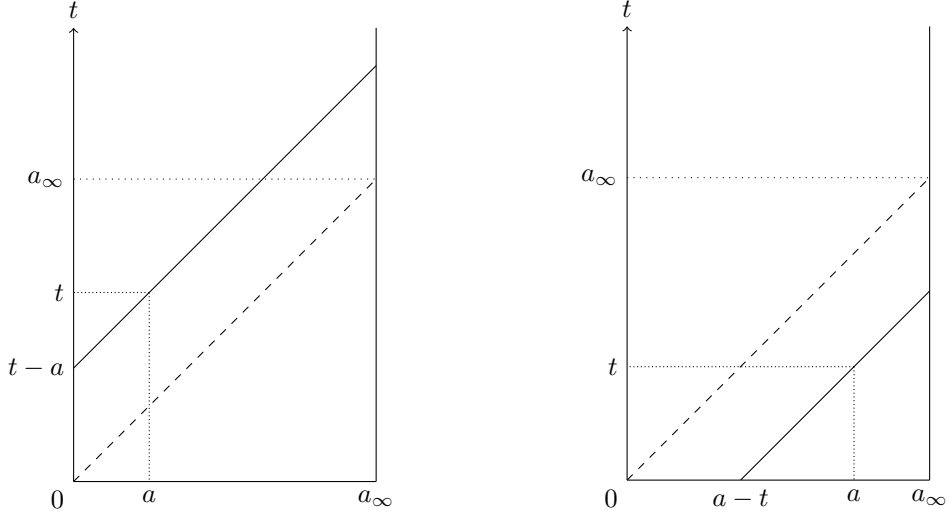


Figure 1: Method of characteristics ( $t > a$  and  $t < a$ )

and

$$\begin{aligned} \|E(a)A(a)S(a,0)\mathcal{E}_{p^0}(t-a)\|_{\mathcal{H}} &\leq \|E(a)\|_{\mathcal{L}(\mathcal{H})} \|S(a,0)\|_{\mathcal{L}(\mathcal{H}_\theta,\mathcal{H}_1)} \|\mathcal{E}_{p^0}(t-a)\|_{\mathcal{H}_\theta} \\ &\leq \frac{C}{a^{1-\theta}} \left( e^{\kappa_\theta t} + \frac{1}{(t-a_\infty)^\theta} \right) \|p^0\|_{L^1(0,a_\infty;\mathcal{H})}. \end{aligned} \quad (2.42)$$

This shows that for  $t > a_\infty$ , the mapping

$$a \mapsto E(a)S(a,0)\mathcal{E}_{p^0}(t-a)$$

is in  $W^{1,1}(0, a_\infty; \mathcal{H})$ . In particular, we can integrate by parts in (2.40) and we obtain

$$\begin{aligned} \mathcal{E}'_{p^0}(t) &= E(0)\mathcal{E}_{p^0}(t) - E(a_\infty)S(a_\infty,0)\mathcal{E}_{p^0}(t-a_\infty) \\ &\quad + \int_0^{a_\infty} (E'(a)S(a,0) + E(a)A(a)S(a,0)) \mathcal{E}_{p^0}(t-a) da \quad (t > a_\infty). \end{aligned} \quad (2.43)$$

Combining the above relation with (2.9), (2.20), (2.38), (2.41) and (2.42), we deduce

$$\|\mathcal{E}'_{p^0}(t)\|_{\mathcal{H}} \leq C \left( e^{\kappa_0 t} + e^{\kappa_\theta t} + \frac{1}{(t-a_\infty)^\theta} \right) \|p^0\|_{L^1(0,a_\infty;\mathcal{H})} \quad (t > a_\infty).$$

Applying (2.23), we deduce that for any  $t > 2a_\infty$ , there exists a constant  $C > 0$  such that for any  $p^0 \in \mathcal{D}(\mathbb{A})$ ,

$$\|\mathbb{S}'(t)p^0\|_{\mathbb{H}} \leq C \left( e^{\kappa_0 t} + e^{\kappa_\theta t} + \frac{1}{(t-2a_\infty)^\theta} \right) \|p^0\|_{\mathbb{H}}. \quad (2.44)$$

Using that  $\mathcal{D}(\mathbb{A})$  is dense in  $\mathbb{H}$ , this allows us to extend  $\mathbb{S}'(t)$  as a bounded operator of  $\mathbb{H}$ . We now show that it is the derivative of  $\mathbb{S}(t)$ , by using standard arguments. By writing that for  $h \neq 0$ ,

$$\frac{\mathbb{S}(t+h)p^0 - \mathbb{S}(t)p^0}{h} = \frac{1}{h} \int_0^h \mathbb{S}'(t+s)p^0 ds$$

we obtain from (2.44) that for any  $t > 2a_\infty$ , there exist constants  $C > 0$  and  $h_0 > 0$  such that for any  $p^0 \in \mathcal{D}(\mathbb{A})$  and for any  $0 < |h| < h_0$ ,

$$\left\| \frac{\mathbb{S}(t+h)p^0 - \mathbb{S}(t)p^0}{h} \right\|_{\mathbb{H}} \leq C \left( e^{\kappa_0 t} + e^{\kappa_\theta t} + \frac{1}{(t-2a_\infty)^\theta} \right) \|p^0\|_{\mathbb{H}}. \quad (2.45)$$

Combining (2.44), (2.45) and the fact that  $\mathcal{D}(\mathbb{A})$  is dense in  $\mathbb{H}$ , we deduce that for any  $p^0 \in \mathbb{H}$  and that for any  $t > 2a_\infty$ , as  $h \rightarrow 0$ ,

$$\frac{\mathbb{S}(t+h)p^0 - \mathbb{S}(t)p^0}{h} \rightarrow \mathbb{S}'(t)p^0 \quad \text{in } \mathbb{H}.$$

□

### 3 The adjoint system

The goal of this section is to characterize the adjoint  $\mathbb{A}^*$  of the operator  $\mathbb{A}$  defined by (1.27), (1.28) and to study the regularization properties of the corresponding semigroup  $(e^{t\mathbb{A}^*})_{t>0}$ .

#### 3.1 Characterization of $\mathbb{A}^*$

First, we introduce the family of unbounded operators

$$\tilde{A}(a) \stackrel{\text{def}}{=} A^*(a_\infty - a) \quad (a \in [0, a_\infty]). \quad (3.1)$$

Due to the hypotheses (1.17)–(1.19), there exists (see, [4, Thm 4.4.1, p.63]) a parabolic evolution operator  $\tilde{S}$  associated with  $\tilde{A}$ : for  $\mathcal{T}$  and  $\mathcal{T}^*$  defined by (2.2)

$$\tilde{S} \in C^0(\mathcal{T}; \mathcal{L}_s(\mathcal{H})) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H}_1^*)) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H})) \cap C^0(\mathcal{T}^*; \mathcal{L}(\mathcal{H}, \mathcal{H}_1^*)), \quad (3.2)$$

$$\forall r \in [0, a_\infty], \quad \tilde{S}(\cdot, r) \in C^1((r, a_\infty]; \mathcal{L}(\mathcal{H})), \quad (3.3)$$

$$\forall a \in (0, a_\infty], \quad \tilde{S}(a, \cdot) \in C^1([0, a]; \mathcal{L}_s(\mathcal{H}_1^*, \mathcal{H})), \quad (3.4)$$

$$\tilde{S}(r, r) = I, \quad \tilde{S}(a_3, a_2)\tilde{S}(a_2, a_1) = \tilde{S}(a_3, a_1) \quad (0 \leq a_1 \leq a_2 \leq a_3 \leq a_\infty) \quad (3.5)$$

$$\forall (a, r) \in \mathcal{T}^*, \quad \frac{\partial}{\partial a} \tilde{S}(a, r) = A^*(a_\infty - a)\tilde{S}(a, r), \quad (3.6)$$

$$\forall (a, r) \in \mathcal{T}^*, \quad \forall p \in \mathcal{H}_1^*, \quad \frac{\partial}{\partial r} \tilde{S}(a, r)p = -\tilde{S}(a, r)A^*(a_\infty - r)p, \quad (3.7)$$

and there exists a constant  $\tilde{M} > 0$  such that for any  $\alpha \in [0, 1]$ ,  $\theta \in [0, 1]$ ,  $\alpha + \theta \leq 1$ ,

$$\sup_{(a,r) \in \mathcal{T}^*} (a-r)^\theta \left\| \tilde{S}(a, r) \right\|_{\mathcal{L}(\mathcal{H}_\alpha^*, \mathcal{H}_{\alpha+\theta}^*)} \leq \tilde{M}. \quad (3.8)$$

This evolution operator allows us to consider the Cauchy problem

$$\frac{\partial \tilde{\psi}}{\partial a} = \tilde{A}\tilde{\psi} + \tilde{f} \quad \text{in } (0, a_\infty), \quad \tilde{\psi}(0) = \tilde{\psi}^0. \quad (3.9)$$

We have the following result similar to Proposition 2.2 (with the similar proof that we skip here).

**Proposition 3.1.** *Let  $\alpha \in [0, 1]$ . For any  $\tilde{f} \in L^2(0, a_\infty; \mathcal{H}_{\alpha-1}^*)$  and for any  $\tilde{\psi}^0 \in [\mathcal{H}_{\alpha-1}^*, \mathcal{H}_\alpha^*]_{1/2}$ , there exists a unique solution*

$$\tilde{\psi} \in L^2(0, a_\infty; \mathcal{H}_\alpha^*) \cap C^0\left([0, a_\infty]; [\mathcal{H}_{\alpha-1}^*, \mathcal{H}_\alpha^*]_{1/2}\right) \cap H^1(0, a_\infty; \mathcal{H}_{\alpha-1}^*)$$

of the system (3.9). Moreover, we have the following estimate:

$$\left\| \tilde{\psi} \right\|_{L^2(0, a_\infty; \mathcal{H}_\alpha^*) \cap H^1(0, a_\infty; \mathcal{H}_{\alpha-1}^*)} \lesssim_\alpha \left\| \tilde{\psi}^0 \right\|_{[\mathcal{H}_{\alpha-1}^*, \mathcal{H}_\alpha^*]_{1/2}} + \left\| \tilde{f} \right\|_{L^2(0, a_\infty; \mathcal{H}_{\alpha-1}^*)}. \quad (3.10)$$

Finally, for  $\alpha = 1$ , the above solution can be written with the Duhamel formula:

$$\tilde{\psi}(a) = \tilde{S}(a, 0)\tilde{\psi}^0 + \int_0^a \tilde{S}(a, r)\tilde{f}(r) dr \quad (a \in [0, a_\infty]). \quad (3.11)$$

Using the above results, we can now characterize  $\mathbb{A}^*$  and its domain:

**Proposition 3.2.** *The adjoint of  $(\mathcal{D}(\mathbb{A}), \mathbb{A})$  in  $\mathbb{H}$  is given by*

$$\begin{aligned} \mathcal{D}(\mathbb{A}^*) &= \{ \psi \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H}) ; \psi(a_\infty) = 0 \}, \\ \mathbb{A}^* : \mathcal{D}(\mathbb{A}^*) &\rightarrow \mathbb{H}, \quad \psi \mapsto \frac{\partial \psi}{\partial a} + A^* \psi + E^* \psi(0). \end{aligned}$$

*Proof.* The proof is divided into 3 steps.

*Step 1.* We consider the operator  $\mathbb{A}_0^\sharp : \mathcal{D}(\mathbb{A}_0^\sharp) \subset \mathbb{H} \rightarrow \mathbb{H}$  defined by

$$\mathcal{D}(\mathbb{A}_0^\sharp) \stackrel{\text{def}}{=} \{ \psi \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H}) ; \psi(a_\infty) = 0 \}, \quad \mathbb{A}_0^\sharp \psi \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial a} + A^* \psi.$$

Let us prove that for any  $\lambda > 0$ , the operator  $\lambda I - \mathbb{A}_0^\sharp : \mathcal{D}(\mathbb{A}_0^\sharp) \rightarrow \mathbb{H}$  is invertible and that

$$\sup_{\lambda > 0} \left\| \lambda(\lambda I - \mathbb{A}_0^\sharp)^{-1} \right\|_{\mathcal{L}(\mathbb{H})} < \infty, \quad \sup_{\lambda > 0} \left\| \mathbb{A}_0^\sharp(\lambda I - \mathbb{A}_0^\sharp)^{-1} \right\|_{\mathcal{L}(\mathbb{H})} < \infty. \quad (3.12)$$

As usual, the above relations are equivalent by using the formula  $\mathbb{A}_0^\sharp(\lambda I - \mathbb{A}_0^\sharp)^{-1} = -I + \lambda(\lambda I - \mathbb{A}_0^\sharp)^{-1}$  and we only need to show the first one. The equation

$$(\lambda I - \mathbb{A}_0^\sharp)\psi = f, \quad (3.13)$$

with  $\psi \in \mathcal{D}(\mathbb{A}_0^\sharp)$  and  $f \in \mathbb{H}$  can be written as

$$\frac{\partial \tilde{\psi}}{\partial a} = \tilde{A}\tilde{\psi} - \lambda\tilde{\psi} + \tilde{f}, \quad \tilde{\psi}(0) = 0 \quad (a \in [0, a_\infty]),$$

where  $\tilde{\psi} \stackrel{\text{def}}{=} \psi(a_\infty - \cdot)$  and  $\tilde{f} \stackrel{\text{def}}{=} f(a_\infty - \cdot)$ . Hence, using Proposition 3.1 with  $\alpha = 1$ , we see that the above equation admits a unique solution  $\tilde{\psi} \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H})$ . This shows that (3.13) admits a unique solution  $\psi \in \mathcal{D}(\mathbb{A}_0^\sharp)$ . Moreover,  $\tilde{\psi}$  is given by the Duhamel formula

$$\tilde{\psi}(a) = \int_0^a e^{-\lambda(a-r)} \tilde{S}(a, r)\tilde{f}(r) dr.$$

Using standard property on the convolution and (3.8), this yields

$$\left\| \tilde{\psi} \right\|_{\mathbb{H}} \leq \frac{\tilde{M}}{\lambda} \left\| \tilde{f} \right\|_{\mathbb{H}}$$

which implies (3.12).

*Step 2.* We now consider the operator  $\mathbb{A}^\sharp : \mathcal{D}(\mathbb{A}^\sharp) \subset \mathbb{H} \rightarrow \mathbb{H}$  defined by

$$\mathcal{D}(\mathbb{A}^\sharp) \stackrel{\text{def}}{=} \mathcal{D}(\mathbb{A}_0^\sharp), \quad \mathbb{A}^\sharp \psi \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial a} + A^* \psi + E^* \psi(0) = \mathbb{A}_0^\sharp \psi + V\psi, \quad (3.14)$$

where  $V$  is defined by

$$V : \mathcal{D}(\mathbb{A}_0^\sharp) \rightarrow \mathbb{H}, \quad V\psi \stackrel{\text{def}}{=} E^*\psi(0).$$

Let us prove that for  $\lambda > 0$  large enough the operator  $\lambda I - \mathbb{A}^\sharp : \mathcal{D}(\mathbb{A}^\sharp) \rightarrow \mathbb{H}$  is invertible. In order to do that, it is sufficient to show that for  $\lambda > 0$  large enough, we have

$$\|V(\lambda I - \mathbb{A}_0^\sharp)^{-1}\|_{\mathcal{L}(\mathbb{H})} < 1. \quad (3.15)$$

This implies that  $I - V(\lambda I - \mathbb{A}_0^\sharp)^{-1}$  is invertible and thus that  $\lambda I - \mathbb{A}^\sharp$  is invertible with

$$(\lambda I - \mathbb{A}^\sharp)^{-1} = (\lambda I - \mathbb{A}_0^\sharp)^{-1} \left( I - V(\lambda I - \mathbb{A}_0^\sharp)^{-1} \right)^{-1}.$$

To show (3.15), we use a trace theorem: for any  $\varepsilon > 0$ ,

$$\|\psi(0)\|_{\mathcal{H}} \lesssim_\varepsilon \|\psi\|_{H^{1/2+\varepsilon}(0, a_\infty; \mathcal{H})}.$$

Using that  $H^{1/2+\varepsilon}(0, a_\infty; \mathcal{H}) = [L^2(0, a_\infty; \mathcal{H}), H^1(0, a_\infty; \mathcal{H})]_{1/2+\varepsilon}$  and an interpolation inequality (see, for instance, [33, Theorem 1.9.3 (3) page 59]) we deduce that for any  $\varepsilon \in (0, 1/2)$ ,

$$\|\psi\|_{H^{1/2+\varepsilon}(0, a_\infty; \mathcal{H})} \lesssim_\varepsilon \|\psi\|_{H^1(0, a_\infty; \mathcal{H})}^{1/2+\varepsilon} \|\psi\|_{L^2(0, a_\infty; \mathcal{H})}^{1/2-\varepsilon}.$$

Combining the above estimates and using (1.24), we deduce that for any  $\varepsilon \in (0, 1/2)$ ,

$$\forall \psi \in \mathcal{D}(\mathbb{A}_0^\sharp), \quad \|V\psi\|_{\mathbb{H}} \lesssim_\varepsilon \|\psi\|_{\mathcal{D}(\mathbb{A}_0^\sharp)}^{1/2+\varepsilon} \|\psi\|_{\mathbb{H}}^{1/2-\varepsilon}.$$

Hence, by combining the above estimates with (3.12), we conclude that for any  $\varepsilon \in (0, 1/2)$ ,

$$\forall f \in \mathbb{H}, \quad \left\| V(\lambda I - \mathbb{A}_0^\sharp)^{-1} f \right\|_{\mathbb{H}} \lesssim_\varepsilon \left\| (\lambda I - \mathbb{A}_0^\sharp)^{-1} f \right\|_{\mathcal{D}(\mathbb{A}_0^\sharp)}^{1/2+\varepsilon} \left\| (\lambda I - \mathbb{A}_0^\sharp)^{-1} f \right\|_{\mathbb{H}}^{1/2-\varepsilon} \lesssim_\varepsilon \frac{\|f\|_{\mathbb{H}}}{\lambda^{1/2-\varepsilon}}.$$

Fixing  $\varepsilon \in (0, 1/2)$  and taking  $\lambda > 0$  large enough, we deduce (3.15) and we conclude that  $\lambda I - \mathbb{A}^\sharp : \mathcal{D}(\mathbb{A}^\sharp) \rightarrow \mathbb{H}$  is invertible for  $\lambda$  large enough.

*Step 3.* Let us now prove that  $(\mathcal{D}(\mathbb{A}^*), \mathbb{A}^*) = (\mathcal{D}(\mathbb{A}^\sharp), \mathbb{A}^\sharp)$ , which will end the proof of the proposition. First, using integrations by parts, one can check that  $\mathcal{D}(\mathbb{A}^\sharp) \subset \mathcal{D}(\mathbb{A}^*)$  and that for  $\psi \in \mathcal{D}(\mathbb{A}^\sharp)$  we have  $\mathbb{A}^*\psi = \mathbb{A}^\sharp\psi$ . Second, let us prove  $\mathcal{D}(\mathbb{A}^*) \subset \mathcal{D}(\mathbb{A}^\sharp)$ . Using Proposition 2.12,  $\mathbb{A}^*$  is the infinitesimal generator of the strongly continuous semigroup  $(\mathbb{S}^*(t))_{t \geq 0}$  in  $\mathbb{H}$ , and thus  $\lambda I - \mathbb{A}^* : \mathcal{D}(\mathbb{A}^*) \rightarrow \mathbb{H}$  is invertible for  $\lambda$  large enough (see [27, Theorem 5.3, p.20]). Let us take  $\lambda > 0$  large enough so that  $\lambda I - \mathbb{A}^*$  and  $\lambda I - \mathbb{A}^\sharp$  are invertible. Let us consider  $\psi \in \mathcal{D}(\mathbb{A}^*)$  and let us set

$$f \stackrel{\text{def}}{=} (\lambda I - \mathbb{A}^*)\psi \in \mathbb{H}, \quad \varphi \stackrel{\text{def}}{=} (\lambda I - \mathbb{A}^\sharp)^{-1} f \in \mathcal{D}(\mathbb{A}^\sharp).$$

In particular,  $\varphi \in \mathcal{D}(\mathbb{A}^*)$  and  $(\lambda I - \mathbb{A}^*)\varphi = (\lambda I - \mathbb{A}^\sharp)\varphi = f$ . Since  $\lambda \in \rho(\mathbb{A}^*)$ , we deduce that  $\psi = \varphi \in \mathcal{D}(\mathbb{A}^\sharp)$  and this ends the proof.  $\square$

### 3.2 Estimates on the semigroup $\{e^{t\mathbb{A}^*}\}_{t \geq 0}$

From Proposition 3.2 we deduce that  $\xi(t, \cdot) \stackrel{\text{def}}{=} e^{t\mathbb{A}^*}\xi^0$  satisfies the system

$$\begin{cases} \frac{\partial \xi}{\partial t}(t, a) - \frac{\partial \xi}{\partial a}(t, a) = A^*(a)\xi(t, a) + E^*(a)\xi(t, 0) & (t, a) \in (0, +\infty) \times (0, a_\infty), \\ \xi(t, a_\infty) = 0 & t \in (0, +\infty), \\ \xi(0, a) = \xi^0(a) & a \in (0, a_\infty). \end{cases} \quad (3.16)$$

Using the method of the characteristics (see Fig. 2), we can show the following result:

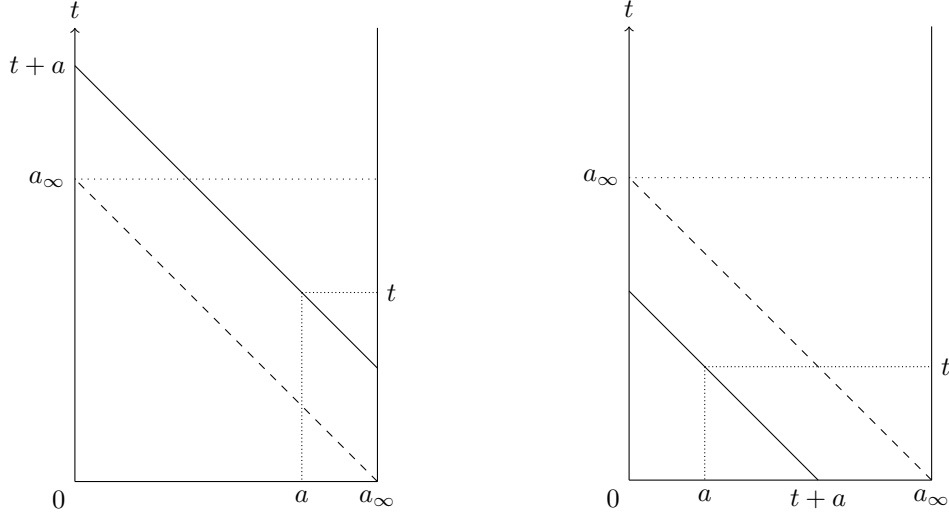


Figure 2: Method of characteristics (for  $t + a > a_\infty$  and  $t + a < a_\infty$ )

**Lemma 3.3.** Assume  $\xi(t, \cdot) \stackrel{\text{def}}{=} e^{t\tilde{A}^*} \xi^0$ . Then  $\xi$  satisfies the following formula:

$$\xi(t, a) = \int_{t+a-a_\infty}^t \tilde{S}(a_\infty - a, a_\infty - a - t + r) E^*(a + t - r) \xi(r, 0) dr \quad (t + a > a_\infty), \quad (3.17)$$

and

$$\xi(t, a) = \tilde{S}(a_\infty - a, a_\infty - a - t) \xi^0(t + a) + \int_0^t \tilde{S}(a_\infty - a, a_\infty - a - t + r) E^*(t + a - r) \xi(r, 0) dr \quad (t + a < a_\infty). \quad (3.18)$$

In particular,  $\xi(\cdot, 0)$  satisfies the following integral equation:

$$\xi(t, 0) = \mathbf{1}_{[0, a_\infty]}(t) \tilde{S}(a_\infty, a_\infty - t) \xi^0(t) + \int_0^t \tilde{S}(a_\infty, a_\infty - t + r) \mathbf{1}_{[0, a_\infty]}(t - r) E^*(t - r) \xi(r, 0) dr. \quad (3.19)$$

*Proof.* We use the classic method of the characteristics: for  $t_0, a_0 \geq 0$ , let us set

$$\phi(s) \stackrel{\text{def}}{=} \xi(t_0 + s, a_\infty - a_0 - s) \quad (s \geq 0, a_0 + s \in [0, a_\infty]).$$

Then we deduce from (3.16) that

$$\phi'(s) = A^*(a_\infty - a_0 - s) \phi(s) + E^*(a_\infty - a_0 - s) \xi(t_0 + s, 0), \quad \phi(0) = \xi(t_0, a_\infty - a_0).$$

Using the evolution operator  $\tilde{S}$  associated with  $\tilde{A}$  defined in (3.1), we can write the Duhamel formula:

$$\phi(s) = \tilde{S}(a_0 + s, a_0) \phi(0) + \int_0^s \tilde{S}(a_0 + s, a_0 + r) E^*(a_\infty - a_0 - r) \xi(t_0 + r, 0) dr.$$

Assume now that  $t + a < a_\infty$ . Then we set  $t_0 \stackrel{\text{def}}{=} 0$ ,  $a_0 \stackrel{\text{def}}{=} a_\infty - a - t$ ,  $s \stackrel{\text{def}}{=} t$  and we obtain (3.18). If we assume that  $t + a > a_\infty$ , then we set  $t_0 \stackrel{\text{def}}{=} t + a - a_\infty$ ,  $a_0 = 0$  and  $s \stackrel{\text{def}}{=} a_\infty - a$  and we obtain (3.17).  $\square$

We deduce the following result

**Lemma 3.4.** Assume  $\theta \in [0, 1/2)$ . Then there exists a constant  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}$ ,  $\xi(t, \cdot) \stackrel{\text{def}}{=} e^{t\Lambda^*} \xi^0$  satisfies

$$\|\xi(t, 0)\|_{\mathcal{H}_\theta^*} \leq C \left( \frac{1}{t^\theta} \mathbf{1}_{[0, a_\infty]}(t) \|\xi^0(t)\|_{\mathcal{H}} + e^{\tilde{\kappa}_0 t} \|\xi^0\|_{\mathbb{H}} \right) \quad (t > 0), \quad (3.20)$$

where  $\tilde{\kappa}_0 \stackrel{\text{def}}{=} \widetilde{M} \|E^*\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}))}$ .

*Proof.* First, applying the Grönwall lemma on (3.19) and using (3.8), we deduce the existence of a constant  $C > 0$  such that

$$\|\xi(t, 0)\|_{\mathcal{H}} \leq C \left( \|\xi^0(t)\|_{\mathcal{H}} \mathbf{1}_{[0, a_\infty]}(t) + e^{\tilde{\kappa}_0 t} \|\xi^0\|_{L^1(0, a_\infty; \mathcal{H})} \right) \quad (3.21)$$

which implies (3.20) for  $\theta = 0$ . Then combining (3.19), (3.21), and (3.8), we deduce the existence of a constant  $C > 0$  such that

$$\begin{aligned} \|\xi(t, 0)\|_{\mathcal{H}_\theta^*} &\leq \frac{\widetilde{M}}{t^\theta} \mathbf{1}_{[0, a_\infty]}(t) \|\xi^0(t)\|_{\mathcal{H}} + \int_0^t \frac{C\tilde{\kappa}_0}{(t-r)^\theta} \mathbf{1}_{[0, a_\infty]}(t-r) \|\xi^0(r)\|_{\mathcal{H}} \mathbf{1}_{[0, a_\infty]}(r) dr \\ &\quad + \int_0^t \frac{C\tilde{\kappa}_0}{(t-r)^\theta} \mathbf{1}_{[0, a_\infty]}(t-r) e^{\tilde{\kappa}_0 r} \|\xi^0\|_{L^1(0, a_\infty; \mathcal{H})} dr. \end{aligned} \quad (3.22)$$

The above relation yields (3.20) for  $\theta \in (0, 1/2)$ .  $\square$

Let us also define, for  $\vartheta \geq 0$ ,

$$\tilde{\kappa}_\vartheta \stackrel{\text{def}}{=} \widetilde{M} \|E^*\|_{L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_\vartheta^*))}.$$

**Theorem 3.5.** Assume

$$E^* \in L^\infty(0, a_\infty; \mathcal{L}(\mathcal{H}_{\vartheta/2}^*)), \quad (3.23)$$

with  $\vartheta \in [0, 1)$ . Then there exists a constant  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}$ ,

$$\|e^{t\Lambda^*} \xi^0\|_{L^2(0, a_\infty; \mathcal{H}_\vartheta^*)} \leq C \left( \frac{1}{t^\vartheta} + e^{\tilde{\kappa}_0 t} \right) \|\xi^0\|_{\mathbb{H}} \quad (t > 0).$$

*Proof.* We set  $\xi(t, \cdot) \stackrel{\text{def}}{=} e^{t\Lambda^*} \xi^0$ . Assume  $t + a > a_\infty$ . Then from (3.8) and (3.17) we have

$$\|\xi(t, a)\|_{\mathcal{H}_\vartheta^*} \leq \int_{t+a-a_\infty}^t \frac{\tilde{\kappa}_{\vartheta/2}}{(t-r)^{\vartheta/2}} \|\xi(r, 0)\|_{\mathcal{H}_{\vartheta/2}^*} dr \quad (t + a > a_\infty).$$

Since  $\vartheta/2 \in [0, 1/2)$ , we can use (3.20) and, combined with the above relation, it yields

$$\|\xi(t, a)\|_{\mathcal{H}_\vartheta^*} \leq C \int_{t+a-a_\infty}^t \frac{1}{(t-r)^{\vartheta/2}} \left( \frac{1}{r^{\vartheta/2}} \mathbf{1}_{[0, a_\infty]}(r) \|\xi^0(r)\|_{\mathcal{H}} + e^{\tilde{\kappa}_0 r} \|\xi^0\|_{\mathbb{H}} \right) dr \quad (t + a > a_\infty).$$

Since  $\vartheta \in [0, 1)$ , this yields

$$\|\xi(t, a)\|_{\mathcal{H}_\vartheta^*} \leq C \left( t^{1/2-\vartheta} + e^{\tilde{\kappa}_0 t} \right) \|\xi^0\|_{\mathbb{H}} \quad (t + a > a_\infty). \quad (3.24)$$

Assume  $t + a < a_\infty$ . Then from (3.8) and (3.18) we have

$$\|\xi(t, a)\|_{\mathcal{H}_\vartheta^*} \leq \frac{\widetilde{M}}{t^\vartheta} \|\xi^0(t+a)\|_{\mathcal{H}} + \int_0^t \frac{\tilde{\kappa}_{\vartheta/2}}{(t-r)^{\vartheta/2}} \|\xi(r, 0)\|_{\mathcal{H}_{\vartheta/2}^*} dr \quad (t + a < a_\infty)$$

and using (3.20), we obtain

$$\|\xi(t, a)\|_{\mathcal{H}_\vartheta^*} \leq \frac{\widetilde{M}}{t^\vartheta} \|\xi^0(t+a)\|_{\mathcal{H}} + C \left( t^{1/2-\vartheta} + e^{\tilde{\kappa}_0 t} \right) \|\xi^0\|_{\mathbb{H}} \quad (t + a < a_\infty).$$

Combining the above relation and (3.24), we deduce the result.  $\square$

## 4 Stabilization of the solutions

This section is devoted to the proof of Theorem 1.3 and Theorem 1.8, that is the existence of feedback operators that stabilize exponentially our system.

### 4.1 Spectral decomposition of $\mathbb{H}$

We recall that (see Proposition 2.6)  $\mathbb{A}$  is with compact resolvents and thus its spectrum  $\sigma(\mathbb{A})$  is reduced to eigenvalues  $(\lambda_k)_{k \geq 1}$  with finite geometric multiplicities,

$$\ell_k \stackrel{\text{def}}{=} \dim \text{Ker}(\mathbb{A} - \lambda_k I) < \infty.$$

We can assume that  $(\text{Re } \lambda_k)_{k \geq 1}$  is a nondecreasing sequence. Let us fix  $\alpha > 0$ . Using Theorem 2.13 and [27, Theorem 4.7, p. 54], we can split the spectrum of  $\mathbb{A}$  into two parts:

$$\text{Re } \lambda_k \geq -\alpha \quad (k \in \{1, \dots, N\}), \quad \text{Re } \lambda_k + \varepsilon < -\alpha \quad (k \geq N + 1), \quad (4.1)$$

for some  $\varepsilon > 0$ . In order to stabilize (1.25), we follow the method detailed in [9] for analytic semigroups. In particular, the framework below is already given in [9] (see also [10]).

First, we consider  $\Gamma$  a contour enclosing  $\{\lambda_k, k = 1, \dots, N\}$  and no other point of  $\sigma(\mathbb{A})$ . We define the projectors (see, for instance, [21, p. 178])

$$\mathbb{K}_+ \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathbb{A})^{-1} d\lambda, \quad \mathbb{K}_- \stackrel{\text{def}}{=} I - \mathbb{K}_+.$$

We also define

$$\mathbb{H}_+ \stackrel{\text{def}}{=} \mathbb{K}_+ \mathbb{H} \subset \mathcal{D}(\mathbb{A}), \quad \mathbb{H}_- \stackrel{\text{def}}{=} \mathbb{K}_- \mathbb{H},$$

so that

$$\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-.$$

Applying [21, Theorem 6.17, p. 178], we can define the parts of  $\mathbb{A}$

$$\mathbb{A}_+ \in \mathcal{L}(\mathbb{H}_+), \quad \mathbb{A}_- : \mathcal{D}(\mathbb{A}) \cap \mathbb{H}_- \rightarrow \mathbb{H}_-,$$

i.e. the restrictions of  $\mathbb{A}$  to  $\mathbb{H}_+$  and  $\mathbb{H}_-$ . Moreover, we have

$$\sigma(\mathbb{A}_+) = \{\lambda_k, k = 1, \dots, N\}, \quad \sigma(\mathbb{A}_-) = \{\lambda_k, k > N\}. \quad (4.2)$$

We can define similarly

$$\mathbb{K}_+^* \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathbb{A}^*)^{-1} d\lambda, \quad \mathbb{K}_-^* \stackrel{\text{def}}{=} I - \mathbb{K}_+^*$$

and we can check that  $\mathbb{K}_+^*$  and  $\mathbb{K}_-^*$  are the adjoint operators of  $\mathbb{K}_+$  and  $\mathbb{K}_-$ . We then set

$$\mathbb{H}_+^* \stackrel{\text{def}}{=} \mathbb{K}_+^* \mathbb{H} \subset \mathcal{D}(\mathbb{A}^*), \quad \mathbb{H}_-^* \stackrel{\text{def}}{=} \mathbb{K}_-^* \mathbb{H},$$

so that

$$\mathbb{H} = \mathbb{H}_+^* \oplus \mathbb{H}_-^*.$$

By a duality argument (see, for instance, [9]), we also deduce that we can extend  $\mathbb{K}_+$  and  $\mathbb{K}_-$  as bounded operators

$$\mathbb{K}_+ : \mathcal{D}(\mathbb{A}^*)' \rightarrow \mathbb{H}_+, \quad \mathbb{K}_- : \mathcal{D}(\mathbb{A}^*)' \rightarrow [\mathcal{D}(\mathbb{A}^*) \cap \mathbb{H}_-^*]',$$

where here  $\mathcal{X}'$  denotes the dual of  $\mathcal{X}$  with respect to  $\mathbb{H}$ . As above, we can use [21, Theorem 6.17, p. 178] to define the parts of  $\mathbb{A}^*$

$$\mathbb{A}_+^* \in \mathcal{L}(\mathbb{H}_+^*), \quad \mathbb{A}_-^* : \mathcal{D}(\mathbb{A}^*) \cap \mathbb{H}_-^* \rightarrow \mathbb{H}_-^*,$$

i.e. the restrictions of  $\mathbb{A}^*$  to  $\mathbb{H}_+^*$  and  $\mathbb{H}_-^*$ . We can check that  $\mathbb{A}_+^*$  and  $\mathbb{A}_-^*$  are the adjoint operators of  $\mathbb{A}_+$  and  $\mathbb{A}_-$ . Moreover, we have

$$\sigma(\mathbb{A}_+^*) = \{\bar{\lambda}_k, k = 1, \dots, N\}, \quad \sigma(\mathbb{A}_-^*) = \{\bar{\lambda}_k, k > N\}. \quad (4.3)$$

We recall that  $\mathbb{B}$  is defined by (1.29). Using Proposition 3.2, we deduce that  $\mathbb{B} : \mathbb{U} \rightarrow \mathcal{D}(\mathbb{A}^*)'$  and thus that its adjoint  $\mathbb{B}^*$  is a bounded operator  $\mathcal{D}(\mathbb{A}^*) \rightarrow \mathbb{U}$ . We define

$$\mathbb{U}_+ \stackrel{\text{def}}{=} \mathbb{B}^* \mathbb{H}_+^*, \quad \mathbb{U}_- \stackrel{\text{def}}{=} \mathbb{B}^* (\mathbb{H}_-^* \cap \mathcal{D}(\mathbb{A}^*)) \quad (4.4)$$

and the orthogonal projections

$$\pi_+ : \mathbb{U} \rightarrow \mathbb{U}_+, \quad \pi_- : \mathbb{U} \rightarrow \mathbb{U}_-. \quad (4.5)$$

We set

$$\mathbb{B}_+ \stackrel{\text{def}}{=} \mathbb{K}_+ \mathbb{B} \pi_+, \quad \mathbb{B}_- \stackrel{\text{def}}{=} \mathbb{K}_- \mathbb{B} \pi_-, \quad (4.6)$$

and we recall (see [9]) that  $\mathbb{B}_+ = \mathbb{K}_+ \mathbb{B}$  and that  $\mathbb{B}_- = \mathbb{K}_- \mathbb{B}$  so that

$$\mathbb{B}_+^* = \mathbb{B}^* \mathbb{K}_+^*, \quad \mathbb{B}_-^* = \mathbb{B}^* \mathbb{K}_-^*. \quad (4.7)$$

First, we deduce from Theorem 2.13 and from [12, Corollary 2.5, p.121], the following result:

**Lemma 4.1.** *Assume (2.38) for some  $\theta \in (0, 1)$ . Then there exists a constant  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}_-^*$ ,*

$$\|e^{t\mathbb{A}_-^*} \xi^0\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}} \quad (t \geq 0). \quad (4.8)$$

Then, combining the above result with Theorem 3.5, we obtain the following result

**Corollary 4.2.** *Assume (2.38) for some  $\theta \in (0, 1)$  and (3.23) for some  $\vartheta \in [0, 1)$ . Then there exists a constant  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}_-^*$ ,*

$$\left\| e^{t\mathbb{A}_-^*} \xi^0 \right\|_{L^2(0, a_\infty; \mathcal{H}_\vartheta^*)} \leq \frac{C}{t^\vartheta} e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}} \quad (t > 0).$$

*Proof.* Using that  $(e^{t\mathbb{A}_-^*})_{t \geq 0} = (e^{t\mathbb{A}^*})_{t \geq 0}$  in  $\mathbb{H}_-^*$  and applying Theorem 3.5, there exists  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}_-^*$ ,

$$\|e^{t\mathbb{A}_-^*} \xi^0\|_{L^2(0, a_\infty; \mathcal{H}_\vartheta^*)} \leq \frac{C}{t^\vartheta} \|\xi^0\|_{\mathbb{H}} \quad (t \in (0, 3a_\infty]). \quad (4.9)$$

Then, using (2.38), we can apply Theorem 2.13 and deduce that the semigroup associated with  $\mathbb{A}$  is differentiable for  $t > 2a_\infty$ . Consequently, the semigroup associated with  $\mathbb{A}^*$  is differentiable for  $t > 2a_\infty$ . In particular, (see, for instance, [27, Lemma 4.2, p.52]), the operator  $\mathbb{A}^* e^{3a_\infty \mathbb{A}^*}$  is a bounded operator in  $\mathbb{H}$ . Moreover, note that  $\mathbb{H}_-^*$  is invariant under the operator  $\mathbb{A}^* e^{3a_\infty \mathbb{A}^*}$ . Combining these properties with (4.8), we deduce the existence of a constant  $C > 0$  such that for any  $\xi^0 \in \mathbb{H}_-^*$  and for  $t \geq 3a_\infty$

$$\left\| \mathbb{A}^* e^{t\mathbb{A}_-^*} \xi^0 \right\|_{\mathbb{H}} = \left\| e^{(t-3a_\infty)\mathbb{A}_-^*} \left( \mathbb{A}^* e^{3a_\infty \mathbb{A}^*} \right) \xi^0 \right\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)(t-3a_\infty)} \left\| \mathbb{A}^* e^{3a_\infty \mathbb{A}^*} \xi^0 \right\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}}.$$

In particular, combining the above estimate with Proposition 3.2, we find that for any  $\xi^0 \in \mathbb{H}_-^*$  and for  $t \geq 3a_\infty$ ,

$$\left\| e^{t\mathbb{A}_-^*} \xi^0 \right\|_{L^2(0, a_\infty; \mathcal{H}_1^*)} \leq C e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}}.$$

Using the above relation and (4.9), we conclude the proof.  $\square$

We use now the projection  $\mathbb{K}_+$  and  $\mathbb{K}_-$  on (1.25) to obtain the decomposition of the solution of (1.25):  $p = p_+ + p_-$ , where

$$\begin{cases} p'_+ = \mathbb{A}_+ p_+ + \mathbb{B}_+ u & t > 0, \\ p_+(0) = p_+^0, \end{cases} \quad \begin{cases} p'_- = \mathbb{A}_- p_- + \mathbb{B}_- u & t > 0, \\ p_-(0) = p_-^0, \end{cases} \quad (4.10)$$

with  $p_+^0 \stackrel{\text{def}}{=} \mathbb{K}_+ p^0$  and  $p_-^0 \stackrel{\text{def}}{=} \mathbb{K}_- p^0$ .



## 4.2 Proof of Theorem 1.3

We are going to use the framework of the above section to show that (1.25) can be stabilized exponentially. More precisely, we will consider a feedback operator  $\mathbb{F}_+ \in \mathcal{L}(\mathbb{H}_+, \mathbb{U}_+)$  that stabilize the first system of (4.10) and using (4.1), we will show that the second system is also exponentially stable. We will thus consider the following systems

$$\begin{cases} p'_+ = \mathbb{A}_+ p_+ + \mathbb{B}_+ \mathbb{F}_+ p_+ & t > 0, \\ p_+(0) = p_+^0, \end{cases} \quad \begin{cases} p'_- = \mathbb{A}_- p_- + \mathbb{B}_- \mathbb{F}_+ p_+ & t > 0, \\ p_-(0) = p_-^0. \end{cases} \quad (4.11)$$

In particular,  $p = p_+ + p_-$  will satisfy the system (1.25) with the feedback operator  $u \stackrel{\text{def}}{=} \mathbb{F}_+ p_+ = \mathbb{F}_+ \mathbb{K}_+ p$ :

$$\begin{cases} p' = (\mathbb{A} + \mathbb{B} \mathbb{F}_+ \mathbb{K}_+) p & t > 0, \\ p(0) = p^0. \end{cases} \quad (4.12)$$

We will thus prove Theorem 1.3 with the feedback operator

$$\mathbb{F} \stackrel{\text{def}}{=} \mathbb{F}_+ \mathbb{K}_+. \quad (4.13)$$

Note that in (4.11),  $p_+ \in C^\infty([0, \infty); \mathbb{H}_+)$  is the classical solution of a system in finite dimension

$$p_+(t) = e^{t(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} p_+^0 \quad (t \geq 0)$$

whereas  $p_-$  is the mild solution of the second system, that is, given by the Duhamel formula:

$$p_-(t) = e^{t\mathbb{A}_-} p_-^0 + \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau) d\tau \quad (t \geq 0). \quad (4.14)$$

In order to prove Theorem 1.3, we need to show the following assertions.

1. The above formulas and  $p = p_+ + p_-$  yield that

$$\|p(t)\|_{\mathbb{H}} \leq C e^{-\alpha t} \|p^0\|_{\mathbb{H}} \quad (t \geq 0). \quad (4.15)$$

2. The following family of maps for  $t \geq 0$  (corresponding to solutions of the above systems) define a strongly continuous semigroup:

$$\mathbb{S}_{\mathbb{F}}(t) : p^0 \in \mathbb{H} \mapsto e^{t(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} \mathbb{K}_+ p^0 + e^{t\mathbb{A}_-} \mathbb{K}_- p^0 + \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ e^{\tau(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} \mathbb{K}_+ p^0 d\tau. \quad (4.16)$$

3. The infinitesimal generator of the above semigroup is the operator  $\mathbb{A}_{\mathbb{F}}$  defined by

$$\mathcal{D}(\mathbb{A}_{\mathbb{F}}) \stackrel{\text{def}}{=} \{f \in \mathbb{H} : (\mathbb{A} + \mathbb{B} \mathbb{F}_+ \mathbb{K}_+) f \in \mathbb{H}\} \quad \text{and} \quad \mathbb{A}_{\mathbb{F}} f \stackrel{\text{def}}{=} (\mathbb{A} + \mathbb{B} \mathbb{F}_+ \mathbb{K}_+) f. \quad (4.17)$$

*Proof of Theorem 1.3.* We divide the proof in 3 steps, as explained above.

Step 1. For the first system in (4.10), we note that  $\dim \mathbb{H}_+ < \infty$  and we can thus use the Fattorini-Hautus test in finite dimension. If (1.32) holds then for any  $\varepsilon > 0$ , there exists a feedback operator  $\mathbb{F}_+ \in \mathcal{L}(\mathbb{H}_+, \mathbb{U}_+)$  with

$$\text{rank } \mathbb{F}_+ \leq \max_{k=1, \dots, N} \ell_k,$$

and such that the corresponding solution  $p_+ \in C^\infty([0, \infty); \mathbb{H}_+)$  satisfies

$$\|p_+(t)\|_{\mathbb{H}} \leq C e^{-(\alpha + \varepsilon)t} \|p_+^0\|_{\mathbb{H}} \quad (t \geq 0). \quad (4.18)$$

We now show that the Duhamel formula (4.14) yields that  $p_-(t) \in \mathbb{H}$  and that it has an exponential decay. First using Lemma 4.1, we deduce that

$$\|e^{t\mathbb{A}_-} p_-^0\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)t} \|p_-^0\|_{\mathbb{H}} \quad (t \geq 0). \quad (4.19)$$

For the other part in (4.14), let us consider  $\xi^0 \in \mathcal{D}(\mathbb{A}^*)$ . Then, from (4.7), for  $t \geq 0$ ,

$$\left\langle \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau) d\tau, \xi^0 \right\rangle_{\mathcal{D}(\mathbb{A}^*)', \mathcal{D}(\mathbb{A}^*)} = \int_0^t \left\langle \mathbb{F}_+ p_+(\tau), \mathbb{B}^* e^{(t-\tau)\mathbb{A}_-} \mathbb{K}_+^* \xi^0 \right\rangle_{\mathbb{U}} d\tau \quad (4.20)$$

and thus from Corollary 4.2 (with  $\vartheta = \gamma$ ), (1.23) and (4.18), for  $t \geq 0$ ,

$$\left| \left\langle \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau) d\tau, \xi^0 \right\rangle \right| \leq C e^{-(\alpha+\varepsilon)t} t^{1-\gamma} \|p_+^0\|_{\mathbb{H}} \|\xi^0\|_{\mathbb{H}}. \quad (4.21)$$

Therefore we deduce that for any  $t \geq 0$ ,  $p_-(t) \in \mathbb{H}$  with

$$\|p_-(t)\|_{\mathbb{H}} \leq C e^{-\alpha t} \|p_-^0\|_{\mathbb{H}} \quad (t \geq 0).$$

Combining this with (4.18), this shows (4.15).

Step 2. Now, to prove that  $p \in C^0(\mathbb{R}_+; \mathbb{H})$ , it is sufficient to show that the function  $\tilde{p}$  defined by

$$\tilde{p}(t) \stackrel{\text{def}}{=} \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau) d\tau \quad (t \geq 0)$$

is continuous in  $[0, \infty)$ . In order to this, we follow the proof of Proposition 4.2.4, p. 117 in [34]. Assume  $t \geq 0$  and  $h > 0$ . Then

$$\tilde{p}(t+h) = e^{h\mathbb{A}_-} \tilde{p}(t) + \int_0^h e^{(h-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau+t) d\tau.$$

Using (4.21), we deduce that  $\tilde{p}(t+h) \rightarrow \tilde{p}(t)$  as  $h \rightarrow 0^+$ . Assume now  $t > 0$  and  $h \in (0, t)$ . Then

$$\tilde{p}(t) = e^{(t-h)\mathbb{A}_-} \int_0^h e^{(h-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau) d\tau + \int_0^{t-h} e^{(t-h-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ p_+(\tau+h) d\tau,$$

and we recall that

$$p_+(\tau+h) = e^{(\tau+h)(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} p_+^0 = e^{\tau(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} p_+(h).$$

Thus, using again (4.21),

$$\|\tilde{p}(t) - \tilde{p}(t-h)\|_{\mathbb{H}} \leq C h^{1-\gamma} \|p^0\|_{\mathbb{H}} + C \|p_+(h) - p_+^0\|_{\mathbb{H}}.$$

We conclude that  $\tilde{p}(t-h) \rightarrow \tilde{p}(t)$  as  $h \rightarrow 0^+$  and this implies that for any  $p^0 \in \mathbb{H}$ ,  $p \stackrel{\text{def}}{=} \mathbb{S}_{\mathbb{F}} p^0 \in C^0([0, \infty); \mathbb{H})$ . Let us show that  $\mathbb{S}_{\mathbb{F}}$  is a continuous semigroup on  $\mathbb{H}$ : first, we note that (4.16) can be written as

$$\mathbb{S}_{\mathbb{F}}(t) p^0 = e^{t\mathbb{A}_-} p^0 + \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ e^{\tau(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} \mathbb{K}_+ p^0 d\tau. \quad (4.22)$$

In particular, if  $t_1, t_2 \geq 0$ , we obtain after some standard computations

$$\mathbb{S}_{\mathbb{F}}(t_1 + t_2) p^0 = e^{t_1 \mathbb{A}_-} \mathbb{S}_{\mathbb{F}}(t_2) p^0 + \int_0^{t_1} e^{(t_1-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_+ e^{\tau(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} e^{t_2(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} \mathbb{K}_+ p^0 d\tau. \quad (4.23)$$

Applying  $\mathbb{K}_+$  on (4.16), we find

$$\mathbb{K}_+ (\mathbb{S}_{\mathbb{F}}(t_2) p^0) = e^{t_2(\mathbb{A}_+ + \mathbb{B}_+ \mathbb{F}_+)} \mathbb{K}_+ p^0.$$

Combining the above relation with (4.23), we deduce that  $\mathbb{S}_{\mathbb{F}}(t_1 + t_2)p^0 = \mathbb{S}_{\mathbb{F}}(t_1)\mathbb{S}_{\mathbb{F}}(t_2)p^0$ .

Step 3. Let us now consider the infinitesimal generator  $\tilde{\mathbb{A}}$  of the semigroup  $(\mathbb{S}_{\mathbb{F}}(t))_{t \geq 0}$ . We need to show that  $\tilde{\mathbb{A}} = \mathbb{A}_{\mathbb{F}}$ , where  $\mathbb{A}_{\mathbb{F}}$  is defined by (4.17).

First, we note that from Proposition 3.2,

$$[\mathbb{H}, \mathcal{D}(\mathbb{A}^*)]_{\gamma} \hookrightarrow [L^2(0, a_{\infty}; \mathcal{H}), L^2(0, a_{\infty}; \mathcal{H}_1^*)]_{\gamma} \hookrightarrow L^2(0, a_{\infty}; \mathcal{H}_{\gamma}^*)$$

and thus  $\mathbb{B}^* \in \mathcal{L}([\mathbb{H}, \mathcal{D}(\mathbb{A}^*)]_{\gamma}, \mathbb{U})$ . On the other hand, since  $\mathbb{A}^*$  is the generator of an infinitesimal semigroup (see Proposition 2.12), we can apply standard results (see, for instance, [27, Theorem 5.3, p.20]): there exist  $\lambda_0 > 0$  and  $C > 0$  such that  $(\lambda_0, \infty) \subset \rho(\mathbb{A}^*)$  and

$$\left\| \lambda (\lambda I - \mathbb{A}^*)^{-1} \right\|_{\mathcal{L}(\mathbb{H})} + \left\| (\lambda I - \mathbb{A}^*)^{-1} \right\|_{\mathcal{L}(\mathbb{H}, \mathcal{D}(\mathbb{A}^*))} \leq C \quad (\lambda > \lambda_0).$$

By interpolation and duality, we deduce that

$$\left\| (\lambda I - \mathbb{A})^{-1} \mathbb{B} \right\|_{\mathcal{L}(\mathbb{U}, \mathbb{H})} \leq \frac{C}{\lambda^{1-\gamma}} \quad (\lambda > \lambda_0).$$

This shows that for  $\lambda > \lambda_0$  large enough,  $\left\| (\lambda I - \mathbb{A})^{-1} \mathbb{B}\mathbb{F}_+\mathbb{K}_+ \right\|_{\mathcal{L}(\mathbb{H})} < 1$  and thus  $I - (\lambda I - \mathbb{A})^{-1} \mathbb{B}\mathbb{F}_+\mathbb{K}_+$  is invertible. This yields that  $\lambda I - \mathbb{A}_{\mathbb{F}} : \mathbb{H} \rightarrow \mathcal{D}(\mathbb{A}_{\mathbb{F}})$  is invertible with

$$(\lambda I - \mathbb{A}_{\mathbb{F}})^{-1} = \left( I - (\lambda I - \mathbb{A})^{-1} \mathbb{B}\mathbb{F}_+\mathbb{K}_+ \right)^{-1} (\lambda I - \mathbb{A})^{-1}. \quad (4.24)$$

Let us fix  $\lambda$  as above. Using the decay of the semigroup  $\mathbb{S}_{\mathbb{F}}$  and [12, Proposition 2.3, p.100], we can write for any  $p^0 \in \mathbb{H}$ ,

$$\int_0^{\infty} e^{-\lambda t} \mathbb{S}_{\mathbb{F}}(t) p^0 dt = (\lambda I - \tilde{\mathbb{A}})^{-1} p^0.$$

Using (4.22), we have for any  $t \geq 0$ ,

$$e^{-\lambda t} \mathbb{S}_{\mathbb{F}_+}^*(t) p^0 = e^{-\lambda t} e^{t\mathbb{A}} p^0 + \int_0^t e^{-\lambda(t-\tau)} e^{(t-\tau)\mathbb{A}} \mathbb{B}\mathbb{F}_+\mathbb{K}_+ (e^{-\lambda\tau} \mathbb{S}_{\mathbb{F}}(\tau) p^0) d\tau. \quad (4.25)$$

Integrating the above relation in  $t \in (0, \infty)$  and using the Fubini theorem, we deduce that for any  $p^0 \in \mathbb{H}$ ,

$$(\lambda I - \tilde{\mathbb{A}})^{-1} p^0 = (\lambda I - \mathbb{A})^{-1} p^0 + (\lambda I - \mathbb{A})^{-1} \mathbb{B}\mathbb{F}_+\mathbb{K}_+ (\lambda I - \tilde{\mathbb{A}})^{-1} p^0.$$

Combining the above relation with (4.24), we deduce that

$$(\lambda I - \tilde{\mathbb{A}})^{-1} = (\lambda I - \mathbb{A}_{\mathbb{F}})^{-1}$$

and therefore  $\tilde{\mathbb{A}} = \mathbb{A}_{\mathbb{F}}$ . This ends the last step of the proof of Theorem 1.3.  $\square$

### 4.3 Proof of Theorem 1.8

We now consider the proof of Theorem 1.8. First we write (1.30) in the form (1.25). In order to this, we only need to define the operator  $\mathbb{B}$ . We consider the following operator

$$\mathbb{D} \in \mathcal{L}(\mathcal{U}, \mathbb{H})$$

defined as follows:  $f \stackrel{\text{def}}{=} \mathbb{D}u$ , for  $u \in \mathcal{U}$  is the weak solution of

$$\begin{cases} \lambda f + \frac{\partial f}{\partial a} = Af & a \in (0, a_\infty), \\ f(0) = \int_0^{a_\infty} E(a)f(a)da + Bu, \end{cases} \quad (4.26)$$

where  $\lambda > 0$  large enough such that  $\lambda \in \sigma(\mathbb{A})$ . To define a weak solution for (4.26), we consider a smooth  $f$  satisfying (4.26) and we multiply the first equation by  $\tilde{q} \in \mathcal{D}(\mathbb{A}^*)$ . Integrating by parts, we obtain

$$\langle f, [\lambda I - \mathbb{A}^*] \tilde{q} \rangle_{\mathbb{H}} = \langle u, B^* \tilde{q}(0) \rangle_{\mathcal{U}}.$$

By setting  $\tilde{f} \stackrel{\text{def}}{=} [\lambda I - \mathbb{A}^*] \tilde{q}$ , we obtain the following weak formulation for (4.26):

$$\langle f, \tilde{f} \rangle_{\mathbb{H}} = \langle u, B^* \left( [\lambda I - \mathbb{A}^*]^{-1} \tilde{f} \right) (0) \rangle_{\mathcal{U}}. \quad (4.27)$$

Note that if  $\tilde{f} \in \mathbb{H}$ , then  $[\lambda I - \mathbb{A}^*]^{-1} \tilde{f} \in \mathcal{D}(\mathbb{A}^*)$  and in particular (see Proposition 3.2),

$$[\lambda I - \mathbb{A}^*]^{-1} \tilde{f} \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H}).$$

We thus deduce that

$$\left( [\lambda I - \mathbb{A}^*]^{-1} \tilde{f} \right) (0) \in \mathcal{H}_{1/2}^*,$$

so that, since  $B^* \in \mathcal{L}(\mathcal{H}_\gamma^*, \mathcal{U})$ ,  $\gamma \in [0, 1/2)$ , the weak formulation (4.27) admits a unique solution  $f \in \mathbb{H}$ . Consequently, the operator  $\mathbb{D}$  is well-defined.

Then, we set

$$\mathbb{B} \stackrel{\text{def}}{=} [\lambda I - \mathbb{A}] \mathbb{D} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\mathbb{A}^*)'),$$

and from the above computation, we remark that its adjoint is given by

$$\mathbb{B}^* : \xi \in \mathcal{D}(\mathbb{A}^*) \mapsto B^* \xi(0) \in \mathcal{U}.$$

*Proof of Theorem 1.8.* The proof is completely similar to the proof of Theorem 1.3, the only difference consists in the estimation of the left-hand side of (4.20). Here, we cannot apply Corollary 4.2, but we use instead Lemma 3.4. More precisely, from (3.20), we obtain that for any  $\xi^0 \in \mathbb{H}$ ,

$$\left\| \left[ e^{t\mathbb{A}^*} \mathbb{K}_-^* \xi^0 \right] (0) \right\|_{\mathcal{H}_\gamma^*} \leq C \left( \frac{1}{t^\gamma} \mathbf{1}_{[0, a_\infty]}(t) \|\mathbb{K}_-^* \xi^0(t)\|_{\mathcal{H}} + \|\xi^0\|_{\mathbb{H}} \right) \quad (t \in (0, 3a_\infty]). \quad (4.28)$$

On the other hand, using that the semigroup associated with  $\mathbb{A}^*$  is differentiable for  $t > 2a_\infty$  and Lemma 4.1, we have for  $t \geq 3a_\infty$  and for  $\xi^0 \in \mathbb{H}$ ,

$$\left\| \mathbb{A}^* e^{t\mathbb{A}^*} \mathbb{K}_-^* \xi^0 \right\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)(t-3a_\infty)} \left\| \mathbb{A}_-^* e^{3a_\infty \mathbb{A}_-^*} \mathbb{K}_-^* \xi^0 \right\|_{\mathbb{H}} \leq C e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}}.$$

Applying Proposition 3.2 and a trace theorem, we deduce that for any  $t \geq 3a_\infty$  and for any  $\xi^0 \in \mathbb{H}$ ,

$$\left\| \left[ e^{t\mathbb{A}_-^*} \mathbb{K}_-^* \xi^0 \right] (0) \right\|_{\mathcal{H}_{1/2}^*} \leq C e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}}.$$

The above relation and (4.28) yield that for  $\xi^0 \in \mathbb{H}$ ,

$$\left\| \left[ e^{t\mathbb{A}_-^*} \mathbb{K}_-^* \xi^0 \right] (0) \right\|_{\mathcal{H}_\gamma^*} \leq C \left( \frac{1}{t^\gamma} \mathbf{1}_{[0, a_\infty]}(t) \|\mathbb{K}_-^* \xi^0(t)\|_{\mathcal{H}} + e^{-(\alpha+\varepsilon)t} \|\xi^0\|_{\mathbb{H}} \right) \quad (t > 0). \quad (4.29)$$

Now we have

$$\left\langle \int_0^t e^{(t-\tau)\mathbb{A}_-} \mathbb{B}_- \mathbb{F}_{+p_+}(\tau) d\tau, \xi^0 \right\rangle = \int_0^t \left\langle \mathbb{F}_{+p_+}(\tau), B^* \left[ e^{(t-\tau)\mathbb{A}_-} \mathbb{K}_-^* \xi^0 \right] (0) \right\rangle_{\mathcal{U}} d\tau, \quad (4.30)$$

and using (1.31), (4.18) and (4.29),

$$\begin{aligned} & \left| \int_0^t \left\langle \mathbb{F}_{+p_+}(\tau), B^* \left[ e^{(t-\tau)\mathbb{A}_-} \mathbb{K}_-^* \xi^0 \right] (0) \right\rangle_{\mathcal{U}} d\tau \right| \\ & \leq C e^{-(\alpha+\varepsilon)t} \int_0^{\alpha_\infty} \frac{e^{(\alpha+\varepsilon)\tau}}{\tau^\gamma} \|\mathbb{K}_-^* \xi^0(\tau)\|_{\mathcal{H}} d\tau \|p_+^0\|_{\mathbb{H}} + C t e^{-(\alpha+\varepsilon)t} \|p_+^0\|_{\mathbb{H}} \|\xi^0\|_{\mathbb{H}}. \end{aligned} \quad (4.31)$$

Since  $\gamma \in [0, 1/2)$ , we deduce

$$\left| \int_0^t \left\langle \mathbb{F}_{+p_+}(\tau), B^* \left[ e^{(t-\tau)\mathbb{A}_-} \mathbb{K}_-^* \xi^0 \right] (0) \right\rangle_{\mathcal{U}} d\tau \right| \leq C e^{-\alpha t} \|p_+^0\|_{\mathbb{H}} \|\xi^0\|_{\mathbb{H}}$$

and we conclude as in the previous proof.  $\square$

#### 4.4 Proof of Theorem 1.9

We prove here Theorem 1.9 by applying Theorem 1.3. We need to introduce the operators  $A$ ,  $B$ ,  $E$  and the corresponding functional framework and show (1.17)–(1.19), (1.23), (1.33) and (1.32).

We set

$$\mathcal{H} \stackrel{\text{def}}{=} L^2(\Omega), \quad \mathcal{H}_1 \stackrel{\text{def}}{=} H^2(\Omega) \cap H_0^1(\Omega), \quad \mathcal{H}_1^* \stackrel{\text{def}}{=} \mathcal{H}_1, \quad (4.32)$$

$$A(a)p \stackrel{\text{def}}{=} L(a, \cdot)p - \mu p \quad (p \in \mathcal{H}_1), \quad (4.33)$$

where  $L$  is given by (1.2). Note that  $A^*$  is given by

$$A(a)^* \xi = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( \sigma_{i,j} \frac{\partial \xi}{\partial x_j} \right) + \sum_{j=1}^d \frac{\partial}{\partial x_j} (m_j \xi) + (r - \mu) \xi \quad (\xi \in \mathcal{H}_1^*). \quad (4.34)$$

Using (1.2), (1.3), (1.6) and (1.7), we deduce that  $A$  and  $A^*$  satisfies (1.17).

In order to obtain (1.18) and (1.19), we apply a general argument of Tanabe [31] (see also, [7]). For sake of completeness, we state in the appendix the corresponding result (Proposition A.1) and give a proof, in order to emphasize that the constants in (1.18) and (1.19) are independent of  $a \in [0, a_\infty]$ .

**Lemma 4.3.** *The operator  $A$  defined above satisfies (1.18) and (1.19).*

*Proof.* In order to apply Proposition A.1, we set

$$\mathcal{V} \stackrel{\text{def}}{=} \mathcal{H}_{1/2} = H_0^1(\Omega),$$

and

$$b_a(p, \xi) \stackrel{\text{def}}{=} (\sigma(a, \cdot) \nabla p, \nabla \xi)_{L^2(\Omega; \mathbb{C}^d)} + (m(a, \cdot) \cdot \nabla p + (r(a, \cdot) - \mu(a, \cdot))p, \xi)_{L^2(\Omega; \mathbb{C})}. \quad (4.35)$$

Using (1.3) and (1.5), there exist constants  $C_\sigma, c_\sigma > 0$  such that

$$\sigma(a, x) \xi \cdot \xi \geq c_\sigma |\xi|^2 \quad \text{and} \quad |\sigma(a, x) \xi| \leq C_\sigma |\xi| \quad (a \in [0, a_\infty], x \in \bar{\Omega}, \xi \in \mathbb{R}^d). \quad (4.36)$$

Here, we use the canonical Euclidean inner product and norm of  $\mathbb{R}^d$ . Using (4.36), (1.6), (1.7) and the Poincaré inequality, we deduce the existence of  $c_b > 0$  and  $\lambda_0 > 0$  such that for any  $p \in \mathcal{V}$  and any  $a \in [0, a_\infty]$ ,

$$\text{Re } b_a(p, p) \geq c_b \|p\|_{\mathcal{V}}^2 - \lambda_0 \|p\|_{\mathcal{H}}^2.$$

Similarly, using (4.36), (1.6) and (1.7), there exists  $C_b > 0$  such that for any  $a \in [0, a_\infty]$ ,

$$|b_a(p_1, p_2)| \leq C_b \|p_1\|_{\mathcal{V}} \|p_2\|_{\mathcal{V}} \quad (p_1, p_2 \in \mathcal{V}).$$

Using integration by parts, we can check that the operator associated with  $b_a$  is

$$\mathcal{A}(a) : \mathcal{V} \rightarrow \mathcal{V}', \quad p \mapsto L(a, \cdot)p - \mu p,$$

and using the elliptic regularity (see, for instance, [15, Theorem 8.12, p. 186]), we see that  $\mathcal{A}(a) : \mathcal{D}(\mathcal{A}(a)) \rightarrow \mathcal{H}$ , with  $\mathcal{D}(\mathcal{A}(a)) = \mathcal{H}_1$ . In particular  $\mathcal{A}(a) = A(a)$ , where  $A(a)$  is defined by (4.33). We can thus apply Proposition A.1 and we deduce the result.  $\square$

For the control operator  $B$ , it corresponds to the Dirichlet control on  $\Gamma \subset \partial\Omega$ . To define it, we use the following standard procedure. First, we consider  $\lambda_1 > 0$  such that for any  $a \in [0, a_\infty]$ ,

$$\lambda_1 I - A(a)^* : \mathcal{H}_1^* \rightarrow \mathcal{H}$$

is invertible. For any  $u \in L^2(\Gamma)$  and  $a \in [0, a_\infty]$ , we consider the weak solution  $p_u$  of

$$\begin{cases} \lambda_1 p_u - L(a, \cdot)p_u + \mu(a, \cdot)p_u = 0 & \text{in } \Omega, \\ p_u = \mathbf{1}_\Gamma u & \text{on } \partial\Omega. \end{cases} \quad (4.37)$$

The weak formulation of the above system is obtained by formally taking the inner product of the first equation with  $\xi \in \mathcal{H}_1^*$ :

$$(p_u, (\lambda_1 I - A(a)^*) \xi)_{\mathcal{H}} = -(u, (\sigma(a, \cdot) \nabla \xi) \cdot n)_{\mathcal{U}}, \quad (4.38)$$

where  $n$  is the unit outward normal vector field on  $\partial\Omega$  and where  $\mathcal{U} \stackrel{\text{def}}{=} L^2(\Gamma)$ . Using a trace property, the weak formulation of (4.37) can be thus written as follows:

$$(p_u, f)_{\mathcal{H}} = - \left( u, \left( \sigma(a, \cdot) \nabla \left( (\lambda_1 I - A(a)^*)^{-1} \right) f \right) \cdot n \right)_{\mathcal{U}} \quad (f \in \mathcal{H}) \quad (4.39)$$

and the Riesz theorem yields the existence and uniqueness of  $p_u \in \mathcal{H}$ . Moreover, the adjoint of the operator  $R_\Gamma : \mathcal{U} \rightarrow \mathcal{H}$ ,  $u \mapsto p_u$  is given by

$$R_\Gamma^* : \mathcal{H} \rightarrow \mathcal{U}, \quad f \mapsto \left( \sigma(a, \cdot) \nabla \left( (\lambda_1 I - A(a)^*)^{-1} \right) f \right) \cdot n$$

and by using a trace property, we see that it can be extended  $R_\Gamma^* \in \mathcal{L} \left( \mathcal{H}_{-1/4+\varepsilon}^*, \mathcal{U} \right)$  for any  $\varepsilon > 0$ . Therefore  $R_\Gamma \in \mathcal{L} \left( \mathcal{U}, \mathcal{H}_{1/4-\varepsilon} \right)$  and we define

$$B(a) \stackrel{\text{def}}{=} \mathbf{1}_{(a_1, a_2)}(a) (\lambda_1 I - A(a)) R_\Gamma \in \mathcal{L} \left( \mathcal{U}, \mathcal{H}_{-3/4-\varepsilon} \right). \quad (4.40)$$

Note that with the above computations,

$$B(a)^* : \mathcal{H}_{3/4+\varepsilon} \rightarrow \mathcal{U}, \quad f \mapsto \mathbf{1}_{(a_1, a_2)}(a) (\sigma(a, \cdot) \nabla f) \cdot n. \quad (4.41)$$

In particular (1.23) is satisfied for any  $\gamma \in (3/4, 1)$ .

For (1.33), we first introduce the operator  $E$ :

$$E(a) : L^2(\Omega) \rightarrow L^2(\Omega), \quad p \mapsto \beta(a, \cdot)p. \quad (4.42)$$

Using (1.9), we see that for any  $a \in [0, \infty]$ ,  $E(a) \in \mathcal{L} \left( L^2(\Omega) \right)$  is a self-adjoint operator. Note that from (4.32), for  $\gamma \in (3/4, 1)$ , (see, for instance, [12, pp. 171-172])

$$\mathcal{H}_{\gamma/2}^* = H_0^\gamma(\Omega).$$

Let us consider  $\gamma \in (3/4, 1)$ ,  $\gamma \leq b$ . Then from [16, Corollary 1.1, p.5],  $E$  satisfies (1.33).

It only remains to show (1.32). In order to do this we apply Lemma 1.7 and we prove that the system associated with  $(A^*, B^*)$  is approximately observable in  $(a_1, a_2)$ . As explained in Remark 1.6, this property is equivalent to the approximate controllability of

$$\begin{cases} \frac{\partial p}{\partial a} - Lp + \mu p = 0 & (a, x) \in (a_1, a_2) \times \Omega, \\ p(a, x) = \mathbf{1}_\Gamma(x)u(a, x) & (a, x) \in (a_1, a_2) \times \partial\Omega, \\ p(a_1, x) = p^0(x) & x \in \Omega. \end{cases} \quad (4.43)$$

The approximate controllability of such a system is well-known and can be obtained by using an extension of the domain  $\Omega$  and an approximate controllability property of a similar system as above but for a distributed control. The corresponding result for the distributed control is a consequence of Carleman estimates: see [14, Lemma 1.2, p.5], whereas the procedure for the extension of the domain is done in the proof of Theorem 2.3, p.29 in [14]. Note that we need the regularity (1.4), (1.6) in the Carleman estimates. □

## 4.5 Proof of Theorem 1.10

We prove here Theorem 1.10 by applying Theorem 1.8. We need to introduce the operators  $A$ ,  $B$ ,  $E$  and the corresponding functional framework and show (1.17)–(1.19), (1.31), (1.38) and (1.37).

We define  $\mathcal{H}$ ,  $\mathcal{H}_1$ ,  $A$  by (4.32), (4.33) and as in the proof of Theorem 1.9, we can check that  $A$  and  $A^*$  satisfies (1.17), and Lemma 4.3 yields that (1.18) and (1.19) hold true.

We also can define the control operator by

$$\mathcal{U} \stackrel{\text{def}}{=} L^2(\omega), \quad B : L^2(\omega) \rightarrow L^2(\Omega), \quad u \mapsto 1_\omega u.$$

In particular, (1.31) holds for  $\gamma = 0$ .

We then define  $E$  as in the proof of Theorem 1.9 by (4.42). Applying [16, Corollary 1.1, p.5], we deduce that (1.9) and (1.14) yield (1.38).

Finally, to prove (1.37), let us consider  $\xi \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H})$  such that

$$\frac{\partial \xi}{\partial a} + \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( \sigma_{i,j} \frac{\partial \xi}{\partial x_j} \right) + \sum_{j=1}^d \frac{\partial}{\partial x_j} (m_j \xi) + (r - \mu)\xi + \beta \xi(0, \cdot) = \lambda \xi \quad \text{in } (0, a_\infty) \times \Omega, \quad (4.44)$$

$$\xi = 0 \quad \text{on } (0, a_\infty) \times \partial\Omega, \quad (4.45)$$

$$\xi(a_\infty, \cdot) = 0 \quad \text{in } \Omega \quad (4.46)$$

and

$$\xi(0, \cdot) = 0 \quad \text{in } \omega. \quad (4.47)$$

Using (1.10), the equation (4.44) implies

$$\frac{\partial \xi}{\partial a} + \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( \sigma_{i,j} \frac{\partial \xi}{\partial x_j} \right) + \sum_{j=1}^d \frac{\partial}{\partial x_j} (m_j \xi) + (r - \mu - \lambda)\xi = 0 \quad \text{in } (0, a_\beta) \times \Omega. \quad (4.48)$$

By using (1.13), we can apply Theorem 4.1 in [11] and deduce that (4.47) implies

$$\xi(0, \cdot) = 0 \quad \text{in } \Omega.$$

Consequently, we can remove the term  $\beta \xi(0, \cdot)$  in (4.44) and from (4.45) and (4.46) we deduce  $\xi \equiv 0$ . We have proven (1.37) and thus Theorem 1.10. □

## A Technical results

### A.1 Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2.

We start by the case  $\alpha = 1$ . First, using (1.19), for any  $s \in [0, a_\infty]$ ,  $p^0 \in \mathcal{H}_{1/2}$  and  $f \in L^2(0, 1; \mathcal{H})$ , there exists a unique solution

$$p \in L^2(0, 1; \mathcal{H}_1) \cap C^0([0, 1]; \mathcal{H}_{1/2}) \cap H^1(0, 1; \mathcal{H})$$

of the system

$$\frac{\partial p}{\partial a} = A(s)p + f \quad \text{in } (0, 1), \quad p(0) = p^0. \quad (\text{A.1})$$

Moreover, there exists a constant  $C > 0$  depending only on the constants in (1.20) and (1.19) such that

$$\|p\|_{L^2(0,1;\mathcal{H}_1)} + \|p\|_{C^0([0,1];\mathcal{H}_{1/2})} + \|p\|_{H^1(0,1;\mathcal{H})} \leq C \left( \|p^0\|_{\mathcal{H}_{1/2}} + \|f\|_{L^2(0,1;\mathcal{H})} \right). \quad (\text{A.2})$$

Note that the above result, with the above estimate holds true if we replace  $(0, 1)$  by any interval of size less than 1 (by extending  $f$  by 0).

Let us fix  $p^0 \in \mathcal{H}_{1/2}$  and  $f \in L^2(0, a_\infty; \mathcal{H})$ . Assume  $a_1 \leq a_\infty$  and  $a_1 < 1$ . For any  $g \in L^2(0, a_1; \mathcal{H})$ , we can consider the system:

$$\frac{\partial p}{\partial a} = A(0)p + f + g \quad \text{in } (0, a_1), \quad p(0) = p^0. \quad (\text{A.3})$$

We obtain a unique solution

$$p \in L^2(0, a_1; \mathcal{H}_1) \cap C^0([0, a_1]; \mathcal{H}_{1/2}) \cap H^1(0, a_1; \mathcal{H}),$$

and we can consider the mapping  $\mathcal{G} : L^2(0, a_1; \mathcal{H}) \rightarrow L^2(0, a_1; \mathcal{H})$  such that

$$(\mathcal{G}g)(a) \stackrel{\text{def}}{=} (A(a) - A(0))p(a) \quad (a \in (0, a_1)).$$

We see that  $g$  is fixed point of  $\mathcal{G}$  if and only if the corresponding solution  $p$  of (A.3) is a solution of (2.1). Using (A.2) and (1.17),

$$\|\mathcal{G}g^1 - \mathcal{G}g^2\|_{L^2(0,a_1;\mathcal{H})} \leq Ca_1' \|g^1 - g^2\|_{L^2(0,a_1;\mathcal{H})}$$

and thus for  $a_1$  small enough,  $\mathcal{G}$  is a strict contraction: applying the Banach fixed-point theorem, we deduce the existence and uniqueness of a fixed point for  $g$ . This implies the existence and uniqueness of a solution of (2.1) in  $(0, a_1)$ . Since the constance  $C$  in the above relation is only depending on  $A$ , we can repeat the argument on  $(a_1, 2a_1)$ , etc. and we deduce the existence and uniqueness of strong solutions for (2.1) in  $(0, a_\infty)$ .

For the general case, let us assume  $\alpha \in [0, 1]$ . By a duality argument (see, for instance, [12, Corollary 2.1, p.207]), for any  $s \in [0, a_\infty]$ ,  $p^0 \in \mathcal{H}_{-1/2}$  and  $f \in L^2(0, 1; \mathcal{H}_{-1})$ , there exists a unique solution

$$p \in L^2(0, 1; \mathcal{H}) \cap C^0([0, 1]; \mathcal{H}_{-1/2}) \cap H^1(0, 1; \mathcal{H}_{-1}) \quad (\text{A.4})$$

of the system (A.1). Moreover, there exists a constant  $C > 0$  depending only on the constants in (1.21) and (1.19) such that

$$\|p\|_{L^2(0,1;\mathcal{H})} + \|p\|_{C^0([0,1];\mathcal{H}_{-1/2})} + \|p\|_{H^1(0,1;\mathcal{H}_{-1})} \leq C \left( \|p^0\|_{\mathcal{H}_{-1/2}} + \|f\|_{L^2(0,1;\mathcal{H}_{-1})} \right). \quad (\text{A.5})$$

By interpolation of (A.2) and (A.5) (see [12, Theorem 2.2, p.208], for any  $s \in [0, a_\infty]$ ,  $p^0 \in [\mathcal{H}_{\alpha-1}, \mathcal{H}_\alpha]_{1/2}$  and  $f \in L^2(0, 1; \mathcal{H}_{\alpha-1})$ , there exists a unique solution

$$p \in L^2(0, 1; \mathcal{H}_\alpha) \cap C^0\left([0, 1]; [\mathcal{H}_{\alpha-1}, \mathcal{H}_\alpha]_{1/2}\right) \cap H^1(0, 1; \mathcal{H}_{\alpha-1})$$



of the system (A.1). Moreover, there exists a constant  $C > 0$  depending only on the constants in (1.20), (1.21) and (1.19) such that

$$\|p\|_{L^2(0,1;\mathcal{H}_\alpha)} + \|p\|_{C^0([0,1];[\mathcal{H}_{\alpha-1},\mathcal{H}_{\alpha}])} + \|p\|_{H^1(0,1;\mathcal{H}_{\alpha-1})} \leq C \left( \|p^0\|_{[\mathcal{H}_{\alpha-1},\mathcal{H}_{\alpha}]} + \|f\|_{L^2(0,1;\mathcal{H}_{\alpha-1})} \right). \quad (\text{A.6})$$

Moreover, using (1.17) and a duality argument we deduce the existence of a constant  $C > 0$  such that

$$\begin{aligned} \|A(a_1)f - A(a_2)f\|_{\mathcal{H}} &\leq C|a_1 - a_2|^\nu \|f\|_{\mathcal{H}_1}, \quad (a_1, a_2 \in [0, a_\infty], f \in \mathcal{H}_1), \\ \|A(a_1)f - A(a_2)f\|_{\mathcal{H}_{-1}} &\leq C|a_1 - a_2|^\nu \|f\|_{\mathcal{H}}, \quad (a_1, a_2 \in [0, a_\infty], f \in \mathcal{H}). \end{aligned}$$

Then an interpolation argument yields

$$\|A(a_1)f - A(a_2)f\|_{\mathcal{H}_{-\alpha}} \leq C|a_1 - a_2|^\nu \|f\|_{\mathcal{H}_{1-\alpha}}, \quad (a_1, a_2 \in [0, a_\infty], f \in \mathcal{H}_{1-\alpha}). \quad (\text{A.7})$$

We can now proceed as in the case  $\alpha = 0$ , by using (A.6) and (A.7) instead of (1.17) and (A.2) and we conclude the proof of the existence and uniqueness of a solution for  $\alpha \in [0, 1]$ .

Finally, the fact that the solution is given by the Duhamel formula (2.10) for  $\alpha = 1$  follows from Proposition 2.1. Both the solution given by the Duhamel formula (2.10) and the solution constructed here (for  $\alpha = 1$ ) are solutions of the system (2.1) with the regularity (A.4). Since such solutions are unique, we deduce the result and conclude the proof of Proposition 2.2.  $\square$

## A.2 Proof of (1.20) and (1.21)

We show here (1.20) from (1.17) and (1.18), the proof for (1.21) is completely similar. First, there exists a constant  $C_0 > 0$  such that

$$\forall p \in \mathcal{H}_1, \quad \|p\|_{\mathcal{H}} + \|A(0)p\|_{\mathcal{H}} \leq C_0 \|p\|_{\mathcal{H}_1}.$$

Then combining the above relation with (1.17), that is

$$\|A(a_1)f - A(a_2)f\|_{\mathcal{H}} \leq C_A|a_1 - a_2|^\nu \|f\|_{\mathcal{H}_1}, \quad (a_1, a_2 \in [0, a_\infty], f \in \mathcal{H}_1),$$

we obtain that for any  $a \in [0, a_\infty]$  and any  $p \in \mathcal{H}_1$ ,

$$\|p\|_{\mathcal{H}} + \|A(a)p\|_{\mathcal{H}} \leq (C_0 + C_A a_\infty^\nu) \|p\|_{\mathcal{H}_1}.$$

For the other estimate of (1.20), we first use (1.18) to deduce the existence of  $\lambda_1 \in \mathbb{R}$  such that for any  $a \in [0, a_\infty]$ ,  $\lambda_1 I - A(a) : \mathcal{H}_1 \rightarrow \mathcal{H}$  is a bounded isomorphism. In particular for any  $a \in [0, a_\infty]$ , there exists a constant  $C(a) > 0$  such that for any  $p \in \mathcal{H}_1$

$$\|p\|_{\mathcal{H}_1} \leq C(a) \|\lambda_1 p - A(a)p\|_{\mathcal{H}}.$$

Now, we deduce from the above relation and (1.17) that for  $a_1, a \in [0, a_\infty]$  and  $p \in \mathcal{H}_1$ ,

$$\|p\|_{\mathcal{H}_1} \leq C(a_1) (\|\lambda_1 p - A(a)p\|_{\mathcal{H}} + C_A |a - a_1|^\nu \|p\|_{\mathcal{H}_1}).$$

In particular, there exists an open neighborhood of  $a_1$  in  $[0, a_\infty]$  such that for any  $p \in \mathcal{H}_1$ ,

$$\|p\|_{\mathcal{H}_1} \leq 2C(a_1) \|\lambda_1 p - A(a)p\|_{\mathcal{H}}.$$

Using the compactness of  $[0, a_\infty]$ , we deduce the existence of a constant  $C > 0$  such that for any  $p \in \mathcal{H}_1$  and  $a \in [0, a_\infty]$ ,

$$\|p\|_{\mathcal{H}_1} \leq C \|\lambda_1 p - A(a)p\|_{\mathcal{H}}$$

and this implies the first estimate of (1.20).  $\square$

### A.3 Proof of Lemma 1.7

Assume that  $\xi \in L^2(0, a_\infty; \mathcal{H}_1^*) \cap H^1(0, a_\infty; \mathcal{H})$  and  $\lambda \in \mathbb{C}$  satisfy

$$\begin{cases} \frac{\partial \xi}{\partial a} + A^* \xi + E^* \xi(0) = \lambda \xi & a \in (0, a_\infty), \\ \xi(a_\infty) = 0, \\ B^* \xi = 0 & a \in (0, a_\infty). \end{cases} \quad (\text{A.8})$$

Then, using the hypothesis on  $E$ ,  $\tilde{\xi}(a) \stackrel{\text{def}}{=} e^{-\lambda a} \xi(a)$  satisfies

$$\frac{\partial \tilde{\xi}}{\partial a} + A^* \tilde{\xi} = 0, \quad B^* \tilde{\xi} = 0 \quad \text{in } (0, a_E).$$

Since  $a_1 < a_E$ , we can use (1.35) in some interval  $(a_1, a_3)$  to obtain that  $\tilde{\xi} \equiv 0$  in  $[a_1, a_3]$ . From the above equation satisfied by  $\tilde{\xi}$ , we find  $\tilde{\xi} \equiv 0$  in  $[0, a_3]$ . In particular  $\xi(0) = 0$  and (A.8) yields that  $\tilde{\xi}$  satisfies

$$\frac{\partial \tilde{\xi}}{\partial a} + A^* \tilde{\xi} = 0 \quad a \in (0, a_\infty), \quad \tilde{\xi}(a_\infty) = 0.$$

Thus,  $\xi \equiv 0$  in  $(0, a_\infty)$ , and we deduce (1.32). □

### A.4 A result of Tanabe

We give here a result obtained in [31] and that allows us to show Lemma 4.3. We first recall the general framework: let us consider two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{V}$  with a continuous and dense inclusion  $\mathcal{V} \hookrightarrow \mathcal{H}$ , so that we can define  $\mathcal{V}'$  the dual of  $\mathcal{V}$  with respect to the pivot space  $\mathcal{H}$ . We also consider  $b : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  a bilinear form,  $c_b, C_b > 0$  and  $\lambda_0 \geq 0$  such that

$$|b(p_1, p_2)| \leq C_b \|p_1\|_{\mathcal{V}} \|p_2\|_{\mathcal{V}} \quad (p_1, p_2 \in \mathcal{V}), \quad \operatorname{Re} b(p, p) \geq c_b \|p\|_{\mathcal{V}}^2 - \lambda_0 \|p\|_{\mathcal{H}}^2 \quad (p \in \mathcal{V}). \quad (\text{A.9})$$

There exists a unique operator  $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$  such that

$$\langle -\mathcal{A}p_1, p_2 \rangle_{\mathcal{V}', \mathcal{V}} \stackrel{\text{def}}{=} b(p_1, p_2) \quad (p_1, p_2 \in \mathcal{V}).$$

We can then consider  $\mathcal{A}$  as an unbounded operator from  $\mathcal{D}(\mathcal{A})$  to  $\mathcal{H}$ , with

$$\mathcal{D}(\mathcal{A}) \stackrel{\text{def}}{=} \{p \in \mathcal{V} : \mathcal{A}p \in \mathcal{H}\}.$$

**Proposition A.1.** *Assume that  $\mathcal{A}$  is the unbounded operator defined through the bilinear map  $b$  as above and assume that  $b$  satisfies (A.9). Then there exist  $N > 0$  and  $\vartheta \in (\pi/2, \pi)$  only depending on  $c_b, C_b$  and  $\lambda_0$  such that*

$$\Sigma_{\lambda_0, \vartheta} \subset \rho(\mathcal{A}) \quad \text{and} \quad \forall \lambda \in \Sigma_{\lambda_0, \vartheta}, \quad \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{N}{|\lambda - \lambda_0|}. \quad (\text{A.10})$$

*Proof.* First, using (A.9) and the Lax-Milgram lemma, we deduce that

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \lambda_0\} \subset \rho(\mathcal{A})$$

and for any  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re} \lambda \geq \lambda_0$  and  $f \in \mathcal{H}$ , the unique solution  $p \in \mathcal{D}(\mathcal{A})$  of

$$(\lambda I - \mathcal{A})p = f$$

satisfies

$$\|p\|_{\mathcal{V}}^2 \leq \frac{1}{c_b} \|f\|_{\mathcal{H}} \|p\|_{\mathcal{H}} \quad \text{and} \quad |\lambda - \lambda_0| \|p\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{H}} \|p\|_{\mathcal{H}} + (C_b + \lambda_0) \|p\|_{\mathcal{V}}^2. \quad (\text{A.11})$$

Combining the two above relations yields

$$\left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{N_0}{|\lambda - \lambda_0|} \quad (\operatorname{Re} \lambda \geq \lambda_0, \lambda \neq \lambda_0), \quad (\text{A.12})$$

with

$$N_0 \stackrel{\text{def}}{=} \left( 1 + \frac{C_b + \lambda_0}{c_b} \right). \quad (\text{A.13})$$

Let us set

$$\vartheta \stackrel{\text{def}}{=} \pi - \arctan(2N_0) \in (\pi/2, \pi). \quad (\text{A.14})$$

Assume now that  $\lambda \in \Sigma_{\lambda_0, \vartheta}$ , with  $\operatorname{Re} \lambda < \lambda_0$ . There exists  $r > 0$  and  $|\theta| \in (\pi/2, \vartheta)$  such that  $\lambda = \lambda_0 + re^{i\theta}$ . We can write

$$\begin{aligned} \lambda I - \mathcal{A} &= (\lambda_0 + ir \sin(\theta)) I - \mathcal{A} + r \cos(\theta) I \\ &= [(\lambda_0 + ir \sin(\theta)) I - \mathcal{A}] \left[ I + r \cos(\theta) [(\lambda_0 + ir \sin(\theta)) I - \mathcal{A}]^{-1} \right]. \end{aligned} \quad (\text{A.15})$$

Using (A.12) and (A.14), we have

$$\left\| r \cos(\theta) [(\lambda_0 + ir \sin(\theta)) I - \mathcal{A}]^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq N_0 |\cot \theta| \leq \frac{1}{2}.$$

Combining the above relation with (A.15), we deduce that  $\lambda \in \rho(\mathcal{A})$  and that

$$\left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{2N_0}{r |\sin \theta|} \leq \frac{N_1}{|\lambda - \lambda_0|},$$

with

$$N_1 \stackrel{\text{def}}{=} \frac{2N_0}{\sin \vartheta}.$$

□

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