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Abstract

The Sterile Insect Technique or SIT is presently one of the most ecological methods for controlling insect pests responsible for disease transmission or crop destruction worldwide. This technique consists of releasing sterile males into the insect pest population. This approach aims at reducing fertility in the population and, consequently, reduce significantly the native insect population after a few generations.

In this work, we study the global stabilization of a pest population at extinction equilibrium by the SIT method. We construct explicit feedback laws that stabilize the model and do numerical simulations to show the efficiency of our feedback laws.

The different feedback laws are also compared taking into account their possible implementation in field interventions.

Keywords: Sterile Insect Technique, Pest control, Dynamical control system, Feedback design, Backstepping feedback, Lyapunov global stabilization, Mosquito population control, Vector borne disease.

Contents

1 Introduction

Mosquitoes are known to transmit a variety of diseases such as malaria, dengue, yellow fever, Zika virus, and others. These diseases are responsible for a significant number of deaths worldwide. According to the World Health Organization (WHO), malaria alone caused approximately 409,000 deaths in 2019, with the majority of deaths occurring in sub-Saharan Africa. Dengue and Zika virus, also transmitted by mosquitoes, are estimated to cause hundreds of thousands of cases and thousands of deaths each year. The precise number of deaths caused by mosquitoes is difficult to determine because many cases are not reported or not diagnosed. Unfortunately, more than half of the world's population is exposed to mosquito-borne diseases. Although there are many effective vector control measures for malaria and arboviroses, some of them can have negative impact on the environment and may result in ecological dammage. For example insecticide spraying can have unintended effects on non-target organisms, including beneficial insects such as bees and butterflies. In addition, repeated use of insecticides often leads to the development of resistance in mosquito populations.

As a possible alternative, the sterile insect technique (SIT) has been proposed as a potential tool for reducing mosquito populations. The technique involves sterilizing male mosquitoes with ionizing radiation and then releasing them into the wild to mate with wild females. In agricultural setting, SIT has been used successfully in controlling a variety of insect pests, including fruit flies, tsetse flies, and moths.

The SIT strategy was first used by R. Bushland and E. Knipling and applied successfully in the early 1950s by nearly eradicating screw-worm fly in North America. Since then, this technique has been considered for different pests and disease vectors [6], [16]. The advantage of using such a technique is that it only targets the desired species and also significantly reduces the degradation of the ecosystem. This is why this technique is increasingly used for the control of insect pests and insect disease vectors.

In order to determine the appropriate releases of sterile males to approach the extinction equilibrium of the population, we use mathematical control theory which provides the necessary tools for constructing such a control. Our work involves starting from the model proposed in [15] without the Allee effect to build this feedback law. Our theoretical results are illustrated with numerical simulations.

Remark 1.1 *While we were finishing writing this work, we learned that the reduced system (system of two ODE studied in [1]) was also recently studied by A. Cristofaro and L. Rossi in [10]. In particular, they were able to construct a feedback law leading to global stabilization of the extinction equilibrium in this setting using a backstepping approach.*

2 Mathematical modeling of mosquito population dynamics

2.1 Mathematical modeling of wild mosquito population dynamics

The life cycle of mosquitoes has many stages but we will consider a simplified model where we just separate an aquatic and an adult phase. The aquatic phase, which includes egg, larva and pupa stages and then the adult phase. In order to lay their eggs, female mosquitoes need not only to be fertilized by males but also to have a blood meal. Thus, every 4-5 days, they will take a blood meal (that can sometimes involve biting several victimes) and lay 100 to 150 eggs in different places (10 to 15 per place). An adult mosquito usually lives for 2 to 4 weeks. The mathematical model we present takes account the two phases: the aquatic phase that we denote by the state E and the adult phase that we split into two sub-compartments, males, M and females, F. We consider the dynamics presented in [15]. Based on this model and neglecting the Allee effect, we obtain the system

$$\dot{E} = \beta_E F\left(1 - \frac{E}{K}\right) - \big(\nu_E + \delta_E\big)E, \tag{2.1}$$

$$\dot{M} = (1-\nu)\nu_E E - \delta_M M, \tag{2.2}$$

$$\dot{F} = \nu\nu_E E - \delta_F F, \tag{2.3}$$

where,

- $E(t) \geq 0$ is the mosquito density in aquatic phase at time t;
- $M(t) \geq 0$ is the wild adult male density at time t;
- $F(t) \geq 0$ is the density of adult females at time t; we have supposed that all females are immediately fertilized in this setting and this equation is only here to use when we add the sterile male in which case only a fraction of the females will be fertilized;
- $\beta_E > 0$ is the oviposition rate;
- $\delta_E, \delta_M, \delta_F > 0$ are the death rates for eggs, wild adult males and fertilized females respectively;
- $\nu_E > 0$ is the hatching rate for eggs;
- $\nu \in (0,1)$ the probability that a pupa gives rise to a female, and $(1-\nu)$ is, therefore, the probability to give rise to a male. And to simplify, we suppose females become fertilized immediately when they emerge from the pupal stage;
- $K > 0$ is the environmental capacity for eggs. It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, it is as if E represents all the aquatic compartments in which case in this term K represents a logistic law's carrying capacity for the aquatic phase that also includes the effects of competition between larvae.

We set $x = (E, M, F)^T$ and $\mathcal{D} = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x \geq 0\}$. The model (2.1)-(2.3) can be written in the form

$$\dot{x} = f(x), \tag{2.4}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^3$ represents the right hand side of (2.1)-(2.3). The map f is continuously differentiable on $\mathbb{R}^3$. Note that if $\dot{x} = f(x)$ and $x(0) \in \mathcal{D}$, then, for every $t \geq 0$, $x(t)$ is defined and belongs to $\mathcal{D}$. Setting the right hand side of (2.1)-(2.3) to zero we obtain the extinction equilibrium $\mathbf{0} = (0,0,0)^T$ and the non-trivial equilibrium $x^* = (E^*, M^*, F^*)^T$ given by

$$E^* = K(1 - \frac{1}{\mathcal{R}_0}), \tag{2.5}$$

$$M^* = \frac{(1-\nu)\nu_E}{\delta_M}E^*, \tag{2.6}$$

$$F^* = \frac{\nu\nu_E}{\delta_F}E^*, \tag{2.7}$$

where

$$\mathcal{R}_0 := \frac{\beta_E \nu \nu_E}{\delta_F(\nu_E + \delta_E)}. \tag{2.8}$$

Note that $x^* \in \mathcal{D}$ if and only if $\mathcal{R}_0 \geq 1$. Let us now recall some definitions connected to the stability of an equilibrium.

Definition 2.1 *Let $x_e \in \mathcal{D}$ be an equilibrium (of (2.4)). The equilibrium x_e is stable in $\mathcal{D}$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$(x_0 \in \mathcal{D} \text{ and } \|x(0) - x_e\| < \delta) \Longrightarrow (\|x(t) - x_e\| < \varepsilon \text{ for all } t > 0)\,. \tag{2.9}$$

The equilibrium x_e is unstable in $\mathcal{D}$ if it is not stable in $\mathcal{D}$. It is a global attractor in $\mathcal{D}$ if, for every initial data in $\mathcal{D}$, $x(t) \to x_e$ as $t \to \infty$. Finally it is globally asymptotically stable in $\mathcal{D}$ if it is both stable and a global attractor in $\mathcal{D}$.

The Jacobian of system (2.1)-(2.3) computed at the extinction equilibrium is

$$J(\mathbf{0}) = \begin{pmatrix} -(\nu_E + \delta_E) & 0 & \beta_E \\ (1-\nu)\nu_E & -\delta_M & 0 \\ \nu\nu_E & 0 & -\delta_F \end{pmatrix}. \tag{2.10}$$

Its characteristic polynomial is

$$\begin{aligned} P(\lambda) = \lambda^3 + (\nu_E + \delta_E + \delta_M + \delta_F)\lambda^2 \\ + ((\nu_E + \delta_E)\delta_F - \beta_E\nu\nu_E + \delta_M(\nu_E + \delta_E))\lambda + \delta_M((\nu_E + \delta_E)\delta_F - \beta_E\nu\nu_E). \end{aligned} \tag{2.11}$$

Its roots are $-\delta_M$ and the roots of equation

$$\lambda^2 + (\nu_E + \delta_E + \delta_F)\lambda + \delta_F(\nu_E + \delta_E)(1 - \mathcal{R}_0) = 0 \tag{2.12}$$

If $\mathcal{R}_0 < 1$, all eigenvalues of $J(\mathbf{0})$ are either negative or have negative real parts, which implies that $\mathbf{0}$ is locally asymptotically stable. If $\mathcal{R}_0 = 1$ the eigenvalues of $J(\mathbf{0})$ are $-\delta_M$, 0, and $-(\nu_E + \delta_E + \delta_F) < 0$. If $\mathcal{R}_0 > 1$, the eigenvalues of $J(\mathbf{0})$ are all real, one is strictly positive, two are strictly negative.

The global stability properties of the extinction equilibrium $\mathbf{0} = (0, 0, 0)^T$ are described in terms of the basic offspring number $\mathcal{R}_0$ of the population. The essential properties of the model (2.1)-(2.3) are summarized in the following theorem similar to [3, Theorem 7] and [4, Theorem 1].

Theorem 2.1 *The following properties hold.*

(P.1) *If $\mathcal{R}_0 \leq 1$, then $\mathbf{0} \in \mathbb{R}^3$ is a globally asymptotically stable equilibrium in $\mathcal{D}$ for (2.4);*

(P.2) *If $\mathcal{R}_0 > 1$, then the system has two equilibria $\mathbf{0}$ and x^* in $\mathcal{D}$ where x^* is stable with basin of attraction $\mathcal{D} \setminus \{x = (E, M, F)^T \in \mathbb{R}^3_+ : E = F = 0\}$ and 0 is unstable in $\mathcal{D}$ with the non negative $M - axis$ being a stable manifold.*

Proof. Let us first prove **(P.1)**. We could proceed as in the proof of [3, Theorem 7 (i)] or [4, 1) in Theorem 1] which are based on properties of monotone operators. We propose a different approach, now based on Lyapunov functions. Let $t \mapsto x(t) = (E(t), M(t), F(t))^T$ be a solution of (2.4) defined at time 0 and such that $(E(0), M(0), F(0))^T \in \mathcal{D}$. One has

$$M(t) = e^{-\delta_M t}M(0) + (1-\nu)\nu_E \int_0^t e^{-\delta_M(t-s)}E(s)\,ds, \tag{2.13}$$

which implies that

$$M(t) \leq M(0) + \frac{(1-\nu)\nu_E}{\delta_M} \sup\{E(s);\ s \geq 0\}, \tag{2.14}$$

$$\begin{aligned} M(t) \leq M(0)e^{-\delta_M t} + \frac{(1-\nu)\nu_E}{\delta_M} e^{-\delta_M t/2} \max\{E(s);\ s \in [0, t/2]\} \\ + \frac{(1-\nu)\nu_E}{\delta_M} \sup\{E(s);\ s \geq t/2\}. \end{aligned} \tag{2.15}$$

Inequality (2.14) shows that $\mathbf{0} \in \mathbb{R}^3$ is a stable equilibrium in $\mathcal{D}$ for (2.4) if $\mathbf{0} \in \mathbb{R}^2$ is a stable equilibrium in $[0,+\infty)^2$ for the subsystem in $(E,F)^T \in [0,+\infty)^2$:

$$\dot{E} = \beta_E F\left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E)\, E, \tag{2.16}$$

$$\dot{F} = \nu\nu_E E - \delta_F F. \tag{2.17}$$

Inequality (2.15) shows that $\mathbf{0} \in \mathbb{R}^3$ is a global attractor in $\mathcal{D}$ for (2.4) if $\mathbf{0} \in \mathbb{R}^2$ is a global attractor in $[0,+\infty)^2$ for the subsystem (2.16)-(2.17) in $(E,F)^T \in [0,+\infty)^2$.

Hence, in order to prove **(P.1)**, it suffices to check that $\mathbf{0} \in [0,+\infty)^2$ is globally asymptotically stable in $[0,+\infty)^2$ for the system (2.16)-(2.17). To prove this last statement, let us consider the Lyapunov function $V : [0,+\infty)^2 \to \mathbb{R}$, $y = (E,F)^T \mapsto V(y)$, defined by

$$V(y) := \delta_F E + \beta_E F. \tag{2.18}$$

Then

$$\text{V is of class } \mathcal{C}^1, \tag{2.19}$$

$$V(y) > V((0,0)^T) = 0,\ \forall y \in [0,+\infty)^2 \setminus \{(0,0)^T\}, \tag{2.20}$$

$$V(y) \to +\infty \text{ when } \|y\| \to +\infty \text{ with } y \in [0,+\infty)^2. \tag{2.21}$$

The time-derivative of V along the trajectories of (2.16)-(2.17) is

$$\dot{V} = -\left(\delta_F\left(\nu_E + \delta_E\right) - \beta_E\nu\nu_E\right) E - \frac{\delta_F\beta_E}{K} EF. \tag{2.22}$$

Let us now assume that

$$\mathcal{R}_0 \leq 1. \tag{2.23}$$

From (2.22) and (2.23) one gets

$$\dot{V} \leq -\frac{\delta_F\beta_E}{K} EF \leq 0. \tag{2.24}$$

We are going to conclude by using the LaSalle invariance principle. Let us assume that we have a trajectory $t \in \mathbb{R} \mapsto y(t) = (E(t), F(t))^T \in [0,+\infty)^2$ of (2.16)-(2.17) such that

$$\dot{V}(y(t)) = 0\ \forall t \in \mathbb{R}. \tag{2.25}$$

Then, using (2.24),

$$E(t)F(t) = 0\ \forall t \in \mathbb{R}. \tag{2.26}$$

Let us assume that there exists $t_0 \in \mathbb{R}$ such that

$$E(t_0) \neq 0. \tag{2.27}$$

Then there exists $\varepsilon > 0$ such that

$$E(t) \neq 0\ \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon), \tag{2.28}$$

which, together with (2.26), implies that

$$F(t) = 0\ \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon). \tag{2.29}$$

Differentiating (2.29) with respect to time and using (2.17) we get

$$E(t) = 0\ \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon), \tag{2.30}$$

in contradiction with (2.28). Hence

$$E(t) = 0 \ \forall t \in \mathbb{R}. \tag{2.31}$$

Differentiating (2.31) with respect to time and using (2.16) we get that

$$F(t) = 0 \ \forall t \in \mathbb{R}. \tag{2.32}$$

With the LaSalle invariance principle, this concludes the proof of **(P.1)**.

Remark 2.1 *In the case where $\mathcal{R}_0 < 1$ a simple linear strict Lyapunov function for the full system* (2.4) *is given in Remark 2.2.*

Let us now prove **(P.2)**. We first note that one has the following lemma, whose proof is obvious and is omitted.

Lemma 2.1 *Let $t \mapsto x(t) = (E(t), M(t), F(t))^T$ be a solution of* (2.4) *defined at time* 0 *and such that $(E(0), M(0), F(0))^T \in \mathcal{D}$. Then it is defined on $[0, +\infty)$. Moreover, if $E(0) \geq K$, then there exists one and only one time $t_0 \geq 0$ such that $E(t_0) = K$ and one has*

$$E(t) < K \ \forall t > t_0. \tag{2.33}$$

Thank to this lemma we are allowed to assume that $E < K$, which we do from now on. We then follow the proof of [3, Theorem 7 (ii)]. To prove the stability and basin of attraction of the non-trivial equilibrium x^* we use [14, Theorem 2.2 in Chapter 2]. This theorem applies to strongly monotone systems. The Jacobian (2.10) associated with (2.4) is not irreducible. Let us consider the subsystem for E and F, that is (2.16)-(2.17), which defines a dynamical system on $\mathbb{R}^2_+$. Its Jacobian

$$j((E,F)^T) = \begin{pmatrix} -(\nu_E + \delta_E) - \frac{\beta_E F}{K} & \beta_E(1 - \frac{E}{K}) \\ \nu\nu_E & -\delta_F \end{pmatrix} \tag{2.34}$$

is irreducible. Applying [14, Theorem 2.2 in Chapter 2] to the two dimensional interval

$$\{(E,F)^T \in \mathbb{R}^2_+ : 0 \leq E \leq E^*, 0 \leq F \leq F^*\}, \tag{2.35}$$

it follows that every solution starting in this interval, excluding the end points, converge either all to $(0,0)^T$ or all to $(E^*, F^*)^T$. As the characteristic equation of $j((0,0)^T)$ is

$$\lambda^2 + (\delta_F + \nu_E + \delta_E)\lambda + \delta_F(\nu_E + \delta_E) - \beta_E\nu\nu_E = 0, \tag{2.36}$$

its discriminant is

$$\Delta = (\nu_E + \delta_E - \delta_F)^2 + 4\beta_E\nu\nu_E \geq 0. \tag{2.37}$$

Therefore, since $\mathcal{R}_0 > 1$, $j((0,0^T))$ has one positive eigenvalue and so $(0,0)^T$ is unstable. Since $j((0,0)^T)$ is a Metzler matrix, it has a strictly positive eigenvector corresponding to the positive eigenvalue. Hence, it is not possible that all solutions converge to $(0,0)^T$. Therefore, they converge to $(E^*, F^*)^T$. The implication for the three dimensional system (2.1)-(2.3) is that all solutions starting in the interval $[\mathbf{0}, x^*]$, excluding the M-axis, converge to x^*. Using the same argument as in [4], any solution starting at a point larger than x^* converges to x^*. Since any point in $\mathcal{D} \setminus \{x = (E,M,F)^T \in \mathbb{R}^3_+ : E = F = 0\}$ can be placed between a point below x^*, but not on the M-axis, and a point above x^*, every solution starting in $\mathcal{D} \setminus \{x = (E,M,F)^T \in \mathbb{R}^3_+ : E = F = 0\}$ converges to x^*. The monotone convergence of the solutions initiated below and above x^* implies the stability of x^* as well. This concludes the proof of **(P.2)** and of Theorem 2.1. □

2.2 SIT model in mosquito population dynamics

The SIT model obtained neglecting the Allee effect from the one presented in [15] is

$$\dot{E} = \beta_E F\left(1 - \frac{E}{K}\right) - \big(\nu_E + \delta_E\big)E, \tag{2.38}$$

$$\dot{M} = (1-\nu)\nu_E E - \delta_M M, \tag{2.39}$$

$$\dot{F} = \nu\nu_E E\frac{M}{M + \gamma_s M_s} - \delta_F F, \tag{2.40}$$

$$\dot{M}_s = u - \delta_s M_s, \tag{2.41}$$

where $M_s(t) \geq 0$ is the sterilized adult male density, $\delta_s > 0$ is the death rate of sterilized adult, $u \geq 0$ is the control (density of sterile males released) at time t, and $0 < \gamma_s \leq 1$ accounts for the fact that females may have a preference for fertile males. Then, the probability that a female mates with a fertile male is $M/(M + \gamma_s M_s)$. From now on we assume that

$$\delta_s \geq \delta_M, \tag{2.42}$$

which is a biologically relevant assumption.

When applying a feedback law $u : \mathcal{D}' \to [0, +\infty)$, the closed-loop system is the system

$$\dot{x} = G(x, u(x)), \tag{2.43}$$

where

$$G(x,u) = \begin{pmatrix} \beta_E F\left(1 - \frac{E}{K}\right) - \big(\nu_E + \delta_E\big)E \\ (1-\nu)\nu_E E - \delta_M M \\ \nu\nu_E E\frac{M}{M+\gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix}. \tag{2.44}$$

Concerning the regularity of the feedback law, we always assume that

$$u \in L^\infty_{\mathrm{loc}}(\mathcal{D}'). \tag{2.45}$$

Note that, even if u is of class $\mathcal{C}^\infty$, the map $x \in \mathcal{D}' \mapsto G(x, u(x)) \in \mathbb{R}^4$ is not continuous and one needs to specify the definition of the solutions for the closed-loop system (2.43). Carathéodory solutions seem to be natural candidates. Roughly speaking, Carathéodory solutions are absolutely continuous curves that satisfy the integral version of the differential equation. These solutions are indeed useful in other contexts. However, if they can lead to robustness for small errors on the control, as shown in [2], they may not be robust with respect to arbitrary small measurement errors on the state, which is crucial for the application. To have a robustness with respect to arbitrary small measurement errors on the state, as shown in [13] (see also [9]), the good definition of the solutions for the closed-loop system (2.43) are the Filippov solutions, i.e. the solution of

$$\dot{x} \in \bigcap_{\varepsilon > 0} \bigcap_{N \in \mathcal{N}} \overline{\mathrm{conv}}\left[X\left(\big((x + \varepsilon B) \cap \mathcal{D}'\big) \setminus N\right)\right] =: Y(x), \tag{2.46}$$

where

- B is the unit ball of $\mathbb{R}^4$;
- for a set A, $\overline{\mathrm{conv}}[A]$ is the smaller closed convex set containing A;
- $\mathcal{N}$ is the set of subsets of $\mathbb{R}^4$ of zero Lebesgue measure.

Let us recall that $x : I \subset \mathbb{R} \to \mathbb{R}^4$, $t \in I \mapsto x(t) \in \mathbb{R}^4$ (where I is an interval of $\mathbb{R}$) is a solution of (2.46) if $x \in W^{1,\infty}_{\mathrm{loc}}(I)$ and is such that

$$\dot{x}(t) \in Y(x(t)) \text{ for almost every } t \in I. \tag{2.47}$$

For references about Filippov solutions, let us mention, in particular, [11,12] and [5, Chapter 1]. For the definition of stability, global attractor and asymptotic stability, we use again Definition 2.1 (with $\mathcal{D}'$ instead of $\mathcal{D}$) and take now into account all the solutions in the Filippov sense in this definition. The motivation for using Filippov solutions is given in [9, Proposition 1.4]. The global asymptotic stability in this Filippov sense implies the existence of a Lyapunov function [7]; see also [9, Lemma 2.2]. This automatically gives some robustness properties with respect to (small) perturbations (including small measurement errors on the state), which is precisely the goal of feedback laws. In fact, for many feedback laws constructed in this article, an explicit Lyapunov function will be given, which allows to quantify this robustness.

Let us emphasize that in our case the Filippov solutions of our closed-loop system enjoy the following properties

$$((E(0),F(0))=(0,0)) \Longrightarrow ((E(t),F(t))=(0,0)\ \forall t\geq 0)\,, \tag{2.48}$$

$$((E(0),F(0))\neq(0,0)) \Longrightarrow (E(t)>0,\ M(t)>0,\ F(t)>0\ \forall t>0)\,. \tag{2.49}$$

From now on, the solutions of the closed-loop systems considered in this article are always the Filippov solutions.

Proposition 2.1 (See [1]: Stability properties of the system (2.38)-(2.41)**)** *Let us assume that*

$$\mathcal{R}_0>1. \tag{2.50}$$

Then the following properties hold.

1. *If $u=0$, we have two equilibria:*
   - *the extinction equilibrium $E^*=F^*=M^*=M_s^*=0$ which is linearly unstable;*
   - *the persistence equilibrium*

$$\overline{E}=K(1-\frac{1}{\mathcal{R}_0}), \tag{2.51}$$

$$\overline{M}=\frac{(1-\nu)\nu_E}{\delta_M}\overline{E}, \tag{2.52}$$

$$\overline{F}=\frac{\nu\nu_E}{\delta_F}\overline{E}, \tag{2.53}$$

$$\overline{M}_s=0, \tag{2.54}$$

   which is locally asymptotically stable.

2. *If $u\geq 0$, then the corresponding solution (E,M,F,M_s) to System* (2.38)-(2.41) *enjoys the following stability property:*

$$\begin{cases} E(0)\in(0,\overline{E}],\\ M(0)\in(0,\overline{M}],\\ F(0)\in(0,\overline{F}],\\ M_s(0)\geq 0, \end{cases} \Longrightarrow \begin{cases} E(t)\in(0,\overline{E}],\\ M(t)\in(0,\overline{M}],\\ F(t)\in(0,\overline{F}],\\ M_s(t)\geq 0, \end{cases} \quad \textit{for all } t\geq 0. \tag{2.55}$$

   Let

$$U^*=\mathcal{R}_0\frac{K(1-\nu)\nu_E\delta_s}{4\gamma_s\delta_M}(1-\frac{1}{\mathcal{R}_0})^2. \tag{2.56}$$

   If $u(.)$ denotes a constant control function equal to some $\overline{U}>U^*$ for all $t\geq 0$, then the corresponding solution $(E(t),M(t),F(t),M_s(t))$ converges to $(0,0,0,\overline{U}/\delta_s)$ as $t\to\infty$.

Concerning the global asymptotic stability of $\mathbf{0}$ for the system (2.38)-(2.41) in $\mathcal{D}' := [0, +\infty)^4$, using a Lyapunov approach, one can get the following theorem.

Theorem 2.2 *Let $u = 0$. If $\mathcal{R}_0 < 1$, then $\mathbf{0}$ is globally asymptotically stable in $\mathcal{D}'$ for the system (2.38)-(2.41).*

Proof. Let $x = (E, M, F, M_s)^T$. We are going to conclude by applying Lyapunov's second theorem. To do so, a candidate Lyapunov function is $V : \mathcal{D}' \to \mathbb{R}_+$, $x \mapsto V(x)$, defined by

$$V(x) := \frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}_0)} F + M + M_s. \tag{2.57}$$

Note that, since $\mathcal{R}_0 < 1$,

$$V(x) > V(\mathbf{0}) = 0, \ \forall x \in \mathcal{D}' \setminus \{\mathbf{0}\}, \tag{2.58}$$

$$V(x) \to +\infty \text{ as } |x| \to +\infty \text{ with } x \in \mathcal{D}'. \tag{2.59}$$

Moreover, along the trajectories of (2.38)-(2.41),

$$\dot{V}(x) = -(\nu\nu_E + \delta_E)E - \frac{\beta_E}{K}\frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} FE - \delta_M M - \beta_E F - \delta_s M_s \\ - \frac{2\beta_E \nu\nu_E}{\delta_F(1 - \mathcal{R}_0)} \frac{\gamma_s M_s}{M + \gamma_s M_s} E \text{ if } M + M_s \neq 0. \tag{2.60}$$

From (2.57) and (2.60), one gets

$$\dot{V}(x) \leq -cV \text{ if } M + M_s \neq 0, \tag{2.61}$$

with

$$c_0 = \min\left\{ \frac{(\nu\nu_E + \delta_E)(1 - \mathcal{R}_0)}{1 + \mathcal{R}_0)}, \frac{\delta_F(1 - \mathcal{R}_0)}{2}, \delta_M, \delta_s \right\} \tag{2.62}$$

Let us point out that, for every solution $t \mapsto x(t) = (E(t), M(t), F(t), M_s(t))^T$ of the closed-loop system (2.38)-(2.41) defined at time 0 and such that $x(0) \in \mathcal{D}'$,

$$(M(0) + M_s(0) > 0) \Longrightarrow (M(t) + M_s(t) > 0, \ \forall t > 0), \tag{2.63}$$

$$(x(0) = 0) \Longrightarrow (x(t) = 0, \ \forall t \geq 0). \tag{2.64}$$

From (2.48), (2.49), (2.58), (2.64) and (2.63), one has, for every solution $t \mapsto x(t) = (E(t), M(t), F(t), M_s(t))^T$ of the closed-loop system (2.38)-(2.41) defined at time 0 and such that $x(0) \in \mathcal{D}'$,

$$V(x(t)) \leq V(x(0))e^{-c_0 t} \ \forall t \geq 0, \tag{2.65}$$

which, together with (2.58) and (2.59), concludes the proof of Theorem 2.2 (and even shows the global exponential stability and provides an estimate on the exponential decay rate c_0 given by (2.62)). □

Remark 2.2 *Note that Theorem 2.2 implies Theorem 2.1 in the case $\mathcal{R}_0 < 1$ and our proof of Theorem 2.2 provides, for this case, a (strict) Lyapunov function which is just*

$$\tilde{V}((E, M, F)^T) := \frac{1 + \mathcal{R}_0}{1 - \mathcal{R}_0} E + \frac{2\beta_E}{\delta_F(1 - \mathcal{R}_0)} F + M. \tag{2.66}$$

It would be interesting to provide Lyapunov functions for the two remaining cases $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 > 1$.

3 Global stabilization by feedback of the extinction equilibrium

3.1 Backstepping feedback

For the backstepping method, the control system has the following structure:

$$\dot{x}_1 = f(x_1, x_2), \tag{3.1}$$

$$\dot{x}_2 = u, \tag{3.2}$$

where the state is $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^m$ and the control is $u \in \mathbb{R}^m$. The key and classical theorem for backstepping is the following one (see, for instance, [8, Theorem 12.24, page 334]).

Theorem 3.1 *Assume that $f \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{R}^m, \mathbb{R}^p)$ and that the control system*

$$\dot{x}_1 = f(x_1, v), \tag{3.3}$$

where the state is $x_1 \in \mathbb{R}^p$ and the control $v \in \mathbb{R}^m$, can be globally asymptotically stabilized by means of a feedback law $x_1 \in \mathbb{R}^p \mapsto v(x_1) \in \mathbb{R}^m$ of class $\mathcal{C}^1$. Then the control system (3.1)-(3.2) can be globally asymptotically stabilized by means of a continuous feedback law $x \in \mathbb{R}^p \times \mathbb{R}^m \mapsto u(x) \in \mathbb{R}^m$.

Let $x := (E, M, F)^T$. One way to rewrite the dynamics (2.38)-(2.41) is

$$\begin{cases} \dot{x} = f(x, M_s), \\ \dot{M}_s = u - \delta_s M_s, \end{cases} \tag{3.4}$$

where

$$f(x, M_s) := \begin{pmatrix} \beta_E F\left(1 - \frac{E}{K}\right) - \left(\nu_E + \delta_E\right)E \\ (1-\nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \end{pmatrix}. \tag{3.5}$$

As f is not of class $\mathcal{C}^1$ and the feedback law has to be non-negative, we cannot directly apply the backstepping theorem. However, to build the feedback law we use the classical Lyapunov approach of the proof of Theorem 3.1 (see, for example, [8, pages 334–335]) allowing us to select an appropriate control. Unfortunately, the control that we get with this approach is not positive all the time. To get around this, using the same Lyapunov function, we propose a new feedback law that is non-negative, decreases the Lyapunov function and leads to global asymptotic stability of the extinction equilibrium.

First, consider the control system $\dot{x} = f(x, M_s)$ with the state being $x \in \mathcal{D}$ and the control being $M_s \in [0, +\infty)$. We assume that M_s is of the form $M_s = \theta M$ and study the closed-loop system

$$\dot{x} = f(x, \theta M). \tag{3.6}$$

We have

$$\begin{cases} \dot{E} = \beta_E F\left(1 - \dfrac{E}{K}\right) - \left(\nu_E + \delta_E\right)E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \dfrac{\nu\nu_E}{1 + \gamma_s\theta} E - \delta_F F. \end{cases} \tag{3.7}$$

It is a smooth dynamical system on $\mathcal{D} = [0, +\infty)^3$ which is also a positively invariant set for this dynamical system.
Setting the right hand side of (3.7) to zero we obtain the equilibrium $\mathbf{0} \in [0, +\infty)^3$ and the non-trivial equilibrium $x^{**} = (E^{**}, M^{**}, F^{**})$ given by

$$E^{**} = K(1 - \frac{1}{\mathcal{R}(\theta)}), \tag{3.8}$$

$$M^{**} = \frac{(1-\nu)\nu_E}{\delta_M} E^{**}, \tag{3.9}$$

$$F^{**} = \frac{\nu\nu_E}{\delta_F(1 + \gamma_s\theta)} E^{**}, \tag{3.10}$$

where the offspring number is now

$$\mathcal{R}(\theta) := \frac{\beta_E \nu \nu_E}{\delta_F(1+\gamma_s\theta)(\nu_E+\delta_E)} = \frac{\mathcal{R}_0}{1+\gamma_s\theta}. \tag{3.11}$$

Note that if $\mathcal{R}(\theta) \leq 1$, $\mathbf{0} \in \mathbb{R}^3$ is the only equilibrium point of the system in $\mathcal{D}$.

Our next proposition shows that the feedback law $M_s = \theta M$ stabilizes our control system $\dot{x} = f((x^T, M_s)^T)$ if $\mathcal{R}(\theta) < 1$.

Proposition 3.1 *Assume that*

$$\mathcal{R}(\theta) < 1. \tag{3.12}$$

Then $\mathbf{0}$ *is globally asymptotically stable in* $\mathcal{D}$ *for system* (3.6).

Proof. We apply Lyapunov's second theorem. To do so, we define $V : [0,+\infty)^3 \to \mathbb{R}_+$, $x \mapsto V(x)$,

$$V(x) := \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)}E + M + \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))}F. \tag{3.13}$$

As (3.12) holds,

$$\text{V is of class } \mathcal{C}^1, \tag{3.14}$$

$$V(x) > V((0,0,0)^T) = 0,\ \forall x \in [0,+\infty)^3 \setminus \{(0,0,0)^T\}, \tag{3.15}$$

$$V(x) \to +\infty \text{ when } \|x\| \to +\infty \text{ with } x \in \mathcal{D}. \tag{3.16}$$

We have

$$\dot{V}(x) = \nabla V(x) \cdot f(x, \theta M) = \begin{pmatrix} \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)} \\ 1 \\ \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))} \end{pmatrix} \cdot \begin{pmatrix} \beta_E F\left(1-\frac{E}{K}\right) - aE \\ cE - \delta_M M \\ \frac{\nu\nu_E}{1+\gamma_s\theta}E - \delta_F F \end{pmatrix}. \tag{3.17}$$

So

$$\dot{V}(x) = -\beta_E F - \delta_M M - \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)}\frac{\beta_E}{K}FE - (\nu\nu_E + \delta_E)E. \tag{3.18}$$

Then, using once more (3.12), we get the existence of $c > 0$ such that

$$\dot{V}(x) \leq -cV(x),\ \forall x \in [0,+\infty)^3. \tag{3.19}$$

This concludes the proof of Proposition 3.1. □

Let us define

$$\psi := \frac{2\beta_E \nu \nu_E}{\delta_F(1-\mathcal{R}(\theta))(1+\gamma_s\theta)}, \tag{3.20}$$

and the map $G : \mathcal{D}' := [0,+\infty)^4 \to \mathbb{R}$, $(x^T, M_s)^T \mapsto G((x^T, M_s)^T)$ by

$$G((x^T, M_s)^T) := \frac{\gamma_s \psi E(\theta M + M_s)^2}{\alpha(M+\gamma_s M_s)(3\theta M + M_s)} + \frac{((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s)}{3\theta M + M_s} + \delta_s M_s + \frac{1}{\alpha}(\theta M - M_s) \text{ if } M + M_s \neq 0, \tag{3.21}$$

$$G((x^T, M_s)^T) := 0 \text{ if } M + M_s = 0. \tag{3.22}$$

Finally, let us define the feedback law $u : \mathcal{D}' \to [0,+\infty)$, $(x^T, M_s)^T \mapsto u((x^T, M_s)^T)$, by

$$u((x^T, M_s)^T) := \max\left(0, G((x^T, M_s)^T)\right). \tag{3.23}$$

Note that u, which is Lebesgue measurable, is not continuous in $\mathcal{D}'$. However

$$\text{there exists } C>0 \text{ such that } |u(y)| \leq C\|y\| \ \forall y \in \mathcal{D}'. \tag{3.24}$$

Property (3.24) is important for the applications since it implies that the density u of sterile males released is going to be small when the state is close to $\mathbf{0}$. For instance, this is essential to reduce the number of mosquitoes necessary for a long term intervention and also to allow using the sterile mosquitoes that are no longer needed in an area where the population is already close to zero, to intervene in other zones.

This is in contrast with the constant control in Proposition 2.1. Property (3.24) also implies that $u \in L^\infty_{\text{loc}}(\mathcal{D}')$, which allows to consider Filippov solutions for the closed-loop system, i.e. the system (2.38)-(2.41) with the feedback law (3.23).

The next theorem shows that the feedback law (3.23) stabilizes the control system (2.38)-(2.41).

Theorem 3.2 *Assume that (3.12) holds. Then $\boldsymbol{0} \in \mathcal{D}'$ is globally asymptotically stable in $\mathcal{D}'$ for system* (2.38)-(2.41) *with the feedback law* (3.23)*.*

Proof. Let $\alpha > 0$ and define $W : \mathcal{D}' \to \mathbb{R}$ by

$$W((x^T, M_s)^T) := V(x) + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s} \text{ if } M + M_s \neq 0, \tag{3.25}$$

$$W((x^T, M_s)^T) := V(x) \text{ if } M + M_s = 0. \tag{3.26}$$

We have

$$W \text{ is continuous,} \tag{3.27}$$

$$W \text{ is of class } \mathcal{C}^1 \text{ on } \mathcal{D}' \setminus \big\{(E, M, F, M_s)^T \in \mathcal{D}';\ M + M_s = 0\big\}, \tag{3.28}$$

$$W((x^T, M_s)^T) \to +\infty \text{ as } \|x\| + M_s \to +\infty, \text{ with } x \in \mathcal{D} \text{ and } M_s \in [0, +\infty), \tag{3.29}$$

$$W((x^T, M_s)^T) > W(\mathbf{0}) = 0,\ \forall (x^T, M_s)^T \in \mathcal{D}' \setminus \{\mathbf{0}\}. \tag{3.30}$$

From now on, and until the end of this proof ,we assume that $(x^T, M_s)^T$ is in $\mathcal{D}'$ and until (3.40) below we further assume that

$$(M, M_s) \neq (0, 0). \tag{3.31}$$

One has

$$\begin{aligned}
\dot W((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, M_s) \\
&\quad + \alpha(\theta M - M_s) \tfrac{2(\theta \dot M - \dot M_s)(\theta M + M_s) - (\theta \dot M + \dot M_s)(\theta M - M_s)}{(\theta M + M_s)^2}, \\
&= \nabla V(x) \cdot f(x, \theta M) + \nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) \\
&\quad + \alpha(\theta M - M_s) \tfrac{\theta \dot M(\theta M + 3M_s) - \dot M_s(3\theta M + M_s)}{(\theta M + M_s)^2}. \\
\nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) &= \begin{pmatrix} \frac{1+\mathcal{R}(\theta)}{1-\mathcal{R}(\theta)} \\ 1 \\ \frac{2\beta_E}{\delta_F(1-\mathcal{R}(\theta))} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \frac{\nu\nu_E\gamma_s E(\theta M - M_s)}{(M + \gamma_s M_s)(1+\gamma_s\theta)} \end{pmatrix} \\
&= \frac{\psi\gamma_s E(\theta M - M_s)}{M + \gamma_s M_s},
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\dot W((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, \theta M) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \\
&\quad \Big[\frac{(\nabla V(x) \cdot (f((x^T, M_s)^T) - f(x, \theta M)))(\theta M + M_s)^2}{\alpha(\theta M - M_s)} \\
&\qquad + \theta \dot M(\theta M + 3M_s) - \dot M_s(3\theta M + M_s)\Big] \\
&= \dot V(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2}\Big[\frac{\psi\gamma_s E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} + ((1-\nu)\nu_E\theta E - \theta\delta_M M)(\theta M + 3M_s) \\
&\quad - u(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s)\Big].
\end{aligned} \tag{3.33}$$

We take u as given by (3.23).
Therefore, in case

$$\frac{\psi\gamma_s E(\theta M+M_s)^2}{\alpha(M+\gamma_s M_s)}+((1-\nu)\nu_E\theta E-\theta\delta_M M)(\theta M+3M_s) \\ +\delta_s M_s(3\theta M+M_s)+\frac{1}{\alpha}(\theta M-M_s)(3\theta M+M_s)>0, \quad (3.34)$$

we have

$$u=\frac{1}{3\theta M+M_s}\Big[\frac{\psi\gamma_s E(\theta M+M_s)^2}{\alpha(M+\gamma_s M_s)}+((1-\nu)\nu_E\theta E-\theta\delta_M M)(\theta M+3M_s) \\ +\delta_s M_s(3\theta M+M_s)+\frac{1}{\alpha}(\theta M-M_s)(3\theta M+M_s)\Big],$$

which, together with (3.33), leads to

$$\dot W((x^T,M_s)^T)=\dot V(x)-\frac{(\theta M-M_s)^2(3\theta M+M_s)}{(\theta M+M_s)^2}. \quad (3.35)$$

Otherwise, i.e. if (3.34) does not hold,

$$\frac{\psi\gamma_s E(\theta M+M_s)^2}{\alpha(M+\gamma_s M_s)}+((1-\nu)\nu_E\theta E-\theta\delta_M M)(\theta M+3M_s) \\ +\delta_s M_s(3\theta M+M_s)+\frac{1}{\alpha}(\theta M-M_s)(3\theta M+M_s)\le 0, \quad (3.36)$$

so, by (3.23),

$$u=0. \quad (3.37)$$

We consider two cases:
Case 1: $\theta M>M_s$
Using (3.33), (3.36) and (3.37)

$$\dot W((x^T,M_s)^T)\le\dot V(x)-\frac{(\theta M-M_s)^2(3\theta M+M_s)}{(\theta M+M_s)^2}. \quad (3.38)$$

Case 2: $\theta M\le M_s$
Using once more (3.33) and (3.37)

$$\dot W((x^T,M_s)^T)=\dot V(x)+\alpha\frac{(\theta M-M_s)}{(\theta M+M_s)^2}\Big[\frac{\psi\gamma_s E(\theta M+M_s)^2}{\alpha(M+\gamma_s M_s)} \\ +\theta((1-\nu)\nu_E E-\delta_M M)(\theta M+3M_s)+\delta_s M_s(3\theta M+M_s)\Big]. \quad (3.39)$$

Using (2.42)

$$-\delta_M M(\theta M+3M_s)+\delta_s M_s(3\theta M+M_s)\ge\delta_M(M_s-\theta M)(M_s+\theta M),$$

which, together with (3.39), implies that

$$\dot W((x^T,M_s)^T)\le\dot V(x)-\alpha\delta_M\frac{(\theta M-M_s)^2}{(\theta M+M_s)}. \quad (3.40)$$

To summarize, using (3.19), (3.35), (3.38) and (3.40), one gets the existence of $c'>0$ independent of $(x^T,M_s)^T\in\mathcal{D}'$, such that

$$\dot W((x^T,M_s)^T)\le -c'W((x^T,M_s)^T) \text{ if } M+M_s\neq 0. \quad (3.41)$$

Since one still has (2.48), (2.49), (2.63) and (2.64) (for $x=(x^T,M_s^T)^T$), this proves Theorem 3.2 as in the proof of Theorem 2.2 (and, again, even gives the global exponential stability and provides an estimate on the exponential decay rate). □

Remark 3.1 *It is important to note that the backstepping feedback control* (3.23) *does not depend on the environmental capacity K, which is also an interesting feature for the field applications.*

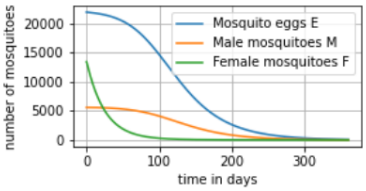
3.1.1 Numerical simulations

The numerical simulations of the dynamics when applying the feedback (3.23) gives the figures 3.1. The parameters we use are set in the following table.

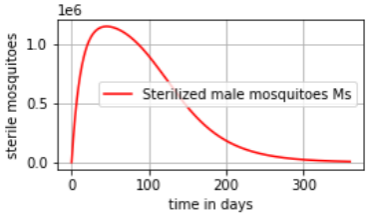
Parameter	Name	Value interval	Chosen value	Unity
β_E	Effective fecundity	7.46-14.85	10	Day^{-1}
γ_s	Mating competitiveness of sterilized males	0-1	1	-
ν_E	Hatching parameter	0.005-0.25	0.05	Day^{-1}
δ_E	Mosquitoes in aquatic phase death rate	0.023-0.046	0.03	Day^{-1}
δ_F	Female death rate	0.033-0.046	0.04	Day^{-1}
δ_M	Males death rate	0.077-0.139	0.1	Day^{-1}
δ_s	Sterilized male death rate		0.12	Day^{-1}
ν	Probability of emergence		0.49	

Table 3.1 *Value intervals of the parameters for the system* (2.38)-(2.41) *(see [15])*

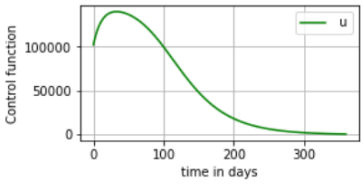
With the parameters given in the table, condition (3.12) is $\theta > 75,67$. We fix $K = 222000$ and we consider the persistence equilibrium as initial condition. That gives $E^0 = 21910, M^0 = 5587, F^0 = 13419$ and $M_s^0 = 0$. We take $\theta = 220$.



(a) Plot of E, M and F



(b) Plot of M_s



(c) Plot of the control u

Figure 3.1 *(a): Plot of E, M and F when applying the feedback* (3.23) *with the initial condition z_0. (b): Plot of M_s. (c): Plot of the feedback control function u.*

In this case, with $t_f = 360$ days,

$$\int_0^{t_f} u(t)\, dt \approx 18702985. \tag{3.42}$$

3.1.2 Robustness test

To analyze the robustness of our feedback against the variations of the parameters, we apply the feedback with the parameters fixed in the first column of tables 3.1 (original parameters). We carry out some variation of the parameters in the second column (new parameters) of table 3.2. The results are summarized in the following tables.

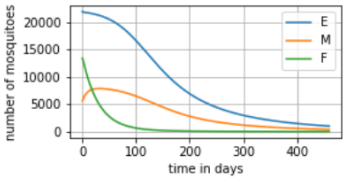
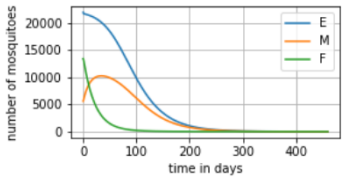
Original parameters	New parameters	Simulation
• $\nu_E = 0.05$ • $\delta_E = 0.03$ • $\delta_F = 0.04$ • $\delta_M = 0.1$ • $\delta_s = 0.12$ • $\beta_E = 8$	• $\nu_E = 0.08$ • $\delta_E = 0.046$ • $\delta_F = 0.033$ • $\delta_M = 0.11$ • $\delta_s = 0.13$ • $\beta_E = 12$	• Simulation $\theta = 220$ 
• ν_E= 0.05 • δ_E= 0.03 • δ_F= 0.04 • $\delta_M = 0.1$ • $\delta_s = 0.12$ • β_E =8	• ν_E= 0.03 • δ_E= 0.023 • δ_F= 0.046 • $\delta_M = 0.08$ • $\delta_s = 0.1$ • β_E =7	• Simulation for $\theta = 220$ 

Table 3.2 *Robustness test*

We observe that the feedback (3.23) is robust: it still stabilizes the dynamics at extinction equilibrium if the changes in the parameters are not too large.

To apply the feedback (3.23) we must estimate the number of male and female mosquitoes and the number of eggs. Some techniques used to measure these parameters are CDC light traps and BG-Sentinel traps. Based on their behavior, such as their attraction to pheromones or light, these traps use different attractants, such as light, CO2, or human odor, to capture mosquitoes. To estimate the population size and the ratio of sterile to fertile mosquitoes a common technique is to do Mark-release-recapture (MRR) studies. It consists in marking a subset of the released mosquitoes with a unique identifier and releasing then into the wild. By comparing the number of marked and unmarked mosquitoes captured in the traps, an estimate of the total population size and the ratio of sterile to fertile mosquitoes can be obtained. Some oviposition traps may be used to capture and count the number of eggs laid by female mosquitoes. In the next sections 3.2 and 3.3, we propose feedback laws depending on less variables.

3.2 Feedback laws depending on total number of male mosquitoes

Some recent adult traps are able to count automatically the number of male mosquitoes that are captured and, even in a more classic setting, there exist traps that use synthetic versions of female insect pheromones to attract and capture male insects. This kind of traps placed at different locations in the field, allow us to determine $M + M_s$ of the target pest population. Our aim in this section is to build a feedback linearly depending on $M + M_s$. Consider the closed-loop system

$$\dot{z} = F(z, u(z)),\ z = (E, M, F, M_s)^T \in \mathcal{D}', \tag{3.43}$$

where

$$u(z) = \alpha(M + M_s), \tag{3.44}$$

$$F(z,u)=\begin{pmatrix}\beta_E F\left(1-\frac{E}{K}\right)-(\nu_E+\delta_E)\,E\\(1-\nu)\nu_E E-\delta_M M\\ \nu\nu_E E\frac{M}{M+\gamma_s M_s}-\delta_F F\\ u-\delta_s M_s\end{pmatrix}, \tag{3.45}$$

and α is a fixed real number. Throughout all this section 3.2, we assume that (2.50) holds and that

$$\alpha\in[0,\delta_s). \tag{3.46}$$

The offspring number related to this system is

$$\mathcal{R}_1(\alpha):=\frac{(\delta_s-\alpha)\beta_E\nu\nu_E}{\delta_F(\nu_E+\delta_E)(\delta_s-(1-\gamma_s)\alpha)}. \tag{3.47}$$

3.2.1 Equilibria of the closed-loop system

Equilibria of the SIT model (3.43) are obtained by solving the system

$$\begin{cases}\beta_E F\left(1-\dfrac{E}{K}\right)-\big(\nu_E+\delta_E\big)E=0,\\ (1-\nu)\nu_E E-\delta_M M=0,\\ \nu\nu_E E\dfrac{M}{M+\gamma_s M_s}-\delta_F F=0,\\ \alpha M-(\delta_s-\alpha)Ms=0.\end{cases} \tag{3.48}$$

We get either the extinction equilibrium

$$\overline{E}=0,\ \ \overline{M}=0,\ \ \overline{F}=0,\ \ \overline{M}_s=0 \tag{3.49}$$

or

$$\overline{E}=K(1-\frac{1}{\mathcal{R}_1(\alpha)}),\ \overline{M}=\frac{(1-\nu)\nu_E}{\delta_M}\overline{E},\ \overline{F}=\frac{(\delta_s-\alpha)\nu\nu_E}{\delta_F((\delta_s-\alpha)+\gamma_s\alpha)}\overline{E},\ \ \overline{M}_s=\frac{(1-\nu)\nu_E\alpha}{(\delta_s-\alpha)\delta_M}\overline{E}. \tag{3.50}$$

Let us assume in the sequel that

$$\mathcal{R}_1(\alpha)<1. \tag{3.51}$$

Using (3.50) and (3.51), one gets $\overline{E}<0$ and therefore the equilibrium given by (3.50) is not relevant. In conclusion the closed-loop system (3.43) has one and only one equilibrium which is the extinction equilibrium $\mathbf{0}$. It is therefore tempting to raise the following conjecture (compare with Theorem 2.1).

Conjecture 3.1 *The extinction equilibrium $\mathbf{0}$ is globally asymptotically stable in $\mathcal{D}'$ for the closed-loop system* (3.43).

We have not been able to prove this conjecture. However

1. In section 3.2.2, we give a positively invariant set for the closed-loop system (3.43) in which, as proved in section 3.2.3, $\mathbf{0}$ is globally asymptotically stable for (3.43);
2. In section 3.2.4, we provide numerical evidence for this conjecture.

3.2.2 Invariant set of the closed-loop system

From (2.50), (3.46), and (3.51), one gets

$$\frac{\beta_E\nu\nu_E-(\nu_E+\delta_E)\delta_F}{\beta_E\nu\nu_E-(1-\gamma_s)(\nu_E+\delta_E)\delta_F}\delta_s<\alpha<\delta_s. \tag{3.52}$$

Let us define, with $z = (E, M, F, M_s)^T$,

$$\mathcal{T}_1 := \{z \in \mathcal{D}' : \beta_E F(1 - \frac{E}{K}) \leq (\nu_E + \delta_E)E\}, \tag{3.53}$$

$$\mathcal{T}_3 := \{z \in \mathcal{D}' : (1-\nu)\nu_E E \leq \delta_M M\}, \tag{3.54}$$

and, for $\kappa > 0$,

$$\mathcal{T}_2(\kappa) = \{z \in \mathcal{D}' : M \leq \kappa M_s\}. \tag{3.55}$$

One has the following theorem.

Theorem 3.3 *Assume that* (3.52) *holds and that*

$$\kappa \leq \frac{\gamma_s \delta_F(\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F(\nu_E + \delta_E)}, \tag{3.56}$$

$$\kappa \geq \frac{\delta_s - \alpha}{\alpha}. \tag{3.57}$$

Then $\mathcal{M}(\kappa) := \mathcal{T}_1 \cap \mathcal{T}_2(\kappa) \cap \mathcal{T}_3$ *is a positively invariant set of the closed-loop system* (3.43).

Remark 3.2 *Note that* (3.52) *implies that*

$$0 < \frac{\delta_s - \alpha}{\alpha} < \frac{\gamma_s \delta_F(\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F(\nu_E + \delta_E)}. \tag{3.58}$$

Hence there are $\kappa > 0$ *such that both* (3.56) *and* (3.57) *hold.*

Proof of Theorem 3.3. Let us first study the case where one starts with $E = F = 0$: we consider the Filippov solution(s) to the Cauchy problem

$$\dot{z} = F(z, u(z)),\ E(0) = 0,\ M(0) = M_0,\ F(0) = 0,\ M_s(0) = M_{s0}, \tag{3.59}$$

where $(M_0, M_{s0})^T \in [0, +\infty)^2$ is such that

$$M_0 \leq \kappa M_{s0}. \tag{3.60}$$

From (3.44), (3.45), and (3.59), one gets

$$E(t) = F(t) = 0,\ \forall t \geq 0, \tag{3.61}$$

$$\dot{M} = -\delta_M M \text{ and } \dot{M}_s = \alpha M - (\delta_s - \alpha)M_s. \tag{3.62}$$

In particular, for every $t \geq 0$, $z(t) \in \mathcal{T}_1 \cap \mathcal{T}_3$. It remains to check that

$$z(t) \in \mathcal{T}_2(\kappa)\ \forall t \geq 0. \tag{3.63}$$

From (3.62), one has

$$\frac{d}{dt}(M - \kappa M_s) = -(\delta_M + \kappa\alpha)(M - \kappa M_s) - \kappa((1+\kappa)\alpha - \delta_s + \delta_M)M_s. \tag{3.64}$$

From (3.57) one has

$$(1+\kappa)\alpha - \delta_s + \delta_M \geq \delta_M. \tag{3.65}$$

Property (3.63) readily follows from (3.60), (3.64) and (3.65).

Let us now deal with the case where $E + F > 0$. Note that, for $z \in \mathcal{M}(\kappa)$, this implies that

$$E > 0 \text{ and } M > 0. \tag{3.66}$$

Until the end of the proof of Theorem 3.3 we assume that $z \in \mathcal{D}'$ and is such that (3.66) holds. Let $h_1 : \mathcal{D}' \to \mathbb{R}$ be defined by

$$h_1(z) := \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E. \tag{3.67}$$

Its time derivative along the solution of the closed-loop system (3.43) is

$$\begin{aligned}\dot{h}_1(z) = \beta_E \nu \nu_E E \frac{M}{M + \gamma_s M_s}(1 - \frac{E}{K}) - \delta_F \beta_E F(1 - \frac{E}{K}) - \frac{\beta_E^2 F^2}{K}(1 - \frac{E}{K}) \\ + \frac{\beta_E(\nu_E + \delta_E)EF}{K} - (\nu_E + \delta_E)\beta_E F(1 - \frac{E}{K}) + (\nu_E + \delta_E)^2 E. \end{aligned} \tag{3.68}$$

For a set $\Sigma \subset \mathcal{D}'$, let us denote by $\partial\Sigma$ its boundary in $\mathcal{D}'$. On $\partial\mathcal{T}_1$, $\beta_E F(1 - E/K) = (\nu_E + \delta_E)E$. Hence

$$\dot{h}_1(z) = \beta_E \nu \nu_E E \frac{M}{M + \gamma_s M_s}(1 - \frac{E}{K}) - \delta_F(\nu_E + \delta_E)E \text{ if } z \in \partial\mathcal{T}_1. \tag{3.69}$$

In particular, using (3.56),

$$\dot{h}_1(z) \leq -\beta_E \nu \nu_E \frac{M}{M + \gamma_s M_s} \frac{E^2}{K} < 0 \text{ if } z \in \partial\mathcal{T}_1 \cap \mathcal{T}_2(\kappa). \tag{3.70}$$

Let us now turn to the behavior of the closed-loop system on the $\partial\mathcal{T}_2(\kappa)$. Let $h_2 : \mathcal{D}' \to \mathbb{R}$ be defined by

$$h_2(z) := M - \kappa M_s. \tag{3.71}$$

Its time derivative along the solution of the closed-loop system (3.43) is

$$\dot{h}_2(z) = (1 - \nu)\nu_E E - \delta_M M - \kappa\left(\alpha M - (\delta_s - \alpha) M_s\right), \tag{3.72}$$

which leads to

$$\dot{h}_2(z) = (1 - \nu)\nu_E E - ((1 + \kappa)\alpha - \delta_s + \delta_M)M \text{ if } z \in \partial\mathcal{T}_2(\kappa). \tag{3.73}$$

From (3.54), (3.65), and (3.73), one gets that

$$\dot{h}_2(z) \leq 0 \text{ if } z \in \mathcal{T}_3 \cap \partial\mathcal{T}_2(\kappa). \tag{3.74}$$

Finally, let us study the behavior of the closed-loop system on the $\partial\mathcal{T}_3$. Let $h_3 : \mathcal{D}' \to \mathbb{R}$ be defined by

$$h_3(z) := (1 - \nu)\nu_E E - \delta_M M. \tag{3.75}$$

Its time derivative along the solution of the closed-loop system (3.43) is

$$\dot{h}_3(z) = \beta_E F\left(1 - \frac{E}{K}\right) - \big(\nu_E + \delta_E\big)E - \delta_M\left((1 - \nu)\nu_E - \delta_M M\right), \tag{3.76}$$

which leads to

$$\dot{h}_3(z) = \beta_E F\left(1 - \frac{E}{K}\right) - \big(\nu_E + \delta_E\big)E \text{ if } z \in \partial\mathcal{T}_3. \tag{3.77}$$

In particular,

$$\dot{h}_3(z) \leq -\beta_E \frac{EF}{K} \leq 0 \text{ if } z \in \mathcal{T}_2(\kappa) \cap \partial\mathcal{T}_3. \tag{3.78}$$

This concludes the proof of Theorem 3.3. □

3.2.3 Global asymptotic stability result

Let

$$\overline{\kappa} := \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}, \tag{3.79}$$

$$\mathcal{M} := \mathcal{M}(\overline{\kappa}). \tag{3.80}$$

Let us recall that, by (3.58), $\overline{\kappa}$, which clearly satisfies (3.56), satisfies also (3.57). In particular, by Theorem 3.3, $\mathcal{M}$ is positively invariant for the closed-loop system (3.43). The main result of this section is the following theorem.

Theorem 3.4 *Assume that* (3.52) *holds. Then* $\boldsymbol{0}$ *is globally asymptotically stable for the closed-loop system* (3.43) *in* $\mathcal{M}$.

Proof. The first step of the proof is the following lemma which shows that Theorem 3.4 holds with $\mathcal{M}$ replaced by $\mathcal{M}(\kappa)$ provided that (3.56) is a strict inequality and that (3.57) holds.

Lemma 3.1 *Let us assume that* (3.57) *holds and that*

$$\kappa < \frac{\gamma_s \delta_F (\nu_E + \delta_E)}{\beta_E \nu \nu_E - \delta_F (\nu_E + \delta_E)}. \tag{3.81}$$

Then $\boldsymbol{0}$ *is globally asymptotically stable for system* (3.43) *in* $\mathcal{M}(\kappa)$.

To prove this lemma we use a Lyapunov approach. Our Lyapunov function is $U : \mathcal{D}' \to \mathbb{R}_+$, $z \mapsto U(z)$,

$$U(z) = \delta_F E + \varepsilon M + \beta_E (1+\varepsilon) F + \varepsilon^2 M_s, \tag{3.82}$$

where $\varepsilon \in (0, 1]$ is a constant which will be chosen later on. One has

$$U \text{ is of class } \mathcal{C}^1, \tag{3.83}$$

$$U(z) > U(\mathbf{0}) = 0, \ \forall z \in \mathcal{D}' \setminus \{\mathbf{0}\}, \tag{3.84}$$

$$U(z) \to +\infty \text{ as } |z| \to +\infty \text{ with } z \in \mathcal{D}'. \tag{3.85}$$

Let us assume for the moment being that

$$M + M_s \neq 0. \tag{3.86}$$

Then, the time derivative of U along the solution of the closed-loop system (3.43) is

$$\begin{aligned} \dot{U}(z) = \delta_F \left(\beta_E F \left(1 - \frac{E}{K} \right) - (\nu_E + \delta_E) E \right) + \varepsilon \left((1-\nu) \nu_E E - \delta_M M \right) \\ + \beta_E (1+\varepsilon) \left(\nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \right) + \varepsilon^2 \left(\alpha M - (\delta_s - \alpha) M_s \right). \end{aligned} \tag{3.87}$$

In particular,

$$\begin{aligned} \dot{U}(z) \leq -\varepsilon \delta_F \beta_E F - \left((\nu_E + \delta_E) - \varepsilon (1-\nu) \nu_E - \beta_E (1+\varepsilon) \nu \nu_E \frac{\kappa}{\kappa + \gamma_s} \right) E \\ - \varepsilon \left(\delta_M - \varepsilon \alpha \right) M - \varepsilon^2 (\delta_s - \alpha) M_s \text{ if } z \in \mathcal{M}(\kappa). \end{aligned} \tag{3.88}$$

Let us now point out that (3.81) implies that

$$\beta_E \nu \nu_E \frac{\kappa}{\kappa + \gamma_s} < (\nu_E + \delta_E). \tag{3.89}$$

From (3.88) and (3.89) one gets that for $\varepsilon > 0$ small enough there exists $c(\varepsilon) > 0$ independent of $z \in \mathcal{M}(\kappa)$ such that

$$\dot{U}(z) \leq -c(\varepsilon)U(z) \text{ if } z \in \mathcal{M}(\kappa). \tag{3.90}$$

It remains to remove assumption (3.86). Let $t \mapsto z(t) = (E(t), M(t), F(t), M_s(t))^T$ be a Filippov solution of the closed loop for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}(\kappa)$. We observe that if $(E_0, F_0) = (0, 0)$, then $z(0) \in \mathcal{M}(\kappa)$ implies that $M_0 > 0$, from which one gets that $M(t) > 0$ for every $t \geq 0$. Hence (3.90) holds for every $t \geq 0$. While, if $(E_0, F_0) \neq (0, 0)$, then $M(t) > 0$ for every $t > 0$. In particular, one still has (3.88) and therefore (3.90) for every $t > 0$. Hence,

$$U(z(t)) \leq e^{-c(\varepsilon)t}U(z(0)),\ \forall t \geq 0, \tag{3.91}$$

which, together with (3.84) and (3.85), concludes the proof of Lemma 3.1.

Let us now deduce from Lemma 3.1 that

$$\mathbf{0} \text{ is a global attractor for the closed-loop system (3.43) in } \mathcal{M}. \tag{3.92}$$

Let $z(t) = (E(t), M(t), F(t), M_s(t))^T$ be a Filippov solution of the closed-loop system (3.43) for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}(\kappa)$. If $(E_0, F_0) = (0, 0)$ then one has (3.61) and (3.62) which leads to $z(t) \to \mathbf{0}$ as $t \to +\infty$ (note that, by (3.58), $\delta_s - \alpha > 0$). Let $h_2 : \mathcal{D}' \to \mathbb{R}$ be defined by

$$\overline{h}_2(z) := M - \overline{\kappa}M_s. \tag{3.93}$$

Note that, if for some $t_0 \geq 0$, $\overline{h}_2(z(t)) < 0$, then there exists $\kappa > 0$ satisfying (3.57) and (3.81) such that $z(t_0) \in \mathcal{M}(\kappa)$. By Lemma 3.1 one then has $z(t) \to \mathbf{0}$ as $t \to +\infty$. If there is no such t_0, then

$$\overline{h}_2(z(t)) = 0 \text{ for every } t \geq 0. \tag{3.94}$$

From (3.73) with $\kappa = \overline{\kappa}$, (3.79), (3.93), and (3.94), one gets that

$$h_3(z(t)) = 0 \text{ for every } t \geq 0, \tag{3.95}$$

which together with (3.78) implies that

$$E(t)F(t) = 0 \text{ for every } t \geq 0. \tag{3.96}$$

Since $z(t) \in \mathcal{T}_1$, (3.53) and (3.96) imply that

$$F(t) = 0 \text{ for every } t \geq 0. \tag{3.97}$$

Then, if for some $t_0 \geq 0$, $E(t_0) = 0$, one has $(E(t_0), F(t_0)) = (0, 0)$, which, as already pointed out above, implies that $z(t) \to \mathbf{0}$ as $t \to +\infty$. It remains to handle the case where

$$E(t) > 0 \text{ for every } t \geq 0. \tag{3.98}$$

In particular, since $z(t) \in \mathcal{T}_3$, one has, using (3.54),

$$M(t) > 0 \text{ for every } t \geq 0. \tag{3.99}$$

Then, differentiating (3.97) with respect to time and using (3.43) and (3.45), one gets

$$E(t) = 0 \text{ for every } t \geq 0, \tag{3.100}$$

which leads to a contradiction with (3.98). This concludes the proof of (3.92).

In order to end up the proof of Theorem 3.4 it just remains to check that

$$\mathbf{0} \text{ is stable for the closed-loop system (3.43) in } \mathcal{M}. \tag{3.101}$$

For that, let $\overline{U} : \mathcal{D}' \to \mathbb{R}_+$, $z \mapsto \overline{U}(z)$, be defined by

$$\overline{U}(z) = \delta_F E + \beta_E F, \tag{3.102}$$

which corresponds to the definition of U given in (3.82) with $\varepsilon = 0$. Let $z(t) = (E(t), M(t), F(t), M_s(t))^T$ be a Filippov solution of the closed loop for the initial condition $z(0) = (E_0, M_0, F_0, M_{s0})^T \in \mathcal{M}$. As above, we maysrestrict our attention to the case where

$$E(t) > 0 \text{ for every } t > 0. \tag{3.103}$$

Let us recall that since $z(t) \in \mathcal{M} \subset \mathcal{T}_3$, (3.54), and (3.103) imply that

$$M(t) > 0 \text{ for every } t \geq 0. \tag{3.104}$$

Then, $\overline{U}(z(t))$ can be differentiated with respect to time and one has, by (3.88) with $\varepsilon = 0$ and $\kappa = \overline{\kappa}$, and (3.79),

$$\dot{\overline{U}}(z(t)) \leq 0, \tag{3.105}$$

which shows that

$$E(t) + F(t) \leq \frac{\max\{\delta_E, \delta_F\}}{\min\{\delta_E, \delta_F\}} \left(E(0) + F(0)\right), \text{ for every } t \geq 0. \tag{3.106}$$

It remains to estimate $M(t)$ and $M_s(t)$. Using $z(t) \in \mathcal{T}_2(\overline{\kappa})$ and (3.55), one already has

$$M(t) \leq \overline{\kappa} M_s(t) \text{ for every } t \geq 0. \tag{3.107}$$

Using (3.43), (3.44), (3.45), (3.58), (3.79), and (3.107), one has

$$\dot{M}_s(t) \leq (\alpha\overline{\kappa} + \alpha - \delta_s) M_s(t) \leq 0 \text{ for every } t \geq 0. \tag{3.108}$$

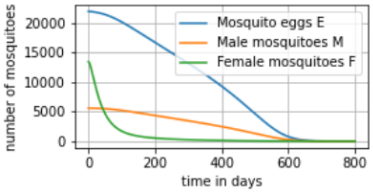
In particular, using also (3.107),

$$M_s(t) \leq M_s(0) \text{ and } M(t) \leq \overline{\kappa} M_s(0) \text{ for every } t \geq 0. \tag{3.109}$$

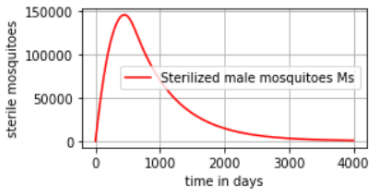
This concludes the proof of (3.101) and, therefore, of Theorem 3.4. □

3.2.4 Numerical simulations

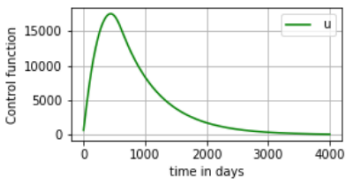
In this section, we will show numerical simulations of the dynamics when we apply feedback (3.44). We fix $z_0 = (21910, 5587, 13419, 0) \notin \mathcal{M}$. We now compute condition (3.52) according to the parameter set in the table 3.1. This gives $0.11843 < \alpha < 0.12$. We take $\alpha = 0.11843$. The following figures show the evolution of the states when condition (3.52) holds.



(a) Plot of E, M and F



(b) Plot of M_s



(c) Plot of the control u

Figure 3.2 *(a): Plot of E, M and F for system (2.38)-(2.41) when applying feedback (3.44). with the initial condition $z_0 \notin \mathcal{M}$ and final time $T = 800$. (b): Plot of M_s for final time $T = 4000$ when we apply the backstepping feedback (3.44). (c): Plot of the feedback control function (3.44).*

Remark 3.3 *We observe that the convergence time of the states E, M and F is longer than when we applied the backstepping feedback control* (3.23)*. In this case, with $t_f = 700$ days,*

$$\int_0^{t_f} u(t)\, dt \approx 17916614. \tag{3.110}$$

We take several initial conditions randomly and plot the resulting dynamics in figure 3.3,

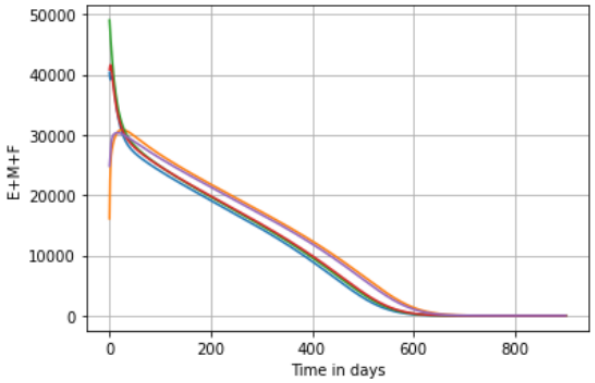


Figure 3.3 *Plot of $\|x(x_0, t)\|_1$ when applying the feedback* (3.44) *with several randomly chosen initial conditions x_0.*

3.2.5 Robustness test

To analyze the robustness of our feedback against variations of the parameters, we carry out some variation of the parameters (new values) in table 3.3. The results are summarized in table 3.3.

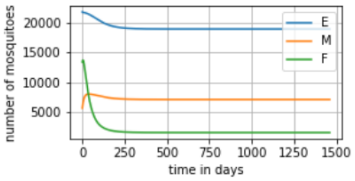
Old parameters	New Parameters	Simulation
• ν_E= 0.05 • δ_E= 0.03 • δ_F= 0.04 • δ_M = 0.1 • δ_s = 0.12 • β_E =8	• ν_E= 0.08 • δ_E= 0.046 • δ_F= 0.033 • δ_M = 0.12 • δ_s = 0.139 • β_E =11	• Plot of E, M and F 

Table 3.3 *Robustness test*

We observe that very small perturbations of the parameters destabilize the origin.

3.3 Feedback laws depending on wild male mosquitoes

In the application of the technique it might also be possible to estimate only fertile males. For instance, in MRR experiments, sterile mosquitoes are identified by the presence of a marker, such as a dye or a fluorescent protein, which has been applied before their release (although, at present, it is not always easy to do this for all the mosquitoes released in field interventions). Nevertheless, since the technology is evolving very fast, it is possible that in can become standard practice in the near future (for instance, we recall that PCR analysis of the captured mosquitoes is already currently

used thanks to genetic bar-coding). Thus, it is interesting to set up the mathematical techniques to deal with this situation. Therefore, we consider in this section the case where the feedback depends only on the state M. Consider the closed-loop system

$$\dot{z} = F(z, u(z)),\ z = (E, M, F, M_s)^T \in \mathcal{D}', \tag{3.111}$$

where

$$u(z) = \lambda M \tag{3.112}$$

and

$$F(z, u(z)) = \begin{pmatrix} \beta_E F\left(1 - \frac{E}{K}\right) - \left(\nu_E + \delta_E\right)E \\ (1-\nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M+\gamma_s M_s} - \delta_F F \\ \lambda M - \delta_s M_s \end{pmatrix}, \tag{3.113}$$

The offspring number related to this system is

$$\mathcal{R}_2(\lambda) := \frac{\delta_s \beta_E \nu \nu_E}{\delta_F(\nu_E + \delta_E)(\delta_s + \gamma_s \lambda)}. \tag{3.114}$$

We assume that

$$\mathcal{R}_2(\lambda) < 1. \tag{3.115}$$

Note that this inequality is equivalent to

$$\lambda > \frac{(\beta_E \nu \nu_E - (\nu_E + \delta_E)\delta_F)\delta_s}{\gamma_s(\nu_E + \delta_E)\delta_F}. \tag{3.116}$$

Let us point out that the closed-loop system (3.111) is exactly the closed-loop system (3.43) if one performs the following change of variables (with natural notations):

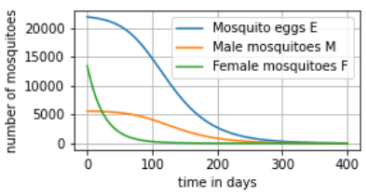
$$\alpha^{(3.43)} = \lambda^{(3.111)} \text{ and } \delta_s^{(3.43)} = \delta_s^{(3.111)} + \lambda^{(3.111)}. \tag{3.117}$$

Hence Theorem 3.3 and Theorem 3.4 lead to the following theorem.

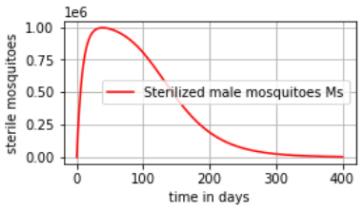
Theorem 3.5 *Assume that* (2.50) *and* (3.116) *hold. Then* $\mathcal{M}$ *is positively invariant for the closed-loop system* (3.111) *and* $\mathbf{0}$ *is globally asymptotically stable for the closed-loop system* (3.111) *in* $\mathcal{M}$.

3.3.1 Numerical simulations

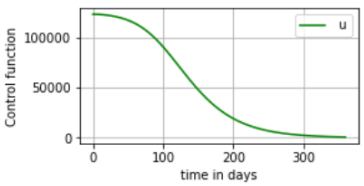
In this section, we show the numerical evolution of the states when we apply feedback (3.44). We fix as initial condition $z_0 = (21910, 5587, 13419, 0) \notin \mathcal{M}$ and $K = 22200$. We now compute condition (3.52) according to the parameters set in table 3.1. This gives $\lambda > 9.06$. We take for the simulation $\lambda = 22$.



(a) Plot of E, M and F



(b) Plot of M_s



(c) Plot of u

Figure 3.4 *(a): The results of the simulation* E, M *and* F *for system* (2.38)-(2.41) *when applying the feedback* (3.112) *with the initial condition* $z_0 \notin \mathcal{M}$ *for final time* $T = 400$ *and* $\lambda = 22$. *(b): Plot of* M_s *for final time* $T = 400$. *(c): Plot of the control function* (3.112).

Remark 3.4 *Notice that with $t_f = 400$ days,*

$$\int_0^{t_f} u(t)\, dt \approx 17289041. \tag{3.118}$$

In figure 3.5 we take several initial conditions randomly for $\lambda = 22$.

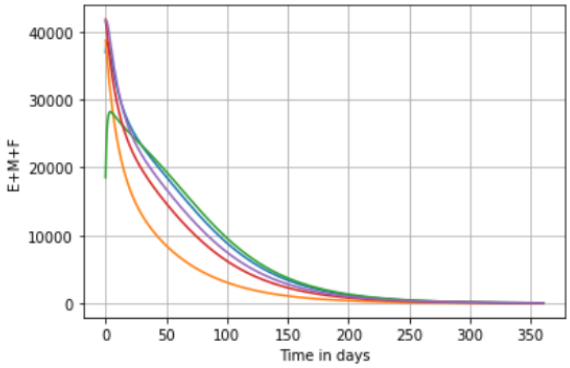


Figure 3.5 *Plot of $\|x(x_0,t)\|_1$ when applying the feedback (3.112) with several randomly chosen initial conditions x_0.*

3.3.2 Robustness test

To analyze the robustness of our feedback against the variations of the parameters, we carry out some variation of the parameters (new parameters of the following table 3.4) in the dynamics. The results are summarized in table 3.4.

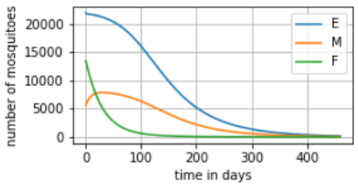
Original parameters	New Parameters	Simulation
• ν_E= 0.05 • δ_E= 0.03 • δ_F= 0.04 • δ_M = 0.1 • δ_s = 0.12 • β_E =8	• ν_E= 0.08 • δ_E= 0.046 • δ_F= 0.033 • δ_M = 0.11 • δ_s = 0.13 • β_E =12	• Plot of E, M and F for λ = 22 

Table 3.4 *Robustness test*

We observe that feedback (3.112) is robust with respect to changes of parameters: for rather large perturbations on the parameters it stills globally stabilizes the dynamics at the extinction equilibrium.

Conclusion

We have built feedback laws that stabilize the SIT dynamical model and have studied their robustness with respect to changes of parameters. We study three types of feedback laws:

1) a backstepping one in section 3.1.

2) one depending linearly on the total number of male mosquitoes, $M + M_s$ in section 3.2.

3) one depending linearly on the number of wild male mosquitoes M in section 3.3.

For the first one we were able to prove the global asymptotic stability. However, it depends on three variables (E, M and M_s) which may be difficult to measure in the field.

For the second one, we proved the global asymptotic stability only in a certain invariant set $\mathcal{M}$. We conjecture that this feedback gives global stability and we show numerical evidence for this conjecture (see figure 3.3). The advantage of this feedback law is that it depends only on the total number of male mosquitoes $M + M_s$ which is a natural quantity to measure in the field.

However, this feedback law has an important drawback due to the narrow interval allowed for the gain α of the feedback in (3.52). This might pose a problem for the robustness of this method relative to the variations of the biological parameters.

For the third one, we proved the global asymptotic stability only in a certain invariant set $\mathcal{M}$. We also conjecture that this feedback gives global stability and we show numerical evidence for this conjecture (see figure 3.5). The main difference w.r.t. the previous feedback law is that now the method is robust w.r.t. variations of the biological parameters. However, the drawback in this case is that M should be harder to measure in the field.

Also in our work, we did not consider the pest population's spatial distribution. This has again an impact in practical terms. In our future works, we will construct for this dynamics an observer that can estimate the state from easily measurable variables and we will also integrate the spatial aspect in this dynamical model.

As stated in the introduction, although the paper is mostly written for the specific case of mosquitoes, our results can be extended to the case of other pests for which the Sterile Insect Technique is pertinent,

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