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Playability of Self-Sustained Musical Instrument Models: Statistical Approaches

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Abstract

Self-sustained musical instruments, such as wind or bowed string instruments, are complex nonlinear systems. They admit a wide variety of regimes, which sometimes coexist for certain values of the control parameters. This phenomenon is known as multistability. With fixed parameters, the selection of a regime and the shape of the transient depend not only on the values of the control parameters, but also on the initial conditions. In this article, we focus on the statistical influence of initial conditions on regime selection and transient duration. An existing sample-based method called basin stability is presented to calculate the probability of occurrence of each regime. A second sample-based method is proposed for the calculation of the probability density function of transient durations. Additionally, a study taking into account specific control scenarios is presented to highlight the influence of the distribution of initial conditions considered for the statistical methods. These methods are presented on a Van der Pol oscillator seen as a prototypical musical instrument model. They are then applied to a physical model of trumpet, to demonstrate their potential for a high dimensional self-oscillating musical instrument. Finally, their interest regarding questions of playability is discussed.

Keywords: Self-sustained musical instruments, Multistability, Basin stability, Transient duration, Playability

1 Introduction

Musical instruments are complex dynamical systems. Some of them are self-sustained oscillators, meaning that a continuous energy supply can make them oscillate. Wind instruments and bowed string instruments belong to this category. These instruments

admit a wide variety of regimes, which sometimes coexist for certain values of control parameters. This is known as multistability. Multistability is a common phenomenon in self-sustained oscillators. It has been observed experimentally, theoretically and numerically on a wide variety of instruments, e.g., single and double reed instruments [1–3], flutes [4], brass instruments [5], vocal folds [6] and bowed string instruments [7]. There are multiple works focusing on mapping the operating regimes of these instruments [8]. We find in [9, 10] among the first curves representing the evolution of the amplitude of a solution as a function of a control parameter. The stability of these solutions is also studied few years later [11]. A seminal work of Dalmont et al. [12] represents bifurcation diagrams of self-sustained musical instruments. These diagrams show the evolution of known solution properties as a function of one or more control parameters. The stability of these solutions is given, as well as some key features, such as amplitude and fundamental frequency for periodic solutions. Thanks to these diagrams, one can predict the behavior of an instrument whose control parameters vary in a quasi-static manner – i.e., slowly in relation to the system’s rate of evolution. In the case of multistability, bifurcation diagrams predict hysteretic behaviors, such as those of reed and brass instruments [13] or flute-like instruments [14]. Some unexpected regimes are also predicted with this method, such as the ghost note in brass instruments [15] or the wolf tone of the cello [16]. Nevertheless, it is much more difficult to predict the regime of a multistable instrument when the control parameters vary rapidly. In particular, the blowing pressure dynamics are shown to have an influence over the regime selection in flute-like instruments [17]. Similar results are shown in [18] for the saxophone.

Similarly, once the control parameters are fixed,

78 the time required to reach the steady state depends
79 on the previous evolution of the control parameters.
80 This phenomenon was observed by [19,20] on the clar-
81 inet. Generally speaking, the duration of attack trans-
82 sients [7,21] or transitions between notes [22] are im-
83 portant topics, since the quality of a transient per-
84 ceived by a musician or a listener depends greatly on
85 its characteristics and duration [23]. The duration of
86 the transients that gives rise to a sound is thus stud-
87 ied on multiple self-sustained instruments, from the
88 trombone [24] to the cristal Baschet [25], using linear
89 stability analysis. Like the prediction of the playing
90 regime, the duration of the transient is estimated for
91 control parameters varying in a quasi-static manner.
92 Linear stability analysis cannot predict dynamic phe-
93 nomena such as those observed by [19,20]. Finally,
94 despite their high efficiency in the quasi-static regime,
95 current methods for analyzing self-oscillating instru-
96 ments are limited when considering rapidly varying
97 control parameters. By taking these rapid variations
98 into account, the dimensionality of the problem be-
99 comes infinite since the parameters can follow any
100 time series. Consequently, it motivates researchers
101 to limit their studies to specific transient scenarios.
102 For example, several authors consider pressure ramps
103 of variable slope and study the influence of the pres-
104 sure rise rate on the selected regime [26,27] or the
105 transient duration [20]. In this article, we study rapid
106 variations in control parameters through stochastic
107 initial conditions, in cases of multistability between
108 an equilibrium and a periodic regime. We are inter-
109 ested in situations where the musician moves quickly
110 from one quasi-static configuration to another, as is
111 the case during an attack [21,28], a break or a change
112 of note [22,29,30]. Quasi-statistical analysis is not
113 valid during this transition and the new quasi-static
114 configuration is studied under unknown initial condi-
115 tions. With this in mind, we study the statistical in-
116 fluence of initial conditions on the selected regime and
117 on the duration of the transient. The probability of
118 obtaining each regime is estimated with an already ex-
119 isting method called basin stability [31], and an orig-
120 inal approach is proposed to determine the probabili-
121 ty of obtaining each transient duration. Finally, the
122 distributions of initial conditions considered for these
123 statistical methods are discussed. In particular, ini-
124 tial conditions generated by specific transient control
125 scenarios are studied.

126 The remainder of the document is organized as fol-
127 lows. The system chosen to introduce the methods
128 is given in Sec.2. Sec.3.1 focuses on the basin stabil-
129 ity metric and its use for musical instruments. The
130 transient duration is investigated in Sec.3.2 and the
131 influence of the distribution of initial conditions is ex-
132 plored in Sec.4. To highlight the interest of the meth-
133 ods on more sophisticated and high dimensional sys-
134 tems, an application to a physical model of trumpet is
135 presented in Sec.5. The different methods and results

are discussed in Sec.6. Perspectives and conclusions
are drawn in Sec.7.

2 Minimal multistable system

Most physical models of self-sustained musical instru-
ments have no exact analytical solutions for transient
or steady-state regimes. Therefore, numerical meth-
ods are often used to study them, such as finite
differences [32] or harmonic balance method [33]. A
number of authors have proposed less detailed models
with exact analytical solutions and only few state
variables. These simplified models help to investigate
the mechanisms behind self-oscillations [34]. In
order to compare our statistical approach with
exact analytical results, we have chosen to illustrate
our methods on such a system. In this article, we
focus on a Van der Pol oscillator with fifth-order
nonlinearity, as described in [35]. This oscillator is
characterized by a region of multistability, where
both an equilibrium and a periodic solution are
stable. This feature can also be observed on several
musical instruments, such as saxophones [27] or brass
instruments [5]. Its phase space is of dimension
two and can therefore be displayed simply, enabling
more detailed dynamic analysis. A technical pub-
lication [36] presents the musical interest of this
system and provides a demonstrator solving it in
real time (<https://zenodo.org/records/8413627>).
Note that even though the global behavior of musical
instruments is varied and rich, the system studied
here can be obtained by change of variables and
polynomial expansions from any system presenting
this type of multistable behavior. It is only used
to illustrate the interest of a statistical approach to
predict the behavior of a multistable system under
transient excitations.

The motion of the considered oscillator is given by
the following equation:

$$\ddot{x} + [-\mu + \sigma(x^2 + \dot{x}^2) + \nu(x^2 + \dot{x}^2)^2] \dot{x} + x = 0. \quad (1)$$

We impose $\sigma = -1.5$ and $\nu = 0.1$ in order to obtain
a multistable saxophone-like or brass-like behavior.
The parameter μ is the control parameter of this sim-
ple "musical instrument" and its effect is analogous to
the blowing pressure for wind instruments [36]. A har-
monic balance method with only one harmonic gives
the amplitude X of solutions of the form:

$$x = X \cos(t + \varphi). \quad (2)$$

There are up to three solutions, depending on the
value of μ :

$$\begin{cases} X_{Eq} = 0, \\ X_{2\pi+} = \sqrt{\frac{-\sigma + \sqrt{\sigma^2 + 4\mu\nu}}{2\nu}} & \text{if } \mu \geq \mu_{SN}, \\ X_{2\pi-} = \sqrt{\frac{-\sigma - \sqrt{\sigma^2 + 4\mu\nu}}{2\nu}} & \text{if } \mu_{SN} \geq \mu \geq \mu_H, \end{cases} \quad (3)$$

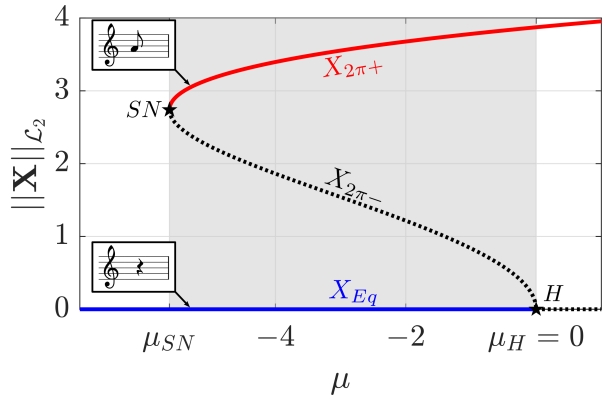


Figure 1: Bifurcation diagram of a fifth order Van der Pol oscillator Eq.(1) with respect to the control parameter μ , with $\sigma = -1.5$ and $\nu = 0.1$. Stable solutions are represented with continuous lines, unstable solutions in broken lines and bifurcations are indicated with star markers. The region of multistability is shaded in grey.

183 with $\mu_H = 0$ and $\mu_{SN} = -\frac{\sigma^2}{4\nu}$. These solutions consist of an equilibrium (X_{Eq}) and two 2π -periodic solutions ($X_{2\pi+}$ and $X_{2\pi-}$), which are represented on a bifurcation diagram in Fig.1. This oscillator shows an inverse Hopf bifurcation at $\mu = \mu_H$ and a saddle-node bifurcation at $\mu = \mu_{SN}$. Between these two bifurcations, the system is multistable: both the equilibrium X_{Eq} and the largest periodic solution $X_{2\pi+}$ are stable. Saxophones [27] and brass instruments [5] are likely to present this bifurcation sequence and therefore this multistability behavior.

194 A traditional bifurcation diagram only depicts the steady state solutions. The influence of initial conditions on the transient and the steady-state behavior is concealed. This influence can be seen on a phase plane for a given value of the control parameters. In Fig.2, we show the phase plane of the fifth order Van der Pol oscillator for $\mu = -3$, which is a situation of multistability. For the remainder of the paper, we use this value each time we need to choose a specific μ for illustration. The equation of motion Eq.(1) is numerically integrated with the function ode45 of Matlab [37], for three initial states very close to one another. In that case, the slight change in initial conditions leads the system either to the equilibrium (in blue) or to the stable limit cycle (in red). The two oscillating trajectories have different transient durations: one is almost instantaneous while the other lasts approximately one period.

212 In short, Fig.1 and 2 illustrate that the steady state and the transient duration of this minimal self-sustained musical instrument are sensitive to the initial conditions and to the control parameter value around a bifurcation. This sensitivity is merely a mark of non linearity and is shared by most non lin-

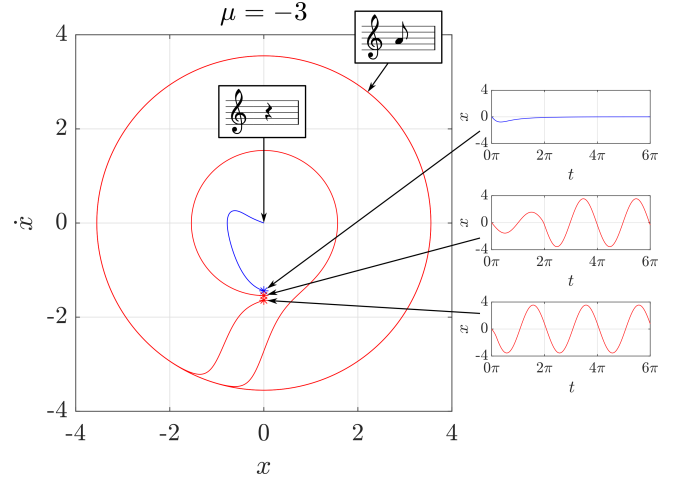


Figure 2: Illustration of the sensitivity to the initial conditions, for the Van der Pol oscillator, regarding the steady state regime and the transient duration. On the left side, the phase plane of the system for $\mu = -3$, with trajectories coming from three close initial states (indicated with star markers). On the right side, the evolution of the position x with respect to time (dimensionless).

ear dynamical systems. In the Sec.3.1 and 3.2, we present two statistical approaches to predict the selected regime and the transient duration for such a system.

3 Methods

3.1 Basin stability

Some self-sustained musical instruments are multistable. In such a case, once their control parameters are settled, their steady state depends exclusively on their initial state. The subset of initial conditions leading to a specific attractor is called its basin of attraction [38, 39]. In Fig.3, we depict the basins of attraction of the fifth order Van der Pol oscillator, for $\mu = -3$. The basin of the equilibrium is shaded in blue and the basin of the periodic solution is shaded in red. In that case, the two basins are separated by the unstable limit cycle of amplitude $X_{2\pi-}$, plotted in broken line and given in Eq.(1) and Eq.(3). In the general case, however, the geometry of a basin of attraction can be more complex and its boundaries difficult to find. We discuss this problem in Sec.6. Moreover, in the scope of this article, we consider that the initial conditions resulting from a quick change in the control parameters are unknown. They are randomly taken within a subset of the initial state space. Since the initial state is unknown, the precise geometry of the basins of attraction does not suffice to predict its playing regime.

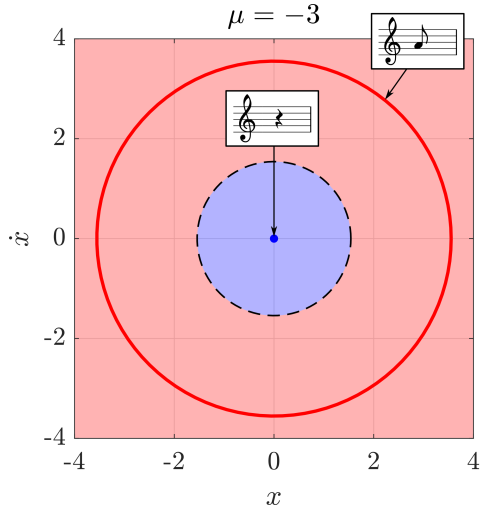


Figure 3: Basins of attraction of the fifth order Van der Pol oscillator ($\mu = -3$). Initial conditions taken in the red shaded area give rise to the periodic solution whereas initial conditions taken in the blue shaded area give rise to the equilibrium solution.

247 To take into account the uncertainty of the initial
 248 state, we propose to use the notion of *basin stability*,
 249 a probabilistic metric introduced by [31]. The prin-
 250 ciple is to evaluate the size of the basins of attrac-
 251 tion with respect to the total size of the considered
 252 space of initial states. The result gives the probability
 253 to reach each attractor, considering a random initial
 254 state. Formally, considering a subset of interest of the
 255 space of initial conditions \mathcal{Q} , an attractor A , its basin
 256 of attraction B and a probability density function ρ ,
 257 the basin stability $\mathcal{S}_B(A)$ is defined as follows:

$$\mathcal{S}_B(A) = \int_{\mathcal{Q}} \mathbf{I}_B(\mathbf{X})\rho(\mathbf{X})d\mathbf{X}, \quad (4)$$

258 where

$$\mathbf{I}_B(\mathbf{X}) = \begin{cases} 1, & \text{if } \mathbf{X} \in B, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

259 The probability density function ρ can follow a spe-
 260 cific distribution if the random choice of initial con-
 261 ditions is not uniform. This aspect is discussed in
 262 Sec.4 and 6. Due to the nature of ρ as a probability
 263 density function over \mathcal{Q} , the basin stability metric is
 264 always comprised between zero and one. $\mathcal{S}_B(A) = 0$
 265 indicates that the basin of attraction of the solution
 266 is statistically never reached. $\mathcal{S}_B(A) = 1$ means that
 267 the basin of attraction of A occupies all of the region
 268 of interest \mathcal{Q} , and that A is reached by every trajec-
 269 tory. All the illustrations of Sec.3 are realized with a
 270 uniform distribution. The volumic integral in the ex-
 271 pression Eq.(4) of \mathcal{S}_B is rarely calculable exactly and
 272 we rather compute the corresponding discrete sum.
 273 In other words, we take N random samples in \mathcal{Q} and
 274 we evaluate the proportion of samples belonging to

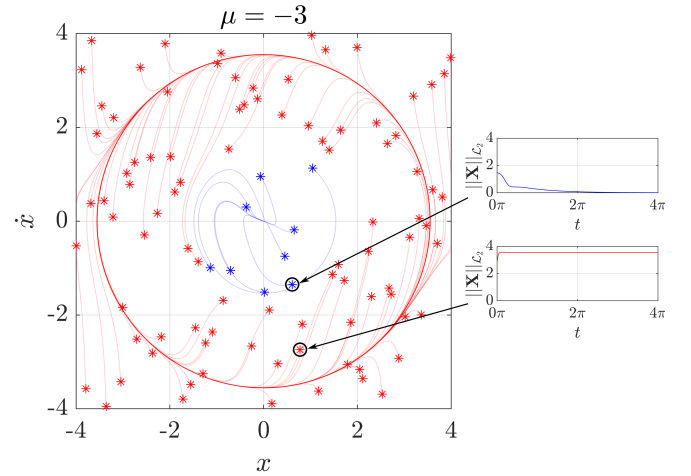


Figure 4: Illustration of the classification method in the general case. The plotted trajectories are obtained with time integrations starting from the stars as initial conditions. Red trajectories are inside the basin of attraction of the stable periodic solution $X_{2\pi+}$ whereas the blue ones belong to the basin of the equilibrium X_{Eq} .

the studied basin of attraction. The basin stability
 estimation, which is noted $\hat{\mathcal{S}}_B$, then writes:

$$\hat{\mathcal{S}}_B(A) = M/N, \quad (6)$$

where M is the number of samples belonging to B , the
 basin of attraction of A . As pointed out in [31,40,41],
 this computation corresponds to N independent trials
 with probability of success \mathcal{S}_B . The resulting standard
 error due to sub sampling writes:

$$\text{err}(\hat{\mathcal{S}}_B) = \sqrt{\frac{\mathcal{S}_B(1 - \mathcal{S}_B)}{N}}. \quad (7)$$

In practice, \mathcal{S}_B is unknown so it is replaced by its
 estimation $\hat{\mathcal{S}}_B$ in Eq.(7). This error only accounts
 for sub-sampling. It is interesting to notice that
 the error does not depend on the dimension of the
 state space. As a result, the basin stability is a
 metric suitable for low dimension problems as well
 as for high dimension ones. In general, for more
 complex systems, the basin boundaries are unknown
 and the classification method would consist in time
 integrating the system and observing toward which
 attractor it would converge. Such strategy is illus-
 trated in Fig.4. However, for this minimal multistable
 system, the basin boundary is analytically known:
 the boundary is the unstable periodic solution whose
 \mathcal{L}_2 -norm $X_{2\pi-}$ is given in Eq.(3). As a result, it
 is much faster to classify the samples with this an-
 alytical expression rather than with time integrations.

Fig.5 gives the basin stability of the Van der Pol os-
 cillator. To compute it, we arbitrarily choose a subset

302 of the phase space \mathcal{Q} which includes all the attractors
303 and which is independent of μ . For $\mu \in [-7, 1]$ as in
304 Fig.1, the subset $\mathcal{Q} = ([-4, 4] \times [-4, 4])$ meets these
305 conditions. To uniformly sample \mathcal{Q} , we use a Latin
306 Hypercube Sampling of 100 samples. The basin stability
307 of the equilibrium is plotted in blue and the one
308 corresponding to the periodic solution in red. The
309 error bars correspond to the standard error given in
310 Eq.(7). In addition, within the multistability region,
311 i.e., for $\mu_{SN} \geq \mu \geq \mu_H$, the exact basin stabilities
312 write:

$$\begin{cases} S_{\mathcal{B}}(X_{Eq}) = \frac{\pi X_{2\pi-}^2}{\text{area}(\mathcal{Q})}, \\ S_{\mathcal{B}}(X_{2\pi+}) = 1 - S_{\mathcal{B}}(X_{Eq}). \end{cases} \quad (8)$$

313 These exact solutions are represented with black lines.
314 They mainly remain inside the error bars estimated
315 with $\hat{S}_{\mathcal{B}}$ (i.e., $S_{\mathcal{B}}$ replaced by $\hat{S}_{\mathcal{B}}$ in Eq.(7)). This
316 validates the error estimation of the sample-based
317 method given in Eq.(7). Outside the multistability
318 area, the only stable solution has a basin stability
319 equal to 1 whereas the other one has a basin stability
320 equal to 0. Regarding this metric, non existing
321 solutions and unstable solutions are identical and
322 have a zero basin stability. Inside the multistability
323 area, both solutions have a non zero basin stability.
324 The stability of the periodic solution increases with
325 μ whereas the stability of the equilibrium decreases.
326 One can observe that $S_{\mathcal{B}}$ is continuous at the Hopf
327 bifurcation but discontinuous at the saddle-node
328 bifurcation. Indeed, in the first case, the basin of
329 attraction of the equilibrium shrinks progressively
330 until it becomes a dot in the phase space for $\mu = \mu_H$.
331 In the second case, the periodic solution suddenly
332 disappears at μ_{SN} , even though its basin of attrac-
333 tion had previously occupied a large part of \mathcal{Q} . The
334 value of $S_{\mathcal{B}}$ corresponds to the probability to obtain
335 the related solution, given a random initial state
336 inside \mathcal{Q} . If a musician were to impose uniformly
337 random initial conditions in $\mathcal{Q} = ([-4, 4] \times [-4, 4])$,
338 Fig.5 indicates that he/she would have at least a
339 60% chance of reaching the periodic regime in the
340 steady state. However, the assumption of a musician
341 peeking a random initial condition uniformly seems
342 oversimplified. This topic is discussed in Sec.4 and 6.

344 Overall, the basin stability indicates which values of
345 the parameters (here, the only parameter is μ) fosters
346 one solution or another. It gives the probability to
347 produce a specific regime, considering a subset \mathcal{Q} of
348 the phase space and a probability density function ρ
349 of \mathcal{Q} . The result depends greatly on the choice of \mathcal{Q}
350 and ρ . The standard error of this metric depends on
351 the number of samples and on the relative size of the
352 studied basin of attraction. It does not depend on the
353 phase space dimension however and thus is very well
354 adapted to high but finite dimensional systems.

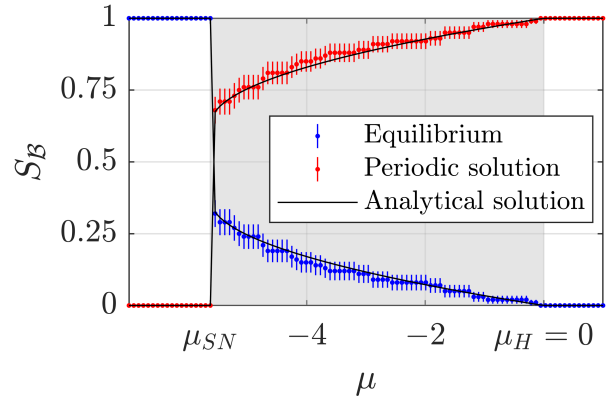


Figure 5: Basin stability of the two stable solutions
computed from 100 samples. In red: basin stability
of the periodic solution. In blue: basin stability of
the equilibrium. Error bars correspond to the stan-
dard error of the size estimation of the Monte-Carlo
method due to sub-sampling Eq.(7). In black: basin
stability computed with the analytic expression of the
basin boundary (circle of known diameter $X_{2\pi-}$ given
in (3)). The region of multistability is shaded in grey.

3.2 Transient duration

355 In music, transient phenomena play a crucial role in
356 sound and instrument discrimination [42–44]. Musicians
357 may also have expectations concerning the charac-
358 teristics of these transients, and in particular con-
359 cerning their duration. For example, Guettler and
360 Askenfelt [23] highlight the importance of the type
361 and duration of violin attacks on their quality as
362 perceived by professional string players. Several au-
363 thors [21, 22, 25, 45] assume that musicians and in-
364 strument makers look for short transients, whether
365 for the onset or for sequences of linked notes. In this
366 paper, we thus consider the reaction time of an instru-
367 ment – i.e., its tendency to produce short transients
368 – as a playability criterion. In self-sustained instru-
369 ments, the transient depends on the parameters of the
370 system but also on the initial conditions, as can
371 be seen on Fig.6. In this figure, the evolution of the
372 fifth order Van der Pol oscillator has been computed
373 for a large number of random initial states, all other
374 things being equal. In Fig.6(a), the trajectories are
375 represented in the phase plane whereas in Fig.6(b),
376 the same trajectories are represented with respect to
377 time. The transient part of a trajectory is defined be-
378 tween its initial state and the moment it reaches the
379 neighborhood of an attractor. The transient dura-
380 tion is denoted τ hereafter. The neighborhoods of the
381 two attractors are shaded in grey and are defined at a
382 distance ε in \mathcal{L}_2 -norm from the attractor. Although
383 the size of these neighborhoods has an impact on the
384 transient duration results, for now it is arbitrarily set
385 to $\varepsilon = 0.1$. Fig.6(b) highlights the diversity of tran-
386

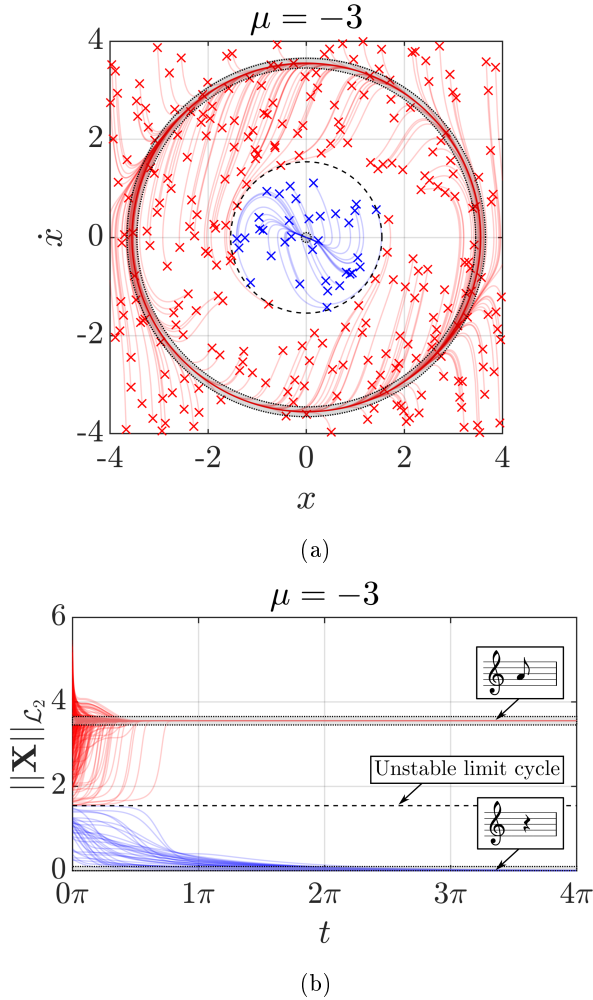


Figure 6: (a) Phase trajectories and (b) time evolution of the associated norm of the system’s state for 300 random initial conditions. Blue trajectories lead to the equilibrium and red trajectories lead to the periodic solution. The neighborhoods of the two stable solutions are shaded in grey. They are defined at a distance $\varepsilon = 0.1$ in \mathcal{L}_2 -norm from the stable solutions.

387 sient durations that can be obtained only by picking
 388 different initial conditions. Some tendencies seem to
 389 appear. For instance, for this distribution of initial
 390 conditions, trajectories leading to the equilibrium are
 391 usually longer than those leading to the periodic so-
 392 lution.

393 These tendencies stand out when the distributions
 394 of the transient durations are represented graphically.
 395 Information on a distribution can be displayed as in
 396 Fig.7. The upper panel gives the cumulative distri-
 397 bution function, i.e., the proportion of trajectories
 398 with a transient duration shorter than the value τ_i
 399 represented in abscissa. The two lower panels depict
 400 the corresponding violin plots [46, 47], where the box
 401 plots give the extreme values (tips of the whiskers),
 402 the median and the two quartiles, and the shaded
 403 curves represent the probability density function of

404 the distributions. To compute the latter, we used the
 405 kernel density estimator of Matlab (ksdensity). This
 406 statistic information is calculated for each attractor
 407 separately. As in Sec.3.1, we use a Latin Hypercube
 408 Sampling to uniformly explore the space of initial
 409 states. With this sampling, we do not control the
 410 exact number of samples in each basin of attraction,
 411 but it can be estimated with the basin stability \mathcal{S}_B
 412 previously computed. As a result, for lowest values
 413 of \mathcal{S}_B , the statistical transient analysis relies on only
 414 few samples and may not be very representative. It
 415 should be remembered that such misrepresentations
 416 occur only on highly improbable regimes. If a specific
 417 case calls for in-depth study of these regimes, the
 418 region of interest \mathcal{Q} can be adapted, leaving the
 419 method otherwise unchanged. The cumulative distri-
 420 bution function lends itself well to interpretation
 421 in musical context, where one could define a longest
 422 acceptable transient duration. The probability of the
 423 transient being shorter than this upper limit can be
 424 read directly on the cumulative distribution function,
 425 under the hypothesis of random initial conditions
 426 following a uniform law.
 427

428 We use the violin plot representation in Fig.8
 429 to depict the evolution of the transient duration
 430 distribution with respect to the control parameter
 431 μ . Plotting the probability density function over the
 432 boxplot is particularly interesting when there are
 433 multiple maxima in the distribution since multiple
 434 maxima are not visible on a box plot. For instance,
 435 some examples of bi-modal distributions can be
 436 observed around the saddle-node bifurcation (for
 437 $\mu \approx \mu_{SN}$).
 438

439 Fig.8 shows that most transients leading to the pe-
 440 riodic solution are very short compared with its period
 441 ($T = 2\pi$). Indeed, the violin plots are centered around
 442 low values of τ (a magnified view for $\mu = -3$ is visible
 443 in Fig.7). However, the maximum transient durations
 444 are markedly larger than the median duration, up to
 445 800 times longer for $\mu = -0.26$. This indicates that,
 446 for most values of μ , a few initial conditions have ex-
 447 tremely long transients. In fact, these particular ini-
 448 tial states are very close to the unstable limit cycle. In
 449 that case, the trajectory follows the unstable limit cy-
 450 cle for some time while slowly diverging from it, which
 451 lengthens the transient duration. These rare and very
 452 long durations have particularly large values in the
 453 vicinity of the Hopf bifurcation. For the equilibrium,
 454 transient durations are longer and distributed more
 455 uniformly for all the values of μ . Outside the vicinity
 456 of the two bifurcations, the transient durations lead-
 457 ing to both solutions statistically decrease as μ in-
 458 creases. This is coherent with the fact that $-\mu$ is the
 459 linear part of the damping. As μ increases, the sys-
 460 tem becomes less and less damped until μ eventually
 461 becomes positive and energy is added to the system.

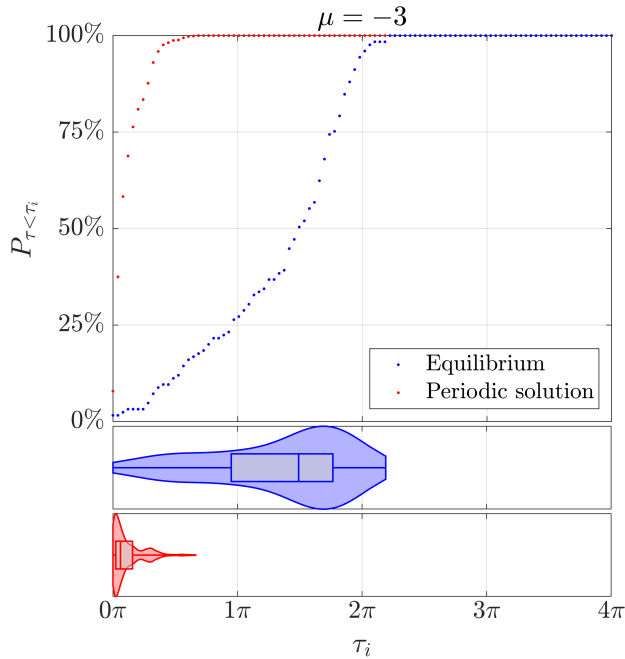


Figure 7: (Top) Cumulative probability function of the transient duration distribution. In ordinate, the proportion of trajectories with a transient duration below the value given in abscissa. (Bottom) Corresponding violin plots. The box plots indicate the extreme values, the median, the upper and lower quartiles whereas the curves around represent the probability density function of the transient duration distribution. A latin hypercube of 1000 samples is used and the transient durations are defined with $\varepsilon = 0.1$.

462 Near the saddle-node bifurcation, this trend does not
 463 apply. The equilibrium becomes the only stable solu-
 464 tion but its transient duration globally increases
 465 around μ_{SN} . This is a usual phenomenon around a
 466 saddle-node bifurcation. It is interpreted by Strogatz
 467 in [38] as the remaining *ghost* of a nearby attractor
 468 that does not exist anymore. Around the Hopf bifur-
 469 cation, the periodic solution becomes the only stable
 470 solution but its maximum transient duration signifi-
 471 cantly increases. However, apart from these very few
 472 long transients, Fig.8(b) shows that the overall trend
 473 does not change significantly for the periodic solution
 474 around the Hopf bifurcation. Considering initial con-
 475 ditions chosen from a uniform distribution, these long
 476 transients are highly improbable.

477 4 Influence of the distribution of 478 initial conditions

479 In order to provide a more meaningful analysis in a
 480 context of musical performance, we consider another
 481 distribution for the initial conditions, informed by
 482 temporal evolutions of the control parameter. More
 483 precisely, we now consider initial conditions result-

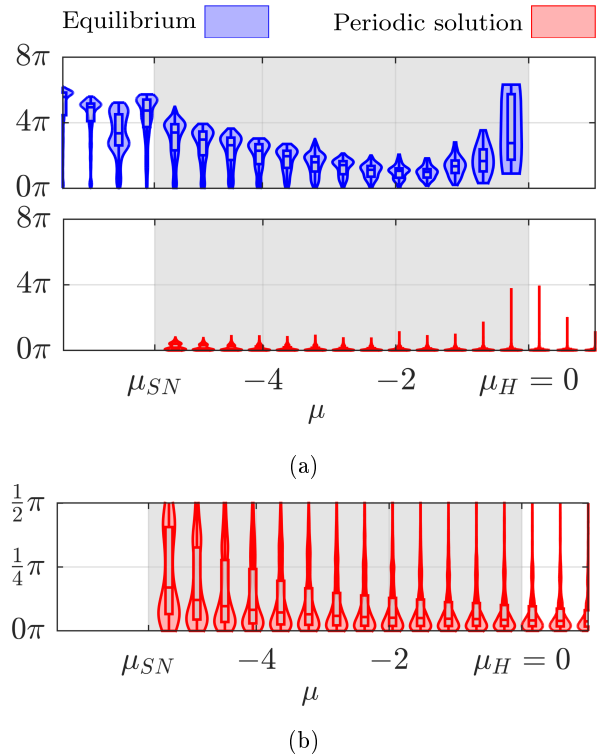


Figure 8: Violin plots of the transient duration with respect to the control parameter μ . In blue: equilibrium. In red: periodic solution. Number of samples: $N = 1000$. Panel (b) is a magnified version of the transients leading to the periodic solution. The region of multistability is shaded in gray.

484 ing from a transient variation of the control parame-
 485 ter μ , just before it reaches a constant target value.
 486 Our goal is therefore to consider what we call here-
 487 after a "transient-informed" distribution and to ana-
 488 lyze how it affects the basin stability results. To
 489 obtain a transient-informed distribution, we compute
 490 the time evolution of the system under several tran-
 491 sient variations of μ . These transient-informed dis-
 492 tributions could also be used to compute the transient
 493 duration statistics, but we only focus on the basin
 494 stability here, since the results are more remarkable.

495 4.1 Control scenarios

496 Systems such as the fifth order Van der Pol oscillator
 497 presenting an inverse Hopf bifurcation followed
 498 by a saddle-node bifurcation produce a sound of
 499 non-zero amplitude when the control parameter
 500 exceeds the Hopf bifurcation (this is the case for
 501 the trumpet and the saxophone for certain sets of
 502 parameters). Nevertheless, it is possible to reduce the
 503 sound amplitude afterward by decreasing the control
 504 parameter under the Hopf bifurcation value, into the
 505 multistability region. To produce a low-volume sound
 506 at the onset, one can attempt to realize this gesture

507 quickly by reducing the control parameter, and move
508 the system into the multistable region before the
509 limit cycle is attained. However, this strategy may
510 fail if the system remains too close to the equilibrium
511 during the transient. It would become trapped in the
512 equilibrium's basin of attraction when the parameter
513 stops varying, which would result in no stable sound
514 being produced. We consider piecewise linear control
515 scenarios, as depicted in Fig.9. These scenarios are
516 defined by four parameters: their initial value μ_0 ,
517 their maximal value μ_{max} that corresponds to what
518 can be called the overshoot value, their final value
519 μ_{end} at which the basin stability is computed and
520 their duration τ_{att} that corresponds to the "attack
521 duration". For simplicity, we choose the same
522 slope (in absolute value) for the increasing and the
523 decreasing phases (before and after the overshoot).

524
525 The order of magnitude of μ_0 is based on the bi-
526 furcation diagrams of the trumpet models presented
527 in [5]. The objective is to give as much importance to
528 the multistability as in these trumpet models. There-
529 fore μ_0 is set to have a similar ratio between the size
530 of the multistability region $[\mu_{SN}, \mu_H]$ and the size of
531 the stability region of the equilibrium $[\mu_0, \mu_H]$ as in
532 the models of [5]. μ_0 is then set to $\mu_0 = -25$. At the
533 end of the attack, $\mu = \mu_{end}$ and the basin stability is
534 computed for this value of μ . The overshoot μ_{max} and
535 the duration of the attack τ_{att} are the two remaining
536 parameters of the control scenarios. In the following,
537 we study the influence of a given overshoot μ_{max} on
538 the regime selection, while the attack duration τ_{att}
539 remains free to take different values. We arbitrarily
540 choose $\mu_{max} \in [1, 10]$ and $\tau_{att} \in [1, 20]$ (dimensionless
541 values). In [21], Ernout and Fabre measured pressure
542 rise times going approximately from 5 to 150 ms, for
543 all fingerings between C5 (1046.5 Hz) and C6 (2093
544 Hz). Therefore, by choosing $\tau_{att} \in [1, 20]$, we keep
545 a ratio between the longest and the shortest attacks
546 with the same order of magnitude as in [21].

547 4.2 Distribution generation

548 To generate a transient-informed distribution of ini-
549 tial conditions for a specific μ_{end} , we apply 10 control
550 scenarios with different durations ($\tau_{att} \in [1, 20]$) and
551 with a unique value of μ_{max} , to $N_0 = 10$ pre-initial
552 conditions chosen close to the equilibrium (Latin Hy-
553 percube Sampling taken inside $x \in [-0.1, 0.1]$ and
554 $\dot{x} \in [-0.1, 0.1]$). As a result, we obtain a transient-
555 informed distribution of $N = 100$ initial conditions
556 depicted in Fig.10. Firstly, it can be observed that
557 this distribution is far from being uniform; most
558 transient-informed initial conditions are very close to
559 either the equilibrium or the periodic solution. It is
560 expected for such a system to leave an unstable equi-
561 librium through its fast eigendirection (i.e., associated
562 with the eigenvalue of highest modulus) and to con-

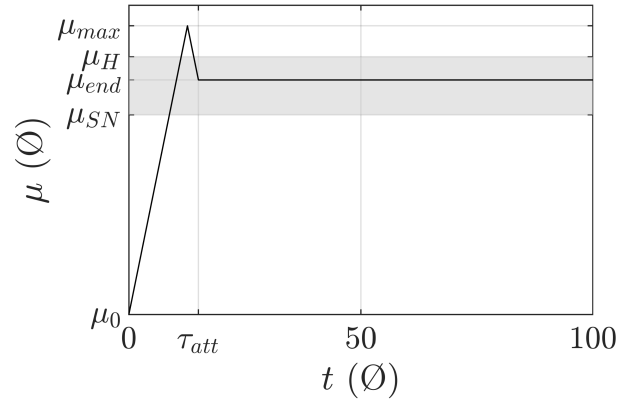


Figure 9: Control scenarios used to compute the transient-informed initial conditions. The multistability region is shaded in gray. μ_0 is the initial value of μ , μ_{max} is the overshoot, μ_{end} is the final value of μ , at which the basin stability is computed, μ_H and μ_{SN} are the value of μ at the Hopf and at the saddle-node bifurcations and τ_{att} is the attack duration.

563 verge towards a stable equilibrium through its slow
564 eigendirection (i.e., associated with the eigenvalue of
565 smallest modulus). Such behavior is explained by
566 Strogatz in example 5.2.3, p.133 of [38]. Precisely,
567 around the equilibrium in Fig.10, the states of the
568 system at $\mu = \mu_{end}$ are gathered along a specific di-
569 rection whereas the states of the system at $\mu = \mu_{max}$
570 are regrouping along another direction. In addition,
571 we plot on the same figure the fast eigendirection of
572 the equilibrium at $\mu = \mu_{max}$ and its slow eigendirec-
573 tion at $\mu = \mu_{end}$. At the overshoot, the system clearly
574 leaves the equilibrium through its fast eigendirection.
575 At the end of the attack, some of the trajectories con-
576 verge toward the equilibrium. If the simulation was
577 performed over a longer duration, they would eventu-
578 ally gather along the slow eigendirection of the equi-
579 librium.

580 4.3 Basin stability with transient- 581 informed distribution

582 The basin stability is then computed using these
583 transient-informed distributions. The results are
584 shown in Fig.11 for three different overshoot values.
585 For the sake of readability, only the basin stability
586 of the periodic solution is represented (we recall
587 that the basin stability of the equilibrium is the
588 complementary to one).

589
590 The result computed with the uniform distribution
591 is represented in black. Firstly, this figure highlights
592 that the basin stability depends greatly on the
593 considered distribution, even though some features
594 are preserved: $\mathcal{S}_B(\mu)$ is discontinuous at $\mu = \mu_{SN}$,
595 continuous at $\mu = \mu_H$ and monotonously increasing
596 for the cases where $\mu_{max} = 3$ and 4. In the case where

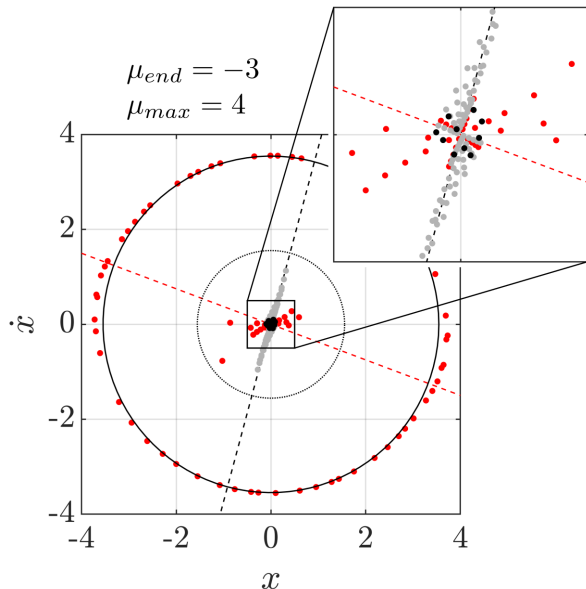


Figure 10: Example of a transient-informed distribution generated with 10 control scenarios of different durations. (●) Pre-initial Latin Hypercube of size $N_0 = 10$; (●) states of the system when $\mu = \mu_{max}$; (●) transient-informed distribution at the end of the attack. The black dotted line is the fast eigendirection of the equilibrium at $\mu = \mu_{max}$ and the red one is its slow eigendirection at $\mu = \mu_{end}$.

597 $\mu_{max} = 2$, the overshoot scenario does not suffice to
 598 leave the basin of the equilibrium and no oscillations
 599 are observed in the multistable region. Secondly, for
 600 the situation considered, this figure shows that the
 601 large-overshoot attacks are more likely to make the
 602 system end up on the periodic solution. Moreover,
 603 the basin stability increases faster when μ_{end} gets
 604 closer to μ_H . In terms of playability, a slowly varying
 605 \mathcal{S}_B can be interpreted as a region where the difficulty
 606 to produce a sound does not depend much on the
 607 final value of the control parameter. For the situation
 608 under study, this means that for a given value of
 609 μ_{max} , all the attacks with μ_{end} in this region of
 610 slow varying \mathcal{S}_B have almost the same probability
 611 to converge towards the periodic solution. As a
 612 result, if we ignore the difficulty for a musician to
 613 maintain $\mu > \mu_{SN}$, there is no additional difficulty
 614 to reducing the sound amplitude to its minimal value.
 615

616 5 Application to a trumpet model

618 In order to evaluate the interest of the methods in-
 619 troduced in previous sections to a more advanced and
 620 high dimensional model of musical instrument, the
 621 analysis of basin stability and transient durations is
 622 applied in this section to a physical model of B_b trum-

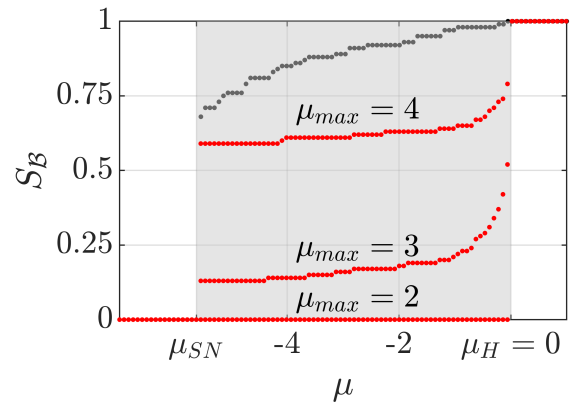


Figure 11: Basin stability of the periodic solution computed with different distributions. The red curves are computed with transient-informed distributions of initial conditions, each curve corresponding to a specific overshoot μ_{max} . The black curve is computed with a uniform distribution (it is the same as in Fig.5 without the error bars).

pet.

623

624 5.1 Presentation of the model

625 The model used is identical to the one considered
 626 in [5]. It is detailed in appendix A. The lips are rep-
 627 resented by a one-degree-of-freedom oscillator. The
 628 air column inside the body of the instrument, also
 629 known as the resonator, is modeled by a modal trun-
 630 cation of its input impedance. These two elements are
 631 coupled by a nonlinear function that models the air
 632 jet through the lip channel. In this article, we con-
 633 sider an 11-mode truncation of the resonator and fix
 634 the resonance frequency of the lips around the sec-
 635 ond impedance peak ($f_L = 200$ Hz and $f_2 = 232.7$
 636 Hz). This peak corresponds to the lowest note of the
 637 instrument, in "normal" playing conditions (the first
 638 peak being associated with the "pedal note"). The
 639 variables of this model are the position and velocity
 640 of the lip as well as the real and imaginary parts of
 641 the 11 modal pressures. This results in a system of
 642 dimensionality 24. The parameter values used in this
 643 article are given in appendix A.

644 5.2 Bifurcation diagram

645 Unlike the 5th-order Van der Pol oscillator, this trum-
 646 pet model has no analytical solutions. To determine
 647 its solutions, numerical continuation can be used.
 648 Here, we use the Manlab software [33]. The bifurca-
 649 tion diagram of the model is shown in figure 12. The
 650 amplitude of the mouthpiece pressure p and the fre-
 651 quency of the sound produced f_{play} are plotted with
 652 respect to the blowing pressure p_0 . In this zone, the
 653 system exhibits an inverse Hopf bifurcation followed
 654 by a saddle-node bifurcation, like the Van der Pol os-

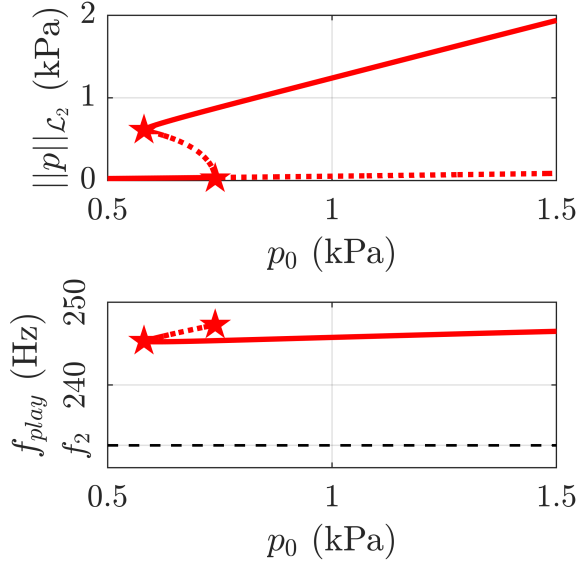


Figure 12: Bifurcation diagram of the 11-mode trumpet model, calculated with Manlab. Top pane: amplitude of the oscillations $\|p\|_{\mathcal{L}_2}$ versus blowing pressure p_0 . Bottom pane: fundamental frequency of the oscillations f_{play} versus blowing pressure p_0 . Stable solutions are shown as solid lines, unstable solutions as dotted lines. Bifurcations are represented by stars.

655 cillator presented in previous sections. The periodic
656 regime is associated with the second mode of the res-
657 onator, represented on the bottom panel of the figure
658 by a horizontal dotted line.

659 5.3 Basin stability and transient dura- 660 tions computed with bSTAB

661 To calculate the system’s basin stability and transient
662 durations, we use the bSTAB toolbox [48]. For our ap-
663 plication, we have made a number of modifications to
664 this toolbox, which are detailed in appendix B. These
665 modifications rely on convergence criteria in order to
666 identify when the system has reached steady state.
667 The numerical integration lasts until these criteria are
668 met, rather than for an arbitrary time which would
669 be identical for all trajectories. These modifications
670 allow us both to reduce the duration of the time in-
671 tegrations required to compute basin stability, and to
672 calculate the duration of transients (which is not pro-
673 vided in the original toolbox). These criteria must be
674 defined carefully, in order to avoid samples missclas-
675 sification, leading to errors in the estimation of basin
676 stability and transient durations.

677 5.3.1 Initial conditions from a uniform distri- 678 bution

679 As for the Van der Pol oscillator, we first consider
680 a uniform random distribution of initial conditions

681 \mathcal{Q}_u selected inside a hyper-rectangle. This hyper-
682 rectangle is chosen large enough to contain all the
683 solutions, whatever the value of p_0 . Its bounds are
684 set as the extrema of the stable periodic solution at
685 $p_0 = 0.8$ kPa, rounded up to the nearest integer value.
686

687 The basin stability calculated with \mathcal{Q}_u for $N = 300$
688 samples is given in figure 13(a), the transient du-
689 ration distributions are given in figure 13(b) and a
690 zoomed-in view is given in figure 13(c). The basin
691 stability shares some features with the Van der Pol
692 oscillator. Indeed, it is discontinuous at the saddle-
693 node bifurcation and remains continuous at the Hopf
694 bifurcation. The observations and comments made
695 about this feature on the simple Van der Pol oscillator
696 translate directly to this more complicated system.
697 In addition, $\mathcal{S}_{\mathcal{B}}$ is monotonous. However, unlike
698 the Van der Pol oscillator, for dynamical systems of
699 dimensionality three or more, the basins’ boundaries
700 are different than the unstable periodic solutions.
701 These objects even have different dimensionalities
702 (the basins’ boundaries are hypersurfaces and the
703 limit cycles are hyperlines). Consequently, the basin
704 stability of the equilibrium cannot be inferred from
705 a size estimate of the unstable periodic solution and
706 the sampling approach is then required.
707

708 Concerning transient durations, the Hopf bifur-
709 cation induces a few long transients in its vicinity,
710 leading both to the equilibrium and to the periodic
711 solution. This is probably related to the increasing
712 timescale of the slowest eigendirection whose stability
713 is reversed at the Hopf bifurcation. Eventually, the
714 timescale of this eigendirection goes to the infinity
715 at the Hopf bifurcation. The ghost effect around the
716 saddle-node bifurcation, where the periodic solutions
717 do not exist yet, is also visible but less than for the
718 Van der Pol oscillator. This may be due to the fact
719 that in a phase space of dimensionality two, for a
720 value of the control parameter slightly below the
721 value of the saddle-node, the trajectories initiated far
722 away from the equilibrium necessarily have to enter
723 the region of influence of this ghost limit cycle. On
724 the contrary, in a phase space of higher dimensionality,
725 these trajectories can pass away from this region.
726 Moreover, the median and the interquartile range of
727 both solutions increase around the Hopf bifurcation
728 (even though it is hardly noticeable for the periodic
729 solution with that scale).
730

731 Finally, this trumpet model sometimes exhibits
732 transients of more than several seconds, which is
733 extremely long compared to usual musical timescales.
734 Beyond a certain duration, these asymptotic regimes
735 can be considered unplayable or at least difficult to
736 play. However, the regime that is heard during one
737 of these extremely long transients can be treated
738 as a playable regime by the musician, even if it is

739 unstable. In this respect, one could see a percussion
 740 instrument as a dynamic system with only one
 741 asymptotic solution, the equilibrium, but whose
 742 transients are long enough to be used to make music.
 743 We could then consider a "transient" category for
 744 samples with a transient duration exceeding a certain
 745 "musical duration", rather than classifying them
 746 according to their asymptotic solution.

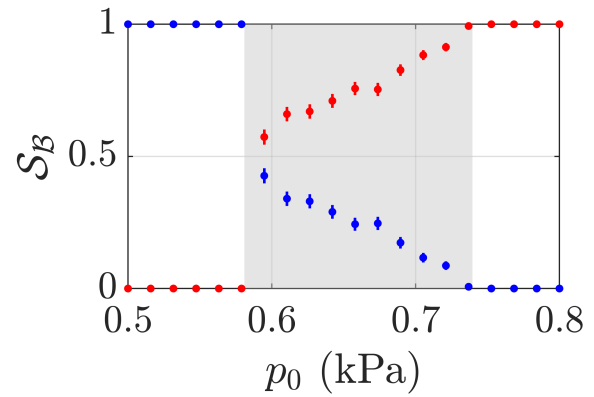
747

748 5.3.2 Initial conditions from a distribution 749 based on archetypal time variations of 750 p_0

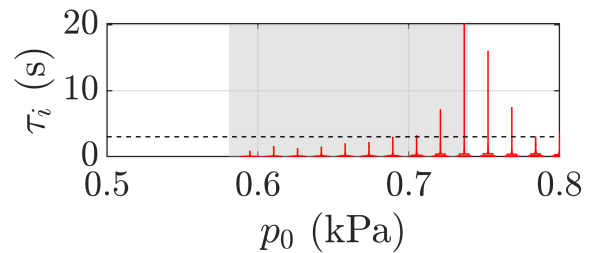
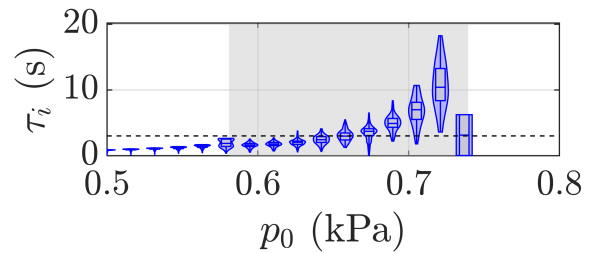
751 In the following, we study the influence of attack
 752 transients on basin stability. We apply the same
 753 method as in Sec.4 to generate distributions \mathcal{Q}_{tb}
 754 based on transient evolutions of p_0 . Again, we
 755 consider pressure rises with overshoot (see Fig.9),
 756 in the manner of a strong tonguing attack or a
 757 sforzando, as can be found in [49]. The configuration
 758 and the parameters chosen here differ from [49].
 759 Consequently, we consider slightly different control
 760 transients than those measured in that article, so
 761 that $p_{0,max} \in [1, 10]$ kPa and $\tau_{att} \in [10, 500]$ ms.
 762 Ten attack transients are applied to each sample
 763 of a uniform distribution of $N_0 = 100$ pre-initial
 764 conditions. This leads to a final distribution of
 765 $N = 1000$ initial conditions. Each variable of the
 766 pre-initial distribution is selected in ranges 0.3 times
 767 smaller than in Sec.5.3.1. These ranges are chosen
 768 arbitrarily and have a significant influence on the
 769 results.

770

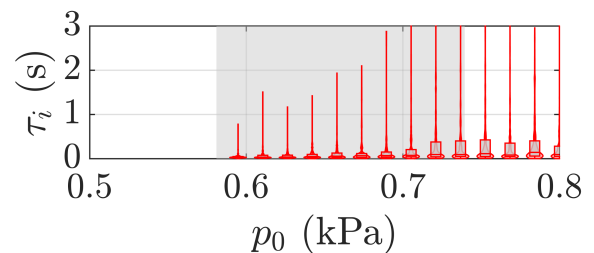
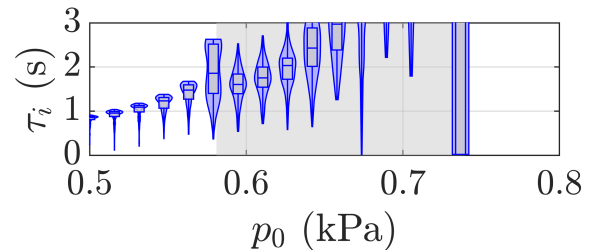
771 In Figure 14(a), we present the influence of the at-
 772 tack duration τ_{att} on the behavior of the system. Each
 773 curve has a fixed value of τ_{att} and 10 different values of
 774 $p_{0,max} \in [1, 10]$ kPa. It is striking to see that the basin
 775 stability does not evolve monotonously with τ_{att} . In-
 776 deed, one would expect that the longer the system
 777 stays in the region of monostability of the periodic
 778 solution, the higher the chances it has to end up in
 779 its basin of attraction. However, when τ_{att} increases
 780 from 10 to 50 ms, $\mathcal{S}_{\mathcal{B}}$ decreases. $\mathcal{S}_{\mathcal{B}}$ does not evolve
 781 much for $\tau_{att} \in [50, 100]$ ms, and it then increases, as
 782 expected, for $\tau_{att} > 100$ ms. These observations can
 783 be analyzed by considering recent results concerning
 784 dynamical bifurcations [50]: how close the system has
 785 come to the equilibrium when the Hopf bifurcation
 786 is crossed is crucial to predict the dynamics above
 787 the bifurcation. This closeness is the result of the
 788 time spent by the system below the bifurcation but
 789 also of the number of significant digits used in the
 790 simulation (numerical noise is therefore inevitable).
 791 This makes direct interpretation not straightforward.
 792 In figure 14(b) we study instead the influence of the
 793 overshoot amplitude $p_{0,max}$ for various attack dura-
 794 tions $\tau_{att} \in [10, 500]$ ms. In that case, $\mathcal{S}_{\mathcal{B}}$ increases



(a)



(b)



(c)

Figure 13: Basin stability (a) and transient duration distributions (b) of the trumpet model calculated with a uniform \mathcal{Q}_u distribution of $N = 300$ initial conditions. The dotted lines in (b) indicate the limit of the scale used for the zoomed view in (c). Error bars are given by the Eq.7.

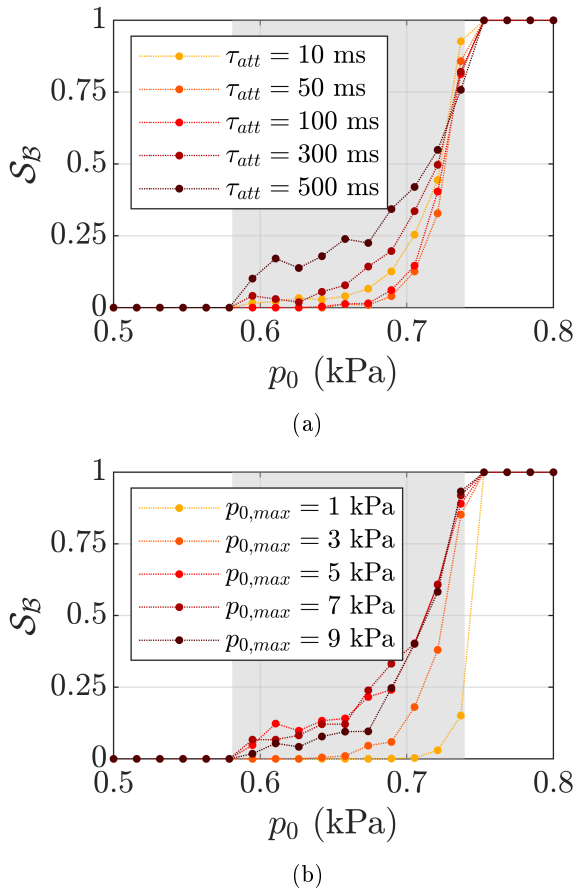


Figure 14: Basin stability of the trumpet model calculated with distributions \mathcal{Q}_{tb} of $N = 1000$ initial conditions based on attack transients (10 scenarios applied to each of the $N_0 = 100$ pre-initial conditions). (a) Each curve corresponds to a specific attack duration τ_{att} . (b) Each curve corresponds to a specific overshoot value $p_{0,max}$. Only the basin stability of the periodic regime is shown.

795 with the overshoot value, until it reaches its maximal
796 value for $p_{0,max} \in [5, 7]$ kPa. Hence, it seems that
797 there are optimal values of transient control param-
798 eters to maximize S_B . Further study would be needed
799 to understand the dynamical reasons leading to this
800 non trivial behavior.

6 Discussion

6.1 Computational cost

The methods presented in this paper allow to compute the basin stability and the transient duration distributions of a system. These metrics describe the global, statistical behavior of a system and could be very useful to study musical instruments, notably their playability. However, in order to take into account all the potential behaviors of a system, their computation can be costly. As a result, the computation cost limits the possible applications of these methods, for parametric studies for example.

Initially, [31] proposed to use Monte-Carlo techniques to numerically estimate the basin stability. Some limitations of these techniques have been drawn [40] when applied on strange basins of attraction (e.g., fractal and riddled/intermingled). However, such basins remain to be exhibited on self-sustained musical instruments. Some authors also proposed enrichment to the metric, to take into account uncertainties and variations of the parameters [51, 52]. Additionally, improvements on classical Monte Carlo approaches, notably using machine learning tools, show promise to accelerate basin stability computation. This could be interesting for more complex musical instrument models. For example, [41] proposed a method based on support vector machines. The idea is to find the basins' boundaries with a limited number of samples. Work still needs to be done to determine which method should be used depending on the situation and on the classification cost of a single sample. Indeed, if the classification of each sample is fast, it may be slower to find basins' boundaries rather than to apply a simple Monte-Carlo size estimation. For instance, it is the case for the Van der Pol oscillator presented in this article.

If the transient duration statistics and the basin stability are both studied, the two metrics can be computed simultaneously, using the same time integrations (as in Sec.5). The cost needed to compute the basin stability with Monte-Carlo methods only depends on the size of the basin as indicated by the absolute standard error given by Eq.(7). For systems with higher dimensions, that cost could increase if the classification cost does. Moreover, the relative standard error, which writes $\text{err}_{\text{rel}}(\hat{S}_B) = \sqrt{\frac{1/S_B - 1}{N}}$, increases as S_B decreases. Higher dimension systems may have more stable solutions which lead to smaller basin stability values. More samples would thus be needed to keep the relative standard error low. This standard error formula stands for independent trials with only two outcomes – the sample is either inside or outside the studied basin of attraction. The

857 calculation of the transient duration distributions
858 does not enter into this category since the outcome is
859 a continuous random variable: a transient duration.
860 With a sparse sampling, some transient behaviors
861 that appear for specific initial conditions might be
862 missed. Consequently, it could be interesting to
863 estimate the convergence of the probability density
864 functions in order to chose an adequate number of
865 samples.

866
867 The statistical methods presented here can be
868 applied applied at low computational cost on the
869 fifth order Van der Pol oscillator, because the two
870 attractors and the boundary of the basins of attraction
871 are analytically known and characterized
872 by their \mathcal{L}_2 -norm. On more complex systems,
873 the attractors and basins' boundaries usually do
874 not have analytic expressions. The main difficulty
875 encountered in carrying out the statistical analysis of
876 the trumpet model presented in Sec.5 was to define
877 adequate classification and stopping criteria. Indeed,
878 non-linear systems sometimes evolve with very varied
879 dynamics, making it difficult to distinguish between
880 a steady state and a transient regime with slow
881 dynamics. This kind of behavior leads to significant
882 differences in transient duration between different
883 initial conditions, especially at a bifurcation, where
884 one eigendirection becomes infinitely slow. These
885 discrepancies in transient durations are particularly
886 visible in Fig.13(b), around the Hopf bifurcation. In
887 this regard, the Manlab solutions were very helpful to
888 define efficient classification and convergence criteria
889 (see appendix B). For systems with a wide range of
890 transient durations, we would strongly recommend to
891 set a convergence criteria rather than to use a fixed
892 integration time, as this is the case in the version
893 of bSTAB currently online. This would improve
894 performances and limit missclassification.

896 6.2 Initial conditions and musical gestures 897

898 In Sec.3, the presented statistical approach is based
899 on a uniform distribution of initial conditions. This
900 is a general distribution and it allows different
901 instrument configurations to be compared. However,
902 the resulting probabilities probably fail to translate
903 the musician's playing experience. As a first attempt
904 to tackle this issue, we study in Sec.4 initial conditions
905 resulting from an archetypal control gesture.
906 This approach highlights the importance of the
907 considered distribution of initial conditions and it
908 can be interpreted in musical terms more directly
909 than random initial conditions. We would like to
910 draw the reader's attention to the fact that the
911 results obtained with this transient-based approach
912 not only depend on the transient control parameters,

913 but also on the choice of the pre-initial conditions.
914 Overall, applicative studies on playability would
915 greatly benefit from a preliminary description of
916 the control scenarios and, where relevant, initial
917 conditions distributions. In that perspective, it could
918 be of great interest to measure transient blowing
919 scenarios provided by humans (musicians and non
920 musicians) in an experimental study. Moreover,
921 as illustrated by Fig.10, dynamical systems are
922 more likely to cross specific regions of their phase
923 space during transients. Choosing distributions
924 of initial conditions based on the slow and fast
925 eigendirections (depending on the solutions that exist
926 during the transient) could also be an interesting idea.

927
928 In this paper, we focus on initial conditions induced
929 by the fast evolution of one control parameter.
930 However, musicians may also impact the initial
931 conditions by acting directly on a state variable. One
932 example among many others is the use of the tongue
933 in reed instruments to impose initial reed positions
934 and velocities. Measurements of tongue-induced
935 reed positions in a clarinet are presented in [53] for
936 example. To improve the statistical approaches, these
937 initial conditions should also be taken into account.

939 6.3 Stability, transient duration and 940 playability

941 Other studies were interested in predicting the
942 transient duration of a self-sustained musical instrument
943 [24, 25]. A common approach is to linearize
944 the system around an unstable equilibrium and to
945 consider the exponential growth of a perturbation.
946 This exponential growth depends on the positive real
947 part of the eigenvalues of the matrix of derivatives
948 (i.e., the jacobian matrix). This method is better
949 adapted for studying initial conditions close to an
950 equilibrium solution. The Floquet theory might
951 be used in the same way for initial conditions next
952 to a periodic solution, although it has yet to be
953 applied in that manner to a musical instrument
954 model (for more details on the Floquet theory,
955 refer to [54, 55]). In this article, we consider initial
956 conditions not necessarily close to any solution.
957 Hence, these linear analyses might not be valid
958 for every sample. On more complex systems with
959 several competing multistable solutions, its local
960 nature prevents it from giving a complete analysis
961 of the transients. However, the systematic and fast
962 nature of linear analysis makes it a good candidate to
963 bolster certain sections of a complete statistical study.

964
965 Finally, the statistical study of the trumpet model
966 presented in Sec.5 highlights several interests of the
967 methods. The long transients observed in Fig.13 raise
968 the question of the playability of a regime as a func-

tion of its transient duration. In a standard musical context, we argue that a regime that takes several seconds to establish cannot really be considered playable. Therefore, certain stable regimes could actually be unplayable due to transient behaviors, which shows the relevance of nuancing the notion of stability using transients when studying playability. The transient-based analysis of Sec. 5.3.2 also indicates that the transient control parameters of a musical gesture may have nontrivial optimal values. Future works could analyze these types of transient control scenarios into more details, relying on the statistical approaches as playability guidelines.

7 Conclusion and perspectives

In this article, we present a sample-based approach that can be used to enrich a bifurcation diagram. In this study we use it to investigate two playability issues: the prediction of the steady state in multi-stable situations and the prediction of the transient duration. The method used for the first issue is called the basin stability [31], whereas the method proposed for describing the responsiveness of a dynamical system is new, to the authors' knowledge. These two methods rely on distributions of initial conditions that represent the transient action of a musician. The results strongly depend on this choice of distribution and we proposed a method based on time integration to generate transient-informed distributions.

The methods are then applied to a physical model of trumpet with the aim to evaluate their interest in a more practical case. The transient-based basin stability of this model present non trivial tendencies regarding some transient control parameters. Moreover, very long transients are highlighted, which raises the question of the playability of such asymptotic regime. Overall, these statistical methods show interest for the analysis of musical instruments, but they can also be extended to a large variety of systems.

Future works will focus on two aspects. First, these methods will be applied on musical instrument models in configurations showing multiple oscillating regimes and rich transients. Then, measurements on musicians will be conducted to evaluate the initial conditions that they induce, depending on the desired musical effect.

8 Data Availability Statement

No new data were created or analyzed in this study.

Appendix

A Trumpet model

We consider the brass instrument model described in [5]. The convention used to represent the lip position is given in figure 15. The dimensioned and unregularized equations are as follows:

$$\begin{cases} \ddot{x} + \frac{\omega_L}{Q_L} \dot{x} + \omega_L^2(x - x_0) = \frac{p_0 - p}{\mu_L}, \\ \dot{p}_n - s_n p_n = Z_c C_n u \quad \forall n \in [1, N_m], \\ p = 2 \sum_{n=1}^{N_m} \Re(p_n), \\ u = W \sqrt{\frac{2|p_0 - p|}{\rho}} \cdot \text{sign}(p_0 - p) \cdot \max(x, 0), \end{cases} \quad (9)$$

with

$$\begin{cases} \omega_L = \sqrt{\frac{k}{m}}, \\ Q_L = \frac{\sqrt{km}}{c}, \\ \mu_L = \frac{m}{S}. \end{cases} \quad (10)$$

S is the surface area of the lip to which the pressures p_0 and p are applied, μ_L is the surface mass of the lip, Q_L its quality factor and ω_L its angular eigenfrequency. The resonator is represented by its input impedance, which is treated as a sum of N_m modes with poles s_n and residues C_n . The lip allows air to pass through a rectangular surface of height x and width W . The system Eq.(9) is scaled as follows:

$$\begin{cases} \ddot{\tilde{x}} + \frac{\omega_L}{Q_L} \dot{\tilde{x}} + \omega_L^2(\tilde{x} - 1) = \omega_L^2(\gamma - \tilde{p}), \\ \dot{\tilde{p}}_n - s_n \tilde{p}_n = C_n \tilde{u} \quad \forall n \in [1, N_m], \\ \tilde{p} = 2 \sum_{n=1}^{N_m} \Re(\tilde{p}_n), \\ \tilde{u} = \zeta \sqrt{|\gamma - \tilde{p}|} \cdot \text{sign}(\gamma - \tilde{p}) \cdot \max(\tilde{x}, 0). \end{cases} \quad (11)$$

Variables affected by this scaling are marked with a $\tilde{\cdot}$ symbol and are defined as follows:

$$\begin{cases} \tilde{x} = \frac{x}{x_0}, \\ p_M = \mu_L \omega_L^2 x_0, \\ \tilde{p} = \frac{p}{p_M}, \\ \gamma = \frac{p_0}{p_M}, \\ \tilde{p}_n = \frac{p_n}{p_M}, \\ u_M = \frac{u}{Z_c}, \\ \tilde{u} = \frac{u}{u_M}, \\ \zeta = Z_c W x_0 \sqrt{\frac{2}{\rho p_M}}. \end{cases} \quad (12)$$

Irregular functions appearing in the flow term are regularized:

$$\begin{cases} |\bullet|_r = \sqrt{\bullet^2 + \epsilon}, \\ \text{sign}(\bullet)_r = \frac{\bullet}{|\bullet|_r}, \\ \max(\bullet, 0)_r = \frac{\bullet + |\bullet|_r}{2}. \end{cases} \quad (13)$$

The regularization term ϵ is arbitrarily set to $\epsilon = 10^{-6}$.

Time is scaled by the first modal angular frequency $\omega_1 = \Im(s_1)$ and the concerned variables are written with a $\hat{\cdot}$ symbol. Finally, we separate the real R_n and

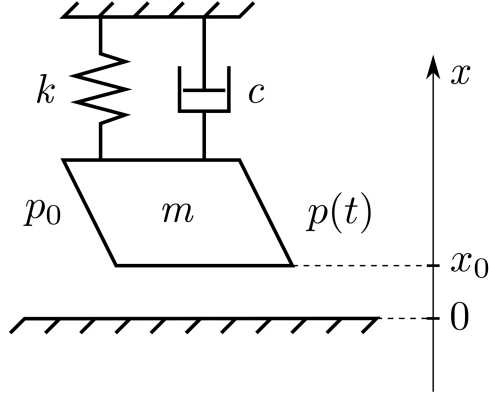


Figure 15: Schematic diagram of the one-degree-of-freedom lip model. Assume $x = 0$ when the lip is in the closed position and note x_0 the position of the lip at rest (the situation shown here).

1043 imaginary I_n parts of the modal pressures p_n , and the
1044 system we solve is as follows:

$$\begin{cases} \ddot{\tilde{x}} + \frac{\hat{\omega}_L}{Q_L} \dot{\tilde{x}} + \hat{\omega}_L^2 (\tilde{x} - 1) = \hat{\omega}_L^2 (\gamma - \tilde{p}), \\ \dot{\tilde{R}}_n = \Re(\hat{s}_n) \tilde{R}_n - \Im(\hat{s}_n) \tilde{I}_n + \Re(\hat{C}_n) \tilde{u} \\ \quad \forall n \in [1, N_m], \\ \dot{\tilde{I}}_n = \Re(\hat{s}_n) \tilde{I}_n + \Im(\hat{s}_n) \tilde{R}_n + \Im(\hat{C}_n) \tilde{u} \\ \quad \forall n \in [1, N_m], \\ \tilde{p} = 2 \sum_{n=1}^{N_m} \Re(\tilde{p}_n), \\ \tilde{u} = \zeta \sqrt{|\gamma - \tilde{p}|_r} \cdot \text{sign}(\gamma - \tilde{p})_r \cdot \max(\tilde{x}, 0)_r. \end{cases} \quad (14)$$

1045
1046 The parameter values chosen for this article are as
1047 follows:

$$\begin{cases} \omega_L = 2\pi \times 200 \text{ rad.s}^{-1}, \\ \mu_L = 2 \text{ kg.m}^{-2}, \\ Q_L = 3, \\ x_0 = 1e - 4 \text{ m}, \\ W = 8.10^{-3} \text{ m}, \\ Zc = 1.83 \text{ kg.s}^{-1} \cdot \text{m}^{-4}, \\ \epsilon = 10^{-6}, \end{cases} \quad (15)$$

Table 1: Resonator modal parameters. Complex residues C_n and poles s_n and resonance frequencies f_n .

n	C_n ($\text{kg.s}^{-1} \cdot \text{m}^{-4}$)	s_n (rad.s^{-1})	f_n (Hz)
1	744.6	$-13.98 + i522.5$	83.15
2	954.5	$-22.42 + i1462$	232.7
3	1335	$-28.64 + i2187$	348.1
4	2582	$-37.64 + i2907$	462.6
5	3140	$-45.82 + i3658$	582.1
6	4191	$-49.82 + i4339$	690.6
7	4013	$-58.42 + i5029$	800.4
8	2602	$-66.77 + i5705$	908.1
9	1278	$-72.24 + i6459$	1028
10	909.7	$-94.40 + i7211$	1148
11	620.7	$-128.6 + i7931$	1262

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B Modifications to bSTAB

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The bSTAB toolbox [48] has been designed to calculate the basin stability of any dynamic system as automatically as possible. It generates initial conditions, performs a time integration for each of them, then classifies the obtained regime by comparing it to reference signals. In our case, we compare the peak-to-peak amplitude of R_2 , the real part of the second modal pressure, to that of the MANLAB solutions. Since classification must be performed on the steady part of the regime, the time integration continues until the following convergence criteria are met.

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During the last duration t^* :

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1. The amplitude of R_2 is within ϵ of that of a reference solution;
2. The difference between the amplitude of R_2 and that of the reference decreases.

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The transient duration is defined as the duration after which the envelope of R_2 stays within ϵ of the amplitude of the reference solution. In this article, we choose $\epsilon = 0.1$.

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The computation of the transient duration is illustrated in Fig.16(a). The corresponding trajectory is also represented in Fig.16(b), with its projection along x , R_2 and R_4 .

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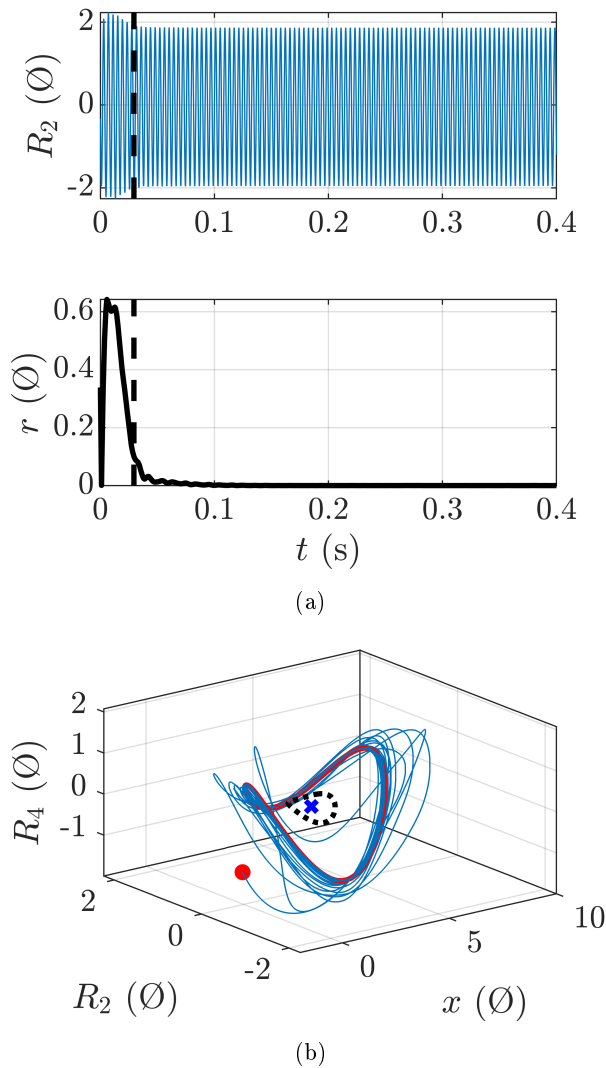


Figure 16: (a) Computation of the transient duration (represented by the dotted line) and (b) projection of the corresponding trajectory along x , R_2 and R_4 . The red dot is the initial state, the blue cross is the equilibrium, the red line is the stable periodic solution and the black broken line is the unstable periodic solution.

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