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Predictor-Based Prescribed-Time Output Feedback for a Parabolic PDE

Salim Zekraoui¹, Nicolas Espitia², Wilfrid Perruquetti³, and Miroslav Krstic⁴

Abstract—In this paper, we consider a 1D reaction-diffusion system with boundary input delay and propose a general method for studying the problem of prescribed-time output boundary stabilization. We first reformulate the system as a PDE-PDE cascade system (i.e., a cascade of a linear transport partial differential equation (PDE) with a linear reaction-diffusion PDE), where the transport equation represents the effect of the input delay. We then apply a time-varying infinite-dimensional backstepping transformation to convert the cascade system and the proposed observer system into two prescribed-time stable (PTS) target systems. The stability analysis is conducted on the target systems, and the desired stability property is transferred back to the closed-loop system and the error system using the inverse transformation. The effectiveness of the proposed approach is demonstrated through numerical simulations.

I. INTRODUCTION

The stabilization of parabolic partial differential equations (PDEs), in particular reaction-diffusion PDEs, with arbitrary levels of instability and arbitrarily long input delay is a challenging problem that was introduced and solved using the so-called *Backstepping method* for PDEs in [8] and generalized for other classes of parabolic PDEs (e.g. [15], [2], [3], [14], [7], [20]). Most of these works provide only asymptotic or exponential convergences, though in several applications (e.g., chemical, biological, or population (epidemiological) processes, etc), where the transient process must occur within a finite given time, and delays need to be perfectly compensated, Non-asymptotic convergences (finite-time, fixed-time, or prescribed-time) are strongly desired.

Contributions on Non-asymptotic stabilization for time-delay systems, and finite-dimensional systems with input delay can be found in e.g. [6], [11], [10], [4], [24], [22]. Prescribed-time stabilization - initially introduced in [16], then extended to 1D Reaction-Diffusion PDEs in [5], [18] [1], [21] - and predictor feedback for compensation of input delay [9] are a perfect match because both techniques deal with finite-time converging dynamics. Intuitively, by applying predictor feedback to a prescribed-time feedback (for either an ODE or PDE plant), the former feedback being time-varying and the latter infinite-dimensional, one should be able to obtain convergence in a time that is the sum of the

prescribed time and the input delay. Combining prescribed-time feedback with predictors is possible as shown in [12], where an ODE-PDE cascade representation of a class of LTI systems with input delay is provided and a time-varying backstepping-based approach is used to design a predictor feedback that ensures delay compensation in prescribed time.

In this paper, we address the problem of achieving prescribed-time boundary output-feedback stabilization for a class of 1D linear reaction-diffusion PDEs with input delay. We propose a novel approach, inspired by [9, Chapter 11], rather than classical PDE backstepping, to solve this problem. This approach is an advantageous alternative, even in the case of exponential stabilization, but requires radical advancements to be adjusted from nonlinear ODEs to linear PDEs. Our contribution builds upon the results of [5] and [18] for the case of delayed input and extends the results of [17] to output-feedback stabilization. Unlike [17], our resulting predictor-based controller does not depend on a spatial derivative of the state and also overcomes the issue of incompatibility of boundary conditions of kernel equations (which arises when considering point-wise damping term in the design).

The main idea of our approach is to transform the original PDE system and the proposed observer into PDE-PDE cascade systems and then apply a time-varying backstepping transformation - depending on a predictor variable - to transform only the hyperbolic PDE states. This transformation leads to stable target systems that ensure the desired prescribed-time convergence. Finally, by inverse transformation, we transfer the stability property and the desired non-asymptotic convergence back to the original closed-loop system and to the error system.

This conference paper is a tutorial version of our journal manuscript submitted 5 months ago to IEEE Transactions on Automatic Control [23]; and it is organized as follows: In Section II, we introduce the unstable 1D reaction-diffusion system with boundary input delay and recall briefly some results related to prescribed-time output boundary feedback stabilization for the delay-free 1D reaction-diffusion system. In Section III, we focus on the problem of prescribed-time output-feedback stabilization of the original delayed system, where we start by reformulating the PDE system as a PDE-PDE cascade system and introducing the used prescribed-time observer; We use an invertible transformation to link the cascade systems to some well-chosen prescribed-time stable target systems (see Figure 1). We perform a stability analysis on the target original and observer system. Then, by inverse transformation, we establish the boundedness of the states of the original system and the error system and their

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convergence to the origin in a prescribed time using a suitable norm equivalence. In Section IV, we consider a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section V.

Notation:

\mathbb{R}_+ denotes the set of non negative real numbers. \mathbb{N}^* denotes the set of natural numbers excluding zero. For all $a, b \in \mathbb{R}$ with $a \leq b$, $L^2(a, b)$ denotes the set $\{f : [a, b] \rightarrow \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty\}$ with the scalar product $\langle f, g \rangle_{L^2(a, b)} := \int_a^b f(x)g(x)dx$, and the norm $\|f\|_{L^2(a, b)} := (\int_a^b f(x)^2 dx)^{\frac{1}{2}}$.

II. PROBLEM STATEMENT AND PRELIMINARIES

Let us consider the following reaction-diffusion PDE with a constant reaction term and a known constant boundary input delay $D > 0$:

$$\begin{aligned} z_t(t, x) &= z_{xx}(t, x) + \lambda z(t, x), & (1) \\ z(t, 0) &= 0, & (2) \\ z(t, 1) &= U(t - D), & (3) \\ y(t) &= z_x(t, 1), & (4) \\ z(t_0, x) &= z_0(x), & (5) \end{aligned}$$

with an initial time $t_0 \geq 0$, a reaction term $\lambda \in \mathbb{R}$, time and space variables $(t, x) \in [t_0, t_0 + D + T] \times [0, 1]$, the state $z(t, x) \in \mathbb{R}$, the control $U(t) \in \mathbb{R}$, with the following initial condition: $U(t_0 + s) = 0$ for all $s \in [-D, 0]$, the collocated output $y(t) \in \mathbb{R}$, and the initial condition $z_0 \in L^2(0, 1)$.

Our main goal is to design a predictor-based output-feedback controller achieving prescribed-time stabilization of the closed-loop system (1)-(5) in the following sense: there exist a positive fractional function $M_1(\cdot)$, a positive constant $M_2 > 0$ and a continuous function $\mu : [t_0, t_0 + D + T] \rightarrow \mathbb{R}_+$ that tends to infinity as t goes to $t_0 + D + T$, such that for any initial condition $z_0 \in L^2(0, 1)$ and for all $t \in [t_0, t_0 + D + T]$, the following estimate holds:

$$\|z(t, \cdot)\|_{L^2(0,1)} \leq M_1(\mu(t - t_0 - D)) e^{-M_2 \mu(t - t_0 - D)} \|z_0\|_{L^2(0,1)}.$$

Remark 1: Note that in our approach, it is necessary to have a prescribed-time boundary controller for the delay-free case of system (1)-(5).

A. Prescribed-time output boundary feedback stabilization in the delay-free case

Before presenting our approach, let us briefly summarize the main results of [18] on prescribed-time output boundary feedback stabilization of the delay-free case system (1)-(5) (i.e. $D = 0$). Consider the following blow-up function:

$$\gamma_m(t - \bar{t}_0) := \frac{\gamma_{m,0}^m T^m}{(\bar{t}_0 + T - t)^m}, \quad (6)$$

for $m \in \mathbb{N}^*$, defined for all $t \in [\bar{t}_0, \bar{t}_0 + T]$ where $\bar{t}_0 \geq 0$ and $T > 0$ is a priori fixed.

1) Observer design: Assuming that $D = 0$, the following observer system was proposed in [18]:

$$\begin{aligned} \hat{z}_t(t, x) &= \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) + P(x, t - \bar{t}_0, T) \\ &\quad \times [z_x(t, 1) - \hat{z}_x(t, 1)], \end{aligned} \quad (7)$$

$$\hat{z}(t, 0) = 0, \quad (8)$$

$$\hat{z}(t, 1) = U(t), \quad (9)$$

$$\hat{z}(\bar{t}_0, x) = \hat{z}_0(x), \quad (10)$$

with observer gain P given by

$$\begin{aligned} P(x, t - \bar{t}_0, T) &:= -\frac{\gamma_3(t - \bar{t}_0)}{2\gamma_{3,0}^2} x \sum_{n=0}^{+\infty} \left(\frac{\gamma_3(t - \bar{t}_0)^{\frac{1}{3}}}{4T\gamma_{3,0}^{\frac{1}{3}}} \right)^n \\ &\quad \times \frac{(-1 - x^2)^n}{(n+1)!} \sum_{j=0}^n \sum_{k=0}^j \frac{1}{j!} \binom{j}{k} \\ &\quad \times \binom{n+2+k}{n-j} \left(\frac{-T\gamma_3(t - \bar{t}_0)^{\frac{2}{3}}}{2\gamma_{3,0}} \right)^j, \end{aligned} \quad (11)$$

and γ_3 defined in (6). The observer state $\hat{z}(t, \cdot)$ converges to $z(t, \cdot)$ within the prescribed terminal time $\bar{t}_0 + T$ provided that

$$\gamma_{3,0} T > \sqrt[3]{4}. \quad (12)$$

More precisely, there exist a positive constant α_1 and a positive polynomial function $Q_1(\cdot)$ in terms of $\gamma_2(t - \bar{t}_0)$ such that, for any initial conditions $z(\bar{t}_0, \cdot)$ and $\hat{z}(\bar{t}_0, \cdot)$, the following inequality holds for all $t \in [\bar{t}_0, \bar{t}_0 + T]$:

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_1(t - \bar{t}_0) \|z(\bar{t}_0, \cdot) - \hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (13)$$

where

$$\zeta_1(t - \bar{t}_0) := Q_1(\alpha_1 \gamma_2(t - \bar{t}_0)) e^{-\alpha_1 \gamma_2(t - \bar{t}_0)}. \quad (14)$$

In particular, $\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$ as $t \rightarrow \bar{t}_0 + T$.

2) Control design: We recall the following time-varying boundary output control:

$$U(t) := \int_0^1 K(1, s, t - \bar{t}_0) \hat{z}(t, s) ds, \quad (15)$$

where the state $\hat{z}(t, \cdot)$ is generated from (7)-(10) and the control gain K is given explicitly as in [5, Lemma 1] by

$$\begin{aligned} K(x, s, t - \bar{t}_0) &= -\frac{1}{2} \gamma_2(t - \bar{t}_0) s \sum_{n=0}^{\infty} \frac{\left(\sqrt{\gamma_2(t - \bar{t}_0)} \right)^n}{(n+1)!} \\ &\quad \times \left(\frac{x^2 - s^2}{4T\gamma_{2,0}} \right)^n L_n^{(1)} \left(-T\gamma_{2,0} \sqrt{\gamma_2(t - \bar{t}_0)} \right), \end{aligned} \quad (16)$$

where $L_n^{(1)}(\cdot)$ are the generalized Laguerre polynomials. In addition, (16) can be simplified using the first-order modified Bessel function $I_1(\cdot)$ to get [5],

$$\begin{aligned} K(x, s, t - \bar{t}_0) &= -\gamma_2(t - \bar{t}_0) s e^{\frac{\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - s^2)}{4T\gamma_{2,0}}} \\ &\quad \times \frac{I_1(\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - s^2))}{\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - s^2)}, \end{aligned} \quad (17)$$

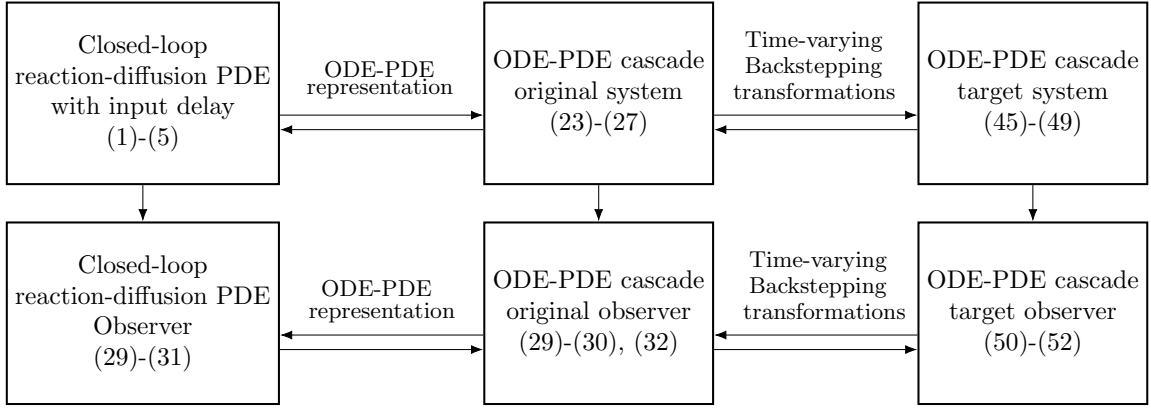


Fig. 1. Overview of the main change of coordinates employed throughout this paper.

for $(x, s, t) \in \mathcal{S} := \{(x, s, t) \in [0, 1]^2 \times [\bar{t}_0, \bar{t}_0 + T) : s \leq x\}$;

Using the control (15) and provided that

$$2\gamma_{2,0}T > 1, \quad (18)$$

the closed-loop PDE system (1)-(5) is prescribed-time stable in the following sense: there exist two positive constants α_2 and α_3 and two positive polynomial functions $Q_2(\cdot)$ and $Q_3(\cdot)$ defined in terms of $\gamma_1(t - \bar{t}_0)$ such that for any initial conditions $z(\bar{t}_0, \cdot)$ and $\hat{z}(\bar{t}_0, \cdot)$ at initial time \bar{t}_0 , the following inequality holds:

$$\|z(t, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_2(t - \bar{t}_0) \left(\|z(\bar{t}_0, \cdot)\|_{L^2(0,1)} + \|\hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)} \right), \quad (19)$$

for all $t \in [\bar{t}_0, \bar{t}_0 + T)$, where

$$\zeta_2(t - \bar{t}_0) := Q_2(\alpha_2 \gamma_1(t - \bar{t}_0)) e^{-\alpha_2 \gamma_1(t - \bar{t}_0)}. \quad (20)$$

Furthermore, we have

$$|U(t)| \leq \zeta_3(t - \bar{t}_0) \|\hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (21)$$

for all $t \in [\bar{t}_0, \bar{t}_0 + T)$, with

$$\zeta_3(t - \bar{t}_0) := Q_3(\alpha_3 \gamma_1(t - \bar{t}_0)) e^{-\alpha_3 \gamma_1(t - \bar{t}_0)}. \quad (22)$$

In particular, $\|\hat{z}(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$, $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$, and $|U(t)| \rightarrow 0$ when $t \rightarrow \bar{t}_0 + T$.

III. PRESCRIBED-TIME STABILIZATION BY OUTPUT FEEDBACK

Let us now come back to (1)-(5) and its PDE-PDE cascade representation:

$$z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), \quad (23)$$

$$z(t, 0) = 0, \quad (24)$$

$$z(t, 1) = v(t, 0), \quad (25)$$

$$v_t(t, y) = v_y(t, y), \quad (26)$$

$$v(t, D) = U(t). \quad (27)$$

with $(t, x, y) \in [t_0, t_0 + D + T) \times [0, 1] \times [0, D]$, and $v(t, \cdot)$ is the transport PDE state whose solution is given by

$$v(t, y) = \begin{cases} 0, & t_0 \leq t + y \leq t_0 + D, \\ U(t + y - D), & t + y \geq t_0 + D. \end{cases} \quad (28)$$

We propose the following observer for (1)-(5):

$$\hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) + P(x, t - t_0, D + T) \times [z_x(t, 1) - \hat{z}_x(t, 1)], \quad (29)$$

$$\hat{z}(t, 0) = 0, \quad (30)$$

$$\hat{z}(t, 1) = U(t - D). \quad (31)$$

with the observer gain P is given as in (11) where we replace T by $D + T$ in the expression of $\gamma_3(t - t_0)$ to ensure that the convergence of $\hat{z}(t, \cdot)$ to $z(t, \cdot)$ is achieved in $t_0 + D + T$ instead of $t_0 + T$. Note that (31) can be always expressed using $v(t, \cdot)$ as it was done in (25), More precisely:

$$z(t, 1) = v(t, 0). \quad (32)$$

Remark 2: As our goal is to design an output-feedback control $U(t)$ for (23)-(27), we do not need to estimate the dynamics of (26)-(27) because it is expressed in terms of the control $U(t)$ which, in turn, is in terms of the observed state $\hat{z}(t, \cdot)$. Consequently, it is clear that, if $\gamma_{3,0}$ satisfies the following condition:

$$\gamma_{3,0}(D + T) > \sqrt[3]{4}, \quad (33)$$

then, from (13), the following inequality holds:

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_1(t - t_0 - D) \|z_0 - \hat{z}_0\|_{L^2(0,1)}, \quad (34)$$

for $t \in [t_0, t_0 + D + T)$ where $z_0 = z(t_0, \cdot)$ and $\hat{z}_0 = \hat{z}(t_0, \cdot)$. In particular, $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow \|\hat{z}(t, \cdot)\|_{L^2(0,1)}$ as $t \rightarrow t_0 + D + T$.

A. Time-varying infinite-dimensional backstepping transformation

We consider the following time-varying infinite-dimensional backstepping transformation:

$$\omega(t, y) = v(t, y) - \mathcal{F}(t + y - t_0 - D, \hat{\phi}(t, \cdot, y)), \quad (35)$$

where \mathcal{F} has the same structure as in (15), i.e.,

$$\mathcal{F}(t+y-t_0-D, \hat{\phi}(t, \cdot, y)) := \int_0^1 K(1, s, t+y-t_0-D) \times \hat{\phi}(t, s, y) ds \quad (36)$$

and $\hat{\phi}$ is chosen to satisfy $\hat{\phi}(t, x, y) = \hat{z}(t+y, x)$, and therefore, is solution of the following parabolic PDE:

$$\hat{\phi}_y(t, x, y) = \hat{\phi}_{xx}(t, x, y) + \lambda \hat{\phi}(t, x, y) + P(x, t+y-t_0, D+T) \times [\phi_x(t, 1, y) - \hat{\phi}_x(t, 1, y)], \quad (37)$$

$$\hat{\phi}(t, 0, y) = 0, \quad (38)$$

$$\hat{\phi}(t, 1, y) = v(t, y), \quad (39)$$

$$\hat{\phi}(t, x, 0) = \hat{z}(t, x). \quad (40)$$

with the predictor ϕ is chosen to satisfy $\phi(t, x, y) = z(t+y, x)$ which means it is the solution of

$$\phi_y(t, x, y) = \phi_{xx}(t, x, y) + \lambda \phi(t, x, y), \quad (41)$$

$$\phi(t, 0, y) = 0, \quad (42)$$

$$\phi(t, 1, y) = v(t, y), \quad (43)$$

$$\phi(t, x, 0) = z(t, x). \quad (44)$$

with $(t, x, y) \in \{(t, x, y) \in [t_0, t_0 + D + T] \times [0, 1] \times [0, D] : t + y \in [t_0 + D, t_0 + D + T]\}$.

B. Target System

Using (35), we transform respectively (23)-(27) and (29)-(31) into the two following target systems:

$$z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), \quad (45)$$

$$z(t, 0) = 0, \quad (46)$$

$$z(t, 1) = \omega(t, 0) + \mathcal{F}(t-t_0-D, \hat{z}(t, \cdot)), \quad (47)$$

$$\omega_t(t, y) = \omega_y(t, y), \quad (48)$$

$$\omega(t, D) = 0. \quad (49)$$

and

$$\hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) + P(x, t-t_0, D+T) \times [z_x(t, 1) - \hat{z}_x(t, 1)], \quad (50)$$

$$\hat{z}(t, 0) = 0, \quad (51)$$

$$\hat{z}(t, 1) = \omega(t, 0) + \mathcal{F}(t-t_0-D, \hat{z}(t, \cdot)). \quad (52)$$

where $\omega : [t_0, t_0 + D + T] \times [0, D] \rightarrow \mathbb{R}$ is the transport PDE state.

Note that using the fact that $\phi(t, x, y) = z(t+y, x)$ and $\hat{\phi}(t, x, y) = \hat{z}(t+y, x)$ for all $(t, x, y) \in [t_0, t_0 + D + T] \times [0, 1] \times [0, D]$, it is clear that (35) verifies (48)-(49).

C. Time-varying infinite-dimensional inverse transformation

The inverse transformation is given by,

$$v(t, y) = \omega(t, y) + \mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)), \quad (53)$$

where

$$\mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)) := \int_0^1 K(1, s, t+y-t_0-D) \times \hat{\psi}(t, s, y) ds \quad (54)$$

where $\hat{\psi}$ is the solution of

$$\hat{\psi}_y(t, x, y) = \hat{\psi}_{xx}(t, x, y) + \lambda \hat{\psi}(t, x, y) + P(x, t+y-t_0, D+T) \times [\psi_x(t, 1, y) - \hat{\psi}_x(t, 1, y)], \quad (55)$$

$$\hat{\psi}(t, 0, y) = 0, \quad (56)$$

$$\hat{\psi}(t, 1, y) = \omega(t, y) + \mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)), \quad (57)$$

$$\hat{\psi}(t, x, 0) = \hat{z}(t, x). \quad (58)$$

and ψ is generated from

$$\psi_y(t, x, y) = \psi_{xx}(t, x, y) + \lambda \psi(t, x, y), \quad (59)$$

$$\psi(t, 0, y) = 0, \quad (60)$$

$$\psi(t, 1, y) = \omega(t, y) + \mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)), \quad (61)$$

$$\psi(t, x, 0) = z(t, x). \quad (62)$$

Similarly, from the inverse transformation (53), we recover (26)-(27).

D. Prescribed-time predictor-based output controller

We recover the expression of the boundary control $U(t)$ as follows:

$$U(t) = v(t, D) = \mathcal{F}(t-t_0, \hat{\phi}(t, \cdot, D)) := \int_0^1 K(1, s, t-t_0) \hat{\phi}(t, s, D) ds, \quad (63)$$

from (35) at $y = D$, where $\hat{\phi}$ is generated from (41)-(44) and K is given in (17). Likewise from (53) at $y = D$, we can get

$$U(t) = v(t, D) = \mathcal{F}(t-t_0, \hat{\psi}(t, \cdot, D)), \quad (64)$$

where $\hat{\psi}$ is generated from (55)-(58).

Remark 3: To achieve exponential output-feedback stabilization, it is sufficient to replace the control gain K and the observer gain P respectively by

$$K_{\text{exp}}(x, s) = -(\lambda + \lambda_0) s \frac{I_1(\sqrt{(\lambda + \lambda_0)(x^2 - s^2)})}{\sqrt{(\lambda + \lambda_0)(x^2 - s^2)}}, \quad (65)$$

$$P_{\text{exp}}(x) = -(\lambda + \lambda_0) x \frac{I_1(\sqrt{(\lambda + \lambda_0)(1 - x^2)})}{\sqrt{(\lambda + \lambda_0)(1 - x^2)}}, \quad (66)$$

for $\lambda_0 \geq 0$ and $(x, s) \in \mathcal{T} := \{(x, s) \in [0, 1]^2 : s \leq x\}$. The expression of the exponential predictor-based output-feedback controller $U_{\text{exp}}(t)$ is then given by

$$U_{\text{exp}}(t) = \int_0^1 K_{\text{exp}}(1, s) \hat{\phi}(t, s, D) ds, \quad (67)$$

or

$$U_{\text{exp}}(t) = \int_0^1 K_{\text{exp}}(1, s) \hat{\psi}(t, s, D) ds, \quad (68)$$

with $\hat{\phi}$ and $\hat{\psi}$ are respectively generated from (37)-(40) and (55)-(58) using the observer gain $P_{\text{exp}}(x)$ in (66).

E. Stability analysis

Proposition 1: For the transport PDE $v(t,x)$ satisfying (26), there exists a positive polynomial function $Q_4(\cdot)$ in terms of $\gamma_1(t-t_0-D)$ such that the following estimate holds: $\|v(t,\cdot)\|_{L^2(0,D)} \leq \zeta_4(t-t_0-D) \left(\|z(t_0+D,\cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0+D,\cdot)\|_{L^2(0,1)} \right)$, for all $t \in [t_0+D, t_0+D+T]$, where $\zeta_4(t-t_0-D)^2 := Q_4(\alpha_3 \gamma_1(t+y-t_0-D)) e^{-\alpha_3 \gamma_1(t+y-t_0-D)}$ with $\gamma_1(\cdot)$ given in (6). In particular, it holds $\|v(t,\cdot)\|_{L^2(0,D)} \rightarrow 0$ for all $t \rightarrow t_0+D+T$.

Proof: The proof is a direct application of (53) with the fact that $\omega(t,y) = 0, \forall (t,y) \in [t_0+D, t_0+D+T] \times [0,D]$ and $\hat{\psi}(t,\cdot,y) = \hat{z}(t+y,\cdot)$, and using (21) for $\bar{t}_0 := t_0+D$. ■

Let us now give our main result,

Theorem 1: Let $\gamma_{2,0} > 0$, and $\gamma_{3,0} > 0$ be chosen such that (18) and (33) are ensured. Let $T > 0, D > 0$ and $t_0 \geq 0$. Then, the solution of the closed-loop system (23)-(27) with the observer (29)-(31) and the prescribed-time time-varying output control (63) (or (64)) is prescribed-time stable in the following sense: For any initial conditions z_0 and \hat{z}_0 , the quantities $\|z(t,\cdot)\|_{L^2(0,1)}$, $\|\hat{z}(t,\cdot)\|_{L^2(0,1)}$, and $\|v(t,\cdot)\|_{L^2(0,D)}$ remain bounded for all $t \in [t_0, t_0+D]$; and for all $t \in [t_0+D, t_0+D+T]$, the following norm $I(t) = \|z(t,\cdot)\|_{L^2(0,1)} + \|\hat{z}(t,\cdot)\|_{L^2(0,1)} + \|v(t,\cdot)\|_{L^2(0,D)}$ satisfies,

$$I(t) \leq L_D \zeta_5(t-t_0-D) \left(\|z_0\|_{L^2(0,1)} + \|\hat{z}_0\|_{L^2(0,1)} \right), \quad (69)$$

for $L_D > 0$ with $\zeta_5(\cdot) := \zeta_2(\cdot) + \zeta_4(\cdot)$. In particular, $I(t) \rightarrow 0$ as $t \rightarrow t_0+D+T$ and $|U(t)| \rightarrow 0$ as $t \rightarrow t_0+T$.

Proof: Due to space limitation, we give only a sketch of the proof. On one hand, seeing that the control input $U(t)$ is initialized by zero on $[t_0-D, t_0]$, the boundedness of the state of (23)-(25) is proven from the explicit computation of the solution from [13, Chapter 3] and consequently the boundedness of the observer state is also ensured from (34). On the other hand, inequality (69) is directly obtained from (19) and Proposition 1. In particular, the convergence $I(t)$ to the origin is ensured when $t \rightarrow t_0+D+T$. Furthermore, the boundedness of the norms $\|z(t,\cdot)\|_{L^2(0,1)}$ and $\|\hat{z}(t,\cdot)\|_{L^2(0,1)}$ is ensured on $[t_0, t_0+D+T]$ and since $v(t,y)$ is given on $[t_0, t_0+D] \times [0,D]$ in terms of $U(s-D)$ for $s \in [t_0, t_0+D]$ (which is bounded because it is expressed in terms of $\hat{z}(s,\cdot)$ and $K(1,\cdot, s-t_0-D)$ that are bounded for all $s \in [t_0, t_0+D]$), the boundedness of $\|v(t,\cdot)\|_{L^2}$ on $[t_0, t_0+D]$ is also deduced. Finally, the convergence of the control $U(t)$ to the origin, in a prescribed time t_0+T , is deduced from the expression (64) and what was proven before. ■

IV. NUMERICAL RESULTS

In this section, we give numerical simulations for the closed-loop system (23)-(27) with prescribed-time predictor-based output-feedback controller $U(t)$ given in (64) used to attain prescribed-time stabilization. We take the delay $D = 0.5s$, the reaction coefficient $\lambda = 11$, the initial time $t_0 = 0$, the prescribed time $T = 1s$ and the initial conditions: $z_0(x) = 10(x-x^2)$, $\hat{z}_0(x) = 0$, $v_0(y) = 0$. We take the initial

conditions of the blow-up functions γ_2 and γ_3 , given in (6), respectively as $\gamma_2(0) = 3.3$ and $\gamma_3(0) = 2.2$.

In the top and middle of Figure 2, we give a comparison between the norm of the closed-loop system (23)-(27) using first the prescribed-time predictor-based output-feedback controller $U(t)$ given in (64) (in red solid line) and afterwards using the exponential predictor-based controller $U_{\text{exp}}(t)$ given in (68) with the constant-gain $\lambda + \lambda_0 = 11$ involved in the control and observer kernels K_{exp} and P_{exp} given in Remark 3 (in black dashed line). In particular, we compare the case with a higher constant-gain $\lambda + \lambda_0 = 28$ (in blue dashed line). On the bottom of Figure 2, we give a comparison between the controllers $U(t)$ (in red solid line) and $U_{\text{exp}}(t)$ with the gain $\lambda + \lambda_0 = 11$ (in black dashed line) and with a higher gain $\lambda + \lambda_0 = 28$ (in blue dashed line).

As it can be observed, at the delay $D = 0.5s$, the norm of the closed-loop system (23)-(27) using the exponential controller $U_{\text{exp}}(t)$ (with the higher gain $\lambda + \lambda_0 = 28$) exhibits the ‘‘peaking phenomenon’’ [19] (see the blue dashed curve in the top plot of Figure 2). After that the delay $D = 0.5s$, the norm outpaces the same norm using the prescribed-time controller $U(t)$. However, as time progresses, the curves of the two norms cross, and from then on the norm of the closed-loop system (23)-(27) using the prescribed-time controller $U(t)$ outperforms the same norm using the exponential controller $U_{\text{exp}}(t)$. This is due to the fact that the exponential controller starts with an aggressive control effort, because of the high gain $\lambda + \lambda_0$, but with time its effort diminishes (see the middle of Figure 2). In contrast, the prescribed-time controller $U(t)$ starts with a moderate effort to avoid peaking and then gradually increases its control effort towards the end of the simulation to ensure that the convergence is completed in the prescribed time.

V. CONCLUSION

This paper dealt with the problem of output-feedback stabilization in prescribed time of a 1D reaction-diffusion system with boundary input delay. The system is rewritten into a PDE-PDE setting (a cascade of a parabolic PDE with a hyperbolic PDE), where the hyperbolic PDE models the effect of the delay on the input. The predictor-based output controller is designed using a time-varying infinite-dimensional backstepping transformation that transforms the PDE-PDE unstable system into a well-chosen target system. The inverse transformation ensures the desired convergence. Numerical simulations are given to illustrate the results. Future work will extend this result to more complex dynamics, e.g., stabilization in prescribed time of parabolic PDEs coupled with hyperbolic PDEs.

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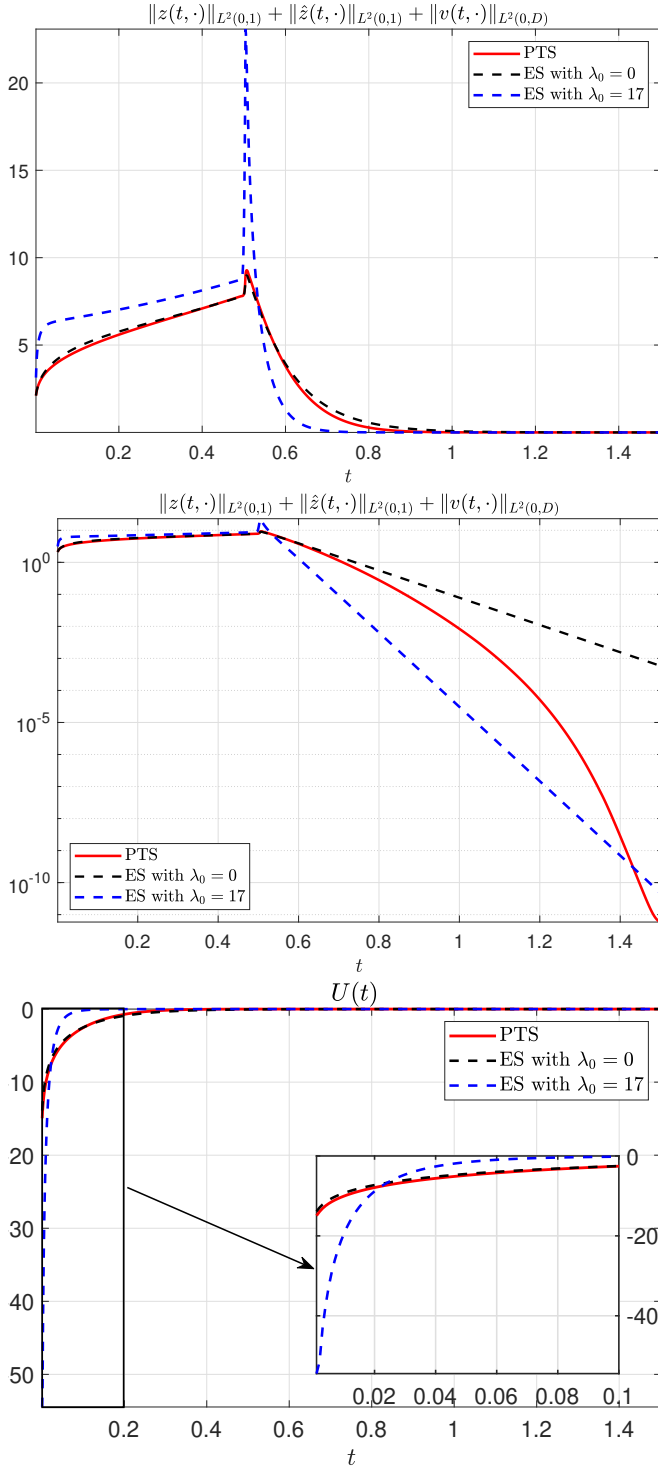


Fig. 2. On the top: The evolution of the norm $\|z(t, \cdot)\|_{L^2(0,1)}^2 + \|\hat{z}(t, \cdot)\|_{L^2(0,1)}^2 + \|v(t, \cdot)\|_{L^2(0,D)}^2$ of the closed-loop system (23)-(27) with prescribed-time predictor-based controller $U(t)$ (64) in red solid line and with exponential predictor-based controller $U_{\text{exp}}(t)$ (68) for $\lambda_0 = 0$ in black dashed line and $\lambda_0 = 17$ in blue dashed line (logarithmic scale in the middle), with $z(t_0, x) = 10(x - x^2)$, $\hat{z}(t_0, x) = 0$ and $D = 0.5\text{s}$. On the bottom, the evolution of the prescribed-time predictor-based controller $U(t)$ (64) in red solid line along with the evolution of the exponential predictor-based controller $U_{\text{exp}}(t)$ (68) for $\lambda_0 = 0$ in black dashed line and $\lambda_0 = 17$ in blue dashed line.

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