



**HAL**  
open science

# The sandpile model on the complete split graph: $q, t$ -Schröder polynomials, sawtooth polyominoes, and a cyclic lemma

Henri Derycke, Mark Dukes, Yvan Le Borgne

► **To cite this version:**

Henri Derycke, Mark Dukes, Yvan Le Borgne. The sandpile model on the complete split graph:  $q, t$ -Schröder polynomials, sawtooth polyominoes, and a cyclic lemma. 2024. hal-04777007

**HAL Id: hal-04777007**

**<https://hal.science/hal-04777007v1>**

Preprint submitted on 12 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The sandpile model on the complete split graph: $q, t$ -Schröder polynomials, sawtooth polyominoes, and a cyclic lemma

Henri Derycke, Mark Dukes, and Yvan Le Borgne

**ABSTRACT.** This paper builds on work initiated in Dukes (2021) that considered the classification of recurrent configurations of the Abelian sandpile model on the complete split graph. We introduce two statistics,  $\text{wtopple}_{CTI}$  and  $\text{wtopple}_{ITC}$ , on sorted recurrent configurations. These statistics arise from two natural but different toppling conventions, CTI and ITC, for Dhar's burning algorithm as it is applied to the complete split graph. In addition, we introduce the bivariate  $q, t$ -CTI and  $q, t$ -ITC *polynomials* that are the generating functions of the bistatistics  $(\text{height}, \text{wtopple}_{ITC})$  and  $(\text{height}, \text{wtopple}_{CTI})$  on the sorted recurrent configurations.

We prove that a modification of the bijection given in Dukes (2021) from sorted recurrent configurations to Schröder paths maps the bistatistic  $(\text{height}, \text{wtopple}_{ITC})$  to the bistatistic  $(\text{area}, \text{bounce})$ . The generating function of the bistatistic  $(\text{area}, \text{bounce})$  on Schröder paths is known in the literature as the  $q, t$ -Schröder polynomial and was introduced by Egge, Haglund, Killpatrick and Kremer (2003). This connection allows us to relate the  $q, t$ -ITC polynomial to the theory of symmetric functions and also establishes symmetry of the  $q, t$ -ITC polynomials.

We also give a characterization of the sorted recurrent configurations as a new class of polyominoes that we call *sawtooth polyominoes*. The CTI and ITC topplings processes on sorted recurrent configurations are proven to correspond to two bounce paths from one side of the sawtooth polyomino to the other. The only difference between the two bounce paths is the initial direction in which they travel. Moreover, and building on the results of Aval et al. (2016), we present a cyclic lemma for a slight extension of stable configurations that allows for an enumeration of sorted recurrent configurations within the framework of the sandpile model. Finally, we conjecture equality of the  $q, t$ -CTI and  $q, t$ -ITC polynomials.

## 1. INTRODUCTION

The complete split graph is a graph consisting of two distinct parts: a *clique part* in which all distinct pairs of vertices are connected by a single edge and an *independent part* in which no two vertices are connected to an edge. In addition, there is precisely one edge between every pair of vertices that lie in different parts. In this paper we denote by  $S_{n,d}$  the complete split graph that consists of the solitary sink part  $\{s\}$  and vertices  $V = \{v_1, \dots, v_n\}$  in the clique part, and vertices  $W = \{w_1, \dots, w_d\}$  in the independent part.<sup>1</sup> The graph  $S_{5,3}$  is illustrated in Figure 1.

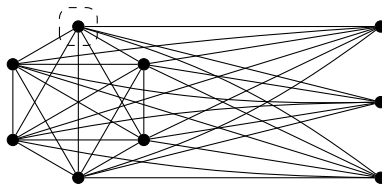


FIGURE 1. The complete split graph  $S_{5,3}$ . The sink is in the dashed rectangle, the 5 vertices below it form the clique part, and the 3 vertically aligned vertices to its right are the independent part.

Dukes [7] recently characterized the recurrent states of the Abelian sandpile model (ASM) on the complete split graph. He presented a bijection from sorted recurrent configurations on  $S_{n,d}$  to  $\text{Schröder}_{n,d}$ , the set of Schröder words consisting of  $n$  up steps,  $n$  down steps, and  $d$  horizontal

YLB was partially supported by ANR Project COMBINÉ ANR-19-CE48-0011.

<sup>1</sup>Our decision to discount mention of the sink in the parameters of  $S_{n,d}$  improves the presentation.

steps.<sup>2</sup> Separately, in Dukes and Le Borgne [8], it was proven that sorted recurrent configurations of the ASM on the complete bipartite graph  $K_{m,n}$  were in one-to-one correspondence with parallelogram polyominoes having an  $m \times n$  bounding box. Moreover, this body of research was extended by Aval et al. [1, 2] wherein a more thorough consideration of statistics on these polyominoes revealed deep connections to the theory of symmetric functions. These connections were then used to prove several symmetry conjectures first stated in [8]. With this in mind, we will delve further into the research presented in Dukes [7] with a view to extending that work in the same manner as [1] to the complete split graph.

In this paper we consider the ASM on the complete split graph in which a clique vertex is the sink. We will only be interested in sorted, or *weakly-decreasing*, recurrent configurations on  $S_{n,d}$ . Beyond the enumerative connection with Schröder paths, this restriction to sorted configurations is also motivated by the natural action of the symmetric group on recurrent configurations, and this features in Section 6 and also in the recent work of D’Adderio et al. [9]). In Section 2 we introduce the sandpile model and recall some necessary terminology. We define two toppling processes, CTI and ITC, on sorted recurrent configurations that will be used throughout the paper. We also introduce two bivariate polynomials that each encode two statistics on these sorted recurrent configurations and call these the  $q, t$ -CTI and  $q, t$ -ITC polynomials.

In Section 3 we consider the bijection from sorted recurrent configurations to Schröder paths given in Dukes [7], and prove a slight modification of this bijection translates the height statistic on configurations to the area statistic on Schröder paths while also translating a toppling statistic to a bounce statistic on the paths. This allows us to prove that the  $q, t$ -ITC polynomial introduced in Section 2 is equal to the  $q, t$ -Schröder polynomial of Egge, Haglund, Killpatrick and Kremer [6], and in so doing relate it to the theory of symmetric functions. Two symmetry properties of the  $q, t$ -ITC polynomials are immediate as a result of this connection. In Section 4 we prove a formula for the  $q, t$ -ITC polynomials that uses the  $q$ -binomial coefficients.

In Section 5 we define a new class of polyominoes that we term *sawtooth polyominoes*. We prove they are in bijection with Schröder words. We also show how two bounce paths within these new polyominoes illustrates the CTI and ITC toppling process of the corresponding recurrent configuration. Moreover, we show how to directly construct the sawtooth polyomino that corresponds to a recurrent configuration. The statistics that form the  $q, t$ -CTI and  $q, t$ -ITC polynomials are also expressible as statistics on the corresponding sawtooth polyominoes.

In Section 6 we introduce a framework for the sandpile model on the complete split graph and within that framework derive a cyclic lemma for sandpile model configurations. This builds on work from Aval et al. [2] in which a cyclic lemma for configurations on the complete bipartite graph was proven. That this framework enables this to be done without having to utilize some planar representations of the configurations as intermediate objects suggests a more general result may hold true. An outcome of this is that it provides another enumeration of sorted recurrent configurations. Finally, in Section 7 equality of the  $q, t$ -CTI and  $q, t$ -ITC polynomials is conjectured.

This paper also adds to the growing body of recent research that has found unexpected connections between recurrent configurations of the Abelian sandpile model on parameterized graph classes and other combinatorial objects [1, 2, 3, 4, 9, 10, 11].

## 2. THE SANDPILE MODEL ON $S_{n,d}$ AND TWO TOPPLING PROCESSES

In this section we first recall some necessary sandpile terminology and concepts, and also introduce two toppling conventions that will be important throughout the paper. The ASM may be defined on any undirected graph  $G$  with a designated vertex  $s$  called the *sink*. A *configuration* on  $G$  is an assignment of non-negative integers to the non-sink vertices of  $G$ :

$$c : V(G) \setminus \{s\} \mapsto \mathbb{N} = \{0, 1, 2, \dots\}.$$

---

<sup>2</sup>In Dukes [7] the correspondence was presented with Mötzkin paths. We have chosen to use the equivalent term Schröder path here due to its relation to papers concerning the  $q, t$ -Schröder polynomial. Moreover notice the change of notations for the split graph parameters:  $n \leftarrow m - 1$  and  $d \leftarrow n$ .

The number  $c(v)$  is referred to as *the number of grains* at vertex  $v$  or the *height* of  $v$ . Given a configuration  $c$ , a vertex  $v$  is said to be *stable* if the number of grains at  $v$  is strictly smaller than the threshold of that vertex, which is the degree of  $v$ , denoted  $\deg(v)$ . Otherwise  $v$  is *unstable*. A configuration is *stable* if all non-sink vertices are stable.

If a vertex is unstable then it may *topple*, which means the vertex donates one grain to each of its neighbors. The sink vertex has no height associated with it and only absorbs grains, thereby modelling grains exiting the system. Starting from any configuration  $c$  and successively toppling unstable vertices one will eventually reach a stable configuration  $c'$ . This final configuration  $c'$  does not depend on the order in which unstable vertices were toppled. We call  $c'$  the *stabilization* of  $c$ .

Starting from the empty configuration, one may indefinitely add any number of grains to any vertices in  $G$  and topple vertices should they become unstable. Certain stable configurations will appear again and again, that is, they *recur*, while other stable configurations will never appear again. These *recurrent configurations* are the ones that appear in the long term limit of the system. Let  $\text{Rec}(G)$  be the set of recurrent configurations on  $G$ . Determining the set  $\text{Rec}(G)$  for a given graph  $G$  is not a straightforward task. In [5, Section 6], Dhar describes the so-called *burning algorithm*, which establishes in linear time whether a given stable configuration is recurrent:

**Proposition 2.1** ([5], Section 6.1). *Let  $G$  be a graph with sink  $s$ , and let  $c$  be a stable configuration on  $G$ . Then  $c$  is recurrent if and only if there exists an ordering  $v_0 = s, v_1, \dots, v_n$  of the vertices of  $G$  such that, starting from  $c$ , for any  $i \geq 1$ , toppling the vertices  $v_0, \dots, v_{i-1}$  causes the vertex  $v_i$  to become unstable. Moreover, if such a sequence exists, then toppling  $v_0, \dots, v_n$  returns the initial configuration  $c$ .*

We will consider two slight variants of the classical parallel chip-firing process that we call CTI toppling and ITC toppling. We first explain CTI toppling. This variant is always only applied to a recurrent configuration  $c \in \text{Rec}(S_{n,d})$  whose sink has just been toppled.

Given an unstable configuration, we first check if there are any unstable clique vertices. If there are then topple these in parallel. Next check if there are any unstable independent vertices. If there are then topple these in parallel. We repeat these two steps successively until there are no remaining unstable vertices. We will refer to this toppling order as *CTI toppling*, an abbreviation for Clique Then Independent. We will also write  $\text{Topple}_{\text{CTI}}(c) = (P_1, Q_1, \dots, P_t, Q_t)$  to indicate the vertices that were toppled in parallel at each step. Here  $P_1$  represents the (non-sink) clique vertices that were initially toppled in parallel.  $Q_1$  is the collection of independent vertices that were next toppled, and so on. Note that these sets can be empty but their unions  $P_i \cup Q_i \neq \emptyset$ . We will find it convenient to replace the comma that separates the clique and independent parts in a configuration with a semi-colon.

**Example 2.2.** Consider the recurrent configuration  $c = (7, 6, 5, 2, 1; 5, 4, 4)$  on  $S_{5,3}$ . Topple the sink to get  $(8, 7, 6, 3, 2; 6, 5, 5)$ . The set of unstable clique vertices is now  $P_1 = \{v_1\}$ . Topple all in  $P_1$  to get the configuration  $(0, 8, 7, 4, 3; 7, 6, 6)$ . The set of unstable independent vertices is now  $Q_1 = \{w_1, w_2, w_3\}$ . Topple all in  $Q_1$  to get the configuration  $(3, 11, 10, 7, 6; 1, 0, 0)$ . The set of unstable clique vertices is  $P_2 = \{v_2, v_3\}$ . Topple all in  $P_2$  to get the configuration  $(5, 4, 3, 9, 8; 3, 2, 2)$ . The set of unstable independent vertices is  $Q_2 = \emptyset$  so the configuration remains unchanged. The set of unstable clique vertices is  $P_3 = \{v_4, v_5\}$ . Topple all in  $P_3$  to get the original configuration  $(7, 6, 5, 2, 1; 5, 4, 4)$ . As there are no further topplings to be done, we will set  $Q_3 = \emptyset$  so that  $P_3$  has a pair. To conclude,  $\text{Topple}_{\text{CTI}}(c) = (\{v_1\}, \{w_1, w_2, w_3\}, \{v_2, v_3\}, \emptyset, \{v_4, v_5\}, \emptyset)$ .

Let  $\text{SortedRec}(S_{n,d})$  be the set of sorted recurrent configurations on  $S_{n,d}$ , i.e. those recurrent configurations  $(a_1, \dots, a_n; b_1, \dots, b_d)$  for which

$$a_1 \geq a_2 \geq \dots \geq a_n \text{ and } b_1 \geq b_2 \geq \dots \geq b_d.$$

Given  $c \in \text{SortedRec}(S_{n,d})$ , we define the *height of  $c$* ,  $\text{height}(c)$ , to be the sum of the configuration entries. Suppose  $\text{Topple}_{\text{CTI}}(c) = (P_1, Q_1, \dots, P_t, Q_t)$ . Define  $\text{topple}_{\text{CTI}}(c) :=$

$(p_1, q_1, \dots, p_t, q_t)$ , and

$$\text{wtopple}_{CTI}(c) := 1(p_1 + q_1) + 2(p_2 + q_2) + \dots + t(p_t + q_t).$$

**Example 2.3.** Consider the sandpile model on  $S_{2,2}$  with sink  $v_3$ . In the table below for every configuration  $c \in \text{SortedRec}(S_{2,2})$  we give the height  $\text{height}(c)$ , the bounce path  $\text{topple}_{CTI}(c)$ , and the bounce  $\text{wtopple}_{CTI}(c)$ .

$c$	$\text{height}(c)$	$\text{topple}_{CTI}(c)$	$\text{wtopple}_{CTI}(c)$	$c$	$\text{height}(c)$	$\text{topple}_{CTI}(c)$	$\text{wtopple}_{CTI}(c)$
(3, 3; 2, 2)	10	(2, 2, 0)	4	(2, 2; 2, 1)	7	(0, 1, 2, 1, 0)	7
(3, 3; 1, 1)	8	(2, 2, 0)	4	(3, 3; 2, 1)	9	(2, 2, 0)	4
(2, 2; 2, 0)	6	(0, 1, 2, 1, 0)	7	(3, 3; 0, 0)	6	(2, 2, 0)	4
(3, 3; 1, 0)	7	(2, 2, 0)	4	(3, 3; 2, 0)	8	(2, 2, 0)	4
(1, 1; 2, 2)	6	(0, 2, 2)	6	(3, 2; 2, 2)	9	(1, 2, 1)	5
(2, 1; 2, 1)	6	(0, 1, 1, 1, 1)	8	(3, 1; 1, 0)	5	(1, 1, 1, 1, 0)	6
(3, 1; 2, 0)	6	(1, 1, 1, 1, 0)	6	(2, 0; 2, 2)	6	(0, 2, 1, 0, 1)	7
(3, 1; 2, 2)	8	(1, 2, 1)	5	(3, 0; 1, 1)	5	(1, 2, 1)	5
(3, 0; 2, 1)	6	(1, 2, 1)	5	(3, 0; 2, 2)	7	(1, 2, 1)	5
(3, 1; 2, 1)	7	(1, 2, 1)	5	(3, 1; 1, 1)	6	(1, 2, 1)	5
(2, 0; 2, 1)	5	(0, 1, 1, 1, 1)	8	(2, 1; 2, 2)	7	(0, 2, 2)	6
(1, 0; 2, 2)	5	(0, 2, 1, 0, 1)	7	(3, 2; 2, 0)	7	(1, 1, 1, 1, 0)	6
(3, 2; 1, 0)	6	(1, 1, 1, 1, 0)	6	(3, 2; 0, 0)	5	(1, 0, 1, 2, 0)	7
(2, 1; 2, 0)	5	(0, 1, 1, 0, 1, 1, 0)	9	(2, 2; 2, 2)	8	(0, 2, 2)	6
(3, 2; 2, 1)	8	(1, 2, 1)	5	(3, 2; 1, 1)	7	(1, 2, 1)	5

Given the set-up above, let us define the following bivariate polynomial that we call the  $q, t$ -CTI polynomial:

$$\mathfrak{F}_{n,d}^{CTI}(q, t) := \sum_{c \in \text{SortedRec}(S_{n,d})} q^{\text{height}(c) - \binom{n+d}{2} + \binom{d}{2}} t^{\text{wtopple}_{CTI}(c) - (n+d)}.$$

**Example 2.4.** The polynomial  $\mathfrak{F}_{2,2}^{CTI}(q, t)$  is readily calculated from the table in Example 2.3.

$$\begin{aligned} \mathfrak{F}_{2,2}^{CTI}(q, t) &= q^5 + t^5 + q^4 t + q t^4 + q^3 t^2 + q^2 t^3 + q^4 + t^4 + 2q^3 t + 2q t^3 + 2q^2 t^2 + 2q^3 + 2t^3 \\ &\quad + 3q^2 t + 3q t^2 + q^2 + t^2 + 2q t + q + t. \end{aligned}$$

Now that CTI toppling and the statistics associated with it have been defined, it is straightforward to define ITC toppling and its associated statistics. Almost everything is the same as in the CTI case, except what happens immediately after toppling the sink. ITC toppling stands for Independent Then Clique toppling: Given the unstable configuration the results from toppling the sink, first identifying those independent vertices that are unstable. Topple those unstable independent vertices in parallel and identify those clique vertices that are unstable. Topple those unstable clique vertices, and so on.

Given  $c \in \text{SortedRec}(S_{n,d})$ ,  $\text{height}(c)$  is the sum of the configuration entries, as before. Suppose  $\text{Topple}_{ITC}(c) = (Q'_1, P'_1, \dots, Q'_t, P'_t)$  and set  $p'_i = |P'_i|$  and  $q'_i = |Q'_i|$ . Define  $\text{topple}_{ITC}(c) := (q'_1, p'_1, \dots, q'_t, p'_t)$ , and

$$\text{wtopple}_{ITC}(c) := 1(q'_1 + p'_1) + 2(q'_2 + p'_2) + \dots + t(q'_t + p'_t).$$

Finally, we define the accompanying polynomial to these two statistics:

$$\mathfrak{F}_{n,d}^{ITC}(q, t) := \sum_{c \in \text{SortedRec}(S_{n,d})} q^{\text{height}(c) - \binom{n+d}{2} + \binom{d}{2}} t^{\text{wtopple}_{ITC}(c) - (n+d)},$$

and call it the  $q, t$ -ITC polynomial.

We will have more to say about the  $\mathfrak{F}_{n,d}^{ITC}(q, t)$  polynomials in the next section and their relationship to other polynomials that appear in the literature. One final piece of notation that is related to ITC toppling and will be of use going forward is the following.

**Definition 2.5.** Suppose  $c \in \text{SortedRec}(S_{n,d})$  with  $\text{topple}_{ITC}(c) := (q'_1, p'_1, \dots, q'_t, p'_t)$ . We will call the pair

$$((q'_0, q'_1, \dots, q'_t), (p'_0, p'_1, \dots, p'_t))$$

an *ITC-toppling sequence*. Here we use the convention  $q'_0 = 0$  and  $p'_0 = 1$ . Note that  $p'_i = 0$  and for all other  $i \geq 1$  it must hold that  $p'_i > 0$  and  $q'_i \geq 0$ .

### 3. SCHRÖDER PATHS, ITC-TOPPLING, AND SYMMETRIC FUNCTIONS

In this section we will use a modification of the bijection from Dukes [7] to show how an area and bounce bivariate on Schröder paths corresponds to a height and weighted-toppling bivariate on sorted recurrent configurations when ITC topplings is employed. Experimentally,  $\mathfrak{F}_{n,d}^{CTI}(q,t)$  and  $\mathfrak{F}_{n,d}^{ITC}(q,t)$  appear to be symmetric in  $q$  and  $t$ . This was the case for the corresponding bivariate polynomials for both the complete and complete bipartite graphs, where connections were found between paths and symmetric functions. In this paper we have also discovered such a connection for  $\mathfrak{F}_{n,d}^{ITC}(q,t)$ .

Let  $\text{Schröder}_{n,d}$  be the set of Schröder words consisting of  $n$  U's,  $n$  D's, and  $d$  H's. The defining property of such words is that for every prefix the number of D's that appear is never more than the number of U's that appear. In Dukes [7] it was shown that configurations in  $\text{SortedRec}(S_{n,d})$  are in one-to-one correspondence with paths in  $\text{Schröder}_{n,d}$  via a bijection

$$\phi : \text{Schröder}_{n,d} \mapsto \text{SortedRec}(S_{n,d}).$$

The bijection  $\phi$  is defined as follows:  $\phi(p) = (a_1, \dots, a_n; b_1, \dots, b_d)$  corresponds to the Schröder word  $p \in \text{Schröder}_{n,d}$  where

- $b_i$  is the number of D's following the  $i$ th H of  $p$ .
- $a_j + 1$  is the number of non-U's following the  $j$ th U of  $p$ .

For example, if  $p = UHUDUHHDUDUDD \in \text{Schröder}_{5,3}$  then

$$\phi(p) = (7, 6, 5, 2, 1; 5, 4, 4) \in \text{SortedRec}(S_{5,3}).$$

Schröder paths are pictorial representations of Schröder words where, in Sections 3 and 4 we identify  $U$ ,  $H$ , and  $D$  with the steps  $(0,1)$ ,  $(1,1)$ , and  $(1,0)$ , respectively. The blue line in Figure 2 illustrates the Schröder path for  $p$ .

Given a Schröder word/path  $p$ , we define its area and bounce in the same way as it is defined in Haglund [13, Sec.1]. Let  $\text{area}(p)$  to be the number of 'lower triangles' (triangles whose vertex set is  $\{(i,j), (i+1,j), (i+1,j+1)\}$ ) between  $p$  and the diagonal  $y = x$ .

To define  $\text{itcbounce}(p)$ , first remove all the  $H$  steps from  $p$  and collapse in the obvious way to form a Catalan path  $C(p)$ . Next construct the bounce path for  $C(p)$  which is the classical Catalan bounce path from  $(a,a)$  to  $(0,0)$ : move west until at the boundary of the path at that level; then move down until meeting  $y = x$  and change direction to west; and repeat the previous two in that order. The bounce,  $\text{bounce}(C(p))$ , is the sum of the  $x$ -coordinates of where the bounce path meets the diagonal (not including the initial point). The  $U$  steps of this bounce path occurring just before  $D$  steps are also  $U$  steps of  $C(p)$ . The corresponding  $U$  steps of  $p$  are called the 'peaks' of  $p$ . For each  $H$  step  $\alpha$  of  $p$  let  $b(\alpha)$  denote the number of peaks above it, and define the bounce of the Schröder path  $p$  to be

$$\text{itcbounce}(p) := \text{bounce}(C(p)) + \sum_{\alpha} b(\alpha),$$

where the sum is over all  $H$  steps of  $p$ .

The statistic  $\text{wtopple}_{ITC}$  is closely related to the  $\text{itcbounce}$  statistic on Schröder path via the bijection  $\phi$  of Dukes [7] composed with the mirror map  $\mu$  on words. Given  $w = w_1 w_2 \dots w_k$ , define  $\mu(w) := w_k w_{k-1} \dots w_1$ .

**Theorem 3.1.** *Given a sorted configuration  $c \in \text{SortedRec}(S_{n,d})$ , let  $w = \mu \circ \phi^{-1}(c) \in \text{Schröder}_{n,d}$ . Then*

$$\text{area}(w) = \text{height}(c) - \left( \binom{n+d}{2} - \binom{d}{2} \right) \text{ and } \text{itcbounce}(w) = \text{wtopple}_{ITC}(c) - (n+d).$$

An illustration of the proof of this theorem is given in Figure 2, and this is applied to the same path that appears in Haglund [13, Sec.1]. An immediate corollary to Theorem 3.1 is the equality of the  $q, t$ -ITC polynomial with the  $q, t$ -Schröder polynomial.

**Corollary 3.2.** *For all  $n \geq 1$  and  $d \geq 0$ , we have*

$$\mathfrak{F}_{n,d}^{ITC}(q, t) = \sum_{w \in \text{Schröder}_{n,d}} q^{\text{area}(w)} t^{\text{itcbounce}(w)} = \langle \nabla e_{n+d}, e_n h_d \rangle$$

which is known to be a polynomial symmetric in  $q$  and  $t$ .

Let us mention that there is an extension to this corollary in the very recent paper D'Adderio, Dukes, Iraci, Lazar, Le Borgne, and Vanden Wyngaerd et al. [9] that is not dependent upon a lattice path interpretation.

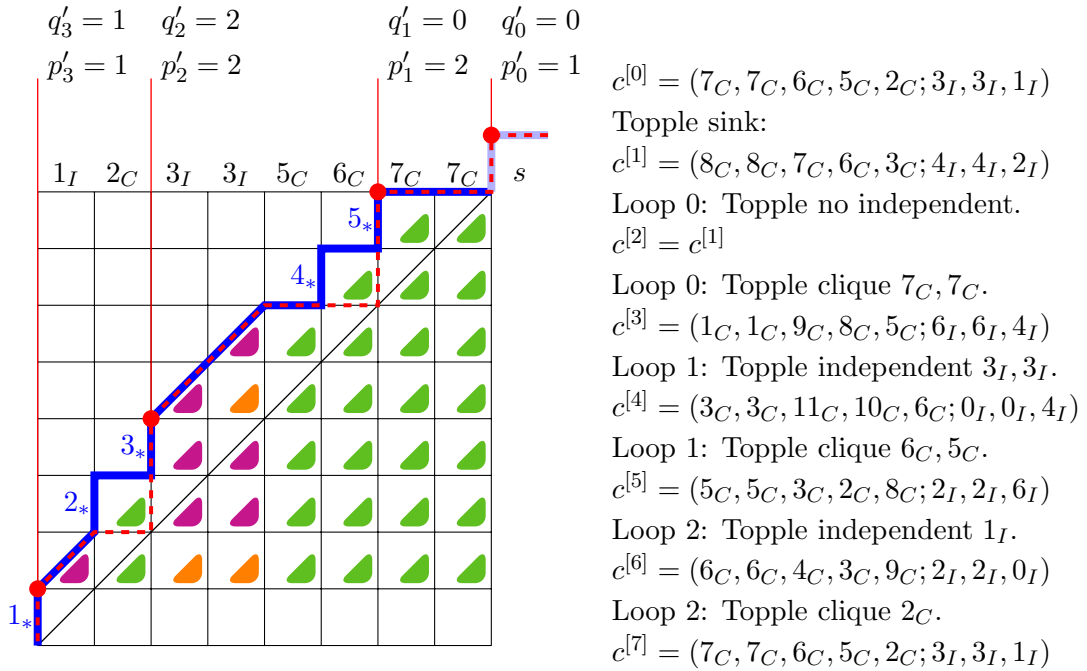


FIGURE 2. Configuration  $c = (7_C, 7_C, 6_C, 5_C, 2_C; 3_I, 3_I, 1_I) \in \text{SortedRec}(S_{5,3})$  and its related Schröder word  $w = UHUDUHHUDUDDD$  (corresponding to the example Schröder path in Haglund [13, Sec.1]). The peaks are indicated with red dots and the bounce path of is the red dotted line. To the right we follow the ITC-toppling process for  $c$ . Note that the degree of a clique vertex is  $n + d = 8$  while the degree of an independent vertex is  $d + 1 = 6$ .

By considering the definition of  $\mu \circ \phi^{-1}$ , it appears that, in the word  $w$ , the step that crosses each column encodes the number of grains on a vertex in the recurrent configuration. More precisely, a column crossed by a  $D$  step corresponds to a clique vertex and its number of grains is the number of rows below this horizontal step, less one. Alternatively, a column crossed by a  $H$  step corresponds to an independent vertex whose number of grains is the number of vertical ( $U$ ) steps, denoted by the blue  $1_*, 2_*, \dots$ , in rows below this diagonal  $H$  step.

We split the proof of Theorem 3.1 into Lemma 3.3 and Lemma 3.4, each being related to one of the two statistic equalities.

**Lemma 3.3.** *Given  $c \in \text{SortedRec}(S_{n,d})$ , let  $w = \mu \circ \phi^{-1}(c) \in \text{Schröder}_{n,d}$ . Then*

$$\text{area}(w) = \text{height}(c) - \left( \binom{n+d}{2} - \binom{d}{2} \right).$$

*Proof.* Let  $c \in \text{SortedRec}(S_{n,d})$  with  $c = (a_1, \dots, a_n; b_1, \dots, b_d)$  and set  $w = \mu \circ \phi^{-1}(c)$ . We will prove that  $\text{level}(\phi \circ \mu(w)) = \text{area}(w)$  where  $\text{level}(c) = \text{height}(c) - \binom{n+d}{2} + \binom{d}{2}$  is the number of grains in  $c$  minus the number of edges non-incident to the sink in  $S_{n,d}$ .

When the Schröder word  $w$  starts with the letter  $U$ , as in the example in Figure 2, we have two ways to count the number of lower triangles in all the squares below the Schröder path excluding those on the first row:

$$\sum_{k=1}^n a_k + \sum_{k=1}^d b_k + \binom{d}{2} = \text{area}(w) + \binom{n+d}{2}.$$

On the left hand side, the first sum corresponds to those lower triangles in columns corresponding to clique vertices (these are the green lower triangles in Figure 2). More precisely, in the column of a clique vertex (which are those with a  $D$  step), the green triangles also count the number of grains given by the definition of  $\mu \circ \phi^{-1}$ .

The remainder of the left-hand-side concerns columns that correspond to independent vertices (those columns crossed by a  $H$  step). There are two cases to consider here.

- If a lower triangle has a  $H$  step beside it, then let us call it a *magenta lower triangle*. Likewise, if a lower triangle has a  $H$  step in its column and a  $U$  step in its row, then also call it a magenta lower triangle.
- If a lower triangle has a  $H$  step in its column and a different  $H$  step in its row, then we call it an *orange lower triangle*.

Note that there will be  $\binom{d}{2}$  orange triangles since there are  $d$   $H$  steps and every pair of distinct  $H$  steps gives rise to an orange lower triangle. The number of magenta triangles corresponds to  $\sum_{k=1}^d b_k$ . On the right-hand side, the lower triangles are either counted by  $\text{area}(w)$  for those above the main diagonal and  $\binom{n+d}{2}$  for those below.

The identity may be rewritten:

$$\begin{aligned} \text{area}(w) &= \sum_{k=1}^n a_k + \sum_{k=1}^d b_k + \binom{d}{2} - \binom{n+d}{2} \\ &= \text{height}(c) - \left( \binom{n+d}{2} - \binom{d}{2} \right) = \text{level}(c). \end{aligned}$$

Alternatively, should  $w$  begin with the letter  $H$  then set  $w = H^j w'$  (where  $j \geq 1$ ) and  $w'$  begins with the letter  $U$ . For this case  $\text{area}(w) = \text{area}(w')$  and there are  $j$  independent vertices having 0 grains that we can simply ignore in  $\sum_{k=1}^d b_k = \sum_{k=1}^{d-j} b_k$ . An example of this case is illustrated in Figure 3.

Again, we can count the number of lower triangles beneath  $w$  in two different ways. The only difference to the previous counting argument is that there are now  $j$  initial  $H$  steps and these give rise to three new lower triangle ‘types’ to which we associate the adjectives blue, brown, and grey. The blue lower triangles are those that are formed as a result of the initial  $j$   $H$  steps (the number of these will be  $\binom{j}{2}$ ) of  $w$ . The brown lower triangles correspond to columns that are traversed by a  $H$  step and have one of the initial  $j$   $H$  steps in the same row. Similarly, the grey triangles correspond to columns that are traversed by a  $D$  step and have one of the initial  $H$  steps in the same row. This gives the equation:

$$\begin{aligned} &\binom{j}{2} + j(n-j) + \sum_{k=1}^n (a_k - j) + jn + \sum_{k=1}^{d-j} b_k + \binom{n-j}{2} \\ &= \text{area}(w') + \binom{j}{2} + j(n-j) + jd + \binom{n+d-j}{2}. \end{aligned}$$



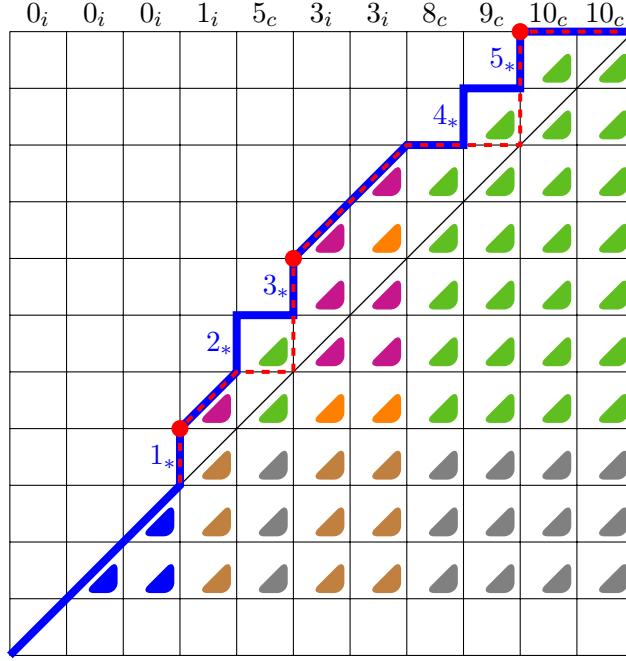


FIGURE 3. A word having prefix  $H$ . Here  $w = HHHw'$  so that  $j = 3$ .

After simplifying we obtain:

$$\sum_{k=1}^n (a_k - j) + \sum_{k=1}^{d-j} b_k + \binom{n-j}{2} = \text{area}(w') + \binom{(n-j) + d}{2}.$$

This expression corresponds to the identity for the previous case for configuration

$$c' := (a_1 - j, \dots, a_n - j; b_1, b_2, \dots, b_{d-j}) = \mu \circ \phi^{-1}(w').$$

By the first case, we have  $\text{level}(c') = \text{area}(w')$ . Since the number of grains in  $c$  and  $c'$  differs by  $jd$ , which is also the difference of the number of edges non-incident to sink between  $S_{n,d}$  and  $S_{n,d-j}$ , we deduce that  $\text{level}(c) = \text{level}(c')$ . We already observed that  $\text{area}(w) = \text{area}(w')$ . Hence  $\text{level}(c) = \text{level}(c') = \text{area}(w') = \text{area}(w)$ .  $\square$

**Lemma 3.4.** *Given  $c \in \text{SortedRec}(S_{n,d})$ , let  $w = \mu \circ \phi^{-1}(c) \in \text{Schroder}_{n,d}$ . Then*

$$\text{wtopple}_{ITC}(u) - (n + d) = \text{itcbounce}(w).$$

*Proof.* Given  $c \in \text{SortedRec}(S_{n,d})$  we consider its ITC-topping sequence  $((p_k, q_k))_{k \geq 0}$ . From the definition of an ITC-topping sequence, we define the peaks  $(\text{Peak}^{\text{loop}}(k))_{k \geq 0}$  of loop iterations as follows:

$$\text{Peak}^{\text{loop}}(k) := (x_k, y_k)$$

where  $y_k$  is the number of untopped vertices in the middle of the  $k$ -th loop iteration (just after the toppling of independent vertices, if any), and  $x_k$  is the number of untopped vertices at the end of the  $k$ -th loop iteration. By definition of the ITC-topping sequence, we have

$$y_k = n + d + 1 - \left( q_k + \sum_{j=0}^{k-1} (q_j + p_j) \right) \text{ and } x_k = n + d + 1 - \sum_{j=0}^k (q_j + p_j),$$

where we simply subtract from all the  $n + d + 1$  vertices the toppled vertices up to that time. These peaks coincide with the peaks  $(\text{Peak}^{\text{bounce}}(k))_k$  defined by Haglund [13, Sec.1] on the Schröder path  $w = \mu \circ \phi^{-1}(c)$ .

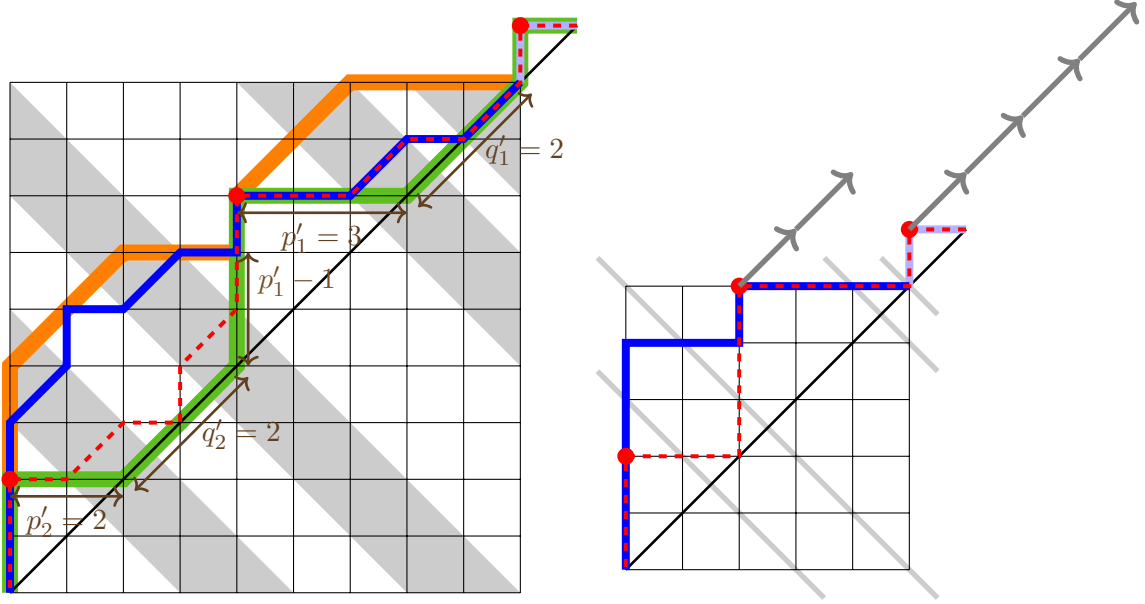


FIGURE 4. Reformulation of the bounce path directly on Schröder paths instead of on the Dyck path subword formed by deleting  $H$  steps. Configuration  $c = (7, 6, 6, 5, 4; 5, 5, 4, 3) \in \text{SortedRec}(S_{4,4})$  maps to  $c' = (4, 4, 4, 3, 3)$  that is recurrent on  $S_{5,0} = K_6$ . Notice  $\text{Peak}_{\text{Dyck}}(2) = (0, p'_2)$  and  $\text{Peak}_{\text{Schröder}}(2) = (0, p'_2)$ ;  $\text{Peak}_{\text{Dyck}}(1) = (p'_2, p'_2 + p'_1)$  and  $\text{Peak}_{\text{Schröder}}(1) = (p'_2 + q'_2, p'_2 + p'_1 + q'_2)$ ;  $\text{Peak}_{\text{Dyck}}(0) = (p'_2 + p'_1, p'_2 + p'_1 + p'_0)$  and  $\text{Peak}_{\text{Schröder}}(0) = (p'_2 + p'_1 + q'_2 + q'_1, p'_2 + p'_1 + p'_0 + q'_2 + q'_1)$

The ITC-topping sequence induces a partition  $(P'_k)_{k=0,\dots,b}$  of sink and clique vertices:

$$\{s\} \cup V = P'_0 \cup P'_1 \cup \dots \cup P'_b$$

where  $P'_0 = \{s\}$  and otherwise

$$P'_k = \left\{ v_\ell : \sum_{j=0}^{k-1} p_j \leq \ell < \sum_{j=0}^k p_j \right\}.$$

These are equivalently defined through the pairs  $((Q'_k, P'_k))_k$  of the ITC-topping process on  $c$ . Notice that  $|P'_k| = p_k$ .

Let  $\pi_{U,D}$  be the map that keeps only the occurrences of letters  $U$  or  $D$  in any word, i.e. it removes all occurrences of  $H$  in Schröder words. We define a compressed configuration  $c'$  on  $S_{n,0} = \{s\} \cup V = K_{n+1}$  as

$$c' := \text{compress}_{K_{n+1}}(c) := \phi \circ \pi_{U,D} \circ \phi^{-1}(c),$$

where  $\phi$  is applied to  $S_{n,d}$  whereas  $\phi^{-1}$  is applied to  $S_{n,0}$ . The configuration  $c'$  is equivalently defined by:  $c'_i + 1$  is the number of occurrences of letters  $U$  after the  $i$ -th occurrence of letter  $D$  when the path is read from north-east to south-west. See Figure 4 for an example.

If  $w = \mu \circ \phi^{-1}(c)$  is the Schröder path of configuration  $c$ , then  $w'$  is the Dyck word of  $c'$  obtained by deletion of counter-diagonals crossed by diagonal steps. Haglund's [13, Sec.1] bounce path for the Schröder path  $w$  is obtained from the bounce path of the Dyck path  $w'$  by inserting diagonal steps in the bounce path parallel to (and in the same counter-diagonal) those of  $w$ .

This implies that the bounce path in  $w'$  has bounces of size  $p_0 = 1, p_1, \dots, p_b$  so that, in particular, the peaks in the bounce path of  $w'$  are

$$\text{Peak}_{Dyck}(k) = \left( n + 1 - \sum_{j=0}^k p_j, n + 1 - \sum_{j=0}^{k-1} p_j \right) = \left( \sum_{j=k+1}^b p_j, \sum_{j=k}^b p_j \right).$$

This leads to the same partition  $(P'_i)_{i \geq 0}$  of  $D$  steps in the Dyck word  $w'$  as in the Schröder word  $w$ . Any peak is an endpoint of a vertical ( $U$ ) step of the Dyck word. The insertion of the deleted counter-diagonals preserves this property in Haglund's definition of peaks for a Schröder word.

It remains to count the number of counter-diagonals inserted below each  $\text{Peak}_{Dyck}(k)$  peak. It appears, by the ITC-toppling process, that the peak ending bounce  $p_k$  admits  $\sum_{j=k+1}^b q_j$  such counter-diagonals. These correspond to the independent vertices toppled after the end of the  $k$ -th loop iteration. Hence

$$\text{Peak}_{Schröder}(k) = \text{Peak}_{Dyck}(k) + \left( \sum_{j=k+1}^b q_j \right) (1, 1) = \left( \sum_{j=k+1}^b p_j + \sum_{j=k+1}^b q_j, \sum_{j=k}^b p_j + \sum_{j=k+1}^b q_j \right)$$

and this coincides with  $\text{Peak}^{loop}(k)$  by rewriting and making use of the facts  $n + 1 = \sum_{j=0}^b p_j$  and  $d = \sum_{j=0}^b q_j$ . From Haglund [13, Eqn.(10)] we have  $\text{bounce}(w) = \text{bounce}(w') + \sum_k b(k)$ . The consideration above allows us to write  $\text{bounce}(w') = \sum_k (k - 1)p_k$  and  $\sum_k b(k) = \sum_k (k - 1)q_k$ , and we have

$$\text{bounce}(w) = \sum_{j=1}^b (j - 1)(p_j + q_j)$$

and, by the definition of the ITC-toppling process  $\text{wtopple}$ ,

$$\text{wtopple}(c) = \sum_{j=0}^b j(p_j + q_j).$$

The two differ by the value  $n + d$ . □

*Proof of Theorem 3.1.* Combine Lemmas 3.3 and 3.4. □

#### 4. AN EXPLICIT SUM FOR $\mathfrak{F}_{n,d}^{ITC}(q, t)$

Let  $\text{ITC}_{n,d}$  be the set of all possible ITC-toppling sequences on  $S_{n,d}$ . The set  $\text{ITC}_{n,d}$  indexes a partition of the set  $\text{Schröder}_{n,d}$  of Schröder words. This partition gives rise to an explicit sum involving the Gaussian binomial coefficients for the  $q, t$ -ITC polynomial.

Recall that a *composition* of an integer  $n$  is a sequence of positive integers  $a = (a_1, \dots, a_k)$  whose sum is  $n$ . We will denote this  $a \vDash n$ , and also use the notation  $a \vDash_k n$  to indicate  $a$  has length  $k$ . A *weak composition* of  $n$  is a sequence of non-negative integers whose sum is  $n$ . We will denote this  $a \vDash^* n$ , and also use the notation  $a \vDash_k^* n$  to indicate a weak-composition  $a$  has length  $k$ .

Define the partial order  $\leq$  on Schröder paths in  $\text{Schröder}_{n,d}$  by  $w \leq w'$  if all lower triangles of  $w$  are also lower triangles of  $w'$  (see Figure 2 for an illustration of lower triangles).

**Theorem 4.1.** *For all  $n \geq 1$  and  $d \geq 0$ ,*

$$\mathfrak{F}_{n,d}^{ITC}(q, t) = \sum_{k=1}^n \sum_{\substack{(b_1, \dots, b_k) \vDash_k^* d \\ (a_1, \dots, a_{k-1}, 0) \vDash_k n}} \prod_{i=1}^k q^{\binom{a_i}{2}} \binom{a_i + b_i}{b_i}_q \binom{a_i + b_i + a_{i-1} - 1}{a_{i-1} - 1}_q t^{(i-1)(a_i + b_i)}.$$

In the above expression we define  $a_0 := 1$ .

The formula in Theorem 4.1 is similar, but slightly different in form to, the one that appears in Egge, Haglund, Killpatrick and Kremer [6, Theorem 1]. Our aim in presenting the formula above is to provide one that has a natural interpretation in terms of the sandpile model through the correspondences we have used.

*Proof of Theorem 4.1.* Lemma 4.2 describes the set  $\text{ITC}_{n,d}$  of all the possible ITC-toppling sequences as being in bijection with pairs  $(b = (b_1, \dots, b_k), a = (a_1, \dots, a_{k-1}))$  where  $b \vDash_k^* d$  and  $a \vDash_{k-1} n$ .

Lemma 4.3 then shows that any ITC-toppling sequence defines two extremal Schröder words  $w^{\text{lower}}$  and  $w^{\text{upper}}$  such that all Schröder paths admitting this ITC-toppling sequence are the Schröder words  $w'$  such that  $w^{\text{lower}} \leq w' \leq w^{\text{upper}}$ . The Schröder words satisfying these inequalities correspond to those Schröder paths geometrically enclosed by a sequence of (possibly degenerate) hexagons that are defined by the pair  $(w^{\text{lower}}, w^{\text{upper}})$ , instead of by the sequences of rectangles as in the usual Dyck word case.

Finally, Lemma 4.4 shows that the set of Schröder words contained between two extremal Schröder words may be obtained from  $w^{\text{lower}}$  whose minimal (in  $q$ ) weight is  $\prod_{i=1}^k q^{\binom{a_i}{2}} t^{(i-1)(b_i+a_i)}$  by commutation of steps inside the same hexagon. Since, inside each hexagon having side lengths  $a_{i-1}-1, b_i$  and  $a_i$  (for a visual aid see see Figure 4), every commutation is possible within each hexagon, it appears that the sum of all possible commutations leads to the  $q$ -multinomial coefficient  $\binom{a_{i-1}-1+b_i+a_i}{a_{i-1}-1, b_i, a_i}_q$  for this hexagon. Since two commutations in distinct hexagons are independent we obtain the expected formula.  $\square$

**Lemma 4.2.** *For all  $n \geq 1, d \geq 0$ ,*

$$\text{ITC}_{n,d} = \{((0, b_1, \dots, b_k), (1, a_1, \dots, a_{k-1}, 0)) : (b_1, \dots, b_k) \vDash^* d \text{ and } (a_1, \dots, a_{k-1}) \vDash n\}.$$

*Proof.* Let  $X_{n,d}$  be the set on the right-hand side of the stated equality. From Definition 2.5 we have

$$\begin{aligned} \text{ITC}_{n,d} = \{ & ((0, b_1, \dots, b_k), (1, a_1, \dots, a_{k-1}, 0)) : \\ & (b_1, a_1, \dots, b_k, a_k) = \text{topple}_{\text{ITC}}(c) \text{ for some } c \in \text{SortedRec}(S_{n,d}) \}. \end{aligned}$$

Since  $b_i \geq 0$  for all  $1 \leq i \leq k$ ,  $a_i > 0$  for all  $1 \leq i < k$ , and  $a_k = 0$ , we have further restrictions on the sequences that arise. In addition to that, Dhar's burning algorithm tells us that for every recurrent configuration  $c$ , every non-sink vertex topples exactly once. Thus  $a_1 + \dots + a_t = n$  and  $b_1 + \dots + b_t = d$  for some  $t$ . Since  $a_t = 0$  we must have  $a_1 + \dots + a_{t-1} = n$ . Therefore

$$\begin{aligned} \text{ITC}_{n,d} \subseteq \{ & ((0, b_1, \dots, b_k), (1, a_1, \dots, a_{k-1}, 0)) : (b_1, \dots, b_k) \vDash^* d \text{ and } (a_1, \dots, a_{k-1}) \vDash n\} \\ =: & X_{n,d}. \end{aligned}$$

Next, consider  $((0, b_1, \dots, b_k), (1, a_1, \dots, a_{k-1}, 0)) \in X_{n,d}$ . Let  $c$  be the configuration on  $S_{n,d}$  defined as follows:

$$c(v_i) := n + d - \left( \sum_{j=0}^{k-1} a_j + b_j \right) - b_k,$$

for all  $n - (a_0 + \dots + a_k) \leq i < n - (a_0 + \dots + a_{k-1})$  and  $1 \leq k \leq t$ , and

$$c(w_i) := n + 1 - (a_0 + a_1 + \dots + a_{k-1}),$$

for all  $d - (b_0 + \dots + b_k) \leq i < d - (b_0 + \dots + b_{k-1})$  and  $1 \leq k \leq t$ . It is straightforward to see that this configuration is indeed recurrent, weakly decreasing, and has  $\text{topple}_{\text{ITC}}(c) = (b_1, a_1, \dots, b_t, a_t)$ . This shows  $X_{n,d} \subseteq \text{ITC}_{n,d}$ , and so  $\text{ITC}_{n,d} = X_{n,d}$  as claimed.  $\square$

Every ITC-toppling sequence  $((b_0, \dots, b_k), (a_0, \dots, a_k))$  defines two extremal Schröder paths: the *lower (Schröder) path*

$$w^{\text{lower}} := \mu \left( H^{b_1} U^{a_1} \left( \prod_{i=2}^k D^{a_{i-1}} H^{b_i} U^{a_i} \right) \right),$$

and the *upper (Schröder) path*

$$w^{\text{upper}} := \mu(U^{a_1} H^{b_1} \left( \prod_{i=2}^k D U^{a_i} H^{b_i} D^{a_{i-1}-1} \right)).$$

Recall that  $\mu$  is the word-reversal operator and for each of the extremal paths the non-commutative product of words in each is given in increasing order of the indices. In Figure 4, the lower path  $w^{\text{lower}}$  is illustrated in green, sometimes hiding the upper path  $w^{\text{upper}}$  illustrated in orange.

**Lemma 4.3.** *Let  $w \in \text{Schröder}_{n,d}$ . The Schröder paths  $w'$  for which  $w^{\text{lower}} \leq w' \leq w^{\text{upper}}$  are precisely those Schröder paths  $w'$  for which  $\text{topple}_{ITC}(\phi(\mu(w'))) = \text{topple}_{ITC}(\phi(\mu(w)))$ .*

*Proof.* Let  $w \in \text{Schröder}_{n,d}$  and  $c = \phi(\mu(w)) \in \text{SortedRec}(S_{n,d})$  be its corresponding recurrent configuration. Suppose that  $\text{topple}_{ITC}(c) = (b_1, a_1, \dots, b_k, a_k)$  so that the corresponding ITC-toppling sequence is

$$ITC(w) = ((b_0, \dots, b_k), (a_0, \dots, a_k)).$$

The recurrent configuration that corresponds to  $w^{\text{lower}}$  is the configuration stated in the proof of Lemma 4.2. Moreover, this recurrent configuration for  $w^{\text{lower}}$  also has ITC-toppling sequence  $ITC(w)$ .

For  $w^{\text{upper}}$  the situation is similar. This is seen by considering the rewriting rule  $D^{\alpha-1} H^\beta U^\gamma \rightarrow U^\gamma H^\beta D^{\alpha-1}$  in every hexagon (between the paths  $w^{\text{lower}}$  and  $w^{\text{upper}}$ ) that corresponds to vertices toppled during one loop-iteration  $(b_i, a_i)$ . Compare to the  $w^{\text{lower}}$  case, the rewriting rule for vertices toppled during the  $k$ -th loop iteration leads to  $\alpha = a_{k-1}, \beta = b_k, \gamma = a_k$ . This means every clique vertex toppled during this loop has  $(a_k - 1) + b_k$  grains more and every independent vertex has  $a_k - 1$  grains more. Hence  $w^{\text{upper}}$  corresponds to the configuration  $c^{\text{upper}}$  as follows:

$$c^{\text{upper}}(v_i) := n + d - \left( \sum_{j=0}^{k-2} a_j + b_j \right) - b_{k-1} - 1,$$

for all  $n - (a_0 + \dots + a_k) \leq i < n - (a_0 + \dots + a_{k-1})$  and  $1 \leq k \leq t$ , and

$$c^{\text{upper}}(w_i) := n + 1 - (a_0 + a_1 + \dots + a_{k-2}) - 1,$$

for all  $d - (b_0 + \dots + b_k) \leq i < d - (b_0 + \dots + b_{k-1})$  and  $1 \leq k \leq t$ . From this it is immediate that  $ITC(w^{\text{upper}}) = ITC(w^{\text{lower}}) = ITC(w)$ .

Any  $w'$  such that  $w^{\text{lower}} \leq w' \leq w^{\text{upper}}$  must have  $ITC(w') = ITC(w)$  since stability of vertices is preserved for  $w' \leq w^{\text{upper}}$  while instability is guaranteed for  $w^{\text{lower}} \leq w'$ . For any  $w'$  that does not satisfy  $w^{\text{lower}} \leq w' \leq w^{\text{upper}}$ , we may consider the last difference which is also the first encounter by the ITC-toppling sequence. If  $w^{\text{lower}} \not\leq w'$  then, by inspection of the case of  $w^{\text{lower}}$  with at least one less grain, there exists one vertex that is not unstable when it ought to be due to a lack of grains. Likewise, if  $w' \not\leq w^{\text{upper}}$  then, by inspection of the case of  $w^{\text{upper}}$  with at least one more grain, there exists one vertex that is unstable during the previous loop iteration due to an excess of grains.

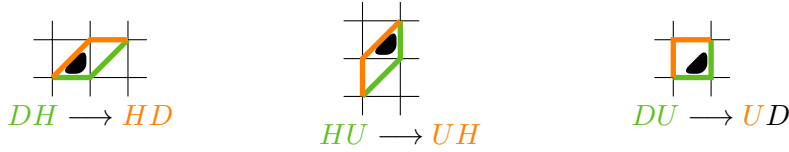
Since the ITC toppling sequences coincide precisely when  $w^{\text{lower}} \leq w' \leq w^{\text{upper}}$ , we may conclude that this is also precisely when  $\text{topple}_{ITC}(\phi(\mu(w'))) = \text{topple}_{ITC}(\phi(\mu(w)))$ .  $\square$

Between their points of intersection, the lower and upper paths forms hexagons having parallel sides, and these opposite sides have side-lengths  $a_{i-1}-1, b_i, a_i$ . Such hexagons become rectangles when  $b_i = 0$ , parallelograms when  $a_{i-1} = 1$ , horizontal lines when both  $b_i = 0$  and  $a_{i-1} = 1$ .

**Lemma 4.4.** *The generating function for a factor in Schröder path enclosed within a hexagon  $H$  defined by  $h^{\text{lower}} = D^a H^b U^c$  and  $h^{\text{upper}} = U^c H^b D^a$  according to the area enclosed in the same hexagon is given by the  $q$ -multinomial coefficient*

$$\sum_{w \in D^a \sqcup H^b \sqcup U^c} q^{\text{area}_H(w)} = \binom{a+b+c}{a, b, c}_q.$$

*Proof.* We notice by inspection of pictures below that each oriented commutation  $XY \rightarrow YX$  that generates all possible words in the (multi-)shuffle  $D^a \sqcup H^b \sqcup U^c$  add one enclosed lower half triangle.



Since  $\text{area}_H(w^{\text{lower}} = D^a H^b U^c) = 0$ , we have a combinatorial interpretation of expected  $q$ -multinomial coefficient. The generating function for the area statistic over all hexagons corresponding to a words consisting of  $a$  D's,  $b$  H's and  $c$  U's (in terms of the lower triangles between it and the path  $D^a H^b U^c$ ) is seen to correspond to the number of inversions of such a word under the natural order  $D \prec H \prec U$ . Therefore, by Foata and Han [12, Theorem 5.1], this is  $\binom{a+b+c}{a,b,c}_q$ . (This is also similar to Egge et al. [6, Lemma 1].)  $\square$

## 5. SORTED RECURRENT STATES AS SAWTOOTH POLYOMINOES

In this section we present a planar characterization of the sorted recurrent configurations from Section 2. Define the unit steps  $s = (0, -1)$ ,  $w = (0, -1)$ ,  $n = (0, 1)$ ,  $e = (1, 0)$  and the two diagonal steps  $nw = (-1, 1)$  and  $se = (1, -1)$ .

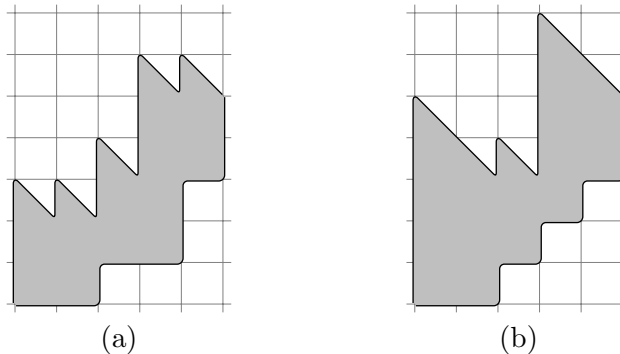
### 5.1. Sawtooth polyominoes and Schröder paths.

**Definition 5.1.** Given positive integers  $n, d$ , define  $\text{Sawtooth}_{n,d}$  to be the set of paths (*Upper*, *Lower*) in the plane with the following properties.

- The upper path *Upper* is a path from  $(0, 0)$  to  $(n, d)$  that takes steps in the set  $\{n, se\}$ .
- The lower path *Lower* is a path from  $(0, 0)$  to  $(n, d)$  that takes steps in the set  $\{e, n\}$ .
- The lower path and upper path only touch at the end-points  $(0, 0)$  and  $(n, d)$ .

We will refer to such a pair of paths, or the planar object they define, as a *sawtooth polyomino* of dimension  $(n, d)$ .

**Example 5.2.** Two sawtooth polyominoes in  $\text{Sawtooth}_{5,5}$ :



Consider the following construction on  $\text{SortedRec}(S_{n,d})$ .

**Definition 5.3.** Let  $w = w_1 w_2 \dots w_{2n+d} \in \text{Schröder}_{n,d}$ . Form a collection of steps in the plane as follows:

**The upper path:** Connect  $(n+1, d)$  to  $(n, d+1)$  with a  $nw$  step. Read  $w$  from left to right. For every U letter draw a  $nw$  step and for every non-U letter draw a  $s$  step. At the end of this process draw a final  $s$  step so as to touch the origin. Let  $\text{Upper}(w)$  be this path from  $(n+1, d)$  to  $(0, 0)$ .

**The lower path:** Read  $w$  from left to right. Starting from  $(n+1, d)$ , for every H letter in  $w$  draw a  $s$  step and for every D letter in  $w$  draw a  $w$  step. At the end of this process connect the final point  $(1, 0)$  to the origin. Let  $\text{Lower}(w)$  be this path from  $(n+1, d)$  to  $(0, 0)$ .

Denote by  $\varphi_{n,d}(w)$  the pair  $(\text{Upper}(w), \text{Lower}(w))$ .

**Example 5.4.** Consider  $w = HUHDHUHDUDUHD \in \text{Schröder}_{4,5}$  which corresponds to the configuration  $c = (7, 4, 2, 1; 4, 4, 3, 3, 1) \in \text{SortedRec}(S_{4,5})$ . The upper and lower paths  $\text{Upper}(w)$  and  $\text{Lower}(w)$  are illustrated in Figure 5.2(a).

**Theorem 5.5.** A word  $w \in \text{Schröder}_{n,d}$  iff  $\varphi_{n,d}(w) \in \text{Sawtooth}_{n+1,d}$ .

*Proof.* Let  $w \in \text{Schröder}_{n,d}$ . By the construction of Definition 5.3, the paths  $\text{Upper}(w)$  and  $\text{Lower}(w)$  are both paths from  $(n+1, d)$  to  $(0, 0)$  whose step types correspond to those Definition 5.1. Let  $\text{Words}_{n,d}$  be the set of words consisting of  $n$  U's,  $n$  D's, and  $d$  H's. We will prove:

- (i) If  $w \in \text{Schröder}_{n,d}$  then  $\varphi_{n,d}(w) \in \text{Sawtooth}_{n+1,d}$ .
- (ii) If  $w \in \text{Words}_{n,d} \setminus \text{Schröder}_{n,d}$  then  $\varphi_{n,d}(w) \notin \text{Sawtooth}_{n+1,d}$ .

For (i), suppose that  $w \in \text{Schröder}_{n,d}$ . Then  $\varphi_{n,d}(w) = (\text{Upper}(w), \text{Lower}(w))$  with

$$\text{Upper}(w) = (a_1, \dots, a_{2n+2+d}) \text{ and } \text{Lower}(w) = (b_1, b_2, \dots, b_{n+1+d}).$$

Note that  $a_i \in \{\text{nw}, \text{s}\}$  and  $b_j \in \{\text{w}, \text{s}\}$ . Let us assume that the upper and lower paths meet at the point one reaches by starting at  $(m, n)$  and following steps  $(a_1, \dots, a_{p+2q})$  and  $(b_1, \dots, b_{p+q})$  where the upper path contains  $q$  nw steps and  $p+q$  s steps and the lower path contains  $p$  s steps and  $q$  w steps. The point at which these two partial paths meet will be  $(n+1-q, d-p)$ . By Definition 5.3,  $a_1 = \text{nw}$  and for  $i \in [1, p+2q-1]$ ,  $a_{1+i} = \text{nw}$  iff  $w_i = U$  and  $a_{1+i} = \text{s}$  iff  $w_i = H$  or  $D$ . Similarly,  $b_j = \text{s}$  iff the  $j$ th non-U letter of  $w$  is  $H$  and  $b_j = \text{w}$  iff the  $j$ th non-U letter of  $w$  is  $H$ . These facts show that for  $(a_2, \dots, a_{p+2q}) = (w_1, \dots, w_{p+2q-1})$  there are precisely  $p$  H steps,  $q$  D steps, and  $(q-1)$  U steps. The number of D's in this prefix of a Schröder word is one more than the number of U's, thereby contradicting the definition of a Schröder word. Hence these two paths cannot meet and  $\varphi_{n,d}(w) \in \text{Sawtooth}_{n+1,d}$ .

For (ii), let  $w \in \text{Words}_{n,d} \setminus \text{Schröder}_{n,d}$ . This means  $w$  is a word consisting of  $n$  U's,  $n$  D's, and  $d$  H's that violates the Schröder path property. In other words, there exists an index  $k$  such that the prefix  $w_1 w_2 \dots w_k$  contains more D's than U's. Let  $k$  be the smallest index that satisfies this property so that there are  $t$  U's,  $t+1$  D's, and  $(k-2t-1)$  H's in the prefix. Consider now  $\text{Upper}(w_1 w_2 \dots w_k) = (a_1, \dots, a_{k+1})$  and  $\text{Lower}(w_1 w_2 \dots w_k) = (b_1, b_2, \dots, b_\ell)$ . The upper path will end at position  $(n, d+1) + t(-1, 1) + (k-2t-1+t+1)(0, -1) = (n-t, d+1+t-(k-t)) = (n-t, d+1+2t-k)$ . The lower path will end at position  $(n+1, d) + (t+1)(-1, 0) + (k-2t-1)(0, -1) = (n-t, d+1+2t-k)$ , and so the upper and lower paths touch. This implies  $\varphi_{n,d}(w) \notin \text{Sawtooth}_{n+1,d}$ .  $\square$

**5.2. Mapping sorted recurrent configurations to sawtooth polyominoes.** The composition of the bijections relating sorted recurrent configurations to Schröder words and Schröder words to sawtooth polyominoes allows us to give a direct mapping from sorted recurrent configurations to sawtooth polyominoes that we now state.

**Definition 5.6.** Let  $c = (a_1, \dots, a_n; b_1, \dots, b_d) \in \text{SortedRec}(S_{n,d})$ .

- (i) For each  $i \in [1, d]$ , draw a vertical line segment from  $(1+b_{d+1-i}, i-1)$  to  $(1+b_{d+1-i}, i)$ .
- (ii) Connect the endpoints of the vertical line segments in the previous step if they have the same ordinate. In the case of an endpoint with ordinate 0, connect it to the origin. In the case of an endpoint with ordinate  $d$ , connect it to the point  $(n+1, d)$ .
- (iii) For each  $j \in [1, n]$ , draw a diagonal line segment from  $(j-1, 3-j+a_{n+1-j})$  to  $(j, 2-j+a_{n+1-j})$ . Draw a diagonal line segment from  $(n, d+1)$  to  $(n+1, d)$ .
- (iv) Connect the endpoints of the diagonal line segments in the previous step if they have the same abscissa. In the case of an endpoint with abscissa 0, connect it to the origin.

Let us denote by  $Upp$  the path that results from steps (i) and (ii), and by  $Low$  the path that results from steps (iii) and (iv). Denote by  $f_{n,d}(c)$  the pair  $(Upp, Low)$ .

**Example 5.7.** Consider  $c = (7, 4, 2, 1; 4, 4, 3, 3, 1) \in \text{SortedRec}(S_{4,5})$ . In this example  $n = 4$  and  $d = 5$ . We first draw the vertical line segments  $(1+1, 0) \rightarrow (1+1, 0)$ ,  $(1+3, 1) \rightarrow (1+3, 1)$ ,  $(1+3, 2) \rightarrow (1+3, 3)$ ,  $(1+4, 3) \rightarrow (1+4, 4)$ , and  $(1+4, 4) \rightarrow (1+4, 5)$ . Then connect those vertical line segments that have endpoints of the same ordinate. Connect  $(1+1, 0)$  to the origin  $(0, 0)$  and connect  $(1+4, 4)$  to  $(m = 5, n = 5)$ . Secondly, draw the diagonal segments  $(0, 2+1) \rightarrow (1, 1+1)$ ,  $(1, 1+2) \rightarrow (2, 0+2)$ ,  $(2, 0+4) \rightarrow (3, -1+4)$ ,  $(3, -1+7) \rightarrow (4, -2+7)$ . Also connect  $(n = 4, d+1 = 6) \rightarrow (n+1 = 5, d = 5)$ . Then connect those diagonal segments whose endpoints have the same abscissa. Connect  $(0, 2+1)$  to the origin. The upper and lower paths  $\text{Upper}(w)$  and  $\text{Lower}(w)$  are illustrated in Example 5.2(a).

**5.3. CTI topplings and bounce paths within sawtooth polyominoes.** The polyomino representation of a sorted recurrent configuration allows us to visualize the CTI toppling process in a rather compact way. This is similar to the bounce path that featured in the paper on parallelogram polyominoes [8].

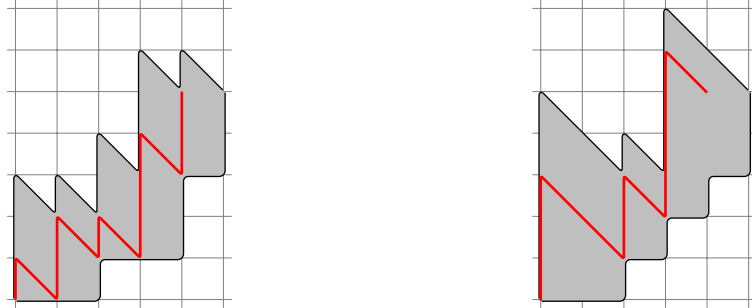
**Definition 5.8.** Given  $P \in \text{Sawtooth}_{n+1,d}$ , let the *CTI-bounce path* of  $P$  be the path from  $(n, d)$  to  $(0, 0)$  that takes steps in  $\{\text{s}, \text{nw}\}$  and defined as follows. Start at  $(n, d)$ .

- (i) If we are currently on the upper path then go to step (ii). Otherwise move in direction  $\text{nw}$  until meeting the upper path.
- (ii) If the point is already on the lower path then go to step (iii). Otherwise move in the direction  $\text{s}$  until meeting the lower path.
- (iii) If the current position is not  $(0, 0)$ , then go to step (i).

The outcome will be a sequence of  $p_1$   $\text{nw}$  steps, followed by  $p_1 + q_1$   $\text{s}$  steps, followed by  $p_2$   $\text{nw}$  steps, followed by  $p_2 + q_2$   $\text{s}$  steps,  $\dots$ , followed by  $p_k$   $\text{nw}$  steps, followed by  $p_k + q_k$   $\text{s}$  steps. Define

$$\text{ctiBounce}(P) := (p_1, q_1, \dots, p_k, q_k) \quad \text{and} \quad \text{ctibounce}(P) := \sum_i i(p_i + q_i).$$

**Example 5.9.** The bounce paths of the two sawtooth polyominoes given in Example 5.2:



(a)  $\text{ctiBounce}(\cdot) = (0, 2, 1, 2, 2, 1, 0, 1, 1, 1, 0)$       (b)  $\text{ctiBounce}(\cdot) = (1, 3, 1, 1, 2, 1)$

**Definition 5.10.** Given  $P \in \text{Sawtooth}_{n+1,d}$ , define the area  $\text{area}(P)$  of  $P$  to be the number of unit squares whose vertices are lattice points that are contained within  $P$ .

The area of the sawtooth polyomino in Example 5.2(a) is 12 while the area of the sawtooth polyomino in Example 5.2(b) is 15.

**Theorem 5.11.** Let  $c \in \text{SortedRec}(S_{n,d})$  and  $P = f_{n,d}(c) \in \text{Sawtooth}_{n+1,d}$ . Then  $\text{height}(c) = \text{area}(P) - (n + d) + \binom{n+d}{2} - \binom{d}{2}$  and  $\text{topple}_{\text{CTI}}(c) = \text{ctiBounce}(P)$ .

*Proof.* Suppose  $c \in \text{SortedRec}(S_{n,d})$  and  $P = (U, L) = f_{n,d}(c) \in \text{Sawtooth}_{n+1,d}$ . From the construction for  $f$ , we find that the total area of the unit squares contained beneath the lower path  $L$  of  $P$  and the  $x$ -axis is

$$d(n+1-1) - (b_1 + b_2 + \dots + b_d).$$

The total area beneath the upper path  $U$  of  $P$  and the  $x$ -axis is

$$d + n - \binom{n}{2} + (a_1 + \dots + a_n).$$



The area of  $P$  is the difference of these:

$$\text{area}(P) = d + n - \binom{n}{2} + (a_1 + \dots + a_n) - dn + (b_1 + b_2 + \dots + b_d).$$

This may be rewritten

$$\begin{aligned} \text{height}(c) &= \text{area}(P) - (d + n) + \binom{n}{2} + dn \\ &= \text{area}(P) - (d + n) + \binom{d+n}{2} - \binom{d}{2}, \end{aligned}$$

where the second follows from the first by noticing that the number of 2-element subsets of a set of size  $d + n$  can be enumerated by  $\binom{d+n}{2} = \binom{d}{2} + \binom{n}{2} + dn$ .

Let  $c \in \text{SortedRec}(S_{n,d})$  and  $P = f_{n,d}(c) \in \text{Sawtooth}_{n+1,d}$ . Consider  $\text{topple}_{CTI}(c) = (p_1, q_1, \dots)$ . In terms of the associated sawtooth polyomino, the effect of toppling the sink corresponds to all diagonal steps (other than the last between  $(n, d+1)$  and  $(n+1, d)$  as it does not represent a height) on the upper path being shifted up by 1 unit. Those that are unstable will be the those that are then on (or above) the line  $x + y = n + 1 + d$ .

Equivalently, instead of shifting each of the diagonal steps by 1, to see which will become unstable as a result of toppling the sink we can start at the point  $(n, d)$  and see which diagonal steps to its left are on or above the line  $x + y = n + 1 + d - 1 = n + d$ . Suppose there are  $p_1$  such steps. These  $p_1$  steps can be seen by following a line from  $(n, d)$  to  $(n - p_1, d + p_1)$  which is where the line  $x + y = n + d$  meets the upper path of the sawtooth polyomino. As  $p_1$  clique vertices are toppled, this means the heights of all independent vertices will now be  $1 + p_1$  more than before the sink was toppled.

Consequently, the set of those that are next toppled are those independent vertices  $w_i$  whose height is now  $b_i + 1 + p_1 \geq n + 1$ . This is the same as those independent vertices whose initial height (before toppling the sink) was  $b_i \geq n - p_1$ , and these independent vertices are precisely those that are strictly to the right of the line that moved vertically down from  $(n - p_1, d + p_1)$  until meeting the lower path of the sawtooth polyomino.

Repeating this argument reveals precisely why the bounce path of the polyomino models the CTI toppling process on the corresponding recurrent configuration. Hence  $\text{topple}_{CTI}(c) = (p_1, q_1, \dots) = \text{ctiBounce}(P)$ .  $\square$

**Example 5.12.** The sawtooth polyomino diagrams that correspond to the 30 sorted recurrent configuration given in Example 2.3 are illustrated in Figure 5. Above each diagram we list the sorted configuration to which it corresponds in parentheses, and below the parentheses we list the bounce  $(p_1, q_1, \dots)$  in angle brackets.

**5.4. ITC topplings and bounce paths within sawtooth polyominoes.** Given the manner in which the bounce path of the previous section emulates the temporal changes to the configuration heights as a result of toppling the sink, it is perhaps unsurprising that ITC topplings also admit a very similar polyomino bounce path description. For the case of ITC topplings, the only difference is that the bounce path initially moves south (if it can) rather than moving north-west from position  $(m - 1, n)$ .

**Definition 5.13.** Given  $P \in \text{Sawtooth}_{n+1,d}$ , let the *ITC-bounce path* of  $P$  be the path from  $(n, d)$  to  $(0, 0)$  that takes steps in  $\{\text{s}, \text{nw}\}$  and defined as follows. Start at  $(n, d)$ .

- (i) If we are currently on the lower path then go to step (ii). Otherwise move in direction  $\text{s}$  until meeting the lower path.
- (ii) If the point is already on the upper path then go to step (iii). Otherwise move in the direction  $\text{nw}$  until meeting the upper path.
- (iii) If the current position is not  $(0, 0)$ , then go to step (i).



While one of the authors has already counted these configurations, via the previously described bijection with Schröder paths [7], our secondary aim is to provide a cyclic lemma that remains within the framework of the sandpile model rather than for some graphical representation of the configurations. Since such a cyclic lemma has already been shown to exist for complete bipartite graphs [2], we suspect that such lemmas are instances of a more general result that holds true for the sandpile model on similar graph classes.

In this section we will decompose a configuration  $u = (u_s; u^{[K]}; u^{[I]})$  on the complete split graph  $S_{n,d}$  into three sub-configurations on three components:

- the isolated sink  $u_s$ ,
- the sub-configuration  $u^{[K]} = (u_0^{[K]}, \dots, u_{n-1}^{[K]})$  on the  $n$  clique ( $K = V$ ) vertices, and
- the sub-configuration  $u^{[I]} = (u_0^{[I]}, \dots, u_{d-1}^{[I]})$  on the  $d$  independent ( $I = W$ ) vertices.

We adopt the convention that the number of grains on the sink is fixed so that

$$u_s := - \left( \sum_{k \in K} u_k^{[K]} + \sum_{i \in I} u_i^{[I]} \right).$$

This allows us to remove explicit reference to the sink part of a configuration and to write  $u = (u^{[K]}; u^{[I]})$  instead of  $u = (u_s; u^{[K]}; u^{[I]})$ .

We now introduce the set of sorted non-negative quasi-stable configurations. The reason for doing so is that this set is easier to count and we will later show it can be partitioned into sets, each of cardinality  $n + 1$ , that are indexed by sorted recurrent configurations.

We will say a clique (resp. independent) sub-configuration  $u^{[K]}$  (resp.  $u^{[I]}$ ) is *quasi-stable* if  $\max_{k \in K} u_k \leq n + d$  (resp.  $\max_{i \in I} u_i \leq n$ ). Notice that the quasi-stable condition corresponds to the stable condition for the sub-configuration  $u^{[I]}$ , whereas a quasi-stable sub-configuration  $u^{[K]}$  may have one more grain on each of its constituent vertices than can feature in a stable sub-configuration. We will call a configuration *quasi-stable* if both its clique and independent sub-configurations are quasi-stable.

A permutation  $\sigma^{[C]}$  on vertices of a component  $C \in \{K, I\}$  acts on a sub-configuration  $u^{[C]}$  by permuting the distribution of its grains:

$$\sigma^{[C]}.u^{[C]} := \left( u_{\sigma^{[C]}(0)}^{[C]}, \dots, u_{\sigma^{[C]}(|C|-1)}^{[C]} \right).$$

We will call a sub-configuration  $u^{[C]}$  on the component  $C$  *sorted* if  $(u_0^{[C]}, \dots, u_{|C|-1}^{[C]})$  is a weakly decreasing sequence. Every sub-configuration  $u^{[C]}$  is equivalent, by some permutation  $\sigma_{\text{sort}}^{[C]}$  of its vertices, to a single *sorted sub-configuration* denoted  $\text{sort}(u^{[C]})$ . By extension, every configuration  $u$  is equivalent to a single *sorted configuration*  $\text{sort}(u)$ , under the action of permutations on  $K$  and  $I$  that sort the individual parts, and we denote this by  $(\sigma_{\text{sort}}^{[K]}, \sigma_{\text{sort}}^{[I]}) \cdot u$ , i.e.

$$\text{sort}(u) = (\sigma_{\text{sort}}^{[K]}, \sigma_{\text{sort}}^{[I]}) \cdot u.$$

We will say that two configurations  $u$  and  $u'$  are *toppling-and-permuting equivalent*, written  $u \equiv_{\sigma, \Delta} u'$ , if

$$u' = (\sigma^{[K]}, \sigma^{[I]}) \cdot u + \sum_{v \in S_{n,d}} a_v \Delta^{(v)}$$

for some permutation  $\sigma^{[K]}$  permuting the vertices of the clique component  $K$  and some permutation  $\sigma^{[I]}$  permuting the vertices of the independent component  $I$ . Here  $\Delta^{(v)}$  is the action of toppling vertex  $v$  and  $(a_v)_v \in \mathbb{Z}^{1+n+d}$  gives the possibly negative number of topplings (or anti-topplings) of each vertex of  $S_{n,d}$  including the sink.

**Theorem 6.1** (Cyclic Lemma). *Every sorted recurrent configuration on  $S_{n,d}$  is toppling-and-permuting equivalent to exactly  $n + 1$  of the  $\binom{2n+d}{n} \binom{n+d}{n}$  sorted quasi-stable non-negative configurations (including itself but no other sorted recurrent configuration).*

The proof of the above result requires several technical lemmas and will be presented at the end of this section. This partition of sorted quasi-stable non-negative configurations into equivalence classes having the same size yields the following corollary.

**Corollary 6.2.** *There are precisely  $\frac{1}{n+1} \binom{2n+d}{n} \binom{n+d}{n} = \frac{1}{2n+1} \binom{2n+1}{n} \binom{2n+d}{d}$  sorted recurrent configurations on  $S_{n,d}$ .*

Next we will analyse the equivalence classes of the toppling-and-permuting relation  $\equiv_{\sigma,\Delta}$  through some new operators on configurations. The *topple-max-then-sort operator*,  $T_C$ , of a component  $C$  acts on a configuration  $u$  as

$$T_C.u = \text{sort}(u + \Delta_c),$$

where  $c$  is one of the vertices for which  $u_c = \max_{c' \in C} u_{c'}$ , i.e. it is one of the vertices having the maximal number of grains in the component  $C$ .

These operators are, in general, not reversible either on configurations (due to sorting considerations) or even on sorted configurations. However, these operators become reversible on the subset of sorted compact configurations (to be defined next) that contain sorted quasi-stable non-negative configurations (which itself includes all the recurrent configurations):

$$\text{sorted recurrent} \subseteq \text{sorted quasi-stable and non-negative} \subseteq \text{sorted compact}.$$

**Definition 6.3.** A configuration  $u$  on  $S_{n,d}$  is *compact* if

$$\text{spread}(u^{[K]}) \leq n + d + 1 \quad \text{and} \quad \text{spread}(u^{[I]}) \leq n + 1,$$

where the *spread*  $\text{spread}(u^{[C]})$  in a component  $C$  of a configuration  $u'$  is defined by

$$\text{spread}(u^{[C]}) := \max_{v \in C} u'_v - \min_{v \in C} u'_v.$$

There are two permutations of vertices of a component  $C$  that will be of special interest to us. We denote by  $\iota_{[C]}$  the identity permutation and by  $\tau_{[C]}$  the cyclic permutation defined by  $\tau_{[C]}(c) := c - 1 \pmod{|C|}$ . The cyclic permutation simply moves the vertex in first position 0 to the last position  $|C| - 1$  and shifts the others.

We see each sub-configuration  $u^{[C]}$  as a vector in  $\mathbb{Z}^{|C|}$ . This allows addition of vectors such as  $u^{[C]} + v^{[C]}$  for two sub-configurations. For any constant  $k$ , we define the constant sub-configuration  $k^{[C]} := (k)_{c \in C}$ . Similarly, for any constant  $k$  and vertex  $c_C \in C$  we define the sub-configuration  $k^{[c_C]}$  that contains  $k$  grains on vertex  $c_C$  and 0 elsewhere.

The following lemma details the behaviour of the three toppling operators on the set of sorted compact configurations. This will be implicitly used in the lemmas that follow it.

**Lemma 6.4.** *For a sorted compact configuration  $u = (u^{[K]}; u^{[I]})$  on  $S_{n,d}$  we have*

$$\begin{aligned} T_s.u &= (u^{[K]} + 1^{[K]}; u^{[I]} + 1^{[I]}) \\ T_K.u &= (\tau_{[K]}.(u^{[K]} - (n + d + 1)^{[0_K]} + 1^{[K]}); u^{[I]} + 1^{[I]}) \\ T_I.u &= (u^{[K]} + 1^{[K]}; \tau_{[I]}.(u^{[I]} - (n + 1)^{[0_I]})), \end{aligned}$$

*and all three of these are themselves compact configurations. These operators restricted to sorted compact configurations are reversible with*

$$\begin{aligned} T_s^{-1}.u &= (u^{[K]} - 1^{[K]}; u^{[I]} - 1^{[I]}) \\ T_K^{-1}.u &= \left( (\tau_{[K]}^{-1}.u^{[K]}) + (n + d + 1)^{[0_K]} - 1^{[K]}; u^{[I]} - 1^{[I]} \right) \\ T_I^{-1}.u &= \left( u^{[K]} - 1^{[K]}; (\tau_{[I]}^{-1}.u^{[I]}) + (n + 1)^{[0_I]} \right). \end{aligned}$$

*Proof.* Let  $u = (u^{[K]}; u^{[I]})$  be a sorted compact configuration on  $S_{n,d}$ . The expression for  $T_s.u$  is simply a reformulation of the description of the configuration  $\text{sort}(u + \Delta^{(s)}) = (\iota_{[K]}, \iota_{[I]}).(u +$

$\Delta^{(s)}$ ). Moreover, the spread of each component is preserved by addition of constant vectors  $1^{[C]}$ , so the configuration remains compact.

Since the configuration  $u$  is sorted, for both sub-configurations  $C \in \{K, I\}$  the first vertex contains the maximal number of grains, i.e.  $u_0^{[C]} = \max_{c \in C} u_c$ . This shows that the expression  $T_{[C]}.u$  is an alternative description for  $u + \Delta^{(0_C)}$ .

The definition of a compact configuration, in particular the threshold on the spread, ensures that the toppled vertex which was maximal in  $u^{[C]}$  becomes minimal in  $(T_C.u)^{[C]}$ . This shows that the sub-configuration is sorted by the (cyclic) permutation  $\tau_{[C]}$ .

Moreover, the maximal resulting spread occurs in the case of a second vertex  $1_C$  having a maximal number of grains such that  $u_1^{[C]} = u_0^{[C]}$ . It coincides, by design, with the bound on spread used in the definition of compact configurations.

In the expressions

$$T_{[K]}.u = (\tau_{[K]}, \iota_{[I]}) \cdot (u + \Delta^{(0_K)}) \quad \text{and} \quad T_{[I]}.u = (\iota_{[K]}, \tau_{[I]}) \cdot (u + \Delta^{(0_I)}),$$

the actions on the configurations do not depend on the particular  $u$ .

Since all the steps are reversible, we deduce that max-topples-then-sort operators are all injective. The reverse map is defined by reversing these descriptions:

$$T_{[K]}.u = \left( (\tau_{[K]}^{-1}, \iota_{[I]}) \cdot u \right) - \Delta^{(0_K)} \quad \text{and} \quad T_{[I]}.u = \left( (\iota_{[K]}, \tau_{[I]}^{-1}) \cdot u \right) - \Delta^{(0_I)}.$$

It remains to check that those reverse operators map a compact configuration to another compact configuration. In a component, the case of maximal spread corresponds to cases for which  $u_{|C|-2}^{[C]} = u_{|C|-1}^{[C]}$ . Similar reasoning to the first part of the proof ensures that the result is also compact.  $\square$

The following lemma shows that the three toppling operators are very well-behaved on the set of sorted compact configurations and summarizes Dhar's criterion in terms of these operators.

**Lemma 6.5.** *The restrictions to sorted compact configurations of the operators  $T_s$ ,  $T_K$  and  $T_I$  commute and*

$$T_s T_K^{[K]} T_I^{[I]} = \text{Id}$$

where  $\text{Id}$  is the identity map/operator.

*Proof.* Let  $X$  and  $Y$  be two distinct components among  $\{K, I, \{s\}\}$ . We show that  $T_X.T_Y.u = T_Y.T_X.u$  corresponds, for any sorted compact configuration  $u$ , to the toppling of the initially maximal vertex  $x(u) = 0_X \in X$  and  $y(u) = 0_Y \in Y$  in respective component followed by the sorting in each component. Indeed, when  $x(u)$  is toppled,  $y(u)$  remains maximal among vertices of  $Y$  since each vertex of this component received the same number of grains since the sorting only act of vertices of component  $X$ . Hence when  $T_Y$  is applied,  $y(u) = 0_Y$  is still a maximal vertex, so it may be toppled and the following sorting acts only on the component  $Y$ .

So  $T_Y.T_X.u$  is equivalent to the parallel toppling of  $x(u)$  and  $y(u)$  followed by sorting. By symmetry of this argument,  $T_X.T_Y.u$  is also equivalent to this parallel toppling then sorting so  $T_Y.T_X = T_X.T_Y$ , and the two operators commute. We remark that if the toppled vertex were not  $0_X$  but another vertex instead, so that  $u_0^{[X]} = u_{x(u)}^{[X]}$ , the result is the same after the sorting of component  $X$ .

The operator  $T_s$  causes the sink  $s$  to topple once. The operator  $T_K^{[K]}$  causes every vertex of  $K$  to topple exactly once in

$$\sum_{k \in K} \tau_{[K]}^k \Delta^{(0_K)} = \sum_{k \in K} \Delta^{(k)}.$$

Similarly, the operator  $T_I^{[I]}$  causes every vertex of  $I$  to topple exactly once in

$$\sum_{i \in I} \tau_{[I]}^i \Delta^{(0_I)} = \sum_{i \in I} \Delta^{(i)}.$$

Finally, every vertex topples exactly once as a result of the operator  $T_s T_K^{|K|} T_I^{|I|}$  as

$$T_s T_K^{|K|} T_I^{|I|} .u = u + \sum_{v \in S_{n,d}} \Delta^{(v)} = u,$$

since in the sandpile model  $\sum_{v \in S_{n,d}} \Delta^{(v)} = 0$ .  $\square$

Next we introduce the notion of the weight of a sub-configuration and introduce a weight operator on configurations. Some necessary properties of these will then be proven in Lemma 6.7 which will be essential in the proof of Lemma 6.8.

**Definition 6.6.**

(a) The *weight* of a sub-configuration  $u^{[K]}$  on the clique component  $K$  of  $S_{n,d}$  is

$$\text{weight}(u^{[K]}) := \sum_{k \in K} \left\lfloor \frac{u_k^{[K]}}{n+d+1} \right\rfloor.$$

(b) The *weight operator*  $T_W$  is

$$T_W := T_K^{|K|+1} T_I^{|I|}.$$

(c) Two configurations  $u$  and  $v$  are  $T_W$ -equivalent,  $u \equiv_{T_W} v$ , if  $u = T_W^t .v$  for some  $t \in \mathbb{Z}$ .

Note that the equivalence in Definition 6.6(c) implies a divisibility by  $(n+d+1)$  of the difference between the number of grains at the sink in both configurations.

**Lemma 6.7.** *Let  $u$  and  $v$  be sorted compact configurations on  $S_{n,d}$ .*

- (a)  $u = T_W .v$  implies that  $u^{[I]} = v^{[I]}$ . More generally,  $u \equiv_{T_W} v$  implies that  $u^{[I]} = v^{[I]}$ .
- (b) The (sorted) compact sub-configuration  $u^{[K]}$  on  $S_{n,d}$  is quasi-stable and non-negative if and only if  $\text{weight}(u^{[K]}) = 0$ .
- (c) Moreover  $\text{weight}(T_W .u^{[K]}) = \text{weight}(u^{[K]}) - 1$  and the quasi-stable and non-negative configuration  $T_W$ -equivalent to  $u^{[K]}$  is  $T_W^{\text{weight}(u^{[K]})} .u^{[K]}$ .

*Proof.* (a) Consider the sub-configuration  $u^{[I]}$ . Since the restriction to the component  $I$  of operators  $T_s$  and  $T_K$  are equivalent, in both simply add one grain to each vertex of  $I$ , we have

$$(T_W .u)^{[I]} = (T_K^{|K|+1} T_I^{|I|} .u)^{[I]} = (T_s T_K^{|K|} T_I^{|I|} .u)^{[I]} = (\text{ld} .u)^{[I]} = u^{[I]},$$

with the penultimate equality coming from Lemma 6.5. Hence  $u^{[I]} = (T_W .u)^{[I]}$ . More generally, and using the same reasoning, we have that  $u \equiv_{T_W} v$  implies  $u^{[I]} = v^{[I]}$ .

(b) As regards the sub-configuration  $u^{[K]}$ , we consider the Euclidean division of each entry of the sub-configuration  $u^{[K]}$ :

$$u_k^{[K]} =: q_k(n+d+1) + r_k, \quad \text{for all } k \in K$$

defining the quotient vector  $(q_k)_{k \in K}$  and the remainder vector  $(r_k)_{k \in K}$ . By design, the operator  $T_W$  satisfies:

$$T_W .u = \tau_{[K]} . \left( u - (n+d+1)^{[0_K]} \right).$$

In terms of the Euclidean division by  $(n+d+1)$ , the action of  $T_W$  may be described directly on the quotient and remainder vectors (in that order) as follows:

$$T_W .(q_k)_{k \in K} = \tau . \left( (q_k)_{k \in K} - 1^{[0_K]} \right) \quad \text{and} \quad T_W .(r_k)_{k \in K} = \tau .(r_k)_{k \in K}.$$

Since  $u^{[K]}$  is compact, we have  $u_{|K|-1}^{[K]} \leq u_0^{[K]} \leq u_{|K|-1}^{[K]} + (n+d+1)$  and so

$$(q_k)_{k \in K} = (q_0, q_0, \dots, q_0, q_0 - 1, \dots, q_0 - 1) = (q_0)^p (q_0 - 1)^{n-p}$$

for some  $p \in \{1, \dots, n\}$ . Hence

$$T_W . \left( (q_0)^p (q_0 - 1)^{n-p} \right) = (q_0)^{p-1} (q_0 - 1)^{n-p+1}.$$

By applying the operator  $T_W$  a total number of  $\sum_{k \in K} q_k = \text{weight}(u^{[K]})$  times, we obtain the configuration  $u' := T_W^{\text{weight}(u^{[K]})}$  such that  $(q'_k)_{k \in K} = (0)_{k \in K}$ . Thus  $u' = \tau_{[K]}^{\text{weight}(u^{[K]})} \cdot (r_k)_{k \in K}$  is also non-negative by definition of the reminders via Euclidean division.

(c) Moreover, for any powers  $t \in \mathbb{Z}$  other than  $t = \text{weight}(u^{[K]})$ , we can consider  $u''^{[K]} := T_W^t \cdot u^{[K]}$ . Since  $(q''_k)_{k \in K} \neq (0)_{k \in K}$ , either we have  $q''_0 > 0$  in which case  $u''^{[K]}$  is not quasi-stable, or  $q''_{|K|-1} < 0$  in which case we have  $u''^{[K]}$  is negative.  $\square$

Finally we are in a position to state the main result of this section, and from which the proof of Theorem 6.1 essentially follows.

**Lemma 6.8.** *Every sorted compact configuration is toppling-and-permuting equivalent to exactly  $n + 1$  sorted quasi-stable and non-negative configurations.*

*Proof.* Let  $u$  be a sorted compact configuration and consider  $v := \text{rec}(u)$ , the sorted recurrent configuration which is toppling-and-permuting equivalent to  $u$ . Apply Dhar's criterion, Prop. 2.1, to  $v$ . We start from  $v^{[0]} = v$  which is a quasi-stable and non-negative configuration. We topple the sink once using the operator  $T_s$  and suppose  $k_0$  applications of  $T_I$  are needed to reach a sub-configuration on  $I$  that is stable, and call the configuration  $v^{[1]}$ . Following this, we topple a clique vertex by applying  $T_K$  and suppose that  $k_1$  applications of  $T_I$  to  $v^{[1]}$  are required to reach a sub-configuration on  $I$  that is stable, and call the configuration  $v^{[2]}$ , which is again quasi-stable. Since the Dhar's criterion for recurrent configurations may be decomposed as

$$T_s T_I^{k_0} \prod_{i=1}^n (T_K T_I^{k_i})$$

we obtain from  $v$ , via each prefix of operators before an occurrence of  $T_s$  or  $T_K$ , precisely  $n + 1$  configurations  $(v^{[i]})_{i=0, \dots, n}$  where the  $I$  component is quasi-stable and non-negative. By the  $T_W$ -equivalence described in Lemma 6.7, we deduce  $n + 1$  quasi-stable non-negative configurations  $(w^{[i]})_{i=0, \dots, n} := (T_W^{\text{weight}(v^{[i]})} \cdot v^{[i]})_{i=0, \dots, n}$  toppling-and-permuting equivalent to  $v$ , and also to  $u$ .

It remains to show that each of these  $n + 1$  configurations are distinct. To do this, we will study the number of grains at the sink modulo  $n + d + 1$  for all the  $n + 1$  configurations. First we notice that since toppling the sink removes  $n + d$  grains from the sink and any of the  $n + d$  topplings of either  $T_K$  or  $T_I$  returns a grain to the sink, all  $(v_s^{[i]} \bmod (n + d + 1))_{i=0, \dots, n}$  are distinct so all configurations  $(v^{[i]})_{i=0, \dots, n}$  are also distinct. Since the operator  $T_W$  contributes  $n + d + 1$  grains to the sink, the  $T_W$ -equivalence preserves the number of grains at the sink modulo  $(n + d + 1)$ . This shows that each  $(w_s^{[i]})_{i=0, \dots, n}$  is distinct, and hence all  $(w^{[i]})_{i=0, \dots, n}$  are distinct (quasi-stable and non-negative configurations).

The remainder of the proof shows that there are no other quasi-stable and non-negative configurations that are toppling-and-permuting equivalent to  $u$ . If  $w$  is a quasi-stable non-negative configuration toppling-and-permuting equivalent to  $u$  then it can be written

$$w = T_s^{\alpha_s} T_K^{\alpha_K} T_I^{\alpha_I} \cdot v,$$

where  $v = \text{rec}(u)$  is the sorted recurrent configuration in that equivalence class.

Using Lemma 6.5 we may write:

$$T_W = T_K^{|K|+1} T_I^{[I]} = T_K T_s^{-1} \left( T_s T_K^{|K|} T_I^{[I]} \right) = T_K T_s^{-1}.$$

Thus  $T^s = T_K T_W^{-1}$  and we can remove the  $T_s$  operator in the description of

$$w = T_W^{-\alpha_s} T_K^{\alpha_s + \alpha_K} T_I^{\alpha_I} v.$$

Now let us define  $q_K$  and  $r_K$  by the Euclidean division of  $\alpha_s + \alpha_K$  by  $n + 1$  :  $\alpha_s + \alpha_K = q_K(n + 1) + r_K$ . Similarly, define  $q_I$  and  $r_I$  by the Euclidean division of  $\alpha_I$  by  $d$  :  $\alpha_I = q_I d + r_I$ . This allows the following factorization of operators for powers of  $T_W$  then  $T_I^d$ :

$$w = T_W^{-\alpha_s} T_K^{q_K(n+1)+r_K} T_I^{q_I d + r_I} \cdot v = T_W^{-\alpha_s} (T_K^{n+1} T_I^d)^{q_K} (T_I^d)^{(q_I - q_K)} (T_K^{r_K} T_I^{r_I}) \cdot v,$$

where we identify the new factors  $T_W = T_K^{n+1}T_I^d$  so we have

$$w = T_W^{q_K - \alpha_s} (T_I^d)^{(q_I - q_K)} (T_K^{r_K} T_I^{r_I}) .v.$$

If we restrict the identity to sub-configurations on the component  $I$ , then  $T_W$  keeps it invariant so we may ignore the factor  $T_W^{q_K - \alpha_s}$ , hence

$$w^{[I]} = (T_I^d)^{(q_I - q_K)} (T_K^{r_K} T_I^{r_I}) .v^{[I]}.$$

Since  $r_I < |I|$ , the value  $v_{|I|-1}^{[I]}$  is not decremented by the operator  $T_I^{r_I}$  hence

$$0 \leq w_{|I|-1-r_I}^{[I]} = v_{|I|-1}^{[I]} + r_K - (n+1)(q_I - q_K) \leq n,$$

where the bounds come from the assumption that  $w^{[I]}$  is quasi-stable and non-negative. As  $0 \leq v_{|I|-1}^{[I]} \leq n$  and  $0 \leq r_K \leq n$ , it follows that  $0 \leq v_{|I|-1}^{[I]} + r_K \leq 2n$  and we compare this value to  $n$ :

- if  $v_{|I|-1}^{[I]} + r_K \leq n$ , in particular when  $r_K = 0$ , then  $(q_I - q_K) = 0$ ;
- otherwise  $v_{|I|-1}^{[I]} + r_K > n$  and  $(q_I - q_K) = 1$ .

By contradiction, we want to exclude the cases  $(q_I - q_K) = 1$  and  $r_I > 0$ . In those cases, we have

$$w^{[I]} = T_K^{r_K} T_I^{d+r_I} .v^{[I]}.$$

Then vertex  $v_0^{[I]}$  topples twice while going from  $v^{[I]}$  to  $w^{[I]}$  so

$$0 \leq w_{|I|-r_I}^{[I]} = v_0^{[I]} + r_K - 2(n+1) \leq n,$$

where the inequalities comes from the assumption that  $w^{[I]}$  is quasi-stable and non-negative. This gives the expected contradiction since  $v_0^{[I]} + r_K \leq 2n$  (because both summands are lower than  $n$ ).

Hence our analysis shows that

$$w^{[I]} = T_K^a T_I^b .v^{[I]}$$

where  $a \in \{0, \dots, n\}$ ,  $b \in \{0, \dots, d\}$ , and more generally there exists  $c \in \mathbb{Z}$  such that

$$w = T_W^c T_K^a T_I^b .v.$$

- If  $a = 0$  then  $w = T_W^c T_I^b .v$  and  $w^{[I]} = T_I^b .v^{[I]}$  so  $b = 0$ . Otherwise  $b \geq 1$  and the toppling of  $v_0^{[I]}$  means that  $w_{|I|-b}^{[I]} = v_0^{[I]} - (n+1) < 0$  so  $w$  would not be non-negative. This gives  $w = T_W^c .v$  and, since  $\text{weight}(w) = c + \text{weight}(v)$  and  $\text{weight}(v) = 0$ , by Lemma 6.7 it follows that  $w^K$  is quasi-stable and non-negative if and only if  $c = 0$ , and so  $w = v$ .
- If  $a \geq 1$ , we apply once the relation  $T_K = T_W T_s$  to obtain  $w = T_W^{c+1} T_s T_K^{a-1} T_I^b .v$ . We will identify the possible choices of  $a$  and  $b$  via the quasi-stability and non-negativity of  $w^{[I]} = T_s T_K^{a-1} T_I^b .v^{[I]}$ . In the sequence of operators from Dhar's criterion,

$$T_s T_I^{k_0} \prod_{i=1}^n (T_K T_I^{k_i}),$$

the  $n$  prefixes of operators before an occurrence of operator  $T_K$  and the appropriate value of  $c$  defined by the weight, gives us the previously identified  $n+1$  quasi-stable and non-negative configurations except the recurrent configuration  $v$  that was found in the first case. Let  $((a_k, b_k))_{k=1, \dots, n}$  the  $n$  possibles values identified via Dhar's criterion. For other values of  $(a, b)$ , either  $T_K$  operators are missing, in which case  $w^{[I]}$  is not non-negative, or  $T_I$  operators are missing and in this case  $w^{[I]}$  is not quasi-stable. More precisely, the decomposition with respect to Dhar's criterion giving priority to toppling vertices in the independent component  $I$  leads to  $((a_k = k, b_k))_{k=1, \dots, n}$ . For a possible  $(a, b)$  pair, we discuss the value of  $b$  with respect to  $b_a$ :



- If  $b < b_a$ , then this means one performs the Dhar criterion but without some expected topplings of unstable vertices in the independent component  $I$ . Here  $w^{[I]}$  will not be (quasi-)stable and such a choice for  $b$  is not possible.
- If  $b > b_a$ , then this signifies the forcing of a toppling of some unstable vertices in the independent component. Here  $w^{[I]}$  will not be non-negative and such a choice for  $b$  is not possible.
- If  $b = b_a$ , then we recover one of the  $n$  expected quasi-stable and non-negative sub-configurations  $w^{[I]}$  of this case and use the weight of  $T_s T_K^{a-1} T_I^b.v$  to identify  $c + 1 = \text{weight}(T_s T_K^{a-1} T_I^b.v)$ . This means we obtain the single power of  $T_W$  leading to a quasi-stable and non-negative  $w^{[K]}$  with the same  $w^{[I]}$ , and hence a quasi-stable and non-negative configuration. □

*Proof of Prop. 6.1.* By Lemma 6.8, we have that every sorted recurrent configuration on  $S_{n,d}$  is one of  $n+1$  sorted quasi-stable and non-negative configurations that are toppling-and-permuting equivalent to it. The set of sorted quasi-stable and non-negative configurations admits a partition into equivalence classes modulo toppling-and-permuting equivalence. The number of sorted quasi-stable and non-negative configurations on  $S_{n,d}$  is easily seen to be  $\binom{2n+d}{n} \binom{n+d}{n}$ . □

## 7. CONCLUSION

There are several outstanding problems from the work presented in this paper. A conjecture that we have been unable to resolve is equality of the  $q, t$ -CTI and the  $q, t$ -ITC polynomials. While we have been able to establish this for some special cases, the general case remains elusive.

**Conjecture 7.1.**  $\mathfrak{F}_{n,d}^{CTI}(q, t) = \mathfrak{F}_{n,d}^{ITC}(q, t)$ .

Since, in Corollary 3.2, we have established equality of the  $q, t$ -Schröder and  $q, t$ -ITC polynomials, Conjecture 7.1 is equivalent to:

**Conjecture 7.2.**  $\mathfrak{F}_{n,d}^{CTI}(q, t) = \sum_{w \in \text{Schröder}_{n,d}} q^{\text{area}(w)} t^{\text{itcbounce}(w)}$ .

We fully expect there to be an explanation for the above correspondences that is realizable in a sandpile setting.

**Conjecture 7.3.** *There exists a bijection  $\Psi : \text{SortedRec}(S_{n,d}) \mapsto \text{SortedRec}(S_{n,d})$  such that for any configuration  $c \in \text{SortedRec}(S_{n,d})$ ,*

$$(\text{height}(c), \text{wtopple}_{CTI}(c)) = (\text{height}(\Psi(c)), \text{wtopple}_{ITC}(\Psi(c))).$$

## REFERENCES

- [1] Jean-Christophe Aval, Michele D’Adderio, Mark Dukes, Angela Hicks, and Yvan Le Borgne. Statistics on parallelogram polyominoes and a  $q, t$ -analogue of the Narayana numbers. *Journal of Combinatorial Theory Series A* **123** (2014), no. 1, 271–286. doi: 10.1016/j.jcta.2013.09.001
- [2] Jean-Christophe Aval, Michele D’Adderio, Mark Dukes, and Yvan Le Borgne. Two operators on sandpile configurations, the sandpile model on the complete bipartite graph, and a Cyclic Lemma. *Advances in Applied Mathematics* **73** (2016) 59–98. doi: 10.1016/j.aam.2015.09.018
- [3] Michele D’Adderio, Alessandro Iraci, Yvan Le Borgne, Marino Romero, and Anna Vanden Wyngaerd. Tiered trees and Theta operators. *Int. Math. Res. Not.* **2023** (2023), no. 24, 20748–20783.
- [4] Michele D’Adderio and Yvan Le Borgne. The sandpile model on  $K_{m,n}$  and the rank of its configurations. *Sém. Lothar. Combin.* **77** (2018), Art. B77h.
- [5] Deepak Dhar. Theoretical studies of self-organized criticality. *Physica A: Statistical Mechanics and its Applications* **369** (2006), no. 1, 29–70. doi: 10.1016/j.physa.2006.04.004
- [6] Erik Egge, James Haglund, Kendra Killpatrick, and Darla Kremer. A Schröder generalization of Haglund’s statistic on Catalan paths. *Electronic Journal of Combinatorics*, 10, 2002. paper #R16
- [7] Mark Dukes. The sandpile model on the complete split graph, Motzkin paths, and tiered parking functions. *Journal of Combinatorial Theory Series A* **180** (2021) Article 105418. doi: 10.1016/j.jcta.2021.105418

- [8] Mark Dukes and Yvan Le Borgne. Parallelogram polyominoes, the sandpile model on a complete bipartite graph, and a  $q, t$ -Narayana polynomial. *Journal of Combinatorial Theory Series A* **120** (2013), no. 4, 816–842. doi: 10.1016/j.jcta.2013.01.004
- [9] Michele D’Adderio, Mark Dukes, Alessandro Iraci, Alexander Lazar, Yvan Le Borgne, and Anna Vanden Wyngaerd. Shuffle theorems and sandpiles *arXiv:2401.06488*.
- [10] Mark Dukes, Thomas Selig, Jason P. Smith, and Einar Steingrímsson. Permutation graphs and the Abelian sandpile model, tiered trees and non-ambiguous binary trees. *Electronic Journal of Combinatorics* **26** (2019), no. 3, P3.29.
- [11] Mark Dukes, Thomas Selig, Jason P. Smith, and Einar Steingrímsson. The Abelian sandpile model on Ferrers graphs – A classification of recurrent configurations. *European Journal of Combinatorics* **81** (2019) 221–241.
- [12] Dominique Foata and Guo-Niu Han.  $q$ -series in Combinatorics; Permutation Statistics (Preliminary version). Preprint, March 2021.
- [13] James Haglund. A proof of the  $q, t$ -Schröder Conjecture. *International Mathematics Research Notices* **2004** (2004), no. 11, 525–560. doi: 10.1155/S1073792804132509

IGM, UNIVERSITÉ GUSTAVE EIFFEL, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE.

*Email address:* `henri.derycke@univ-eiffel.fr`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND.

*Email address:* `mark.dukes@ucd.ie`

LABRI, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE.

*Email address:* `borgne@labri.fr`