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Persistent generatexes: unveiling the dynamical paths in complex systems

Gisela D. Charó*, Davide Faranda & Denisse Sciamarella (*)gdcharo@lsce.ipsl.fr

Laboratoire des Sciences du Climat et de l'Environnement, CEA Saclay l'Orme des Merisiers, UMR 8212 CEA-CNRS-UVSQ, Université Paris-Saclay & IPSL, 91191, Gif-sur-Yvette, France. CNRS – IRD – CONICET – UBA. Institut Franco-Argentin d'Études sur le Climat et ses Impacts (IRL 3351 IFAECI), 6 Centro de Investigaciones del Mar y la Atmósfera, Universidad de Buenos Aires, Argentina.

Flows in phase space

In dynamical systems theory, a *flow* or *phase flow*, is a diffeomorphism $\phi(t, \mathbf{x})$ that satisfies the following equation:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad (1)$$

where time is the only independent variable, $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{f}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ a smooth differential map that generates the flow. Phase space is the space spanned by the dependent variables \mathbf{x} .

Lorenz Attractor & Rössler Attractor

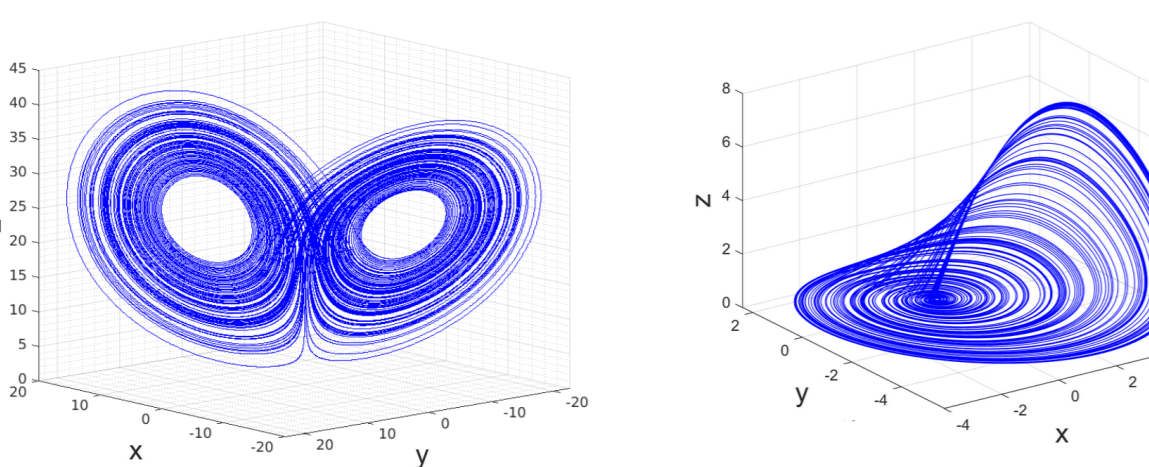
$$(2) \begin{cases} \dot{x} = \sigma(-x + y) \\ \dot{y} = -x(z - r) - y \\ \dot{z} = xy - az \end{cases} \quad (3) \begin{cases} \dot{x} = -y - z \\ \dot{y} = ay + x \\ \dot{z} = b + z(x - c) \end{cases}$$


Figure 1: The systems of equations that generate the Lorenz (2) ($\sigma = 10, r = 24.74, \alpha = 8/3$) and the Rössler (3) attractor ($a = 0.43295, b = 2, c = 4$).

Templex

• When a cell complex K is used to describe a structure of a flow in phase space, the flow will restrict the possibility of circulating along the cells of a complex.

• In order to take the flow on the complex into account, the cell complex K will be endowed with a directed graph $D = (N, E)$ that prescribes the flow direction between its highest-dimensional cells [2].

Definition 1. A **templex** $T \equiv (K, D)$ of dimension d is made of a d -complex K approximating a branched d -manifold associated with a flow in phase space, and a digraph $D = (N, E)$ such that (i) the nodes N are the d -cells of K and (ii) the edges E are the flow-allowed connections between the d -cells.

Definition 2. Let K be a two-dimensional complex and let γ_i denote its 2-cells. The **joining line** is formed by the 1-cells shared by at least three 2-cells. The 1-cells and 2-cells with this property are called **joining 1-cells** and **joining 2-cells** respectively, and the set of joining cells are called the **joining locus** (or *loci*).

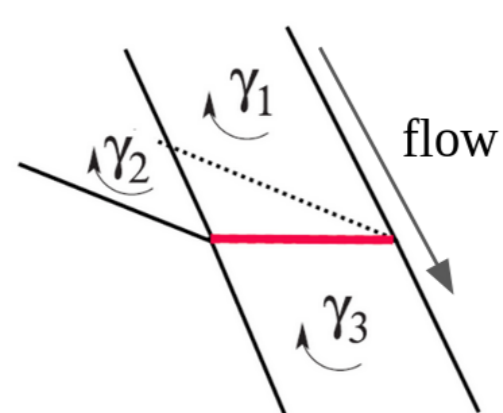


Figure 2: The joining locus is formed by the 1-joining line (in red), the two ingoing 2-cells γ_1, γ_2 , and the outgoing 2-cell γ_3 .

Definition 3. A **generatex** $G = (K^s, D^s)$ is a subtemplex in $T = (K, D)$, where D^s is a cycle of the digraph D , and $K^s \subset K$.

Definition 4. Generatexes G_1 and G_2 are equivalent if they have the same set of ingoing and outgoing cells.

The generatexes stand for the non-equivalent ways of travelling through the complex by the flow. The generatexes provide the 'topological fingerprinting' of the dynamics of the system.

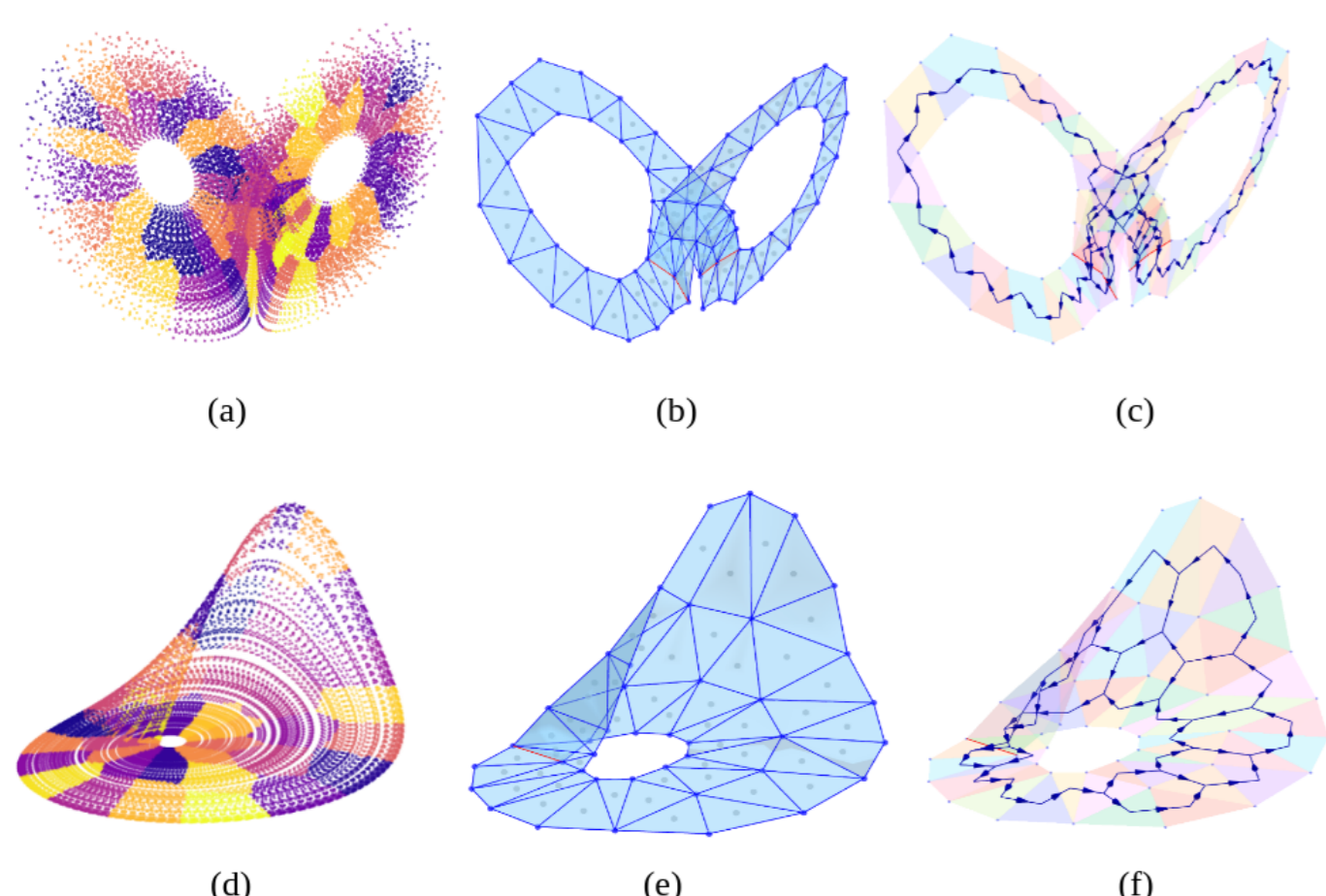


Figure 3: Segmentation of the point cloud for the Lorenz (a) and Rössler (d) point-clouds. Cell complexes (b), (e). Digraphs (c), (f). They were all computed using the algorithm in [1].

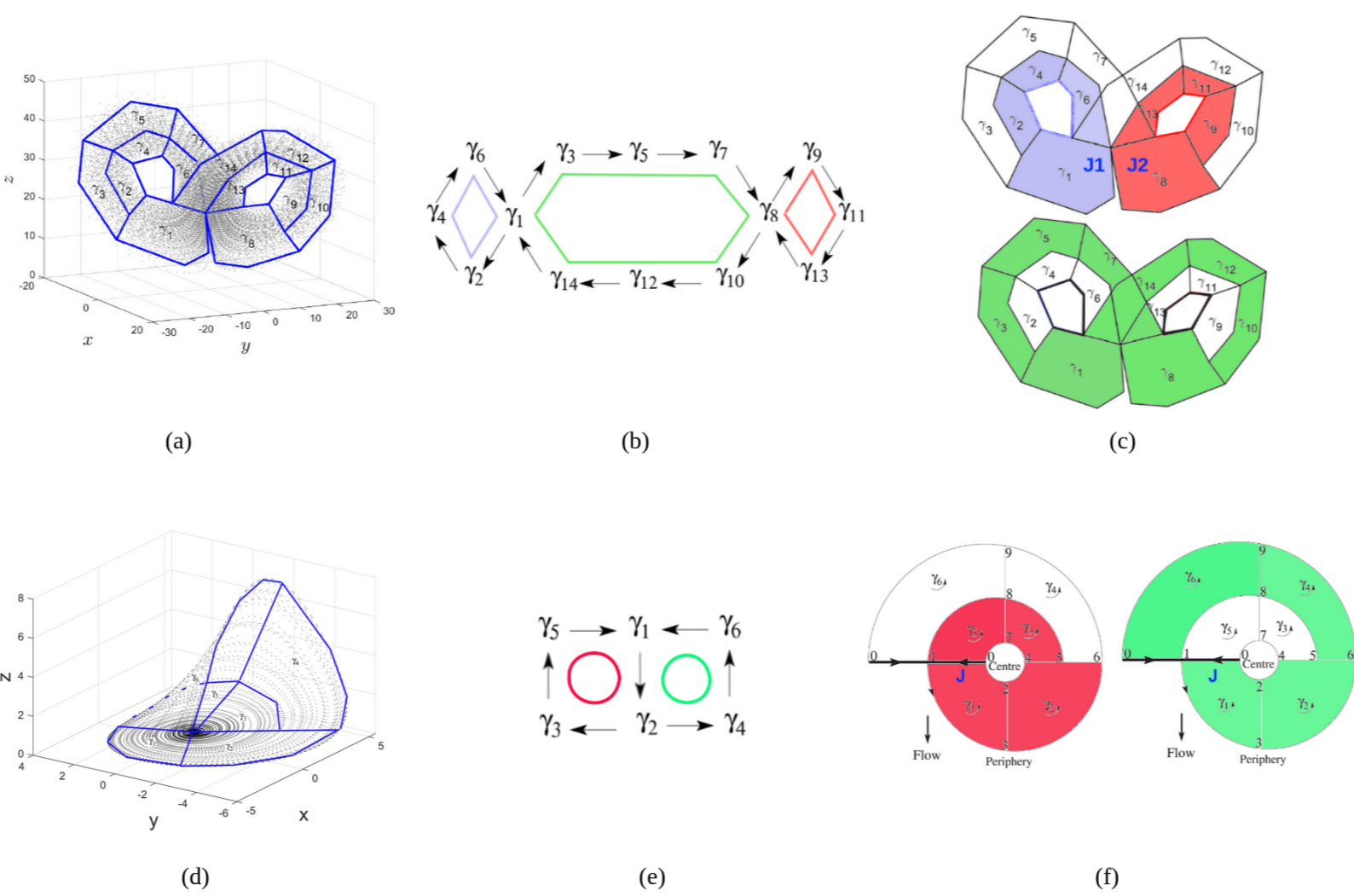


Figure 4: Simplified cell complexes K_L (Lorenz) (a) and K_R (Rössler) (b). Digraphs with coloured cycles D_L (b) and D_R (c). (c) 3 generatexes and a 2-component joining line $J_1 \cup J_2$ (Lorenz), and (f) 2 generatexes and 1-component of the joining line J (Rössler). The templex $T_L = (K_L, D_L)$ has three generatexes $\{G_1, G_2, G_3\}$ where: $K_L(G_1) = \{\gamma_1, \gamma_2, \gamma_4, \gamma_6\}$, and $D_L(G_1) = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 1\}$ (light blue); $K_L(G_2) = \{\gamma_8, \gamma_9, \gamma_{11}, \gamma_{13}\}$, and $D_L(G_2) = \{8 \rightarrow 9 \rightarrow 11 \rightarrow 13 \rightarrow 8\}$ (red); and $K_L(G_3) = \{\gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_8, \gamma_{10}, \gamma_{12}, \gamma_{14}\}$, and $D_L(G_3) = \{1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 10 \rightarrow 12 \rightarrow 14 \rightarrow 1\}$ (green). The templex $T_R = (K_R, D_R)$ has two generatexes $\{G_1, G_2\}$ where: $K_R(G_1) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_5\}$ and $D_R(G_1) = \{1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 1\}$ (red), and $K_R(G_2) = \{\gamma_1, \gamma_2, \gamma_4, \gamma_6\}$ and $D_R(G_2) = \{1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 1\}$ (green).

Local dimension

The dynamics of an attractor can be characterised by studying the recurrence of a state ζ on the attractor. This recurrence is measured by two scalars: D_1 the local dimension and θ the extremal index. D_1 and θ are computed using the Poincaré recurrence theorem and the Extreme Value Theory [4, 3]. Extreme values of the function $g(t, \zeta) = -\log(\text{dist}(x(t), \zeta))$ measure the recurrences of a state ζ on the attractor. Let $P(g(t, \zeta) > q, \zeta) \approx e^{-(x-\mu(\zeta))/\sigma(\zeta)}$ be the distribution of the extremes. For a state ζ , the local dimension is defined as: $D_1(\zeta) = 1/\sigma(\zeta)$ and it measures the density of neighbouring points (similar configurations). The stability of the state ζ is measured by the extremal index $\theta(\zeta)$, defined as the inverse of the average persistence time of trajectories near ζ . The more persistent the state ζ , the more the previous and subsequent states of the system will resemble ζ .

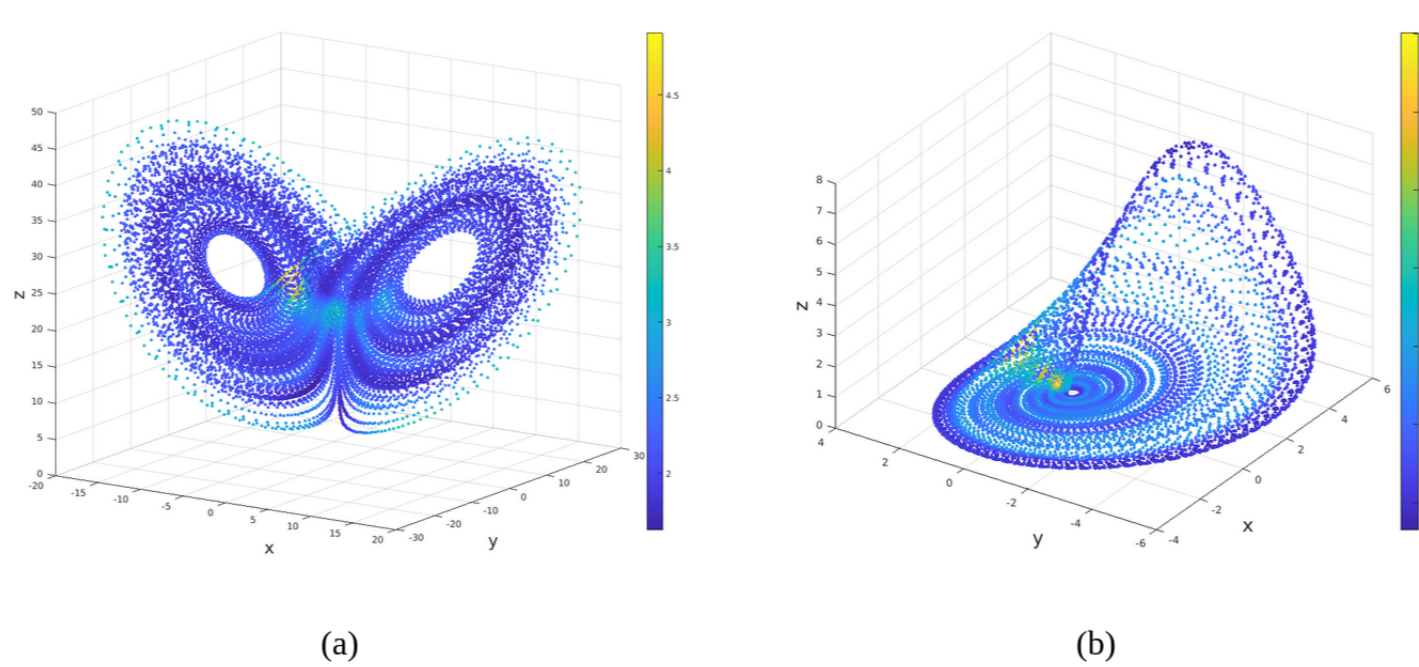


Figure 5: Local dimension D_1 for the Lorenz (a) and Rössler (b) attractors. Note that regions with high values of D_1 are associated with the joining locus at each cell complex.

Persistent Generatexes

Let $\{t_j\}_{1 \leq j \leq N}, N \in \mathbb{N}$ be a monotonically increasing sequence of times and $\{x(t_j) \in \mathbb{R}^n\}$ a point cloud in the n -dimensional phase space, where $\{x(t_j)\}$ is a trajectory obtained from a dynamical system. Let $S = \cup_{m=1}^k S_m$ be a segmentation of the point cloud, such that: $S_m = \{x(t_j) : j_i \in \{1 \dots N\}\}$ and $S_{m_i} \cap S_{m_j} = \emptyset, \forall i, j$.

Filtration is defined as a nested subsequence of digraphs,

$$G_0 \supseteq G_1 \supseteq \dots \supseteq G_{p_f} \supseteq \dots,$$

where $G_{p_i} = (N_{p_i}, E_{p_i})$ and $p_i \in \mathbb{R}_{\geq 0}$ is the parameter of filtration with $0 \leq p_0 \leq p_1 \leq \dots \leq p_{f-1} \leq p_f \leq 1$.

For a segmentation $S, N_{p_i} = N_{p_j} = N \forall i, j$, and each node in N represents a segment in S . The set of edges E_{p_i} are the connections between the nodes depending on a filtration parameter p_i : $(i, j) \in E_{p_i}$ if there is a proportion $\geq p$ of points $\in S_i$ such that $x_i(t + dt) \in S_j$, where dt is a time interval.

This flow-filtration law leads to a sequence of nested set of edges:

$$E_{p_0} \supseteq E_{p_1} \supseteq \dots \supseteq E_{p_f} \supseteq \dots$$

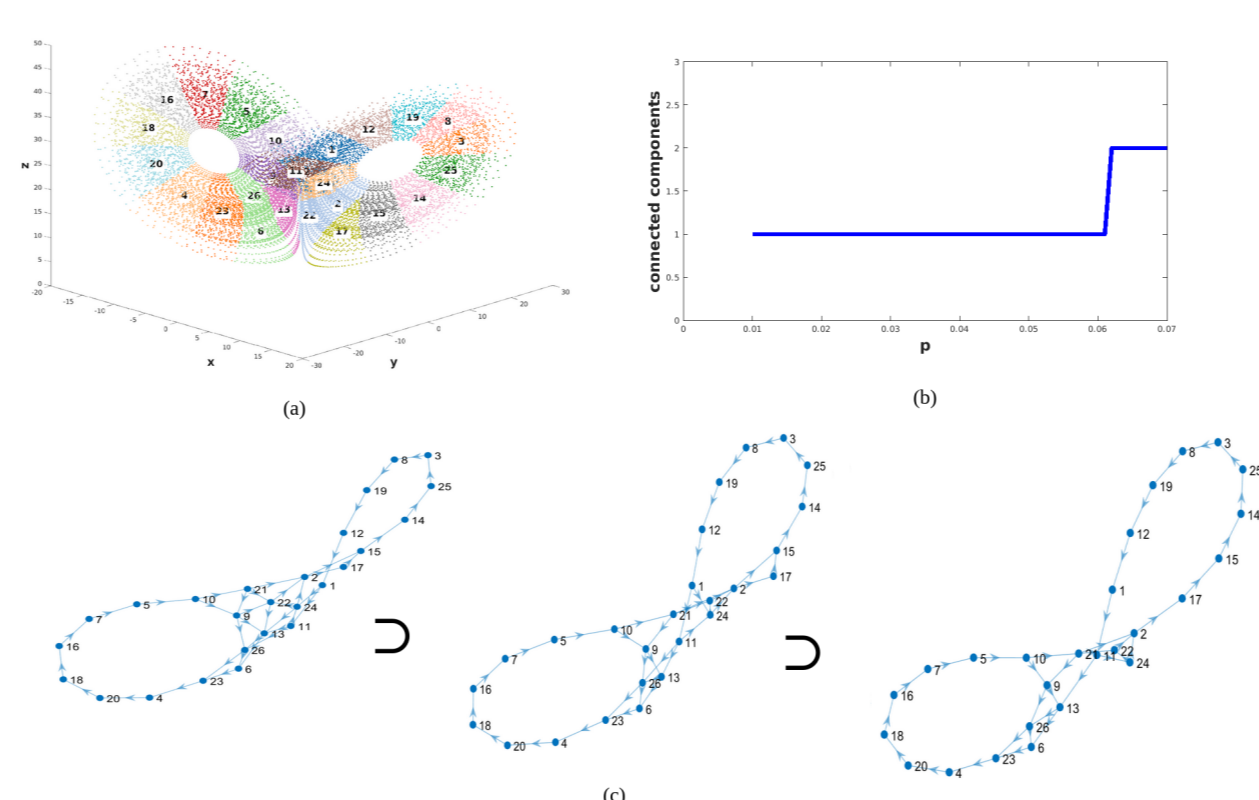


Figure 6: (a) Segmentation of the Lorenz point-cloud, (b) connected components of the digraph as the parameter p increases, and (c) example of a nested sequence of digraphs.

Definition 5. A **templex filtration** $\{T_{p_i} \equiv (S, K, G_{p_i})\}_{i \in \{1 \dots f\}}$ is defined as a sequence of templexes where S is a segmentation of a point cloud, K is a cell

complex associated with the point-cloud, and $G_{p_i} = (N, E_{p_i})$ is a digraph such that (i) the nodes N are segments in S which represent the k -cells of K and (ii) the edges E_{p_i} are the connections between these k -cells associated with the flow-filtration parameter p_i .

Definition 6. The **joining locus** of a segmentation S is formed by the set of outgoing and ingoing nodes. A node $n \in N$ in the digraph $G_{p_i} = (N, E_{p_i})$ is defined as an **outgoing node** if: (i) $\text{indegree}(n) \geq 2$ and (ii) there is $m \in N$ such that $(m, n) \in E_{p_i}$, and $D_1(S_m) \geq \tau$, where S_m is the segment associated with the node m , and τ is a threshold on the local dimension D_1 . This node m will be called an **ingoing node**.

For each filtration parameter p_i , the subset of cycles of G_{p_i} , where in each cycle a pair of ingoing-outgoing node is present, will be noted as $C(p_i)$.

Definition 7. Each cycle of $C(p_i)$, is associated with a chain of segments and k -cells which forms a sub-templex called **pseudogeneratex**. Two pseudogeneratexes are equivalent if they have the same set of outgoing-ingoing nodes.

Definition 8. The set of **persistent generatexes** is the non equivalent set of pseudogeneratexes that persists the most in the barcode of pseudogeneratexes.

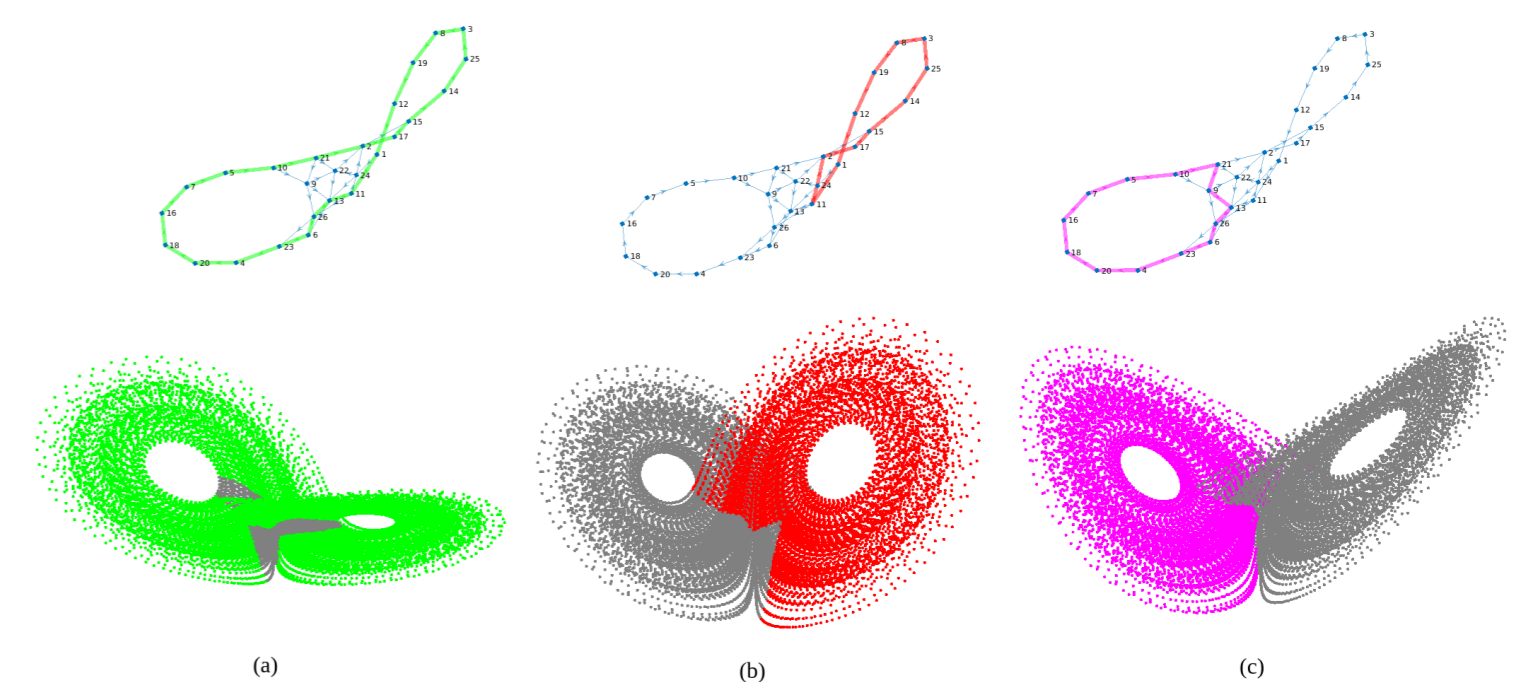
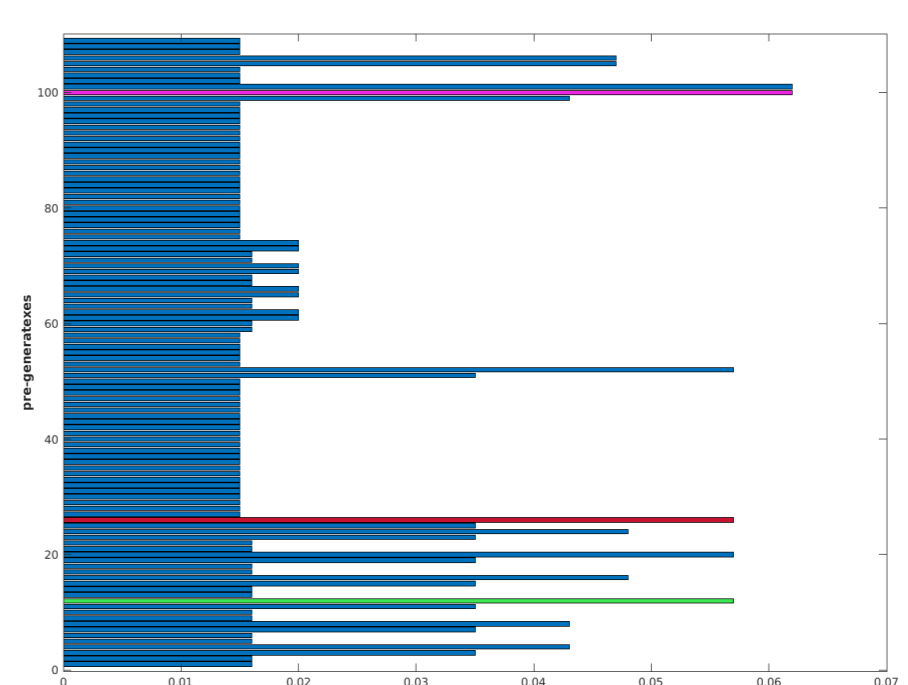


Figure 7: Barcode of pseudogeneratexes (top) for the Lorenz attractor. There are three persistent generatexes coloured in green (a), red (b) and magenta (c) (bottom).

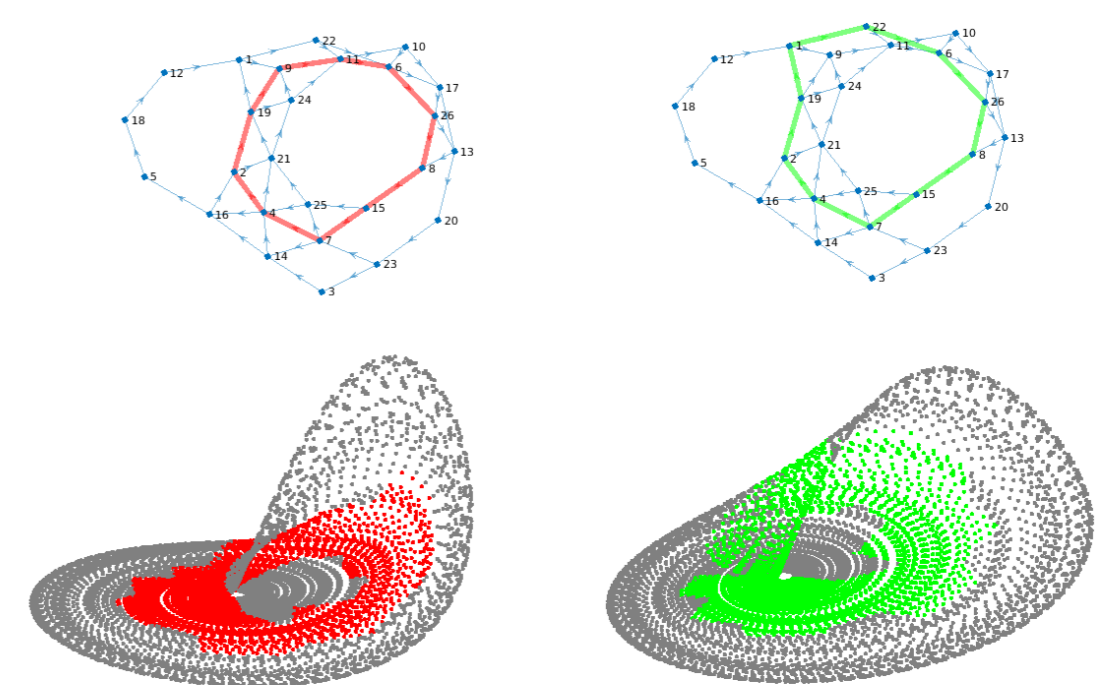
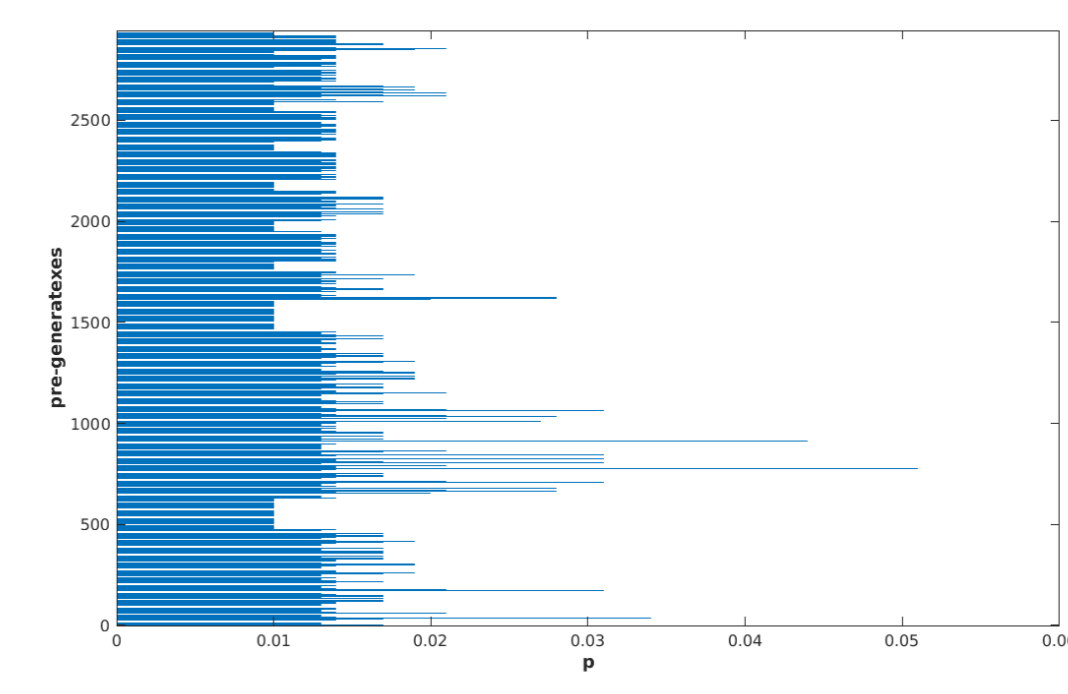


Figure 8: Barcode of pseudogeneratexes (top) for the Rössler attractor. There are two persistent generatexes coloured in red (left) and green (right) (bottom).

Conclusions & Future work

- The templex approach allows to determine the non-equivalent paths compatible with the flow along the structure and the topological structure of the branched manifold.
- In the case of flows where the dimension of the space space is high (atmospheric flows), we may obtain a dynamical characterisation by using persistent generatexes.
- The examples shown here are all three-dimensional systems, but four-dimensional cases are being treated as well.
- Persistent generatexes approach allows us to determine the non-equivalent paths and the joining locus between the clusters of a point cloud.
- It is an ongoing work: exploring the use of different types of segmentations (kmeans, BraMaH), measuring the robustness of the method, etc.

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