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## ► To cite this version:

Gregorio Ricci Curbastro, Edoardo Niccolai. On Some Applications of Absolute Differential Calculus to the Theory of Binary Quadratic Differential Forms and Systems in Two Variables. 2024. hal-04776516

**HAL Id: hal-04776516**

**<https://hal.science/hal-04776516v1>**

Preprint submitted on 11 Nov 2024

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# On Some Applications of Absolute Differential Calculus to the Theory of Binary Quadratic Differential Forms and Systems in Two Variables

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ABSTRACT. The document first appears in Atti del Regio Istituto Veneto, Tomo LI, Serie settima – Tomo quarto, Dispensa VII, 1892-1893, Venezia, pp. 1336-1364, Adunanza ordinaria del giorno 16 Luglio 1893, with the title *Di alcune applicazioni del calcolo differenziale assoluto alla teoria delle forme differenziali quadratiche binarie e dei sistemi a due variabili*.

The mathematical roots of Einstein's general theory of relativity—along with other sets of invariant principles with algebraic and geometric pretensions—are here. And here begins the algebro-geometric fiction that attempts to get Nature (of space-time) talking, which is deaf to our prayers. But that is another story.<sup>a</sup>

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<sup>a</sup> The reference is to a sentence by G. Galilei: «[L]a Natura [è] sorda, & inesorabile à nostri preghi», in *Istoria e dimostrazioni intorno alle macchie solari e loro accidenti comprese in tre lettere scritte all'illustrissimo Signor M. Velserei linceo*, G. Mascardi, Roma, MDCXIII (1613), p. 131.

ON SOME APPLICATIONS  
OF ABSOLUTE DIFFERENTIAL CALCULUS  
TO THE THEORY OF BINARY QUADRATIC DIFFERENTIAL FORMS  
AND SYSTEMS IN TWO VARIABLES<sup>a</sup>

[1336][1] The results that will be found in this work are, for the most part, not new: what is new are the methods that lead to them and the form that they assume. After having repeatedly insisted on the advantages that the methods of absolute differential calculus offer in research in which properties independent of the choice of coordinates are concerned, I still deem it appropriate to show some of their fundamental applications in a field that, precisely because it has been explored in every part by geometers, is better suited to highlighting the advantages mentioned by means of comparisons. Furthermore, it will be found in this paper an introduction necessary for the understanding of other advantages that, in due course, will be made public.

For the moment, I shall limit myself to considering the binary positive quadratic differential forms [1337][2] in themselves, that is, irrespective of their common origin from ternary forms of the same nature and with constant coefficients, and I will linger mainly over the theory of systems of lines traced on manifolds represented by them. It will be seen that some concepts, such as that of geodetic curvature, which have originated from geometric considerations, present themselves under an analytical aspect, i.e., as the most elementary absolute invariants common to the fundamental form and to the covariant forms, which we will be gradually led to consider; while other concepts, such as that of line bundle, which have appeared here and there almost incidentally in the research of geometers, reveal the necessary constraints with the theory outlined above.

Another concept closely connected with the methods of absolute differential calculus, and which will be usefully applied here, is that of coordinate systems for the determination of systems of lines traced on surfaces. It is not new since it consists in making this determination depend on differential equations rather than on equations in finite terms; the elegant applications given by Professor Beltrami in his *Researches in Analysis Applied to Geometry* are well-known. Here, moreover, the differential equations are chosen in a form, say, canonical, which proves to be very appropriate, and allows us to establish a univocal correspondence between the systems of lines and their coordinate systems.

Although, as indicated in the title, this work is essentially analytical, I have not avoided using a geometric nomenclature, associated with geometric interpretations, which are so conducive to conciseness and clarity. However, because of the importance of the subject, I have added a hint of an analytical theory of transformations of binary quadratic differential forms, which does not require the introduction of special systems of coordinates, but it comes directly from the general theory of forms and from well-known [1338][3] theorems on the integration of complete or unconditionally integrable systems.

I will adopt here some notations established in a brief summary of some of my works published in *Bulletin des Sciences Mathématiques*, June 1892, by Messrs. Darboux and Tannery. But I will explicitly recall that, since we have to consider only one fundamental form  $\varphi$ , if a letter with upper and lower  $m$ -indices represents a multiple covariant or contravariant system, the same letter with an additional index will represent the system derived from it covariantly or contravariantly, according to  $\varphi$ , and that if a letter with lower  $m$ -indices represents a covariant system, the same letter with upper  $m$ -indices will represent its reciprocal with respect to  $\varphi$  and vice versa. I would add that the summations relating to  $p$ -indices are to be understood as always extended to all the dispositions with repetitions  $p$  to  $p$  of the indices 1 and 2, and that, when appropriate, it will be permitted to replace an index  $r$  with an index  $s$ , provided that  $r$  and  $s$  are even or odd together.

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<sup>a</sup> For the sake of brevity, I will use the name *absolute differential calculus* to refer to the set of methods I have previously called covariant and contravariant derivation, as they are applicable to every fundamental form regardless of the choice of independent variables and indeed require that these [variables] be completely general and arbitrary.

1. — Let us assume as a *fundamental form* a binary quadratic differential form

$$\varphi = \sum_{rs} a_{rs} dx_r dx_s,$$

which is irreducible and positive at least in a field, to which the variability of the independent variables  $x_1$  and  $x_2$  will be considered limited. Let  $\mu_r$  be any simple covariant system in two variables  $x_1$  and  $x_2$ . Likewise, the same will be true for the system  $\bar{\mu}_r$ ; it is defined, regardless of the sign, by

$$\sum_r \mu^{(r)} \bar{\mu}_r = 0, \quad \sum_r \bar{\mu}^{(r)} \bar{\mu}_r = \sum_r \mu^{(r)} \mu_r,$$

which give

$$\bar{\mu}_r = (-1)^r \sqrt{a} \mu^{(r+1)}. \quad (1)$$

<sup>[1339|4]</sup> Assuming that  $\sqrt{a}$  is positive, the system  $\bar{\mu}_r$  will be completely determined on the basis of the system  $\mu_r$  and the form  $\varphi$ . Now,  $\bar{\mu}_r$  will be called the *canonical system orthogonal* to the system itself with respect to this form, and it could be written like this:

$$\bar{\mu}_r \equiv N_\varphi(\mu_r).$$

Since Eq. (1) is equivalent to

$$\bar{\mu}^{(r)} = \frac{(-1)^r}{\sqrt{a}} \mu_r + 1, \quad (2)$$

we can also say that  $\bar{\mu}^{(r)}$  is the canonical system orthogonal to the system  $\mu^{(r)}$  with respect to the form  $\varphi$ , putting

$$\bar{\mu}^{(r)} = N_\varphi(\mu^{(r)}).$$

In this paper we will always consider only one fundamental form  $\varphi$ , so we shall designate by  $\bar{\mu}_r$  or  $\bar{\mu}^{(r)}$  the canonical system orthogonal to  $\varphi$  for a system  $\mu_r$  or  $\mu^{(r)}$ , respectively. We will then observe that by Eq. (1) (2) one gets

$$-\mu_r \equiv N_\varphi(\bar{\mu}_r), \quad -\mu^{(r)} \equiv N_\varphi(\bar{\mu}^{(r)}),$$

to wit, it results that the system  $\bar{\mu}_r$  (or the system  $\bar{\mu}^{(r)}$ , respectively) has as its canonical system orthogonal to  $\varphi$  the system  $-\mu_r$  ( $-\mu^{(r)}$ , respectively).

Let us also recall that, if  $\mu_r$  and  $\nu_r$  are any two simple covariant systems, then

$$I = \sum_r \mu^{(r)} \nu_r$$

is an invariant, and, if

$$I_s = \frac{dI}{dx_s},$$

one has<sup>a</sup> the formulas <sup>[1340|5]</sup>

$$I_s = \sum_r \mu^{(r)} \nu_{rs} + \sum_r \nu^{(r)} \mu_{rs}. \quad (3)$$

For any simple system  $\mu_r$  it will be true that

$$\mu_{rs+1s} - \mu_{rss+1} = \sum_q a_{qr,s+1s} \mu^{(q)},$$

or

$$a_{rr,s+1s} = 0, \quad a_{r+1r,s+1s} = (-1)^{r+s} aG, \quad (4a)$$

$$\mu_{rs+1s} - \mu_{rss+1} = (-1)^s \sqrt{a} G \bar{\mu}_r, \quad (4b)$$

where  $G$  is the Gauss invariant relative to the fundamental form.

2. — Consider a simply infinite system of lines drawn on a surface whose linear element is expressed by  $\sqrt{\varphi}$ , which we will briefly call *surface*  $\varphi$ . Once the positive directions for these lines

<sup>a</sup> See § 2 of the above-mentioned summary.

are established, let  $dx_1$  and  $dx_2$  denote the variations that the coordinates of any point undergo for an infinitesimal [punctual] displacement in the positive direction of a line of the system. Let

$$\lambda^{(r)} = \frac{dx_r}{\sqrt{\varphi}}, \quad (5)$$

where  $\sqrt{\varphi}$  represents an absolute value. The simple contravariant system  $\lambda^{(r)}$  will then satisfy

$$\sum_r \lambda^{(r)} \lambda_r = I. \quad (6)$$

If a simple contravariant system  $\lambda^{(r)}$  satisfies mutually this equation, then Eq. (5) defines a system of lines<sup>a</sup> drawn on a surface <sup>[1341][6]</sup>  $\varphi$ . If therefore a system of lines drawn on a surface is considered as fully determined only if its positive directions are fixed, for every system of lines drawn on certain surfaces  $\varphi$  there exists a system  $\lambda^{(r)}$  defined by Eq. (5) satisfying Eq. (6), and reciprocally every system  $\lambda^{(r)}$ , which satisfies Eq. (6), completely determines on the surfaces  $\varphi$  a system of lines, of which Eq. (5) can be considered as a differential equation. For this reason I will call the system  $\lambda^{(r)}$  *contravariant coordinate system* of the system of lines defined by Eq. (5), whilst the system  $\lambda_r$  reciprocal to  $\lambda^{(r)}$  with respect to the fundamental form  $\varphi$  is the *covariant coordinate system*. Every system  $\lambda^{(r)}$  (or  $\lambda_r$ ), which satisfies Eq. (6), can be considered as the contravariant (or covariant) coordinate system of a system of lines—defined by differential Eq. (5)—drawn on a surface  $\varphi$ . Often, for brevity, the lines appertaining to the covariant coordinate system will be called *lines*  $\lambda_r$ .

3. — If  $\lambda_r$  is the covariant coordinate system of a system of lines drawn on a surface  $\varphi$ , the canonical system  $\bar{\lambda}_r$  orthogonal to it with respect to  $\varphi$  has the same meaning for the system of trajectories orthogonal to the lines  $\lambda_r$ ; and it is easy to recognize that the positive directions of the lines  $\bar{\lambda}_r$  are to the lines  $\lambda_r$  as the positive axis of the ordinates is to the axis of the abscissas.

In accordance with the aforementioned conventions, I will designate by  $f_r$  the derivatives with respect to  $x_r$  of any function  $f$  of  $x_1$  and  $x_2$ , and I will put

$$(\Delta_1 f)^2 = \sum_r f^{(r)} f_r.$$

If  $\mu$  and  $\nu$  are any two parameters of the systems <sup>[1342][7]</sup> of lines  $\lambda_r$  and  $\bar{\lambda}_r$ , and by  $ds$  and  $\delta s$  one designates the linear elements of these lines, then it will be possible to specify

$$\Delta_1 \mu = \frac{d\mu}{\delta s}, \quad \Delta_1 \nu = \frac{d\nu}{ds}. \quad (7)$$

By means of the same definitions of the systems  $\lambda_r$  and  $\bar{\lambda}_r$  one can write the identity

$$\sum_r \mu^{(r)} \lambda_r = 0, \quad \sum_r \nu^{(r)} \bar{\lambda}_r = 0,$$

and

$$\lambda_r = \frac{\nu_r}{\Delta_1 \nu}, \quad \bar{\lambda}_r = \frac{\mu_r}{\Delta_1 \mu},$$

taking into account Eq. (7).

The identities

$$\begin{aligned} \varphi &= \left( \frac{d\mu}{\Delta_1 \mu} \right)^2 + \left( \frac{d\nu}{\Delta_1 \nu} \right)^2, \\ d\mu &= \sum_r \mu_r dx_r, \quad d\nu = \sum_r \nu_r dx_r, \end{aligned}$$

lead to

$$\varphi = \sum_{rs} (\lambda_r \lambda_s + \bar{\lambda}_r \bar{\lambda}_s) dx_r dx_s,$$

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<sup>a</sup> Unless the contrary is expressly stated, when we talk about systems of lines traced on a surface, we always would refer to simply infinite systems.

which is equivalent to

$$a_{rs} = \lambda_r \lambda_s + \bar{\lambda}_r \bar{\lambda}_s. \quad (8)$$

From

$$\bar{\lambda}_r = (-1)^r \sqrt{a} \lambda^{(r+1)}, \quad (9)$$

which defines the system  $\bar{\lambda}_r$ , it can be easily deduced

$$\lambda_{r+1} \bar{\lambda}_r - \lambda_r \bar{\lambda}_{r+1} = (-1)^r \sqrt{a}. \quad (10)$$

If now we indicate by  $\mu_r$  a simple covariant system, Eqq. (1) and (8), together with <sup>[1343][8]</sup>

$$\mu_r = \sum_p a_{pr} \mu^{(p)},$$

easily lead to the identities

$$\begin{cases} \mu_r = \lambda_r \sum_p \mu^{(p)} \lambda_p + \bar{\lambda}_r \sum_p \mu^{(p)} \bar{\lambda}_p, \\ \bar{\mu}_r = \bar{\lambda}_r \sum_p \mu^{(p)} \lambda_p - \lambda_r \sum_p \mu^{(p)} \bar{\lambda}_p. \end{cases} \quad (11a)$$

$$\quad (11b)$$

If we combine these identities with the expressions relating to a second simple covariant system  $\nu_r$ , we obtain the formulas

$$\sum_r \mu^{(r)} \nu_r = \sum_r \bar{\mu}^{(r)} \bar{\nu}_r, \quad \sum_r \mu^{(r)} \bar{\nu}_r = - \sum_r \nu^{(r)} \bar{\mu}_r. \quad (12)$$

In particular, if  $\lambda_r$  and  $\xi_r$  are the covariant coordinate systems of two systems of lines drawn on the surfaces  $\varphi$ , and if we consider as positive the angles described in the opposite direction to the movement of clock indices,<sup>a</sup> we have

$$\cos \alpha = \sum_r \lambda^{(r)} \xi_r = \sum_r \bar{\lambda}^{(r)} \bar{\xi}_r, \quad \sin \alpha = \sum_r \xi^{(r)} \bar{\lambda}_r = - \sum_r \lambda^{(r)} \bar{\xi}_r, \quad (13)$$

where  $\alpha$  is the angle that, for positive directions, goes from  $\lambda_r$  to  $\xi_r$ .

4. — If we differentiate the identity (6) according to the rule contained in formula (3), we arrive at

$$\sum_r \lambda^{(r)} \lambda_{rs} = 0.$$

It turns out that the double system  $\lambda_{rs}$  can be considered as the product<sup>b</sup> of the simple system  $\bar{\lambda}_r$  by another simple system, which will be designated by  $\varphi_s$ , so that we have

$$\lambda_{rs} = \bar{\lambda}_r \varphi_s, \quad (14)$$

<sup>[1344][9]</sup> and consequently

$$\varphi_s = \sum_r \bar{\lambda}^{(r)} \lambda_{rs}. \quad (15)$$

If in accordance with the rule mentioned above we differentiate the identity

$$\sum_r \lambda^{(r)} \bar{\lambda}_r = 0,$$

we obtain the formulas

$$\sum_r \lambda^{(r)} \bar{\lambda}_{rs} = - \sum_r \bar{\lambda}^{(r)} \lambda_{rs} = -\varphi_s$$

and thence

$$\bar{\lambda}_{rs} = -\lambda_r \varphi_s. \quad (16)$$

Via Eq. (14), with a further covariant derivation according to  $\varphi$ , the equalities

$$\lambda_{rst} = \bar{\lambda}_r \varphi_{st} - \lambda_r \varphi_s \varphi_t,$$

<sup>a</sup> *TN.* The most obvious reference is the clock face, but it is not movable, otherwise a device (for measuring time) could not indicate hours and minutes.

<sup>b</sup> See § 1 of the above-mentioned summary.

and

$$\lambda_{rs+1s} - \lambda_{rss+1} = \bar{\lambda}_r (\varphi_{s+1s} - \varphi_{ss+1}),$$

are achieved. This last formula, if compared with Eq. (4), gives

$$\varphi_{12} + \varphi_{21} = \sqrt{a}G. \quad (17)$$

As can be seen from the theory of total differential equations, this equation represents the necessary and sufficient condition for the system of equations resulting from (6) and (14) to be complete or unconditionally integrable, in which  $\varphi_r$  is considered as given and  $\lambda_r$  as unknown. Once this condition is verified, the general integral system contains an arbitrary constant, to wit, on the surfaces  $\varphi$  there exists a simply infinite number of systems of lines whose coordinate system satisfies (14). We will soon recognize the very simple relation that links together all these systems.

<sup>[1345][10]</sup> 5. — Let  $\lambda_r$  and  $\xi_r$  be the coordinate systems of two systems of lines drawn on the surfaces  $\varphi$ , and let

$$\xi_{rs} = \bar{\xi}_r \psi_s \quad (18)$$

be an equation analogous to (14) satisfied by the system  $\xi_r$ . Any  $\psi_s$  and  $\varphi_s$  will satisfy Eq. (17), ergo

$$(\varphi_1 - \psi_1)_2 = (\varphi_2 - \psi_2)_1,$$

which tells us that the system  $\varphi_s - \psi_s$  results from the derivatives of a function with respect to  $x_s$ . We can arrive at the same result by another way, which will lead us to a very simple geometric interpretation of this function.

If we differentiate Eq. (13), with the help of Eq. (3), and through Eq. (14) (16) and (18) we use Eq. (13) to eliminate  $\sin \alpha$  and  $\cos \alpha$ , we arrive at

$$\alpha_s = \psi_s - \varphi_s.$$

Now it is enough to remember the meaning of  $\alpha$  established in § 3 to see that the system  $\psi_s - \varphi_s$  has as elements the derivatives of the angle, which must be described to pass from the positive directions of the lines  $\lambda_r$  to those of the lines  $\xi_r$ .

I will call *bundle systems* the set of systems in an infinite number of lines drawn on the surfaces  $\varphi$ —note that here two lines belonging to two given systems of the bundle intersect each other at a constant angle. It turns out then that

$$\varphi_s = \psi_s$$

expresses the necessary and sufficient conditions thanks to which the two systems  $\lambda_r$  and  $\xi_r$  of lines drawn on the surfaces  $\varphi$  are regarded as belonging to the same bundle. We can therefore conclude that, if a simple covariant system  $\varphi_s$  satisfies <sup>[1346][11]</sup> Eq. (17), then the infinite systems of lines drawn on the surfaces  $\varphi$  constitute a bundle. In other words, to every system  $\varphi_s$ , which satisfies Eq. (17), there corresponds a bundle  $F$  resulting from the systems of lines whose coordinate systems satisfy (14), and to every bundle  $F$  of systems of lines there corresponds a simple covariant system  $\varphi_s$  satisfying Eq. (17). For this reason the system  $\varphi_s$  (and its reciprocal  $\varphi^{(s)}$ ) will be called *covariant (contravariant) coordinate system* of the bundle  $F$ . The doubly infinite system, which results from all lines belonging to the systems of the bundle, will be called *bundle of lines*.

Summing up, we can state the following theorems:

(1) *For a simple system  $\varphi_s$  to be the covariant coordinate system of a bundle of systems of lines drawn on the surfaces  $\varphi$ , it is necessary and sufficient that the condition  $\varphi_{12} - \varphi_{21} = \sqrt{a}G$  is verified. Once this condition is verified, the bundle for a coordinate system  $\varphi_s$  results from the systems of lines whose covariant coordinate systems  $\lambda_r$  satisfy the first-order partial differential equation  $\lambda_{rs} = \bar{\lambda}_r \varphi_s$ .*

(2) *If  $\varphi_r$  and  $\psi_r$  are the coordinate systems of two bundles of systems of lines drawn on the surfaces  $\varphi$ , the system  $\psi_r - \varphi_r$  has as elements the derivatives of the angle, the description of which is convenient to make around any point of the surfaces on the tangent plane, in order to pass from the lines of a system of the first bundle to the lines of a system of the second bundles.*

6. — We must move on to the general theory of absolute invariants,<sup>a</sup> which can be obtained by associating one or more covariant systems to a fundamental quadratic <sup>[1347][12]</sup> differential form, when this form is applied to a binary form  $\varphi$  with a covariant coordinate system  $\lambda_r$  of a system of lines drawn on the surfaces  $\varphi$ . From such a theory it follows that

(1) *There is only one absolute invariant  $\sum_r \lambda^{(r)} \lambda_r$  identically equal to unity.*

(2) *All the absolute invariants of the first order will be obtained by algebraically associating to the form  $\varphi$ —in addition to the linear form of coefficients  $\lambda_r$ —the linear form of coefficients  $\varphi_r$ , i.e., the coordinate system of the bundle, to which the lines  $\lambda_r$  belong.*

These first-order invariants are thereby

$$\gamma = \sum_r \lambda^{(r)} \varphi_r, \quad (\gamma) = \sum_r \bar{\lambda}^{(r)} \varphi_r \quad (19)$$

$$\rho^2 = \sum_r \varphi^{(r)} \varphi_r. \quad (20)$$

These invariants are not independent but, as shown by Eq. (8), related by

$$\rho^2 = \gamma^2 + (\gamma)^2. \quad (21)$$

(3) *We will obtain all the absolute invariants of any order  $n > 1$ , by algebraically associating to  $\varphi$  the two linear forms mentioned above and then all the forms whose coefficients are the elements of the covariant systems that can be achieved from the system  $\varphi_r$  with  $n - 1$  successive covariant derivations according to  $\varphi$ .*

From Eq. (17) it appears that in this way the Gauss invariant  $G$  is obtained, and the resulting systems from  $\varphi_r$  can be replaced by  $G$ , with respect to the double symmetric system  $\frac{\varphi_{rs} + \varphi_{sr}}{2}$  and their systems, for  $n > 2$ . Concerning the second-order, as we shall see, the invariant

$$\theta = \frac{1}{2} \sum_{rs} a^{(rs)} \varphi_{sr} \quad (22)$$

has a notable place.

<sup>[1348][13]</sup> The invariants with which we are dealing are evidently analytical expressions (i) of the system of lines  $\lambda_r$  if they actually contain the elements of the system  $\lambda_r$ , (ii) of the bundle if they contain  $\varphi_r$  or the elements of the system  $\frac{\varphi_{rs} + \varphi_{sr}}{2}$  but without explicitly containing  $\lambda_r$ , and finally (iii) of the surfaces  $\varphi$  if, in addition to the coefficients of the fundamental form, they contain only  $G$ . For example, in the first case the invariants are  $\gamma$  and  $(\gamma)$ , in the second case  $\rho$  and  $\theta$ .

7. — Let us first propose to determine the geometric nature of the systems of lines characterized by the differential equation  $\gamma = 0$ . Since Eq. (19) is equivalent to

$$\varphi_r = \gamma \lambda_r + (\gamma) \bar{\lambda}_r, \quad (23)$$

in this case Eq. (14) will assume the form

$$\lambda_{rs} = (\gamma) \bar{\lambda}_r \bar{\lambda}_s,$$

and consequently the identity

$$\lambda_{12} = \lambda_{21}.$$

For Eq. (14) (23) one arrives at  $\gamma = 0$ . The identity represents the necessary and sufficient condition that  $\lambda_r$  is the derivative of a function  $\lambda$  with respect to  $x_r$ . According to Eq. (6), it follows  $(\Delta_1 \lambda)^2 = 1$ , so (§ 3)  $\lambda$  will be a parameter of the lines  $\bar{\lambda}_r$ . We can conclude that

*The equation  $\gamma = 0$  is the characteristic differential equation of systems of geodetic lines,*

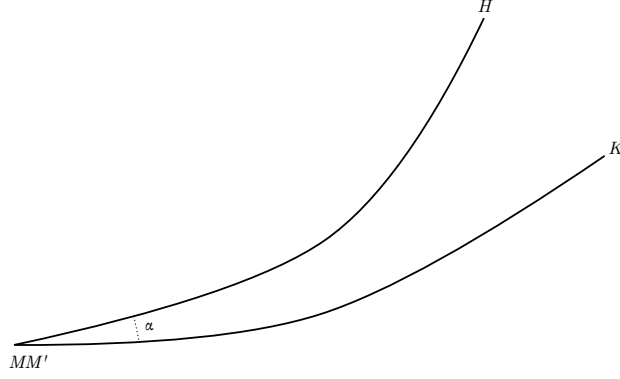
*and consequently*

*The equation  $(\gamma) = 0$  is the characteristic differential equation of systems of parallel lines.*

<sup>[1349][14]</sup> Now, let  $\lambda_r$  be the coordinate system of any system of lines,  $MH$  a line of this system, and  $MM'K$  the geodetic line tangent to it at a point  $M$ , which therefore also contains the point  $M'$  of  $MH$  very close to  $M$ . Let us indicate by  $\xi(r)$  the contravariant coordinate system of the geodetic line  $x_1 = \cos t$ . We will then have (§ 2)  $\xi^{(1)} = 0$ ,  $\sum_r \psi_r \xi^{(r)} = \psi_2 \xi^{(2)} = 0$ , namely  $\psi_2 = 0$ .

<sup>a</sup> See § 3 of the above-mentioned summary.





Supposing that the line  $MM'K$  belongs to the system  $\xi^{(r)}$ , at the point  $M$  we will also have  $\lambda^{(1)} = 0$ ,

$$\gamma = \lambda^{(2)}\varphi_2.$$

If we now designate by  $\alpha$  the angle that must be described to pass from the lines  $\xi_r$  to  $\lambda_r$ , one gets (§ 5)  $\varphi_r = \psi_r + \alpha_r$ , and  $\varphi_2 = \alpha_2 = \frac{d\alpha}{dx_2}$  at the point  $M$ , and then also

$$\gamma = \frac{d\alpha}{dx_2} \frac{dx_2}{ds} = \frac{d\alpha}{ds}.$$

Since  $d\alpha$  is the angle arising from the meeting point  $M'$  of two lines,  $MM'H$  and the geodetic  $MM'K$ , we can conclude that

*The invariant  $\gamma$  represents the geodetic curvature of the lines  $\lambda_r$ , and analogously the invariant  $(\gamma)$  is the geodetic curvature of their orthogonal trajectories  $\bar{\lambda}_r$ .*

By Eq. (20) we can also assert that

*In a bundle of lines the sum of the squares of the geodetic curvatures of any two orthogonal lines has a value independent of the choice of the lines. If  $\varphi_r$  is the coordinate system of the bundle, the sum is represented by the invariant  $\sum_r \varphi^{(r)}\varphi_r$ .*

<sup>[1350][15]</sup> As is it known and as it results from Eqq. (17) and (19) the two invariants  $\gamma$  and  $(\gamma)$  can cancel each other identically together only on developable surfaces and only for the bundle of systems, which in the plane results from the doubly infinite rectilinear system. Leaving aside this case, the equations

$$\rho = \sqrt{\gamma^2 + (\gamma)^2} \quad (24)$$

$$\varphi_r = \rho \lambda'_r \quad (25)$$

define on the surfaces  $\varphi$  a vector, i.e. a quantity  $\rho$  determined in length and direction for each point of the surface, the first being given by  $\rho$  and the second by the lines that have  $\lambda'_r$  as a covariant coordinate system. This vector on the surfaces  $\varphi$  for each bundle of systems  $\varphi_r$  will be called *curvature of the bundle*.

Letting

$$\cos \alpha = \frac{\gamma}{\rho}, \quad \sin \alpha = \frac{(\gamma)}{\rho}, \quad (26)$$

the comparison of Eq. (25) with Eq. (23) gives

$$\lambda'_r = \cos \alpha \lambda_r + \sin \alpha \bar{\lambda}_r, \quad (27)$$

from which it follows (§ 3) that  $\alpha$  is the angle that must be described to pass from the lines  $\lambda_r$  to  $\lambda'_r$ , and  $\gamma$  is the projection of the curvature of the bundle on the lines  $\lambda_r$ .

We conclude that:

*If we consider a system of lines drawn on the surfaces  $\varphi$ , the geodetic curvature of any line of the system at a certain point is the projection of the curvature of the bundle to which the system belongs.*

We will call the lines  $\lambda'_r$  *curvature lines* of the bundle, and it will be easy to recognize that:

*The necessary and sufficient condition for a bundle  $\varphi_r$  to contain a system of geodetic lines consists in the fact that its curvature lines belong to the <sup>[1351][16]</sup> bundle. Verify this condition, the trajectories orthogonal to these lines of curvature are geodetic.*

The validity of what has just been said is guaranteed by a simple observation: if the lines  $\lambda_r$  are assumed to be geodetic, from Eqq. (26) and (27) one has  $\alpha = \frac{\pi}{2}$  and consequently  $\lambda'_r = \bar{\lambda}_r$ .

I will also observe that, if the surfaces  $\varphi$  are developable, the bundle—that in the plane coincides with the doubly infinite rectilinear system—can be considered as a bundle with zero curvature whose lines are all lines of curvature of the bundle.

8. — Let us apply the rules for differentiating product systems, sums and composite systems to Eq. (11) in § 3. Keeping in mind Eqq. (14), one thus arrives at

$$\bar{\mu}_{rs} = \bar{\lambda}_r \sum_p \lambda^{(p)} \mu_{ps} - \lambda_r \sum_p \bar{\lambda}^{(p)} \mu_{ps},$$

from which, remembering Eq. (9), one gets

$$\bar{\mu}_{21} - \bar{\mu}_{12} = \sqrt{a} \sum_{rs} a^{(rs)} \mu_{rs}.$$

One therefore also has

$$\mu_{21} - \mu_{12} = \sqrt{a} \sum_{rs} a^{(rs)} \bar{\mu}_{rs}.$$

Recalling Eq. (17), we can conclude that

(1) *The necessary and sufficient condition for a simple two-variable system  $\mu_r$  to have as elements the derivatives of a function  $\mu$  depends on the possibility that the canonical system  $\bar{\mu}_r$  orthogonal to it with respect to  $\varphi$  satisfies the equation  $\sum_{rs} a^{(rs)} \bar{\mu}_{rs} = 0$ .*

(2) *The necessary and sufficient condition for  $\mu_r$  to be the coordinate system of a bundle of systems of lines drawn on the surfaces  $\varphi$  consists in the fact that the system  $\bar{\mu}_r$  satisfies the equation  $\sum_{rs} a^{(rs)} \bar{\mu}_{rs} = G$ .*

<sup>[1352][17]</sup> Let  $\mu_r$  be any simple covariant system,  $\lambda_r$  the covariant coordinate system of a system of lines drawn on the surfaces  $\varphi$ ,  $\gamma$  their geodetic curvature, and  $\varphi_r$  the coordinate system of the bundle to which they belong. We set

$$\sum_r \mu^{(r)} \lambda_r = \alpha, \quad \sum_r \mu^{(r)} \bar{\lambda}_r = \beta, \quad (28)$$

i.e.

$$\mu_r = \alpha \lambda_r + \beta \bar{\lambda}_r.$$

Via Eqq. (12) we will also have

$$\bar{\mu}_r = -\beta \lambda_r + \alpha \bar{\lambda}_r,$$

from which—using Eq. (14)—follows

$$\bar{\mu}_{rs} = \alpha_s \bar{\lambda}_r - \beta_s \lambda_r - \mu_r \varphi_s,$$

and, thanks to Eq. (19),

$$\sum_{rs} \alpha^{(rs)} \bar{\mu}_{rs} = \sum_r \alpha^{(r)} \bar{\lambda}_r - \sum_r \beta^{(r)} \lambda_r - \alpha \gamma - \beta(\gamma). \quad (29)$$

If we assume that the system  $\mu_r$  is the result of the derivatives of a function  $\mu$  with respect to  $x_r$ , from Eqq. (5) and (28) it follows that  $\alpha$  and  $\beta$  are the ratios of the increases that  $\mu$  undergoes for infinitesimal displacements of the point  $(x_1, x_2)$  along the lines  $\lambda_r$  and  $\bar{\lambda}_r$ .

Since we define

$$\alpha^2 + \beta^2 = \Delta_1^2 \mu$$

with Eqq. (8) and (28), we see that, if we attribute to  $\Delta_1 \mu$  the direction of the trajectories orthogonal to the lines having a parameter  $\mu$ , then  $\alpha$  and  $\beta$  are also the projections onto the lines  $\lambda_r$  and  $\bar{\lambda}_r$  of the first-order differential parameter  $\Delta_1 \mu$ , which will be called *first order parameter* of the function  $\mu$ .

<sup>[1353][18]</sup> If instead we suppose that  $\mu_r$  is the covariant coordinate system of a bundle of systems, we see without difficulty that  $\alpha$  and  $\beta$  are the projections of the curvature of the bundle on the lines  $\lambda_r$  and  $\bar{\lambda}_r$ . The two theorems demonstrated above and the formula (29) allow us to establish two Corollaries:

(1) For  $\alpha$  and  $\beta$  to be the projections of the first-order parameter of a function of  $x_1$  and  $x_2$  on two orthogonal systems  $\lambda_r$  and  $\bar{\lambda}_r$  of geodetic curvatures  $\gamma$  and  $(\gamma)$ , it is necessary and sufficient that the equation

$$\sum_r \beta^{(r)} \lambda_r - \sum_r \alpha^{(r)} \bar{\lambda}_r + \alpha \gamma + \beta(\gamma) = 0$$

is satisfied.

(2) For  $\alpha$  and  $\beta$  to be the projections of the curvature of a bundle of systems on two orthogonal systems  $\lambda_r$  and  $\bar{\lambda}_r$  of geodetic curvatures  $\gamma$  and  $(\gamma)$ , it is necessary and sufficient that the equation

$$\sum_r \beta^{(r)} \lambda_r - \sum_r \alpha^{(r)} \bar{\lambda}_r + \alpha \gamma + \beta(\gamma) + G = 0$$

is satisfied.

In these equations the first two terms represent, as has been said, the projections of the first-order parameters of the functions  $\beta$  and  $\alpha$  on the lines  $\lambda_r$  and  $\bar{\lambda}_r$ , that is, the ratios of the increases of these functions due to infinitesimal displacements of the point  $(x_1, x_2)$  along the lines  $\lambda_r$  and  $\bar{\lambda}_r$ . If we now suppose that the lines  $\lambda_r$  belong to the bundle considered in Corollary (2), from this Corollary one obtains the formula

$$\sum_r (\gamma)^{(r)} \lambda_r - \sum_r \gamma^{(r)} \bar{\lambda}_r + \rho^2 + G = 0,$$

where  $\rho$  designate the curvature.

Indicating by  $ds$  and  $\delta s$  the linear elements of the lines  $\lambda_r$  and  $\bar{\lambda}_r$ , and with  $d\psi$  and  $\delta\psi$  the corresponding variations of any function  $\psi$ , the previous equation can be put in the form

$$\frac{d(\gamma)}{ds} - \frac{\delta\gamma}{\delta s} + \rho^2 + G = 0.$$

<sup>[1354][19]</sup> And this echoes a well-known theorem due to Liouville, from which we can define the following Corollary:

If in any bundle of lines drawn on the surfaces  $\varphi$  we consider two orthogonal systems  $\lambda_r$  and  $\bar{\lambda}_r$  of linear elements  $ds$  and  $\delta s$  and of geodetic curvatures  $\gamma$  and  $(\gamma)$ , the difference  $\frac{\delta\gamma}{\delta s} - \frac{d(\gamma)}{ds}$  is equal to the absolute curvature of the surfaces  $\varphi$  increased by the square of the curvature of the bundle, and is hence the same for all the double orthogonal systems belonging to the bundle.

9. — Now let  $\mu$  be any function of the variables  $x_1$  and  $x_2$ , and  $\lambda_r$  the coordinate system of the lines with a  $\mu$  parameter. We will have (§ 3)

$$\Delta_1 \mu \cdot \bar{\lambda}_r = \mu_r.$$

Indicating by  $\varphi_r$  the covariant coordinate system of the bundle, to which the system  $\lambda_r$  belongs, via Eq. (16), one can write

$$\Delta_1 \mu \lambda_r \varphi_s = \bar{\lambda}_r (\Delta_1 \mu)_s - \mu_{rs},$$

so

$$\Delta_1 \mu \cdot \varphi_s = - \sum_r \lambda^{(r)} \mu_{rs}.$$

Finally, from (19), one gets

$$\Delta_1 \mu \cdot \gamma = - \sum_{rs} \lambda^{(r)} \lambda^{(s)} \mu_{rs}, \quad \Delta_1 \mu \cdot (\gamma) = - \sum_{rs} \bar{\lambda}^{(r)} \lambda^{(s)} \mu_{rs}. \quad (30)$$

Let

$$2\Delta_1 \mu \cdot \sigma = \sum_{rs} a^{(rs)} \mu_{rs}, \quad (31)$$

from which this identity arises <sup>[1355][20]</sup>

$$\frac{1}{\Delta_1 \mu} \mu_{rs} = \sigma a_{rs} - (\sigma + \gamma)(\lambda_r \lambda_s - \bar{\lambda}_r \bar{\lambda}_s) - (\gamma)(\lambda_r \bar{\lambda}_s + \bar{\lambda}_r \lambda_s), \quad (32)$$

i.e., through Eq. (8),

$$\frac{1}{\Delta_1 \mu} \mu_{rs} = (2\sigma + \gamma) \bar{\lambda}_r \bar{\lambda}_s - \gamma \lambda_r \lambda_s - (\gamma)(\lambda_r \bar{\lambda}_s + \bar{\lambda}_r \lambda_s). \quad (33)$$

These formulas express the second derivatives of any function  $\mu$  of  $x_1$  and  $x_2$  by means of its first derivatives and the three second-order differential invariants  $\gamma$ ,  $(\gamma)$  and  $\sigma$ . We have already established the meanings of  $\gamma$  and  $(\gamma)$ . In order to recognize the meaning of  $\sigma$  it is enough to remember that the expression  $\sum_{rs} a^{(rs)} \mu_{rs}$  is nothing but the second-order differential parameter of the function  $\mu$ .

I will call the three invariants  $\gamma$ ,  $(\gamma)$  and  $\sigma$  *fundamental* second-order invariants of the function  $\mu$  associated with the fundamental form  $\varphi$ , by means of which one can evidently express all the other invariants of the same order, which are obtained by associating  $\mu$  with  $\varphi$ . There is, however, an essential difference between  $\gamma$  and  $(\gamma)$ , on the one hand, and  $\sigma$ , on the other:  $\gamma$  and  $(\gamma)$ , as is patent from their geometric meanings and their expressions (19), depend only on the nature of the lines  $\lambda_r$ , and this means that—thanks also to the suggestion of Eq. (30)—they do not change if instead of  $\mu$  we consider any function  $\psi(\mu)$ ;  $\sigma$  by contrast changes. In fact, if we put

$$2\sigma_1 = \frac{\Delta_2 \psi}{\Delta_1 \psi},$$

and by  $\psi'$  and  $\psi''$  we designate the first and second derivatives of  $\psi$  with respect to  $\mu$ , one has

$$\sigma_1 = \sigma + \frac{\psi''}{2\psi'} \Delta_1 \mu.$$

For this reason  $\gamma$  and  $(\gamma)$  can be called <sup>[1356][21]</sup> *first and second geometric invariant*, and  $\sigma$  *functional invariant* of second order of  $\mu$ .

From Eq. (14) and (23) one gets

$$\lambda_{rs} = \bar{\lambda}_r (\gamma \lambda_s + (\gamma) \bar{\lambda}_s),$$

and from the latter, recalling the theorem contained in Eq. (4), it is possible to set a differential relation between the two geometric invariants  $\gamma$  and  $(\gamma)$ . We arrive at a differential relation connecting the three fundamental second-order differential invariants if we start from the formula

$$\frac{1}{\Delta_1 \mu} (\Delta_1 \mu)_s = -(\gamma) \lambda_s + (2\sigma + \gamma) \bar{\lambda}_s, \quad (34)$$

and we multiply Eq. (32) by

$$\mu^{(r)} = \Delta \mu \bar{\lambda}^{(r)},$$

adding with respect to the index  $r$ . In fact, using these equalities and the theorem (1) in the previous paragraph [p. 9] we write

$$\sum_r \left( 2\sigma^{(r)} + \gamma^{(r)} \right) \lambda_r + \sum_r (\gamma)^{(r)} \bar{\lambda}_r + 2(\gamma) \sigma = 0. \quad (35)$$

If we maintain the notations presented in § 8, Eq. (35) can also be presented in the form

$$\frac{d(2\sigma + \gamma)}{ds} + \frac{\delta(\gamma)}{\delta s} + 2(\gamma) \sigma = 0, \quad (36)$$

which lends itself to an easy geometric interpretation.

Which can also be achieved in another way, by applying the theoretical content in Eq. (4) to Eq. (32).

10. — From Eq. (23) it follows

$$\varphi_{rs} = \gamma_s \lambda_r + (\gamma)_s \bar{\lambda}_r + \bar{\varphi}_r \varphi_s,$$

<sup>[1357][22]</sup> and, via Eq. (22),

$$2\theta = \sum_r \gamma^{(r)} \lambda_r + \sum_r (\gamma)^{(r)} \bar{\lambda}_r, \quad (37)$$

then, via Eq. (36),

$$-\theta = \sum_r \sigma^{(r)} \lambda_r + (\gamma) = \sigma. \quad (38)$$

By designating by  $ds$  and  $\delta s$  the linear elements of the lines  $\lambda_r$  and  $\bar{\lambda}_r$ , Eqq. (37) (38) can also be put under the form

$$2\theta = \frac{d\gamma}{ds} + \frac{\delta(\gamma)}{\delta s}, \quad (39)$$

$$-\theta = \frac{d\sigma}{ds} + (\gamma)\sigma. \quad (40)$$

Since, as already observed,  $\theta$  is an invariant relative to the bundle  $\varphi_r$ , of which the system  $\lambda_r$  is a part, the last expressions written for  $\theta$  lend themselves to easy geometric interpretations for this invariant. It will suffice for me to observe that from Eq. (39) and from a well-known theorem it results immediately that:

(1) *The cancellation of the invariant  $\theta$  is the necessary and sufficient condition for the lines  $\lambda_r$  to be isothermal.*

(2) *If a system of lines is isothermal, all the systems of the bundle are isothermal.*

Theorem (2) was already known,<sup>a</sup> while for the first it will be appropriate to give a demonstration based directly on the property that serves to define isothermal systems.

According to this definition, for a system of lines  $\lambda_r$  to be isothermal it is necessary and sufficient that there exists a function  $\nu$  of  $x_1$  and  $x_2$  such that, setting

$$\psi_r = e^\nu \lambda_r, \quad (41)$$

<sup>[1358][23]</sup> all  $\psi_r$  are derivatives of a function  $\psi$ , so one has identically

$$\Delta_2 \psi = 0.$$

Via Eq. (41), since

$$\bar{\psi}_r = e^\nu \bar{\lambda}_r,$$

always maintaining the notations of the previous paragraphs, one gets at the same time

$$\psi_{rs} = e^\nu (\nu_s \lambda_r + \bar{\lambda}_r \varphi_s),$$

and

$$\bar{\psi}_{rs} = e^\nu (\nu_s \bar{\lambda}_r - \lambda_r \varphi_s),$$

on the basis of which, according to theorem (1) in § 8, the analytical translation of the conditions stated above gives us

$$\sum_r \nu^{(r)} \lambda_r = -(\gamma), \quad \sum_r \nu^{(r)} \bar{\lambda}_r = \gamma. \quad (42)$$

For a system  $\lambda_r$  to be isothermal it is necessary and sufficient that there exists a function  $\nu$  whose first-order parameter has projections onto the lines and their orthogonal trajectories  $-(\gamma)$  and  $\gamma$ , respectively; in other words, thanks again to theorem (1) in § 8 and Eq. (37), it is necessary and sufficient that  $\theta = 0$ .

Naturally, the bundles of isothermal systems will be called *isothermal* bundles, whilst the bundle  $\varphi_r$  related to an invariant  $\theta$  will be called *anisothermal* bundle. While Eqq. (39) and (40) of this invariant give us, as we have already noted, two geometric interpretations of the invariant itself, Eq. (22) gives us a very important analytical interpretation. In fact, starting from this expression and still recalling our theorem (1) in § 8, we can conclude that:

*The necessary and sufficient condition for a bundle of systems (of which  $\varphi_r$  is the coordinate system) to be isothermal consists in the fact that the canonical system  $\bar{\varphi}_r$  orthogonal to <sup>[1359][24]</sup> it results from the derivatives of a function with respect to the independent variables  $x_r$ .*

<sup>a</sup> See § 17 of the above-mentioned *Memoria* by Prof. Beltrami.

From Eq. (12) and (23) one has

$$\sum_r \bar{\varphi}^{(r)} \bar{\lambda}_r = \gamma, \quad \sum_r \bar{\varphi}^{(r)} \lambda_r = -(\gamma),$$

and, from the comparison of these equalities with Eq. (42), one recognizes that, when the condition exposed above is satisfied, the system  $\bar{\varphi}_r$  results from the derivatives of the function  $\nu$ , which appears in Eq. (41). Afterwards, one just needs to set the function  $\nu$  whose derivatives are the elements of the system  $\bar{\varphi}_r$ , and the possibility of integrating our expressions will emerge; this integration will give us the isometric parameter  $\psi$  of the system  $\bar{\lambda}_r$ . And this parameter contains, as it is known, two arbitrary constants linearly. In a similar way the isometric parameter  $\chi$  of the lines  $\lambda_r$  will be obtained by integrating

$$\chi_r = e^\nu \bar{\lambda}_r.$$

Since the system  $\varphi_r$ , and consequently the system  $\bar{\varphi}_r$ , can be deduced from the system  $\lambda_r$  with simple derivations, the determination of the functions  $\psi$  and  $\chi$  requires only quadratures. We can conclude that:

*Given the coordinate system of a system of isothermal lines, its isometric parameters and those of its orthogonal trajectories are obtained by simple quadratures.*

If  $\varphi_r$  and  $f_r$  are the coordinate systems of two isothermal bundles, and we designate by  $\alpha$  the angle that the lines of any system of the first bundle create with those of any system of the second bundle, it is calculated (§ 5) that

$$\alpha_r = f_r - \varphi_r,$$

and

$$\alpha_{rs} = f_{rs} - \varphi_{rs}.$$

If by hypothesis

$$\sum_{rs} a^{(rs)} \varphi_{rs} = \sum_{rs} a^{(rs)} f_{rs} = 0,$$

[1360|25] then

$$\Delta_2 \alpha = 0.$$

According to a well-known theorem, the angle generated between the lines of two isothermal systems not belonging to the same bundle is the isometric parameter of a new isothermal system. Attention: this theorem is nothing but a corollary of the following proposition:

*Given the coordinate system  $\varphi_r$  of an isothermal bundle, the coordinate system  $f_r$  of any other isothermal bundle is obtained by setting  $f_r = \varphi_r + \alpha_r$ , if by  $\alpha$  we designate an arbitrary function, which satisfies the equation  $\Delta_2 \alpha = 0$ , that is, the isometric parameter of an arbitrary isothermal system.*

In fact, we will have  $\bar{f}_r = \bar{\varphi}_r + \bar{\alpha}_r$ , and, remembering theorems (1) and (2) in § 8,

$$\sum_{rs} a^{(rs)} \bar{\varphi}_{rs} = G, \quad \sum_{rs} a^{(rs)} \bar{\alpha}_{rs} = 0.$$

And here we conclude that

$$\sum_{rs} a^{(rs)} \bar{f}_{rs} = G.$$

Ergo the system  $f_r$  is the coordinate system of a bundle, and since, in compliance with our hypotheses,

$$\sum_{rs} a^{(rs)} \varphi_{rs} = \sum_{rs} a^{(rs)} \alpha_{rs} = 0,$$

then  $\sum_{rs} a^{(rs)} f_{rs} = 0$ , namely, the bundle will be isothermal.

11. — Given two irreducible binary quadratic differential forms,

$$\varphi = \sum_{rs} a_{rs} dx_r dx_s, \quad \psi = \sum_{pq} (a_{pq}) dy_p dy_q,$$

let us try to recognize whether one is transformable into the other, or whether it is possible to determine  $x_1$  and  $x_2$  as a function of  $y_1$  and  $y_2$  in such a way as to satisfy <sup>[1361][26]</sup>

$$(a_{pq}) = \sum_{rs} a_{rs} x_r^{(p)} x_s^{(q) \text{ a}} \quad (43)$$

$[\dots]$ .

As it results from the general theory of quadratic differential forms, a first derivation with respect to  $y_1$  and  $y_2$  leads from Eq. (43) to a new system, which can be put under the form

$$x_r^{(pq)} = \sum_{st} \left( a^{(st)} \right) (a_{pq,t}) x_r^{(s)} - \sum_{stu} a^{(ur)} a_{st,u} x_s^{(p)} x_t^{(q)}. \quad (44)$$

If we find out the expressions of the third derivatives of  $x$  for  $x$  and for their first derivatives, and if we eliminate the third derivatives, we arrive at the Gauss equation:

$$[G] = G, \quad (45)$$

where  $[G]$  is the Gauss invariant relating to  $\psi$ .

It follows that Eq. (43) cannot be satisfied if the two quantities  $G$  and  $[G]$  are not constant and do not have the same value. If (45) is satisfied identically, i.e., if  $G$  and  $[G]$  have the same constant value, the system resulting from (43) and (44) is complete and has the form of the previous expressions, which Lie carefully investigated in his *Theorie der Transformationsgruppen*, first part, tenth chapter. This system is consequently unconditionally integrable, and its general integral system admits, in addition to two additive constants, three arbitrary constants. We conclude that:

*Once the condition (45) is verified identically, there is a number of substitutions, equal to  $\infty^3$ , which transform the form  $\varphi$  into <sup>[1362][27]</sup>  $\psi$ , and in order to determine both of them it is convenient to integrate the complete system resulting from Eqq. (43) and (44). In particular, each form  $\varphi$ , with constant Gauss invariant, admits  $\infty^3$ -transformations in itself, which will be obtained by integrating the system of Eqq. (43) and (44) after having placed  $(a_{pq}) = a_{pq}$  in it.*

If  $G$  and  $[G]$  are both variables of (43) and (44), it is convenient to add (45) and

$$[G_p] = \sum_r G_r x_r^{(p)}, \quad (46)$$

where  $[G_p]$  represents the derivative of  $[G]$  with respect to  $y_p$ . Eliminating  $x_r^{(p)}$  one arrives at

$$\Delta_1[G] = \Delta_1 G, \quad (47)$$

where  $\Delta_1$  is the first-order differential parameter of  $[G]$  with respect to  $\psi$ .

Let us first assume that (47) is a consequence of (45), namely, that we have

$$\Delta_1[G] = F([G]), \quad \Delta_1 G = F(G),$$

where  $F$  is the symbol of any function. Indicating by  $k_r$  the coordinate system of lines  $G = \text{constant}$ , we will have

$$\frac{1}{\Delta_1 G} (\Delta_1 G)_r = F'(G) \bar{k}_r.$$

By Eq. (32), if we indicate by  $g$ ,  $(g)$  and  $h$  the fundamental geometric second-order differential invariants and the functional of  $G$ , one gets

$$\frac{1}{\Delta_1 G} (\Delta_1 G)_r = -(g) k_r + (2h + g) \bar{k}_r,$$

<sup>[1363][28]</sup> so  $(g) = 0$ , and

$$2h + g = F'(G).$$

Similarly, if  $[g]$  and  $[h]$  are the first geometric invariant and the second-order functional invariant of  $[G]$  relative to  $\psi$ , one obtains

$$2[h] + [g] = F'([G]),$$

---

<sup>a</sup> I point out that by  $x_r^{(p)}$ ,  $x_r^{(pq)}$ , etc. I designate the derivatives of the successive orders of  $x_r$  with respect to  $y_p, y_q$ , etc.

and since  $F'(G) = F'([G])$  is a consequence of Eq. (45) we will also have

$$2[h] + [g] = 2h + g. \quad (48)$$

If we now differentiate Eq. (46) with respect to  $y_1$  and  $y_2$ , and for the second derivatives we replace values given in (44), letting

$$[G_{pq}] \equiv D_\psi[G_p],$$

one arrives at

$$[G_{pq}] = \sum_{rs} G_{rs} x_r^{(p)} x_s^{(q)}. \quad (49)$$

Thanks to Eq. (32) and by virtue of the fact that  $(g) = 0$ , one has

$$\frac{1}{\Delta_1 G} G_{rs} h a_{rs} - (h + g)(k_r k_s - \bar{k}_r \bar{k}_s),$$

hence Eq. (49) is reduced to

$$[h] = h, \quad [g] = g, \quad (50)$$

which, by (48), are consequently interconnected. If for example the first equality is a consequence of (45), i.e., if  $F_1$  represents any function, we simultaneously get

$$h = F_1(G), \quad [h] = F_1([G]).$$

So we arrive at a complete system of differential equations if Eqq. (43) (44) (45) are completed with (46), and its integral system contains an arbitrary constant. Here is the result: <sup>[1364][29]</sup>

*If  $G$  and  $[G]$  are variables and if, by associating them respectively with the forms  $\varphi$  and  $\psi$ , then denoting by  $\Delta_1 G$  and  $\Delta_1 [G]$  their first-order differential parameters, with  $h$  and  $[h]$  their second-order functional invariants, and if  $F$  and  $F_1$  symbolize two arbitrary functions, we simultaneously have*

$$\Delta_1 G = F(G), \quad \Delta_1 [G] = F([G]), \quad h = F_1(G), \quad [h] = F_1([G]).$$

*And now there is a simply infinite number of substitutions that transform  $\varphi$  into  $\psi$ . In order to determine these forms we need to integrate the complete system, which results from Eqq. (43) (44) (45) (46). In particular, the necessary and sufficient conditions for a form  $\varphi$  to be transformable into itself are given by,*

$$\Delta_1 G = F(G), \quad h = F_1(G), \quad (51)$$

*which need to be verified, so as to allow the transformation in question with a simply infinite number of substitutions.*

I will not dwell on the cases in which one of the Eqq. (47) or (50) is incompatible with (45), so that they are not consequently interconnected, since this falls back on well-known and obvious considerations.

I will observe however that, from what has been said above, there do not exist binary quadratic differential forms that admit a number  $\infty^2$  of transformations in themselves; furthermore, for all forms  $\varphi$  that admit a simply infinite number of transformations, it is true not only that  $(g) = 0$ , namely, that on the surfaces  $\varphi$  the lines of equal curvature are parallel, but also that they are isothermal: in fact  $\sum_r h^{(r)} \lambda_r = 0$  established by (51), and thanks to Eq. (38) in § 10 the anisothermal state of the bundle to which the system of those lines belongs is zero.