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# A Critical Review of Asymptotic Numerical Methods

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## Summary

Various sorts of asymptotic-numerical methods have been proposed in the literature: the reduced basis technique, direct computation of series or the use of Padé approximants. The efficiency of the method may also depend on the chosen path parameter, on the order of truncature and on alternative parameters. In this paper, we compare the three classes of asymptotic- numerical method, with a view to define the “best” numerical strategy.

## 1. INTRODUCTION

We are interested by the numerical computation of a solution path  $\mathbf{U}(\lambda)$  of a non-linear continuous problem, where  $\mathbf{U}$  is the unknown and  $\lambda$  is a scalar parameter. Within asymptotic-numerical methods, the basic idea is to search a parametric representation of the solution path  $\mathbf{U}(a)$ ,  $\lambda(a)$  in the form of integro-power series (Damil and Potier-Ferry, 1990). For instance, the unknown is represented as follows

$$\mathbf{U}(a) - \mathbf{U}_0 = \sum_{n=1}^p a^n \mathbf{U}_n \quad (1)$$

The vector fields  $\mathbf{U}_n$  are the solutions of a recurrent sequence of linear problems, with a single tangent operator to be inverted. Next, these linear problems are solved by a classical discretisation technique, generally by the finite element method, but finite differences have been also used recently for application in fluid dynamics in the presence of a free surface (Mordane, 1995).

Of course, the range of validity of the representation is limited by the radius of convergence of the series. It is worth to choose a sufficiently large order of truncature  $p$ , because this yields an excellent accuracy inside the radius of convergence. That is why Cochelin (1994) has been able to propose an efficient continuation method within this framework and

without any iterative correction step. Because one has to invert only one matrix, the computation time is similar as the one required by one step of the modified Newton-Raphson method, and the range of validity is much greater (Cochelin *et al.*, 1994-b).

Such methods have been proposed a long time ago, for instance by (Masur and Schreyer, 1967; Thompson and Walker, 1968; Connor and Morin, 1971; Gallagher, 1975; Glaum *et al.*, 1975). The first attempts to compute many terms seem to be due to Kawahara *et al.* (1976) for flows of viscous fluids and to Noor (1981), Noor and Peters (1981, 1983) for equilibrium paths of non-linear elastic structures, but their conclusions were rather pessimistic. Thus the second authors have suggested to use the previously computed vectors as a basis in a Rayleigh-Ritz approximation. More precisely the solution path is searched in the following form

$$\mathbf{U} - \mathbf{U}_0 = \sum_{n=1}^p r_n \mathbf{U}_n \quad (2)$$

and the reduced unknowns  $r_n(\lambda)$  are obtained by solving the  $p$  equations deduced from the Rayleigh-Ritz procedure. The range of validity is necessarily larger than with the series, because the polynomial approximation (1) is a special case of (2) and because nothing is a priori prescribed about the functions  $r_n(\lambda)$ . A detailed evaluation of this second type of perturbation method has been done by Riks (1984). He concluded that they are less efficient than the more classical incremental-iterative method, especially since the time to get the reduced system increases fastly with  $p$ .

These pessimistic conclusions have been partially removed by our recent studies. First, the radius of convergence depends strongly on the path parameter  $a$  and the natural choice  $a = \lambda$ , that has often been made in the first studies, is not the best one. Second it is possible to replace the polynomial approximation (1) by a rational one

$$\mathbf{U}(a) - \mathbf{U}_0 = \sum_{n=1}^p f_n(a) \mathbf{U}_n \quad (3)$$

where  $f_n(a)$  are rational fractions that can be deduced from formula (1) in a cheap and simple way (Cochelin *et al.*, 1994-a). Those rational fractions are called Padé approximants (Padé, 1892; Baker and Morris, 1980). Third, many recent tests have established the efficiency of the simplest technique (1) for post-buckling of elastic structures (Azrar *et al.*, 1993) in fluid dynamics (Tri *et al.*, 1996) or to detect bifurcation points (Boutyourn *et al.*, 1993) and bifurcating branches (Vannucci *et al.*, 1997).

The aim of this paper is to compare those three classes of asymptotic- numerical methods: the representation by power series, the one by Padé approximants and the Rayleigh-Ritz procedure. At present, the only robust and efficient continuation method is based on power series, because computation time blows up for large  $p$  within Rayleigh-Ritz method and because it remains difficult to control the appearance of poles with Padé approximants. So a full comparison between step by step procedures is yet untimely. Here we limit ourselves to evaluate the extent of the domain of validity obtained by each method and to analyse the quality of the approximation inside this domain. We shall also discuss the influence of the order of truncature, of the choice of the path parameter and the efficiency of the projection technique previously used in Cochelin *et al.* (1994-a), Damil *et al.* (1994).

We shall consider classical benchmarks of elastic cylindrical shells, that involve several limit points and that are known as valuable to evaluate path following techniques.

Also in this paper, we present a new way to build up Padé approximants, with a view to limit the number of poles and therefore to increase the robustness of the algorithm. As for the Rayleigh-Ritz procedure, it is the classical one, but a new algorithm to get the reduced system has been tested, in order to try to decrease the corresponding computation time.

A last point will be discussed, i.e. the accuracy of the computed vector fields at high orders. Indeed, the supporters of the pure asymptotic techniques recommend large orders of truncature, while those of the Rayleigh-Ritz procedure prefer moderate orders (Noor,1981), what is a bit contradictory. That is why we shall compare series and Padé approximants computed by finite elements with the exact values obtained by a symbolic software.

## 2. THREE SORTS OF ASYMPTOTIC NUMERICAL METHODS

The three basic algorithms will be evaluated. Only the polynomial representation has been applied previously in the same way. The rational representation is completely new and the algorithm in the Rayleigh-Ritz procedure differs slightly from the standard one. The details of the implementation in finite element codes have been presented in Cochelin et al (1994-b) and will be not repeated here.

We had established that within geometrically non-linear elasticity, the governing equations can be written under the simple following form (Azrar *et al.*, 1993; Cochelin *et al.*, 1994-b) :

$$\mathbf{L}(\mathbf{U}) + \mathbf{Q}(\mathbf{U}, \mathbf{U}) = \lambda \mathbf{F} \quad (4)$$

where  $\mathbf{L}$  is a linear operator and  $\mathbf{Q}$  a bilinear one. Such a simple framework can be introduced in many cases, for instance in fluid dynamics (Mordane 1995, Tri *et al.* 1996) or in plasticity (Braikat *et al.*, 1997). Problems involving strong nonlinearities can also be put in a simple quadratic framework, for instance for problems involving contact and viscoplasticity (Potier-Ferry *et al.*, 1997). We assume that a solution point  $\mathbf{U}_0, \lambda_0$  of equation (4) is known and that there exists a solution path  $\mathbf{U}(a), \lambda(a)$  starting from this point. We are going to present and to compare three sorts of asymptotic numerical methods to compute such a branch.

### 2.1 The Classical Polynomial Representation

The first idea consists in seeking truncated power series to represent the branch  $\mathbf{U}(a), \lambda(a)$  in the form (1) and (5)

$$\lambda - \lambda_0 = \sum_{n=1}^p a^n \lambda_n \quad (5)$$

with respect to a parameter  $a$ , that will be defined later. By introducing (1) and (5) into (4), we obtain the following set of linear problems

$$\begin{aligned} \mathbf{L}_t(\mathbf{U}_1) &= \lambda_1 \mathbf{F} \\ \mathbf{L}_t(\mathbf{U}_2) &= \lambda_2 \mathbf{F} - \mathbf{Q}(\mathbf{U}_1, \mathbf{U}_1) \\ \mathbf{L}_t(\mathbf{U}_p) &= \lambda_p \mathbf{F} - \sum_{r=1}^{p-1} \mathbf{Q}(\mathbf{U}_r, \mathbf{U}_{p-r}) \end{aligned}$$

where the tangent operator  $\mathbf{L}_t$  is defined as  $\mathbf{L}_t(\cdot) = \mathbf{L}(\cdot) + 2\mathbf{Q}(\mathbf{U}_0, \cdot)$ . The first problem is the linearisation of (4) at the starting point  $\mathbf{U}_0, \lambda_0$ . Hence the first solution  $\mathbf{U}_1, \lambda_1$  corresponds to the tangent of the branch at the starting point. Usually, one defines the path parameter  $a$  as the load increment  $\lambda - \lambda_0$ , or as one component of the displacement increment  $\mathbf{U} - \mathbf{U}_0$ , or as a linearized arc-length parameter defined by the projection of the pair  $(\mathbf{U} - \mathbf{U}_0, \lambda - \lambda_0)$  on the tangent direction  $(\mathbf{U}_1, \lambda_1)$ . Within elasticity, the problem at order  $p$  is similar as a linearized elastic problem, where the right-hand sides can be seen both as prescribed forces (or equivalently residuals for equilibrium equations) and as prescribed strains or stresses (or equivalently residuals for constitutive equations). Solving numerically these linear problems (by finite element method for example), we get the vectors  $\mathbf{U}_i$  and the coefficient  $\lambda_i$ . Since

all these linear problems involve the same linear operator  $\mathbf{L}_t$ , only one matrix triangulation per step is needed.

## 2.2 Padé Approximants for Series of Vectors

To extend the domain of convergence of the polynomial representation (1), we have used in (Cochelin *et al.*, 1994-a) a rational representation called Padé approximant. As the representation (1) is not a scalar series, we have first built up an orthonormal basis  $\mathbf{U}_i^*$  from the vectors  $\mathbf{U}_i$  by a classical Gram-Schmidt orthogonalisation procedure. After this orthogonalisation, the representation (1) can be rewritten in the form

$$\mathbf{U}(a) - \mathbf{U}_0 = \sum p_n(a) \mathbf{U}_n^*$$

where  $p_n(a)$  are polynomials. Second, this new representation is truncated at order about  $p/2$  and each polynomial  $p_n(a)$  is replaced by a suitable rational fraction. Despite of some spectacular improvements, these rational approximations present sometimes the disadvantage to have a number of poles close to the radius of convergence. To avoid this drawback, we propose an alternative way of replacing the polynomials  $p_n(a)$  by rational fractions with a single denominator (see Appendix 1). Finally, we arrive at a new representation by rational fraction of the solution path  $\mathbf{U}(\lambda)$

$$\mathbf{U}(a) - \mathbf{U}_0 = a \mathbf{U}_1 \frac{\Delta_{p-2}}{\Delta_{p-1}} + a^2 \mathbf{U}_2 \frac{\Delta_{p-3}}{\Delta_{p-1}} + \dots + a^{p-1} \mathbf{U}_{p-1} \frac{1}{\Delta_{p-1}} \quad (6)$$

$$\lambda(a) - \lambda_0 = a \lambda_1 \frac{\Delta_{p-2}}{\Delta_{p-1}} + a^2 \lambda_2 \frac{\Delta_{p-3}}{\Delta_{p-1}} + \dots + a^{p-1} \lambda_{p-1} \frac{1}{\Delta_{p-1}} \quad (7)$$

The set of polynomials  $\Delta_i$  depends only on  $p-1$  coefficients  $\{d_1, d_2, \dots, d_{p-1}\}$ , that can be computed from the coefficients  $\alpha_{ij}$  of the Gram-Schmidt procedure (see Appendix 1)

$$\begin{aligned} \Delta_i &= 1 + ad_1 + a^2 d_2 + \dots + a^i d_i \\ d_1 &= -\frac{\alpha_{p(p-1)}}{\alpha_{(p-1)(p-1)}} \quad d_i = -\frac{\alpha_{p(p-i)}}{\alpha_{(p-i)(p-i)}} - \sum_{j=1}^{i-1} \frac{\alpha_{(p-j)(p-i)}}{\alpha_{(p-i)(p-i)}} d_j \quad \text{for } i \leq (p-1) \end{aligned} \quad (8)$$

Let us remark that the computation time to get (6), (7) from (1), (5) is negligible. Because the denominator  $\Delta_{p-1}$  is the same for the  $p-1$  fractions we hope that, by this way, we shall get much less poles and defects than with the previous version.

## 2.3 Rayleigh-Ritz (R-R) Method

An attractive alternative way had been proposed and tested by Noor (1981), Noor and Peters (1981, 1983), which consists in using the vector fields  $\mathbf{U}_n$  as a basis in a R-R approximation. If such an approximation (2) would be applied to Navier-Stokes equations, we would get a reduced problem in a quadratic form

$$\mathbf{l}_{ij} \mathbf{r}_j + \mathbf{q}_{ijk} \mathbf{r}_j \mathbf{r}_k - \lambda \mathbf{f}_i = 0 \quad i, j, k = 1, \dots, p$$

The previous authors have applied this procedure to the displacement variational equation of nonlinear elasticity, which leads to a cubic reduced system

$$\mathbf{l}'_{ij} \mathbf{r}_j + \mathbf{q}'_{ijk} \mathbf{r}_j \mathbf{r}_k + \mathbf{m}_{ijkl} \mathbf{r}_j \mathbf{r}_k \mathbf{r}_l - \lambda \mathbf{f}_i = 0 \quad i, j, k, l = 1, \dots, p$$

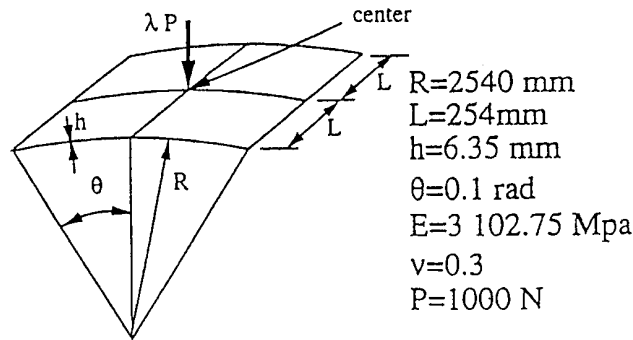
It is cheap to compute a solution path  $\mathbf{r}(\lambda)$  of the reduced system, but unfortunately the number of coefficients  $m_{ijkl}$  increases very rapidly with the order of truncature, as well as the computation time to get those coefficients. That is why such procedure is not efficient for  $p$  sufficiently large and requires a huge CPU time for  $p$  larger than 10.

In Appendix 2, the same algorithm has been rewritten in a slightly different way, in order to obtain a quadratic reduced system and it will be applied in what follows.

## 2.4 Comparison of the Three Methods

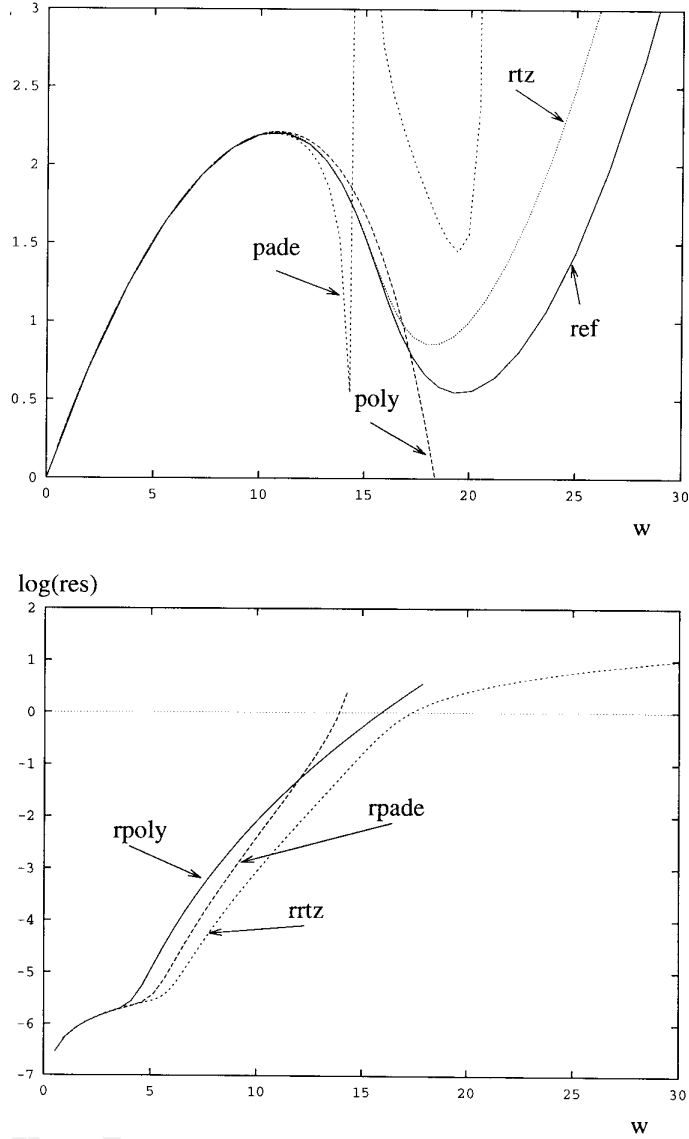
The main point of this paper is to compare polynomial and rational representations with R-R methods, what is presented now. The comparison is restricted to the analysis of one step and the main criterion is the step length. Let us recall that the R-R method is necessarily the best one in terms of step length at a given order  $p$ . It has been established, Cochelin *et al.* (1994-b), that, if the problem has a sufficiently large number of degrees of freedom, the computation time to get the polynomial representation is mainly governed by the matrix inversion. That is why, in a continuation method, it is more interesting to use large orders of truncature (say 15 to 25) than small orders (say 5 to 10). The same conclusion could be extended to the case of the rational representation because the required computation time to get (6) (7) from (1) (5) is negligible. As for the R-R method, it can be applied only for small orders  $p$ , because of the increasing computation time to get the reduced system.

The first test that we consider is the classical cylinder shallow shell loaded by a single force. The characteristics of the shell are given in Figure 1. We used a  $10 \times 10$  mesh for a quarter of the shell, and 200 triangular DKT elements. The second test is the same problem but with a smaller (the half) thickness. This second test presents various limit points. In this section, the expansion parameter  $a$  has been chosen as the deflection at the center.



**Figure 1.** Elastic cylindrical shallow shell loaded by a concentrated force  $\lambda P$ . For the first test  $h = 12.7 \text{ mm}$  and for the second test  $h = 6.35 \text{ mm}$

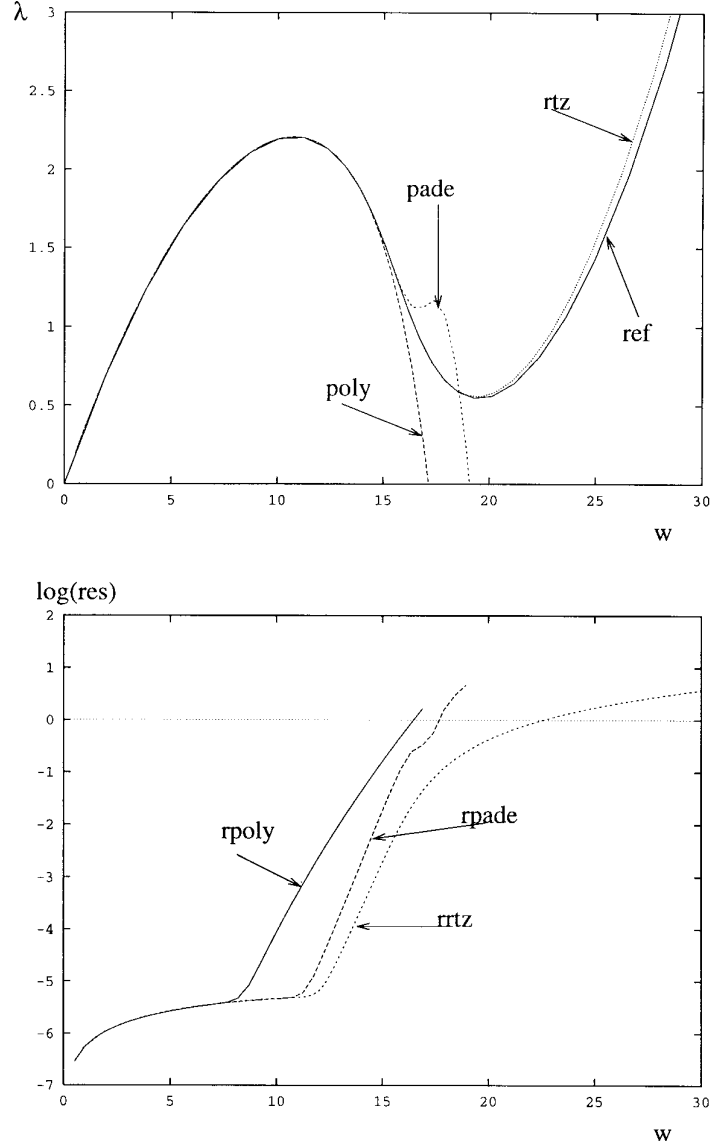
The results of the first test, for a small order  $p = 8$ , are presented on Figure 2. A curve load-deflection in the center is given in Figure 2a. The curve of reference has been obtained by a continuation method. It seems that the range of validity ( r.o.v. ) of the polynomial and of the fraction is about the same (up to  $w = 11$ ) while the r.o.v. with R-R seems much larger (up to  $w = 16$ ). Nevertheless the best criterion to define the end of the step length is to assign a maximal value for the residual vector. So we have plotted in Figure 2b the logarithm of this residue as a function of the deflection. For instance, with a maximal residue of  $10^{-3}$ , the r.o.v. is much smaller, about 7.5, 9 and 10.5, respectively. As expected, the largest r.o.v. is obtained by the R-R method, but the use of Padé approximants increases significantly this r.o.v.



**Figure 2.** Test 1, order 8. The polynomial and the rational representations, the approximation obtained by Rayleigh-Ritz method and the reference solution are designated respectively by poly, pade, rtz and ref. (a) Load as a function of the displacement at the center (response curve), (b) Logarithm of the norm of residual vector versus the displacement at the center (residue curve)

The same pictures are presented on Figure 3 and 4 for an order of truncature  $p$  of 16 and 24 respectively. Considering the response curves 2a, 3a and 4a, the r.o.v. grows much more with  $p$ , as we had established previously. It seems even that, for  $p = 24$ , the R-R method permits to get the whole response curve in one step, at least up to  $w = 30$ . However, if the criterion is a maximal residue of  $10^{-3}$ , the r.o.v. is only about 11, 13, 14.5 for  $p = 16$  and 13, 15, 17 for  $p = 24$ . In any case the largest r.o.v. is obtained with the R-R method while the smallest is obtained with the polynomials.

Perhaps the most important feature is the fact that one gets better results with the polynomials at order 16 or 24, than with the R-R method at order 8. Remember that in



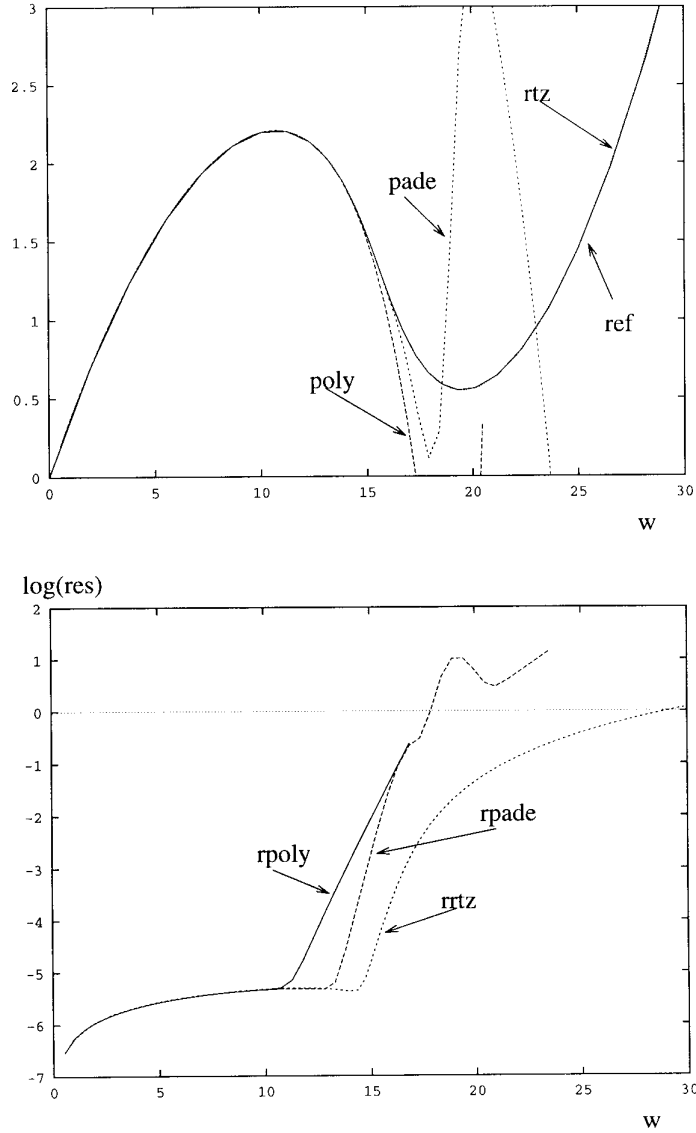
**Figure 3.** Test 1, order 16. (a) Response curve, (b) Residue curve

terms of CPU time, the computation of the polynomials and of the rational approximation at high order is not very expensive, while R-R method cannot reasonably be used beyond  $p = 8$ . So it appears quite clearly that the R-R method is worse than the two others ones, with regards to numerical efficiency.

If one combines the requirements of large step length and of small CPU time, the rational approximation is the most attractive way. Nevertheless, the rational representation has sometimes poles close to the radius of convergence, that are manifest in the three presented cases.

The second example (thinner shell) is presented on Figures 5 and 6, for orders of truncature  $p = 8, 24$ . The problem is more difficult, because the response curve is not so regular and it has two limit points with respect to displacement. So, the R-R method is never able to yield the response curve up to  $w = 30$ . The conclusions are similar as with the first example. The R-R method yields a larger step length than the rational approximation,

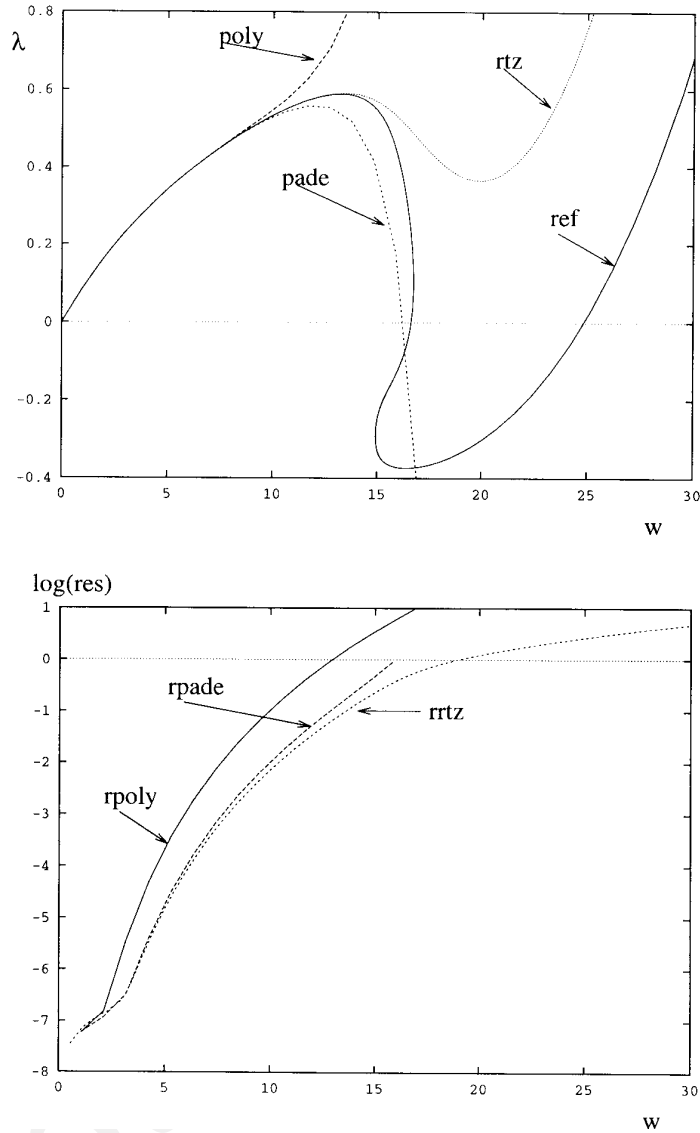




**Figure 4.** Test 1, order 24. (a) Response curve, (b) Residue curve

that itself is better than the polynomial approximation. With the criterion of a maximal residue of  $10^{-3}$ , the step length is about 6, 7.5 and 7.5 for  $p = 8$  and 9, 14.5 and 15 for  $p = 24$ . As compared to the first example, the most remarkable point is the fact that the Padé approximants give results that are very close to those of the R-R method. So, the used rational approximation yields, in this case, an explicit representation of the solution curve that is almost the best possible one by seeking the solution as a combination of the vectors  $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_p$ .

Clear conclusions can be deduced from those two tests. First, it is very interesting to truncate at a large order. Second, so long as a cheap algorithm to compute the coefficients of the reduced problem has not been found, the R-R method is not the most efficient, because it is restricted to small orders of truncature. The most attractive method is the rational approximation, that has the same computational cost as the polynomial and that



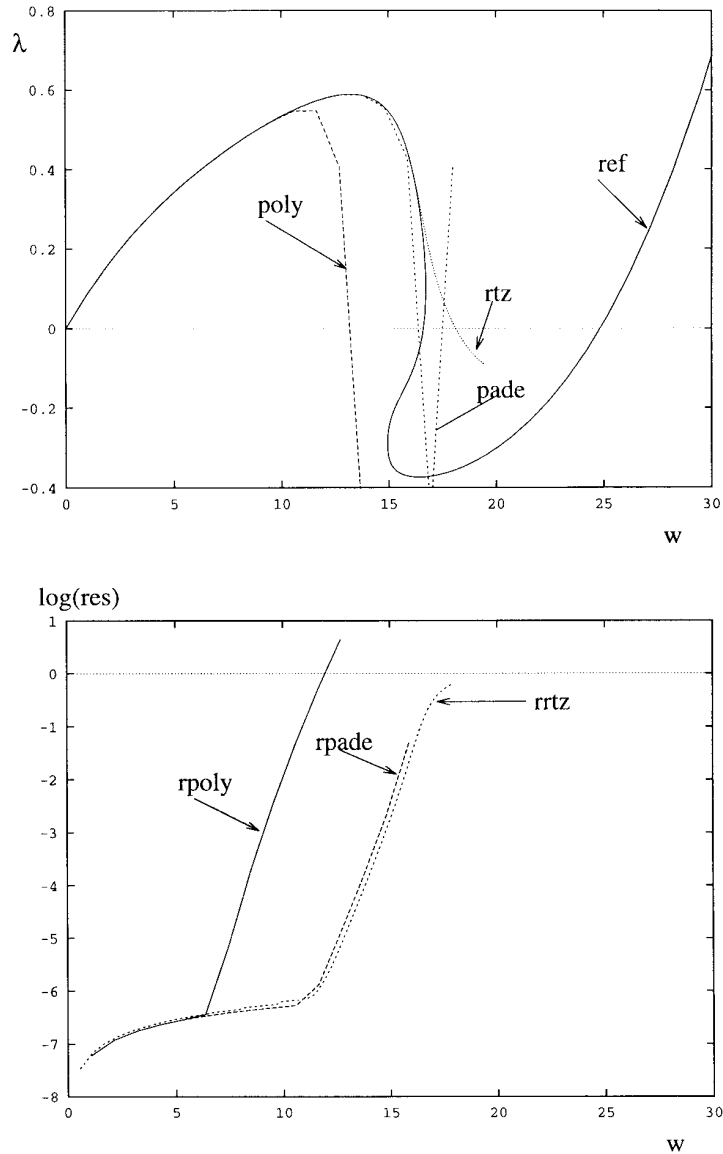
**Figure 5.** Test 2, order 8. (a) Response curve, (b) Residue curve

is compatible with large orders of truncature. Nevertheless, up to now, a reliable and automatic continuation method has not been tested with that rational approximation.

### 3. ADDITIONAL VARIANTS AND REMARKS

The order of truncature is probably the most important parameter to define the best strategy within asymptotic numerical method. We have just established that the best choice is a sufficiently large order, at least for smoothly nonlinear problems what is in agreement with previous results. But, Braikat *et al.* (1997) have obtained similar conclusions for elastoplastic structures, which makes ones to hope an extended application field for ANM and for the strategy of large orders.

A spectacular example of the efficiency of that strategy is presented on Figure 7a, in the previously presented case of the thick shell. The entire response curve seems to have

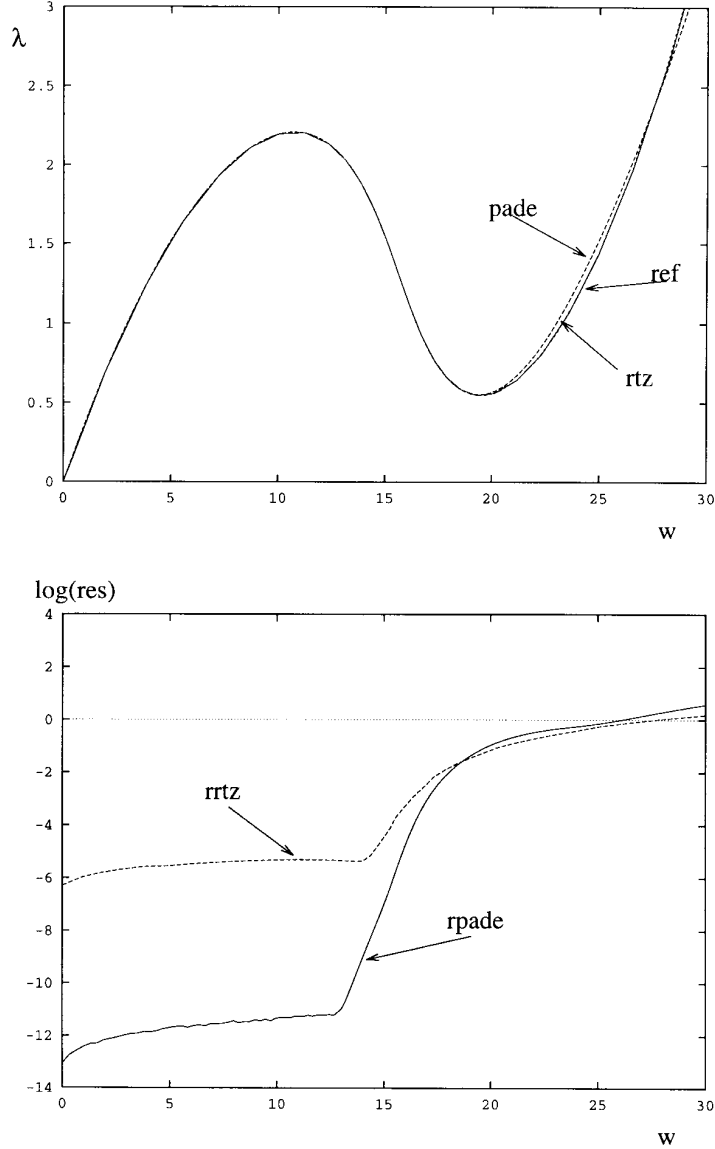


**Figure 6.** Test 2, order 24. (a) Response curve, (b) Residue curve

been obtained up to  $w = 30$  by the R-R method at order 24 in one step, i.e. only with the inversion of the initial stiffness matrix. The rational representation at order 40 gives about the same result, but with a much smaller computation time. As usual, the criterion of a maximal residue is more restrictive and it yields a r.o.v. up to the second limit point at  $w = 18$  (Figure 7b). This example illustrates once more our affirmation, that the old criticisms against perturbation methods are not justified because they do not account for the best asymptotic numerical algorithms.

To a less extent, the efficiency of ANM depends on the choice of the path parameter and, sometimes, spectacular improvements have been obtained by a simple and cheap projection technique (Cochelin *et al.*, 1994-a). Those two points will be rediscussed here.

Where does the interest of large order truncatures come from? The corresponding improvement can be due to the increase of the degree of the polynomials or of the fractions. It can also be a consequence of the addition of new vector fields, which increases the dimension

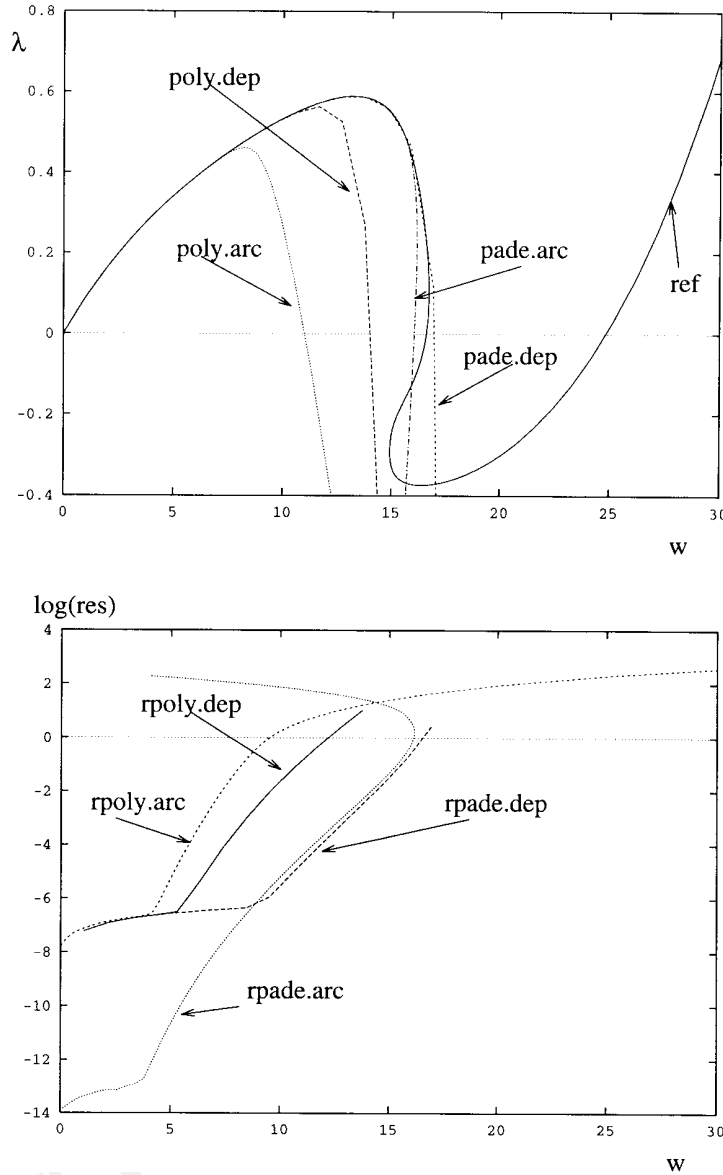


**Figure 7.** Test 1. The rational representation at order 40 and the Rayleigh- Ritz approximation at order 24. (a) Response curve, (b) Residue curve

of the subspace where the solution path lies. This question will be illustrated in Section 3.2, by a comparison of ANM with an analytic asymptotic expansion.

### 3.1 Alternatives Strategies

We had previously shown, that a good choice of the path parameter can increase the step length in a very important way. We rediscuss briefly this point, by considering the thin shell example. We do not present expansions with respect to the load parameter, what is clearly the worst choice in this case as in many others. We consider, first expansions with respect to the displacement at the loaded point ( $a = w$ ), what will be referred as “dep” in figures, second expansions with respect to a sort of arc-length parameter (Cochelin, 1994), what will be referred as “arc” in the figures.



**Figure 8.** Test 2, order 18. Two expansion parameters are compared within polynomial (poly) and rational (Padé) representations: the displacement at the center (dep) and an arc-length parameter (arc). (a) Response curve, (b) Residue curve

According to the response curve of Figure 8a, the displacement parameter is better than the arc-length parameter, but the choice of the best expansion parameter is less influent than the use of Padé approximants. These remarks are corroborated by the residue curve (Figure 8b) and, moreover, the discrepancy between the two parameters becomes very small with the rational representation.

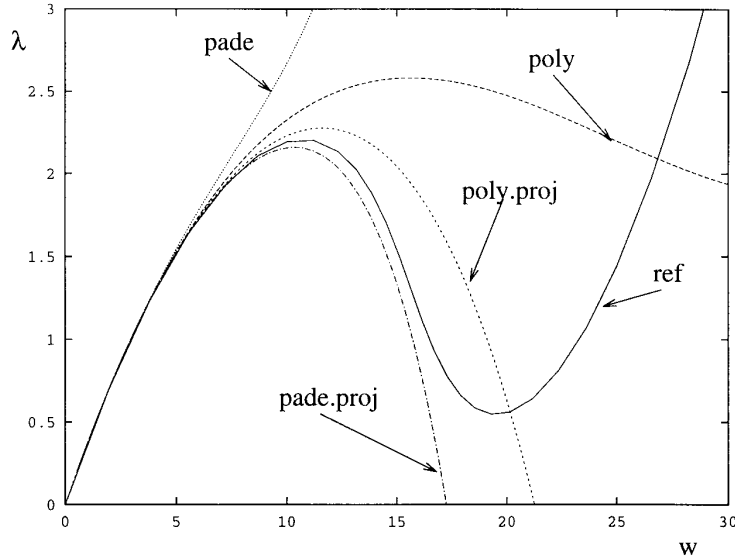
In fact, the step length is not the only criterion for the choice of the path parameter. The arc-length parameter permits to pass through limit points in load or in displacement, which makes the algorithm very robust.

Another possible improvement is the projection technique presented in (Cochelin *et al.*, 1994-a), (Damil *et al.*, 1994). The method consists, first in calculating a representation of the unknown  $\mathbf{U}(a)$  that can be polynomial (1) or rational (6), second to recalculate the

scalar function  $\lambda(a)$  by projecting the functional equation (4) on a given vector, for instance  $\mathbf{U}_1$ , which leads to

$$\lambda(a) - \lambda_0 = \frac{\langle \mathbf{Q}(\mathbf{U} - \mathbf{U}_0, \mathbf{U} - \mathbf{U}_0), \mathbf{U}_1 \rangle + \langle \mathbf{U} - \mathbf{U}_0, \mathbf{F} \rangle}{\langle \mathbf{F}, \mathbf{U}_1 \rangle} \quad (9)$$

In Figure 9, one sees the contribution of the projection technique in the case of a small ( $p = 3$ ) order of truncature and for the thick shell example. The use of the projection does not change very much the domain of coincidence with the reference curve, but the response curve is brought nearer from its exact value, even beyond the radius of convergence of the series, especially if the projection is coupled with the rational approximation. For large orders of truncature, this technique does not change considerably the range of validity, at least for the two examples considered here. So this projection technique does not seem very efficient, at least when it is coupled with the strategy of large orders. Nevertheless, the use of a projection may be interesting in a more classical prediction-correction technique, where the ANM and the projection are applied to define an improved prediction step (Braikat, 1995).

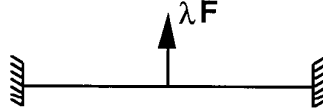


**Figure 9.** Test 1, order 3. Improvement of the response curve by a projection technique for the polynomial (poly.proj) and rational (pade.proj) representation

### 3.2 Accuracy of the Computation of Higher Order Vector Fields

The efficiency of the strategy of high orders of truncature seems to indicate that the ANM is able to compute accurately both the vectorial space generated by the vector fields  $\mathbf{U}_n$  and the functions  $r_n(\lambda)$  that appear in (3). However, it has been observed that, when a new term  $\mathbf{U}_n$  is computed, it is nearly parallel to the preceding ones so that the solution path lies more or less in a subspace of low dimension (Cochelin *et al.*, 1994-a). The vector  $\mathbf{U}_n$  can be splitted into its projection on the subspace generated by  $\{\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{n-1}\}$  and into a new vector  $\alpha_{nn}\mathbf{U}_n^*$ , by a Gram-Schmidt orthogonalisation. Since these new vectors become smaller and smaller, one guesses that they may be computed with a lower and lower accuracy.

To discuss this point, we consider an elastic beam subjected to a bending force concentrated in its middle (Figure 10). In this case the exact expansions into series (1) and the corresponding vector fields  $\mathbf{U}_n$  can be obtained analytically with the help of a symbolic software (we used MAPLE). In other words, we are going to compare an asymptotic method, (the vector fields being calculated analytically) with an asymptotic numerical method (the vector fields being computed numerically by finite element discretisation). In fact, we shall measure the sought accuracy by analysing the coefficients  $\alpha_{ij}$  of the Gram-Schmidt orthogonalization (see Appendix 1). For symmetry reason, only the half beam has been discretized by using 100 plate elements, which limits discretization errors.



**Figure 10.** Elastic beam subjected to a bending concentrated force

$\alpha_{ij}$	Finite element	MAPLE
1,1	0.43097476	0.43094580037
3,1	-0.31165012	-0.3116069657
3,3	0.27573765 E-02	0.2755432378 E-02
9,1	6.6375576	6.635000048
9,5	0.26880778 E-03	0.2682090308 E-03
9,7	-0.67503012 E-06	-0.5027537087 E-06
9,9	0.16343121E-06	0.0003398077304 E-06
11,5	-0.10256109 E-02	-0.1023244124 E-02
11,9	0.37382889 E-05	-0.0002272947509 E-05

**Table 1.** Some Gram-Schmidt coefficients  $\alpha_{ij}$  computed by finite elements and by a symbolic software (MAPLE)

The comparison between the exact coefficients and those obtained by finite elements is presented in Table 1. The accuracy of the coefficients  $\alpha_{11}$ ,  $\alpha_{13}$ ,  $\alpha_{33}$  is very good, which shows that the discretization is sufficiently fine. In fact, the numerical procedure gives a good value of  $\alpha_{ij}$  so long as one of the two indices is lower than 7. For instance,  $\alpha_{21,5}$  (not in the table) is equal to 1.213 and the ANM gives a value of 1.216. On the contrary  $\alpha_{21,9}$  (not in the table) is equal 0.53 E-5, but -0.504 E-2 according to the numerical method. This means that the ANM computes correctly the projection of the vector  $\mathbf{U}_{21}$  on the first vectors (up to 7), but it gives wrong results beyond. Thus numerical errors are accumulated to such extent that the new vectors  $\mathbf{U}_n^*$  are completely false beyond a rather small order ( $n = 7$  in the present example).

The rational approximation defined in (6) (7) (8) is strongly dependent of the high order coefficients  $\alpha_{ij}$ . As expected, all the coefficients  $d_i$  obtained by ANM are also completed false, see Table 2. Thus it would be consistent that the Padé approximants lead to wrong and random results beyond order 9 or 11. Curiously and fortunately, we do not observe such a catastrophic behavior. On the contrary, according to the tests, the new rational approximation presented in Section 2.b seems very efficient and very robust, especially with large orders. We have not yet been able to explain clearly this contradiction, but likely, a sort of coherence should be preserved in the computation of high order terms.

Coeff. of Padé: order 11	Finite elements	MAPLE
$d_2 = -\alpha_{12,10}/\alpha_{10,10}$	398.87712	7.490494939
$d_4 = -\alpha_{11,7}/\alpha_{7,7} - \alpha_{9,7}/\alpha_{7,7}d_2$	2329.8569	16.06360552
$d_6$	3264.3199	9.407144122
$d_8$	701.90779	0.6372487335
$d_{10}$	-1.05007241	0.0001875988059

**Table 2.** Coefficients  $d_i$  of Padé approximants computed by finite elements and by a symbolic software (MAPLE). The series have been truncated at order 12. With  $d_1 = d_3 = d_5 = d_7 = d_9 = 0$

#### 4. CONCLUSION

We have compared the numerical efficiency of three types of asymptotic-numerical method: the polynomial representation, a new rational representation and the reduced basis technique. In the three cases, the largest step length has been obtained with large orders of truncature. According to our tests, the reduced basis technique is the least efficient because it is limited to small orders by the hugely increasing computation time to get the reduced system. At the moment, the best way seems the use of Padé approximants, that can be completed by alternative methods to improve the convergence (see Baker and Graves Morris (1981) or Van Dyke (1984)). It would be very interesting to establish a robust continuation method with those rational approximations, as it has been done with polynomial approximations.

The efficiency of the strategy of large orders of truncature and of rational approximations still seems a bit mysterious and in contradiction with the comparison between asymptotic and asymptotic-numerical expansions, we made in Section 3.2. It is worth continuing to look at this question.

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## APPENDIX 1: PADÉ APPROXIMANTS FOR SERIES OF VECTOR

Since H. Padé's thesis, 1892, we know that rational fractions are more adapted than polynoms to represent a function. On can find a modern presentation of Padé approximants in Baker and Graves-Morris, 1981. We use Padé approximants in the ANM in the following manner:

1. From the vectors  $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_p$ , we build up an orthogonal basis by the classical Gram-Schmidt procedure (We detail the computations for  $p = 6$ )

$$\begin{aligned}\mathbf{U}_1 &= \alpha_{11} \mathbf{U}_1^* \\ \mathbf{U}_2 &= \alpha_{21} \mathbf{U}_1^* + \alpha_{22} \mathbf{U}_2^* \\ \mathbf{U}_3 &= \alpha_{31} \mathbf{U}_1^* + \alpha_{32} \mathbf{U}_2^* + \alpha_{33} \mathbf{U}_3^* \\ \mathbf{U}_4 &= \alpha_{41} \mathbf{U}_1^* + \alpha_{42} \mathbf{U}_2^* + \alpha_{43} \mathbf{U}_3^* + \alpha_{44} \mathbf{U}_4^* \\ \mathbf{U}_5 &= \alpha_{51} \mathbf{U}_1^* + \alpha_{52} \mathbf{U}_2^* + \alpha_{53} \mathbf{U}_3^* + \alpha_{54} \mathbf{U}_4^* + \alpha_{55} \mathbf{U}_5^* \\ \mathbf{U}_6 &= \alpha_{61} \mathbf{U}_1^* + \alpha_{62} \mathbf{U}_2^* + \alpha_{63} \mathbf{U}_3^* + \alpha_{64} \mathbf{U}_4^* + \alpha_{65} \mathbf{U}_5^* + \alpha_{66} \mathbf{U}_6^*\end{aligned}$$

Then, we introduce this into the polynomial representation, which introduces 6 polynomials with a decreasing degree (from 5 to 0) as factors of the vector fields  $\mathbf{U}_k$

$$\begin{aligned}\mathbf{U} - \mathbf{U}_0 &= a \mathbf{U}_1^* (\alpha_{11} + a \alpha_{21} + a^2 \alpha_{31} + a^3 \alpha_{41} + a^4 \alpha_{51} + a^5 \alpha_{61}) \\ &\quad + a^2 \mathbf{U}_2^* (\alpha_{22} + a \alpha_{32} + a^2 \alpha_{42} + a^3 \alpha_{52} + a^4 \alpha_{62}) \\ &\quad + a^3 \mathbf{U}_3^* (\alpha_{33} + a \alpha_{43} + a^2 \alpha_{53} + a^3 \alpha_{63}) \\ &\quad + a^4 \mathbf{U}_4^* (\alpha_{44} + a \alpha_{54} + a^2 \alpha_{64}) \\ &\quad + a^5 \mathbf{U}_5^* (\alpha_{55} + a \alpha_{65}) \\ &\quad + a^6 \mathbf{U}_6^* (\alpha_{66})\end{aligned}$$

2. We replace the five first polynomials by five rational fractions having the same denominator  $\Delta_5 = 1 + ad_1 + a^2d_2 + \dots + a^5d_5$  ( $d_0 = 1$ ) in the following way

$$\begin{aligned}\alpha_{11} + a\alpha_{21} + a^2\alpha_{31} + a^3\alpha_{41} + a^4\alpha_{51} + a^5\alpha_{61} &\approx \frac{b_0 + ab_1 + a^2b_2 + a^3b_3 + a^4b_4}{\Delta_5} \\ \alpha_{22} + a\alpha_{32} + a^2\alpha_{42} + a^3\alpha_{52} + a^4\alpha_{62} &\approx \frac{c_0 + ac_1 + a^2c_2 + a^3c_3}{\Delta_5} \\ \alpha_{33} + a\alpha_{43} + a^2\alpha_{53} + a^3\alpha_{63} &\approx \frac{e_0 + ae_1 + a^2e_2}{\Delta_5} \\ \alpha_{44} + a\alpha_{54} + a^2\alpha_{64} &\approx \frac{f_0 + af_1}{\Delta_5} \\ \alpha_{55} + a\alpha_{65} &\approx \frac{g_0}{\Delta_5}\end{aligned}$$

The coefficients  $b_i, c_i, e_i, f_i$  and  $g_i$  are computed by the same principles as with the classical Padé approximants: we require that each fraction has the same Taylor expansions as the corresponding polynomials up to order 5, 4, 3, 2, 1, respectively. This results in the followings:

$$\begin{aligned}b_0 &= \alpha_{11} \\ b_1 &= \alpha_{21} + \alpha_{11}d_1 & c_0 &= \alpha_{22} \\ b_2 &= \alpha_{31} + \alpha_{21}d_1 + \alpha_{11}d_2 & c_1 &= \alpha_{32} + \alpha_{22}d_1 & e_0 &= \alpha_{33} \\ b_3 &= \alpha_{41} + \alpha_{31}d_1 + \alpha_{21}d_2 + \alpha_{11}d_3 & c_2 &= \alpha_{42} + \alpha_{32}d_1 + \alpha_{22}d_2 & e_1 &= \alpha_{43} + \alpha_{33}d_1 \\ b_4 &= \alpha_{51} + \alpha_{41}d_1 + \alpha_{31}d_2 + \alpha_{21}d_3 + \alpha_{11}d_4 & c_3 &= \alpha_{52} + \alpha_{42}d_1 + \alpha_{32}d_2 + \alpha_{22}d_3 & e_2 &= \alpha_{53} + \alpha_{43}d_1 + \alpha_{33}d_2 \\ f_0 &= \alpha_{44} \\ f_1 &= \alpha_{54} + \alpha_{44}d_1 & g_0 &= \alpha_{55}\end{aligned}$$

and the coefficients of  $\Delta_5$  are solutions of the triangular system

$$\begin{aligned}\alpha_{61} + \alpha_{51}d_1 + \alpha_{41}d_2 + \alpha_{31}d_3 + \alpha_{21}d_4 + \alpha_{11}d_5 &= 0 \\ \alpha_{62} + \alpha_{52}d_1 + \alpha_{42}d_2 + \alpha_{32}d_3 + \alpha_{22}d_4 &= 0 \\ \alpha_{63} + \alpha_{53}d_1 + \alpha_{43}d_2 + \alpha_{33}d_3 &= 0 \\ \alpha_{64} + \alpha_{54}d_1 + \alpha_{44}d_2 &= 0 \\ \alpha_{65} + \alpha_{55}d_1 &= 0\end{aligned}$$

3. After some rearrangements, we arrive at a new form of the previous rational representation, that involves only the initial vectors  $\mathbf{U}_k$  and the coefficients  $d_i$

$$\begin{aligned}\mathbf{U} - \mathbf{U}_0 &= a\mathbf{U}_1 \frac{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + a^2\mathbf{U}_2 \frac{1 + ad_1 + a^2d_2 + a^3d_3}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + \\ &+ a^3\mathbf{U}_3 \frac{1 + ad_1 + a^2d_2}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + a^4\mathbf{U}_4 \frac{1 + ad_1}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + \\ &+ a^5\mathbf{U}_5 \frac{1}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} \\ \lambda - \lambda_0 &= a\lambda_1 \frac{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + a^2\lambda_2 \frac{1 + ad_1 + a^2d_2 + a^3d_3}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + \\ &+ a^3\lambda_3 \frac{1 + ad_1 + a^2d_2}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + a^4\lambda_4 \frac{1 + ad_1}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5} + \\ &+ a^5\lambda_5 \frac{1}{1 + ad_1 + a^2d_2 + a^3d_3 + a^4d_4 + a^5d_5}\end{aligned}$$

## APPENDIX 2: RAYLEIGH-RITZ METHOD

For elastic bodies, the governing equations are deduced from the mixed Hellinger-Reisner formulation

$$\begin{aligned}\int_{\Omega_0} (\mathbf{S} : \delta\boldsymbol{\gamma}(\mathbf{u}) + \delta\mathbf{S} : \boldsymbol{\gamma}(\mathbf{u}) - \mathbf{S} : \mathbf{D}^{-1} : \delta\mathbf{S}) dv - \lambda P_2(\delta\mathbf{u}) &= 0 \quad \forall \delta\mathbf{u} \in CA, \forall \delta\mathbf{S} \\ \boldsymbol{\gamma}(\mathbf{u}) &= \boldsymbol{\gamma}^l(\mathbf{u}) + \boldsymbol{\gamma}^{nl}(\mathbf{u}, \mathbf{u}) \quad \text{and} \quad \delta\boldsymbol{\gamma}(\mathbf{u}) = \boldsymbol{\gamma}^l(\delta\mathbf{u}) + 2\boldsymbol{\gamma}^{nl}(\mathbf{u}, \delta\mathbf{u})\end{aligned}\tag{2.1}$$

where  $\mathbf{u}$  is the displacement,  $\boldsymbol{\gamma}$  the Green-Lagrange strain tensor and  $\mathbf{S}$  the second Piola-Kirchhoff stress tensor. With respect to the mixed unknown  $\mathbf{U} = (\mathbf{u}, \mathbf{S})$ , this problem is quadratic and can be written in the form (4).

We suppose that  $\mathbf{U}_0, \lambda_0$  is a solution and we introduce the change of variables

$$\mathbf{U} = \mathbf{U}_0 + \hat{\mathbf{U}} \quad \text{and} \quad \lambda = \lambda_0 + \hat{\lambda}$$

Then the new variables (that we rename  $\mathbf{U}$  and  $\lambda$ ) are solution of the following problem

$$\mathbf{L}_t(\mathbf{U}) + \mathbf{Q}(\mathbf{U}, \mathbf{U}) = \lambda \mathbf{F}\tag{2.2}$$

The truncature of the asymptotic-numerical solution of (2.2) at the order  $p$  gives a mixed basis  $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_p$  that can be normalised and orthogonalized. One can choose to use the mixed basis  $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_p$  or only the displacement basis  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p$  as a Ritz basis to solve the non-linear mixed problem (2.1). We propose the second choice, but in order to get a quadratic problem, we introduce additional variables. We seek the displacement  $\mathbf{u}$  in the form

$$\mathbf{u} = r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \dots + r_p\mathbf{u}_p\tag{2.3}$$

where the coefficients  $r_1, r_2, \dots, r_p$  are the new unknowns. In this procedure, it is convenient to split the mixed equations (2.1) into equilibrium equations and constitutive equations

$$\int_{\Omega_0} \mathbf{S}: (\gamma^l(\delta \mathbf{u}) + 2\gamma^{nl}(\mathbf{u}_0, \delta \mathbf{u})) dv + \int_{\Omega_0} \mathbf{S}_0: 2\gamma^{nl}(\mathbf{u}, \delta \mathbf{u}) dv + \int_{\Omega_0} \mathbf{S}: 2\gamma^{nl}(\mathbf{u}, \delta \mathbf{u}) dv - \lambda P_e(\delta \mathbf{u}) = 0 \quad (2.4)$$

$$\mathbf{S} = \mathbf{D}(\gamma^l(\mathbf{u}) + 2\gamma^{nl}(\mathbf{u}_0, \mathbf{u}) + \gamma^{nl}(\mathbf{u}, \mathbf{u}))$$

We define  $p(p+1)/2$  additional variables

$$r_k = r_i r_j \quad \text{with} \quad k = ip + j - \frac{i(i-1)}{2} \quad (2.5)$$

and introduce the following notations

$$\mathbf{S}_i = \mathbf{D}(\gamma^l(\mathbf{u}_i) + 2\gamma^{nl}(\mathbf{u}_0, \mathbf{u}_i)) \quad (2.6)$$

$$\mathbf{S}_k = 2\mathbf{D}\gamma^{nl}(\mathbf{u}_i, \mathbf{u}_j) \quad \text{for } i \neq j \quad \text{or } \mathbf{S}_k = \mathbf{D}\gamma^{nl}(\mathbf{u}_i, \mathbf{u}_j) \quad \text{for } i = j$$

so that the following expansions can be deduced exactly from (2.3) and from the constitutive equations ( $N = p(p+3)/2$ )

$$\mathbf{S} = \sum_{j=1}^N r_j \mathbf{S}_j$$

One gets the  $p$  lacking equations by putting  $\delta \mathbf{u} = \mathbf{u}_i$  in (2.4a) as in the classical Rayleigh-Ritz method which leads to

$$\int_{\Omega_0} \mathbf{S}(\mathbf{u}): (\gamma^l(\mathbf{u}_i) + 2\gamma^{nl}(\mathbf{u}_0, \mathbf{u}_i)) dv + \int_{\Omega_0} \mathbf{S}_0: 2\gamma^{nl}(\mathbf{u}, \mathbf{u}_i) dv + \int_{\Omega_0} \mathbf{S}(\mathbf{u}): 2\gamma^{nl}(\mathbf{u}, \mathbf{u}_i) dv - \lambda P_e(\mathbf{u}_i) = 0$$

Then, if we introduce (2.3) and (2.5) into this equation, we get the following reduced problem verified by  $r_i$  and  $r_k$ , where we have introduced more variables in order to get a quadratic problem

$$\mathbf{l}(\mathbf{r}) + \mathbf{q}(\mathbf{r}, \mathbf{r}) - \lambda \mathbf{f} = 0 \quad \text{with } \mathbf{r} = \langle r_1, r_2, \dots, r_p, r_{p+1}, \dots, r_N \rangle \quad (2.7)$$

$$l_{ij} r_j + q_{ijk} r_j r_k - \lambda f_i = 0 \quad i = 1, p \quad j = 1, N \quad k = 1, p$$

where

$$l_{ij} = \int_{\Omega_0} \mathbf{S}_j: (\gamma^l(\mathbf{u}_i) + 2\gamma^{nl}(\mathbf{u}_0, \mathbf{u}_i)) dv + \int_{\Omega_0} \mathbf{S}_0: 2\gamma^{nl}(\mathbf{u}_j, \mathbf{u}_i) dv \quad \text{for } i = 1, p \quad \text{and } j = 1, p$$

$$l_{ij} = \int_{\Omega_0} \mathbf{S}_j: (\gamma^l(\mathbf{u}_i) + 2\gamma^{nl}(\mathbf{u}_0, \mathbf{u}_i)) dv \quad \text{for } i = 1, p \quad \text{and } j = n+1, N$$

$$q_{ijk} = \int_{\Omega_0} \mathbf{S}_j: 2\gamma^{nl}(\mathbf{u}_k, \mathbf{u}_i) dv \quad \text{for } i = 1, p \quad \text{and } j = 1, N \quad \text{and } k = 1, p$$

$$f_i = P_e(\mathbf{u}_i) \quad \text{for } i = 1, p$$

By introducing additional variables  $r_k$ , the computations of the coefficients  $q_{ijk}$  become a bit simpler. This is due to the use of a mixed formulation. Indeed, if we should applied a displacement formulation we should get a cubic reduced problem in the form

$$l'_{ij} r_j + q'_{ijkl} r_j r_k + m_{ijkl} r_j r_k r_l - \lambda f_i = 0 \quad \text{for } i, j, k = 1, p$$

Let us remark that the reduced problem (2.7) has rigorously the same quadratic form as the initial problem (4). Then we can apply for this problem the asymptotic-numerical techniques as developed for (4). We have solved it by the asymptotic-numerical continuation method. Hence we keep all the advantages of the method: the computation is automatic.

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