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The relationship between general equilibrium models with infinitely-lived agents and overlapping generations models, and some applications*

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Abstract

We prove that a two-cycle equilibrium in a general equilibrium model with infinitely-lived agents (GEILA) constitutes an equilibrium in an overlapping generations (OLG) model. Conversely, an equilibrium in an OLG model that satisfies additional conditions is part of an equilibrium in a GEILA model. Our framework, which includes three assets (physical capital, a Lucas tree, and fiat money), encompasses both exchange and production economies. As an application, we demonstrate that equilibrium indeterminacy and rational asset price bubbles can arise not only in OLG models but also in GEILA models.

Keywords: infinite-horizon, general equilibrium, infinitely-lived agent, overlapping generations, asset price bubble, fiat money, equilibrium indeterminacy.

JEL Classifications: D51, E32, E44.

1 Introduction

General equilibrium models with infinitely-lived agents (GEILA) and overlapping generations (OLG) models are two workhorses in macroeconomics. A vast body of literature explores these two frameworks.¹ This raises a natural question: what is the relationship

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¹See [de la Croix and Michel \(2002\)](#) for an introduction to OLG models, and [Becker \(2006\)](#), [Magill and Quinzii \(2008\)](#), [Le Van and Pham \(2016\)](#), among others, for an introduction to GEILA models.

between these two classes of models? If so, can this relationship help us to address economic questions?

Looking back to history, [Woodford \(1986\)](#) considered an economy with capital accumulation and money, where there are two classes of infinitely-lived agents (capitalists and workers). [Woodford \(1986\)](#) studied a special setup in which capitalists have logarithmic utility, never hold money, and face a single trade-off between consumption and investment. Workers, on the other hand, never purchase capital and face a trade-off between consumption and leisure. Then, he obtained an equilibrium system, which is similar to those in an OLG model with two-period-lived workers.² Following [Woodford \(1986\)](#), [Kocherlakota \(1992\)](#) wrote, in his footnote 4, that "In both examples, short sales constraints that bind in alternating periods serve to make the infinite-horizon economy look like an overlapping generations economy".

To date, neither [Woodford \(1986\)](#), [Kocherlakota \(1992\)](#), nor the broader literature has formally established a connection between these two classes of models in a general framework. Our paper seeks to address this gap.

Our contribution is two-fold. First, we prove that (1) a two-cycle equilibrium in a GEILA model is also an equilibrium in an OLG model, and (2) conversely, an equilibrium in an OLG model is part of a two-cycle equilibrium in a GEILA model if and only if it satisfies additional conditions including the transversality conditions.

Compared to [Woodford \(1986\)](#) and [Kocherlakota \(1992\)](#), we establish the observational connection in more general frameworks (including general utility functions, general endowments, and multiple assets). It should be noted that an equilibrium in an OLG model is not automatically part of an equilibrium in GEILA models. In particular, it is necessary to check the transversality conditions.

The existing literature also highlights a connection between standard OLG models and dynamic programming frameworks. [Aiyagari \(1985\)](#) demonstrates that the dynamics of capital in a standard OLG model (Diamond's model) can be derived from a discounted dynamic programming framework. [Hou \(1987\)](#) considers pure exchange economies and establishes an observational equivalence between an OLG model with agents living for two periods and a cash-in-advance economy with a single infinitely-lived representative agent. [Lovo and Polemarchakis \(2010\)](#) depart from a model with an infinitely-lived representative agent and show how the qualitative properties of OLG economies can be replicated by introducing a certain level of myopia.³

²Budget constrains (1.1b) in [Woodford \(1986\)](#) writes $p_t((c_t^w + (k_t^w - dk_{t-1}^w)) + M_{t+1}^w = M_t^w + r_t k_{t-1}^w + w_t n_t$. He also imposes constraints $k_t^w \geq 0$, $M_{t+1}^w \geq 0$, and borrowing constraint $p_t((c_t^w + (k_t^w - dk_{t-1}^w)) \leq M_t^w + r_t k_{t-1}^w$. He focuses on the case in which workers choose $k_t^w = 0$ for any t in optimal.

³It is also known that, in some cases, an OLG model with positive bequests can be reformulated as

Our paper focuses on general equilibrium models with a finite number of infinitely-lived households, which are more general than models with a single representative household. Notice that the results in [Aiyagari \(1985\)](#) and [Hou \(1987\)](#) cannot be applied to our models because our framework includes endowments, physical capital, and long-lived assets (both with and without dividends), while the model in [Aiyagari \(1985\)](#) features only physical capital (similar to a one-sector optimal growth model), and [Hou \(1987\)](#) considers an exchange economy.

As the second contribution, we apply our results to show how equilibrium indeterminacy and rational asset price bubbles can arise in both classes of models.

Our first application concerns equilibrium indeterminacy. Looking back at history, [Kehoe and Levine \(1985\)](#) consider two stationary pure exchange economies: the first involves a finite number of infinitely-lived consumers, and the second (an OLG model) features an infinite number of finitely-lived consumers. They argue that these two models have different implications: in the first model, equilibria are generically determinate, whereas this is not the case in the second model.⁴

The models in our paper are more general than those in [Kehoe and Levine \(1985\)](#) in the sense that we incorporate capital accumulation and imperfect financial markets (in forms of borrowing constraints). Different from [Kehoe and Levine \(1985\)](#), we show that equilibria may be indeterminate in both models. Precisely, we demonstrate that in a non-stationary exchange economy with a finite number of infinitely-lived consumers, equilibrium indeterminacy can arise. The intuition is that in such an economy, the equilibrium system can be supported by an OLG model, which creates room for indeterminacy.

The second application concerns the issue of rational asset price bubbles which has attracted significant attention from scholars in recent years.⁵ Since [Tirole \(1985\)](#), it has become relatively straightforward to build OLG models with bubbles. However, in infinite-horizon general equilibrium models, it is well known that constructing a model where rational asset price bubbles exist is more challenging, particularly when assets yield dividends ([Tirole, 1982](#); [Kocherlakota, 1992](#); [Santos and Woodford, 1997](#)).⁶ A key difficulty, as proved in [Bosi, Le Van and Pham \(2022\)](#)'s Proposition 2, is that, in

an optimal growth model à la Ramsey (see [Barro \(1974\)](#), [Aiyagari \(1992\)](#), [Michel et al. \(2006\)](#) among others).

⁴See [Farmer \(2019\)](#) for an overview of equilibrium indeterminacy in macroeconomics.

⁵For detailed surveys, see [Brunnermeier and Oehmke \(2013\)](#), [Miao \(2014\)](#), [Martin and Ventura \(2018\)](#), [Hirano and Toda \(2024, 2025b\)](#).

⁶[Le Van and Pham \(2016\)](#), [Bosi, Le Van and Pham \(2017a,b, 2018\)](#); [Bosi et al. \(2018\)](#); [Bosi, Le Van and Pham \(2022\)](#) construct models where assets with positive dividends exhibit bubbles. Inspired by [Wilson \(1981\)](#) and [Tirole \(1985\)](#) (Proposition 1.c), [Hirano and Toda \(2025a\)](#) construct some models and provide conditions under which any equilibrium (if it exists) is bubbly.

general, the existence of bubbles in such models requires that the asset holdings of at least two agents fluctuate over time and that the borrowing constraints of at least two agents bind at infinitely many periods.⁷

This property leads to the notion of a two-cycle equilibrium in GEILA models, as introduced above (note that this two-cycle structure is the simplest one of the GEILA models that can generate rational asset price bubbles). Building on our observational connection, this two-cycle equilibrium can be supported by an equilibrium in an OLG model. Thus, if the latter equilibrium exhibits a bubble, we can apply our results and impose additional conditions (which hold under reasonable assumptions) to prove that it is part of a bubbly equilibrium in a GEILA model.

Thanks to our observational connection, constructing models with infinitely-lived agents where asset bubbles exist is no longer a difficult task. This insight allows us not only to recover but also to extend many models of rational bubbles found in the literature. For instance, Example 1 in [Kocherlakota \(1992\)](#) presents an equilibrium where the fiat money has a positive price. However, by applying our result, we go further by showing that, in his model, there exists a continuum of equilibria where the fiat money's price is strictly positive.

The rest of the paper is organized as follows. Section 2 introduces both GEILA and OLG models. Section 3 formally establishes the connection between these two models. Section 4 presents applications of our results to the study of equilibrium indeterminacy and asset price bubbles. Technical proofs are presented in Appendix A.

2 Two models

2.1 An overlapping generations model

We present an OLG framework, which can be considered as a unified model of [Tirole \(1985\)](#) and [Weil \(1990\)](#).⁸ This is a discrete time model and the set of times is $\{0, 1, 2, \dots\}$. There is a consumption good, which is taken as numéraire.

In each period t , there is a representative firm (without market power) that maximizes its profit $\max_{K_t, L_t \geq 0} \{F(K_t, L_t) - r_t K_t - w_t L_t\}$ by choosing the physical capital K_t and the labor L_t , where r_t is the rental rate and w_t is the wage.

The consumer born in period t lives for two periods (young and old) and has $e_t^y \geq 0$ units of consumption as endowments at date when young and $e_{t+1}^o \geq 0$ when old. Endowments are exogenous. We assume that there is no population growth and

⁷See also discussions in Section 4.1 in [Bosi, Le Van and Pham \(2018\)](#).

⁸See [de la Croix and Michel \(2002\)](#) for an introduction of OLG models.

the population size N_t on date t is normalized to 1.

This consumer can invest/save using three assets: the physical capital, a long-lived asset that pays dividends (Lucas' tree), and a pure bubble asset. The structure of the long-lived asset (Lucas' tree) is the following: if the consumer buys 1 unit of this asset with price q_t on date t , she will receive d_{t+1} units of consumption good as dividend and she will be able to resell the asset with price q_{t+1} on date $t + 1$. The positive sequence of real dividends (d_t) is exogenous.

Regarding the pure bubble asset (or fiat money), if the consumer buys a_t units of this asset with the price p_t on date $t + 1$, then he/she will resell this asset with the price p_{t+1} on date $t + 1$ to receive $p_{t+1}a_t$ units of consumption good. As in the traditional literature (Tirole, 1985), the only reason why people buy this asset is to be able to resell it in the future.

Households born at date $t \geq 0$ choose consumptions c_t^y, c_{t+1}^o , investment in physical capital s_t , investment in a long-lived asset a_t (Lucas' tree) and pure bubble asset b_t in order to maximize her intertemporal utility $u(c_t^y) + \beta u(c_t^o)$ subject to the following constraints

$$c_t^y + s_t + q_t a_t + p_t b_t \leq e_t^y + w_t, \quad (1a)$$

$$c_{t+1}^o \leq e_{t+1}^o + (1 - \delta + r_{t+1})s_t + (q_{t+1} + d_{t+1})a_t + p_{t+1}b_t, \quad (1b)$$

$$s_t, a_t, b_t, c_t^y, c_t^o \geq 0,$$

where $\delta \in [0, 1]$ is the depreciation rate of physical capital.

Households born at date -1 just consume, that is $c_0^o = e_0^o + (1 - \delta + r_0)s_{-1} + (q_0 + d_0)a_{-1} + p_0b_{-1}$, where s_{-1}, a_{-1}, b_{-1} are exogenous.

Denote $R_t \equiv 1 - \delta + r_t$. Let us provide a formal definition of equilibrium.

Definition 1. *An intertemporal equilibrium of the two-period OLG economy is a non-negative list $(s_t, a_t, b_t, c_t^y, c_t^o, K_t, L_t, w_t, R_t, q_t, p_t)_{t \geq 0}$ satisfying three conditions: (1) given $R_{t+1}, q_t, q_{t+1}, p_t, p_{t+1}$ and w_t , the list $(s_t, a_t, b_t, c_t^y, c_t^o)_{t \geq 0}$ is a solution to the household's problem and the couple (K_t, L_t) is a solution to the firm's problem, (2) markets clear: $L_t = 1, K_{t+1} = s_t, a_t = 1, b_t = 1$ and $c_t^y + c_t^o + s_t = F(K_t, 1) + (1 - \delta)K_t + e_t^y + e_t^o + d_t$, and (3) $w_t > 0, R_t > 0, q_t > 0, p_t \geq 0 \forall t \geq 0$.*

Our framework encompasses both exchange and production economies. Indeed, a pure exchange economy corresponds to the special case of our model with $\delta = 1$, $s_{-1} = 0$, and $F(K, L) = 0$ for all $(K, L) \in \mathbb{R}_+^2$ (in this case, we remove variables s_t, K_t, L_t, w_t, R_t and conditions $L_t = 1, K_{t+1} = s_t, w_t > 0, R_t > 0$ from Definition 1). Moreover, when $e_t^y = e_t^o = 0$ for all t , our framework reduces to the standard

overlapping generations model with production and a constant population.

Standard assumptions are required.

Assumption 1 (utility function). *The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ is concave, strictly increasing, continuously differentiable and $u'(0) = +\infty$.*

Assumption 2 (production function). *The production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is assumed to be constant return to scale (CRS), concave, increasing in each component, continuously differentiable on $(0, \infty)^2$. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by $f(k) \equiv F(k, 1) \forall k \geq 0$, is concave, strictly increasing, continuously differentiable, $f(0) = 0$. The depreciation rate $\delta \in [0, 1]$.*

Assumption 3 (dividends and endowments). *Dividends and endowments satisfy $0 < d_t < \infty, e_t^y \geq 0, e_t^o \geq 0 \forall t$, and $a_{-1} = 1, b_{-1} = 1, s_{-1} \geq 0, (s_{-1}, e_0^y) \neq (0, 0)$. When we consider the exchange economy, we assume that $e_t^y > 0 \forall t$.*

To simplify our exposition, let us denote the two-period OLG economy by $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (e_t^y, e_t^o)_t, f, \delta, (d_t)_t)$.

Let us focus on interior equilibria in the sense that $K_t > 0 \forall t$ (this is ensured, for instance, by the Inada condition $f'(0) = +\infty$). In equilibrium, we also have $L_t = 1 > 0$. By consequence, the first order conditions (FOC) of the firm's problem give

$$w_t = f(K_t) - K_t f'(K_t) \text{ and } r_t = f'(K_t). \quad (2)$$

Since $a_t, b_t > 0$ and $s_t = K_{t+1} > 0$ in any interior equilibrium, we have the following FOCs of households:

$$u'(c_t^y) = \beta R_{t+1} u'(c_{t+1}^o), \quad (3a)$$

$$q_t R_{t+1} = q_{t+1} + d_{t+1}, \quad (3b)$$

$$p_t R_{t+1} = p_{t+1}. \quad (3c)$$

Note also that under conditions (3b), (3c) and $e_t^y + w_t > 0$, the list $(s_t, a_t, b_t, c_t^y, c_t^o)_{t \geq 0}$ is a solution to the household's maximization problem if (i) $a_t = b_t = 1, s_t > 0$, (ii) condition (3) holds, and (iii) budget constraints (1a), (1b) bind.

By using market clearing conditions $K_{t+1} = s_t, L_t = 1, a_t = 1, b_t = 1$, the FOC (3a) can be rewritten as

$$u'(e_t^y + w_t - K_{t+1} - q_t - p_t) = \beta R_{t+1} u'(e_{t+1}^o + R_{t+1}(K_{t+1} + q_t + p_t)). \quad (4)$$

To summarize our above arguments, we state the following result.

Lemma 1. *Let Assumptions 1-3 be satisfied. A non-negative list $(s_t, a_t, b_t, c_t^y, c_t^o, K_t, L_t, w_t, R_t, q_t, p_t)_{t \geq 0}$ is an interior intertemporal equilibrium of the OLG economy if and only if (1) conditions (2), (3b), (3c), (4) and market clearing conditions in Definition 1 are satisfied, (2) the budget constraints (1a) and (1b) bind and $K_t > 0 \forall t \geq 0$.*

According to Lemma 1, an interior equilibrium can be uniquely determined via the sequence $(q_t, p_t, K_{t+1})_{t \geq 0}$. So, we also call $(q_t, p_t, K_{t+1})_{t \geq 0}$ an equilibrium. When we consider an exchange economy, we call $(q_t, p_t)_{t \geq 0}$ an equilibrium (in this case, we define R_{t+1} by (3b) instead of $R_{t+1} = 1 - \delta + r_{t+1}$).

2.2 A general equilibrium model with infinitely-lived agents

We now develop the model in Le Van and Pham (2016) by adding two ingredients: endowments and a pure bubble asset, allowing us to cover both exchange and production economies. Consider an infinite-horizon general equilibrium model without uncertainty and discrete time ($t \in \{0, 1, 2, \dots\}$). There are m heterogeneous households and a representative firm without market power. There is a single consumption good, which is the numéraire.

For each period t , the representative firm takes prices (r_t, w_t) as given and maximizes its profit by choosing physical capital K_t and labor L_t .

$$(P(r_t, w_t)) : \quad \pi_t \equiv \max_{K_t, L_t \geq 0} (F(K_t, L_t) - r_t K_t - w_t L_t), \quad (5)$$

where the function F satisfies Assumption 2.

Each household i has an endowment $e_{i,t} \geq 0$ units of consumption good and supplies $L_{i,t} \geq 0$ units of labor supply at each date t .⁹

Households invest in physical capital and/or financial assets and consume. In each period t , agent i consumes $c_{i,t}$ units of consumption good. If agent i buys $k_{i,t+1} \geq 0$ units of capital in period t , she will receive $(1-\delta)k_{i,t+1}$ units of old capital in period $t+1$, after being depreciated (δ is the depreciation rate), and $r_{t+1}k_{i,t+1}$ units of consumption good.

As in the OLG model in Section 2.1, there are the so-called a pure bubble asset and a long-lived asset that brings dividends (Lucas' tree). Each household i takes the sequence $(q, p, r, w) \equiv (q_t, p_t, r_t, w_t)_{t \geq 0}$ as given and chooses the sequences of capital $(k_{i,t})_{t \geq 0}$, of the long-lived asset $(a_{i,t})_{t \geq 0}$, of fiat money $(b_{i,t})_{t \geq 0}$ and of consumption

⁹Becker et al. (2014) consider the case $L_{i,t} = 1/m$.

$(c_{i,t})_{t \geq 0}$ in order to maximize her intertemporal utility.

$$(P_i(q, p, r, w)) : \max_{(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \right] \quad (6)$$

subject to constraints $k_{i,t+1}, a_{i,t}, b_{i,t} \geq 0$,¹⁰ and budget constraint

$$\begin{aligned} c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t} + q_t a_{i,t} + p_t b_{i,t} \\ \leq r_t k_{i,t} + (q_t + d_t) a_{i,t-1} + p_t b_{i,t-1} + w_t L_{i,t} + e_{i,t}. \end{aligned} \quad (7)$$

Denote \mathcal{E}_{GEILA} the economy characterized by a list

$$\mathcal{E}_{GEILA} = \left((u_i, \beta_i, (e_{i,t}, L_{i,t})_t, k_{i,0}, a_{i,-1}, b_{i,-1})_{i=1}^m, f, (d_t)_t, \delta \right).$$

Definition 2. *A sequence of prices and quantities*

$(\bar{q}_t, \bar{p}_t, \bar{r}_t, \bar{w}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t}, \bar{b}_{i,t})_{i=1}^m, \bar{K}_t, \bar{L}_t)_{t \geq 0}$ is an intertemporal equilibrium of the economy \mathcal{E}_{GEILA} if the following conditions are satisfied. (i) *Price positivity:* $\bar{w}_t, \bar{q}_t, \bar{r}_t > 0, \bar{p}_t \geq 0$ $\forall t \geq 0$. (ii) *Market clearing:* $\bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t}, \bar{L}_t = \sum_{i=1}^m L_{i,t}, \sum_{i=1}^m \bar{a}_{i,t} = 1, \sum_{i=1}^m \bar{b}_{i,t} = 1$, and

$$\sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1 - \delta)\bar{k}_{i,t}) = e_t + f(\bar{K}_t) + d_t \quad \forall t \geq 0,$$

where $e_t \equiv \sum_{i=1}^m e_{i,t}$ is the aggregate endowment; (iii) *Optimal consumption plans:* for all i , $(\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t}, \bar{b}_{i,t})_{t=0}^{\infty}$ is a solution to the problem $(P_i(\bar{q}, \bar{p}, \bar{r}, \bar{w}))$. (iv) *Optimal production plan:* for all $t \geq 0$, (\bar{K}_t, \bar{L}_t) is a solution to the problem $(P(\bar{r}_t, \bar{w}_t))$.

We impose the standard assumptions on the households' characteristics.

Assumption 4 (endowments). (1) $k_{i,0}, a_{i,-1}, b_{i,-1}, e_{i,t}, L_{i,t} \geq 0$, and $(k_{i,0}, a_{i,-1}, e_{i,0}) \neq (0, 0, 0) \forall i \in \{1, \dots, m\}$. Moreover, $\sum_{i=1}^m L_{i,t} = 1, \sum_{i=1}^m a_{i,-1} = 1, \sum_{i=1}^m b_{i,-1} = 1$.

Assumption 5. For all i , $\beta_i \in (0, 1)$ and the utility function $u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfies (1) $\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty$, where $(W_t)_t$ is defined by $W_0 \equiv f(K_0) + d_0 + \sum_{i=1}^m e_{i,0}$ and $W_t = f(W_{t-1}) + d_t + \sum_{i=1}^m e_{i,t} \forall t \geq 1$ and (2) there exist $\theta, x \in \mathbb{R}$ such that $\frac{u_i(c) - u_i(\lambda c)}{1 - \lambda} \leq \theta u_i(c) + x \forall \lambda \in (\underline{\lambda}, 1), \forall c \in \{z : u_i(z) > -\infty\}$, where $\underline{\lambda} \in (0, 1)$.

Assumption 5.(2) is a variant of Assumption (ii) in Proposition 5.1 in [Ekeland and Scheinkman \(1986\)](#), which plays an important role in proving transversality conditions.

¹⁰We may eventually introduce a short-sale constraint as in [Le Van and Pham \(2016\)](#), [Bosi, Le Van and Pham \(2022\)](#) but it is not the main aim of the present paper.

This assumption is satisfied under standard setups, for instance, $u_i(c) = c^{1-\sigma}/(1-\sigma)$, where $0 < \sigma \neq 1$ or $u_i(c) = \ln(c)$. It also holds when $u_i(0) > -\infty$. Indeed, by the concavity of u_i , we have $u_i(\lambda c) \geq \lambda u_i(c) + (1-\lambda)u_i(0)$, which implies $\frac{u_i(c)-u_i(\lambda c)}{1-\lambda} \leq u_i(c) - u_i(0) \forall \lambda \in (0, 1)$. Then, we take $\theta = 1$ and $x = -u_i(0)$.

Remark 1. Under Assumptions 2 and 4, we have $L_t = 1$, $r_t = f'(K_t)$ and $w_t = f(K_t) - f'(K_t)K_t$ in equilibrium. Hence, we also call $(q_t, p_t, (c_{i,t}, k_{i,t+1}, b_{i,t}, a_{i,t})_{i \in I}, K_t)_{t \geq 0}$ an intertemporal equilibrium.

We now introduce the notion of two-cycle economy and two-cycle equilibrium.

Definition 3 (two-cycle economy). The economy \mathcal{E} is called a two-cycle economy if (1) there are 2 consumers, called 1 and 2,¹¹ with $u_i = u$, $\beta_i = \beta \in (0, 1) \forall i = \{1, 2\}$, (2) their endowments are $k_{1,0} = 0, a_{1,-1} = 0, b_{1,-1} = 0, k_{2,0} \geq 0, a_{2,-1} = 1, b_{2,-1} = 1$, and (3) their labor endowments are $L_{1,2t} = 1, L_{1,2t+1} = 0, L_{2,2t} = 0, L_{2,2t+1} = 1 \forall t$.

Denote this two-cycle economy by $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f, \delta, (d_t)_t)$.

Definition 4. An intertemporal equilibrium $(q_t, p_t, (c_{i,t}, k_{i,t+1}, b_{i,t}, a_{i,t})_{i \in I}, K_t)_{t \geq 0}$ is called a two-cycle equilibrium of the economy \mathcal{E}_{GEILA2} if

$$k_{1,2t} = a_{1,2t-1} = b_{1,2t-1} = 0, \quad k_{1,2t+1} = K_{2t+1}, a_{1,2t} = b_{1,2t} = 1, \quad (8a)$$

$$k_{2,2t} = K_{2t}, a_{2,2t-1} = b_{2,2t-1} = 1, \quad k_{2,2t+1} = a_{2,2t} = b_{2,2t} = 0. \quad (8b)$$

Observe that in a two-cycle equilibrium, we have

$$c_{1,2t-1} = e_{1,2t-1} + R_{2t-1}K_{2t-1} + q_{2t-1} + d_{2t-1} + p_{2t-1}, \quad (9a)$$

$$c_{1,2t} = e_{1,2t} + w_{2t} - K_{2t+1} - q_{2t} - p_{2t}, \quad (9b)$$

$$c_{2,2t-1} = e_{2,2t-1} + w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1}, \quad (9c)$$

$$c_{2,2t} = e_{2,2t} + R_{2t}K_{2t} + q_{2t} + d_{2t} + p_{2t}, \quad (9d)$$

where denote $R_t \equiv r_t + 1 - \delta$ as in Section 2.1.

We have the following key result characterizing the two-cycle equilibrium.

Proposition 1. Consider a two-cycle economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f, \delta, (d_t)_t)$. Let Assumptions 1-4 be satisfied. Denote

$$e_{2t}^o \equiv e_{2,2t}, \quad e_{2t+1}^o \equiv e_{1,2t+1}, \quad e_{2t}^y \equiv e_{1,2t}, \quad e_{2t+1}^y \equiv e_{2,2t+1} \quad \forall t. \quad (10)$$

¹¹Some papers name *odd* and *even* agents.

Let $E_t \equiv (q_t, p_t, (c_{i,t}, k_{i,t+1}, b_{i,t}, a_{i,t})_{i \in \{1,2\}}, K_t)_{t \geq 0}$ be a positive list satisfying (8) and (9).

1. If E_t is a two-cycle equilibrium of the economy \mathcal{E}_{GEILA2} , then, for any t ,

$$q_t R_{t+1} = (q_{t+1} + d_{t+1}), \quad p_t R_{t+1} = p_{t+1}, \quad (11a)$$

$$\frac{1}{R_{t+1}} = \frac{\beta u'(e_{t+1}^o + R_{t+1} K_{t+1} + q_{t+1} + d_{t+1} + p_{t+1})}{u'(e_t^y + w_t - K_{t+1} - q_t - p_t)}, \quad (11b)$$

$$\frac{1}{R_{t+1}} \geq \frac{\beta u'(e_{t+1}^y + w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1})}{u'(e_t^o + R_t K_t + q_t + d_t + p_t)}. \quad (11c)$$

If we require, in addition, Assumption 5 and $\sum_{t=0}^{\infty} \beta^t |u(c_{i,t})| < \infty \forall i \in \{1, 2\}$, then the following transversality conditions hold.

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(e_{2t}^y + w_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0, \quad (12a)$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(e_{2t-1}^y + w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1} + p_{2t-1}) = 0. \quad (12b)$$

2. E_t is a two-cycle equilibrium of the economy \mathcal{E}_{GEILA2} if FOCs (11a-11c) and TVCs (12a-12b) hold, and $\sum_{t=0}^{\infty} \beta^t u(c_{i,t}) \in (-\infty, \infty) \forall i \in \{1, 2\}$.

Proof. See Appendix A.1. □

Conditions (11a-11c) are first-order conditions while (12a-12b) are transversality conditions. These conditions ensure that our positive list constitutes a two-cycle equilibrium. It should be noticed that we allow for $u(0) = -\infty$ and $u(c)$ may be negative.

3 Relationship between GEILA vs OLG models

We now present our main result which shows the connection between the equilibrium in an OLG model and that in a two-cycle economy.

Theorem 1. Let $((u_i, \beta_i, (e_{i,t}, L_{i,t})_t, k_{i,0}, a_{i,-1}, b_{i,-1})_{i=1}^m, (e_t^y, e_t^o)_t, f, \delta, (d_t)_t)$ be a list of fundamentals satisfying Assumptions 1-4.

1. (GEILA \Rightarrow OLG) If $(q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}, K_t)_{t \geq 0}$ is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f, \delta, (d_t)_t)$, then the sequence $(K_{t+1}, q_t, p_t)_{t \geq 0}$ is an equilibrium of the OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (e_t^y, e_t^o)_t, f, \delta, (d_t)_t)$, where the sequence $(e_t^y, e_t^o)_{t \geq 0}$ is defined by (10).

2. (OLG \Rightarrow GEILA) Assume that a positive sequence $(q_t, p_t, K_{t+1})_{t \geq 0}$ is an equilibrium of the two-period OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (e_t^y, e_t^o)_t, f, \delta, (d_t)_t)$. Consider a list $E_t \equiv (q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}, K_t)_{t \geq 0}$ where $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}$ satisfy (8) and (9) for all t .

2.(a) E_t is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{i,t})_t, f, \delta, (d_t)_t)$, where endowments $(e_{i,t})_{t \geq 0}$ are defined by (10), if the following conditions hold.

$$\sum_{t=0}^{\infty} \beta^t u(c_{i,t}) \in (-\infty, \infty) \quad \forall i \in \{1, 2\}, \quad (13a)$$

$$\frac{1}{R_{t+1}} \geq \frac{\beta u'(e_{t+1}^y + w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1})}{u'(e_t^o + R_t K_t + q_t + d_t + p_t)} \quad \forall t, \quad (13b)$$

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(c_{1,2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0, \quad (13c)$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(c_{2,2t-1})(K_{2t} + q_{2t-1} + p_{2t-1}) = 0. \quad (13d)$$

2.(b) Conversely, if E_t is a two-cycle equilibrium of the economy \mathcal{E}_{GEILA2} , then (13a) and (13b) hold. Moreover, if we require, in addition, $\sum_{t=0}^{\infty} \beta^t |u(c_{i,t})| < \infty$ $\forall i \in \{1, 2\}$ and Assumption 5, then the transversality conditions (13c) and (13d) hold.

Proof. Part 1 is a consequence of Lemma 1 and Proposition 1's point 1. Part 2 is a consequence of Lemma 1 and Proposition 1's point 2. The last statement of Theorem 1 follows Proposition 1's point 1 and the transversality conditions (12a-12b). \square

The intuition behind this result is the two-cycle structure of the economy \mathcal{E}_{GEILA2} with infinite-lived agents, which resembles the structure of the OLG economy \mathcal{E}_{OLG} with two-period-lived agents.

Our result leads to interesting implications. First, point 1 shows that analyzing two-cycle equilibria requires us to understand the properties of equilibrium in a two-period OLG model. Second, point 2 provides a way to construct a two-cycle equilibria from an equilibrium in a two-period OLG model. However, we need to impose additional conditions (13b-13d) which are satisfied in many standard setups.

Now, let us focus on two particular cases: a pure exchange economy (i.e., there is no production) and a production economy (i.e., $e_t^y = e_t^o = e_{i,t} = 0 \forall i, \forall t$). By applying parts 1 and 2.(a) of Theorem 1 for each of these two cases, we obtain the following results.

Proposition 2 (exchange economy). *Let $((u_i, \beta_i, (e_{i,t}, a_{i,-1}, b_{i,-1})_{i=1}^m, (e_t^y, e_t^o)_t, (d_t)_t)$ be a list of fundamentals satisfying Assumptions 1-4.*

1. (GEILA \Rightarrow OLG) If $(q_t, p_t, (c_{i,t}, a_{i,t}, b_{i,t})_{i \in \{1,2\}})_{t \geq 0}$ is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{1,t}, e_{2,t})_t, (d_t)_t)$, then the sequence $(q_t, p_t)_{t \geq 0}$ is an equilibrium of the OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (e_t^y, e_t^o)_t, (d_t)_t)$, where $(e_t^y, e_t^o)_{t \geq 0}$ is defined by (10).
2. (OLG \Rightarrow GEILA) Assume that the positive sequence $(q_t, p_t)_{t \geq 0}$ is an equilibrium of the two-period OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, (e_t^y, e_t^o)_t, (d_t)_t)$. A list $(q_t, p_t, (c_{i,t}, a_{i,t}, b_{i,t})_{i=1,2})_{t \geq 0}$, where $((c_{i,t}, a_{i,t}, b_{i,t})_{i \in \{1,2\}})_{t \geq 0}$ is given by (8) and (9), is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, (e_{1,t}, e_{2,t})_t, (d_t)_t)$, where endowments $(e_{i,t})_{t \geq 0}$ are defined by (10), if $\sum_{t=0}^{\infty} \beta^t u_i(c_{i,t}) \in (-\infty, \infty) \forall i \in \{1, 2\}$ and

$$u'(e_t^o + q_t + d_t + p_t) \geq \beta R_{t+1} u'(e_{t+1}^y - q_{t+1} - p_{t+1}) \quad \forall t, \quad (14a)$$

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(e_{2t}^y - q_{2t} - p_{2t})(q_{2t} + p_{2t}) = 0, \quad (14b)$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(e_{2t-1}^y - q_{2t-1} - p_{2t-1})(q_{2t-1} + p_{2t-1}) = 0. \quad (14c)$$

Proposition 3 (production economy). Let $((u_i, \beta_i, (L_{i,t})_t, k_{i,0}, a_{i,-1}, b_{i,-1})_{i=1}^m, f, \delta, (d_t)_t)$ be a list of fundamentals satisfying Assumptions 1-4.

1. (GEILA \Rightarrow OLG) If $(q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}, K_t)_{t \geq 0}$ is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$, then the sequence $(K_{t+1}, q_t, p_t)_{t \geq 0}$ is an equilibrium of the OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f, \delta, (d_t)_t)$.
2. (OLG \Rightarrow GEILA) Assume that the positive sequence $(q_t, p_t, K_{t+1})_{t \geq 0}$ is an equilibrium of the two-period OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f, \delta, (d_t)_t)$. A list $(q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}, K_t)_{t \geq 0}$, where $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in \{1,2\}}$ is determined by (8) and (9), is a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$ if $\sum_{t=0}^{\infty} \beta^t u_i(c_{i,t}) \in (-\infty, \infty) \forall i \in \{1, 2\}$ and

$$u'(R_t K_t + q_t + d_t + p_t) \geq \beta R_{t+1} u'(w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1}) \quad \forall t, \quad (15a)$$

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(w_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0, \quad (15b)$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1} + p_{2t-1}) = 0. \quad (15c)$$

4 Applications: indeterminacy and asset price bubbles

In this section, we present some applications of our results for studying the issue of indeterminacy and asset price bubble. First, we provide a formal definition of asset

price bubble (Tirole, 1982, 1985; Kocherlakota, 1992; Santos and Woodford, 1997; Huang and Werner, 2000; Le Van and Pham, 2016). Assume that we have an asset pricing equation

$$q_t = \frac{q_{t+1} + d_{t+1}}{R_{t+1}}. \quad (16)$$

Solving recursively (16), we obtain an asset price decomposition in two parts

$$q_t = Q_{t,t+\tau}q_{t+\tau} + \sum_{s=1}^{\tau} Q_{t,t+s}d_{t+s}, \text{ where } Q_{t,t+s} \equiv \frac{1}{R_{t+1} \dots R_{t+s}} \quad (17)$$

is the discount factor of the economy from date t to $t + s$.

Definition 5. 1. *The fundamental value of 1 unit of asset at date t is the sum of discounted values of future dividends:*

$$FV_t \equiv \sum_{s=1}^{\infty} Q_{t,t+s}d_{t+s}. \quad (18)$$

2. *We say that there is a bubble at date t if $q_t > FV_t$.*
3. *When $d_t = 0$ for any $t \geq 0$ (the Fundamental Value is zero), we say that there is a pure bubble if $q_t > 0$ for any t (or the fiat money's price is strictly positive).*

Lemma 2 (Montrucchio (2004), Proposition 7). *Consider the case $d_t > 0 \forall t \geq 0$. There is a bubble if and only if $\sum_{t=1}^{\infty} \frac{d_t}{q_t} < \infty$.*

Letting τ in (17) tend to infinity and using (18), we obtain $q_t = FV_t + \lim_{\tau \rightarrow \infty} Q_{t,t+\tau}q_{t+\tau}$. Thus, $q_t - FV_t > 0$ if and only if $q_0 - FV_0 > 0$. Therefore, if a bubble exists at date 0, it exists forever. Moreover, we also see that $q_{t+1} - FV_{t+1} = R_{t+1}(q_t - FV_t) \forall t$.

We now apply our results in Section 3 to study the issue of rational asset prices and equilibrium indeterminacy.

4.1 Exchange economy

First, we focus on the exchange economy. Let us define the sequence $(R_t)_{t \geq 1}$ by

$$\frac{1}{R_{t+1}} \equiv \frac{\beta u'(e_{t+1}^o + q_{t+1} + d_{t+1} + p_{t+1})}{u'(e_t^y - q_t - p_t)} \quad \forall t \geq 0, \quad (19)$$

and summarize our equilibrium system in Proposition 2 as follows:

$$q_t R_{t+1} = (q_{t+1} + d_{t+1}), \quad p_t R_{t+1} = p_{t+1} \quad (20a)$$

$$\frac{1}{R_{t+1}} \geq \frac{\beta u'(e_{t+1}^y - q_{t+1} - p_{t+1})}{u'(e_t^o + q_t + d_t + p_t)}, \quad (20b)$$

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(e_{2t}^y - q_{2t} - p_{2t})(q_{2t} + p_{2t}) = 0, \quad (20c)$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(e_{2t-1}^y - q_{2t-1} - p_{2t-1})(q_{2t-1} + p_{2t-1}) = 0. \quad (20d)$$

According to Proposition 2, condition (20a) is used to characterize the intertemporal equilibrium in an OLG model. Moreover, all conditions (20a-20d) characterize the two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2}(u, \beta, (e_{1,t}, e_{2,t})_t, (d_t)_t)$.

We will use the system (20a-20d) to show that equilibrium indeterminacy and asset price bubbles can exist along a two-cycle equilibrium.¹²

Example 1 (unique equilibrium with or without bubble). *Assume that $u(c) = \ln(c) \forall c$, and $e_t^o = 0 \forall t$. Consider a particular case where there is no fiat money (i.e., $p_t = 0 \forall t$). In this case, condition (20a) implies that there is a unique equilibrium in the OLG model. Moreover, the asset price is $q_t = \frac{\beta}{1+\beta} e_t^y$. This is also part of a two-cycle equilibrium in the economy $\mathcal{E}_{GEILA2}(u, \beta, (e_{1,t}, e_{2,t})_t, (d_t)_t)$ because FOCs and TCVs (20a-20d) hold.*

According to Lemma 2, the equilibrium is bubbly if and only if $\sum_t d_t/q_t < \infty$, or, equivalently, $\sum_t d_t/e_t^y < \infty$.¹³ In words, this requires that the dividend would be very small with respect to the endowment of the economy.¹⁴

We now consider a case where the fiat money may have the strictly positive price (that is, $p_t > 0 \forall t$).¹⁵

Example 2 (continuum of equilibria with fiat money). *Consider an economy with only fiat money (that is $q_t = d_t = 0$ for any t). Assume that $u'(c) = c^{-\sigma}$, where $\sigma > 0$. Assume also that $e_t^y > e_t^o \forall t$ and $\lim_{t \rightarrow \infty} \beta^t (e_t^y)^{1-\sigma} = 0$.*

¹²Solving the non-autonomous system (20a-20d) is far from trivial (see Bosi, Le Van and Pham (2022)'s Section 4, Hirano and Toda (2025a)'s Section IV and Bosi, Le Van and Pham (2025) for detailed analyses in the case $p_t = 0 \forall t$).

¹³A key condition for the existence of bubble $\sum_t d_t/e_t^y < \infty$ is also appeared in Section 9.3.2 in Bosi, Le Van and Pham (2017b), Section 5.1.1 and Section 5.2 in Bosi, Le Van and Pham (2018), Example 5 in Bosi, Le Van and Pham (2021), and Proposition 1 in Hirano and Toda (2025a).

¹⁴Bosi, Le Van and Pham (2022)'s Proposition 7 focuses on the case $q_t > 0, p_t = 0 \forall t$, and provide conditions under which there exists a continuum equilibria of the long-lived asset. Note that their analyses still apply for the case with only fiat money (their Section 4.1.1.)

¹⁵See also Weil (1990) for a detailed analysis of fiat money in a stochastic OLG model.

Any sequence $(p_t)_{t \geq 0}$ satisfying the following system

$$e_t^y - e_t^o \geq 2p_t \geq 0, \quad p_t = \beta p_{t+1} \left(\frac{e_t^y - p_t}{e_{t+1}^o + p_{t+1}} \right)^\sigma, \quad (21)$$

is a sequence of prices of a two-cycle equilibrium of the economy $\mathcal{E}_{GEILA2}(u, \beta, (e_{1,t}, e_{2,t})_t, (d_t)_t)$, where the endowments $(e_{i,t})_{t \geq 0}$ is defined by (10).

Proof. See Appendix A.2. □

Let us consider two particular cases of Example 2.

1. Observe that $p_t = 0 \forall t$ is a solution of the system (21). This is a no trade equilibrium.
2. Focus on the case where $e_t^y = ye^t, e_t^o = de^t$ where $y, d, e > 0$ where $d < y$ and $\beta e^{1-\sigma} < 1$ (to ensure that $e_t^y > e_t^o \forall t$ and $\lim_{t \rightarrow \infty} \beta^t (e_t^y)^{1-\sigma} = 0$). Assume that

$$1 < \beta e \left(\frac{y}{de} \right)^\sigma < \left(\frac{y}{d} \right)^\sigma. \quad (22)$$

Let p be determined by $1 = \beta e \left(\frac{y-p}{(d+p)e} \right)^\sigma$. Then the sequence (p_t) defined by $p_t = pe^t, \forall t \geq 0$, is a two-cycle equilibrium. In this equilibrium, the fiat money's price is strictly positive.

By combining with point 1, we observe that two sequences $((p_t)_{t \geq 0})$ with $p_t = 0, \forall t$, and $(pe^t)_{t \geq 0}$ are two solutions to the system (21). By using the same argument in the proof of Proposition 5 in [Bosi, Le Van and Pham \(2022\)](#), we can prove that any sequence $(p_t)_{t \geq 0}$ defined by $0 < p_0 < p$ and $p_t = \beta p_{t+1} \left(\frac{e_t^y - p_t}{e_{t+1}^o + p_{t+1}} \right)^\sigma \forall t$, is a solution to the system (21). Consequently, there exists a continuum of two-cycle equilibria in which the price of fiat money is strictly positive.

Remark 2. *Example 1 in [Kocherlakota \(1992\)](#) is a special case of our Example 2 with $\sigma = 2, \beta = 7/8, e = 8/7, p = 14, y = 70, d = 35$. An added value with respect to Example 1 in [Kocherlakota \(1992\)](#) is that we show a continuum of two-cycle equilibria whose fiat money's price is strictly positive while [Kocherlakota \(1992\)](#) only presents one equilibrium.*

4.2 Production economy with financial assets

Applying Proposition 3 for a particular where $u(c) = \ln(c) \forall c > 0$, we obtain the following result.

Corollary 1. Let $u(c) = \ln(c) \forall c > 0$ and $\beta \in (0, 1)$. Assume that there is no endowment, i.e., $e_{i,t} = 0 \forall i, \forall t$. Assume that $(q_t, p_t, K_{t+1})_{t \geq 0}$ is an equilibrium of the two-period OLG economy $\mathcal{E}_{OLG} \equiv \mathcal{E}_{OLG}(u, \beta, f, \delta, (d_t)_t)$, i.e.,

$$K_{t+1} + q_t + p_t = \frac{\beta}{1 + \beta} w_t = \frac{\beta}{1 + \beta} (f(K_t) - K_t f'(K_t)), \quad (23a)$$

$$q_t R_{t+1} = (q_{t+1} + d_{t+1}), \quad (23b)$$

$$p_t R_{t+1} = p_{t+1}, \quad (23c)$$

$$K_{t+1} > 0, q_t \geq 0, p_t \geq 0. \quad (23d)$$

If

$$w_{t-1} \beta^2 (1 - \delta + f'(K_t)) (1 - \delta + f'(K_{t+1})) \leq w_{t+1} \quad \forall t \quad (24)$$

then $(q_t, p_t, K_{t+1})_{t \geq 0}$ are asset prices and aggregate capital stock of a two-cycle equilibrium of the two-cycle economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$.

Proof. Under logarithmic utility function, the Euler equation (4) becomes (23a). By consequence, the TVCs (15b) and (15c) are satisfied. Last, thanks to (23a), condition (15a) becomes (24). \square

We now apply Corollary 1 to construct two-cycle equilibria with bubbles in general equilibrium models with two infinitely-lived agents $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$.¹⁶ To make clear our applications, we consider two standard cases: Linear and Cobb-Douglas production functions.

4.2.1 Cobb-Douglas production function

The following result is an application of Corollary 1.

Example 3 (pure bubble in a model with Cobb-Douglas production function). Let $u(c) = \ln(c)$, $\beta \in (0, 1)$, $\delta = 1$, the Cobb-Douglas production function $f(k) = Ak^\alpha$, where $\alpha \in (0, 1)$. Let us focus on the model with only the pure bubble asset and physical capital.

Denote K^* the capital in the bubbleless steady state, that is the steady state without pure bubble asset $K^* = \rho^{1/(1-\alpha)}$, where $\rho \equiv \gamma\alpha A$.

Denote $\gamma \equiv \frac{\beta}{1+\beta} \frac{1-\alpha}{\alpha}$. Observe that $\gamma = 1/f'(K^*)$.

¹⁶Providing a complete analysis of the system (23) is quite hard because it is a non-autonomous two-dimensional system with infinitely many parameters, including the dividend sequence (d_t) . See Tirole (1985), Bosi et al. (2018), Hirano and Toda (2025a), Pham and Toda (2026a,b) for the interplay between dividend-paying asset and capital accumulation in OLG models.

Assume that $\gamma > 1$ (i.e., $f'(K^*) < 1$; this is a so-called "low interest rate condition").

Then, there exists a two-cycle equilibrium with non-negligible bubble (i.e., p_t does not converge to zero) in the two-cycle economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$. In such an equilibrium, the aggregate capital and the asset price are determined by

$$K_t = (\alpha A)^{\frac{1-\alpha^t}{1-\alpha}} K_0^{\alpha^t} \quad \forall t \geq 1 \quad \text{and} \quad p_t = (\gamma - 1)K_{t+1} \quad \forall t \geq 0. \quad (25)$$

Moreover, $\lim_{t \rightarrow \infty} K_t = (\alpha A)^{1/(1-\alpha)} < K^*$ and $\lim_{t \rightarrow \infty} p_t = (\gamma - 1)(\alpha A)^{1/(1-\alpha)} > 0$.

Proof. See Appendix A.2. □

In terms of implications, Example 3 shows that a standard model with pure bubble asset as in Tirole (1985) can be embedded in a GEILA model. Note that under specifications in Example 3, as we prove in Lemma 4 in Appendix, the sequence $(p_t, K_{t+1})_{t \geq 0}$ determined by (25) is the unique equilibrium satisfying two conditions: (i) condition (23) with $q_t = d_t = 0 \quad \forall t$, and (ii) the asset price p_t does not converge to zero.

4.2.2 Linear technology

Let us now consider a linear production function: $F(K, L) = AK + wL$, where $A > 0$ and $w > 0$ represent, respectively, the capital and labor productivities. According to Corollary 1, an equilibrium $(q_t, p_t, K_{t+1})_{t \geq 0}$ of the two-period OLG economy are asset prices and aggregate capital stocks of a two-cycle equilibrium of the two-cycle economy $\mathcal{E}_{GEILA2} \equiv \mathcal{E}_{GEILA2}(u, \beta, f, \delta, (d_t)_t)$ if and only if $\beta(1 - \delta + A) \leq 1$.¹⁷

According to (23b) and (23c), we can compute that

$$p_t = R^t p_0, \quad q_0 = \sum_{s=1}^t \frac{d_s}{R^s} + \frac{q_t}{R^t}, \quad \text{which implies that } q_t = R^s \left(q_0 - \sum_{s=1}^t \frac{d_s}{R^s} \right).$$

To sum up, we obtain the following result.

Example 4. Assume that (1) $u(c) = \ln(c)$, $\beta \in (0, 1)$, (2) there is no endowment, i.e., $e_{i,t} = 0 \quad \forall i, \forall t$, (3) $F(K, L) = AK + wL$, (4) $R \equiv 1 - \delta + A \leq 1$,

$$\frac{\beta}{1 + \beta} w > \sum_{s=1}^t \frac{d_s}{R^s} \quad \text{and} \quad \frac{\beta}{1 + \beta} w > R^t \left(\frac{\beta}{1 + \beta} w - \sum_{s=1}^t \frac{d_s}{R^s} \right) \quad \forall t. \quad (26)$$

¹⁷Le Van and Pham (2016)'s Section 6.1 corresponds to this model with $p_t = 0, \forall t$. This case is also related to Proposition 5 in Bosi et al. (2018).

Then, any sequence $(q_t, p_t, K_{t+1})_{t \geq 0}$ satisfying the following system

$$p_0 \geq 0, \quad p_t = R^t p_0, \quad \sum_{s=1}^{\infty} \frac{d_s}{R^s} \leq q_0 < \frac{\beta}{1+\beta} w - p_0, \quad (27a)$$

$$q_t = R^t \left(q_0 - \sum_{s=1}^t \frac{d_s}{R^s} \right), \quad (27b)$$

$$nk_{t+1} + q_t + p_t = \frac{\beta}{1+\beta} w, \quad (27c)$$

is part of a two-cycle equilibrium in the two-cycle economy \mathcal{E}_{GEILA2} . Moreover, the following statements hold.

1. Fiat money has a positive price if $p_0 > 0$. Moreover, the supremum value \bar{p}_0 of initial fiat price p_0 , which ensures that $p_t > 0 \forall t$, is determined by $\bar{p}_0 = \frac{\beta}{1+\beta} w - \sum_{s=1}^{\infty} \frac{d_s}{R^s}$.
2. If $q_0 = \sum_{s=1}^{\infty} \frac{d_s}{R^s}$, then there is no bubble of the long-lived asset. In this case, we have $p_0 \geq 0$. There exists a continuum of equilibria with pure bubble, indexed by p_0 .
3. If $q_0 > \sum_{s=1}^{\infty} \frac{d_s}{R^s}$, then there is a bubble of the long-lived asset. Moreover, in this case, $\lim_{t \rightarrow \infty} b_t > 0$ if and only if $R = 1$.

Example 4 shows that there exists a continuum of equilibria with a strictly positive price of fiat money (pure bubble asset) and/or with bubbles of the long-lived assets. Bubbles of the long-lived asset and fiat money can co-exist. Indeed, take $p_0 > 0$ so that $\sum_{s=1}^{\infty} \frac{d_s}{R^s} < \frac{\beta}{1+\beta} w - p_0$. Then, take q_0 so that $\sum_{s=1}^{\infty} \frac{d_s}{R^s} < q_0 < \frac{\beta}{1+\beta} w - p_0$. Last, take $k_{t+1} = \frac{\beta}{1+\beta} w - q_t - p_t$. Then, the sequence $(k_{t+1}, q_t, p_t)_{t \geq 0}$ is strictly positive and satisfies (27). By consequence, it is part of an equilibrium whose fiat money's prices are strictly positive (i.e., $p_t > 0 \forall t$) and the long-lived asset has a bubble (i.e., $q_0 > \sum_{s=1}^{\infty} \frac{d_s}{R^s}$).

In Example 4, when $R < 1$, we have $\lim_{t \rightarrow \infty} q_t = \lim_{t \rightarrow \infty} p_t = 0$. When $R = 1$, we have $\lim_{t \rightarrow \infty} p_t = p_0$ and $\lim_{t \rightarrow \infty} q_t = q_0 - \sum_{s=1}^{\infty} \frac{d_s}{R^s}$. This shows that the growth rate and the dividend's size play an important role on the asset prices.

5 Conclusion

This paper bridges two foundational macroeconomic models: the infinite-horizon general equilibrium model with infinitely-lived agents (GEILA) and the overlapping generations (OLG) model. By establishing the connection between the two models, we have

provided a unified view that deepens our understanding of phenomena like equilibrium indeterminacy and rational asset price bubbles in both models. Our results also allow us to construct general equilibrium models with infinitely-lived agents, where asset price bubbles exist. Moreover, we have shown that a cycle of exogenous parameters, which generates a two-cycle economy (Definition 3), can create equilibrium indeterminacy and asset price bubbles (see Section 4).

A Appendix

A.1 Proof of Proposition 1

To prove Proposition 1, we need the following result.

Lemma 3. *Let Assumptions 1-4 be satisfied.*

Part A (necessary conditions). If a sequence $(q_t, p_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in I}, K_t)_{t \geq 0}$ is an equilibrium, then there exists non-negative sequences $((\lambda_{i,t}, \sigma_{i,t}, \mu_{i,t}, \nu_{i,t})_{i \in I})_{t \geq 0}$ satisfying the following conditions for any t, i :

$$(i) \quad c_{i,t} > 0, k_{i,t+1} \geq 0, a_{i,t} \geq 0, b_{i,t} \geq 0, K_t \geq 0, q_t > 0, r_t > 0, p_t \geq 0.$$

$$(ii) \quad K_t = \sum_{i \in I} k_{i,t}, \sum_{i \in I} a_{i,t} = 1, \sum_{i \in I} b_{i,t} = 1.$$

$$(iii) \quad f(K_t) - r_t K_t = w_t = \max\{f(K) - r_t K : k \geq 0\}.$$

$$(iv) \quad c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t} + q_t a_{i,t} + p_t b_{i,t} = r_t k_{i,t} + (q_t + d_t)a_{i,t-1} + p_t b_{i,t-1} + L_{i,t} w_t + e_{i,t}.$$

(v) *First order conditions:*

$$\lambda_{i,t} = \beta_i^t u'_i(c_{i,t}), \quad \lambda_{i,t} \geq R_{t+1} \lambda_{i,t+1} + \sigma_{i,t}, \quad \sigma_{i,t} k_{i,t+1} = 0, \quad (A.1)$$

$$\lambda_{i,t} q_t = (q_{t+1} + d_{t+1}) \lambda_{i,t+1} + \mu_{i,t}, \quad \mu_{i,t} a_{i,t} = 0, \quad (A.2)$$

$$\lambda_{i,t} p_t = \lambda_{i,t+1} p_{t+1} + \nu_{i,t}, \quad \nu_{i,t} b_{i,t} = 0. \quad (A.3)$$

If we require, in addition, Assumption 5 and $\sum_{t=0}^{\infty} \beta^t |u(c_{i,t})| < \infty$, then we have

$$(vi) \quad \text{transversality conditions: } \lim_{t \rightarrow \infty} \beta_i^t u'_i(c_{i,t}) (k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t}) = 0. \quad (A.4)$$

Part B (sufficient conditions). If sequences $(q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in I}, K_t)_{t \geq 0}$ and $((\lambda_{i,t}, \sigma_{i,t}, \mu_{i,t}, \nu_{i,t})_{i \in I})_{t \geq 0}$ satisfy conditions (i-vi) above, then $(q_t, p_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{i \in I}, K_t)_{t \geq 0}$ is an intertemporal equilibrium.

Proof of Lemma 3. For pedagogical purposes and to make the paper self-contained, we provide an elementary proof.

Part B (sufficient condition). We use the classic approach in the optimal control theory (see [Bosi, Le Van and Pham \(2022\)](#) for instance). It suffices to prove the optimality of the allocation $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{t \geq 0}$. Take an arbitrary feasible allocation $(c'_{i,t}, k'_{i,t+1}, a'_{i,t}, b'_{i,t})_{t \geq 0}$. We need to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \geq \limsup_{T \rightarrow \infty} \sum_{t=0}^T \beta_i^t u_i(c'_{i,t})$. Without loss of generality, assume that the budget constraint is binding, i.e., $c'_{i,t} + k'_{i,t+1} + q_t a'_{i,t} + p_t b'_{i,t} = R_t k'_{i,t} + (q_t + d_t) a'_{i,t-1} + p_t b'_{i,t-1} + w_t L_{i,t} + e_{i,t}$. Denote $E_{i,t} \equiv w_t L_{i,t} + e_{i,t}$. We have

$$\lambda_{i,t}(c'_{i,t} + k'_{i,t+1} + q_t a'_{i,t} + p_t b'_{i,t}) = \lambda_{i,t}(E_{i,t} + R_t k'_{i,t} + (q_t + d_t) a'_{i,t-1} + p_t b'_{i,t-1}). \quad (\text{A.5})$$

From the FOCs, we have

$$\lambda_{i,t} k'_{i,t+1} = R_{t+1} \lambda_{i,t+1} k'_{i,t+1} + \sigma_{i,t} k'_{i,t+1}, \quad (\text{A.6a})$$

$$\lambda_{i,t} q_{i,t} a'_{i,t} = \lambda_{i,t+1} (q_{t+1} + d_{t+1}) a'_{i,t} + \mu_{i,t} a'_{i,t}, \quad \lambda_{i,t} p_{i,t} b'_{i,t} = \lambda_{i,t+1} p_{t+1} b'_{i,t} + \nu_{i,t} a'_{i,t}. \quad (\text{A.6b})$$

By [\(A.5\)](#), we get

$$\lambda_{i,t}(c'_{i,t} - E_{i,t}) = \lambda_{i,t}(R_t k'_{i,t} + (q_t + d_t) a'_{i,t-1} + p_t b'_{i,t-1}) - \lambda_{i,t}(k'_{i,t+1} + q_t a'_{i,t} + p_t b'_{i,t}).$$

Then, by taking the sum over t and using [\(A.6\)](#), we obtain

$$\begin{aligned} \sum_{t=0}^T \lambda_{i,t}(c'_{i,t} - E_{i,t}) &= \lambda_{i,0}(R_0 k'_{i,0} + (q_0 + d_0) a'_{i,-1} + p_0 b'_{i,-1}) - \lambda_{i,T}(k'_{i,T+1} + q_T a'_{i,T} + p_T b'_{i,T}) \\ &\quad - \sum_{t=0}^{T-1} \lambda_{i,t}(\sigma_{i,t} k'_{i,t+1} + \mu_{i,t} a'_{i,t} + \nu_{i,t} b'_{i,t}). \end{aligned}$$

Applying this formula for the allocation $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{t \geq 0}$ and using $\sigma_{i,t} k_{i,t+1} = \mu_{i,t} a_{i,t} = \nu_{i,t} b_{i,t} = 0$, we get

$$\sum_{t=0}^T \lambda_{i,t}(c_{i,t} - E_{i,t}) = \lambda_{i,0}(R_0 k_{i,0} + (q_0 + d_0) a_{i,-1} + p_0 b_{i,-1}) - \lambda_{i,T}(k_{i,T+1} + q_T a_{i,T} + p_T b_{i,T}).$$

Taking the difference between $\sum_{t=0}^T \lambda_{i,t}(c'_{i,t} - E_{i,t})$ and $\sum_{t=0}^T \lambda_{i,t}(c'_{i,t} - E_{i,t})$, we obtain

$$\begin{aligned} \sum_{t=0}^T \lambda_{i,t}(c_{i,t} - c'_{i,t}) &= \sum_{t=0}^{T-1} \lambda_{i,t}(\sigma_{i,t}k'_{i,t+1} + \mu_{i,t}a'_{i,t} + \nu_{i,t}b'_{i,t}) + \lambda_{i,T}(k'_{i,T+1} + q_t a'_{i,T} + p_t b'_{i,T}) \\ &\quad - \lambda_{i,T}(k_{i,T+1} + q_t a_{i,T} + p_t b_{i,T}) \\ &\geq -\lambda_{i,T}(k_{i,T+1} + q_t a_{i,T} + p_t b_{i,T}). \end{aligned}$$

Since u_i is concave, we have $u_i(c_{i,t}) - u_i(c'_{i,t}) \geq u'_i(c_{i,t})(c_{i,t} - c'_{i,t})$. Then,

$$\begin{aligned} \sum_{t=0}^T \left(\beta_i^t u_i(c_{i,t}) - \beta_i^t u_i(c'_{i,t}) \right) &\geq \sum_{t=0}^T \beta_i^t u'_i(c_{i,t})(c_{i,t} - c'_{i,t}) = \sum_{t=0}^T \lambda_{i,t}(c_{i,t} - c'_{i,t}) \\ &\geq -\lambda_{i,T}(k_{i,T+1} + q_t a_{i,T} + p_t b_{i,T}). \end{aligned}$$

Thanks to the transversality condition $\lim_{T \rightarrow \infty} \lambda_{i,T}(k_{i,T+1} + q_t a_{i,T} + p_t b_{i,T}) = 0$, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \geq \limsup_{T \rightarrow \infty} \sum_{t=0}^T \beta_i^t u_i(c'_{i,t})$. We have finished our proof.

Part A (necessary condition).

Step 1 (first order conditions). Let us prove the first order condition (A.3) (conditions (A.1) and (A.2) can be proved by using the same argument). To do so, it suffices to prove that (i) $\lambda_{i,t} p_t \geq \lambda_{i,t+1} p_{t+1}$ and (ii) if $b_{i,t} > 0$, then $\lambda_{i,t} p_t = \lambda_{i,t+1} p_{t+1}$.

Point (i). Fix a date t . Obviously, if $p_{t+1} = 0$, then $\lambda_{i,t} p_t \geq \lambda_{i,t+1} p_{t+1}$. Suppose now that $p_{t+1} > 0$. Then, we have $p_t > 0$. Indeed, if $p_t = 0$, then in optimal $a_{i,t} > 1$ because agent i can take $a_{i,t}$ arbitrary large to get more consumption in date $t + 1$ (because $p_{t+1} > 0$) but he/she does not need to pay any thing in date t (because $p_t = 0$). This is a contradiction because $a_{i,t} \leq \sum_j a_{j,t} = 1$.

So, we focus on the case in which $p_t > 0$ and $p_{t+1} > 0$. By Assumption 4 and $r_0 > 0$, we have $r_0 k_{i,0} + (q_0 + d_0) a_{i,-1} + p_0 b_{i,-1} + w_0 L_{i,0} + e_{i,0} > 0$. Then, by Inada's condition, we have $c_{i,t} > 0$ and $c_{i,t+1} > 0$. Consider an allocation $(c'_{i,t}, k'_{i,t+1}, a'_{i,t}, b'_{i,t})_{t \geq 0}$ defined by $(k'_{i,\tau+1}, a'_{i,\tau}) = (k_{i,\tau+1}, a_{i,\tau}) \forall \tau$, $c'_{i,\tau} = (c'_{i,\tau}) \forall \tau \in \{t, t+1\}$, $b'_{i,\tau} = b_{i,\tau} \forall \tau \neq t$, and

$$c'_{i,t} = c_{i,t} - \epsilon, \quad b'_{i,t} = b_{i,t} + \frac{\epsilon}{p_t}, \quad c'_{i,t+1} = c_{i,t+1} + \frac{p_{t+1}}{p_t} \epsilon.$$

where $\epsilon \in (0, c_{i,t})$. Clearly, this allocation is feasible. By the optimality of $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{t \geq 0}$, we have $\beta_i^t u_i(c_{i,t}) + \beta_i^{t+1} u_i(c_{i,t+1}) \geq \beta_i^t u_i(c'_{i,t}) + \beta_i^{t+1} u_i(c'_{i,t+1})$. This means that

$$\frac{u_i(c_{i,t}) - u_i(c_{i,t} - \epsilon)}{\epsilon} \geq \beta_i \frac{u_i\left(c_{i,t+1} + \frac{p_{t+1}}{p_t} \epsilon\right) - u_i(c_{i,t+1})}{\frac{p_{t+1}}{p_t} \epsilon} \frac{p_{t+1}}{p_t}.$$

Let ϵ tend to 0, we get $\lambda_{i,t}p_t \geq \lambda_{i,t+1}p_{t+1}$.

Point (ii). We now suppose $b_{i,t} > 0$, then by doing the same argument as above, but with $\epsilon < 0$ satisfying $b_{i,t} + \frac{\epsilon}{p_t} > 0$ and $c_{i,t+1} + \frac{p_{t+1}}{p_t}\epsilon > 0$, we get $\lambda_{i,t}p_t \leq \lambda_{i,t+1}p_{t+1}$. So, we obtain $\lambda_{i,t}p_t = \lambda_{i,t+1}p_{t+1}$ if $b_{i,t} > 0$.

Step 2 (transversality condition). We prove the transversality condition (A.4) by using the approach of Ekeland and Scheinkman (1986) and Kamihigashi (2000).

Fix an agent i and a date t . Denote $x_i \equiv (x_{i,s})_{s \geq 0} \equiv (k_{i,s+1}, a_{i,s}, b_{i,s})_{s \geq 0}$. For $\lambda \in (\underline{\lambda}, 1)$, define $x_i(\lambda)$ and $c_{i,t}(\lambda)$ by

$$\begin{aligned} x_i(\lambda) &= (x_{i,0}, \dots, x_{i,t-1}, \lambda x_{i,t}, \lambda x_{i,t+1}, \dots), \\ c_{i,t}(\lambda) &= c_{i,t} + (1 - \lambda)(k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t}), \quad c_{i,s}(\lambda) = c_{i,s} \quad \forall s < t, \quad c_{i,s}(\lambda) = \lambda c_{i,s} \quad \forall s > t. \end{aligned}$$

It is clear that $(c_{i,s}(\lambda), x_{i,s}(\lambda))_{s \geq 0}$ is a feasible allocation of the maximization problem of agent i . So, the optimality of $(c_{i,t}, k_{i,t+1}, a_{i,t}, b_{i,t})_{t \geq 0}$ implies that $\limsup_{T \uparrow \infty} \sum_{s=0}^T \beta_i^s (u_i(c_{i,s} - u_i(c_{i,s}(\lambda))) \geq 0$.¹⁸ Then, we obtain

$$\begin{aligned} \beta_i^t \frac{u_i(c_{i,t} + (1 - \lambda)(k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t})) - u_i(c_{i,t})}{1 - \lambda} &\leq \limsup_{T \uparrow \infty} \sum_{s=t+1}^T \beta_i^s \frac{u_i(c_{i,s}) - u_i(\lambda c_{i,s})}{1 - \lambda} \\ &\leq \limsup_{T \uparrow \infty} \left(\sum_{s=t+1}^T \beta_i^s \theta u_i(c_{i,s}) + \beta_i^s x \right), \end{aligned}$$

where we use Assumption 5 in the last inequality. Let λ increasingly tend to 1, we have

$$\beta_i^t u_i'(c_{i,t})(k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t}) \leq \limsup_{T \uparrow \infty} \left(\sum_{s=t+1}^T \beta_i^s |\theta| |u_i(c_{i,s})| + \beta_i^s |x| \right).$$

Let t tend to infinity and use $\sum_{t=0}^{\infty} \beta_i^t |u(c_{i,t})| < \infty$, the right hand side converges to zero and hence we obtain $\limsup_{t \rightarrow \infty} \beta_i^t u_i'(c_{i,t})(k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t}) \leq 0$. Since $k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t} \geq 0 \quad \forall t$, we have $\lim_{t \rightarrow \infty} \beta_i^t u_i'(c_{i,t})(k_{i,t+1} + q_t a_{i,t} + p_t b_{i,t}) = 0$. \square

¹⁸Here, " \uparrow " means "increases to".

Proof of Proposition 1. According to Lemma 3, first order conditions become

$$\frac{1}{r_{2t} + 1 - \delta} = \frac{q_{2t-1}}{q_{2t} + d_{2t}} = \frac{\beta_2 u'_2(c_{2,2t})}{u'_2(c_{2,2t-1})} \geq \frac{\beta_1 u'_1(c_{1,2t})}{u'_1(c_{1,2t-1})}, \quad (\text{A.7a})$$

$$\frac{1}{r_{2t+1} + 1 - \delta} = \frac{q_{2t}}{q_{2t+1} + d_{2t+1}} = \frac{\beta_1 u'_1(c_{1,2t+1})}{u'_1(c_{1,2t})} \geq \frac{\beta_2 u'_2(c_{2,2t+1})}{u'_2(c_{2,2t})}, \quad (\text{A.7b})$$

$$p_{2t-1} = \frac{\beta_2 u'_2(c_{2,2t})}{u'_2(c_{2,2t-1})} p_{2t} \geq \frac{\beta_1 u'_1(c_{1,2t})}{u'_1(c_{1,2t-1})} p_{2t}, \quad (\text{A.7c})$$

$$p_{2t} = \frac{\beta_1 u'_1(c_{1,2t+1})}{u'_1(c_{1,2t})} p_{2t+1} \geq \frac{\beta_2 u'_2(c_{2,2t+1})}{u'_2(c_{2,2t})} p_{2t+1}. \quad (\text{A.7d})$$

According to (9a-9c) and $\beta_1 = \beta_2 = \beta, u_1 = u_2 = u$, the inequalities in FOCs are rewritten as follows:

$$\frac{\beta u'(e_{2,2t} + R_{2t}K_{2t} + q_{2t} + d_{2t} + p_{2t})}{u'(e_{2,2t-1} + w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})} \geq \frac{\beta u'(e_{1,2t} + w_{2t} - K_{2t+1} - q_{2t} - p_{2t})}{u'(e_{1,2t-1} + R_{2t-1}K_{2t-1} + q_{2t-1} + d_{2t-1} + p_{2t-1})},$$

$$\frac{\beta u'(e_{1,2t+1} + R_{2t+1}K_{2t+1} + q_{2t+1} + d_{2t+1} + p_{2t+1})}{u'(e_{1,2t} + w_{2t} - K_{2t+1} - q_{2t} - p_{2t})} \geq \frac{\beta u'(e_{2,2t+1} + w_{2t+1} - K_{2t+2} - q_{2t+1} - p_{2t+1})}{u'(e_{2,2t} + R_{2t}K_{2t} + q_{2t} + d_{2t} + p_{2t})}.$$

Transversality conditions become

$$\lim_{t \rightarrow \infty} \beta_1^{2t} u'_1(c_{1,2t})(k_{1,2t+1} + q_{2t}a_{1,2t} + p_{2t}b_{1,2t}) = 0,$$

$$\lim_{t \rightarrow \infty} \beta_1^{2t+1} u'_1(c_{1,2t+1})(k_{1,2t+2} + q_{2t+1}a_{1,2t+1} + p_{2t+1}b_{1,2t+1}) = 0,$$

$$\lim_{t \rightarrow \infty} \beta_2^{2t} u'_2(c_{2,2t})(k_{2,2t+1} + q_{2t}a_{2,2t} + p_{2t}b_{2,2t}) = 0,$$

$$\lim_{t \rightarrow \infty} \beta_2^{2t+1} u'_2(c_{2,2t+1})(k_{2,2t+2} + q_{2t+1}a_{2,2t+1} + p_{2t+1}b_{2,2t+1}) = 0.$$

These are rewritten as follows:

$$\lim_{t \rightarrow \infty} \beta_1^{2t} u'_1(c_{1,2t})(K_{2t+1} + q_{2t} + p_{2t}) = 0, \quad (\text{A.9a})$$

$$\lim_{t \rightarrow \infty} \beta_2^{2t+1} u'_2(c_{2,2t+1})(K_{2t+2} + q_{2t+1} + p_{2t+1}) = 0. \quad (\text{A.9b})$$

Since $\beta_1 = \beta_2 = \beta, u_1 = u_2 = u$, TVCs become

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(e_{1,2t} + w_{2t} - K_{2t+1} - q_{2t} - p_{2t})(K_{2t+1} + q_{2t}) = 0, \quad (\text{A.10a})$$

$$\lim_{t \rightarrow \infty} \beta^{2t-1} u'(e_{2,2t-1} + w_{2t-1} - K_{2t} - q_{2t-1} - p_{2t-1})(K_{2t} + q_{2t-1}) = 0. \quad (\text{A.10b})$$

Remark 3. With the notations $e_{2t}^o \equiv e_{2,2t}, e_{2t+1}^o \equiv e_{1,2t+1}$ and $e_{2t}^y \equiv e_{1,2t}, e_{2t+1}^y \equiv e_{2,2t+1}$,

the inequalities in FOCs become

$$\frac{\beta u'(e_{t+1}^o + R_{t+1}K_{t+1} + q_{t+1} + d_{t+1} + p_{t+1})}{u'(e_t^y + w_t - K_{t+1} - q_t - p_t)} \geq \frac{\beta u'(e_{t+1}^y + w_{t+1} - K_{t+2} - q_{t+1} - p_{t+1})}{u'(e_t^o + R_t K_t + q_t + d_t)}.$$

□

A.2 Proofs for Section 4

Proof of Example 2. The system (20a-20d) becomes.

$$p_{t+1} = p_t R_{t+1} \geq 0, \quad (\text{A.11a})$$

$$\frac{1}{R_{t+1}} \equiv \frac{\beta u'(e_{t+1}^o + p_{t+1})}{u'(e_t^y - p_t)} \geq \frac{\beta u'(e_{t+1}^y - p_{t+1})}{u'(e_t^o + p_t)}, \quad (\text{A.11b})$$

$$\lim_{t \rightarrow \infty} \beta^{2t} u'(e_{2t}^y - p_{2t}) p_{2t} = \lim_{t \rightarrow \infty} \beta^{2t-1} u'(e_{2t-1}^y - p_{2t-1}) p_{2t-1} = 0. \quad (\text{A.11c})$$

Then, we can check these conditions under assumptions in Example 2. For instance, let us look at (A.11c). Since $u'(c) = c^{-\sigma}$, condition (A.11c) is equivalent to $\lim_{t \rightarrow \infty} \beta^t (e_t^y - p_t)^{-\sigma} p_t = 0$. This is satisfied because $p_t/e_t^y \leq 1/2$ and $\lim_{t \rightarrow \infty} \beta^t (e_t^y)^{1-\sigma} = 0$ with $\sigma > 0$. □

Proof of Example 3. According to Corollary 1, it suffices to show that the sequence $(K_{t+1}, p_t)_{t \geq 0}$ satisfies the equilibrium system

$$\begin{cases} w_{t-1} \beta^2 f'(K_t) f'(K_{t+1}) \leq w_{t+1}, \\ K_1 + p_0 = \frac{\beta}{1+\beta} w_0, \quad K_{t+1} + p_t = \gamma \alpha A K_t^\alpha \quad \forall t \geq 0, \quad \text{where } \gamma \equiv \frac{\beta}{1+\beta} \frac{1-\alpha}{\alpha}, \\ p_{t+1} = \alpha A K_{t+1}^{\alpha-1} p_t, \quad K_{t+1} > 0, p_t \geq 0. \end{cases} \quad (\text{A.12})$$

It is easy to verify the last four conditions in the system (A.12). Let us check the first condition.

Let us focus on the case 2.(b) (that is, $\gamma > 1$ and $p_0 = \bar{b}$) in Lemma 4 below. We have $K_{t+1} = \alpha A K_t^\alpha$. Since $\delta = 1$, condition (24) becomes

$$\begin{aligned} w_{t-1} \beta^2 f'(K_t) f'(K_{t+1}) &\leq w_{t+1} \\ \Leftrightarrow (1-\alpha) A K_{t-1}^\alpha \beta^2 \alpha A K_t^{\alpha-1} \alpha A K_{t+1}^{\alpha-1} &\leq (1-\alpha) A K_{t+1}^\alpha \Leftrightarrow \beta^2 \leq \frac{K_{t+1}}{\alpha A K_t^\alpha} \frac{K_t}{\alpha A K_{t-1}^\alpha} = 1, \end{aligned}$$

which is satisfied because $\beta < 1$

□

The following result completes the proof of Example 3.

Lemma 4. Let K_0 be exogenous and strictly positive. Consider the following system:

$$K_1 + b_0 = \frac{\beta}{1 + \beta} w_0, \quad K_{t+1} + p_t = \gamma \alpha A K_t^\alpha \quad \forall t \geq 0, \quad \text{where } \gamma \equiv \frac{\beta}{1 + \beta} \frac{1 - \alpha}{\alpha}, \quad (\text{A.13a})$$

$$p_{t+1} = \alpha A K_{t+1}^{\alpha-1} p_t, \quad (\text{A.13b})$$

$$K_{t+1} > 0, p_t \geq 0. \quad (\text{A.13c})$$

1. If $\gamma \leq 1$ (i.e., $f'(K^*) \geq 1$), the system (A.13) has a unique solution given by

$$p_t = 0, \quad K_t = \rho^{\frac{1-\alpha t}{1-\alpha}} K_0^{\alpha t} \quad \forall t \geq 1, \quad (\text{A.14})$$

where $\rho \equiv \gamma \alpha A$. Moreover, $\lim_{t \rightarrow \infty} K_t = K^*$.

2. If $\gamma > 1$ (i.e., $f'(K^*) < 1$), the system (A.13) is indeterminate: The set of solutions is any sequence $(K_{t+1}, p_t)_{t \geq 0}$ defined by (A.13a), (A.13b), and $p_0 \in [0, \bar{b}]$, where the so-called bubble critical value \bar{b} is defined by

$$\bar{b} \equiv w_0 \frac{\beta}{1 + \beta} \frac{\gamma - 1}{\gamma} = (1 - \alpha) A K_0^\alpha \left[1 - \frac{1 + \alpha \beta}{(1 - \alpha)(1 + \beta)} \right], \quad (\text{A.15})$$

which is positive because $\gamma > 1$.

Moreover, the following properties hold.

(a) (bubbleless solution) If $p_0 = 0$, then $p_t = 0 \quad \forall t$. The sequence $(K_t)_{t \geq 1}$ is determined by (A.14).

(b) (bubbly solution) If $p_0 > 0$, then $p_t > 0 \quad \forall t$.

When $p_0 < \bar{b}$, we have $\lim_{t \rightarrow \infty} p_t = 0$ and $\lim_{t \rightarrow \infty} K_t = K^*$.

When $p_0 = \bar{b}$, we have $\lim_{t \rightarrow \infty} p_t > 0$. Precisely, we have

$$p_t = (\gamma - 1) K_{t+1} \quad \forall t \geq 0, \quad K_t = \rho_1^{\frac{1-\alpha t}{1-\alpha}} K_0^{\alpha t} \quad \forall t \geq 1, \quad (\text{A.16})$$

where $\rho_1 \equiv \alpha A$. Moreover, $\lim_{t \rightarrow \infty} K_t = (\alpha A)^{1/(1-\alpha)} < K^*$ and $\lim_{t \rightarrow \infty} p_t = (\gamma - 1)(\alpha A)^{1/(1-\alpha)} > 0$.

Proof of Lemma 4. The proof presented here is similar to the proof in the literature (see Proposition 4 in Bosi et al. (2018) among others).

If $p_0 > 0$, then we have $p_t > 0, \forall t$. Combining (A.13a) and (A.13b), we have

$$\frac{K_{t+1}}{p_t} + 1 = \frac{\gamma \alpha A K_t^\alpha}{p_t} = \frac{\gamma \alpha A K_t^\alpha}{\alpha A K_t^{\alpha-1} p_{t-1}} = \gamma \frac{K_t}{p_{t-1}} \quad \forall t \geq 1. \quad (\text{A.17})$$

Denote $z_t \equiv K_{t+1}/p_t \forall t \geq 0$. We get a difference equation: $z_{t+1} = \gamma z_t - 1 \forall t \geq 0$. If $\gamma \neq 1$, the solution of this difference equation must satisfy

$$z_t = \gamma^t z_0 - \frac{1 - \gamma^t}{1 - \gamma} \forall t \geq 1$$

We now consider different cases.

1. When $\gamma \leq 1$, there is no bubble. Indeed, suppose that there is a pure bubble. Since $\gamma \leq 1$, condition $z_{t+1} = \gamma z_t - 1$ implies that z_t becomes negative soon or later: this leads to a contradiction. In this case, capital transition becomes $k_{t+1} = \rho k_t^\alpha$, where $\rho \equiv \gamma \alpha A$. Solving recursively, we find the explicit solution (A.14).

2. Let $\gamma > 1$. If $p_t = 0$, then (A.14) follows immediately.

If $p_t > 0$. Then, we obtain

$$z_t = \frac{[(\gamma - 1)z_0 - 1]\gamma^t + 1}{\gamma - 1}. \quad (\text{A.18})$$

A positive solution exists if and only if $z_0 \geq 1/(\gamma - 1)$. Hence, the existence of a positive solution requires

$$p_0 \leq (\gamma - 1)K_1 = (\gamma - 1) \left[\frac{\beta}{1 + \beta} w_0 - p_0 \right].$$

Solving this inequality for p_0 , we find $0 < p_0 \leq \bar{b}$.

We now observe that for $p_0 \in (0, \bar{b}]$ given, the sequence (K_{t+1}, p_t) constructed by (A.13a) and (A.13b) is a solution with $p_t > 0$ for any t .

When $p_0 < \bar{b}$ (that is $z_0 > 1/(\gamma - 1)$), thanks to (A.18), we get $\lim_{t \rightarrow \infty} z_t = \infty$. According to (A.13a), K_t is uniformly bounded from above, which implies that $\lim_{t \rightarrow \infty} p_t = 0$. Thus, $\lim_{t \rightarrow \infty} K_t = K^*$.

When $p_0 = \bar{b}$, we have $z_t = 1/(\gamma - 1) \forall t \geq 0$. In this case, $k_{t+1} = \rho_1 k_t^\alpha$, where $\rho_1 \equiv \alpha A$, and $p_t = (\gamma - 1)k_{t+1}$. Solving recursively, we get (A.16). □

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