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Clarence Kineider, Roméo Troubat. Connected components of the space of triples of transverse partial flags in $SO(p,q)$ and Anosov representations. 2024. hal-04775698

HAL Id: hal-04775698

<https://hal.science/hal-04775698v1>

Preprint submitted on 12 Nov 2024

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CONNECTED COMPONENTS OF THE SPACE OF TRIPLES OF TRANSVERSE PARTIAL FLAGS IN $SO_0(p, q)$ AND ANOSOV REPRESENTATIONS

CLARENCE KINEIDER AND ROMÉO TROUBAT

ABSTRACT. We count the number of connected components in the space of triples of transverse flags in any flag manifold of $SO_0(p, q)$. We compute the effect the involution of the unipotent radical has on those components and deduce that for certain parabolic subgroups P_Θ , any P_Θ -Anosov subgroup is virtually isomorphic to either a surface group of a free group. We give examples of Anosov subgroups which are neither free nor surface groups for some sets of roots which do not fall under the previous results. As a consequence of the methods developed here, we get an explicit algorithm based on computation of minors to check if a unipotent matrix in $SO_0(p, q)$ belong to the Θ -positive semigroup $U_\Theta^{>0}$ when $p \neq q$.

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Key words and phrases. Anosov representations, Θ -positivity, Total positivity, Flag manifolds, Einstein universe.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 101018839), as well as from the École Normale Supérieure.

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INTRODUCTION

In his celebrated paper ([Lab06]), Labourie introduced Anosov representations of a surface group as a way to generalize Hitchin representations, introduced by Hitchin in [Hit92]. The notion of Anosov representations was then expanded to any hyperbolic group by Guichard-Wienhard in [GW12]. Known examples of Borel-Anosov representations include representations of surface groups and of free groups, for instance through the inclusion of co-compact lattices and Schottky groups of $SL(2, \mathbb{R})$ via the irreducible representation $SL(2, \mathbb{R}) \rightarrow SL(d, \mathbb{R})$. A question was then asked by Andrés Sambarino regarding the abstract structure of groups admitting Borel-Anosov representations in $SL(d, \mathbb{R})$.

Question. *Let Γ be a Borel Anosov subgroup of $SL(d, \mathbb{R})$. Is Γ virtually isomorphic to either a free group of a surface group ?*

This question was answered positively for certain values of d , first by Canary-Tsouvalas and Tsouvalas in [CT20], [Tso20] for $d = 3, 4$ and $d = 2 \pmod{4}$, then by Dey in [Dey24] for $d \neq 5$ and $d \neq \pm 1 \pmod{8}$.

For F_0 and F_∞ two transverse elements in the space of full flags $SL(d, \mathbb{R})/B$, where B is the Borel subgroup of $SL(d, \mathbb{R})$, the space $\Omega(F_\infty)$ is parametrised by the unipotent radical U of B via the map $g \in U \mapsto g \cdot F_0$. This parametrisation induces a natural involution defined by $i(g \cdot F_0) = g^{-1} \cdot F_0$ which preserves transversality with F_0 , thus inducing a map $i : \Omega(F_0) \cap \Omega(F_\infty)$. In [Dey24], Dey uses this involution to obtain a bound on the dimension of boundaries of Borel-Anosov subgroups of $SL(d, \mathbb{R})$. In [DGR23], Dey-Greenberg-Riestenberg generalize this method to any pair (G, P_Θ) where G is a semi-simple Lie group and $P_\Theta \subset G$ is a parabolic group such that G/P_Θ is self-opposite, thus obtaining the following theorem :

Theorem A (Dey-Greenberg-Riestenberg, [DGR23]). *Assume that*

$$i : \pi_0(\Omega(F_0) \cap \Omega(F_\infty)) \rightarrow \pi_0(\Omega(F_0) \cap \Omega(F_\infty))$$

does not have any fixed point. Then any P_Θ -Anosov subgroup of G is virtually isomorphic to either a free group or a surface group.

They then apply this result to $G = Sp(2n, \mathbb{R})$, obtaining that any P_Θ -Anosov subgroup of $Sp(2n, \mathbb{R})$ is virtually isomorphic to either a surface group or a free group when Θ contains an odd root. Our goal in this paper will be to apply this theorem when possible when $G = SO_0(p, q)$ is the connected component of the identity in the group $SO(p, q)$ of isometries preserving a quadratic form of signature (p, q) , and when P_Θ is any parabolic subgroup of $SO_0(p, q)$. The first step in this endeavor will be to count the number of connected components of $\Omega(F_0) \cap \Omega(F_\infty)$, the space of triples of transverse flags up to the action of $SO_0(p, q)$, as well as the action of the involution i .

Theorem B. *The number of connected components of $\Omega(F_0) \cap \Omega(F_\infty) \subset SO_0(p, q)/P_\Theta$ is as follows :*

- *If $p > q$, Θ does not contain q and is not equal to $\{1, \dots, q-1\}$, there are $2^{|\Theta|}$ connected components, all of which are stable by i .*
- *If $q = 2$, $p > 2$, and $\Theta = \{1\}$, there are 3 connected components, two of which are Θ -positive. One of them is stable by i while the two Θ -positive are exchanged.*

- If $q = 3$, $p > 3$ and $\Theta = \{1, 2\}$, there are 11 connected components, four of which are Θ -positive. None of the Θ -positive components are stable by i , and among the remaining seven, three of them are stable by i and four are unstable.
- If $q > 3$, $p > q$, and $\Theta = \{1, \dots, q-1\}$, there are $3 \times 2^{q-1}$ connected components, 2^{q-1} of which are Θ -positive. None of the Θ -positive components are stable by i while among the remaining 2^q components, 2^{q-1} are stable and 2^{q-1} are not.
- If $p > q+1$, Θ contains q and is not equal to $\{1, \dots, q\}$, there are $2^{|\Theta|-1}$ connected components, all of which are stable by i .
- If $p > q+1$ and $\Theta = \{1, \dots, q\}$, there are the same number of connected components as for $\Theta = \{1, \dots, q-1\}$ with the same effect for the involution.
- If $p = q+1$, Θ contains q and is not equal to $\{1, \dots, q\}$, there are $2^{|\Theta|}$ connected components. When $q = 1$ or $2 \pmod 4$, none of them are stable by i . When $q = 3$ or $0 \pmod 4$, all of them are stable by i .
- If $q = 2$, $p = 3$ and $\Theta = \{1, 2\}$, there are 8 connected components, four of which are totally positive. None of them is stable by i .
- If $q = 3$, $p = 4$ and $\Theta = \{1, 2, 3\}$, there are 30 connected components, 8 of which are totally positive. Six of them are stable by i while the other 24 are unstable.
- If $p = q+1$ and $\Theta = \{1, \dots, q\}$, there are $(q+5)2^{q-1}$ connected components, 2^q of which are totally positive. When $q = 1$ or $2 \pmod 4$, none of them are stable by i . When $q = 3$ or $0 \pmod 4$, 2^q are stable by i and $(q+1)2^{q-1} + 2^q$ are not.
- $p = q$ and Θ does not contain any of the last two roots, there are $2^{|\Theta|}$ connected components, all of which are stable.
- If $p = q$, q is even and Θ contains only one of the two last roots, there are 2 connected components. When $q = 0 \pmod 4$, they are stable by i while when $q = 2 \pmod 4$, they are exchanged.
- If $p = q$ and q is odd, the parabolics P_q and P'_q associated to the last two roots are not self-opposite.

When $p = q+1$ the group $\mathrm{SO}(p, q)$ is split and when $\Theta = \{1, \dots, q\}$, P_Θ is its Borel subgroup. In that case, the number of connected components in $\Omega(F_0) \cap \Omega(F_\infty)$ was already known from work of Gekhtman-Shapiro-Vainshtein [GSV03].

Combining those results with theorem 4.2, we obtain the following theorems:

Theorem C. *Assume that $q = 1$ or $q = 2 \pmod 4$ and let Γ be a P_Θ -Anosov subgroup of $\mathrm{SO}_0(q+1, q)$ containing the last root. Then Γ is virtually isomorphic to either a free group or a surface group.*

Theorem D. *Assume that $q = 2 \pmod 4$ and let Γ be a P_Θ -Anosov subgroup of $\mathrm{SO}_0(q, q)$ containing one of the last two roots. Then Γ is virtually isomorphic to either a free group or a surface group.*

The groups $\mathrm{SO}_0(q+1, q)$ and $\mathrm{SO}_0(q, q)$ are split. When taking P_Θ to be the Borel subgroups of either of those two, the two theorem give the same result as [Dey24] via the inclusions $\mathrm{SO}_0(q+1, q) \subset \mathrm{SL}(2q+1, \mathbb{R})$ and $\mathrm{SO}_0(q, q) \subset \mathrm{SL}(2q, \mathbb{R})$.

We show that to determine the connected component of $\Omega(F_0) \cap \Omega(F_\infty)$ to which a given flag $g \cdot F_0$ belong to, one only has to check the signs of some explicit minors of the matrix g . In particular, we give an explicit algorithm to check whether a matrix in $\mathrm{SO}_0(p, q)$ is Θ -positive in the sense of Guichard-Wienhard in [GW18]. This algorithm yield a parametrization of the Θ -positive semigroup $U_\Theta^{\geq 0}$ that is different from the one given in [GW18].

Finally, we give an example of a P_Θ -Anosov subgroup in $\mathrm{SO}_0(p, q)$ for $\Theta = \{1, \dots, q-2\}$ which is virtually isomorphic neither to a surface group nor to a free group.

Theorem E. *There exists a $P_{1,\dots,q-2}$ -Anosov subgroup of $\mathrm{SO}_0(p, q)$ which is isomorphic to the free product of a surface group by \mathbb{Z} .*

In section 1, we define and give a parametrization of the various flag manifolds associated to $\mathrm{SO}_0(p, q)$, as well as recall basic terminology of the geometry of the pseudo-Riemannian Einstein universe. In section 2, we establish the equations characterizing transversality to a given point in an affine chart of any flag manifold, both via a geometric approach and an computational approach. In section 3, we count the number of connected components in $\Omega(F_0) \cap \Omega(F_\infty)$. In section 4, we compute the action of the involution on those connected components and establish theorems C and D. In section 5, we construct the Anosov subgroups of theorem E.

ACKNOWLEDGMENTS

We would like to thank Olivier Guichard, Subhadip Dey and Max Riestenberg for their helpful comments during the preparation of this article. We are also grateful towards Olivier Schiffmann for organizing the math camp during which most of the ideas presented here arose.

1. PARABOLIC SPACES OF $\mathrm{SO}_0(p, q)$

1.1. Flag varieties of $\mathrm{SO}_0(p, q)$. Let $p \geq q \geq 0$ two non-negative integers with $p > 0$, and let $(e_1, \dots, e_q, x_{q+1}, \dots, x_{p-q}, \tilde{e}_q, \dots, \tilde{e}_1)$ be a basis of \mathbb{R}^{p+q} . Let $Q_{p,q}$ be the non-degenerate quadratic form on \mathbb{R}^{p+q} of signature (p, q) which matrix in the basis \mathcal{B} is

$$M_{p,q} = \begin{pmatrix} 0 & 0 & J \\ 0 & I_{p-q} & 0 \\ J & 0 & 0 \end{pmatrix}$$

where is the following q -by- q square matrix:

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

When the context is clear we will write Q instead of $Q_{p,q}$, and we will denote by $B_{p,q}$ (or just B) the associated bilinear form, and for any subspace F of $\mathbb{R}^{p,q}$, we will denote by F^\perp its orthogonal:

$$F^\perp = \{v \in \mathbb{R}^{p,q} \mid \forall w \in F, B(v, w) = 0\}.$$

A subspace F will be called *isotropic* if $Q|_F = 0$. In particular, note that $\mathrm{Span}(e_1, \dots, e_q)$ is a maximal isotropic subspace and that the quadratic form Q restricted to $\mathrm{Span}(x_{q+1}, \dots, x_{p-q})$ is positive definite.

We denote by $\mathbb{R}^{p,q}$ the space \mathbb{R}^{p+q} endowed with the quadratic form $Q_{p,q}$. The group $\mathrm{SO}_0(p, q)$ is the group of orientation-preserving isometries of $\mathbb{R}^{p,q}$, and $\mathrm{SO}_0(p, q)$ will denote the identity component of $\mathrm{SO}_0(p, q)$. The Lie algebra of $\mathrm{SO}_0(p, q)$ is the space

$$\mathfrak{so}(p, q) = \{A \in \mathcal{M}_{p+q}(\mathbb{R}) \mid A^T M_{p,q} + M_{p,q} A = 0\}.$$

For any non-negative integer $k \geq 0$, a totally isotropic subspace of $\mathbb{R}^{p,q}$ of dimension $k + 1$ will be called a k -*photon*. Sometime a 1-photon will simply be called a *photon*, and a 0-photon will be called a *point*. A vector $v \in \mathbb{R}^{p,q}$ will be called *timelike* if $Q(v) < 0$, *spacelike* if $Q(v) > 0$ and *lightlike* if $Q(v) = 0$. The set of all lightlike vector form a cone, called the *lightcone* (of the origin), denoted by $C(0)$. Given $v \in \mathbb{R}^{p,q}$, the lightcone of v is the translation of $C(0)$:

$$C(v) = \{w \in \mathbb{R}^{p,q} \mid Q(w - v) = 0\}.$$

The lightcone $C(0)$ divide $\mathbb{R}^{p,q}$ into two parts: the set of timelike vectors and the set of spacelike vectors. The set of spacelike vectors is a connected component of $\mathbb{R}^{p,q} \setminus C(0)$ if $p > 1$, and has two connected component if $p = 1$. The set of timelike vectors is a connected component of $\mathbb{R}^{p,q} \setminus C(0)$ if $q > 1$, is empty if $q = 0$, and has two connected components if $q = 1$. When the space of timelike vectors has two connected components, we will call a vector in the component containing e_1 of *future type* and a vector in the component of $-e_1$ of *past type*.

Definition 1.1. Let $k \geq 0$ be a non-negative integer and let $\mathbf{i} = (i_1, \dots, i_k)$ be a k -tuple of integers satisfying $0 \leq i_1 < \dots < i_k \leq q$. A \mathbf{i} -flag in $\mathbb{R}^{p,q}$ is a k -tuple $F = (F^{i_1}, \dots, F^{i_k})$ such that for all $1 \leq \ell \leq k$, F^{i_ℓ} is a $(i_\ell - 1)$ -photon and for all $1 \leq \ell \leq k - 1$, $F^{i_\ell} \subset F^{i_{\ell+1}}$. The space of all \mathbf{i} -flag of $\mathbb{R}^{p,q}$, called the \mathbf{i} -flag variety, will be denoted by $\mathcal{F}^{\mathbf{i}}$.

When $p \neq q$ and $\mathbf{i} = (1, 2, \dots, q)$, we will call $\mathcal{F}^{\mathbf{i}}$ the *full flag variety*.

Remark 1.2. When $p = q$, every $(q - 2)$ -photon is contained in exactly 2 distinct $(p - 1)$ -photons, and conversely any pair of distinct $(p - 1)$ -photons determine a unique $(q - 2)$ -photon given by their intersection. Hence a full flag in $SO(p, p)$ is an $(1, \dots, p - 1)$ -flag.

Let $\mathbf{i} = (i_1, \dots, i_k)$, we define the two *standard \mathbf{i} -flags* $F_0^{\mathbf{i}}$ and $F_\infty^{\mathbf{i}}$, or just F_0 and F_∞ when the context is clear, as follows:

$$F_0^{\mathbf{i}} = (\text{Span}(\tilde{e}_1, \dots, \tilde{e}_{i_\ell}))_{1 \leq \ell \leq k}$$

and

$$F_\infty^{\mathbf{i}} = (\text{Span}(e_1, \dots, e_{i_\ell}))_{1 \leq \ell \leq k}.$$

The group $SO(p, q)$ acts smoothly and transitively on $\mathcal{F}^{\mathbf{i}}$, and the stabilizer of $F_0^{\mathbf{i}}$ (resp. $F_\infty^{\mathbf{i}}$) is a parabolic subgroup denoted by $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}}^{opp}$). Thus, the space $\mathcal{F}^{\mathbf{i}}$ is diffeomorphic to $SO(p, q)/P_{\mathbf{i}}$ which is a smooth manifold. When $\mathbf{i} = (1, 2, \dots, q)$, $P_{\mathbf{i}}$ is a Borel subgroup of $SO(p, q)$. The subgroup $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}}^{opp}$) is the intersection of $SO(p, q)$ with the set of block-lower triangular (resp. upper triangular) matrices with the square blocks on the diagonal of size $(i_1, i_2 - i_1, \dots, i_k - i_{k-1}, p - q + 2(q - i_k), i_k - i_{k-1}, \dots, i_2 - i_1)$. The intersection $L_{\mathbf{i}} = P_{\mathbf{i}} \cap P_{\mathbf{i}}^{opp}$ is the subgroup of block-diagonal matrices with block sizes $(i_1, i_2 - i_1, \dots, i_k - i_{k-1}, p - q + 2(q - i_k), i_k - i_{k-1}, \dots, i_2 - i_1)$ and is called the Levi subgroup of $P_{\mathbf{i}}$. Let $U_{\mathbf{i}}$ be the unipotent radical of $P_{\mathbf{i}}^{opp}$ and let $\mathfrak{u}_{\mathbf{i}}$ be the Lie algebra of $U_{\mathbf{i}}$. When $\mathbf{i} = (1, 2, \dots, k - 1, k)$, we will write an element of $\mathfrak{u}_{\mathbf{i}}$ in the following way:

$$(1.1) \quad \left(\begin{array}{cccc|ccc|cccc} 0 & -a_1^{k-1} & -a_1^{k-2} & \cdots & -a_1^1 & \bar{v}_1^0 & -b_1^1 & \cdots & -b_1^{k-2} & -b_1^{k-1} & 0 \\ & 0 & -a_2^{k-2} & \cdots & -a_2^1 & \bar{v}_2^0 & -b_2^1 & \cdots & -b_1^{k-1} & 0 & b_1^{k-1} \\ & & 0 & \ddots & \vdots & \vdots & \vdots & \ddots & 0 & b_2^{k-2} & b_1^{k-2} \\ & & & \ddots & -a_{k-1}^1 & \bar{v}_{k-1}^0 & -b_{k-1}^1 & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & \bar{v}_k^0 & 0 & b_{k-1}^1 & \cdots & b_2^1 & b_1^1 \\ \hline & & & & & 0_{p+q-2k} & v_k^0 & v_{k-1}^0 & \cdots & v_2^0 & v_1^0 \\ \hline & & & & & & 0 & a_{k-1}^1 & \cdots & a_2^1 & a_1^1 \\ & & & & & & & \ddots & \ddots & \vdots & \vdots \\ & & & & & & & & 0 & a_2^{k-2} & a_1^{k-2} \\ & & & & & & & & & 0 & a_1^{k-1} \\ & & & & & & & & & & 0 \end{array} \right)$$

where all $(a_j^i)_{1 \leq j \leq k, 1 \leq i \leq k-j}$ and $(b_j^i)_{1 \leq j \leq k, 1 \leq i \leq k-j}$ are real numbers, and the $(v_j^0)_{1 \leq j \leq k}$ are (column) vectors in $\mathbb{R}^{p-k, q-k}$. The line vectors \bar{v}_j^0 are defined such that $\bar{v}_j^0 v_j^0 = -Q(v_j^0)$. For any other choice of $\mathbf{i} = (i_1, \dots, i_k)$, \mathbf{u}_i is a subspace of $\mathbf{u}_{1, \dots, i_k}$ where a subset of the a_j^i vanish. For any $1 \leq j \leq k$ and for any $0 \leq i \leq k-j$ we define

$$v_j^i = (b_j^i, b_j^{i-1}, \dots, b_j^1, v_j^0, a_j^1, \dots, a_j^{i-1}, a_j^i) \in \mathbb{R}^{p-k+i-1, q-k+i-1}.$$

We will also sometime denote the first (resp. the last) coordinate of v_j^0 by b_j^0 (resp. a_j^0), and the vector obtained by removing the first and the last entries of v_j^0 by v_j^{-1} . For $1 \leq j \leq k$, let $u_j(v_j^{k-j})$ be the square matrix of size $p+q$ whose j -th line is $(0, \dots, 0, -a_j^{k-j}, \dots, -a_j^1, \bar{v}_j^1, -b_j^1, \dots, -b_j^{k-j}, 0, \dots, 0)$ and whose $(p+q+1-j)$ -th column is $(0, \dots, 0, b_j^{k-j}, \dots, b_j^1, v_j^1, a_j^1, \dots, a_j^{k-j}, 0, \dots, 0)$, with all other entries being 0. A matrix in \mathbf{u}_i then uniquely writes as

$$u = u_1(v_1^{k-1}) + u_2(v_2^{k-2}) + \dots + u_k(v_k^0).$$

Proposition 1.3. *The map*

$$\Psi_{\mathbf{i}} : \begin{array}{ccc} \mathbf{u}_i & \rightarrow & U_{\mathbf{i}} \\ u = u_1(v_1^{k-1}) + \dots + u_k(v_k^0) & \mapsto & U = \exp(u_1(v_1^{k-1})) \dots \exp(u_k(v_k^0)) \end{array}$$

is a diffeomorphism.

Proof. First note that for all $1 \leq j \leq k$, the subspace

$$V_j = \left\{ u_i(v_j^{k-j}) \mid v_j^{k-j} \in \mathbb{R}^{p-j, q-j} \right\}$$

is an abelian Lie subalgebra of \mathbf{u}_i . Let $f_j = (f_j^1, \dots, f_j^{p+q-2j})$ be a basis of V_j . Then if $v_j^{k-j} = \sum \lambda_i f_j^i$, we have

$$\exp(u_j(v_j^{k-j})) = \exp(\lambda_1 f_j^1) \dots \exp(\lambda_{p+q-2j} f_j^{p+q-2j}).$$

Let $f = f_1 \cup \dots \cup f_k$ be the corresponding basis of \mathbf{u}_i as a vector space. This basis has the property that for any $1 \leq i \leq \dim \mathbf{u}_k$, the first i vectors of f span a subalgebra of \mathbf{u}_k . Then by [CG90] Prop. 1.2.8, the map $\Psi_{\mathbf{i}}$ is a diffeomorphism. \square

1.2. Transversality.

Definition 1.4. Let $\mathbf{i} = (i_1, \dots, i_k)$. Two flags $F_1, F_2 \in \mathcal{F}^{\mathbf{i}}$ are called *transverse* if for all $1 \leq \ell \leq k$, $F_1^{i_\ell} \oplus (F_2^{i_\ell})^\perp = \mathbb{R}^{p,q}$.

The two standard flags F_0 and F_∞ are transverse. Given a flag $F \in \mathcal{F}^{\mathbf{i}}$, we will denote by $\Omega(F) \subset \mathcal{F}^{\mathbf{i}}$ the subset of flags that are transverse to F .

Proposition 1.5. *Let $\mathbf{i} = (i_1, \dots, i_k)$. For any flag $F \in \Omega(F_\infty^{\mathbf{i}})$ transverse to $F_\infty^{\mathbf{i}}$ there exists a unique $U \in U_{\mathbf{i}}$ such that $F = U \cdot F_0^{\mathbf{i}}$. Thus $\Omega(F_\infty^{\mathbf{i}})$ is diffeomorphic to $U_{\mathbf{i}}$, hence diffeomorphic to \mathbf{u}_i .*

Let $U \in U_{\mathbf{i}}$. The flag $F = U \cdot F_0$ is transverse to F_∞ and we now want to express under which conditions it is transverse to F_0 . Since for any $i \in \mathbf{i}$ we have $F_0^i = \text{Span}(\tilde{e}_1, \dots, \tilde{e}_i)$, its orthogonal is

$$(F_0^i)^\perp = \text{Span}(\tilde{e}_1, \dots, \tilde{e}_q, x_{q+1}, \dots, x_{p-q}, e_q, \dots, e_{i-1}).$$

For $F^i = U \cdot F_0^i$ to be transverse to F_0^i , the projection of F^i to $\text{Span}(e_1, \dots, e_i)$ parallel to $(F_0^i)^\perp$ must be surjective. This corresponds to the matrix $U \in U_{\mathbf{i}}$ having its $i \times i$ upper right minor to be non-zero. We will denote this minor by $\det_i(U)$ in the following. We thus obtain the first description of $\Omega(F_0^{\mathbf{i}}) \cap \Omega(F_\infty^{\mathbf{i}})$:

$$\Omega(F_0^{\mathbf{i}}) \cap \Omega(F_\infty^{\mathbf{i}}) \stackrel{\text{homeo}}{\simeq} \{U \in U_{\mathbf{i}} \mid \forall i \in \mathbf{i}, \det_i(U) \neq 0\}.$$

1.3. Einstein universe. Let $p \geq q \geq 1$. The space $\text{SO}_0(p, q)/P_1$ of photons in $\mathbb{R}^{p, q}$ is called *Einstein universe* and will be denoted by $\text{Ein}_{p-1, q-1}$. This is a smooth manifold of dimension $p + q - 2$ embedded in $\mathbb{R}P^{p+q-1}$, on which the quadratic form Q induces a conformal class of pseudo-Riemannian metric of signature $(p - 1, q - 1)$. When $q = 1$ the Einstein universe $\text{Ein}_{p-1, 0}$ is identified the boundary of the hyperbolic space \mathbb{H}^{p-1} which is a sphere \mathbb{S}^{p-2} with its conformal structure. When $q = 2$, the space $\text{Ein}_{p-1, 1}$ is called *Lorentzian*. Since Einstein universe only has a conformal structure, only the sign of the quadratic form on a tangent vector make sense. In particular, the type (time, space or light) of a tangent vector is well defined, and so is the lightcone of a point. The lightcone of a point $x \in \text{Ein}_{p-1, q-1}$ is given by:

$$C(x) = \mathbb{P}_{\text{Iso}}(x^\perp) = \mathbb{P}\{v \in x^\perp \mid Q(v) = 0\}.$$

The complement in $\text{Ein}_{p-1, q-1}$ of any lightcone is an affine space called *Minkowski space*, denoted by $\text{Mink}_{p-1, q-1}$. A choice of any point in $\text{Mink}_{p-1, q-1} = \text{Ein}_{p-1, q-1} \setminus C(x)$ identifies it with $\mathbb{R}^{p-1, q-1}$. The choice of any two points $x, x' \in \text{Ein}_{p-1, q-1}$ such that $x' \notin C(x)$ defines a (unique up to linear isometry) *affine chart* $\text{Ein}_{p-1, q-1} \setminus C(x) \simeq \mathbb{R}^{p-1, q-1}$ in which x' is the origin.

Proposition 1.6. *Let $q \geq 2$. Let $x, x' \in \text{Ein}_{p-1, q-1}$, with $x' \notin C(x)$. The space of photons passing through x is identified with $C(x) \cap C(x')$ which is conformally equivalent to $\text{Ein}_{p-2, q-2}$.*

Proof. The set of photons passing through x is the set of isotropic planes in $\mathbb{R}^{p, q}$ containing the line x . Let Δ be a photon passing through x . Since $x \notin C(x')$, Δ is not contained in x'^\perp . However since $\dim x'^\perp = p + q - 1$, the intersection of Δ and x'^\perp is an isotropic line in $x^\perp \cap x'^\perp$, hence defines a point in $C(x) \cap C(x')$. Conversely, any point $y \in C(x) \cap C(x')$ give rise to a unique photon passing through x defined by $\Delta = x \oplus y$. Since the quadratic form Q restricted to the space $x^\perp \cap x'^\perp$ is of signature $(p - 1, q - 1)$, the set $C(x) \cap C(x')$ of isotropic lines in $x^\perp \cap x'^\perp$ is identified with $\text{Ein}_{p-2, q-2}$. \square

2. EQUATIONS OF TRANSVERSALITY

2.1. Pointed photons. Let $p \geq q \geq 2$. We will consider the space $\mathcal{F} = \mathcal{F}^{1,2}$ of pointed photons in $\mathbb{R}^{p, q}$. Let $F_0 = (x_0, \Delta_0)$ and $F_\infty = (x_\infty, \Delta_\infty)$ be the two standard flags defined in Section 1.2. From Proposition 1.5, any pointed photon $F = (x, \Delta) \in \mathcal{F}$ transverse to F_∞ can be written as $U \cdot (x_0, \Delta_0)$ where

$$U = \exp \begin{pmatrix} 0 & -a & \bar{v}_1 & -b & 0 \\ & 0 & 0 & 0 & b \\ & & 0 & 0 & v_1 \\ & & & 0 & a \\ & & & & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & \bar{v}_2 & 0 & 0 \\ & & 0 & v_2 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \\ = \begin{pmatrix} 1 & -a & \bar{v}_1 - a\bar{v}_2 & -b - \frac{a}{2}Q(v_2) + B(v_1, v_2) & -\frac{Q(w_1)}{2} \\ & 1 & \bar{v}_2 & -\frac{Q(v_2)}{2} & b \\ & & I & v_2 & v_1 \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}.$$

Compared to the parametrization 1.1, to alleviate the notations we will write $v_j \in \mathbb{R}^{p-2, q-2}$ instead of v_j^0 for $j = 1, 2$, a (resp. b) instead of a_1^1 (resp. b_1^1) and $w_1 = v_1^1 = (b, v_1, a) \in \mathbb{R}^{p-1, q-1}$. For the flag F to be transverse to F_0 , we first need to have $x \oplus x_0^\perp = \mathbb{R}^{p, q}$. In other words, x needs not to lie on the lightcone of x_0 in $\text{Ein}_{p-1, q-1}$. Since the coordinates of x in the affine chart defined by x_∞ and x_0 are precisely given by w_1 , the transversality condition is $Q(w_1) \neq 0$ which is equivalent to $\det_1(U) = -\frac{Q(w_1)}{2} \neq 0$. The second transversality condition is given by

$\det_2(U) \neq 0$. We can interpret this condition geometrically as follows: there is a unique photon Δ'_0 passing through x and intersecting Δ_0 since Δ_0 intersect $C(x)$ in one point, which is given by $\text{Span}(b\tilde{e}_1 + Q(w_1)\tilde{e}_2)$. Both Δ and Δ'_0 are photons passing through x , so we can see them as two points in $E = C(x) \cap C(x_\infty) \simeq \text{Ein}_{p-2, q-2}$. In this space, the transversality of photons means that $\Delta \cap C(x_\infty)$ must not lie on the lightcone of $\Delta'_0 \cap C(x_\infty)$. The point $\Delta'_0 \cap C(x_\infty) \in E$ is on the lightcone of $\Delta_\infty \cap E$ precisely when $Q(v_1) = 0$. Otherwise, it lies in the affine chart $E \setminus C(\Delta_\infty \cap E)$. When $Q(v_1) \neq 0$, we use the affine chart $\mathbb{R}^{p-2, q-2} = E \setminus C(\Delta_\infty \cap E)$ with origin $\Delta'_0 \cap E$. In this chart, the coordinate of $\Delta \cap E$ is $v_2^{(1)} = v_2 + \frac{2b}{Q(v_1)}v_1$. This means that when $Q(v_1) \neq 0$, the transversality condition on Δ is $Q(v_2^{(1)}) \neq 0$. We now look at the equation given by $\det_2(U) \neq 0$:

$$\begin{aligned} \det_2(U) &= \det \begin{pmatrix} -b - \frac{a}{2}Q(v_2) + B(v_1, v_2) & -\frac{Q(w_1)}{2} \\ -\frac{Q(v_2)}{2} & b \end{pmatrix} \\ &\stackrel{L_1 \leftarrow L_1 + aL_2}{=} \det \begin{pmatrix} -b + B(v_1, v_2) & -\frac{Q(v_1)}{2} \\ -\frac{Q(v_2)}{2} & b \end{pmatrix} \\ &\stackrel{L_1 \leftarrow L_2 + \frac{2b}{Q(v_1)}L_1}{=} \det \begin{pmatrix} -b + B(v_1, v_2) & -\frac{Q(v_1)}{2} \\ -\frac{1}{2}Q(v_2 + \frac{2b}{Q(v_1)}v_1) & 0 \end{pmatrix} \end{aligned}$$

The equation $\det_2(U) \neq 0$ is then $-\frac{1}{4}Q(v_1)Q(v_2 + \frac{2b}{Q(v_1)}v_1) \neq 0$ and we retrieve the geometric interpretation. Note however that $Q(v_1)$ may vanish. In that case, the point $\Delta'_0 \cap E$ does not lie in the affine chart $E \setminus C(\Delta_\infty \cap E)$. The intersection of the lightcone of $\Delta'_0 \cap E$ with the affine chart is then a hyperplane.

2.2. (1,2,3)-flags. Let $p > q \geq 3$. We will now consider the space $\mathcal{F} = \mathcal{F}^{1,2,3}$ of (1, 2, 3)-flags. Let $F_0 = (x_0, \Delta_0, \Phi_0)$ and $F_\infty = (x_\infty, \Delta_\infty, \Phi_\infty)$ be the two standard flags defined in Section 1.2. From Proposition 1.5, any pointed photon $F = (x, \Delta, \Phi) \in \mathcal{F}$ transverse to F_∞ can be written as $U \cdot (x_0, \Delta_0, \Phi_0)$ where

$$U = \exp(u_1(v_1^2)) \exp(u_2(v_2^1)) \exp(u_3(v_3^0)).$$

Now we observe that the fibers of the map

$$\begin{aligned} \mathcal{F}^{1,2,3} &\rightarrow \text{Ein}_{p-1, q-1} \\ (x, \Delta, \Phi) &\mapsto x \end{aligned}$$

above a point x transverse to x_0 is identified to the space of pointed photons in $\text{Ein}_{p-2, q-2}$. Our goal is to use the description given in the Section 2.1 to describe this fiber.

We start with $x = U \cdot x_0$ which is transverse to x_0 exactly when $Q(v_1^2) \neq 0$, as seen in Section 2.1. We now want to see Δ_0 and Φ_0 in the space $E = C(x) \cap C(x_\infty) \simeq \text{Ein}_{p-2, q-2}$. As in Section 2.1, we define Δ'_0 to be the only photon passing through x that intersect Δ_0 , and in the same spirit we define Φ'_0 to be the set of all photons passing through x and intersecting Φ_0 . To alleviate the notations we will write Δ'_0 (resp. Δ , Φ , Φ'_0 , Δ_∞ , Φ_∞) instead of their respective intersection with E . The point Δ'_0 lies on the lightcone of Δ_∞ exactly when $Q(v_1^1) = 0$. In the following, we suppose $Q(v_1^2) \neq 0$. To use the computations done in Section 2.1, we need to use a basis $e' = (e'_2, \dots, e'_q, x'_{q+1}, \dots, x'_{p-q}, \tilde{e}'_q, \dots, \tilde{e}'_2)$ of $x^\perp \cap x_\infty^\perp \subset \mathbb{R}^{p, q}$ in which $\Delta'_0 = \text{Span}(\tilde{e}'_2)$, $\Phi'_0 = \text{Span}(\tilde{e}'_2, \tilde{e}'_3)$, $\Delta_\infty = \text{Span}(e'_2)$ and $\Phi_\infty = \text{Span}(e'_2, e'_3)$. Such a basis is obtained by applying

the matrix

$$\begin{pmatrix} 1 & -\frac{2b_1^2 a_1^1}{Q(v_1^1)} & \frac{-2b_1^2}{Q(v_1^1)} \bar{v}_1^0 & -\frac{2b_1^2 b_1^1}{Q(v_1^0)} & -\frac{4(b_1^2)^2}{Q(v_1^0)} \\ & 1 & \frac{-2b_1^1}{Q(v_1^0)} \bar{v}_1^0 & -\frac{4(b_1^1)^2}{Q(v_1^0)} & \frac{2b_1^1 b_1^2}{Q(v_1^1)} \\ & & I & \frac{-2b_1^1}{Q(v_1^0)} v_1^0 & \frac{-2b_1^2}{Q(v_1^1)} v_1^0 \\ & & & 1 & \frac{2b_1^2 a_1^1}{Q(v_1^1)} \\ & & & & 1 \end{pmatrix} \in \text{SO}_0(p-1, q-1)$$

to the basis $(e_2, \dots, \tilde{e}_2)$. The coordinates of Δ in the affine chart $E \setminus C(\Delta_\infty)$ are then

$$\Delta = b_2^{1,(1)} e'_3 + v_2^{0,(1)} + a_2^{1,(1)} \tilde{e}'_3$$

where

$$b_2^{1,(1)} = b_2^1 - \frac{2b_1^1 b_1^2}{Q(v_1^0)}$$

2.3. General case. Let us consider the case of the Grassmanian of k dimensional isotropic subspaces of $\mathbb{R}^{p,q}$. For F_0 and F_∞ the two standard $(k-1)$ -photons, the equation of transversality to F_0 in the affine map defined by F_∞ and F_0 is $\det_k(U) = (-1)^k \det(S) \neq 0$ where $S = (s_{i,j})$ is the anti-transpose (i.e. the transpose with respect to the anti-diagonal) of the upper-right $k \times k$ submatrix of U , defined by $s_{i,j} = b_j^{k-i+1}$ when $i > j$, $s_{i,i} = -\frac{1}{2}Q(v_i^{k-i})$ and

$$s_{i,j} = -b_i^{k-j+1} - Q(v_i^{k-j+1}, v_j^{k-j+1}) - \sum_{l=1}^{j-i} a_i^{k-l} s_{i+l,j}.$$

We are going to perform a succession of change of variables. In order to keep coherent notations, the values before any change of variable will be denoted with an exponent (0). Let us perform elementary operations on the matrix S in order to eliminate the coefficient on the first column assuming that for all i , $Q(v_1^{i,(0)}) \neq 0$. We apply the following transformations :

$$\begin{aligned} L_1 &\leftarrow L_1 + a_1^{k-1,(0)} L_2, \\ L_2 &\leftarrow L_2 + \frac{2b_1^{k-1,(0)}}{Q(v_1^{k-2,(0)})} L_1, \\ &\dots \\ L_1 &\leftarrow L_1 + a_1^{1,(0)} L_k, \\ L_k &\leftarrow L_k + \frac{2b_1^{1,(0)}}{Q(v_1^{0,(0)})}. \end{aligned}$$

We then obtain the matrix S' where for all $i > 1$, $s'_{i,1} = 0$. Our goal will be to make the necessary change of variables such that the sub-matrix S_2 defined by $S_2 = (S')_{i>1, j>1}$ is of the same form as S .

For all $i > j$, let us write

$$\begin{aligned}
b_j^{k-i+1,(1)} &= s'_{i,j} \\
&= b_j^{k-i+1,(0)} - \frac{2b_1^{k-i+1,(0)}}{Q(v_1^{k-i+1,(0)})} b_1^{k-j+1,(0)} - \frac{2b_1^{k-i+1,(0)}}{Q(v_1^{k-i+1,(0)})} Q(v_1^{k-i-1,(0)}, v_j^{k-i-1,(0)}) \\
&\quad - \sum_{l=1}^{i-j} \frac{2b_1^{k-i+1,(0)} a_j^{k-j-l,(0)} b_1^{k-j-l,(0)}}{Q(v_1^{k-i+1,(0)})}
\end{aligned}$$

and $v_i^{k-i,(1)} = v_i^{k-i,(0)} + \frac{2b_1^{k-i+1,(0)}}{Q(v_1^{k-i,(0)})} v_1^{k-i,(0)}$. We can then see that

$$s'_{i,i} = -\frac{1}{2} Q(v_i^{k-i,(0)}) - \frac{2b_1^{k-i+1,(0)}}{Q(v_1^{k-i,(0)})} - \frac{2(b_1^{k-i+1,(0)})^2}{Q(v_1^{k-i,(0)})} = -\frac{1}{2} Q(v_i^{k-i,(1)}).$$

Let us then write $a_j^{k-j-1,(1)} = a_j^{k-j-1,(0)} + \frac{2b_1^{k-1,(0)} a_j^{k-j-1,(0)}}{Q(v_1^{k-2,(0)})}$ and for $i < k-j-1$,

$$a_j^{i,(1)} = a_j^{i,(0)} + \frac{2b_1^{k-j+1,(0)} a_j^{i,(0)}}{Q(v_1^{k-j,(0)})} + \sum_{l=i+1}^{k-j} \frac{2a_j^{l,(1)} b_1^{l,(0)} a_1^{i,(0)}}{Q(v_1^{l-1,(0)})},$$

as well as, for $i \leq k-j$,

$$v_j^{i,(1)} = v_j^{i,(0)} + \frac{2b_1^{k-j+1,(0)}}{Q(v_1^{k-j,(0)})} v_1^{i,(0)} + \sum_{l=i}^{k-j-1} \frac{a_j^{l,(1)} b_1^{l,(0)}}{Q(v_1^{l,(0)})} v_1^{i,(0)}.$$

By some elementary but tedious computations which we will not explicit, we then get that for $j > i$,

$$s'_{i,j} = -b_i^{k-j+1,(1)} - Q(v_i^{k-j+1,(1)}, v_j^{k-j+1,(1)}) - \sum_{l=1}^{j-i} a_i^{k-l,(1)} u'_{i+l,j}$$

and that for all $i \leq k-j$,

$$Q(v_j^{i,(1)}) = Q(v_j^{i-1,(1)}) + 2a_j^{i,(1)} b_j^{i,(1)},$$

thus showing that the submatrix S_2 is indeed of the same form as S with the change of variable $v_j^{i,(0)} \leftarrow v_j^{i,(1)}$, $a_j^{i,(0)} \leftarrow a_j^{i,(1)}$ and $b_j^{i,(0)} \leftarrow b_j^{i,(1)}$. Since $Q(v_1^{1,(0)})$ is assumed to be non-zero, the transversality to F_0^2 then becomes $\det(S_2) \neq 0$ and we can re-apply this process, thus getting new variables $v_j^{i,(m)}$ for $1 \leq m \leq k-1$. Assuming that for all i, j, m , $Q(v_j^{i,(m)})$ is non-zero, we get the new variables for $i \leq k-j$ and $m \leq j-1$:

$$\begin{aligned}
v_j^{i,(m+1)} &= v_j^{i,(m)} + \frac{2b_{m+1}^{k-m-j+1,(m)}}{Q(v_{m+1}^{k-m-j,(m)})} v_{m+1}^{i,(m)} + \sum_{l=i}^{k-m-j-1} \frac{a_j^{l,(m+1)} b_{m+1}^{l,(m)}}{Q(v_{m+1}^{l,(m)})} v_{m+1}^{i,(m)}, \\
a_j^{i,(m+1)} &= a_j^{i,(m)} + \frac{2b_{m+1}^{k-m-j+1,(m)} a_j^{i,(m)}}{Q(v_{m+1}^{k-m-j,(m)})} + \sum_{l=i+1}^{k-m-j} \frac{2a_j^{l,(m+1)} b_{m+1}^{l,(m)} a_{m+1}^{i,(m)}}{Q(v_{m+1}^{l-1,(m)})}, \\
b_j^{i,(m+1)} &= b_j^{i,(m)} - \frac{2b_{m+1}^{i,(m)}}{Q(v_{m+1}^{i,(m)})} b_{m+1}^{k-j+1,(m)} - \frac{2b_{m+1}^{i,(m)}}{Q(v_{m+1}^{i,(m)})} Q(v_{m+1}^{i,(m)}, v_j^{i,(m)}) \\
&\quad - \sum_{l=1}^{k-m-i+1-j} \frac{2b_{m+1}^{i,(m)} a_j^{k-m-j-l,(m)} b_{m+1}^{k-m-j-l,(m)}}{Q(v_{m+1}^{i,(m)})}.
\end{aligned}$$

In the end, the equations of transversality for the $(1, \dots, k)$ -flags become

$$\begin{aligned}
\det_1(U) &= \frac{-1}{2} Q(v_1^{k-1,(0)}) \neq 0, \\
\det_2(U) &= \frac{1}{4} Q(v_1^{k-2,(0)}) Q(v_2^{k-2,(1)}) \neq 0, \\
&\dots \\
\det_k(U) &= \frac{(-1)^{k+1}}{2^k} Q(v_1^{0,(0)}) Q(v_2^{0,(1)}) \dots Q(v_k^{0,(k-1)}) \neq 0.
\end{aligned}$$

2.4. Realization as minors. Let $(F_0, u \cdot F_0, F_\infty)$ be a triple of transverse $(1, \dots, k)$ -flags, for $k \leq q$. In order to determine the connected component of $\Omega(F_0) \cap \Omega(F_\infty)$ in which $U \cdot F_0$ lies, we will need the following data:

- For all $1 \leq j \leq k$ and $0 \leq i \leq k-j$, we need to know the sign of $Q(v_j^{i,(j-1)})$. Moreover, if $k-i = q-1$ (this happens when $k = q-1$ and $i = 0$ or when $k = q$ and $i = 1$) or $k-i = q-p-1$ (this happens when $k = q-p-1$ and $i = 0$) we need to know the sign of the first coordinate $(v_j^{i,(j-1)})_1$ of $v_j^{i,(j-1)}$ (to differentiate between future and past vectors when $k-i = q-1$ and between positive and negative numbers when $k-i = q-p-1$)
- For all $1 \leq j \leq k-1$ and $1 \leq i \leq k-j$, we need to know the sign of $b_j^{i,(j-1)}$.

The purpose of this section is to show that this data can be retrieved only using explicit minors of U . For I, J two subsets of $\{1, \dots, p+q\}$ of same cardinal, we will denote by $\Delta_{I,J}(U)$ the (I, J) -minor of U , i.e. the determinant of the submatrix $(u_{i,j})_{i \in I, j \in J}$. We then claim the following equalities: $\forall 1 \leq j \leq k, \forall 0 \leq i \leq k-j$,

$$\frac{(-1)^j}{2^j} Q(v_1^{i,(0)}) Q(v_2^{i,(1)}) \dots Q(v_j^{i,(j-1)}) = \Delta_{\{1, \dots, k-i\}, \{j+1, \dots, k-i, p+q-j+1, \dots, p+q\}}(U)$$

and $\forall 1 \leq j \leq k, \forall 1 \leq i \leq k-j$,

$$\frac{(-1)^{j-1}}{2^{j-1}} Q(v_1^{i,(0)}) Q(v_2^{i,(1)}) \dots Q(v_{j-1}^{i,(j-2)}) b_j^{i,(j-1)} = \Delta_{\{1, \dots, k-i+1\}, \{j, \dots, k-i, p+q-j+1, \dots, p+q\}}(U).$$

In both cases, the quantities we are interested in are ratios of two minors of U . The data of all the signs of the numbers stated above is equivalent to the data of all the signs of the minors above.

When $k-i = q-1$ or $k-i = q-p-1, \forall 1 \leq j \leq k$:

$$\frac{(-1)^{j-1}}{2^{j-1}} Q(v_1^{i-1,(0)}) \dots Q(v_{j-1}^{i-1,(j-2)}) (v_j^{i-1,(j-1)})_1 = \Delta_{\{1, \dots, k-i+2\}, \{j, \dots, k-i+1, p+q-j+1, \dots, p+q\}}(U)$$

In both cases this is well defined, and $v_j^{-1,(j-1)}$ belong to a space on which the restriction of Q is positive definite, hence the sign of $(v_j^{i-1,(j-1)})_1$ is always the same as the one of $\Delta_{\{1,\dots,k-i+2\},\{j,\dots,k-i+1,p+q-j+1,\dots,p+q\}}(U)$. This observation is crucial because the number

$$Q(v_1^{-1,(0)})Q(v_2^{-1,(1)})\dots Q(v_{j-1}^{-1,(j-2)})$$

does not arise as one of the previously computed minors.

In the case when $k = q - 1$ which correspond to the Θ -positive structure of $\mathrm{SO}_0(p, q)$, the total number of minors to compute is $p(p - 1)$, which is the same number as the Θ -length of the longest word $w_{\mathbb{G}}^{\max}$ of the Θ -Weyl group defined in [GW24]. This result is analogous to what happens in the case of full flags in a semisimple split Lie group: the number of (generalized) minors required to determine the connected component is the length of the longest element of the Weyl group.

3. COUNTING CONNECTED COMPONENTS OF TRIPLES OF TRANSVERSE FLAGS

3.1. In $\mathrm{SO}_0(3, 2)$. There are only three parabolic spaces associated to $\mathrm{SO}_0(3, 2)$: the space of isotropic lines in $\mathbb{R}^{3,2}$ also known as the Einstein space $\mathrm{SO}_0(3, 2)/P_1 = \mathrm{Ein}_{2,1}$, the set $\mathrm{SO}_0(3, 2)/P_2$ of photons of $\mathrm{Ein}_{2,1}$ and the set $\mathrm{SO}_0(3, 2)/P_{1,2}$ of pointed photons in $\mathrm{Ein}_{2,1}$. The Lie group $\mathrm{SO}_0(3, 2)$ is split and the parabolic subgroup $P_{1,2}$ is its Borel subgroup, therefore we already know from [Lus94] that the space of flags transverse to F_0 and F_∞ must contain four connected components (called *totally positive*) for which the associated spaces in $U_{1,2}$ are semi-groups. Since $\mathrm{Ein}_{2,1}$ is three-dimensional, those components can be seen geometrically.

Proposition 3.1. *In $\mathrm{SO}_0(3, 2)/P_1$, the space $\Omega(F_0) \cap \Omega(F_\infty)$ has three connected components.*

Proof. The choice of F_∞ and F_0 gives an affine chart of $\mathrm{Ein}_{2,1}$, $\Omega(F_\infty) \simeq \mathbb{R}^{2,1}$ in which F_0 is the origin. The set $\Omega(F_0) \cap \Omega(F_\infty)$ then corresponds to the points in $\mathbb{R}^{2,1}$ which do not lie in the lightcone of the origin. This splits $\mathbb{R}^{2,1}$ in the space, future and past connected components, hence the result. \square

Let us now study the space of pointed photons of $\mathrm{Ein}_{2,1}$. Let us write $F_0 = (x_0, \Delta_0)$ and $F_\infty = (x_\infty, \Delta_\infty)$. Let us consider the affine chart from x_∞ where x_0 is the origin. The space of pointed photons for which the point is in the affine chart is a trivial bundle $\mathbb{R}^{2,1} \times \mathbb{S}^1$ over $\mathbb{R}^{2,1}$ for which the fiber over $x \in \mathbb{R}^{2,1}$ is the set of photons going through x which is $\mathrm{Ein}_{1,0} \simeq \mathbb{S}^1$.

Lemma 3.2. *In this affine chart, the set of pointed photons transverse to $(x_\infty, \Delta_\infty)$ is $\mathbb{R}^{2,1} \times (\mathbb{S}^1 \setminus \{N\})$, where N is a point of \mathbb{S}^1 .*

Proof. Let $x \in \mathbb{R}^{2,1}$. There exists a unique photon Δ going through x which intersects Δ_∞ in the boundary of the affine chart. Let $N \in \mathbb{S}^1$ such that $(x, \Delta) = (x, N)$. Let y be a different point in $\mathbb{R}^{2,1}$. If $y \in (\Delta - x)^\perp$, the photons Δ and $y - x + \Delta$ have the same endpoint at infinity which lies in Δ_∞ . If not, the endpoint of $y - x + \Delta$ lies in the unique photon from x_∞ to the endpoint of Δ , i.e Δ_∞ . In both cases, the unique photon going through y which is not transverse to Δ_∞ is (y, N) , hence the result. \square

Thus for any point $x \in \mathbb{R}^{2,1} \setminus \Delta_0$, there exists exactly one photon through x intersecting Δ_∞ which is (x, N) and one photon through x intersecting Δ_0 which is $(x, S(x))$. The coordinates of this second photon in \mathbb{S}^1 depend on x . One must then follow the variations of $S(x)$ when x varies in $\mathbb{R}^{2,1}$. Let Δ be the only photon through the origin intersecting Δ_∞ , i.e the photon of the pointed photon $(0, N)$, and let $V = \Delta_0 \oplus \Delta$.

Lemma 3.3. *The two points $S(x)$ and N in \mathbb{S}^1 are equal if and only if $x \in V$.*

Proof. Assume that $x \in V$. Then $x + \Delta$ is the only photon through x which intersects Δ_0 , thus we must have $S(x) = N$. When x is not in V , the photon $x + \Delta$ does not intersect Δ_0 , thus $S(x) \neq N$ which gives us the result. \square

We can now count geometrically the number of connected components of $\Omega(F_\infty) \cap \Omega(F_0)$ in $\text{SO}_0(3,2)/P_{1,2}$. The transversality of x with x_0 already separates the connected components into three groups.

Proposition 3.4. *In the space $\text{SO}_0(3,2)/P_{1,2}$,*

- *there are two connected components of $\Omega(F_\infty) \cap \Omega(F_0)$ where x is spacelike,*
- *there are three connected components of $\Omega(F_\infty) \cap \Omega(F_0)$ where x is in the future cone, two of which are totally positive in the sense of [Lus94],*
- *there are three connected components of $\Omega(F_\infty) \cap \Omega(F_0)$ where x is in the past cone, two of which are totally positive.*

In total, this gives us 8 connected components for $\text{SO}_0(3,2)/P_{1,2}$, four of which are totally positive.

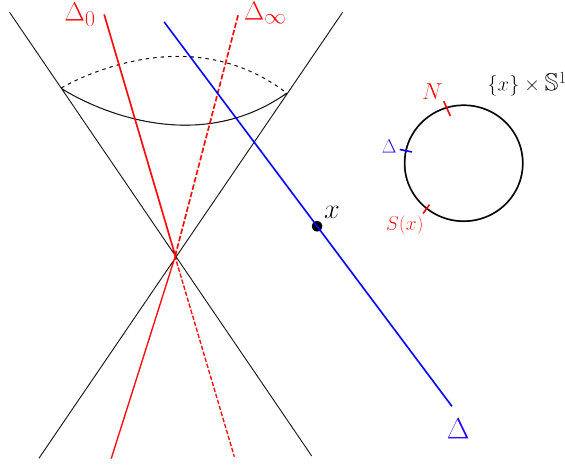


FIGURE 3.1. The fiber bundle of pointed photons when $Q(x) < 0$.

Proof. Let's assume that $Q(x) > 0$. When x is in the orthogonal of V , Δ_0 and $x + \Delta_0$ have the same accumulation point at the boundary, thus $S(x) = -N$. When $x \notin V$, $S(x) \neq N$ and the condition of transversality to both Δ_0 and Δ_∞ separates the fiber $\{x\} \times \mathbb{S}^1$ in two connected components. The vector space V is of dimension 2 and thus separates $\mathbb{R}^{2,1}$ in two parts. Let $x = (a, b, v)$ be the decomposition of x in $\mathbb{R}^{2,1} = \Delta_\infty \oplus \Delta_0 \oplus V^\perp$. When $Q(x) < 0$, x may cross the vector space V with both $b > 0$ or $b < 0$. Doing so one way or the other makes $S(x)$ go to N clockwise or counterclockwise. Adding all of this up, one eventually counts two connected components.

Assume now that x lies in the future cone of the origin. The same reasoning applies, but we must then always have $b > 0$. When x is on one side of V , the fiber $\{x\} \times \mathbb{S}^1$ separates into two connected components, one of which disappears when x crosses V . We must then have three connected components, two of whom correspond to the totally positive components. The same reasoning applies when x lies in the past cone, hence the result. \square

Now that we have counted the connected components for the pointed photons, it becomes easy to count those in $\text{SO}_0(3, 2)/P_2$, the non-pointed photons in $\text{Ein}_{2,1}$.

Proposition 3.5. *In $\text{SO}_0(3, 2)/P_2$, the space $\Omega(F_\infty) \cap \Omega(F_0)$ has two connected components.*

Proof. Let $x_\infty \in \Delta_\infty$, $x_0 \in \Delta_0$, $\mathbb{R}^{2,1}$ be an affine chart of x_∞ with x_0 in its center and let $P = \mathbb{R}^{2,0}$ in $\mathbb{R}^{2,1}$. A photon transverse to Δ_∞ cannot be contained in the light cone of x_∞ , thus any photon of $\Omega(F_\infty) \cap \Omega(F_0)$ must be within the affine chart. For each photon Δ in the affine chart, there exists a unique x on Δ such that $x \in P$, meaning that the photons in the affine chart can be identified to the restriction of the previous fiber bundle on P . The pointed photons with points in P transverse to Δ_0 and Δ_∞ have two connected components, we get the intended result. \square

We may also find those results using the change of variables introduced in the previous section. Let us write $x = v_1^1 = (a, v_1^0, b)$. We must first choose whether v_1^1 is picked to be spacelike, future oriented or past oriented. We then have an easier equation on \mathbb{R} defined when $v_1^0 \neq 0$ which becomes undefined when v_1^0 goes from positive to negative. When it is defined, the connected components are separated by the sign of $v_2^{0,(1)} = v_2^0 - \left(-\frac{2b_1^1}{(v_1^0)^2}v_1^0\right)$. Assume that v_1^1 is spacelike. When v_1^0 is positive and goes to zero, assuming that $b_1^1 > 0$, $-\frac{2b_1^1}{(v_1^0)^2}v_1^0$ goes to $-\infty$ and the component $v_2^0 < -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ cannot cross over, leaving only $v_2^0 > -\frac{2b_1^1}{(v_1^0)^2}v_1^0$. Inversely, when v_1^0 is negative and goes to zero, $-\frac{2b_1^1}{(v_1^0)^2}v_1^0$ goes to $+\infty$. The component $v_2^0 > -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ cannot cross over, leaving only $v_2^0 < -\frac{2b_1^1}{(v_1^0)^2}v_1^0$. This tells us that the component $v_2^0 < -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ when $v_1^0 < 0$ and $v_2^0 > -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ when $v_1^0 > 0$ are actually the same component. The same can be done taking $b_1^1 < 0$, which tells us that $v_2^0 < -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ when $v_1^0 > 0$ and $v_2^0 > -\frac{2b_1^1}{(v_1^0)^2}v_1^0$ when $v_1^0 < 0$ are the same connected component. We thus obtain via computations the two connected components of $\text{SO}_0(3, 2)/P_{1,2}$ where v_1^1 is spacelike.

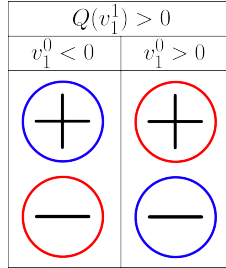


FIGURE 3.2. In red and blue the two connected components.

When assuming that v_1^1 is in one of the two timelike cones, the reasoning stays the same except for the fact that the sign of b_1^1 becomes fixed. This gives us three connected components as two of them cannot cross over from $v_1^0 < 0$ to $v_1^0 > 0$.

3.2. Pointed photons in $\text{SO}_0(p, 3)$. We will now compute the number of connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ for the space of pointed photons in $\text{SO}_0(p, 3)$, meaning for the flag manifolds of the form $\text{SO}_0(p, 3)/P_{1,2}$ for some $p > 3$.





$Q(v_1^1) < 0, b_1^1 > 0$	
$v_1^0 < 0$	$v_1^0 > 0$
	
	

FIGURE 3.3. In red the connected component which exists in both $v_1^1 < 0$ and $v_1^1 > 0$; in blue and green the positive components.

The main difference with the pointed photons in $SO_0(3, 2)$ is that the elements v_1^0 and v_2^0 both live in a Lorentzian space $\mathbb{R}^{p-2,1}$. This yields another set of rules when changing the sign of $Q(v_1^0)$. Let's assume for instance that $Q(v_1^0) > 0$ and $b_1^1 > 0$. The point v_2^0 must not be on the lightcone of $-\frac{2b_1^1}{Q(v_1^0)}v_1^0$ and is therefore confined into three areas, namely space, future and past. When v_1^0 goes from the space part to the future part, $-\frac{2b_1^1}{Q(v_1^0)}v_1^0$ is negatively colinear to v_1^0 and goes to infinity along the past orientation of a photon; therefore the past of $-\frac{2b_1^1}{Q(v_1^0)}v_1^0$ does not cross over during the transition, leaving only the space and future parts. This is illustrated in figure 3.4.

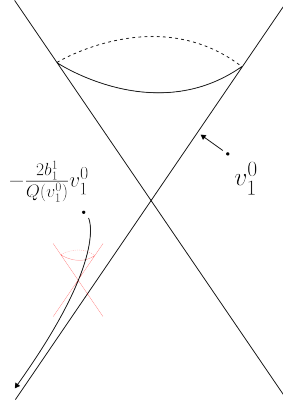


FIGURE 3.4. The transition from v_1^0 spacelike to v_1^0 future oriented when $b_1^1 > 0$. In red, the light cone of $-\frac{2b_1^1}{Q(v_1^0)}v_1^0$.

Inversely, when going from v_1^0 future to v_1^0 spacelike with b_1^1 positive the future component does not cross over, leaving only the space and past parts at the limit. When comparing, one then sees that the future for v_1^0 spacelike becomes the space for v_1^0 future and the space for v_1^0 spacelike becomes the past for v_1^0 future. When doing every possible transition, one can then count the exact number of connected components, see figure 3.5.

Proposition 3.6. *The space $\Omega(F_0) \cap \Omega(F_\infty)$ in $SO_0(p, 3)/P_{1,2}$ contains 11 connected components, 4 of which are positive.*

$Q(v_1^1) < 0$			$Q(v_1^1) > 0$		
$Q(v_1^0) < 0$		$Q(v_1^0) > 0$	$Q(v_1^0) < 0$		$Q(v_1^0) > 0$
P	F	b_1^1	P	F	
\textcircled{S}	\textcircled{S}	+	\textcircled{S}	\textcircled{P}	\textcircled{S}
\textcircled{P}	\textcircled{P}		\textcircled{S} \textcircled{P} \textcircled{F}	\textcircled{S} \textcircled{P} \textcircled{F}	\textcircled{P}
\textcircled{F}	\textcircled{F}	-	\textcircled{S} \textcircled{P} \textcircled{F}	\textcircled{S} \textcircled{P} \textcircled{F}	\textcircled{F}

FIGURE 3.5. The eleven connected components of $\Omega(F_0) \cap \Omega(F_\infty)$, one group of four when $Q(v_1^0) < 0$ and one group of seven when $Q(v_1^0) > 0$. The pink components are isolated and are the Θ -positive components.

3.3. **(1, ..., k)-flags in $\text{SO}_0(p, q)$.** We will now be dealing with the case of complete flags of photons up to dimension $k < q - 1$ in $\text{SO}_0(p, q)$, $\text{SO}_0(p, q)/P_{1, \dots, k}$. As in the previous sections, we know that when for all $0 \leq m \leq k - 1$ and $i < k - j$, $Q(v_j^{i, (m)}) \neq 0$, the equations of transversality become

$$\begin{aligned}
Q(v_1^{k-1, (0)}) &\neq 0, \\
Q(v_1^{k-2, (0)})Q(v_2^{k-2, (1)}) &\neq 0, \\
&\dots \\
Q(v_1^{0, (0)})Q(v_2^{0, (1)})\dots Q(v_k^{0, (k-1)}) &\neq 0.
\end{aligned}$$

It is easy to check what happens when all $Q(v_j^{i, (j-1)})$ are non-zero except $Q(v_k^{0, (k-1)})$. In this case the equation becomes $Q(v_k^{0, (k-1)}) \neq 0$ and since $v_k^{0, (k-1)}$ is in the pseudo-Riemannian space $\mathbb{R}^{p-k, q-k}$, $v_k^{0, (k-1)}$ can either be spacelike or timelike. This holds for every open set of $\text{SO}_0(p, q)/P_{1, \dots, k}$ where all relevant vectors are of non-zero norm, however since the transversality equations do not forbid $Q(v_j^{i, (j-1)}) = 0$ when $i \neq k - j$, we must track what happens to the space and time parts when those change signs. We only have to explicit what happens when the $Q(v_1^{i, (0)})$ change signs as when those are constant we can deduce the rules for the $Q(v_j^{i, (j-1)})$, $j > 1$ by induction on j .

Lemma 3.7. *We have that*

$$\begin{aligned}
a_j^{i, (1)} &= a_j^i + \frac{2b_1^{k-1}a_1^i}{Q(v_1^i)} + \sum_{i+1}^{k-j} \frac{2a_j^l b_1^l a_1^i}{Q(v_1^i)}, \\
v_j^{i, (1)} &= v_j^i + \frac{2b_1^{k-j+1}}{Q(v_1^i)} v_1^i + \sum_{l=i}^{k-j} \frac{a_j^l b_1^l}{Q(v_1^i)} v_1^i.
\end{aligned}$$

Proof. Let us fix j and prove by decreasing induction on i that

$$\frac{2b_1^{k-1}}{Q(v_1^{k-2})} + \sum_{l=i+1}^{k-j} \frac{2a_j^{l,(1)}b_1^l}{Q(v_1^{l-1})} = \frac{2b_1^{k-1}}{Q(v_1^i)} + \sum_{l=i+1}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^i)}.$$

The result is true for $i = k - j$ since $a_j^{k-j,(1)} = a_j^{k-j} + \frac{2b_1^{k-1}a_1^{k-j}}{Q(v_1^{k-j})}$. Assume that it is true for any $i' > i$. Then

$$\begin{aligned} \frac{2b_1^{k-1}}{Q(v_1^{k-2})} + \sum_{l=i+1}^{k-j} \frac{2a_j^{l,(1)}b_1^l}{Q(v_1^{l-1})} &= \frac{2b_1^{k-1}}{Q(v_1^{i+1})} + \sum_{l=i+2}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^{i+1})} + \frac{2a_j^{i+1,(1)}b_1^{i+1}}{Q(v_1^i)} \\ &= \frac{2a_j^{i+1}b_1^{i+1}}{Q(v_1^i)} + \frac{2b_1^{k-1}}{Q(v_1^{i+1})} + \frac{4b_1^{k-1}a_1^{i+1}b_1^{i+1}}{Q(v_1^{i+1})Q(v_1^i)} \\ &\quad + \sum_{l=i+2}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^{i+1})} + \sum_{l=i+2}^{k-j} \frac{4a_j^l b_1^l a_1^{i+1}b_1^{i+1}}{Q(v_1^{i+1})Q(v_1^i)} \\ &= \frac{2a_j^{i+1}b_1^{i+1}}{Q(v_1^i)} + \frac{2b_1^{k-1}}{Q(v_1^{i+1})} \left(\frac{Q(v_1^i) + 2a_1^{i+1}b_1^{i+1}}{Q(v_1^i)} \right) \\ &\quad + \sum_{l=i+2}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^{i+1})} \left(\frac{Q(v_1^i) + 2a_1^{i+1}b_1^{i+1}}{Q(v_1^i)} \right) \\ &= \frac{2a_j^{i+1}b_1^{i+1}}{Q(v_1^i)} + \frac{2b_1^{k-1}}{Q(v_1^i)} + \sum_{l=i+2}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^i)} \\ &= \frac{2b_1^{k-1}}{Q(v_1^i)} + \sum_{l=i+1}^{k-j} \frac{2a_j^l b_1^l}{Q(v_1^i)}, \end{aligned}$$

using that $Q(v_1^{i+1}) = Q(v_1^i) + 2a_1^{i+1}b_1^{i+1}$. This gives us the intended results. \square

Remark 3.8. In particular, when $Q(v_1^{l,(0)})$ goes to zero for $l \neq i$, $a_j^{i,(1)}$ and $v_j^{i,(1)}$ stay bounded.

Lemma 3.9. *When $Q(v_1^{0,(0)})$ goes to zero, we have for all $j \geq 2$,*

$$b_j^{1,(1)} \sim -\frac{Q(v_j^{0,(1)})}{2a_j^{1,(1)}}.$$

Proof. We know that $Q(v_j^{1,(1)}) = Q(v_j^{0,(1)}) + 2a_j^{1,(1)}b_j^{1,(1)}$, meaning that

$$b_j^{1,(1)} = \frac{Q(v_j^{1,(1)}) - Q(v_j^{0,(1)})}{2a_j^{1,(1)}}.$$

From lemma 3.7 we know that $Q(v_j^{1,(1)})$ and $a_j^{1,(1)}$ are both bounded when $Q(v_1^{0,(0)})$ goes to zero, meaning that we get the equivalence

$$b_j^{1,(1)} \sim -\frac{Q(v_j^{0,(1)})}{2a_j^{1,(1)}}.$$

\square

Lemma 3.10. For all i , when $Q(v_1^{i,(0)})$ goes to zero, $Q(v_j^{l,(j-1)})$ stays bounded for all $(j, l) \neq (2, i)$.

Proof. This lemma relies on similar computations to those presented in the proof of Lemma 3.7. \square

Let us then write down the effect of changing the signs of $Q(v_1^{l,(0)})$.

Proposition 3.11. When $Q(v_1^{i,(0)})$ goes to zero with $i \geq 1$,

$$v_2^{i,(1)} \sim \frac{2a_2^{i,(1)}b_1^i}{Q(v_1^i)}v_1^i,$$

$$a_2^{i,(1)} \sim \frac{2a_1^{i+1,(1)}b_1^{i+1,(1)}a_1^i}{Q(v_1^i)},$$

while all else stays bounded. When $Q(v_1^{0,(0)})$ goes to zero,

$$v_2^{0,(1)} \sim \frac{2a_2^{1,(1)}b_1^{1,(0)}}{Q(v_1^0)}v_1^0,$$

$$b_2^{1,(1)} \sim -\frac{Q(v_2^{0,(1)})}{2a_2^{1,(1)}}$$

while all else stays bounded.

This result shows that two different phenomena appear depending of whether $Q(v_1^{0,(0)})$ changes signs or any of the $Q(v_1^{i,(0)})$ for $i \geq 1$. Let us explicit what happens in those two cases.

Lemma 3.12. When $Q(v_1^{i,(0)})$ changes sign, the space and time parts in $Q(v_2^{i,(1)}) \neq 0$ are reversed. Furthermore $a_2^{i,(1)}$ must have the sign of $\frac{a_1^{i+1,(1)}b_1^{i+1,(1)}a_1^{i,(0)}}{Q(v_1^{i,(0)})}$ before and after the change of signs. For all others i, j , the time and space parts of $Q(v_j^{i,(j-1)}) \neq 0$ are unchanged.

Proof. Since $Q(v_2^{i,(1)}) \sim \frac{2a_2^{i,(1)}b_1^i}{Q(v_1^{i,(0)})}$, $v_2^{i,(1)} \neq 0$ implies that $v_2^{1,(0)}$ is outside the isotropic cone of a point $v_2^{1,(0)} - v_2^{i,(1)}$ which goes to infinity along the isotropic line $\text{Span}(v_1^{0,(0)})$ in one direction or the other depending on the sign of $-\frac{a_2^{i,(1)}b_1^i}{Q(v_1^{i,(0)})}$. Since $Q(v_1^{i,(0)})$ changes sign, this implies that the direction of $\text{Span}(v_1^{i,(0)})$ along which $v_2^{i,(1)} - v_2^{i,(0)}$ goes to infinity is reversed when $Q(v_1^{i,(0)})$ goes to zero positively or negatively, which implies that the time and space parts of $Q(v_2^{i,(1)}) \neq 0$ are reversed, hence the result. The second part follows immediately from $a_2^{i,(1)} \sim \frac{2a_1^{i+1,(1)}b_1^{i+1,(1)}a_1^{i,(0)}}{Q(v_1^{i,(0)})}$.

The third part follows from the fact that all other values remain bounded when $Q(v_1^{i,(0)})$ goes to zero. \square

Lemma 3.13. When $Q(v_1^{0,(0)})$ changes sign, the space and time parts in $Q(v_2^{0,(1)}) \neq 0$ are reversed. Furthermore $b_2^{1,(1)}$ must have the sign of $-\frac{Q(v_2^{0,(1)})}{2a_2^{1,(1)}}$ before and after the change of signs. For all others i, j, m , the time and space parts of $Q(v_j^{i,(j-1)}) \neq 0$ are unchanged.

Proof. Since $k \leq q-2$, the space $\mathbb{R}^{p-k, q-k}$ to which v_2^0 belongs to remains non-Lorentzian and the same argument as before applies. The second part follows immediately from $b_2^{1,(1)} \sim -\frac{Q(v_2^{0,(1)})}{2a_2^{1,(1)}}$. The third part follows from the fact that all other values remain bounded when $Q(v_1^0)$ goes to zero. \square

Example 3.1. Let us discuss the case of pointed photons in $\text{SO}_0(p, q)$ for $q > 3$. Using the change of variables obtained in section 2, the situation becomes similar to the $\text{SO}_0(3, 2)$ case except that v_1^0 no long lives in a euclidian line but in a non-Lorentzian pseudo-Riemannian space $\mathbb{R}^{p-2, q-2}$. The change of variable is non-defined when $Q(v_1^1) = 0$ and when it is defined, the fact that $v_2^{0, (1)} = v_2^0 + \frac{2b_1^1}{Q(v_1^0)}v_1^0$ has non-zero norm implies that v_2^1 must not be on the light cone of $-\frac{2b_1^1}{Q(v_1^0)}v_1^0$. The transversality of points split the connected components into two groups, whether $Q(v_1^1)$ is positive or negative. Let us assume it is positive. When $Q(v_1^0)$ is negative, since $Q(v_1^1) = Q(v_1^0) + 2a_1^1b_1^1$ we cannot have $b_1^1 = 0$, thus splitting into two choices for the sign of b_1^1 . This gives us the following situation which yields two connected components :

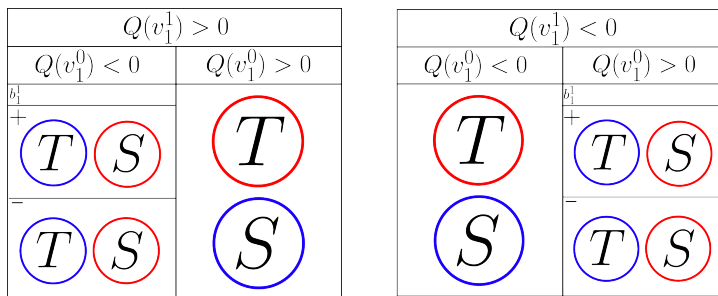


FIGURE 3.6. In red and blue the four connected components of $\Omega(F_0) \cap \Omega(F_\infty)$.

The case where $Q(v_1^1) < 0$ is symmetric. We thus have the total number of connected components.

Proposition 3.14. *The space $\Omega(F_0) \cap \Omega(F_\infty)$ in $\text{SO}_0(p, q)/P_{1,2}$ contains 4 connected components when $q > 3$.*

Remark 3.15. Those connected components exactly correspond to the components delimited by the signs of the two upper-left minors of S .

Proposition 3.16. *The space $\Omega(F_0) \cap \Omega(F_\infty) \subset \text{SO}_0(p, q)/P_{1, \dots, k}$ has 2^k connected components, determined by the signs of the upper left minors of size i of S for $1 \leq i \leq k$.*

Proof. By applying the previous rules when changing the signs of the $Q(v_j^{i, (j-1)})$ whose sign is not fixed by the equation of transversality, one can then see that every change only reverses the space and time parts of the corresponding equations, and thus that every equation of transversality only split $\text{SO}_0(p, q)/P_{1, \dots, k}$ into two, thus giving 2^k connected components as there are k equations, hence the result. \square

3.4. $(1, \dots, q-1)$ -flags in $\text{SO}_0(p, q)$. In this section we will only treat the case where $p > q$, the case where $p = q$ will be discussed in a following section. We know from the works of Guichard-Wienhard on Θ -positivity (see [GW18]) that there must be 2^{q-1} Θ -positive connected components in $\Omega(F_0) \cap \Omega(F_\infty)$. We apply the same reasoning as before, except that the elements $v_j^{0, (j-1)}$ now belong to the Lorentzian space $\mathbb{R}^{p-q+1, 1}$, meaning that we now have to distinguish between the space part, future part and past part of the complement of a lightcone. When $Q(v_1^{i, (0)})$ changes sign for $i > 0$, the same happens as in the previous case as $v_2^{i, (1)}$ belongs to a non-Lorentzian pseudo-Riemannian space.

Lemma 3.17. *When $v_1^{0,(0)}$ goes from the space part to the future part and $a_2^{1,(1)}b_1^{1,(0)}$ is positive, the past part in $Q(v_2^{0,(1)}) \neq 0$ does not cross over, the space part becomes the past part and the future part becomes the space part. When either the sign of $a_2^{1,(1)}b_1^{1,(0)}$ changes or $v_1^{0,(0)}$ goes from the space part to the past part, the future part does not cross over, the space part becomes the future part and the past part becomes the space part. Furthermore $b_2^{1,(1)}$ must have the sign of $-\frac{Q(v_2^{0,(1)})}{2a_2^{1,(1)}}$ before and after the change of signs. For all others i, j , the time and space parts of $Q(v_j^{i,(j-1)}) \neq 0$ are unchanged.*

Proof. As before, since $Q(v_2^{0,(1)}) \sim \frac{2a_2^{1,(1)}b_1^{1,(0)}}{Q(v_1^{0,(0)})}$, $v_2^{0,(1)} \neq 0$ implies that $v_2^{0,(0)}$ is outside the isotropic cone of a point $v_2^{0,(0)} - v_2^{0,(1)}$ which goes to infinity along the isotropic line $\text{Span}(v_1^{0,(0)})$ in one direction or the other depending on the sign of $-\frac{a_2^{1,(1)}b_1^{1,(0)}}{Q(v_1^{0,(0)})}$. When $v_1^{0,(0)}$ crosses from the space part to the future part, $Q(v_1^{0,(0)})$ is originally positive. Furthermore, since $a_2^{1,(1)}b_1^{1,(0)}$ is assumed to be positive, $v_2^{0,(0)} - v_2^{0,(1)}$ goes to infinity while being negatively colinear to $v_1^{0,(0)}$ which is going to the future part, meaning that the past cone of $v_2^{0,(0)} - v_2^{0,(1)}$ disappears as $v_1^{0,(0)}$ goes from space to future. Inversely, when $v_1^{0,(0)}$ goes from future to space the sign of $Q(v_1^{0,(0)})$ is originally negative, meaning that everything is reversed and it is the future cone of $v_2^{0,(0)} - v_2^{0,(1)}$ that does not cross over. In the end, one sees that when $v_1^{0,(0)}$ goes from space to future, the past part does not cross over, the space part becomes the past part and the future part becomes the space part, hence the result. \square

Remark 3.18. The transversality equations are $\det_i(S) \neq 0$ for $1 \leq i \leq q-1$, so the signs of the $\det_i(S)$ separates $\Omega(F_0) \cap \Omega(F_\infty)$ into at least 2^{q-1} connected components. We will show that for a given choice of the signs of the $\det_i(S)$ there are 2 connected components (except in the case when Θ -positive components arise, which we will also detail).

As previously, we start with an initial set of connected components each given by the choice of the sign of all $Q(v_j^{i,(j-1)})$ for $1 \leq j \leq q-1$ and $0 \leq i \leq k-j$. When $Q(v_j^{i,(j-1)})Q(v_j^{i-1,(j-1)}) < 0$, we also need to fix the sign of $b_j^{i-1,(j-1)}$. These component are obtained when restricting to the open subset of $\Omega(F_0) \cap \Omega(F_\infty)$ where all $Q(v_j^{i,(j-1)})$ are all non-zero. We then "glue" them together along the locus where exactly one of the $Q(v_j^{i,(j-1)})$ vanish. The locus where two or more vanish at the same time is of codimension greater than 2 hence does not change the number of connected components. Contrary to the previous sections, the combinatorics of the gluing process for $(1, \dots, q-1)$ -flags is more involved. To proceed with the study of this gluing process, we first describe a synthetic model describing the situation.

We represent an initial connected component by a anti-triangular $(q-1) \times (q-1)$ matrix $M = (m_{i,j})_{1 \leq i \leq q-1, 0 \leq j \leq q-1-i}$ with entries valued in $\mathbb{Z}/3\mathbb{Z}$ identified with the set $\{*, +, -\}$ (with $* = 0, + = 1, - = 2$), with every entry having one of the two colors red or blue. The way to read this data is as follows: the color of $m_{i,j}$ corresponds to the sign of $Q(v_i^{j,(i-1)})$ (blue for negative and red for positive), $m_{i,j} = *$ if and only if the (i, j) and $(i, j-1)$ entries are of the same color (the entry $(i, -1)$ count as red as $Q(v_i^{-1,(i-1)})$ is always positive), and if $m_{i,j} \neq *$ then it corresponds to the sign of $b_i^{j,(i-1)}$. We now want to describe the effect changing the sign of $Q(v_i^{j,(i-1)})$ has on this matrix. We will call such a modification a *alteration* of M .

For each (i, j) such that $i < q-1$ and $j < q-1-i$, we define

$$\mu_{i,j} = (-1)^{\varepsilon_1} (-1)^{\varepsilon_2} m_{i,j} m_{i+1,j} m_{i,j+1} m_{i+1,j+1}$$

where

$$\varepsilon_1 = \begin{cases} 1 & \text{if } m_{i+1,j} \text{ and } m_{i+1,j+1} \text{ are of the same color} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \varepsilon_2 = \begin{cases} 1 & \text{if } m_{i,j} = * \\ 0 & \text{otherwise} \end{cases}$$

When $\mu_{i,j} \neq +$ we say that the (i,j) entry of M is *alterable*. When $m_{i,j}$ is alterable, one can alter it as follows: choose a sign (+ or -) for every * among the four entries $m_{i,j}, m_{i+1,j}, m_{i,j+1}, m_{i+1,j+1}$ such that after replacing every * by the sign chosen, we get $s_{i,j} = -$. Then change the color of $m_{i,j}$ and $m_{i+1,j}$, and replace the entries that were not * at the beginning with *. There might be multiple sign choices that result in an admissible alteration, hence a alterable entry may have several possible alterations. These rules are obtained by translating the result of Proposition 3.11 in this setup, and the correction factor given by ε_1 and ε_2 are due to the presence of $a_i^{j,(i-1)}$ in the formulas whereas we are only working with the $b_i^{j,(i-1)}$ here. To get coherent formulas when $j = q - 2 - i$ (meaning that the $(i+1, j+1)$ entry does not exist), we need to make the following fix: we set $m_{i+1, q-1-i}$ to be a red + when $m_{i+1, q-2-i}$ is blue and a blue - when $m_{i+1, q-2-i}$ is red.

Example 3.2. Let us explicit an example. Assume $q = 4$. The transversality in $\text{SO}_0(p, 4)/P_{1,2,3}$ is given by three equations :

$$\begin{aligned} Q(v_1^{2,(0)}) &\neq 0, \\ Q(v_1^{1,(0)})Q(v_2^{1,(1)}) &\neq 0, \\ Q(v_1^{0,(0)})Q(v_2^{0,(1)})Q(v_3^{0,(2)}) &\neq 0. \end{aligned}$$

Since $v_1^{2,(0)}, v_1^{1,(1)}$ and $v_2^{1,(1)}$ all live in non-Lorentzian space, the first two equations split the space $\Omega(F_0) \cap \Omega(F_\infty)$ in four parts. For each cell in which $Q(v_1^{2,(0)}), Q(v_1^{1,(0)}), Q(v_1^{0,(0)}), Q(v_2^{1,(1)}), Q(v_2^{0,(1)})$ are all non-zero, the third equation of transversality becomes $Q(v_3^{0,(2)}) \neq 0$. The vector $v_3^{0,(2)}$ lives in a Lorentzian space, meaning it can be either spacelike, future or past. When going from one cell to another, the nature of the vector $v_3^{0,(2)}$ can change.

Let us start by considering the matrix

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline + & + & \\ \hline + & & \\ \hline \end{array}$$

Let us then change the nature of $v_1^{0,(0)}$ from spacelike to future with $b_1^{1,(0)} > 0$. Let us assume that $a_2^{1,(1)}$ is negative. Since $a_2^{1,(1)}b_1^{1,(0)}$ is negative, the future part in $Q(v_2^{0,(1)})$ does not cross over. The space part must then become future and the past part space. Assuming that $Q(v_2^{0,(1)})$ is negative, $-\frac{Q(v_2^{0,(1)})}{2a_2^{1,(1)}}$ must then be positive, meaning that $b_2^{1,(1)}$ must be negative. Finally, since all other values remain bounded, we get the following alteration

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline + & + & \\ \hline + & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline + & + & * \\ \hline * & * & \\ \hline + & & \\ \hline \end{array}$$

Let us now change $v_1^{0,(0)}$ from future to spacelike with $b_2^{1,(1)} > 0$. By applying the same method, one then get

$$\begin{array}{|c|c|c|} \hline + & + & * \\ \hline * & * & \\ \hline + & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & * & * \\ \hline - & - & \\ \hline + & & \\ \hline \end{array}$$

By the applying the same reasoning, one gets the following alterations

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline - & + & \\ \hline - & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline + & + & * \\ \hline * & * & \\ \hline - & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & * & * \\ \hline + & - & \\ \hline - & & \\ \hline \end{array}$$

We are going to show that these two alteration class of matrices are part of the same connected component. Let us start with the matrix

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline + & - & \\ \hline - & & \\ \hline \end{array}$$

and change the nature of $v_1^{1,(0)}$ from spacelike to timelike. We know that crossing from $v_1^{1,(0)}$ spacelike to $v_1^{1,(0)}$ timelike changes $v_2^{1,(1)}$ from spacelike to timelike. Let us assume that $b_1^{2,(0)}$ and $b_1^{1,(0)}$ are positive. Since $a_2^{1,(1)}$ must have the sign of $\frac{b_1^{2,(0)} a_1^{1,(0)}}{Q(v_1^{1,(0)})}$ and $b_1^{1,(0)}$ positive implies $a_1^{1,(0)}$ negative, we know that $a_2^{1,(1)}$ has to be positive on departure and negative on arrival. On departure, $Q(v_2^{1,(1)}) > 0$ so when $Q(v_2^{0,(1)})$ is negative, $a_2^{1,(1)}$ positive implies $b_2^{1,(1)}$ negative. From all this we deduce the following alteration:

$$\begin{array}{|c|c|c|} \hline * & * & * \\ \hline + & - & \\ \hline - & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & + & + \\ \hline + & * & \\ \hline - & & \\ \hline \end{array}$$

We will now change the nature of $v_2^{0,(1)}$ from past to spacelike with $b_2^{1,(1)} > 0$. Since $v_3^{0,(2)} \sim -\frac{2b_2^{1,(1)}}{Q(v_2^{0,(1)})}$, the past part does not cross over, the space part becomes the past part and the future part becomes the space part. Let us then change back the nature of $v_2^{0,(1)}$ from spacelike to future. By applying the same rules, we then get the following alterations :

$$\begin{array}{|c|c|c|} \hline * & + & + \\ \hline + & * & \\ \hline - & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & + & + \\ \hline * & + & \\ \hline * & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & + & + \\ \hline - & * & \\ \hline + & & \\ \hline \end{array}$$

Finally, let us change $v_1^{1,(0)}$ from timelike to spacelike. With the same reasoning as before, we get the following alteration :

$$\begin{array}{|c|c|c|} \hline * & + & + \\ \hline - & * & \\ \hline + & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline * & * & * \\ \hline - & - & \\ \hline + & & \\ \hline \end{array}$$

We have shown that all these open sets of $\Omega(F_0) \cap \Omega(F_\infty)$ were part of the same connected component.

The Θ -positive components correspond to matrices M that have no admissible alteration. We then call the matrix M a Θ -positive sign matrix.

Lemma 3.19. *In the alteration class of any matrix M there is a matrix M' such that for all $2 \leq i \leq q-1$ and for all $0 \leq j \leq q-1-i$, the (i, j) entry of M' is blue when j is even and red when j is odd. We call such a matrix a striped sign matrix.*

Remark 3.20. The color of the first line entries of a striped matrix are determined entirely by the signs of the $\det_i(U)$, hence are invariants of the alteration class of M .

Proof. We prove the result by induction on the column index. If an entry $(i, 0)$ with $i \geq 2$ of the first column is red, it is a $*$ by definition, hence the entry $(i-1, 0)$ is alterable. Doing any alteration of $(i-1, 0)$ will result in $(i, 0)$ being blue. By applying this process starting from

bottom to the top, we can change the whole first column to blue except for the entry $(1, 0)$. Once the column j is either blue or red except for $(1, j)$, the same argument allow us to color the column $j + 1$ with the other color, once again except for $(1, j + 1)$. \square

Remark 3.21. When M is a striped sign matrix and (i, j) is an alterable entry with $i \geq 2$, the only possible alteration at (i, j) takes all four entries (i, j) , $(i + 1, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$ to $*$, obtaining a matrix that is no longer striped. Altering again the same entry, one get back a striped matrix, and every possible such alteration result in a striped matrix M' such that among the entries (i, j) , $(i + 1, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$ of M' , an even number of them have their opposite signs compared to M . We will use this operation of altering twice to change an even number of signs a lot in the following proofs.

Remark 3.22. Because of the phenomenon described in Remark 3.21, any two sign matrices M and M' which have the same $*$ entries but whose total number of $-$ entries differ mod 2 can not be in the same alteration class. In particular, there are at least two alteration classes of striped matrix having the same signs for all $Q(v_1^i)$ for $0 \leq i \leq q - 2$.

Lemma 3.23. *If a striped matrix M is not Θ -positive, there is a alteration-equivalent striped matrix $M' = (m'_{i,j})$ such that:*

$$m'_{i,j} = \begin{cases} + & \text{if } j \text{ or } q - 1 - i \text{ is odd} \\ - & \text{if } j \text{ and } q - 1 - i \text{ are even} \end{cases}$$

unless for all $0 \leq j \leq q - 2$, $m_{1,j} = *$, in which case

$$m'_{i,j} = \begin{cases} \pm & \text{if } i = 2 \text{ and } j = q - 3 \\ + & \text{else if } j \text{ or } q - 1 - i \text{ is odd} \\ - & \text{else if } j \text{ and } q - 1 - i \text{ are even} \end{cases}$$

We call a striped matrix of this form a normalized striped matrix.

Proof. We start by showing that we can alter M to a striped matrix whose $(2, 0)$ entry is alterable if q is odd and whose $(1, 0)$ entry is alterable if q is even. If there are no alterable entry with $i \geq 2$, it means that there is one alterable entry $(1, j)$ on the first line. Altering it twice allow us to change the signs of $(2, j)$ and $(2, j + 1)$, thus making $(2, j - 1)$ alterable because exactly one of the four signs determining if it is alterable changed, and it was not alterable before by hypothesis. There are two edge cases when we can not apply this result, first is when $j = q - 2$ and $m_{1,q-2} = m_{1,q-1} = *$:

$$\begin{array}{ccc} \dots & * & * \\ \dots & \pm & \end{array}$$

in which case $(1, j - 1)$ is also alterable and we can apply the previous construction to $(1, j - 1)$. The second edge case is when $q = 4$ and the first line is all $*$:

$$\begin{array}{ccc} * & * & * \\ \pm & \pm & \\ \pm & & \end{array}$$

We showed how to get to a normal striped form in Example 3.2.

Now we have a alterable entry (i, j) with $i \geq 2$. We will "move" it to make the upper left corner $(2, 0)$ alterable. For this, notice that altering (i, j) twice changing the signs of (i, j) and $(i, j + 1)$ make $(i, j - 1)$ alterable, and altering (i, j) twice changing the signs of (i, j) and $(i + 1, j)$ make $(i - 1, j)$ alterable. Iterating this process allow us to alter into a striped matrix whose $(2, 0)$ entry is alterable. If q is even and the $(1, 0)$ is not alterable yet, we can make it alterable by altering twice $(2, 0)$ changing the sign of $(2, 0)$ and $(3, 0)$.

We want now to have a striped matrix for which every (i, j) entry with $q - i - 1$ even and j even is alterable. We start with the one on the top left, which we already made alterable,

and we will make the one two cells to the right and the one two cells below also alterable. By iterating this process, we will make all of them alterable. If a cell (i, j) is alterable, we can make $(i + 2, j)$ and $(i, j + 2)$ (provided they exist) alterable by making $(i + 1, j + 1)$ alterable: if it is not already the case, we can change signs of (i, j) and $(i + 1, j + 1)$, thus making $(i + 1, j + 1)$ alterable. Then, if neither $(i + 2, j)$ nor $(i, j + 2)$ are alterable, we can make them both alterable by altering $(i + 1, j + 1)$ twice changing signs of $(i + 2, j + 1)$ and $(i + 1, j + 2)$. If both $(i + 2, j)$ and $(i, j + 2)$ are alterable, there is nothing to do. If only $(i + 2, j)$ (resp. $(i, j + 2)$) is alterable and if $(i + 2, j + 2)$ is not out of bounds, we can make $(i, j + 2)$ (resp. $(i + 2, j)$) alterable by altering twice $(i + 1, j + 1)$ changing signs of $(i + 1, j + 2)$ and $(i + 2, j + 2)$ (resp. $(i + 2, j + 1)$ and $(i + 2, j + 2)$). If $(i + 2, j + 2)$ is out of bounds and only exactly one of $(i + 2, j)$ or $(i, j + 2)$ is alterable, we just ensure that $(i + 2, j)$ is alterable by changing signs of $(i + 2, j + 1)$ and $(i + 1, j + 2)$. Applying this process starting with the top-left corner and expanding from it ensures that the only the one in the top right. That entry is either $(2, q - 4)$ if q is even or $(1, q - 3)$ if q is odd. In the latter case, we actually just need to make the $(2, q - 3)$ be a $+$ if the first row is not only made of $*$. For this, we distinguish two cases: either $(1, q - 4)$ is a \pm or it is a $*$. When $(1, q - 4)$ is a \pm , we first make it alterable by changing the signs of $(1, q - 5)$ and $(1, q - 4)$ (since $(1, q - 5)$ is alterable) if it is not. Then changing signs of $(1, q - 4)$ and $(2, q - 3)$ give us the result. When $(1, q - 4)$ is a $*$, notice that $(1, q - 3)$ and $(1, q - 2)$ must be \pm since $(1, q - 3)$ is not alterable. Then by hypothesis there is an entry $(1, j)$ which is not a $*$, and we can assume j to be maximal for this property. Then by altering twice successively $(1, j)$, $(1, j + 1), \dots, (1, q - 4)$, we can change only the signs of $(1, j)$ and $(2, q - 3)$, hence the result. When we need to make $(2, q - 3)$ alterable, we use the same arguments to change its sign thus making it alterable.

At this point, we have a striped matrix whose entries (i, j) with $q - 1 - i$ and j even are alterable (except for $(2, q - 4)$ or $(1, q - 3)$ when the first row is all $*$ which we will discuss later). We can alter twice each of these entries to make all of them be $-$ and all of their left, bottom and bottom-left neighbors $+$, thus resulting in a normalized striped matrix. When the first row is all $*$, the only entry that can not be changed this way is the last one of the second row which stays as a sign \pm . \square

Note that a normalized striped matrix has all its entries (i, j) with $i \geq 2$ alterable.

Lemma 3.24. *In the alteration class of any matrix that is not Θ -positive, there is a unique normalized striped matrix such that the first row is either all red $*$ or a sequence of \pm and $*$ such that only the last sign \pm of the row may be $-$.*

Proof. Let M be a normalized striped matrix. Let j_0 such that $m_{1, j_0} = -$ and there exists a $j_1 > j_0$ such that $m_{1, j_1} = \pm$. We can suppose that j_1 is minimal for this property. Then for all $j_0 < j < j_1$, we have $m_{1, j} = *$. Then by altering twice $(1, j_0), (1, j_0 + 1), \dots, (1, j_1 - 1)$, we can change only the signs of $(1, j_0)$ and $(1, j_1)$. When $j_1 = j_0 + 1$, it may happen that $(1, j_0)$ is not alterable. If that is the case, altering twice $(2, j_0)$ changing the signs of $(2, j_0)$ and $(2, j_0 + 1)$ makes $(1, j_0 + 1)$ alterable, then altering twice $(1, j_0 + 1)$ changing the signs of $(1, j_0 + 1)$ and $(1, j_0 + 2)$ makes $(1, j_0)$ alterable, and finally altering twice again $(2, j_0)$ takes the matrix back to being normalized striped form, except that now $(1, j_0)$ is alterable and we can proceed. \square

Remark 3.25. From the previous lemmas we get that the the alteration class of a sign matrix (hence a connected component of $\Omega(F_0) \cap \Omega(F_\infty)$) that is not Θ -positive is entirely determined by the sequence of blue/red entries of the first line and the parity of the number of $-$ entries of any striped sign matrix in its alteration class.

From these lemmas we obtain the following count of the number of connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ for all p and q :

Proposition 3.26. *In $\mathrm{SO}(p, q)/P_{1, \dots, q-1}$, for $p > q$:*

- *When $q = 2$, there are 3 connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 2 of which are Θ -positive.*
- *When $q = 3$, there are 11 connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 4 of which are positive.*
- *When $q \geq 4$, there are $3 \times 2^{q-1}$ connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 2^{q-1} of which are Θ -positive.*

Remark 3.27. The two cases we have already treated, $\mathrm{SO}_0(p, 2)/P_1$ and $\mathrm{SO}_0(p, 3)/P_{1,2}$, are actually special cases where the general formula does not apply.

Remark 3.28. The Θ -positive components can be obtained in the following way: the corresponding sign matrix must not contain any $*$, hence is striped, with a first line being also striped, i.e. $Q(v_i^{j, (i-1)})$ is positive when j is odd and negative when j is even for all $1 \leq i \leq q-1$. Then, the component is determined by the signs \pm of the first line: indeed, since no entry is alterable, fixing any signs in the first line determines uniquely the $(2, q-3)$ entry by $\mu_{2, q-3} = +$, which in turn determines $(2, q-4)$ etc. By computing line by line starting with the second line, going from right to left on each line, one can find a unique Θ -positive striped matrix having the prescribed first line, thus describing the 2^{q-1} Θ -positive components. Computing all the vectors $v_i^{j, (i-1)}$ and checking that no entry of the corresponding sign matrix is alterable allow for an explicit algorithm to decide whether a matrix in U_Θ is Θ -positive. Also note that with our conventions on the quadratic form Q , the sign matrix with only $+$ entries is *not* Θ -positive.

3.5. **(1, ..., q)-flags in $\mathrm{SO}_0(p, q)$.** Assume that $p > q+1$. Let $F \in \Omega(F_0) \cap \Omega(F_\infty) \subset \mathrm{SO}_0(p, q)/P_{1, \dots, q}$. The first $(q-1)$ equations already separates $\Omega(F_0) \cap \Omega(F_\infty)$ into $3 \times 2^{q-1}$ connected components. When all relevant elements are non-zero, the equation added by the transversality of the maximal isotropic spaces is $Q(v_q^{0, (q-1)}) \neq 0$. However $v_q^{0, (q-1)}$ belongs in the Euclidean space $\mathbb{R}^{p-q, 0}$ which is of dimension at least 2 since $p > q+1$, meaning that the space $Q(v_q^{0, (q-1)}) \neq 0$ is actually connected.

Proposition 3.29. *The connected components of $\Omega(F_0) \cap \Omega(F_\infty) \subset \mathrm{SO}_0(p, q)/P_{1, \dots, q}$ are exactly those coming from the transversality in flags of $\mathrm{SO}_0(p, q)/P_{1, \dots, q-1}$, meaning that the map induced on connected components by $\pi : \mathrm{SO}_0(p, q)/P_{1, \dots, q} \rightarrow \mathrm{SO}_0(p, q)/P_{1, \dots, q-1}$ is bijective.*

Proof. By combining proposition 3.11 with the fact that the $v_j^{0, (j-1)}$ live in a connected space, one gets the result. \square

3.6. **(1, ..., q)-flags in $\mathrm{SO}_0(q+1, q)$.** Let us now deal with the case of maximal flags when $p = q+1$. Since the group $\mathrm{SO}_0(q+1, q)$ is a split group for which $P_{1, \dots, q}$ is a Borel subgroup, we know from [Lus94] that there must be 2^q totally positive connected components in $\Omega(F_0) \cap \Omega(F_\infty)$. This stems from the fact that the $v_j^{0, (j-1)}$ now live in the Euclidean space \mathbb{R} and that the equation $Q(v_q^{0, (q-1)}) \neq 0$ separates \mathbb{R} into two connected components determined by the sign of $v_q^{0, (q-1)}$.

Proposition 3.30. *In $\mathrm{SO}_0(q+1, q)/P_{1, \dots, q}$:*

- *When $q = 2$, there are 8 connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 4 of which are totally positive.*
- *When $q = 3$, there are 30 connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 8 of which are totally positive.*
- *When $q \geq 4$, there are $(q+5)2^{q-1}$ connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, 2^q of which are totally positive.*

Proof. Let $(v_1^{q-1, (0)}, \dots, v_q^{0, (0)})$ be an element of u_Θ . As proven before, the first $q-1$ equations of transversality already splits $\Omega(F_0) \cap \Omega(F_\infty)$ into $3 \times 2^{q-1}$ connected components, 2^{q-1} of which are Θ -positive. Let us assume that $(v_1^{q-1, (0)}, \dots, v_{q-1}^{1, (0)})$ lives in a component which is not Θ -positive.

The last equation of transversality is

$$[v_1^{0,(0)}]^2 \dots [v_q^{0,(q-1)}]^2 \neq 0,$$

with the cells where the change of variables are defined being those where all the $v_j^{0,(j-1)}$ are positive or negative for $j < q$. The sign of $v_q^{0,(q-1)}$ separates each cell into two connected components. Depending on the cell of in which $(v_1^{0,(0)}, \dots, v_{q-1}^{0,(0)})$ is, the signs of the $b_j^{1,(j-1)}$ may be fixed or not. Assume that it is not fixed and let us change the sign of $v_j^{0,(j-1)}$ from positive to negative. Let us first take $b_j^{0,(j-1)} > 0$. Since $v_{j+1}^{0,(j)} \sim \frac{2b_j^{1,(j-1)}}{v_j^{0,(j-1)}}$, we have $v_{j+1}^{0,(j)} \rightarrow +\infty$ which means that the space $v_{j+1}^{0,(j)} < 0$ does not cross over. However, when taking $b_j^{0,(j-1)}$ negative, the opposite happens, meaning that in the end both $v_{j+1}^{0,(j)} < 0$ and $v_{j+1}^{0,(j)} > 0$ cross over. When assuming that the sign of $b_j^{0,(j-1)}$ is fixed, one then cannot connect all the parts of the cells into two different connected components.

However, since it was assumed that $(v_1^{q-1,(0)}, \dots, v_{q-1}^{1,(q-2)})$ was not in a Θ -positive component, it is possible the get into a different cell within the same connected component of the sub-maximal flags. Doing this can always make it so that the sign of $b_j^{0,(j-1)}$ is no longer fixed. Furthermore, switching the cell of $(v_1^{q-1,(0)}, \dots, v_{q-1}^{1,(q-2)})$ does not change the signs of $v_1^{0,(0)}, \dots, v_q^{0,(q-1)}$, meaning that the connected component of the sub-maximal flags is split into two connected components when adding the last equation.

Let us now assume that $(v_1^{q-1,(0)}, \dots, v_{q-1}^{1,(q-2)})$ is in a Θ -positive component. The sign of the $b_j^{1,(j-1)}$ are now all fixed. One then sees from switching the signs of the $v_j^{0,(j-1)}$ that the last equation split the component into $q + 1$ connected components, two of which are positive. Combining those two cases gives the result.

$b_1^{1,(0)} > 0, b_2^{1,(1)} > 0, b_3^{1,(2)} > 0, b_4^{1,(3)} > 0$															
$v_1^{0,(0)} < 0$								$v_1^{0,(0)} > 0$							
$v_2^{0,(1)} < 0$				$v_2^{0,(1)} > 0$				$v_2^{0,(1)} < 0$				$v_2^{0,(1)} > 0$			
$v_3^{0,(2)} < 0$		$v_3^{0,(2)} > 0$		$v_3^{0,(2)} < 0$		$v_3^{0,(2)} > 0$		$v_3^{0,(2)} < 0$		$v_3^{0,(2)} > 0$		$v_3^{0,(2)} < 0$		$v_3^{0,(2)} > 0$	
$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$	$v_4^{0,(3)} < 0$	$v_4^{0,(3)} > 0$

FIGURE 3.7. The six connected component in a Θ -positive cell of the sub-maximal flags in $\mathrm{SO}_0(6, 5)$. The components in pink and yellow are totally positive.

□

Remark 3.31. This result was already known due to Gekhtman-Shapiro-Vainshtein, see [GSV03].

Remark 3.32. When all necessary vectors are non-zero, the last equation of transversality becomes

$$[v_1^{0,(0)}]^2 \dots [v_q^{0,(q-1)}]^2 \neq 0.$$

However, since $[v_1^{0,(0)}]^2 \dots [v_q^{0,(q-1)}]^2 = [v_1^{0,(0)} \dots v_q^{0,(q-1)}]^2$. Let us define the map $f : u_\Theta \rightarrow \mathbb{R}$ as

$$f(v_1^{q-1}, \dots, v_q^0) \mapsto v_1^{0,(0)} \dots v_q^{0,(q-1)}.$$

One may check that f is actually polynomial and thus may be defined on the whole u_Θ , regardless of the fact that the change of variables may sometimes be undefined. One then gets that $\det(S) = f^2$, and that the sign of f splits $\Omega(F_0) \cap \Omega(F_\infty)$ into two groups of connected components. In fact, this precisely gives the two connected components composing the groups defined by the first $(q-1)$ equations of transversality which are not Θ -positive.

3.7. Incomplete flags in $\mathrm{SO}_0(p, q)$. From the computations we have done we can deduce the number of connected components of $\Omega(F_0) \cap \Omega(F_\infty) \subset \mathrm{SO}_0(p, q)/P_\Theta$ where Θ is a subset of the set of roots of $\mathrm{SO}_0(p, q)$ which has not been already covered and $p > q$. In order to do this, one only has to take the description of connected components in $\mathrm{SO}_0(p, q)/P_{1, \dots, k}$ where $k = \max(\Theta)$ and identify the connected components which are no longer separated by an equations $Q(v_j^{k-j, (j-1)}) \neq 0$. We will give a quasi-complete list of all the cases barring those involving the biggest roots of $\mathrm{SO}_0(q, q)$ which will require its own separate study.

Proposition 3.33. *In $\Omega(F_0) \cap \Omega(F_\infty)$:*

- When Θ does not contain the biggest root and is not equal to $\{1, \dots, q-1\}$, there are $2^{|\Theta|}$ connected components determined by the signs of the significant minors of S .
- When Θ contains the biggest root, is not equal to $\{1, \dots, q\}$ and $p > q+1$, there are $2^{|\Theta|-1}$ connected components determined by the signs of the significant minors of S except the determinant of S itself which is always non-negative and does not add any connected components.
- When Θ contains the biggest root, is not equal to $\{1, \dots, q\}$ and $p = q+1$, there are $2^{|\Theta|}$ connected components determined by the signs of the significant minor of S except the determinant of S itself which is always non-negative but split in two connected components nonetheless.

Proof. Let us first suppose that Θ does not contain the last root and $p > q$. Then $\Theta \subset \{1, \dots, q-1\}$ and the projection from $\Omega(F_0^{1, \dots, q-1}) \cap \Omega(F_\infty^{1, \dots, q-1})$ to $\Omega(F_0^\Theta) \cap \Omega(F_\infty^\Theta)$ obtained by forgetting the subspaces whose dimensions do not belong to Θ is surjective. However since $\Theta \neq \{1, \dots, q-1\}$ there is $j_0 \in \{1, \dots, q-1\} \setminus \Theta$, and the alteration classes of matrices defined in 3.23 are identified if they differ only from their j_0 column. This identifies the two different classes of normalized striped matrices having the same signs of all $\det_i(U)$, showing that there are at most $2^{|\Theta|}$ connected components in $\Omega(F_0^\Theta) \cap \Omega(F_\infty^\Theta)$. However since there are still $|\Theta|$ minors that have to be non-zero, there are at least $2^{|\Theta|}$ connected components, hence the result. \square

3.8. Maximal photons in $\mathrm{SO}_0(q, q)$. We are only going to discuss partially the case $p = q$. In this case, the group $\mathrm{SO}_0(q, q)$ is split.

Proposition 3.34. *The space of maximal photons of $\mathrm{Ein}_{q-1, q-1}$ is the union of two connected components.*

Proof. The space $\mathrm{Ein}_{q-1, q-1}$ is conformally equivalent to $\mathbb{S}^{q-1} \times \mathbb{S}^{q-1}$ and the maximal photons are exactly the graphs of isometries $f : \mathbb{S}^{q-1} \rightarrow \mathbb{S}^{q-1}$ for the round Riemannian metric. The space of isometries $\mathrm{Isom}(\mathbb{S}^{q-1}) \simeq O(q)$ is the union of two connected components, hence the result. \square

This stems from the fact the the root diagram of $\mathrm{SO}_0(q, q)$ has two highest roots, each corresponding to a connected component of the space of maximal photons in $\mathrm{SO}_0(q, q)$. We will denote by P_q and P'_q the parabolic subgroups associated to those two roots.

Proposition 3.35. *For each sub-maximal photon Q in $\mathrm{SO}_0(q, q)/P_{q-1}$, there exists a unique pair of maximal photons in $\mathrm{SO}_0(q, q)/P_q \times \mathrm{SO}_0(q, q)/P'_q$ each containing Q .*

Proof. Let Q be an isotropic sub-vector space in $\mathbb{R}^{q,q}$ of dimension $q-1$. The orthogonal of Q is of dimension $q+1$ and is the orthogonal sum of Q with a space of signature $(1, 1)$, $Q^\perp = Q \oplus^\perp \mathbb{R}^{1,1}$. Taking an isotropic space of dimension q containing Q is equivalent to taking an isotropic line in $\mathbb{R}^{1,1}$. There are exactly two isotropic lines in $\mathbb{R}^{1,1}$, hence the result. \square

In particular, P_q and P'_q may not be self-opposite, while every other parabolic space associated with $\mathrm{SO}_0(p, q)$ is self-opposite.

Proposition 3.36. *The spaces $\mathrm{SO}_0(q, q)/P_q$ and $\mathrm{SO}_0(q, q)/P'_q$ are self-opposite if and only if $q = 0 \pmod{2}$.*

Proof. Let $F_0 \in \mathrm{SO}_0(q, q)/P_q$ be the graph of the identity map in \mathbb{S}^{q-1} for a conformal model $\mathrm{Ein}_{q-1, q-1} \simeq \mathbb{S}^{q-1} \times \mathbb{S}^{q-1}$. Two maximal photons in $\mathrm{Ein}_{q-1, q-1}$ are transverse if and only if they do not intersect. Any other maximal photon in $\mathrm{Ein}_{q-1, q-1}$ can be seen as the graph of an isometry $g : \mathbb{S}^{q-1} \rightarrow \mathbb{S}^{q-1}$. When q is odd, any direct isometry in $\mathrm{SO}_0(q)$ must have a fixed point, meaning that its graph in $\mathrm{Ein}_{q-1, q-1}$ must intersect F_0 . Therefore any maximal photon transverse to F_0 must be within $\mathrm{SO}_0(q, q)/P'_q$. Inversely, when q is even, any maximal photon transverse to F_0 must be within $\mathrm{SO}_0(q, q)/P_q$, hence the result. \square

The question of counting and parametrizing connected components in $\Omega(F_0) \cap \Omega(F_\infty)$ can then only make sense when q is even.

Proposition 3.37. *When q is even, the subset $\Omega(F_0) \cap \Omega(F_\infty)$ of $\mathrm{SO}_0(q, q)/P_q$ has two connected components.*

Proof. As before, the equation of transversality is given by the determinant of the matrix S , $\det(S) \neq 0$. However in $\mathrm{SO}_0(q, q)/P_q$, all the a_j^i and v_j^i are equal to zero and the matrix S is skew-symmetric. In particular, when q is even, $\det(S)$ is always non-negative. It is then well known that the space $\det(S) \neq 0$ is the union of two connected components determined by the sign of the Pfaffian of S , hence the result. \square

4. OBSTRUCTIONS ON THE STRUCTURE OF P_Θ -ANOSOV SUBGROUPS OF $\mathrm{SO}_0(p, q)$

The aim of this section is to answer a variant of a question by Sambarino regarding the structure of Anosov subgroups.

Question 4.1. *Let Γ be a Borel Anosov subgroup of $\mathrm{SL}(d, \mathbb{R})$. Is Γ virtually isomorphic to either a free group of a surface group ?*

In [DGR23], Dey-Greenberg-Riestenberg generalize the arguments developped in [Dey24] to any P_Θ -Anosov subgroup of a semi-simple Lie group G as long as P_Θ is self-opposite. Let P_Θ be a parabolic subgroup of a semi-simple group G such that G/P_Θ is self-opposite. Let F_0, F_∞ be two transverse points in G/P_Θ . The unipotent radical U_Θ acts freely and transitively on $\Omega(F_\infty)$, which gives a parametrization $g \in U_\Theta \mapsto g \cdot F_0$. This parametrization of $\Omega(F_\infty)$ endows it with a continuous involution i inherited from U_Θ defined by $i(g \cdot F_0) = g^{-1} \cdot F_0$. Since F_0 is fixed by i and i preserves transversality, the involution acts on $\Omega(F_0) \cap \Omega(F_\infty)$ and on $\pi_0(\Omega(F_0) \cap \Omega(F_\infty))$.

Theorem 4.2 (Dey-Greenberg-Riestenberg, [DGR23]). *Assume that*

$$i : \pi_0(\Omega(F_0) \cap \Omega(F_\infty)) \rightarrow \pi_0(\Omega(F_0) \cap \Omega(F_\infty))$$

does not have any fixed point. Then any P_Θ -Anosov subgroup of G is virtually isomorphic to either a free group or a surface group.

Our aim is to compute the action of the involution on the connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ for any parabolic subgroups P_Θ of $\mathrm{SO}_0(p, q)$, to answer positively to Question 4.1 (when replacing $\mathrm{SL}(d, \mathbb{R})$ with $\mathrm{SO}(p, q)$) when possible and to provide counter-examples in some other cases.

4.1. Computing the involution. In order to parametrise the space $\Omega(F_\infty)$ in $\mathrm{SO}_0(p, q)/P_{1, \dots, k}$, we used the diffeomorphism from u_Θ to U_Θ ,

$$(v_1^{k-1}, \dots, v_k^0) \in u_\Theta \longmapsto \exp(A_1(v_1^{k-1})) \dots \exp(A_k(v_k^0)) \cdot F_0 \in \mathrm{SO}_0(p, q)/P_{1, \dots, k}.$$

We wish to compute the effect of the involution on the coordinates in u_Θ . For each $(v_1^{k-1}, \dots, v_k^0)$ in u_Θ , there exists an element $(i(v_1^{k-1}), \dots, i(v_k^0))$ in u_Θ such that

$$\begin{aligned} [\exp(A_1(v_1^{k-1})) \dots \exp(A_k(v_k^0))]^{-1} &= \exp(-A_k(v_k^0)) \dots \exp(-A_1(v_1^{k-1})) \\ &= \exp(A_1(i(v_1^{k-1}))) \dots \exp(A_k(i(v_k^0))). \end{aligned}$$

We will write

$$\begin{aligned} i(v_1^{k-1}) &= (i(a_1^{k-1}), \dots, i(a_1^1), i(v_1^0), i(b_1^1), \dots, i(b_1^{k-1})), \\ &\dots \\ i(v_{k-1}^1) &= (i(a_{k-1}^1), i(v_{k-1}^0), i(b_{k-1}^1)), \\ i(v_k^0) &= i(v_k^0). \end{aligned}$$

Proposition 4.3. *The involution on u_Θ is as follows :*

$$\begin{aligned} i(a_j^i) &= -a_j^i + \sum_{l=1}^{k-i-j} i(a_j^{i+l}) i(a_{j+l}^i), \\ i(v_j^0) &= -v_j^0 + \sum_{l=1}^{k-j} i(a_{j+l}^1) i(v_{j+l}^0), \\ i(b_j^i) &= -b_j^i - Q(v_j^i, v_j^i) - \sum_{l=1}^{k-j-i+1} a_j^{k-i-l+1} i(b_{j+l}^i). \end{aligned}$$

Proof. This is a straightforward computation. Of particular note is that the involution acts by transposition on the matrix S , providing further evidence that i preserves the transversality to F_0 . \square

4.2. $(1, \dots, k)$ -flags in $\mathrm{SO}_0(p, q)$. As explained in proposition 3.16, the connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ in $\mathrm{SO}_0(p, q)/P_{1, \dots, k}$, $k < q - 1$ are determined by the signs of the successive upper-left minors of S . Since the involution acts on S by transposition, those minors are left unchanged and every connected component of $\Omega(F_0) \cap \Omega(F_\infty)$ is preserved by the involution.

Proposition 4.4. *When $\Theta = \{1, \dots, k\}$ and $k < q - 1$, the action of i on $\pi_0(\Omega(F_0) \cap \Omega(F_\infty))$ is trivial.*

4.3. $(1, \dots, q-1)$ -**flags in $SO_0(p, q)$** . In order to compute the action of i on the connected components of $\Omega(F_0) \cap \Omega(F_\infty)$, one only has to exhibit one element of each connected component on which the involution is easily computable. Let us first assume that for each i, j , $a_j^i = 0$. We deduce from the formulas of the change of variables that for each i, j, m , $a_j^{i,(m)} = 0$. This implies that for each j , $Q(v_j^{k-j,(j-1)}) = Q(v_j^{k-j-1,(j-1)}) = \dots = Q(v_j^{0,(j-1)})$. Since only the signs of $Q(v_1^{k-1,(0)}), Q(v_2^{k-2,(1)}), \dots, Q(v_{q-2}^{0,(q-3)})$ are fixed, it is possible to find a cell of any connected components determined by the first $q-2$ equations in which all the a_j^i are equal to zero. Since the Θ -positive connected components are never stable by the involution, we will not discuss them.

As the involution preserves to signs of the upper-left minors, and since all the a_j are equal to zero, it must also preserve each equation set ($Q(v_j^{0,(j-1)}) \neq 0$). Let us determine the action of the involution on its connected components.

Proposition 4.5. *For each j , the action of I on $(Q(v_j^{0,(j-1)}) \neq 0)$ always sends the space component on itself. When j is even, the future and past components are preserved. When j is odd, the future and past components are exchanged.*

Proof. For $j = 1$, the result is clear since $i(v_1^0) = -v_1^0$. For $j = 2$, we have $v_2^{0,(1)} = v_2^0 + \frac{2b_1^{q-2}}{Q(v_1^0)}v_1^0$. Let us take $v_2^0 = 0$. Since the involution acts by $i(b_1^{q-2}) = -b_1^{q-2} - Q(v_1^0, v_2^0)$, we now have $i(b_1^{q-2}) = -b_1^{q-2}$. Since $i(v_1^0) = -v_1^0$, this gives us $i(v_2^{0,(1)}) = v_2^{0,(1)}$, hence the result.

Let us take $v_\ell^0 = 0$ and $b_\ell^i = 0$ for each ℓ even. For $j = 3$, we have

$$v_3^{0,(2)} = v_3^0 + \frac{2b_1^{q-3}}{Q(v_1^0)}v_1^0 + \frac{2b_2^{q-3,(1)}}{Q(v_2^{0,(1)})}v_2^{0,(1)}.$$

Since $v_2^0 = 0$ and $b_2^{q-3} = 0$, we have

$$b_2^{q-3,(1)} = -\frac{2b_1^{q-2}b_1^{q-3}}{Q(v_1^0)},$$

as well as $v_2^{0,(1)} = \frac{2b_1^{q-2}}{Q(v_1^0)}v_1^0$. This gives us

$$\begin{aligned} v_3^{0,(2)} &= v_3^0 + \frac{2b_1^{q-3}}{Q(v_1^0)} - \frac{2b_1^{q-3}}{Q(v_1^0)} \\ &= v_3^0. \end{aligned}$$

By repeating this process, we obtain that when j is even, $v_j^{0,(j-1)} = \frac{2b_{j-1}^{k-j+1}}{Q(v_{j-1}^{0,(j-2)})}v_{j-1}^{0,(j-2)}$ and when j is odd, $v_j^{0,(j-1)} = v_j^0$. This gives us the intended result. \square

The first $q-2$ equations of transversality split $\Omega(F_0) \cap \Omega(F_\infty)$ into 2^{q-2} groups of connected components. When all relevant variables are non-zero, the last equation of transversality is

$$Q(v_1^{0,(0)})Q(v_2^{0,(1)}), \dots, Q(v_{q-1}^{0,(q-2)}) \neq 0.$$

The sign of the determinant of S once again splits each of the 2^{q-2} groups into two groups. Finally, as proven in 3.26, the parity of the number of timelike vectors in the $v_1^{0,(0)}, \dots, v_{q-1}^{0,(q-2)}$ which are in the future cone splits each of these groups into two connected components.

Proposition 4.6. *Among the $3 \times 2^{q-1}$ connected components in $\Omega(F_0) \cap \Omega(F_\infty)$, the 2^{q-1} Θ -positive components are all exchanged, 2^{q-1} of the remaining components are exchanged and the rest is stable by i .*

Proof. Assume that q is even. The involution thus acts on $(v_{q-1}^{0,(q-2)} \neq 0)$ by exchanging the past and future parts. Let us chose the vectors $v_1^{0,(0)}, \dots, v_{q-2}^{0,(q-3)}$. The choice of $v_{q-1}^{0,(q-2)}$ determines three connected components. If the parity of the number of future vectors among the timelike vectors of the $v_1^{0,(0)}, \dots, v_{q-2}^{0,(q-3)}$ is unchanged by the involution, the component where $v_{q-1}^{0,(q-2)}$ is spacelike must be sent to itself and the two components where $v_{q-1}^{0,(q-2)}$ is past and future must be sent to each other. Since i is an involution, the fourth connected component must be stable by i . If the parity of the number of future vectors among the timelike vectors of the $v_1^{0,(0)}, \dots, v_{q-2}^{0,(q-3)}$ is changed by the involution, the component where $v_{q-1}^{0,(q-2)}$ is spacelike must not be stable by the involution while the two components defined by $v_{q-1}^{0,(q-2)}$ past and future are. This tells us that half of the non-positive components must be stable by i while the other must be unstable, hence the result.

Assume that q is odd. The involution thus acts on $(v_{q-1}^{0,(q-2)} \neq 0)$ by preserving the past and future parts. If the parity of the number of future vectors among the timelike vectors of the $v_1^{0,(0)}, \dots, v_{q-2}^{0,(q-3)}$ is unchanged by the involution, the three components given by the choice of $v_{q-1}^{0,(q-2)}$ must be preserved by i . If the parity of the number of future vectors among the timelike vectors of the $v_1^{0,(0)}, \dots, v_{q-2}^{0,(q-3)}$ is changed by the involution, the components given by the choice of $v_{q-1}^{0,(q-2)}$ must not be stable by i . This gives us the result. \square

Remark 4.7. In particular, there always exists at least one connected component which is stable by involution. The component where all the $v_j^{0,(j-1)}$ are spacelike is of particular interest as it is easy to exhibit antipodal spheres of higher dimensions contained inside. Let us take all a_j^i, b_j^i equal to zero and embed $\mathbb{R}^{p-q+1,0}$ in $\mathbb{R}^{p-q+1,1}$ as a positive sub-vector space. The map

$$\varphi : v \in \mathbb{R}^{p-q+1} \longmapsto \exp(A_1(v)) \dots \exp(A_{q-1}(v)) \cdot F_0$$

satisfies that for each distinct pair v, v' , the elements $\varphi(v)$ and $\varphi(v')$ are transverse. Adding F_∞ as the point at infinity gives us an antipodal sphere of dimension $p - q + 1$. It is not known to us whether this sphere may be realised as the boundary of a $P_{1,\dots,q-1}$ -Anosov subgroup, except when $q = 2$ where those are the boundaries of fuchsian representations of uniform lattices of $\text{SO}_0(p, 1)$.

Remark 4.8. When considering the complete flags of photons $\text{SO}_0(p, q)/P_{1,\dots,q}$ for $p > q + 1$, since the last equation does not change the count of the connected components, the computation of the involution remains the same.

4.4. Maximal photons in $\text{SO}_0(q + 1, q)$. Let us compute the involution for the space of maximal photons $\text{SO}_0(q + 1, q)/P_q$. The space of complete flags of photons will be discussed at the end of the section.

Dealing only with the maximal photons is formally equivalent to taking all the a_j^i equal to zero, which brings us back to proposition 4.5. The equation of transversality is given by $f^2 \neq 0$, where f is defined in remark 3.32 and the two connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ are determined by the sign of f . When all necessary vectors are non-zero, f is equal to $v_1^{0,(0)} \dots v_q^{0,(q-1)}$. The same reasoning as in 4.5 tells us that depending on the parity of j , $(v_j^{0,(j-1)} < 0)$ and $(v_j^{0,(j-1)} > 0)$ are either preserved or sent to each other. One may easily determine in which case the sign of f is reversed and in which case it is invariant.

Proposition 4.9. *When $q = 1$ or $q = 2 \pmod{4}$, the two connected components of $\text{SO}_0(q + 1, q)/P_q$ are swapped by i . When $q = 3$ or $q = 0 \pmod{4}$, the two connected components of $\text{SO}_0(q + 1, q)/P_q$ are stable by i .*

From this we may deduce the following theorem using Theorem 4.2:

Theorem 4.10. *Assume that $q = 1$ or $q = 2 \pmod{4}$ and let Γ be a P_Θ -Anosov subgroup of $\mathrm{SO}_0(q+1, q)$ where Θ contains the last root. Then Γ is virtually isomorphic to either a free group or a surface group.*

This applies in particular to Borel-Anosov subgroups of $\mathrm{SO}_0(q+1, q)$; however this result was already known due to Dey ([Dey24]).

4.5. Maximal photons in $\mathrm{SO}_0(q, q)$. As stated in proposition 3.36, when q is even, the space $\Omega(F_0) \cap \Omega(F_\infty)$ is the union of two connected components. Since all a_j^i and v_j^i are equal to zero, the involution becomes $i(b_j^i) = -b_j^i$.

Proposition 4.11. *When $q = 0 \pmod{4}$, the two connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ are stable under the involution. When $q = 2 \pmod{4}$, the two connected components of $\Omega(F_0) \cap \Omega(F_\infty)$ are swapped by the involution.*

Proof. The two connected components are determined by the sign of the Pfaffian of S . The effect of the involution on S is $i(S) = -S$. When the size of S is equal to $0 \pmod{4}$, $\mathrm{Pf}(-S) = \mathrm{Pf}(S)$ while when it is equal to $2 \pmod{4}$, $\mathrm{Pf}(-S) = -\mathrm{Pf}(S)$, hence the result. \square

Using Theorem 4.2, we obtain the following result:

Theorem 4.12. *Assume that $q = 2 \pmod{4}$ and let Γ be a $P_q\Theta$ -Anosov subgroup of $\mathrm{SO}_0(q, q)$ containing one of the two last roots. Then Γ is virtually isomorphic to either a free group or a surface group.*

5. CONSTRUCTING HIGHER DIMENSIONAL ANOSOV SUBGROUPS

This section will be dedicated to the construction of P_Θ -Anosov subgroups of $\mathrm{SO}_0(p, q)$ which are not virtually isomorphic to a surface group or a free group for some of the Θ which do not fall under the assumptions of Theorems 4.12 or ??.

Lemma 5.1. *Let $P_\Theta \subset \mathrm{SO}_0(p, q)$ be a parabolic subgroup and let x^+, x^- be two transverse points in $\mathrm{SO}_0(p, q)/P_\Theta$. Then there exists $\delta \in \mathrm{SO}_0(p, q)$ such that $\langle \delta \rangle$ is P_Θ -Anosov with boundary $\{x^+, x^-\}$.*

Proof. Since x^+ and x^- are transverse, they can be completed into two transverse complete flags $\bar{x}^+, \bar{x}^- \in \mathrm{SO}_0(p, q)/P_{1, \dots, q}$. We thus have $\bar{x}^\pm = (F_1^\pm, \dots, F_q^\pm)$ with $F_i^- \oplus F_i^+ \simeq \mathbb{R}^{i, i}$ for each i . Let us decompose orthogonally $F_q^- \oplus F_q^+ = \mathbb{R}_1^{1,1} \oplus \dots \oplus \mathbb{R}_q^{1,1}$ such that for each i , $F_i^- \oplus F_i^+ = \mathbb{R}_1^{1,1} \oplus \dots \oplus \mathbb{R}_i^{1,1}$ and let $V = (F_q^- \oplus F_q^+)^\perp$. Finally, let δ be the transformation acting trivially on V and

$$\delta|_{\mathbb{R}_i^{1,1}} = \begin{pmatrix} \lambda^i & 0 \\ 0 & \lambda^{-i} \end{pmatrix},$$

with $\lambda > 1$. The subgroup $\langle \delta \rangle$ is $P_{1, \dots, q}$ -Anosov with boundary $\{\bar{x}^+, \bar{x}^-\}$ in $\mathrm{SO}_0(p, q)/P_{1, \dots, q}$. In particular, $\langle \delta \rangle$ is also P_Θ -Anosov with boundary $\{x^+, x^-\}$, hence the result. \square

Proposition 5.2. *Let P_Θ be a parabolic subgroup of $\mathrm{SO}_0(p, q)$, Γ a P_Θ -Anosov subgroup of $\mathrm{SO}_0(p, q)$ and x^+, x^- two points of $\mathrm{SO}_0(p, q)/P_\Theta$ which are transverse to each other and such that for all $y \in \partial\Gamma \subset \mathrm{SO}_0(p, q)/P_\Theta$, x^\pm and y are transverse. Let $\delta \in \mathrm{SO}_0(p, q)$ associated to x^+ and x^- by the previous lemma. Then there exists a finite index subgroup $\Gamma' \subset \Gamma$ and $k \in \mathbb{N}$ such that $\langle \Gamma', \delta^k \rangle$ is P_Θ -Anosov and isomorphic to the free product $\Gamma' * \mathbb{Z}$.*

Proof. This is a direct application of a result by Dey and Kapovich ([DK23], [DK24]). Since x^+ and x^- are transverse to the whole boundary of Γ , there must exist pairwise transverse compacts B, A^+, A^- such that $\partial\Gamma \subset B$ and $x^\pm \in A^\pm$. Since A^\pm is transverse to $\partial\Gamma$, there exists $C > 0$ such that if $\gamma \in \Gamma$ satisfies $|\gamma| > C$, $\gamma \cdot A^\pm \subset B$, thus there exists a finite index subset $\Gamma' \subset \Gamma$ such that for all non trivial $\gamma \in \Gamma'$, $\gamma \cdot A^\pm \subset B$. Inversely, there exists k such that $\delta^k \cdot B \subset A^+$ and $\delta^{-k} \cdot B \subset A^-$. By applying the result from Dey and Kapovich, we then get that $\langle \Gamma', \delta^k \rangle$ is P_Θ -Anosov and isomorphic to $\Gamma' * \mathbb{Z}$. \square

Remark 5.3. When Γ is a surface group, any finite index subgroup $\Gamma' \subset \Gamma$ is also a surface group.

Since the free product of a surface group with \mathbb{Z} is not virtually isomorphic to either a surface group or a free group, this means that in order to construct a suitable P_Θ -Anosov subgroup one only needs to find a P_Θ -Anosov subgroup Γ isomorphic to a surface group and two transverse points x^+ and x^- in $\text{SO}_0(p, q)/P_\Theta$ which are transverse to the boundary of Γ .

Proposition 5.4. *Let $P_\Theta = P_{1, \dots, q-2}$ be the stabilizer of a sub-sub-maximal flag of isotropic spaces in $\mathbb{R}^{p, q}$. There exists a P_Θ -Anosov subgroup of $\text{SO}_0(p, q)$ which is isomorphic to the free product of a surface group and a cyclic group.*

Proof. Let Γ be a Θ -positive subgroup of $\text{SO}_0(q-1, q-1)$. Since the Borel subgroup of $\text{SO}_0(q-1, q-1)$ is the stabilizer of a sub-maximal flag of isotropic spaces in $\mathbb{R}^{q-1, q-1}$, the inclusion $\Gamma \subset \text{SO}_0(q-1, q-1) \subset \text{SO}_0(p, q)$ with $p \geq q$ gives a $P_{1, \dots, q-2}$ -Anosov subgroup of $\text{SO}_0(p, q)$. For each $k \leq q-2$, let $\pi_k : \text{SO}_0(p, q)/P_{1, \dots, q-2} \rightarrow \text{SO}_0(p, q)/P_k$ be the projection on the k -th factor. Let x_0 and x_∞ be two distinct points in $\partial\Gamma$. In the affine chart of $\text{Ein}_{p-1, q-1}$ defined by $\pi_1(\xi(x_0))$ and $\pi_1(\xi(x_\infty))$, the image of $\pi_1 \circ \xi$ is contained within a sub-vector space V of $\mathbb{R}^{p-1, q-1}$ of signature $(p-2, q-2)$. Furthermore, depending on whether q is even or odd, all the elements of the image are of positive or negative norm. Let x_1^+ and x_1^- be two elements of the orthogonal of V such that $x_1^- = -x_1^+$ and of the same sign as the image of $\pi_1 \circ \xi$. For each $x \in \partial\Gamma$, x_1^\pm is transverse to $\pi_1(\xi(x))$.

Let us consider the set of flags of affine isotropic spaces in $\mathbb{R}^{p-1, q-1}$ of dimension $1, \dots, q-3$ containing x^\pm . Since the intersection E of the isotropic cones of x^\pm and x_∞ is a copy of $\text{Ein}_{p-2, q-2}$, this space can be identified with $\text{SO}_0(p, q)/P_{1, \dots, q-3}$, and since x^\pm is orthogonal to V , the boundary of V in the isotropic cone of x_∞ is contained within E . We can then do the same procedure in the affine chart of E defined by $\pi_2(\xi(x_\infty))$ and $\pi_2(\xi(x_0))$ to get two flags $(x_1^\pm, \dots, x_{q-2}^\pm)$ which are transverse to each other and transverse to the image of ξ . This yields the result via proposition 5.2. \square

Remark 5.5. It is actually not necessary to take a finite index subgroup of Γ . For each $\gamma \neq e$ in Γ , the set O_γ of points x in $\text{SO}_0(p, q)/P_\Theta$ which are transverse to $\gamma \cdot x$ is an open and dense subset of $\text{SO}_0(p, q)/P_\Theta$. Since Γ is countable, the set $\bigcap_{\gamma \neq e} O_\gamma$ is still dense, meaning that up to a small perturbation of x^+ and x^- one may assume that they are both in $\bigcap_{\gamma \neq e} O_\gamma$. We may then take B to be the reunion of a small compact containing $\partial\Gamma$ and of small enough compacts containing the $\gamma \cdot x^\pm$. There is only a finite number of those compacts which are not contained within the compact containing $\partial\Gamma$, thus B is also compact. We then get that A^\pm and B are transverse to each other, hence the result.

Remark 5.6. Iterating this procedure allows one to construct a $P_{1, \dots, q-2}$ -Anosov subgroup which is isomorphic to $\Gamma * F_n$ for each n . Indeed, theorem A from [DK23] tells us that the boundary of $\Gamma * \mathbb{Z}$ is contained within $B \cup A^\pm$; by taking A^\pm small enough, it is possible to find another pair of transverse points transverse to $B \cup A^\pm$ and thus to the boundary of $\Gamma * \mathbb{Z}$. The result follows by repeating the process for $\Gamma * F_{n-1}$.

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