Light Scattering by a Sphere according to A.F. Stevenson's method

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Abstract: This paper takes up the calculation of the electromagnetic field scattered by a sphere composed of a material with any relative permittivity and permeability, according to the method of A.F. Stevenson (1953). Unlike Stevenson's original publication, the calculations are detailed and completed. This work completes that achieved for an ellipsoid (Gruy, hal-04655313). However, the result is not obtained by replacing (a,b,c) by (R,R,R) into the expressions of the electromagnetic field, revealing technical problems, but by directly applying Stevenson's method. The solution then uses the spherical harmonic functions in place of the ellipsoidal harmonic functions. The solution obtained by Stevenson's method has been successfully compared to that obtained by Mie's theory. The interest of this result, which has also been well known for a long time, lies in its use as a tool for partial validation of the solution obtained for the ellipsoid.

Content

1. Introduction

The interaction between an electromagnetic wave and an object results in the existence of an electromagnetic field inside the object and outside the object (the scattered field). The electromagnetic field is obtained by solving Maxwell's equations associated with boundary conditions (the surface of the object and the surface of a sphere of infinite radius and the same center as the object). The corresponding analytical solutions exist for very few objects, including the sphere (Mie theory, 1908).

For a particle whose size is small compared with the wavelength of the incident radiation , it can be interesting to develop the electromagnetic field, which is a function of the wave vector, in the form of a series in $(ik)^n$, where *k* is the modulus of the wave vector. This approach was formally discussed by Stevenson (1953) and applied to the simplest non-spherical object: the ellipsoid (1953).

Stevenson's article on ellipsoids is difficult to read, as various researchers have noted, and for good reason:

- no intermediate calculation is provided, only the final result, with this indication: « The calculations are somewhat long and involved, but are fairly straightforward applications of ellipsoidal harmonics. »
- The calculation shows the result to order *n*=0 and 1, and very partially to order 2, bearing in mind that the higher the order, the more complex the calculations. Stevenson confined himself strictly to the calculations needed to obtain the far scattered field.

In a previous paper (Gruy, hal-04655313), we presented, based on Stevenson's paper, the detailed calculations leading to the development of the electromagnetic field to order 2 (complete). This allows the reader to calculate the field at any point (inside, outside the ellipsoid).

The developments and results obtained for the ellipsoid could be applied to the special case of the sphere by taking $a_1 = a_2 = a_3 = R$. However, a number of technical problems arise during this direct application, such as the definitions of ν, ν' , of X_ν and therefore the expression of U_1 for a sphere. We will therefore follow the methodology proposed by Stevenson, but with the emphasis on the use of spherical harmonics instead of ellipsoidal harmonics.

2. Fundamentals on Stevenson's method

Using Stevenson's notations, the incident plane polarized electromagnetic wave is characterized by the three sets of director cosines :

$u = (l, m, n)$ for k vector

$u_1 = (l_1, m_1, n_1)$ for $E^{(0)}$ vector

$u_2 = (l_2, m_2, n_2)$ for $H^{(0)}$ vector

u , u_1 , u_2 are therefore the unit vectors respectively carrying the vectors k , $E^{(0)}$ et $H^{(0)}$.

This means:

$$\begin{aligned} E^{(0)} &= \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} e^{ik(lx_1 + mx_2 + nx_3)} \\ H^{(0)} &= \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} e^{ik(lx_1 + mx_2 + nx_3)} \end{aligned} \tag{2.1}$$

Incident fields and scattered fields can be expressed as a power series :

$$\begin{aligned} E^{(0)} &= \sum_{p=0} E_p^{(0)} (ik)^p \quad E = \sum_{p=0} E_p (ik)^p \\ H^{(0)} &= \sum_{p=0} H_p^{(0)} (ik)^p \quad H = \sum_{p=0} H_p (ik)^p \end{aligned} \tag{2.2}$$

The power series truncated at the order 2 for the incident wave is:

$$\begin{aligned} E_0^{(0)} &= \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} \quad E_1^{(0)} = \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} (lx_1 + mx_2 + nx_3) \quad E_2^{(0)} = \frac{1}{2} \begin{pmatrix} l_1 & m_1 & n_1 \end{pmatrix} (lx_1 + mx_2 + nx_3)^2 \\ H_0^{(0)} &= \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} \quad H_1^{(0)} = \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} (lx_1 + mx_2 + nx_3) \quad H_2^{(0)} = \frac{1}{2} \begin{pmatrix} l_2 & m_2 & n_2 \end{pmatrix} (lx_1 + mx_2 + nx_3)^2 \end{aligned} \tag{2.3}$$

According to Stevenson, we distinguish the field inside the ellipsoid, denoted $E^{(i)}, H^{(i)}$, and the scattered or external field (E, H). They obey the equations (coming directly from the Maxwell's equations):

$$
\begin{aligned}
&E_0 = \nabla V_0 \quad E_0^{(i)} = \nabla V_0^{(i)} \\
&H_1 = G_1 + \nabla U_1 \quad H_1^{(i)} = G_1^{(i)} + \nabla U_1^{(i)} \\
&E_2 = F_2 + \nabla V_2 \quad E_2^{(i)} = F_2^{(i)} + \nabla V_2^{(i)} \\
&\nabla \wedge G_1 = -E_0 \quad \nabla \wedge G_1^{(i)} = -\varepsilon E_0^{(i)} \\
&\nabla \wedge F_2 = H_1 \quad \nabla \wedge F_2^{(i)} = \mu H_1^{(i)} \\
&\nabla \bullet G_1 = \nabla \bullet G_1^{(i)} = \nabla \bullet F_2 = \nabla \bullet F_2^{(i)} = 0
\end{aligned}
\tag{2.4}
$$

$\wedge$ is the cross product, $\bullet$ is the scalar product.

with the boundary conditions (surface of the sphere) :

$$
\begin{aligned}
&n \wedge \nabla \left(V_0 - V_0^{(i)} \right) = -n \wedge E_0^{(0)} \\
&n \bullet \nabla \left(V_0 - \varepsilon V_0^{(i)} \right) = -n \bullet E_0^{(0)} \\
&n \wedge \nabla \left(U_1 - U_1^{(i)} \right) = -n \wedge \left(G_1 - G_1^{(i)} + H_1^{(0)} \right) \\
&n \bullet \nabla \left(U_1 - \mu U_1^{(i)} \right) = -n \bullet \left(G_1 - \mu G_1^{(i)} + H_1^{(0)} \right) \\
&n \wedge \nabla \left(V_2 - V_2^{(i)} \right) = -n \wedge \left(F_2 - F_2^{(i)} + E_2^{(0)} \right) \\
&n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) = -n \bullet \left(F_2 - \varepsilon F_2^{(i)} + E_2^{(0)} \right)
\end{aligned}
\tag{2.5}
$$

n is the outward pointing unit normal.

with

$$
\int n \bullet E_0 dS = \int n \bullet H_1 dS = \int n \bullet E_2 dS = 0 \tag{2.6}
$$

It should be mentioned that, according to Stevenson, the functions V_0, U_1, V_2 and $V_0^{(i)}, U_1^{(i)}, V_2^{(i)}$ are harmonic functions in the mathematical sense and not electrical potentials ($E_0 = \nabla V_0$ and not $E_0 = -\nabla V_0$ in physics).

To simplify the calculations without loss of generality due to spherical symmetry, we will take $u\left(1 \quad 0 \quad 0\right) \quad u_1\left(0 \quad 1 \quad 0\right) \quad u_2\left(0 \quad 0 \quad 1\right)$. These particular values will be used mainly when the benefit of a simpler calculation is obvious, e.g. the second order term.

Maxwell's equations are invariant if the following substitutions are made: $E \to H, \quad H \to -E, \quad \varepsilon \leftrightarrow \mu$. So $H_0, H_0^{(i)}, -E_1, -E_1^{(i)}, H_2, H_2^{(i)}$ are straightly obtained from $E_0, E_0^{(i)}, H_1, H_1^{(i)}, E_2, E_2^{(i)}$ if we carry out the following substitutions for the director cosines (of the incident electric and magnetic fields) and the material properties:

$$\begin{aligned}
&(l_1 \quad m_1 \quad n_1) \rightarrow (l_2 \quad m_2 \quad n_2) \\
&(l_2 \quad m_2 \quad n_2) \rightarrow -(l_1 \quad m_1 \quad n_1) \\
&\varepsilon \leftrightarrow \mu
\end{aligned} \tag{2.7}$$

3. Real spherical harmonics

The first real spherical harmonics are shown below:

degree 0 : $Y_{0,0} = \sqrt{\frac{1}{4\pi}}$

degree 1 : $Y_{1,-1} = \sqrt{\frac{3}{4\pi}}\frac{x_2}{r} \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\frac{x_3}{r} \quad Y_{1,1} = \sqrt{\frac{3}{4\pi}}\frac{x_1}{r}$

degree 2 :

$$Y_{2,-2} = \sqrt{\frac{15}{4\pi}}\frac{x_1x_2}{r^2} \quad Y_{2,-1} = \sqrt{\frac{15}{4\pi}}\frac{x_2x_3}{r^2} \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}}\frac{3x_3^2 - r^2}{r^2} \quad Y_{2,1} = \sqrt{\frac{15}{4\pi}}\frac{x_1x_3}{r^2} \quad Y_{2,2} = \sqrt{\frac{15}{16\pi}}\frac{x_1^2 - x_2^2}{r^2}$$

degree 3 :

$$Y_{3,-3} = \frac{1}{4}\sqrt{\frac{35}{2\pi}}\frac{x_2\left(3x_1^2 - x_2^2\right)}{r^3} \quad Y_{3,-2} = \frac{1}{2}\sqrt{\frac{105}{\pi}}\frac{x_1x_2x_3}{r^3} \quad Y_{3,-1} = \frac{1}{4}\sqrt{\frac{21}{2\pi}}\frac{x_2\left(5x_3^2 - r^2\right)}{r^3}$$

$$Y_{3,0} = \frac{1}{4}\sqrt{\frac{7}{\pi}}\frac{x_3\left(5x_3^2 - 3r^2\right)}{r^3}$$

$$Y_{3,1} = \frac{1}{4}\sqrt{\frac{21}{2\pi}}\frac{x_1\left(5x_3^2 - r^2\right)}{r^3} \quad Y_{3,2} = \frac{1}{4}\sqrt{\frac{105}{\pi}}\frac{x_3\left(x_1^2 - x_2^2\right)}{r^3} \quad Y_{3,3} = \frac{1}{4}\sqrt{\frac{35}{2\pi}}\frac{x_1\left(x_1^2 - 3x_2^2\right)}{r^3} \tag{3.1}$$

Next, we introduce the parameter $g_{l,m}$ which is the numerical coefficient into $Y_{l,m}$; for instance $g_{3,1} = \frac{1}{4}\sqrt{\frac{21}{2\pi}}$. This gives:

$$Y_{l,m} = g_{l,m}Z_{l,m} \tag{3.2}$$

where $Z_{l,m}$ is a function of Cartesian coordinates, for instance $Z_{3,1} = \frac{x_1\left(5x_3^2 - r^2\right)}{r^3}$.

Next, we need to express the following monomials as a function of $Z_{l,m}$:

degree 1 :

$$x_2 / r = Z_{1,-1} \quad x_3 / r = Z_{1,0} \quad x_1 / r = Z_{1,1} \tag{3.3}$$

degree 2 :

$$x_1x_2 / r^2 = Z_{2,-2} \quad x_2x_3 / r^2 = Z_{2,-1} \quad x_1x_3 / r^2 = Z_{2,1}$$

$$x_1^2 / r^2 = \frac{1}{3} - \frac{1}{6}Z_{2,0} + \frac{1}{2}Z_{2,2} \quad x_2^2 / r^2 = \frac{1}{3} - \frac{1}{6}Z_{2,0} - \frac{1}{2}Z_{2,2} \quad x_3^2 / r^2 = \frac{1}{3} + \frac{1}{3}Z_{2,0} \tag{3.4}$$

degree 3 :

$$
\begin{aligned}
x_1x_2^2 / r^3 &= -\frac{1}{4}Z_{3,3} - \frac{1}{20}Z_{3,1} + \frac{1}{5}Z_{1,1} \\
x_2x_1^2 / r^3 &= \frac{1}{4}Z_{3,-3} - \frac{1}{20}Z_{3,-1} + \frac{1}{5}Z_{1,-1} \\
x_3x_1^2 / r^3 &= \frac{1}{2}Z_{3,2} - \frac{1}{10}Z_{3,0} + \frac{1}{5}Z_{1,0} \\
x_3x_2^2 / r^3 &= -\frac{1}{2}Z_{3,2} - \frac{1}{10}Z_{3,0} + \frac{1}{5}Z_{1,0} \\
x_1x_3^2 / r^3 &= \frac{1}{5}Z_{3,1} + \frac{1}{5}Z_{1,1} \\
x_2x_3^2 / r^3 &= \frac{1}{5}Z_{3,-1} + \frac{1}{5}Z_{1,-1} \\
x_1^3 / r^3 &= \frac{1}{4}Z_{3,3} - \frac{3}{20}Z_{3,1} + \frac{3}{5}Z_{1,1} \\
x_2^3 / r^3 &= -\frac{1}{4}Z_{3,-3} - \frac{3}{20}Z_{3,-1} + \frac{3}{5}Z_{1,-1} \\
x_3^3 / r^3 &= \frac{1}{5}Z_{3,0} + \frac{3}{5}Z_{1,0} \\
x_1x_2x_3 / r^3 &= Z_{3,-2}
\end{aligned}
\tag{3.5}
$$

Note that the transition from ellipsoidal to spherical coordinates is not so straightforward (see the work of Panou, 2015).

At degree l, harmonic functions, respecting the boundary conditions (center of the sphere and point at infinity) are put in the form :

$$
\begin{aligned}
R_l(r,\theta,\phi) &= \sum_{m=-l}^{l} L_{l,m} r^{-(l+1)} Y_{l,m}(\theta,\phi) \\
R_l^{(i)}(r,\theta,\phi) &= \sum_{m=-l}^{l} K_{l,m} r^{l} Y_{l,m}(\theta,\phi)
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
R_l &= \sum_{m=-l}^{l} L'_{l,m}\, r^{-(l+1)} Z_{l,m} \\
R_l^{(i)} &= \sum_{m=-l}^{l} K'_{l,m}\, r^{l} Z_{l,m}
\end{aligned}
\tag{3.7}
$$

with $L'_{l,m} = L_{l,m} g_{l,m} \quad K'_{l,m} = K_{l,m} g_{l,m}$

With Stevenson's notations, the harmonic potentials will appear in the manuscript:

$$U_1 = R_2 \quad U_1^{(i)} = R_2^{(i)} \quad V_2 = R_3 \quad V_2^{(i)} = R_3^{(i)}$$

We will use certain results obtained for the ellipsoid (Gruy, hal-04655313), the application of which to the particular case of the sphere is not problematic. Obviously, the calculations are much easier with

the sphere and some of the parameters used by Stevenson have simpler analytical forms. Thus, for those that we will use in the manuscript :

$$r^2 = x_1^2 + x_2^2 + x_3^2 = \xi + R^2$$
$$I(\xi) = 2\left(\xi + R^2\right)^{-1/2} \quad I = 2R^{-1}$$
$$I_a(\xi) = \frac{2}{3}\left(\xi + R^2\right)^{-3/2} \quad I_a = \frac{2}{3}R^{-3}$$
$$I_{aa}(\xi) = \frac{2}{5}\left(\xi + R^2\right)^{-5/2} \quad I_{aa} = \frac{2}{5}R^{-5} \tag{3.8}$$
$$f_1 = \frac{3R^3}{2(\varepsilon+2)}$$
$$A_1 = \frac{3R^3(1-\varepsilon)}{2(\varepsilon+2)}l_1 \quad A_1^{(i)} = \frac{3}{(\varepsilon+2)}l_1 \quad \ldots\ldots$$

4. Zeroth order term

We remind the well-known result, so named static field (see, for instance,, Stratton, 1941) :

$$V_0 = \frac{R^3}{r^3}\frac{1-\varepsilon}{2+\varepsilon}\left(l_1x_1 + m_1x_2 + n_1x_3\right) = \frac{R^3}{r^3}\frac{1-\varepsilon}{2+\varepsilon}x_2 \tag{4.1}$$

$$V_0^{(i)} = \frac{3}{2+\varepsilon}\left(l_1x_1 + m_1x_2 + n_1x_3\right) = \frac{3}{2+\varepsilon}x_2 \tag{4.2}$$

$$E_0 = \frac{1-\varepsilon}{2+\varepsilon}\frac{R^3}{r^3}\left(u_1 - 3x_2\frac{\nabla r}{r}\right) \tag{4.3}$$

$$E_0^{(i)} = \frac{3}{2+\varepsilon}u_1 \tag{4.4}$$

with $u_1 = (0,1,0)$ et $\nabla r = \left(x_1/r, x_2/r, x_3/r\right)$

We easily check that the condition 2.6 from Stevenson holds: $\int n \bullet E_0 dS = 0$.

5. First order term

from Gruy, (hal-04655313 Eq. 5.12) :

$$G_1 = \frac{R^3}{r^3}\frac{1-\varepsilon}{2+\varepsilon}\begin{vmatrix} l_1 \\ m_1 \\ n_1 \end{vmatrix} \wedge \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} \quad G_1^{(i)} = -\frac{3\varepsilon}{2(2+\varepsilon)}\begin{vmatrix} l_1 \\ m_1 \\ n_1 \end{vmatrix} \wedge \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

Hence,

$$G_1 - \mu^e G_1^{(i)} + H_1^{(0)} = \left[\frac{R^3}{r^3} \frac{1-\varepsilon}{2+\varepsilon} + \frac{3\varepsilon}{2(2+\varepsilon)} \mu^e \right] \begin{vmatrix} l_1 \\ m_1 \\ n_1 \end{vmatrix} \wedge \begin{vmatrix} x_1 \\ x_2 + (lx_1 + mx_2 + nx_3) \\ x_3 \end{vmatrix} \begin{vmatrix} l_2 \\ m_2 \\ n_2 \end{vmatrix} \quad (5.1)$$

with $e = 0,1$

One denotes : $\alpha = \frac{1-\varepsilon}{2+\varepsilon} + \frac{3\varepsilon}{2(2+\varepsilon)} \mu^e$

Hence, on the surface of the sphere,

$$G_1 - \mu^e G_1^{(i)} + H_1^{(0)} = \alpha \begin{vmatrix} l_1 \\ m_1 \\ n_1 \end{vmatrix} \wedge \begin{vmatrix} x_1 \\ x_2 + (lx_1 + mx_2 + nx_3) \\ x_3 \end{vmatrix} \begin{vmatrix} l_2 \\ m_2 = [B] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ n_2 \end{vmatrix} \quad (5.2)$$

with

$$[B] = \begin{bmatrix} ll_2 & ml_2 - \alpha n_1 & nl_2 + \alpha m_1 \\ lm_2 + \alpha n_1 & mm_2 & nm_2 - \alpha l_1 \\ n_2 l - \alpha m_1 & mn_2 + \alpha l_1 & nn_2 \end{bmatrix} \quad (5.3)$$

The trace of $[B]$ vanishes. if $e = 0$, $\alpha = \frac{1}{2}$ and $[B]$ is symmetric given the orthogonality of the unit vectors of the incident electromagnetic field.

The expression of the boundary conditions on the sphere surface implies that the solution for U_1 and $U_1^{(i)}$ obeys 3.6 with $l = 2$, therefore

$$\nabla \left(U_1 - \mu^e U_1^{(i)} \right) = \begin{vmatrix} \lambda_{-2} x_2 - \lambda_0 2x_1 + \lambda_1 x_3 + \lambda_2 2x_1 \\ \lambda_{-2} x_1 + \lambda_{-1} x_3 - \lambda_0 2x_2 - \lambda_2 2x_2 \\ \lambda_{-1} x_2 + \lambda_0 4x_3 + \lambda_1 x_1 \end{vmatrix} - 5R^{-4} n \sum_{m=-2}^{2} L_{2,m} Y_{2,m} \quad (5.4)$$

which can be rewritten as :

$$\nabla \left(U_1 - \mu^e U_1^{(i)} \right) = [A_U] Rn - 5R^{-4} n \sum_{m=-2}^{2} L_{2,m} Y_{2,m} \quad (5.5)$$

with

$$[A_U] = \begin{bmatrix} 2(\lambda_2 - \lambda_0) & \lambda_{-2} & \lambda_1 \\ \lambda_{-2} & -2(\lambda_0 + \lambda_2) & \lambda_{-1} \\ \lambda_1 & \lambda_{-1} & 4\lambda_0 \end{bmatrix} \quad (5.6)$$

and

$$\lambda_m = \left(R^{-5} L_{2,m} - \mu^e K_{2,m} \right) g_{2,m} \quad (5.7)$$

$R^{-5}L_{2,m} - \mu^e K_{2,m}$ is called P_m if $e=0$ and P_m^μ if $e=1$.

$\left[A_U\right]$ is symmetric and has a null trace. if $e=0$, the matrix $\left[A\right]=\left[A_U\right]+\left[B\right]$ is symmetric as well. The matrices A, A_U, B will be denoted A^μ, A_U^μ, B^μ if $e=1$.

The previous results will enable us to reformulate the boundary conditions (in $r=R$), starting with the cross product ($e=0$):

$$n \wedge \left[A\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{R}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \wedge \left[A\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (5.8)$$

Each component *l* of this vector is a quadratic form. Since the products $x_i x_j$ can be expressed as linear combinations of spherical harmonics (Eq. 3.4), so can the components of the cross product.

The elements of the cross product for each component *l* and each spherical harmonics *m* are given in table 1:

l *m*	-2	-1	0	1	2	$Y_{0,0}$
1	$A_{3,1}$	$A_{3,3}-A_{2,2}$	$-\frac{1}{6}A_{3,2}-\frac{1}{3}A_{2,3}$	$-A_{2,1}$	$-\frac{1}{2}A_{3,2}$	$\frac{A_{3,2}}{3}-\frac{A_{2,3}}{3}$
2	$-A_{3,2}$	$A_{1,2}$	$\frac{A_{1,3}}{3}+\frac{A_{3,1}}{6}$	$A_{1,1}-A_{3,3}$	$-\frac{1}{2}A_{3,1}$	$\frac{A_{1,3}}{3}-\frac{A_{3,1}}{3}$
3	$A_{2,2}-A_{1,1}$	$-A_{1,3}$	$-\frac{1}{6}A_{2,1}+\frac{1}{6}A_{1,2}$	$A_{2,3}$	$\frac{1}{2}A_{2,1}+\frac{1}{2}A_{1,2}$	$\frac{A_{2,1}}{3}-\frac{A_{1,2}}{3}$

Table 1: components of the cross product CP expressed by means of spherical harmonics $CP_l = \sum_m CP_{l,m} Z_{l,m}$

The constant contribution appears in the column $Y_{0,0}$. $\left[A\right]$ being a symmetric matrix, the equation 5.8 is equivalent to the system of 5 equations (one for each spherical harmonic function) :

$$A_{1,2}=0 \quad A_{1,3}=0 \quad A_{2,3}=0 \quad A_{1,1}=A_{2,2} \quad A_{1,1}=A_{3,3} \qquad (5.9)$$

or

$$\begin{aligned}
P_{-2} &= R^{-5}L_{2,-2} - K_{2,-2} = -\frac{1}{2}\left(ml_2 + lm_2\right)g_{2,-2}^{-1} \\
P_{-1} &= R^{-5}L_{2,-1} - K_{2,-1} = -\frac{1}{2}\left(nm_2 + mn_2\right)g_{2,-1}^{-1} \\
P_0 &= R^{-5}L_{2,0} - K_{2,0} = -\frac{1}{4}nn_2 g_{2,0}^{-1} \qquad (5.10) \\
P_1 &= R^{-5}L_{2,1} - K_{2,1} = -\frac{1}{2}\left(nl_2 + \ln_2\right)g_{2,1}^{-1} \\
P_2 &= R^{-5}L_{2,2} - K_{2,2} = \frac{1}{4}\left(mm_2 - ll_2\right)g_{2,2}^{-1}
\end{aligned}$$

The scalar product ($e=1$) is written as :

$$n\bullet\left[A^{\mu}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-5R^{-4}\sum_{m=-2}^{2}L_{2,m}Y_{2,m}=\frac{1}{R}\sum_{i,j}A^{\mu}_{i,j}x_ix_j-5R^{-4}\sum_{m=-2}^{2}L_{2,m}Y_{2,m}=0 \tag{5.11}$$

or

$$\begin{aligned}0&=\left(A^{\mu}_{1,2}+A^{\mu}_{2,1}\right)Z_{2,-2}+\left(A^{\mu}_{1,3}+A^{\mu}_{3,1}\right)Z_{2,1}+\left(A^{\mu}_{2,3}+A^{\mu}_{3,2}\right)Z_{2,-1}\\&+\left(-\frac{A^{\mu}_{1,1}}{6}-\frac{A^{\mu}_{2,2}}{6}+\frac{A^{\mu}_{3,3}}{3}\right)Z_{2,0}+\left(\frac{A^{\mu}_{1,1}}{2}-\frac{A^{\mu}_{2,2}}{2}\right)Z_{2,2}+\frac{1}{3}\sum_{i=1}^{3}A^{\mu}_{i,i}-5R^{-5}\sum_{m=-2}^{2}L_{2,m}Y_{2,m}\end{aligned} \tag{5.12}$$

This leads to :

$$\begin{aligned}&3R^{-5}L_{2,-2}+2\mu K_{2,-2}=\left(ml_2+lm_2\right)g^{-1}_{2,-2}\\&3R^{-5}L_{2,1}+2\mu K_{2,1}=\left(nl_2+\ln_2\right)g^{-1}_{2,1}\\&3R^{-5}L_{2,-1}+2\mu K_{2,-1}=\left(mn_2+nm_2\right)g^{-1}_{2,-1}\\&3R^{-5}L_{2,0}+2\mu K_{2,0}=\frac{1}{2}nn_2g^{-1}_{2,0}\\&3R^{-5}L_{2,2}+2\mu K_{2,2}=\frac{1}{2}\left(ll_2-mm_2\right)g^{-1}_{2,2}\end{aligned} \tag{5.13}$$

We note that $\sum_{i=1}^{3}A^{\mu}_{i,i}=0$.

finally, with 5.10,

$$\begin{aligned}&\left(2\mu+3\right)R^{-5}L'_{2,-2}=\left(ml_2+lm_2\right)\left(1-\mu\right)\quad K'_{2,-2}=\left(ml_2+lm_2\right)\frac{5}{2\left(2\mu+3\right)}\\&\left(2\mu+3\right)R^{-5}L'_{2,1}=\left(nl_2+\ln_2\right)\left(1-\mu\right)\quad K'_{2,1}=\left(nl_2+\ln_2\right)\frac{5}{2\left(2\mu+3\right)}\\&\left(2\mu+3\right)R^{-5}L'_{2,-1}=\left(mn_2+nm_2\right)\left(1-\mu\right)\quad K'_{2,-1}=\left(mn_2+nm_2\right)\frac{5}{2\left(2\mu+3\right)}\\&\left(2\mu+3\right)R^{-5}L'_{2,0}=\frac{1}{2}\left(1-\mu\right)nn_2\quad K'_{2,0}=nn_2\frac{5}{4\left(2\mu+3\right)}\\&\left(2\mu+3\right)R^{-5}L'_{2,2}=\frac{1}{2}\left(ll_2-mm_2\right)\left(1-\mu\right)\quad K'_{2,2}=\left(ll_2-mm_2\right)\frac{5}{4\left(2\mu+3\right)}\end{aligned} \tag{5.14}$$

with $L'_{l,m}=L_{l,m}g_{l,m}\quad K'_{l,m}=K_{l,m}g_{l,m}$

we can see that

$$L'_{2,m}=\frac{2}{5}\left(1-\mu\right)R^5K'_{2,m} \tag{5.15}$$

if $\mu=1$:

$$
\begin{aligned}
&L'_{2,-2}=0 \quad K'_{2,-2}=\left(ml_2+lm_2\right)/2\\
&L'_{2,1}=0 \quad K'_{2,1}=\left(nl_2+\ln_2\right)/2\\
&L'_{2,-1}=0 \quad K'_{2,-1}=\left(mn_2+nm_2\right)/2\\
&L'_{2,0}=0 \quad K'_{2,0}=nn_2/4\\
&L'_{2,2}=0 \quad K'_{2,2}=\left(ll_2-mm_2\right)/4
\end{aligned}
\tag{5.16}
$$

Ultimately, the two harmonic potentials can be expressed as:

$$
\begin{aligned}
&U_1^{(i)}=\frac{1}{2}\frac{5}{2\mu+3}\left(ll_2x_1^2+mm_2x_2^2+nn_2x_3^2+\left(ml_2+lm_2\right)x_1x_2+\left(nl_2+\ln_2\right)x_1x_3+\left(mn_2+nm_2\right)x_2x_3\right)\\
&U_1=\frac{2}{5}\left(1-\mu\right)\frac{R^5}{r^5}U_1^{(i)}
\end{aligned}
\tag{5.17}
$$

or (choosing the particular set of unit vectors, see §2)

$$
\begin{aligned}
&U_1^{(i)}=\frac{1}{2}\frac{5}{2\mu+3}x_1x_3\\
&U_1=\frac{2}{5}\left(1-\mu\right)\frac{R^5}{r^5}U_1^{(i)}=\frac{1-\mu}{2\mu+3}\frac{R^5}{r^5}x_1x_3
\end{aligned}
\tag{5.18}
$$

finally,

$$
\begin{aligned}
&H_1^{(i)}=-\frac{3\varepsilon}{2\left(2+\varepsilon\right)}\begin{vmatrix}x_3\\0\\-x_1\end{vmatrix}+\frac{1}{2}\frac{5}{2\mu+3}\begin{vmatrix}x_3\\0\\x_1\end{vmatrix}\\
&H_1=\frac{R^3}{r^3}\frac{1-\varepsilon}{2+\varepsilon}\begin{vmatrix}x_3\\0\\-x_1\end{vmatrix}+\frac{1-\mu}{2\mu+3}\frac{R^5}{r^5}\left[\begin{vmatrix}x_3\\0\\x_1\end{vmatrix}-5x_1x_3\frac{\nabla r}{r}\right]
\end{aligned}
\tag{5.19}
$$

if $\mu=1$, then

$$
\begin{aligned}
&H_1^{(i)}=-\frac{3\varepsilon}{2\left(2+\varepsilon\right)}\begin{vmatrix}x_3\\0\\-x_1\end{vmatrix}+\frac{1}{2}\begin{vmatrix}x_3\\0\\x_1\end{vmatrix}\\
&H_1=\frac{R^3}{r^3}\frac{1-\varepsilon}{2+\varepsilon}\begin{vmatrix}x_3\\0\\-x_1\end{vmatrix}
\end{aligned}
\tag{5.20}
$$

We easily check that the condition 2.6 from Stevenson holds: $\int n\bullet H_1dS=\int n\bullet\left(G_1+\nabla U_1\right)dS=0$.

6. Second order term

6.1. Calculation of $F_2, F_2^{(i)}$

6.1.1 calculation of F_2

We easily show that,

outside the sphere (hal-04655313) :

$$F_{2,G} = \frac{2}{3}\frac{A}{r} \tag{6.1}$$

however, the calculation of $F_{2,U}$ ($\nabla \wedge F_{2,U} = \nabla U_1$) requires more attention:

Given the expression of U_1, we assume that $F_{2,U}$ obeys the general expression:

$$F_{2,U} = \frac{D_0}{r^3} + \sum_{m=-2}^{2} \frac{d_m}{r^3} Z_{2,m} \tag{6.2}$$

where D_0, d_m are constant vectors to be determined.

We easily show that :

$$r^4 \nabla \wedge F_{2,U} = 3D_0 \wedge \nabla r + \sum_m d_m \wedge \left(5Z_{2,m}\nabla r - H_m\right) \tag{6.3}$$

with

$$H_{-2} = \begin{vmatrix} x_2/r \\ x_1/r \\ 0 \end{vmatrix} \quad H_{-1} = \begin{vmatrix} 0 \\ x_3/r \\ x_2/r \end{vmatrix} \quad H_0 = \begin{vmatrix} -2x_1/r \\ -2x_2/r \\ 4x_3/r \end{vmatrix} \quad H_1 = \begin{vmatrix} x_3/r \\ 0 \\ x_1/r \end{vmatrix} \quad H_2 = \begin{vmatrix} 2x_1/r \\ -2x_2/r \\ 0 \end{vmatrix} \quad \nabla r = \begin{vmatrix} x_1/r \\ x_2/r \\ x_3/r \end{vmatrix}$$

and

n \m	-2	-1	0	1	2
1	$x_2 x_1^2 / r^3$	$Z_{3,-2}$	$\frac{3}{5}Z_{3,1} - \frac{2}{5}Z_{1,1}$	$x_3 x_1^2 / r^3$	$\frac{1}{2}Z_{3,3} - \frac{1}{10}Z_{3,1} + \frac{2}{5}Z_{1,1}$
2	$x_1 x_2^2 / r^3$	$x_3 x_2^2 / r^3$	$\frac{3}{5}Z_{3,-1} - \frac{2}{5}Z_{1,-1}$	$Z_{3,-2}$	$\frac{1}{2}Z_{3,-3} + \frac{1}{10}Z_{3,-1} - \frac{2}{5}Z_{1,-1}$
3	$Z_{3,-2}$	$x_2 x_3^2 / r^3$	$\frac{3}{5}Z_{3,0} + \frac{4}{5}Z_{1,0}$	$x_1 x_3^2 / r^3$	$Z_{3,2}$

Table 2: expressions of $Z_{2,m} x_n / r$ (components of $Z_{2,m}\nabla r$) as a function of $Z_{l,m}$. Expressions of $x_n x_l^2 / r^3$ have been previously reported (Eqs. 3.5).

One deduce for $5Z_{2,m}\nabla r - H_m$:

m=-2	m=-1	m=0	m=1	m=2
$\begin{vmatrix} \frac{5}{4}Z_{3,-3}-\frac{1}{4}Z_{3,-1} \\ -\frac{5}{4}Z_{3,3}-\frac{1}{4}Z_{3,1} \\ 5Z_{3,-2} \end{vmatrix}$	$\begin{vmatrix} 5Z_{3,-2} \\ -\frac{5}{2}Z_{3,2}-\frac{1}{2}Z_{3,0} \\ Z_{3,-1} \end{vmatrix}$	$\begin{vmatrix} 3Z_{3,1} \\ 3Z_{3,-1} \\ 3Z_{3,0} \end{vmatrix}$	$\begin{vmatrix} \frac{5}{2}Z_{3,2}-\frac{1}{2}Z_{3,0} \\ 5Z_{3,-2} \\ Z_{3,1} \end{vmatrix}$	$\begin{vmatrix} \frac{5}{2}Z_{3,3}-\frac{1}{2}Z_{3,1} \\ \frac{5}{2}Z_{3,-3}+\frac{1}{2}Z_{3,-1} \\ 5Z_{3,2} \end{vmatrix}$

Table 3 : $5Z_{2,m}\nabla r - H_m$ vector

therefore,

$$\nabla \wedge F_{2,U} = \nabla U_1 \quad \Rightarrow \quad \left(3D_0 \wedge \nabla r + \sum_m d_m \wedge \left(5Z_{2,m}\nabla r - H_m \right) \right) r^{-4}$$

$$= \nabla\left(\frac{\beta}{r^5} x_1 x_3 \right) = \beta r^{-4} \begin{vmatrix} x_3/r - 5x_3 x_1^2/r^3 \\ 0 - 5x_1 x_2 x_3/r^3 \\ x_1/r - 5x_1 x_3^2/r^3 \end{vmatrix} = \beta r^{-4} \begin{vmatrix} \frac{1}{2}Z_{3,0} - \frac{5}{2}Z_{3,2} \\ -5Z_{3,-2} \\ -Z_{3,1} \end{vmatrix} \qquad (6.4)$$

with $\beta = \frac{1-\mu}{2\mu+3} R^5$

$Z_{l,m}$ form a system of orthogonal (but non orthonormal) functions; we immediately deduce that $D_0 = 0$. We can also deduce a system of linear equations whose unknowns are the components of d_m, called $d_{m,k}$. The analysis of the system shows that the equations are not independent. This means that the components of d_m may be expressed as a function of the components of, for instance, d_0 :

m=-2	m=-1	m=0	m=1	m=2
$\begin{vmatrix} 2\beta - 6d_{0,2} \\ -6d_{0,1} \\ 0 \end{vmatrix}$	$\begin{vmatrix} 0 \\ 3d_{0,3} \\ \beta - 6d_{0,2} \end{vmatrix}$	$\begin{vmatrix} d_{0,1} \\ d_{0,2} \\ d_{0,3} \end{vmatrix}$	$\begin{vmatrix} 3d_{0,3} \\ 0 \\ -6d_{0,1} \end{vmatrix}$	$\begin{vmatrix} -3d_{0,1} \\ -\beta + 3d_{0,2} \\ 0 \end{vmatrix}$

Table 4: d_m vector

We can then check that $\nabla \bullet F_{2,U} = 0$.

Let us take the particular case $d_0 = 0$:

$$F_{2,U} = \frac{\beta}{r^3} \begin{vmatrix} 2Z_{2,-2} \\ -Z_{2,2} \\ Z_{2,-1} \end{vmatrix} = \frac{\beta}{r^3} \begin{vmatrix} 2x_1 x_2/r^2 \\ \left(x_2^2 - x_1^2 \right)/r^2 \\ x_2 x_3/r^2 \end{vmatrix} \qquad (6.5)$$

We have $F_2 = F_{2,G} + F_{2,U}$ to within one additive constant ; its value will be discussed later on. therefore,

$$\nabla \bullet F_2 = \nabla \bullet \left(\frac{2A}{3r} \right) \tag{6.10}$$

F_2 does not satisfy the zero divergence condition (Eq. 2.4). To overcome this difficulty, we look for a scalar function (denoted V_2" by Stevenson), such as $F_{2,new} = F_{2,old} + \nabla V_2$", with $F_{2,old} = F_{2,G} + F_{2,U}$. We verify that $\nabla \wedge F_{2,new} = 0$ (with $\nabla \wedge F_{2,old} = 0$ yet), because of $\nabla \wedge \nabla V_2" = 0$.

To ensure $\nabla \bullet F_{2,new} = 0$, V_2" has to obey :

$$\Delta V_2" = -\nabla \bullet \left(\frac{2A}{3r} \right) \tag{6.11}$$

V_2" may be expressed as:

$$V_2" = \sum_j A_j \phi_j \tag{6.12}$$

Using the Laplacian in spherical coordinates, we can easily obtain an expression equivalent to 6.56 (hal-04655313) with an indeterminate constant J_j :

$$\phi_j = x_j f_j(r) \tag{6.13}$$

with

$$f_j(r) = \frac{R^2 J_j}{r^3} - \frac{1}{3r} \tag{6.14}$$

Conjecture : $F_2 + \nabla V_2$ does not depend on the choice of J_j.

This conjecture will be proved after the calculation of V_2.

6.1.2. calculation of $F_2^{(i)}$

Inside the sphere :

$$\mu^{-1} \nabla \wedge F_2^{(i)} = G_1^{(i)} + \nabla U_1^{(i)} \tag{6.15}$$

$$\nabla \bullet F_2^{(i)} = 0 \tag{6.16}$$

6.15 is written as :

$$\mu^{-1}\nabla \wedge F_2^{(i)} = [M]r = ([A]+[B])r \tag{6.17}$$

with

$$A = \frac{5}{2\mu+3}\begin{bmatrix} ll_2 & (ml_2+lm_2)/2 & (nl_2+\ln_2)/2 \\ (ml_2+lm_2)/2 & mm_2 & (mn_2+nm_2)/2 \\ (nl_2+\ln_2)/2 & (mn_2+nm_2)/2 & nn_2 \end{bmatrix} \quad B = -\frac{\varepsilon}{2}\begin{bmatrix} 0 & -A_3^{(i)} & A_2^{(i)} \\ A_3^{(i)} & 0 & -A_1^{(i)} \\ -A_2^{(i)} & A_1^{(i)} & 0 \end{bmatrix}$$

A is symmetric, B is antisymmetric.

Given the expression 6.17, we assume that :

$$\mu^{-1}F_{2,k}^{(i)} = \sum_{i,j} D_{i,j}^k x_i x_j \tag{6.18}$$

The matrix D^k is symmetric by definition.

This gives :

$$2\sum_i \left(D_{i,j}^k \delta_{i,l} - D_{i,j}^l \delta_{i,k}\right) = M_{m,j} \tag{6.19}$$

where the relationship between m and (k, l) is the following :

$$(k,l) \to m \quad (3,2) \to 1 \quad (1,3) \to 2 \quad (2,1) \to 3$$

Equation 6.16 becomes :

$$\sum_{k,j} D_{k,j}^k x_j = \sum_j x_j \sum_k D_{k,j}^k = 0 \tag{6.20}$$

or $\sum_k D_{k,i}^k = 0 \quad \forall i$ (6.21)

The equation set linear in $D_{i,j}^k$ contains 18 unknowns $D_{i,j}^k$ and 12 equations. Therefore, we add the condition :

$$D_{k,i}^k = 0 \quad i \neq k \tag{6.22}$$

Conjecture : $F_2^{(i)} + \nabla V_2^{(i)}$ does not depend on the choice of $D_{k,i}^k \quad i \neq k$.

The issue is still open.

One easily deduces that:

$$D^1_{1,1}=0 \quad D^1_{2,2}=-\frac{1}{4}\left[\frac{5}{2\mu+3}\left(mn_2+nm_2\right)-\varepsilon A^{(i)}_1\right] \quad D^1_{3,3}=\frac{1}{4}\left[\frac{5}{2\mu+3}\left(mn_2+nm_2\right)+\varepsilon A^{(i)}_1\right]$$
$$D^2_{1,1}=\frac{1}{4}\left[\frac{5}{2\mu+3}\left(nl_2+\ln_2\right)+\varepsilon A^{(i)}_2\right] \quad D^2_{2,2}=0 \quad D^2_{3,3}=-\frac{1}{4}\left[\frac{5}{2\mu+3}\left(nl_2+\ln_2\right)-\varepsilon A^{(i)}_2\right] \qquad (6.23)$$
$$D^3_{1,1}=-\frac{1}{4}\left[\frac{5}{2\mu+3}\left(ml_2+lm_2\right)-\varepsilon A^{(i)}_3\right] \quad D^3_{2,2}=\frac{1}{4}\left[\frac{5}{2\mu+3}\left(ml_2+lm_2\right)+\varepsilon A^{(i)}_3\right] \quad D^3_{3,3}=0$$

Determining the unknowns $D^1_{2,3}, D^2_{1,3}, D^3_{1,2}$ is a little more complicated. They have to satisfy :

$$M_{1,1}=2\left(D^3_{1,2}-D^2_{1,3}\right)=\frac{5}{2\mu+3}ll_2$$
$$M_{2,2}=2\left(D^1_{2,3}-D^3_{1,2}\right)=\frac{5}{2\mu+3}mm_2 \qquad (6.24)$$
$$M_{3,3}=2\left(D^2_{1,3}-D^1_{2,3}\right)=\frac{5}{2\mu+3}nn_2$$

The solution of which is :

$$D^1_{2,3}=\frac{1}{6}\frac{5}{2\mu+3}\left(mm_2-nn_2\right) \quad D^2_{1,3}=\frac{1}{6}\frac{5}{2\mu+3}\left(nn_2-ll_2\right) \quad D^3_{1,2}=\frac{1}{6}\frac{5}{2\mu+3}\left(ll_2-mm_2\right) \qquad (6.25)$$

At the end the components $F^{(i)}_{2,l}$ of $F^{(i)}_2$ are:

$$\mu^{-1}F^{(i)}_{2,1}=\frac{1}{4}\varepsilon A^{(i)}_1\left(x_2^2+x_3^2\right)+\frac{1}{4}\frac{5}{2\mu+3}\left(mn_2+nm_2\right)\left(x_3^2-x_2^2\right)+\frac{1}{3}\frac{5}{2\mu+3}\left(mm_2-nn_2\right)x_2x_3$$
$$\mu^{-1}F^{(i)}_{2,2}=\frac{1}{4}\varepsilon A^{(i)}_2\left(x_1^2+x_3^2\right)+\frac{1}{4}\frac{5}{2\mu+3}\left(ln_2+nl_2\right)\left(x_1^2-x_3^2\right)+\frac{1}{3}\frac{5}{2\mu+3}\left(nn_2-ll_2\right)x_1x_3 \qquad (6.26)$$
$$\mu^{-1}F^{(i)}_{2,3}=\frac{1}{4}\varepsilon A^{(i)}_3\left(x_1^2+x_2^2\right)+\frac{1}{4}\frac{5}{2\mu+3}\left(lm_2+ml_2\right)\left(x_2^2-x_1^2\right)+\frac{1}{3}\frac{5}{2\mu+3}\left(ll_2-mm_2\right)x_1x_2$$

Stevenson's expression (Eq. 2.24 in his article) is indeed found for a sphere, but with an expression of $C^{(i)}_1$ that cannot be deduced from that of the ellipsoid.

With the particular values of the unit vectors, 6.26 reduces to :

$$F^{(i)}_{2,1}=0$$
$$F^{(i)}_{2,2}=\frac{1}{4}\varepsilon\mu A^{(i)}_2\left(x_1^2+x_3^2\right)+\frac{1}{4}\frac{5\mu}{2\mu+3}\left(x_1^2-x_3^2\right) \qquad (6.27)$$
$$F^{(i)}_{2,3}=0$$

6.2. Calculation of $V_2, V_2^{(i)}$

Let us write the boundary condition on the surface of the sphere (Eq. 2.5).

A straightforward calculation of the right-hand side yields (by denoting $J = J_2$)

$$n \bullet \left(F_2 ' + \nabla V_2 \text{ "} + E_2^{(0)} - \varepsilon F_2^{(i)} \right)$$
$$= R \left(\left(1-3J\right) \frac{1-\varepsilon}{2+\varepsilon} + \left(\frac{-3\varepsilon^2 \mu}{\varepsilon+2} + 2 - \frac{5\mu\varepsilon}{2\mu+3} \right) \frac{x_1^2}{4R^2} + \left(\frac{-3\varepsilon^2 \mu}{\varepsilon+2} + \frac{5\mu\varepsilon}{2\mu+3} \right) \frac{x_3^2}{4R^2} \right) x_2 + R \frac{1-\mu}{2\mu+3} x_2 \tag{6.28}$$

$$n \wedge \left(F_2 ' + \nabla V_2 \text{ "} + E_2^{(0)} - F_2^{(i)} \right) =$$
$$R^2 \left(\frac{1}{2}\left(1+3J\right) \frac{1-\varepsilon}{2+\varepsilon} + \left(\frac{-3\varepsilon\mu}{\varepsilon+2} + 2 - \frac{5\mu}{2\mu+3} \right) \frac{x_1^2}{4R^2} + \left(\frac{-3\varepsilon\mu}{\varepsilon+2} + \frac{5\mu}{2\mu+3} \right) \frac{x_3^2}{4R^2} \right) \begin{vmatrix} -x_3 / R \\ 0 \\ x_1 / R \end{vmatrix} \tag{6.29}$$
$$+ R^2 \frac{1-\mu}{2\mu+3} \begin{vmatrix} x_3 x_1^2 / R^3 \\ x_1 x_2 x_3 / R^3 \\ -x_1 \left(x_1^2 + x_2^2 \right) / R^3 \end{vmatrix}$$

The x_k and $x_k x_n^2$ monomials may be expressed as function of the spherical harmonics (Eqs. 3.3 and 3.5). Hence,

$$n \bullet \left(F_2 ' + \nabla V_2 \text{ "} + E_2^{(0)} - \varepsilon F_2^{(i)} \right)$$
$$= R^2 \left(\left(1-3J\right) \frac{1-\varepsilon}{2+\varepsilon} + \left(\frac{-3\varepsilon^2 \mu}{\varepsilon+2} + 1 \right) \frac{1}{10} + \frac{1-\mu}{2\mu+3} \right) Z_{1,-1}$$
$$+ R^2 \frac{1}{80} \left(\left(\frac{-9\varepsilon^2 \mu}{\varepsilon+2} + \frac{25\mu\varepsilon}{2\mu+3} - 2 \right) \right) Z_{3,-1} + R^2 \frac{1}{16} \left(\frac{-3\varepsilon^2 \mu}{\varepsilon+2} + 2 - \frac{5\mu\varepsilon}{2\mu+3} \right) Z_{3,-3} \tag{6.30}$$

$$n \wedge \left(F_2 ' + \nabla V_2 \text{ "} + E_2^{(0)} - F_2^{(i)} \right) = R^2 H \begin{vmatrix} -Z_{1,0} \\ 0 \\ Z_{1,1} \end{vmatrix} + R^2 \begin{vmatrix} p_0 Z_{3,0} + p_2 Z_{3,2} \\ p_{-2} Z_{3,-2} \\ p_1 Z_{3,1} + p_3 Z_{3,3} \end{vmatrix} \tag{6.31}$$

with

$$H = \frac{1}{2} \frac{1-\varepsilon}{2+\varepsilon} \left(3J+1\right) + \frac{1+9\mu}{10\left(2\mu+3\right)} - \frac{3\varepsilon\mu}{5\left(\varepsilon+2\right)} \tag{6.32}$$

$$
\begin{aligned}
p_0 &= \frac{1}{40}\left(\frac{3\varepsilon\mu}{\varepsilon+2}+\frac{2-7\mu}{2\mu+3}\right)\\
p_1 &= -p_0/2\\
p_2 &= \frac{1}{8}\left(\frac{3\varepsilon\mu}{\varepsilon+2}-\frac{2+3\mu}{2\mu+3}\right)\\
p_3 &= \frac{1}{16}\left(-\frac{3\varepsilon\mu}{\varepsilon+2}+\frac{6-\mu}{2\mu+3}\right)\\
p_{-2} &= \frac{1-\mu}{2\mu+3}
\end{aligned}
\tag{6.33}
$$

Let's look at the left-hand side of the boundary condition (Eq.2.5).

The form of the boundary condition on the surface of the sphere (right-hand sides 6.30-6.31) suggests a general solution for V_2 and $V_2^{(i)}$ involving spherical harmonics of degree $l=1$ and $l=3$. We begin by expressing the contribution of $l=3$ to $\nabla\left(V_2-\varepsilon^e V_2^{(i)}\right)$ with $e=0,1$. To simplify the writing, we will note, as we did for $l=2$:

$$
\lambda_m = \left(R^{-7}L_{3,m}-\varepsilon^e K_{3,m}\right)g_{3,m} \tag{6.34}
$$

Following Eq. 3.6, we have ($r=R$):

$$
\nabla\left(V_2-\varepsilon^e V_2^{(i)}\right) = [A]\lambda - 7R^{-5}n\sum_{m=-3}^{3} L_{3,m}Y_{3,m}
$$

with

$$
\lambda = {}^T\left(\lambda_{-3}\quad \lambda_{-2}\quad \lambda_{-1}\quad \lambda_0\quad \lambda_1\quad \lambda_2\quad \lambda_3\right)
$$

$$
A = \begin{bmatrix}
6x_1x_2 & x_2x_3 & -2x_1x_2 & -6x_1x_3 & 5x_3^2-r^2-2x_1^2 & 2x_1x_3 & 3x_1^2-3x_2^2\\
3x_1^2-3x_2^2 & x_1x_3 & 5x_3^2-r^2-2x_2^2 & -6x_2x_3 & -2x_1x_2 & -2x_2x_3 & -6x_1x_2\\
0 & x_1x_2 & 8x_2x_3 & 5x_3^2-3r^2+4x_3^2 & 8x_1x_3 & x_1^2-x_2^2 & 0
\end{bmatrix}
$$

The cross product $n\wedge\nabla\left(V_2-V_2^{(i)}\right)$ can be written as :

$$
n\wedge\nabla\left(V_2-V_2^{(i)}\right) = n\wedge[A]\lambda = [B]\lambda \tag{6.35}
$$

$$B=\frac{1}{R}\begin{bmatrix} -3x_3\left(x_1^2-x_2^2\right) & x_1\left(x_2^2-x_3^2\right) & 10x_2^2x_3-5x_3^3+x_3r^2 \\ 6x_1x_2x_3 & x_2\left(x_3^2-x_1^2\right) & -10x_1x_2x_3 \\ 3x_1\left(x_1^2-3x_2^2\right) & x_3\left(x_1^2-x_2^2\right) & 5x_1x_3^2-x_1r^2 \end{bmatrix}$$

$$\begin{bmatrix} 15x_2x_3^2-3x_2r^2 & 10x_1x_2x_3 & x_2x_1^2-x_2^3+2x_2x_3^2 & 6x_1x_2x_3 \\ -15x_1x_3^2+3x_1r^2 & -10x_1^2x_3+5x_3^3-x_3r^2 & x_1x_2^2-x_1^3+2x_1x_3^2 & x_3\left(3x_1^2-3x_2^2\right) \\ 0 & -5x_2x_3^2+x_2r^2 & -4x_1x_2x_3 & -9x_1^2x_2+3x_2^3 \end{bmatrix} \tag{6.36}$$

$$B=R^2\begin{bmatrix} -3Z_{3,2} & -\left(Z_{3,1}+Z_{3,3}\right)/4 & -5Z_{3,2}-2Z_{3,0} & 3Z_{3,-1} & 10Z_{3,-2} & \frac{1}{2}\left(Z_{3,-3}+Z_{3,-1}\right) & 6Z_{3,-2} \\ 6Z_{3,-2} & \left(Z_{3,-1}-Z_{3,-3}\right)/4 & -10Z_{3,-2} & -3Z_{3,1} & -5Z_{3,2}+2Z_{3,0} & \frac{1}{2}\left(-Z_{3,3}+Z_{3,1}\right) & 3Z_{3,2} \\ 3Z_{3,3} & Z_{3,2} & Z_{3,1} & 0 & -Z_{3,-1} & -4Z_{3,-2} & -3Z_{3,-3} \end{bmatrix} \tag{6.37}$$

We see that B does not contain spherical harmonics with degree one.

The scalar product $n\bullet\nabla\left(V_2-\varepsilon V_2^{(i)}\right)$ can be written as :

$$n\bullet\nabla\left(V_2-\varepsilon V_2^{(i)}\right)=n\bullet\left[A\right]\lambda^{e=1}-7R^{-5}\sum_{m=-3}^{3}L_{3,m}Y_{3,m} \tag{6.38}$$

with

$$A=\begin{bmatrix} 6x_1x_2 & x_2x_3 & -2x_1x_2 & -6x_1x_3 & 5x_3^2-r^2-2x_1^2 & 2x_1x_3 & 3x_1^2-3x_2^2 \\ 3x_1^2-3x_2^2 & x_1x_3 & 5x_3^2-r^2-2x_2^2 & -6x_2x_3 & -2x_1x_2 & -2x_2x_3 & -6x_1x_2 \\ 0 & x_1x_2 & 8x_2x_3 & 5x_3^2-3r^2+4x_3^2 & 8x_1x_3 & x_1^2-x_2^2 & 0 \end{bmatrix}$$

Hence,

$$\begin{aligned} &Rn\bullet\left[A\right]\lambda^{e=1}=\left(9x_1^2x_2-3x_2^3\right)\lambda_{-3}^{e=1}+\left(3x_1x_2x_3\right)\lambda_{-2}^{e=1}+\left(-2x_1^2x_2+13x_2x_3^2-r^2x_2-2x_2^3\right)\lambda_{-1}^{e=1} \\ &+\left(-6x_1^2x_3-6x_2^2x_3-3r^2x_3+9x_3^3\right)\lambda_0^{e=1}+\left(-2x_2^2x_1+13x_1x_3^2-r^2x_1-2x_1^3\right)\lambda_1^{e=1} \\ &+\left(3x_1^2x_3-3x_2^2x_3\right)\lambda_2^{e=1}+\left(3x_1^3-9x_1x_2^2\right)\lambda_3^{e=1} \\ &=3R^3\left(Z_{3,-3}\lambda_{-3}^{e=1}+Z_{3,-2}\lambda_{-2}^{e=1}+Z_{3,-1}\lambda_{-1}^{e=1}+Z_{3,0}\lambda_0^{e=1}+Z_{3,1}\lambda_1^{e=1}+Z_{3,2}\lambda_2^{e=1}+Z_{3,3}\lambda_3^{e=1}\right) \end{aligned} \tag{6.39}$$

Finally,

$$n\bullet\nabla\left(V_2-\varepsilon\ V_2^{(i)}\right)=\sum_{m=-3}^{3}\left(3R^2Z_{3,m}\lambda_m^{e=1}-7R^{-5}L_{3,m}Y_{3,m}\right) \tag{6.40}$$

If V_2 and $V_2^{(i)}$ are expressed with spherical harmonics with degree 3, then $n\wedge\nabla\left(V_2-V_2^{(i)}\right)$ and $n\bullet\nabla\left(V_2-\varepsilon V_2^{(i)}\right)$ only contain spherical harmonics with degree 3. However,

$n \bullet \left(F_2 '+ \nabla V_2 "+ E_2^{(0)} - \varepsilon F_2^{(i)} \right)$ and $n \wedge \left(F_2 '+ \nabla V_2 "+ E_2^{(0)} - F_2^{(i)} \right)$ also contain spherical harmonics with degree one ; we deduce that V_2 and $V_2^{(i)}$ must contain spherical harmonics with degree one :

$$\begin{aligned} V_2 &= \sum_{m=-1}^{1} L_{1,m} r^{-2} Y_{1,m} = r^{-3} \left(L_{1,-1} g_{1,-1} x_2 + L_{1,0} g_{1,0} x_3 + L_{1,1} g_{1,1} x_1 \right) \\ V_2^{(i)} &= \sum_{m=-1}^{1} K_{1,m} r Y_{1,m} = K_{1,-1} g_{1,-1} x_2 + K_{1,0} g_{1,0} x_3 + K_{1,1} g_{1,1} x_1 \end{aligned} \tag{6.41}$$

hence ($r = R$)

$$\begin{aligned} \nabla \left(V_2 - \varepsilon^e V_2^{(i)} \right) \Big|_{r=R} &= \begin{vmatrix} \gamma_1 \\ \gamma_{-1} \\ \gamma_0 \end{vmatrix} - 3R^{-4} \left(L_{1,-1} g_{1,-1} x_2 + L_{1,0} g_{1,0} x_3 + L_{1,1} g_{1,1} x_1 \right) n \\ \gamma_m &= \left(R^{-3} L_{1,m} - \varepsilon^e K_{1,m} \right) g_{1,m} \end{aligned} \tag{6.42}$$

The corresponding cross product and scalar product are :

$$n \wedge \nabla \left(V_2 - V_2^{(i)} \right) \Big|_{r=R} = \frac{1}{R} \begin{vmatrix} \gamma_0 x_2 - \gamma_{-1} x_3 \\ \gamma_1 x_3 - \gamma_0 x_1 \\ \gamma_{-1} x_1 - \gamma_1 x_2 \end{vmatrix} = \begin{vmatrix} \gamma_0 Z_{1,-1} - \gamma_{-1} Z_{1,0} \\ \gamma_1 Z_{1,0} - \gamma_0 Z_{1,1} \\ \gamma_{-1} Z_{1,1} - \gamma_1 Z_{1,-1} \end{vmatrix} \tag{6.43}$$

$$\begin{aligned} & n \bullet \nabla \left(V_2 - \varepsilon V_2^{(i)} \right) \Big|_{r=R} = \frac{1}{R} \left(\gamma_1^{e=1} x_1 + \gamma_{-1}^{e=1} x_2 + \gamma_0^{e=1} x_3 \right) - 3R^{-4} \left(L_{1,-1} g_{1,-1} x_2 + L_{1,0} g_{1,0} x_3 + L_{1,1} g_{1,1} x_1 \right) \\ & = \gamma_1^{e=1} Z_{1,1} + \gamma_{-1}^{e=1} Z_{1,-1} + \gamma_0^{e=1} Z_{1,0} - 3R^{-3} \left(L_{1,-1} g_{1,-1} Z_{1,-1} + L_{1,0} g_{1,0} Z_{1,0} + L_{1,1} g_{1,1} Z_{1,1} \right) \end{aligned} \tag{6.44}$$

The $K_{1,m}$ and $L_{1,m}$ constants are obtained from the boundary condition on the sphere surface (cross product and scalar product) , knowing that the $Z_{l,m}$ form a set of orthogonal functions:

$$\begin{vmatrix} \gamma_0 Z_{1,-1} - \gamma_{-1} Z_{1,0} \\ \gamma_1 Z_{1,0} - \gamma_0 Z_{1,1} \\ \gamma_{-1} Z_{1,1} - \gamma_1 Z_{1,-1} \end{vmatrix} = -R^2 H \begin{vmatrix} -Z_{1,0} \\ 0 \\ Z_{1,1} \end{vmatrix} \Rightarrow \begin{cases} \gamma_{-1} = -R^2 H \\ \gamma_0 = 0 \\ \gamma_1 = 0 \end{cases} \tag{6.45}$$

$$\begin{aligned} & \gamma_1^{e=1} Z_{1,1} + \gamma_{-1}^{e=1} Z_{1,-1} + \gamma_0^{e=1} Z_{1,0} - 3R^{-3} \left(L_{1,-1} g_{1,-1} Z_{1,-1} + L_{1,0} g_{1,0} Z_{1,0} + L_{1,1} g_{1,1} Z_{1,1} \right) \\ & = -R^2 H^{e=1} Z_{1,-1} \\ & \Rightarrow \begin{cases} \gamma_{-1}^{e=1} - 3R^{-3} L_{1,-1} g_{1,-1} = -R^2 H^{e=1} \\ \gamma_0^{e=1} - 3R^{-3} L_{1,0} g_{1,0} = 0 \\ \gamma_1^{e=1} - 3R^{-3} L_{1,1} g_{1,1} = 0 \end{cases} \end{aligned} \tag{6.46}$$

with

$$H^{e=1} = (1-3J)\frac{1-\varepsilon}{2+\varepsilon} + \left(\frac{-3\varepsilon^2\mu}{\varepsilon+2}+1\right)\frac{1}{10} + \frac{1-\mu}{2\mu+3} \tag{6.47}$$

hence

$$\begin{aligned} &K'_{1,-1} = K_{1,-1}g_{1,-1} = R^2\frac{2H+H^{e=1}}{2+\varepsilon} \quad L'_{1,-1} = L_{1,-1}g_{1,-1} = R^5\frac{H^{e=1}-\varepsilon H}{2+\varepsilon} \\ &K'_{1,0} = 0 \quad L'_{1,0} = 0 \\ &K'_{1,1} = 0 \quad L'_{1,1} = 0 \end{aligned} \tag{6.48}$$

Therefore the contribution of terms with degree 1 to V_2 and $V_2^{(i)}$ is :

$$\begin{aligned} V_2 &= \frac{R^5}{r^3}\frac{H^{e=1}-\varepsilon H}{2+\varepsilon}x_2 \\ V_2^{(i)} &= R^2\frac{2H+H^{e=1}}{2+\varepsilon}x_2 \end{aligned} \tag{6.49}$$

$V_2^{(i)}$ does not contain J whereas V_2 can be written as:

$$V_2 = \frac{R^5}{r^3}\frac{3}{2}\frac{1-\varepsilon}{2+\varepsilon}\left(-J+F(\varepsilon,\mu)\right)x_2 = \frac{R^2}{r^3}A_2x_2\left(-J+F(\varepsilon,\mu)\right) \tag{6.50}$$

It is the sum $V_2+V_2"$ and not separately V_2 and $V_2"$ that has a physical meaning (as present in E_2). We note that, following 6.12-6.14 and 6.50, $V_2+V_2" = A_2x_2F'(R,r,\varepsilon,\mu)$ does not depend on J ; therefore, it is not necessary to determine J .

The same procedure is used for the constants $K_{3,m}$ and $L_{3,m}$:

Starting from the cross product (6.31 and 6.35-37) :

$$R^2\begin{bmatrix} -3Z_{3,2} & -(Z_{3,1}+Z_{3,3})/4 & -5Z_{3,2}-2Z_{3,0} & 3Z_{3,-1} & 10Z_{3,-2} & \frac{1}{2}(Z_{3,-3}+Z_{3,-1}) & 6Z_{3,-2} \\ 6Z_{3,-2} & (Z_{3,-1}-Z_{3,-3})/4 & -10Z_{3,-2} & -3Z_{3,1} & -5Z_{3,2}+2Z_{3,0} & \frac{1}{2}(-Z_{3,3}+Z_{3,1}) & 3Z_{3,2} \\ 3Z_{3,3} & Z_{3,2} & Z_{3,1} & 0 & -Z_{3,-1} & -4Z_{3,-2} & -3Z_{3,-3} \end{bmatrix}\begin{pmatrix} \lambda_{-3} \\ \\ \\ \\ \\ \\ \lambda_3 \end{pmatrix}$$

$$= -R^2\begin{vmatrix} p_0Z_{3,0}+p_2Z_{3,2} \\ p_{-2}Z_{3,-2} \\ p_1Z_{3,1}+p_3Z_{3,3} \end{vmatrix} \tag{6.51}$$

$$\Rightarrow \left| \begin{array}{l} 10\lambda_1 + 6\lambda_3 = 0 \quad -\frac{1}{4}\lambda_{-2} = 0 \quad -\frac{1}{4}\lambda_3 = 0 \quad 3\lambda_0 + \frac{1}{2}\lambda_2 = 0 \quad \lambda_2 = 0.... \\ .....-3\lambda_{-3} - 5\lambda_{-1} = -p_2 \quad -2\lambda_{-1} = -p_0 \\ \quad 6\lambda_{-3} - 10\lambda_{-1} = -p_{-2} \quad \frac{1}{4}\lambda_{-2} = 0 \quad -\frac{1}{4}\lambda_{-2} = 0 \quad -3\lambda_0 + \frac{1}{2}\lambda_2 = 0..... \\ .....-5\lambda_1 + 3\lambda_3 = 0 \quad 2\lambda_1 = 0 \quad -\frac{1}{2}\lambda_2 = 0 \\ \quad 3\lambda_{-3} = -p_3 \quad \lambda_{-2} = 0 \quad \lambda_{-1} = -p_1 \quad -\lambda_1 = 0 \quad -4\lambda_2 = 0 \quad -3\lambda_3 = 0 \end{array} \right. \tag{6.52}$$

We check that these equations are compatible with each other (20 equations for 7 unknowns), taking into account redundancies and relationships between parameters p_m .

$$\Rightarrow \begin{cases} \lambda_m = 0 \quad m = -2,0,1,2,3 \\ \lambda_{-3} = \frac{1}{48}\left(\frac{3\varepsilon\mu}{\varepsilon+2} - \frac{6-\mu}{2\mu+3} \right) \\ \lambda_{-1} = \frac{1}{80}\left(\frac{3\varepsilon\mu}{\varepsilon+2} + \frac{2-7\mu}{2\mu+3} \right) \end{cases} \tag{6.53}$$

Starting from the scalar product (6.30 and 6.40):

$$\begin{aligned} &\sum_{m=-3}^{3} \left(3R^2 Z_{3,m} \lambda_m^{e=1} - 7R^{-5} L_{3,m} g_{3,m} Z_{3,m} \right) \\ &= -R^2 \frac{1}{80}\left(\frac{-9\varepsilon^2\mu}{\varepsilon+2} + \frac{25\mu\varepsilon}{2\mu+3} - 2 \right) Z_{3,-1} - R^2 \frac{1}{16}\left(\frac{-3\varepsilon^2\mu}{\varepsilon+2} + 2 - \frac{5\mu\varepsilon}{2\mu+3} \right) Z_{3,-3} \end{aligned} \tag{6.54}$$

$$\Rightarrow \begin{cases} 3R^2 \lambda_m^{e=1} - 7R^{-5} L_{3,m} g_{3,m} = 0 \quad m = -2,0,1,2,3 \\ 3R^2 \lambda_{-3}^{e=1} - 7R^{-5} L_{3,-3} g_{3,-3} = -R^2 \frac{1}{16}\left(\frac{-3\varepsilon^2\mu}{\varepsilon+2} + 2 - \frac{5\mu\varepsilon}{2\mu+3} \right) = -R^2 T_{-3} \\ 3R^2 \lambda_{-1}^{e=1} - 7R^{-5} L_{3,-1} g_{3,-1} = -R^2 \frac{1}{80}\left(\frac{-9\varepsilon^2\mu}{\varepsilon+2} + \frac{25\mu\varepsilon}{2\mu+3} - 2 \right) = -R^2 T_{-1} \end{cases} \tag{6.55}$$

We directly deduce that :

$$\begin{aligned} &K'_{3,m} = L'_{3,m} = 0 \quad m = -2,0,1,2,3 \\ &K'_{3,m} = \frac{T_m - 4\lambda_m}{4+3\varepsilon} \quad L'_{3,m} = \frac{T_m + 3\varepsilon\lambda_m}{4+3\varepsilon} R^7 \quad m = -1,-3 \end{aligned} \tag{6.56}$$

Finally the contribution of terms with degree one and three to V_2 and $V_2^{(i)}$ is :

$$\begin{aligned} V_2 &= \frac{R^5}{r^3}\frac{H^{e=1}-\varepsilon H}{2+\varepsilon}x_2 + \frac{T_{-1}+3\varepsilon\lambda_{-1}}{4+3\varepsilon}\frac{R^7}{r^7}x_2\left(5x_3^2-r^2\right)+\frac{T_{-3}+3\varepsilon\lambda_{-3}}{4+3\varepsilon}\frac{R^7}{r^7}x_2\left(3x_1^2-x_2^2\right) \\ V_2^{(i)} &= R^2\frac{2H+H^{e=1}}{2+\varepsilon}x_2 + \frac{T_{-1}-4\lambda_{-1}}{4+3\varepsilon}x_2\left(5x_3^2-r^2\right)+\frac{T_{-3}-4\lambda_{-3}}{4+3\varepsilon}x_2\left(3x_1^2-x_2^2\right) \end{aligned} \tag{6.57}$$

7. Complete solution to second order

We directly deduce the internal field from Eqs 6.27 and 6.57 :

$$E_2^{(i)} = \frac{T_{-1}-4\lambda_{-1}}{4+3\varepsilon}X_{-1}+\frac{T_{-3}-4\lambda_{-3}}{4+3\varepsilon}X_{-3}+\left(\frac{1}{4}\varepsilon\mu A_2^{(i)}\left(x_1^2+x_3^2\right)+\frac{1}{4}\frac{5\mu}{2\mu+3}\left(x_1^2-x_3^2\right)+R^2\frac{2H+H^{e=1}}{2+\varepsilon}\right)u_1 \tag{6.58}$$

and the external field :

$$\begin{aligned} E_2 &= \frac{R^3}{r}\frac{(1-\varepsilon)}{(\varepsilon+2)}u_1+\frac{\beta}{r^3}\begin{vmatrix} 2x_1x_2/r^2 \\ \left(x_2^2-x_1^2\right)/r^2 \\ x_2x_3/r^2 \end{vmatrix} - \frac{R^3}{r}\frac{(1-\varepsilon)}{(\varepsilon+2)}\frac{1}{2}\left(u_1-x_2\frac{\nabla r}{r}\right)+\frac{R^5}{r^3}Q\left(u_1-3x_2\frac{\nabla r}{r}\right) \\ &+\frac{T_{-1}+3\varepsilon\lambda_{-1}}{4+3\varepsilon}\frac{R^7}{r^7}\left(X_{-1}-7x_{-1}\frac{\nabla r}{r}\right)+\frac{T_{-3}+3\varepsilon\lambda_{-3}}{4+3\varepsilon}\frac{R^7}{r^7}\left(X_{-3}-7x_{-3}\frac{\nabla r}{r}\right) \end{aligned} \tag{6.59}$$

with

$$\begin{aligned} Q &= \frac{1}{2+\varepsilon}\left[\frac{1-\varepsilon}{2+\varepsilon}(1-\varepsilon/2)+\frac{1}{10}\left(1+\frac{3\varepsilon^2\mu}{\varepsilon+2}\right)+\frac{1-\mu}{2\mu+3}-\frac{\varepsilon}{10}\frac{1+9\mu}{2\mu+3}\right] \\ x_{-1} &= x_2\left(5x_3^2-r^2\right) \\ x_{-3} &= x_2\left(3x_1^2-x_2^2\right) \\ X_{-1} &= \nabla\left(r^3Z_{3,-1}\right)=\nabla\left(x_2\left(5x_3^2-r^2\right)\right)=\begin{vmatrix} -2x_1x_2 \\ 4x_3^2-x_1^2-3x_2^2 \\ 8x_2x_3 \end{vmatrix} \\ X_{-3} &= \nabla\left(r^3Z_{3,-3}\right)=\nabla\left(x_2\left(3x_1^2-x_2^2\right)\right)=\begin{vmatrix} 6x_1x_2 \\ 3\left(x_1^2-x_2^2\right) \\ 0 \end{vmatrix} \end{aligned} \tag{6.60}$$

if $\mu=1$,

$$\begin{aligned} E_2^{(i)} &= \frac{1}{20}\left(\frac{1-\varepsilon}{\varepsilon+2}\right)\frac{1+\varepsilon}{4+3\varepsilon}X_{-1}+\frac{1}{12}\left(\frac{1-\varepsilon}{\varepsilon+2}\right)\frac{5+3\varepsilon}{4+3\varepsilon}X_{-3} \\ &+\left(\frac{3}{4}\frac{\varepsilon}{2+\varepsilon}\left(x_1^2+x_3^2\right)+\frac{1}{4}\left(x_1^2-x_3^2\right)+R^2\frac{3(1-\varepsilon)(1+\varepsilon/10)}{(\varepsilon+2)^2}\right)u_1 \end{aligned} \tag{6.61}$$

$$E_2 = \frac{R^3}{r}\frac{(1-\varepsilon)}{(\varepsilon+2)}u_1 - \frac{R^3}{r}\frac{(1-\varepsilon)}{(\varepsilon+2)}\frac{1}{2}\left(u_1 - x_2\frac{\nabla r}{r}\right) + \frac{R^5}{r^3}\frac{3}{5}\frac{2-\varepsilon}{2+\varepsilon}\frac{1-\varepsilon}{2+\varepsilon}\left(u_1 - 3x_2\frac{\nabla r}{r}\right)$$
$$-\frac{1}{40}\frac{1-\varepsilon}{4+3\varepsilon}\frac{R^7}{r^7}\left(X_{-1} - 7x_{-1}\frac{\nabla r}{r}\right) + \frac{1-\varepsilon}{8(4+3\varepsilon)}\frac{R^7}{r^7}\left(X_{-3} - 7x_{-3}\frac{\nabla r}{r}\right) \qquad (6.62)$$

Expressions 6.61 and 6.62 have been validated by comparison with the solution derived from Mie's theory (see Van de Hulst).

We easily show that the above expressions satisfy the condition 2.6 of Stevenson : $\int n \bullet E_2 dS = \int n \bullet \left(F_2' + \nabla V_2'' + \nabla V_2\right) dS = 0$. F_2' is defined within an additive constant (vector), denoted d ; however, $\int n \bullet E_2 dS = 0$ is yet valid, because of $\int n \bullet d\, dS = 0$.

We have just computed E_0, H_1, E_2 at a (x_1, x_2, x_3) point. In order to obtain $H_0, -E_1, H_2$, we need apply 2.7. Applying 2.7 to E_0 and H_1 poses no problem; the same cannot be stated for E_2, since its expression has been obtained for particular unit vectors (of the incident field). To calculate H_2, we propose an alternative to 2.7, consisting of the following steps:

- do the transform : $x_1' = x_1 \quad x_2' = x_3 \quad x_3' = -x_2$
- do the exchange $\varepsilon \leftrightarrow \mu$
- apply the expressions for E_2 at (x_1', x_2', x_3') point, the corresponding vector is called E_2'
- deduce $H_{2,1} = E'_{2,1} \quad H_{2,2} = -E'_{2,3} \quad H_{2,3} = E'_{2,2}$

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