An O(T 3) Algorithm for the Lot-Sizing Problem with Constant Capacities and Large Inventory Bounds

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An $\mathcal{O}(T^3)$ Algorithm for the Lot-Sizing Problem with Constant Capacities and Large Inventory Bounds

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Abstract

This work aims at designing a new algorithm of improved complexity to solve the lot-sizing problem with constant capacities and inventory bounds, in cases where those bounds are higher than twice the production capacity at all time. We build upon previous works by investigating the decomposition of this optimization problem into smaller subplans. New results on the feasibility and optimality of a subplan are proposed that lead to an improved algorithm of cubic worst-case complexity.

Keywords: Lot-Sizing, Constant Capacities, Inventory Bounds, Computational Complexity

1. Introduction

The Lot-Sizing problem covers a broad range of application for the optimal planification of production, and its size often requires efficient algorithms for its resolution. It represents, across T consecutive periods of time, the interaction between a demand for an item, the production generating such item and the use of an inventory to store items produced in advance. The cost of production are considered to be linear, however a setup cost must be accounted for at any period of production. Holding costs linearly penalize the amount of item in inventory at each period. The objective is to find a feasible production plan that minimizes total costs.

This work tackles the Single-Item Capacitated Lot-Sizing problem with constant production capacity, linear costs and inventory upper bounds. Existing polynomial algorithms for the resolution of this problem are of $\mathcal{O}(T^4)$ complexity at best. We undertake an in-depth study of the feasibility and optimality conditions of such problems and we derive properties that lead to an improved algorithm of $\mathcal{O}(T^3)$ complexity for instances whose potential for storage is at least twice the production capacity.

1.1. Related work

The Lot-Sizing problem has been widely studied for decades as its applications and variations are numerous (Brahimi et al., 2017). There has been precise studies of its complexity in order to classify which models could be solved in polynomial time, or were otherwise NP-hard (Brahimi et al., 2006). The Capacitated Lot-Sizing Problem (CLSP) may be classified by a system first introduced by Bitran and Yanasse (1981), where each problem belongs to a particular category depending of its setup costs, holding costs, production costs and production capacity being non-increasing (NI), non-decreasing (ND), of none value (Z) or "general" (G) otherwise. In our case, we study the G/G/G/C CLSP with general setup costs, holding costs, production costs and constant production capacity, with an added upper bound constraint on the maximum inventory capacity. The standard G/G/G/C CLSP has been shown to be solved in polynomial time, first with an $\mathcal{O}(T^4)$ algorithm (Florian and Klein, 1971) then later with an $\mathcal{O}(T^3)$ algorithm (van Hoesel and Wagelmans, 1993).

The G/G/G/C Lot-Sizing Problem with Constant Capacities and Storage Upper Bounds, denoted LS-CC-SUB in the following, was first introduced by Erenguc and Aksoy (1990). It has been studied by Wolsey

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(2005), demonstrating equivalences between that problem and certain cases of Lot-Sizing problem with time-windowed demand and showing it can be solved with a polynomial $\mathcal{O}(T^4)$ algorithm. A special case of the LS-CC-SUB was solved in $\mathcal{O}(T^3)$ complexity by Akbalik et al. (2015) for a problem with Wagner-Whitin cost structure. An optimality property for the LS-CC-SUB with lower and upper bounds on the production and storage capacity is given in Love (1973), although the study is restricted afterward to the uncapacitated lot-sizing problem (without upper bound on production capacity) with inventory bounds.

1.2. Contributions

In the following, an introduction to the studied mathematical model is first proposed in section 2 where general assumptions on the model are made and the notion of subplan is developed. We then focus on the feasibility conditions of a subplan for the LS-CC-SUB in section 3, and on its optimality conditions in section 4. The resulting properties on feasibility and optimality allow for the design in sections 5 and 6 of an efficient dynamic resolution that improves the overall complexity.

2. The Lot-Sizing Problem with Constant Capacities and Storage Upper Bounds

2.1. Mathematical model

We present a general model for the LS-CC-SUB. Let C denote the constant production capacity in each period. For any period $t \in [|1, T|]$, we define the following parameters:

- I^0 the initial inventory value;
- S_t the inventory capacity in t;
- s_t: the setup cost in t;
- c_t: the production cost in t;
- h_t: the holding cost in t;
- d_t: the demand in t;

Furthermore for any periods $1 \leq t_1 \leq t_2 \leq T$, the cumulative demand in $[|t_1, t_2|]$ is denoted by d_{t_1,t_2}. Then for each period $t \in [|1, T|]$, we define the following variables:

- x_t: the production value in t;
- I_t: the inventory value in t
- z_t: the choice to produce in t

A general model for the LS-CC-SUB is then:

$$\min \sum_{t\in[|1,T|]} z_t \cdot s_t + x_t \cdot c_t + I_t \cdot h_t \tag{1}$$

$$\text{s.t. } x_t \leq z_t \cdot C \qquad \forall t \in [|1, T|] \tag{2}$$

$$I_1 = I^0 + x_1 - d_1 \tag{3}$$

$$I_t = I_{t-1} + x_t - d_t \qquad \forall t \in [|2, T|] \tag{4}$$

$$I_t \leq S_t \qquad \forall t \in [|1, T|] \tag{5}$$

$$I_t,\, x_t \geq 0 \qquad \forall t \in [|1, T|] \tag{6}$$

$$z_t \in \{0, 1\} \qquad \forall t \in [|1, T|] \tag{7}$$

2.2. Simplifying assumptions

We present three assumptions made without loss of generality to simplify the mathematical model of the LS-CC-SUB and derive necessary properties for the design of our novel algorithm.

2.2.1. Assumptions on holding costs

In the following, we assume the holding costs to be of zero value. If it is not the case, an equivalent problem is obtained by integrating the holding costs to the production costs (see, *e.g.*, Wagelmans et al. (1992)). Suppose we define new production costs c' such that $c'_t = c_t + \sum_{t'=1}^{t} h_{t'}$ for all periods $t \in [|1, T|]$. Then an equivalent model to (1)-(7) without holding costs would be:

$$\min \sum_{t \in [|1,T|]} z_t \cdot s_t + x_t \cdot c'_t - \sum_{t \in [|1,T|]} h_t \cdot d_{1,t} \tag{8}$$

$$\text{s.t. } (2)\text{-}(7) \tag{9}$$

Hence in the following, we always assume the holding costs h to be of zero value.

2.2.2. Assumptions on the demand

We also assume the demand to be upper-bounded by the constant production capacity C at any period $t \in [|1, T|]$. If it is not the case, an equivalent problem with modified demand and inventory value can be found for which this property applies. This equivalent model is obtained by displacing any demand above the production capacity to earlier periods, so that it can be stored in advance.

Applying this technique is trivial for the unbounded inventory problem (see, *e.g.*, van Hoesel and Wagelmans (1993)), however in the bounded case the inventory capacity must also be modified. Although it has been alluded to by Wolsey (2005), we propose to precisely define how to transform such instances of the LS-CC-SUB. A necessary condition for the LS-CC-SUB to be feasible is that the inventory capacity must be higher than the inventory value needed to meet current and future demand. Let $\underline{I}_t$ represent the minimum necessary inventory value at the end of period $t \in [|1, T|]$:

$$\begin{cases} \underline{I}_T = 0 \\ \underline{I}_t = \max\{0,\, \underline{I}_{t+1} + d_{t+1} - C\}, \quad \forall t \leq T \end{cases}$$

Then a solution to the LS-CC-SUB problem is feasible only if $I_t \geq \underline{I}_t$, for all periods $t \in [|1, T|]$. It follows that the LS-CC-SUB problem is feasible only if $S_t \geq \underline{I}_t$ at all periods t.

Suppose that at a given period t, the demand d_t is greater than the production capacity C. Obviously, production at period t is not sufficient to supply the demand in full, and production at previous periods must be stored in prevision of the demand at period t. This additional necessary production is represented, at the closest periods previous to t, by the maximum possible increase in the demand and a corresponding decrease in the inventory capacity.

Therefore an equivalent instance of the problem may be built as such, with a demand d' and an inventory capacity S' defined by $d'_t = \min\{C,\, d_t + \underline{I}_t\}$ and $S'_t = S_t - \underline{I}_t$, for all periods $t \in [|1, T|]$. In this equivalent problem, the demand d' is obviously lower than the production capacity C at all time. Hence in the following, we always assume the demand to be lower than the production capacity C.

2.2.3. Assumptions on inventory bounds

Finally, we assume that for any period t there always exists a feasible production plan such that the inventory I_t may attain its upper bound S_t. If it is not the case, an equivalent model can be obtained by tightening the inventory bounds.

First, the inventory value cannot exceed the overall inventory obtained from a production at all period while respecting the inventory capacity. Therefore, an equivalent model can be formulated with its inventory capacity S' defined as:

$$\begin{aligned} S'_1 &= \min\{S_1,\, I^0 + C - d_1\} \\ S'_t &= \min\{S_t,\, S'_{t-1} + C - d_t\}, \quad \forall t > 0 \end{aligned}$$

Furthermore, since the supply cannot exceed the sum of demand and inventory capacity, some inventory value upper bounds are unattainable as the inventory would not be sustained in future periods. Therefore, an equivalent model can be formulated with its inventory capacity S' defined as:

$$S'_t = \min\{S_t,\, S'_{t+1} + d_{t+1}\}, \quad \forall t < T$$

Hence we always assume:

$$S_1 \leq I^0 + C - d_1 \tag{10}$$
$$S_t \leq S_{t-1} + C - d_t \qquad \forall t > 0 \tag{11}$$
$$S_t \leq S_{t+1} + d_{t+1} \qquad \forall t < T \tag{12}$$

Inequalities (10)-(12) ensure that at any period $t \in [|1, T|]$ and for any inventory value $I_t^* \in [0, S_t]$, a solution to the LS-CC-SUB problem between period t and T with entry stock I_t^* may be found.

2.3. Polynomial algorithms for the LS-CC-SUB and the CLSP

We present the concept of subplans as first introduced by Florian and Klein (1971) to solve the CLSP and employed by Wolsey (2005) to design an $\mathcal{O}(T^4)$ algorithm for the LS-CC-SUB.

Let $cf(t)$ denote the cost of producing at full capacity at period $t \in [|1, T|]$:

$$cf(t) = s_t + C \cdot c_t, \quad \forall t \in [|1, T|]$$

The production at period t is called fractional if it belongs in $[0, C[$. A subplan denoted (t_1, t_2, I^1, I^2) with $1 \leq t_1 \leq t_2 \leq T$ and $I^1, I^2 \in \{0, 1\}$ is defined as a set of consecutive periods between t_1 and t_2 with at most one period of fractional production and of entry and ending inventory values $I_{t_1-1} = I^1 \cdot S_{t_1-1}$ and $I_{t_2} = I^2 \cdot S_{t_2}$.

Between any pair of fractional production periods in an optimal schedule, there always exists one period with an inventory value at its lower or upper bound. Hence to obtain an optimal solution to the LS-CC-SUB, it is only necessary to consider feasible solutions that can be subdivided into subplans and choose the best combinations of subplans forming a feasible and optimal solution.

For any subplan (t_1, t_2, I^1, I^2), the production needed for any solution of that subplan to be feasible is:

$$d_{t_1,t_2} + I^2 \cdot S_{t_2} - I^1 \cdot S_{t_1-1} = K \cdot C + f$$

In other words, any feasible solution to that subplan contains a number $K \in \mathbb{N}$ of full production periods and a fractional production period of $f \in [0, C[$ value. If $f = 0$ then it is sufficient to select the K periods of full production needed for the solution to be feasible and optimal. Otherwise, one may enumerate every feasible period of fractional production and determine the optimal set of full production period once the fractional one is fixed. This can be done in $\mathcal{O}(T^2)$ by dynamically solving the associated linear problem (Wolsey, 2005).

Thus in order to solve the LS-CC-SUB one must consider every possible subplan (t_1, t_2, I^1, I^2), and for each subplan one must determine the optimal set of full production periods for each possible fractional production period in $[|t_1, t_2|]$. Therefore as there are $\mathcal{O}(T^2)$ subplan and each subplan resolution is of $\mathcal{O}(T^2)$ complexity, the LS-CC-SUB is of $\mathcal{O}(T^4)$ complexity.

Our novel $\mathcal{O}(T^3)$ algorithm for the LS-CC-SUB derives from the work of van Hoesel and Wagelmans (1993) for the CLSP. In that work, the authors dynamically solve all subplans starting at period $t_1 \in [|1, T|]$ by incrementally decreasing the ending period t_2 from T to t_1. Therefore only the solutions of subplans ending at period T are required. Since there are only $\mathcal{O}(T)$ of such subplans to be solved, their algorithm for the CLSP is of $\mathcal{O}(T^3)$ complexity.

In order to use this paradigm in our case, it is necessary to also account for the time-varying inventory bounds. In particular, the implementation of that dynamical scheme to solve a feasible subplan (t_1, t_2, I^1, I^2) requires that all subplans from t_1 to periods higher than t_2 are also feasible. That property was true in the unbounded inventory case but remains to be proved in ours.

3. Results on the feasibility of a subplan

Let (t_1, t_2, I^1, I^2) be a feasible subplan with K full production period and a fractional period of non-zero production f. Suppose the fractional period is fixed to $t_0 \in [|t_1, t_2|]$, we denote by $P(t_0)$ the problem to determine the corresponding K full production periods. In order to solve the subplan, one has to find for all fractional production periods the optimal solution to problem P (if it exists) and select the cheapest solution among them.

Let $P'(t_0)$ denote the problem to determine the corresponding K full production periods with the added possibility to place a full production on period t_0. We demonstrate in this section that problems $P(t_0)$ and $P'(t_0)$ are feasible for any fractional period t_0 belonging in bounds defined in the following.

Let I_t and J_t at any period t be defined as:

$$I_t = \sum_{t'=t_1}^{t} x_t + I^1 \cdot S_{t_1-1} - d_{t_1,t} \qquad \forall t \in [|t_1, t_2|]$$

$$J_t = \sum_{t'=t}^{t^2} x_t + S_{t-1} - I^2 \cdot S_{t_2} - d_{t,t_2} \qquad \forall t \in [|t_1, t_2|]$$

The following property applies:

$$I_t + J_{t+1} = S_t, \quad \forall t \in [|t_1, t_2|]$$

The inventory lower and upper bound constraints for that subplan are equivalent to:

$$I_t \geq 0 \qquad \forall t \in [|t_1, t_2|]$$
$$I_t \leq S_t \iff J_{t+1} \geq 0 \qquad \forall t \in [|t_1, t_2|]$$

A production schedule x solution of $P'(t_0)$ is feasible if and only if this set of constraints is respected and if the ending storage value is $I^2 \cdot S_{t_2}$ for an entry storage value $I^1 \cdot S_{t_1-1}$.

Variables I represent the inventory evolution throughout time according to production:

$$I_{t_1} = I^1 \cdot S_{t_1-1} + x_{t_1} - d_{t_1}$$
$$I_{t+1} = I_t + x_{t+1} - d_{t+1} \qquad \forall t \in [|t_1, t_2 - 1|]$$

Variables I must be non-negative, have an entry value $I^1 \cdot S_{t_1-1}$ and attain at period t_2 the value $I_{t_2} = I^2 \cdot S_{t_2}$.

One may notice the symmetry between I and J. Indeed, J represents a similar process in reverse time from period t_2 to period t_1. Let us denote by d' the demand defined aeach period t by $d'_t = d_t - S_{t-1} + S_t$. Then J can be described as follow:

$$J_{t_2} = (1 - I^2) \cdot S_{t_2} + x_{t_2} - d'_{t_2}$$
$$J_{t-1} = J_t + x_{t-1} - d'_{t-1} \qquad \forall t \in [|t_1 + 1, t_2|]$$

Variables J must be non-negative, have an entry value $(1 - I^2) \cdot S_{t_2}$ and attain at period t_1 the value $J_{t_1} = (1 - I^1) \cdot S_{t_1-1}$.

Variables J must respect properties identical to those of variables I while representing the same process in reverse time. Moreover as per inequalities (10)-(11), one has $d'_t \leq C$. In other words the demand associated to J is lower than the production capacity C, and all feasibility properties defined in section 3 also applies to the process described by J in reverse time.

We now use this symmetry between I and J to define bounds on the feasible fractional period in $[|t_1, t_2|]$ for problem P', by proving that there exists a production schedule such that both I and J are positive everywhere given their starting and ending values only if the fractional production period is within these bounds .

First, notice that there can be no fractional production periods as long as production at full capacity is mandatory to respect demand. Thus there are no fractional production period before a period t^I_{min} defined as:

$$t^I_{min} = \min\{t \in [|t_1, t_2|];\ d_{t_1,t} \leq (t - t_1) \cdot C + f + I^1 \cdot S_{t_1-1}\} \tag{13}$$

Secondly, there can be no fractional production period as long as demand does not exceed the fractional production value as it would result in unused leftover. Thus there are no fractional production period after a period t^I_{max} defined as:

$$t^I_{max} = \max\{t \in [|t_1, t_2|];\ I^2 \cdot S_{t_2} + d_{t,t_2} \geq f\} \tag{14}$$

Thanks to the symmetry property highlighted above between I and J in reverse time, similar bounds may be defined as necessary conditions for J to be positive, denoted t^J_{min} and t^J_{max}:

$$t^J_{min} = \min\{t \in [|t_1, t_2|];\ (1 - I^1) \cdot S_{t_1-1} + d'_{t_1,t} \geq f\}$$
$$t^J_{max} = \max\{t \in [|t_1, t_2|];\ d'_{t,t_2} \leq (t_2 - t) \cdot C + f + (1 - I^2) \cdot S_{t_2}\}$$

Here t^J_{max} represents the latest period before production must be at full capacity until period t_2: in other words it is the latest possible period for fractional production. The parameter t^J_{min} represents the earliest period for which the inventory remains lower than its upper bound despite fractional production.

Hence the bounds (t_{min}, t_{max}) on the value of the fractional production period as necessary conditions for P to be feasible are:

$$t_{min} = \max(t^I_{min},\ t^J_{min})$$
$$t_{max} = \min(t^I_{max},\ t^J_{max})$$

Similar bounds may be found for the problem P' to be feasible. In that case the lower bound defined by t^I_{min} does not represent a necessary condition anymore: since a single time period can be both a fractional and a full production period for the P' problem, the demand can always be satisfied at the fractional production period. By symmetry, the upper bound t^J_{max} is not necessary either. Hence P' is feasible only if the fractional production period is within bounds (t'_{min}, t'_{max}) for $t'_{min} = t^J_{min}$ and $t'_{max} = t^I_{max}$.

We now prove that problems P and P' are feasible if and only if the fractional production period belongs in the respective bounds (t_{min}, t_{max}) and (t'_{min}, t'_{max}).

Lemma 1. *Consider a subplan with a storage capacity higher than its constant production capacity at all time. If there are no fractional production and a number of full production period lower than or equal to the number of time period, then that subplan is feasible.*

Proof. We prove this lemma by induction on the number K of required full production period. First, suppose the subplan requires no full production period nor any fractional production, in other words no production at all, relying only on its starting inventory to satisfy demand and meet the ending inventory constraint. It is obviously feasible.

Suppose now that the lemma is proved for any number lower than K of full production period. Let us prove the lemma is also valid for any subplan (t_1, t_2, I^1, I^2) with a number K of full production period lower than the number of time period in the subplan. First, we derive a useful property from assumptions (10)-(11):

$$S_t \geq I^2 \cdot S_{t_2} + d_{t+1,t_2} - (t_2 - t) \cdot C, \quad \forall t[|t_1, t_2 - 1|] \tag{15}$$

This property guarantees that for any subplan there always exists a sufficient inventory capacity to build the target ending inventory value $I^2 \cdot S_{t_2}$.

Let t' denote the earliest period for which the inventory value in case of full production is below its upper bound:

$$t' = \min\{t \in [|t_1, t_2|];\ I^1 \cdot S_{t_1-1} - d_{t_1,t} + C \leq S_t\}$$

If $t' = t_1$, a full production period can be placed in t_1, and $K - 1$ full production period must be placed in subplan $(t_1 + 1, t_2, I^1, I^2)$ to build a feasible solution. By induction property, there exists a feasible solution for that subplan since it has one less required production period for one less time period. Thus a feasible solution can be retrieved for subplan (t_1, t_2, I^1, I^2).

Otherwise $t' > t_1$ and the inventory balance in case of full production at time period t lower than t' is higher than S_t: $I^1 \cdot S_{t_1-1} - d_{t_1,t} + C > S_t$. It means that in case of no production before t', the inventory value would remain positive, and since $I^1 \cdot S_{t_1-1} \leq S_{t_1-1}$ and as per (10)-(11), it would remain below its upper bound. By definition of t' one has $I^1 \cdot S_{t_1-1} - d_{t_1,t'-1} + C > S_{t'-1}$. We deduce:

$$I^1 \cdot S_{t_1-1} - d_{t_1,t'} + C > S_{t'-1} - d_{t'} \geq 0 \tag{16}$$

Hence inventory value remains within its bound between t_1 and t' in case of full production in t' and none before. The number of full production period to be placed after t' is $K' = K - 1$ such that:

$$K' = \frac{I^2 \cdot S_{t_2} + d_{t'+1,t_2} - (I^1 \cdot S_{t_1-1} - d_{t_1,t'} + C)}{C}$$

Let us prove that the remaining number of full production period K' is lower than or equal to the number of remaining time period $1 + t_2 - t'$. As per inequality(16), one has:

$$K' < \frac{I^2 \cdot S_{t_2} + d_{t'+1,t_2} - (S_{t'-1} - d_{t'})}{C}$$

Then as per inequality (15), one has:

$$K' < (t_2 - t') + \frac{S_{t'} + d_{t'} - S_{t'-1}}{C}$$

Finally as per inequality (11), one has: $K' < (t_2 - t') + C$.

Hence the number K' of production period to be placed between $t' + 1$ and t_2 is lower than or equal to the number of time period $t_2 - t'$. According to the induction property, there exists a feasible solution to a subplan spanning from $t' + 1$ to t_2 with entry and ending storage value $I^1 \cdot S_{t_1-1} + C - d_{t_1}$ and $I^2 \cdot S_{t_2}$ from which we can derive a feasible solution to subplan (t_1, t_2, I^1, I^2).

Therefore this proof by induction is completed. □

Theorem 1. *Consider a subplan with a storage capacity twice higher than its constant production capacity at all time. If that subplan has a number of required production period lower than the number of time period, then the corresponding problem P is feasible for any fractional production period within bounds (t_{min}, t_{max}).*

Proof. Let t_0 be the fractional production period of subplan (t_1, t_2, I^1, I^2) within bounds (t_{min}, t_{max}).

Consider now a modified subplan without period t_0 nor the demand satisfied by the corresponding fractional production. This subplan is obtained as follows from the original subplan (t_1, t_2, I^1, I^2). Let τ denotes the earliest period after t_0 for which the fractional production no longer fully meets the demand. Such a period exists since t_0 is lower than or equal to t_{max}.

Suppose that $\tau > t_0$. The demand d' of the modified subplan in periods $[t_0, \tau - 1|]$ would then be none and in period τ would correspond to the part of the demand not met by the fractional production, such that:

$$\begin{aligned} d'_t &= 0, && \forall t \in [|t_0, \tau - 1|] \\ d'_\tau &= d_\tau - (f - d_{t_0,\tau-1}) \end{aligned}$$

While the demand d' is equal to the original demand d for all other periods. The storage capacity S' of the modified subplan is modified accordingly, since the fractional production in t_0 is stored to meet demand for periods in $[|t_0 + 1, \tau|]$. Hence it is defined as:

$$S'_t = S_t - (f - d_{t_0,t}), \qquad \forall t \in [|t_0, \tau - 1|]$$

While the storage capacity S' is equal to the original storage capacity S for all other periods. Note that in that case, the storage capacity is higher than the production capacity at all time.

Suppose that $\tau = t_0$, then the demand $d't_0$ is non-zero and of value $d_{t_0} - f$. Hence it must be met by storage constructed from full production in periods preceding t_0. Hence, it is possible to displace that demand at the closest previous periods and modify the storage capacity accordingly. Let ζ denotes the closest period before t_0 before which full production is not needed to meet demand in t_0. Such a period exists since t_0 is higher than or equal to t_{min}.

The demand d' of the modified subplan in periods $[\zeta + 1, t_0 - 1|]$ would then correspond to the maximum production capacity, would be none in period t_0 and correspond in period ζ to the part of the demand not

met by the fractional production, such that:

$$\begin{aligned} &d'_{t_0} = 0 \\ &d'_t = C, && \forall t \in [|\zeta + 1, t_0 - 1|] \\ &d'_\zeta = d_\zeta + ((t_0 - \zeta) \cdot C - d_{\zeta,t0}) \end{aligned}$$

While the demand d' is equal to the original demand d for all other periods. The storage capacity S' of the modified subplan is modified accordingly, since the full production in $[|\zeta, t_0 - 1|]$ is stored to meet demand in t_0. Hence it is defined as:

$$S'_t = S_t - ((t + 1 - \zeta) \cdot C - d_{\zeta,t}), \qquad \forall t \in [\zeta, t_0 - 1|]$$

While the storage capacity S' is equal to the original storage capacity S for all other periods. Note that in that case, the storage capacity is higher than the production capacity at all time.

Thus it is possible to obtain an equivalent subplan to (t_1, t_2, I^1, I^2) with fractional production in t_0, such that there is no demand nor production in t_0. In other words, that subplan has no fractional production and a number of full production period lower than or equal to the number of time period, with a storage capacity higher than the production capacity at all time. Hence, the modified subplan is feasible and a feasible solution to subplan (t_1, t_2, I^1, I^2) with fractional production in t_0 may be retrieved by integrating back the fractional production period and the corresponding demand it satisfies. □

A similar reasoning leads the following property:

Corollary 1. *Consider a subplan with a storage capacity twice higher than its constant production capacity at all time. If that subplan has a number of required production period lower than the number of time period, then the corresponding problem P' is feasible for any fractional production period within bounds (t'_{min}, t'_{max}).*

We have proved that problem P and P' are feasible for the fractional production period belonging within bounds (t_{min}, t_{max}) and (t'_{min}, t'_{max}) respectively. This allows for the design of a dynamical programming scheme to solve the subplan, by iteratively solving P' at each fractional production period and deducing the corresponding solution to P at each step.

4. Dynamically retrieving feasible and optimal solutions

let (t_1, t_2, I^1, I^2) be a valid subplan with K full production period and a fractional period of non-zero production f. We define properties to efficiently implement a dynamical programming scheme to solve problem P' and P for all fractional production period within bounds (t_{min}, t_{max}).

Let the fractional production period t_0 be fixed in $[|t_1, t_2|]$, we introduce two functions $A(t)$, $B(t)$ for $t \in [|t_1, t_2|]$ that represents respectively the minimum and maximum number of full production periods in $[|t_1, t|]$ for any feasible solution of $P(t_0)$:

$$A(t) = \begin{cases} \left\lceil \frac{d_{t_1,t} - I^1 \cdot S_{t_1 - 1}}{C} \right\rceil & \forall t < t_0 \\ \left\lceil \frac{d_{t_1,t} - f - I^1 \cdot S_{t_1 - 1}}{C} \right\rceil & \forall t \geq t_0 \end{cases} \qquad B(t) = \begin{cases} \left\lfloor \frac{S_t + d_{t_1,t} - I^1 \cdot S_{t_1 - 1}}{C} \right\rfloor & \forall t < t_0 \\ \left\lfloor \frac{S_t + d_{t_1,t} - f - I^1 \cdot S_{t_1 - 1}}{C} \right\rfloor & \forall t \geq t_0 \end{cases}$$

Then, the constraints on the lower and upper bounds of the inventory value for a fractional production period in t_0 can be expressed as:

$$\begin{aligned} &\sum_{t'=t_1}^{t'=t} z_{t'} \geq A(t) && \forall t \in [|t_1, t_2|] \\ &\sum_{t'=t_1}^{t'=t} z_{t'} \leq B(t) && \forall t \in [|t_1, t_2|] \end{aligned}$$

A solution of $P(t_0)$ is feasible if and only if for any $t \in [|t_1, t_2|]$ the number of full production periods in $[|t_1, t|]$ is at least $A(t)$ and at most $B(t)$. Functions A and B are integral, non-decreasing except on period t_0 as per inequality (12).

For all $k \in [|1, K|]$, let $\underline{w}_k$, $\bar{w}_k \in [|t_1, t_2|]$ respectively be the lower and upper bound for the k-th period of production. They are defined as follow:

$$\begin{aligned} \underline{w}_k &= \max\{t_1\} \cup \{t \in [|t_1 + 1, t_2|];\ B(t-1) \leq k-1\} && \forall k \in [|1, K|] \\ \bar{w}_k &= \min\{t \in [|t_1, t_2|];\ A(t) = k\} && \forall k \in [|1, K|] \end{aligned}$$

We introduce for all $k \in [|1, K|]$ the parameter $t^k \in [|1, T|]$ representing the chosen k-th period of full production. Then a solution of $P(t_0)$ is feasible if and only if:

$$\begin{aligned} & t^k \in [|\underline{w}_k, \bar{w}_k|] && \forall k \in [|1, K|] \\ & t^k > t^{k-1} && \forall k \in [|2, K|] \\ & t^k \neq t_0 && \forall k \in [|1, K|] \end{aligned}$$

Interval $[|\underline{w}_k, \bar{w}_k|]$ represents the window of available production for the k-th period of full production.

One may notice that the value of lower and upper bounds $\underline{w}_k$ and $\bar{w}_k$ for all full production periods $k \leq K$ increase when displacing the fractional production period from $t_0 + 1$ to t_0. Indeed, an early fractional production period means less restriction to meet the demand but an inventory value closer to its maximum capacity, hence the latest and earliest time to place the k-th full production period increase.

First, we define conditions to retrieve an optimal solution to problem $P(t_0)$ from the optimal solution to problem $P'(t_0)$, for any fractional period $t_0 \in [|t_{min}, t_{max}|]$.

Theorem 2. *Let (t_1, t_2, I^1, I^2) be a feasible subplan and $t_0 \in [|t_{min}, t_{max}|]$ a candidate fractional production period. Problem $P(t_0)$ admits an optimal solution within one change in the full production periods of the optimal solution to problem $P'(t_0)$ if any is feasible, and is unfeasible otherwise.*

Proof. We prove this theorem by induction on the number K of required full production periods.

Suppose there are no required full production periods, in other words $K = 0$ and no full production period may be moved as there is none. Then either $P(t_0)$ is unfeasible or the only possible solution to problem $P(t_0)$ is identical to the solution of $P'(t_0)$, *i.e.* the only production period is fractional at period t_0.

Suppose now that the theorem is verified for any feasible subplan with at most $K-1$ full production periods. If the optimal solution to $P'(t_0)$ has no full production at period t_0, then it is obviously also an optimal solution to $P(t_0)$. Otherwise, we suppose that the k_0-th full production period is placed in t_0, and must be moved to another period available for production in order to obtain a feasible solution to $P(t_0)$.

Let us denote by k^+ the smallest index $k > k_0$ of full production for which the associated period t^k is strictly higher than the upper bound $\bar{w}_{k-1}$ of the previous full production. If such index does not exist, it is given the value $K+1$ with $t^{K+1} = t_2$. Similarly, let us denote by k^- the greatest index $k < k_0$ of full production for which the associated period t^k is strictly lower than the lower bound $\underline{w}_{k+1}$ of the next full production. If such index does not exist, it is given the value 0 with $t^0 = t_1$. Index k^+ and k^- may be defined as:

$$\begin{aligned} k^+ &= \min\left\{\{k \in [|k_0, K-1|];\ t^{k+1} > \bar{w}_k\} \cup \{K+1\}\right\} \\ k^- &= \max\left\{\{k \in [|2, k_0|];\ t^{k-1}k < \underline{w}_k\} \cup \{0\}\right\} \end{aligned}$$

Suppose that there are no available period of production in interval $[|\underline{w}_{k^-}, \bar{w}_{k^+}|]$. It is not possible to place a full production period from t_0 to a period beyond $\bar{w}_{k^+}$ within one move as there would be less than $k^+ - 1$ production period in $[|t_1, \bar{w}_{k^+}|]$. Similarly, it is not possible to place a full production period from t_0 to a period under $\underline{w}_{k^-}$ within one move. Hence to obtain a feasible solution, one would have first to move inside $[|\underline{w}_{k^-}, \bar{w}_{k^+}|]$ a $k^+ + 1$-th or higher (resp. $k^- - 1$-th or lower) indexed production period to extend the range of possible available period replacing the k_0-th one. Since it is impossible because there are no available period in that interval, we deduce that problem $P(t_0)$ is feasible only if the full production at period t_0 can be moved to an available period within the interval $[|\underline{w}_{k^-}, \bar{w}_{k^+}|]$.

If $k^+ < K+1$ or $k^- > 0$ then the choice of full production periods in intervals $[|t_1, \underline{w}_{k^-}|[$ and $]|\bar{w}_{k^+}|, t_2|]$ is already optimal, and problem $P(t_0)$ is solved by placing strictly less than K production periods in interval $[|\underline{w}_{k^-}, \bar{w}_{k^+}|]$. Hence the theorem is verified by induction.

Otherwise, any solution obtained by moving the full production period from t_0 to any available time period inside $[|\underline{w}_1, \bar{w}_K|]$ is feasible, and any solution obtained by moving the full production period from t_0 to an available time period outside $[|\underline{w}_1, \bar{w}_K|]$ is unfeasible.

Hence if we cannot move the full production from t_0 to any other period in interval $[|\underline{w}_1, \bar{w}_K|]$, problem $P(t_0)$ is unfeasible, otherwise an optimal solution to problem $P(t_0)$ is retrieved by moving the full production from t_0 to the cheapest available period in that interval. Hence the theorem is also verified in that case.

□

We now define conditions to retrieve an optimal solution to problem $P'(t_0)$ from the optimal solution to problem $P'(t_0+1)$, for any fractional production period $t_0 \in [|t'_{min}, t'_{max}-1|]$.

Theorem 3. *Let (t_1, t_2, I^1, I^2) be a feasible subplan and $t_0 \in [|t'_{min}, t'_{max}-1|]$ a candidate fractional production period. An optimal solution to $P'(t_0)$ can be deduced from an optimal solution to $P'(t_0+1)$ by moving at most one full production period.*

Proof. First, remember that the value of lower and upper bounds $\underline{w}_k$ and $\bar{w}_k$ for all full production periods $k \leq K$ increase when displacing the fractional production period from t_0+1 to t_0.

Suppose that the upper bound $\bar{w}_{k_1}$ of the k_1-th full production period increases when displacing the fractional production period from t_0+1 to t_0. The values of A remains constant everywhere except on period t_0, hence $\bar{w}_{k_1}$ may only increase if one had $A(t_0) = k_1$ in $P'(t_0+1)$ and $A(t_0) < k_1$ in $P'(t_0)$. Since there are no other changes in value of A, only the upper bound for period k_1 increases from a value in $[|t_1, t_0|]$ to a value in $[|t_0+1, t_2|]$.

Similarly, suppose that the lower bound $\underline{w}_{k_2}$ of the k_2-th full production period increases when displacing the fractional production period from t_0+1 to t_0. The values of B remains constant everywhere except on period t_0, and $B(t_0)$ may only decrease by 1 when displacing the fractional production period from t_0+1 to t_0. Hence $\underline{w}_{k_2}$ may only increase if one had $B(t_0) = k_2$ in $P'(t_0+1)$ and $B(t_0) = k_2 - 1$ in $P'(t_0)$. Since there are no other changes in value of B, only the lower bound for period k_2 increases from a value in $[|t_1, t_0|]$ to the value t_0+1.

Furthermore, note that if both such full production periods k_1 and k_2 exist and are of different value, then we always have $k_1 < k_2$. Otherwise one would have $\underline{w}_{k_1} \geq \underline{w}_{k_2}$, hence both lower bounds would increase, which is impossible as explained above.

If there are no full production periods with any bounds increase, then all production windows in problem $P'(t_0)$ are identical to those of problem $P'(t_0+1)$, and the optimal solution is the same for both problems. Similarly, suppose there exists a full production period k_1 (resp. k_2) with an upper (resp. lower) bound increase, and no full production period with a lower (resp. upper) bound increase. An optimal solution of problem $P'(t_0)$ may be deduced from the one of problem $P'(t_0+1)$, by identifying the optimal time period of the full production period k_1 (resp. k_2) in its new production window. therefore only one full production period is moved.

Finally, suppose there exists both a full production period k_1 with an upper bound increase, and a full production period k_2 with a lower bound increase. As mentioned before, the upper bound for period k_1 and lower bound for period k_2 increase from a value in $[|t_1, t_0|]$ to respectively a value in $[|t_0+1, t_2|]$ and to the value t_0+1. Hence, in order to remain feasible, a full production period must be moved from $[|t_1, t_0|]$ to $[|t_0+1, t_2|]$. It is then feasible and optimal to move the k_1-th full production period to the cheapest available period in $[|t_0+1, \bar{w}_{k_2}|]$. □

We are now able to implement an efficient dynamic resolution of a feasible subplan. Suppose the latest fractional period t_{max} is known, as well as an optimal solution to the problem $P'(t_{max})$. Then it is possible to iteratively solve problems P' and P for all fractional production period within bounds (t_{min}, t_{max}), by moving at most two full production periods at each step.

5. Dynamic resolution of a subplan for the LS-CC-SUB

Let us consider a feasible subplan (t_1, t_2, I^1, I^2). Suppose the upper bound t_{max} of the fractional production period is known, as well as an optimal solution to the problem $P'(t_{max})$. By extending the work of van Hoesel and Wagelmans (1993) to the inventory bounded problem, we propose an $\mathcal{O}(T)$ algorithm to find the optimal solution to all problems $P'(t_0)$ and $P(t_0)$ (if any) for a fractional period t_0 in $[|t_{min}, t'_{max}|]$.

Suppose that for a fractional production period $t_0 \in [|t_{min}, t'_{max} - 1|]$, an optimal solution to $P'(t_0 + 1)$ is known. In order to solve problem $P'(t_0)$, the fractional production period is first moved from $t_0 + 1$ to t_0, while all full production periods remain the same. Consequently, I_{t_0} increases and J_{t_0+1} decreases by f units. Then as per 3, an optimal solution to problem $P'(t_0)$ can be retrieved from the current solution with at most two moves. Afterward and as per theorem 2, an optimal solution to problem $P(t_0)$ can be retrieved from the current solution with at most one move, if any.

5.1. Dynamic resolution of the P' problem

As explained in section 4, the lower and upper bounds for every ordered full production periods are non-decreasing when moving the fractional production period from $t_0 + 1$ to t_0. Hence there are new available production periods at later time in problem $P'(t_0)$ compared to problem $P'(t_0 + 1)$, while some earlier production period become unfeasible. Therefore, a feasible and optimal solution of $P'(t_0)$ can be deduced from the one of problem $P'(t_0 + 1)$ with at most two moves forward in time.

For all fractional production period t_0 and a solution x to problem $P'(t_0)$ with a corresponding inventory value I, let u^{t_0} and s^{t_0} be the parameters defined as follow:

- Period u^{t_0} is the earliest period in $[|t_0, t_2|]$ such that the inventory value is below the production capacity:

$$u^{t_0} = \min\{t \in [|t_0, t_2|];\ I_t < C\} \cup \{t_2\}$$

- Period s^{t_0} is the earliest period such that all following inventory values in $[|t_1, t_0|]$ are above the production capacity:

$$s^{t_0} = \max\{t \in [|t_1, t_0|];\ I_{t-1} < C\}$$

Similarly, let δ^{t_0} and γ^{t_0} be the parameters defined as follow:

- Period δ^{t_0} is the earliest and cheapest period in $[|t_0, u|]$ of available full production:

$$\delta^{t_0} = \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|t_0 + 1, u|]\}$$

- Period $\gamma_r^{t_0}$ is the most expensive production period in $[|s, r|]$, for all $r \in [|s, t_0|]$:

$$\gamma_r^{t_0} = \arg\max\{cf(t');\ x_{t'} = 1,\ t' \in [|s, r|]\}, \quad \forall r \in [|s, t_0|]$$

Suppose that an optimal solution to problem $P'(t_0 + 1)$ is known, as well as parameters $(u^{t_0}, s^{t_0}, \delta^{t_0}, \gamma^{t_0})$ for the corresponding optimal solution to $P'(t_0 + 1)$. A feasible and optimal solution to problem $P'(t_0)$ may be retrieved from that solution to $P'(t_0 + 1)$ with at most one move of full production. Note that moving production forward in time decreases (resp. increases) the inventory value I (resp. J) on all intermediate periods by C. Hence feasibility conditions impose that any move of production forward in time must be from $[|s^{t_0}, t_0]$ to $[|t_0 + 1, u^{t_0}|]$.

An optimal solution may be retrieved by moving a production period from the most expensive one in $[|s^{t_0}, t_0]$, indicated by $\gamma_{t_0}^{t_0}$, to the cheapest available in $[|t_0 + 1, u^{t_0}|]$ indicated by δ^{t_0}. If $J_{t_0+1} \geq 0$ then the current solution for $P'(t_0)$ is already feasible, and this move is done only if it results in a cheaper production plan, *i.e.* if $cf(\gamma_{t_0}^{t_0}) > cf(\delta^{t_0})$. In the contrary if $J_{t_0+1} < 0$, the current solution for $P'(t_0)$ is not feasible and that move must be done regardless to obtain a feasible and optimal solution.

An optimal solution to problem P' may thus be dynamically found for all fractional production periods bounded by (t'_{min}, t'_{max}) from the knowledge of an optimal solution to problem $P'(t_{max})$. Finding an optimal solution to all problem P' takes at most $t'max - t'min$ full production moves in total, in other words the number of required moves increase linearly with the number of periods in the subplan. However each full

production move is done with the full knowledge of parameters (u, s, δ, γ) at the corresponding fractional period and for the current solution. Those parameters also have to be dynamically updated in constant time at each step of the dynamical process in order for the algorithm to be of linear complexity.

First, we demonstrate how to efficiently deduce parameters (u, s, δ, γ) at fractional period t_0 from those at fractional period $t_0 + 1$ for a known optimal solution to problem $P'(t_0 + 1)$:

- If $I_{t_0} < C$, then u^{t_0}. Otherwise it takes the value u^{t_0+1}.
- If $I_{t_0} < C$, δ^{t_0} take the value 0. Otherwise if $x_{t_0+1} = 0$, and if $\delta^{t_0+1} = 0$ or $cf(t_0 + 1) < cf(\delta^{t_0+1})$, it takes the value $t_0 + 1$. Finally in any other case, it is equal to δ^{t_0+1}.
- If $t_0 \geq s^{t_0+1}$, then parameter s^{t_0} and γ^{t_0} are equal to s^{t_0+1} and γ^{t_0+1}. Otherwise, s^{t_0} and γ^{t_0} are recomputed.

Note that the update of u and δ is done in constant time, while the overall successive computation of s and γ across the iterations is done in linear time (van Hoesel and Wagelmans, 1993). Therefore all operations to obtain parameters (u, s, δ, γ) at fractional period t_0 from those at fractional period $t_0 + 1$ for a known optimal solution to problem $P'(t_0 + 1)$ is done in linear time.

The following algorithm represent all the related update operations.

Algorithm 1 Update of (x, I, J) and (u, s, δ, γ)

Input: Fractional period t_0, Optimal solution (x, I, J) to $P'(t_0 + 1)$, Parameters (u, s, δ, γ)

$x_{t_0} \leftarrow x_{t_0} + f$
$x_{t_0+1} \leftarrow x_{t_0+1} - f$
$I_{t_0} \leftarrow I_{t_0} + f$
$J_{t_0+1} \leftarrow J_{t_0+1} - f$
Update δ
if $I_{t_0} < C$ **then**
    $u \leftarrow t_0$
    $\delta \leftarrow 0$
else if $x_{t_0+1} = 0$ and $(\delta = 0$ or $cf(t_0 + 1) < cf(\delta))$ **then**
    $\delta \leftarrow t_0 + 1$
end if
if $t_0 = s - 1$ **then**
    $s \leftarrow \max\{t' \in [|t_1, t|];\ I_{t'-1} < C\}$
    $\gamma_r \leftarrow \arg\max\{cf(t');\ x_{t'} = 1,\ t' \in [|s, r|]\}, \quad \forall r \in [|s, t_0|]$
end if
return $(x, I, J, u, s, \delta, \gamma)$

We now prove that updating parameters $(u^{t_0}, ^{t_0} s, \delta^{t_0}, \gamma^{t_0})$ when a full production period is moved can be done in constant time. Such a result is proven to be true for the case without inventory bounds by van Hoesel and Wagelmans (1993). We extend it to the case with inventory bounds.

Lemma 2. *Suppose that an optimal solution to problem $P'(t_0+1)$ is known, as well as parameters $(u^{t_0}, s^{t_0}, \delta^{t_0}, \gamma^{t_0})$ for the corresponding optimal solution to $P'(t_0 + 1)$. If production is moved, then u^{t_0} and δ^{t_0} takes the value t_0.*

Proof. Remember that a full production period may only be moved from period $\gamma_{t_0}^{t_0}$ to period δ^{t_0}.

If that move is profitable, then u is updated to t_0 because we must have $I_{t_0} < C + f$ in the initial solution to $P'(t_0)$, otherwise that move would have been feasible and profitable in $P'(t_0+1)$ (van Hoesel and Wagelmans, 1993).

If that move is not profitable and is made because the inherited solution is unfeasible, it means all production periods in $[|s, t_0|]$ are cheaper than any available periods in $[|t_0 + 1, u|]$. Moving forward a production period from γ_{t_0} to δ ensures that the solution remains feasible for any fractional production period in $[|\gamma_{t_0}, t_0|]$ since it increases J by C on interval $[|\gamma_{t_0} + 1, t_0|]$. Therefore, once such a non-profitable feasibility move is made, only

optimality moves are considered for the fractional period t'_0 in that interval $[|\gamma_{t_0}, t_0 - 1|]$, and from cheaper production period since γ_{t_0} was the most expensive.

Since all available period in $[|t_0, u|]$ are also more expensive than γ_{t_0} and period u decreases when the fractional production decreases, any profitable move for $t'_0 \in [|\gamma_{t_0}, t_0 - 1|]$ must be targeting an available period lower than t_0. Finally since there are no feasibility move for fractional production in $[|\gamma_{t_0}, t_0 - 1|]$, any move for $t'_0 \in [|\gamma_{t_0}, t_0 - 1|]$ must be targeting an available period lower than t_0.

Since γ_{t_0} is an available period after the feasible move from γ_{t_0} to δ, the cheapest available period in $[|\gamma_{t_0-1}, u|]$ is lower than t_0. Hence any feasibility or optimality move to find an optimal solution to problem P' for the fractional period in interval $[|t_{min}, \gamma_{t_0-1}|]$ would target that cheapest available period lower than $[t_0$. If it is an optimality move, u is updated to the current fractional period. Otherwise if it is a non-profitable feasibility move, the same reasoning can be applied once again until an optimality move is made or the last iteration is attained.

Therefore by induction, u^{t_0} always takes the value t_0 if a full production period is moved to retrieve an optimal solution to problem $P'(t_0)$. Hence δ^{t_0} also takes the value t_0 in that case. □

We have proved parameters u and δ are updated in constant time after a production move. We now propose a property that leads to a similar observation for parameters s and γ. Such a result is proven to be true for the case without inventory bounds by van Hoesel and Wagelmans (1993). We extend it to the case with inventory bounds.

Lemma 3. *Suppose that an optimal solution to problem $P'(t_0+1)$ is known, as well as parameters $(u^{t_0}, s^{t_0}, \delta^{t_0}, \gamma^{t_0})$ for the corresponding optimal solution to $P'(t_0+1)$. If production is moved to solve $P'(t_0)$, then the resulting optimal solution is also optimal for all problems $P'(t)$ such that $t \in [|\gamma^{t_0}_{t_0}, t_0|]$.*

Proof. If production has been moved for optimality purpose, there are no further optimality move for any P' problem until fractional production in $s - 1$ is reached (van Hoesel and Wagelmans, 1993). Moreover there are no feasibility move until fractional production reaches $\gamma_{t_0} - 1$ since all J values on interval $[|\gamma_{t_0} + 1, t_0|]$ are increased by C. Hence the lemma is verified in case of an optimality move of full production.

Otherwise, if production has been moved for non-profitable feasibility, there are no further feasibility move until fractional production in $\gamma_{t_0} - 1$ is reached since all J values on interval $[|\gamma_{t_0} + 1, t_0|]$ are increased by C. We prove in what follows that there are no optimality move either until fractional production in $\gamma_{t_0} - 1$ is reached.

The non-profitable feasibility move has to be from the period $\gamma^{t_0}_{t_0}$ to the more expensive period δ^{t_0}. Suppose now that after that move, there exists a fractional period $t'_0 \in [|\gamma^{t_0}_{t_0}, t_0 - 1|]$ such that $\gamma^{t'_0}_{t'_0}$ is more expensive than $\delta^{t'_0}$. Obviously one has $\delta^{t'_0} \leq t_0$, otherwise there would have existed a profitable feasibility move instead of the non-profitable one.

Note that the inventory value of the optimal solution to problem $P'(t_0 + 1)$ between period $\gamma^{t_0}_{t_0}$ and t_0 was higher than C, since the non-profitable feasibility move that followed did not cause unfeasibility. Therefore there was the possibility of a profitable full production move from $\gamma^{t'_0}_{t'_0}$ to $\delta^{t'_0}$. Such a move did not happen, despite considering the solution to $P'(t_0 + 1)$ as optimal, which is absurd.

Therefore such fractional period t'_0 does not exist. Hence in case of a non-profitable feasibility move at fractional period t_0, there cannot be an optimal move until fractional period reaches value $\gamma^{t_0}_{t_0} - 1$.

We conclude than when production has been moved, there are no further move until fractional production in $\gamma_{t_0} - 1$ is reached. □

We have proved that after a production move from period $\gamma^{t_0}_{t_0}$ to period δ^{t_0}, there are no further production move for the fractional production period in $[|\gamma^{t_0}_{t_0}, t_0|]$. Hence there is no need to update the parameters s and γ on those intervals.

Moreover note that, after that move, the inventory values in interval $[|s^{t_0}, \gamma^{t_0}_{t_0} - 1|]$ do not change. In other words, we have for all period $t \in [|s^{t_0}, \gamma^{t_0}_{t_0} - 1|]$, the values s^t and γ^t that remain identical to s^{t_0} and γ^{t_0}, at the exception of $\gamma^t_{t_0}$ which takes the value $\gamma^t_{t_0-1}$ since there are no full production at period $\gamma^{t_0}_{t_0}$ anymore.

Hence, we have proved parameters s and γ are updated in constant time after a production move.

By leveraging all previous observations, we propose a linear-time algorithm to solve all problem P' from the initial knowledge of the optimal solution to problem $P'(t'_{max})$.

Algorithm 2 Dynamic resolution of the P' problems

Input: Subplan (t_1, t_2, I^1, I^2), bounds (t'_{min}, t'_{max}), optimal solution (x, I, J) to $P'(t'_{max})$

$t_0 \leftarrow t'_{max}$
$u \leftarrow t_0$
$\delta \leftarrow \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|t_0 + 1, u|]\}$
$s \leftarrow \max\{t' \in [|t_1, t_0|];\ I_{t'-1} < C\}$
$\gamma_r \leftarrow \arg\max\{cf(t');\ x_{t'} = 1,\ t' \in [|s, r|]\}, \quad \forall r \in [|s, t_0|]$
OptMove $\leftarrow$ "Yes"
while $t_0 > t'_{min}$ **do**
  $t_0 \leftarrow t_0 - 1$
  Update $(x, I, J, u, s, \delta, \gamma)$ with Algorithm 1
  if $t_0 < s$ **then**
    OptMove $\leftarrow$ "Yes"
  end if
  if (*OptMove* = "Yes" and $cf(\gamma_{t_0}) > cf(\delta)$) or $J_{t_0+1} < 0$ **then**
    $x_{\gamma_{t_0}} \leftarrow x_{\gamma_{t_0}} - C$
    $x_\delta \leftarrow x_\delta + C$
    for $t' \in [|\gamma_{t_0}, t_0|]$ **do**
      $I_{t'} \leftarrow I_{t'} - C$
      $J_{t'+1} \leftarrow J_{t'+1} + C$
    end for
    $u \leftarrow t_0$
    if $cf(\gamma_{t_0}) \leq cf(\delta)$ **then**
      $\gamma_{\gamma_{t_0}} \leftarrow \gamma_{\gamma_{t_0}-1}$
    else
      OptMove $\leftarrow$ "No"
    end if
    $\delta \leftarrow t_0 \cdot \mathbf{1}_{x_{t_0}=0}$
  end if
  Current plan of production x is an optimal solution to problem $P'(t_0)$
end while
return Solutions

This algorithm is almost identical to the one proposed by van Hoesel and Wagelmans (1993), except for the management of the feasibility production move. Beside the update of I and J after each production move, it only requires additional operations of constant-time complexity in the while-loop. As for the update of I and J, it corresponds to an enumeration of all period from γ_{t_0} to t_0. As per Lemma 3, there is not other production move until the fractional production period is of value γ_{t_0}. Hence throughout the iterations, the update of I and J enumerates at most once all periods in $[|t_1, t_2|]$, and Algorithm 2 is of linear complexity.

5.2. Retrieving the optimal solution of the P problem

Let t_0 be the current fractional production period, and q^{t_0} and η^{t_0} be the parameters defined as follow:

- Period q^{t_0} is the latest period in $[|t_1, t_0|]$ such that the remaining storage capacity is below the production capacity:
$$q^{t_0} = \max\{t_1\} \cup \{t;\ t \in [|t_1, t_0|],\ J_{t_0} < C\}$$
- Period $\eta^{t_0}_{q,v}$ is the earliest and cheapest period in $[|q, v|]$ of available full production, for all $v \in [|q, t_0|]$:
$$\eta^{t_0}_{q,v} = \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|q, v|]\}, \quad \forall v \in [|q, t_0|]$$

In order to find an optimal solution to $P(t_0)$ from the optimal solution to $P'(t_0)$, at most one full production move must be done as per theorem 2. If there is no full production period at period t_0 then the optimal solution to $P'(t_0)$ is an optimal solution to $P(t_0)$. Otherwise, that production period must be moved to the cheapest feasible available period. If such a period does not exist, then problem $P(t_0)$ is unfeasible. If it exists, then it is indicated by parameter $\zeta = \arg\min\{cf(\eta_{q,t_0-1}),\, cf(\delta)\}$ for the current solution to problem $P'(t_0)$.

If parameter q is known for problem $P'(t_0+1)$, then it can be determined for problem $P'(t_0)$ as follow. Since full production periods are only moved forward in time, the value of J_t for $t < t_0$ does not decrease. Hence q needs only to be updated if $q = t_0 + 1$, by finding the latest period $t \in [|t_1, t_0|]$ such that $J_t < C$. Therefore, re-evaluating all values of η and updating q is only necessary when the current value of q drops below the fractional period t_0.

However, the value of q may change when a full production period is moved forward in time, as it may be the case when determining the optimal solution to problem $P'(t_0)$ from the optimal solution to problem $P'(t_0+1)$ by moving a production period from γ_{t_0} to δ. Hence, the values of q and η_q must also be updated in that case, and again in linear complexity. First, if $\gamma_{t_0} > q$ then period γ_{t_0} becomes a candidate cheapest available period after the full production is moved. Therefore η_q must be updated accordingly with the following algorithm.

Algorithm 3 Update of η_q after a production move

Input: Fractional period t_0, Parameters $(\gamma_{t_0}, q, \eta_q)$

```
if q ≤ γ_{t_0} then
    for t ∈ [|γ_{t_0}, t_0|] do
        if η_{q,t_0} = 0 or cf(η_{q,t_0}) > cf(γ_{t_0}) then
            η_{q,t_0} ← γ_{t_0}
        end if
    end for
end if
return η_q
```

Then, one must also consider that when a full production period is moved forward in time from γ_{t_0} to δ, then the value of parameters J_t for $t \in [|\gamma_{t_0}+1, \delta|]$ increases by C. Hence the value of q may decrease. In order to preserve the linear complexity, we introduce a parameter q' initialized to $q' = q$. That parameter q' and $\eta_{q'}$ are updated as follows after a full production period is moved:

Algorithm 4 Update of q' $\eta_{q'}$ after a production move

Input: Fractional period t_0, Parameters $(\gamma_{t_0}, q, q', \eta_{q'})$

```
q' ← min(q, γ_{t_0})
η_{q',v} ← arg min{cf(t'); x_{t'} = 0, t' ∈ [|q', v|]},    ∀v ∈ [|q', q|]
return q', η_{q'}
```

Notice that the optimal period ζ to which a full production period is moved in order to retrieve the optimal solution to $P(t_0)$ may be determined such that $\zeta \leftarrow \arg\min\{cf(\eta_{q',q}),\, cf(\eta_{q,t_0-1}),\, cf(\delta)\}$. Finally, we demonstrate how to efficiently update parameters q and η at fractional period t_0 from those at fractional period $t_0 + 1$ for a known optimal solution to problem $P'(t_0 + 1)$. The following algorithm represent all the related update operations.

Algorithm 5 Update of q, q' and η

Input: Fractional period t_0, Current solution (x, I, J), Parameters (u, q, η)

if $q = t + 1$ **then**
  $q \leftarrow \max\{t_1\} \cup \{t';\ t' \in [|t_1, t_0|],\ J_{t'} < C\}$
  $\eta_{q,v} \leftarrow \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|q, v|]\}, \quad \forall v \in [|q, t_0|]$
  $q' \leftarrow q$
end if
return (q, q', η)

Using all previously constructed algorithm and building upon Algorithm 2, we propose the following algorithm solving the subplan. It enumerates all feasible fractional production period, solving at each iteration the problem P' and deducing the corresponding to problem P. Note it requires as input the knowledge of an initial optimal solution to problem $P'(t'_{max})$ which is discussed in section 6.

Algorithm 6 Dynamic resolution of the subplan problem

Input: Subplan (t_1, t_2, I^1, I^2), bounds (t_{min}, t_{max}), optimal solution (x, I, J) to $P'(t'_{max})$

$t_0 \leftarrow t_{max}$
$u \leftarrow t_0$
$\delta \leftarrow \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|t_0 + 1, u|]\}$
$q \leftarrow \max\{t_1\} \cup \{t';\ t' \in [|t_1, t_0|],\ J_{t'} < C\}$
$\eta_v \leftarrow \arg\min\{cf(t');\ x_{t'} = 0,\ t' \in [|q, v|]\}, \quad \forall v \in [|q, t_0|]$
$s \leftarrow \max\{t' \in [|t_1, t_0|];\ I_{t'-1} < C\}$
$\gamma_r \leftarrow \arg\max\{cf(t');\ x_{t'} = 1,\ t' \in [|s, r|]\}, \quad \forall r \in [|s, t_0|]$
$q' \leftarrow q$
$OptMove \leftarrow$ "Yes"
while $t_0 > t_{min}$ **do**
  $t_0 \leftarrow t_0 - 1$
  Update $(x, I, J, u, s, \delta, \gamma)$ with Algorithm 1
  Update (q, q', η) with Algorithm 5
  if $t_0 < s$ **then**
    $OptMove \leftarrow$ "Yes"
  end if
  if ($OptMove =$ "Yes" and $cf(\gamma_{t_0}) > cf(\delta)$) or $J_{t_0+1} < 0$ **then**
    $x_{\gamma_{t_0}} \leftarrow x_{\gamma_{t_0}} - C$
    $x_\delta \leftarrow x_\delta + C$
    for $t' \in [|\gamma_{t_0}, t_0|]$ **do**
      $I_{t'} \leftarrow I_{t'} - C$
      $J_{t'+1} \leftarrow J_{t'+1} + C$
    end for
    Update η_q with Algorithm 3
    Update $(q', \eta_{q'})$ with Algorithm 4
    $u \leftarrow t_0$
    if $cf(\gamma_{t_0}) \leq cf(\delta)$ **then**
      $\gamma_{\gamma_{t_0}} \leftarrow \gamma_{\gamma_{t_0}-1}$
    else
      $OptMove \leftarrow$ "No"
    end if
    $\delta \leftarrow t_0 \cdot \mathbf{1}_{x_{t_0}=0}$
  end if
  if Full production in period t_0 **then**
    $\zeta \leftarrow \arg\min\{cf(\eta_{q',q}),\ cf(\eta_{q,t_0-1}),\ cf(\delta)\}$
    Determine solution to $P(t_0)$ by moving full production from t_0 to ζ
  else
    The solution to $P(t_0)$ is equivalent to the solution to $P'(t)$
  end if
end while
return Solutions

The presented algorithm is an extension of algorithm 2, with the added consideration of parameters q and η to retrieve the optimal solution of problem P throughout the enumeration of all fractional production periods. Beside the call to algorithms 5, 3 and 4, all added operations are done in constant-time complexity.

Algorithm 5 is only called when the current value of parameter q is equal to $t+1$, for t the fractional production period enumerated from t_{max} to t_{min}. The new value of q is then evaluated by enumerating period from the current period t to t_1 and stopping at the first period t' such that $J_{t'} < C$. Similarly, η_q is updated by an enumeration of periods from t to the new value of q. Hence the successive update of q and η_q only requires to enumerate all periods from t_2 to t_1 throughout the iterations, and is done in linear time.

Remember that according to Lemma 3, after production is moved from period $\gamma^{t_0}_{t_0}$ to period δ^{t_0}, there are no further production to be moved to solve problem P' for the fractional production period in $[|\gamma^{t_0}_{t_0}, t_0|]$. Moreover

note that Algorithm 3 and Algorithm 4 are only called after a full production period is moved.

Algorithm 3 enumerates all periods from the current value of γ_{t_0} to t_0. As per Lemma 3 after production is moved, Algorithm 3 is not called again until the fractional production period is equal to γ_{t_0}. Therefore, that algorithm enumerates at most once all periods from t_2 to t_1 and its calls throughout the while-loop is of linear complexity.

Algorithm 4 updates parameter q' and $\eta_{q'}$. More specifically, parameter q' takes the value q if it is higher than γ_{t_0}, or the value γ_{t_0} otherwise. In the first case, the update of $\eta_{q'}$ is in constant-time complexity. In the second case, it requires the enumeration of all periods in interval $[\gamma_{t_0}, q]$. As per Lemma 3, this algorithm is not called again until the fractional production period is equal to γ_{t_0}. Since we consider in that case that γ_{t_0} is lower than q, Algorithm 5 would have been called before the fractional production period reaches the value γ_{t_0}, updating the value of q in the process to a period lower than or equal to q'. Hence, Algorithm 4 enumerates at most once all periods from t_2 to t_1 and its calls throughout the while-loop is of linear complexity.

Therefore Algorithm 6 is of linear complexity, and solving all subplans is done in cubic time, given the knowledge of an initial optimal solution to problem $P'(t'_{max})$ for each subplan. We discuss in the following section this initialization for all subplans in overall cubic time, so that the LS-CC-SUB may be solved in $\mathcal{O}(T^3)$ worst-case complexity.

6. Initialization to the dynamic resolution of all subplans

For a given starting period $t_1 \in [|1, T|]$ and starting, ending inventory value $I^1, I^2 \in \{0, 1\}$, the initialization is done dynamically for all subplans (t_1, t_2, I^1, I^2) with $t_2 \in [|t_1, T|]$ by solving all corresponding $P'(t'_{max})$ problems.

We extends results from van Hoesel and Wagelmans (1993) to present how to efficiently obtain an optimal solution to problem $P'(t'_{max})$ for subplan (t_1, t_2, I^1, I^2) from the one for subplan $(t_1, t_2 + 1, I^1, I^2)$

Theorem 4. *For any $t_2 \in [|t_1, T-1|]$, the optimal solution of problem $P'(t'_{max})$ for subplan (t_1, t_2, I^1, I^2) can be deduced from the one of subplan $(t_1, t_2 + 1, I^1, I^2)$ with at most one change in full production periods.*

Proof. In order to simplify the notation, we denote by t'_{max} and t''_{max} the respective upper bound value of the fractional period for subplan (t_1, t_2, I^1, I^2) and $(t_1, t_2 + 1, I^1, I^2)$.

The difference in total required production between subplans (t_1, t_2, I^1, I^2) and (t_1, t_2+1, I^1, I^2) is $d_{t_2+1} + I^2 \cdot (S_{t_2+1} - S_{t_2})$. Hence as per inequalities (10)-(12), this difference in production is lower than the production capacity and there is as much or one less full production period in subplan (t_1, t_2, I^1, I^2) than in subplan $(t_1, t_2 + 1, I^1, I^2)$.

Period t''_{max} is an upper bound on the value of all full production period for both subplan $(t_1, t_2 + 1, I^1, I^2)$ and subplan (t_1, t_2, I^1, I^2), because of inequalities (10)-(11). Hence, any full production period in the optimal solution to problem $P'(t''_{max})$ for subplan $(t_1, t_2 + 1, I^1, I^2)$ that is under t'_{max} remains optimal for problem $P'(t'_{max})$ of subplan (t_1, t_2, I^1, I^2). Indeed from one subplan to another, it remains below its current upper bound, while the lower bound remain constant despite the change in fractional production period as per results obtained in section 4. Therefore, there are no changes needed to the value of full production periods under t'_{max} when the optimal solution of problem $P'(t'_{max})$ for subplan (t_1, t_2, I^1, I^2) is deduced from the one of subplan $(t_1, t_2 + 1, I^1, I^2)$.

Moreover, consider subplan $(t_1, t_2 + 1, I^1, I^2)$ with K required full production period and a fractional production of value f. Suppose that in the optimal solution to the latter, the latest period of full production is denotes t^C. Then according to inequalities (10)-(11), we have: $I^2 \cdot S_{t_2} + d_{t^C, t_2} \geq f$. In other word, in the optimal solution to problem $P'(t''_{max})$ for subplan $(t_1, t_2 + 1, I^1, I^2)$, there cannot be more than one full production period higher than the latest period of fractional production for subplan (t_1, t_2, I^1, I^2).

Therefore, at most one full production period must be moved when deducing an optimal solution to $P'(t'_{max})$ for subplan (t_1, t_2, I^1, I^2) from the one for subplan $(t_1, t_2 + 1, I^1, I^2)$.

□

When deducing the optimal solution of problem $P'(t'_{max})$ for subplan (t_1, t_2, I^1, I^2) from the one of subplan $(t_1, t_2 + 1, I^1, I^2)$, the procedure is straight forward if no move of full production period is required, either because the latest production period is deleted or because all production periods are under t'_{max}.

Otherwise, a production periods higher than t'_{max} must be reallocated to a feasible available period under t'_{max}, or a production period under t'_{max} must be deleted. Similarly to section 4, we introduce a parameter k^- such that:

$$k^- = \max\left\{\{k \in [|2, K|];\ t^{k-1} < \underline{w}_k\} \cup \{0\}\right\}$$

Then the reallocation or the deletion must be done in interval $[|\underline{w}_{k^-}, t'_{max}|]$. This procedure is done in linear time by enumerating all periods in that interval.

Hence performing the initialization for all subplan (t_1, t_2, I^1, I^2) takes as much time as solving the problem $P'(t^I_{max})$ for subplan (t_1, T, I^1, I^2) and linearly deduce the optimal solution for all subplan (t_1, t_2, I^1, I^2) with $t_2 \in [|t_1, T - 1|]$. Therefore all initializations can be done in $\mathcal{O}(T^3)$.

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