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Abstract

The article investigates the fine structure of free cocommutative Hopf algebras, which are unshuffle Hopf algebras, that is enveloping algebras of free Lie algebras (possibly up to completion), and of the dual commutative cofree Hopf algebras, which are shuffle Hopf algebras. We introduce and study in particular base changes related to Lie idempotents families. The article also develops a categorical and duality framework to address the non graded case that includes for example quasi-shuffle Hopf algebras. Lastly, we survey various classical examples. We develop in detail the one of finite topologies that illustrates how one can take advantage of various extra algebraic structures such as infinitesimal bialgebras or double bialgebras structures in this context.

1 Introduction

The purpose of the present article is to systematically develop the theory of free cocommutative and the dual commutative cofree Hopf algebras. In concrete terms, a free cocommutative Hopf algebra is a cocommutative Hopf algebra H which is freely generated as an associative algebra by a subset S of elements (the set of generators). We will also consider the case where H is complete; H is then, up to isomorphism, the algebra of noncommutative formal power series over S. When the elements of S are primitive, H identifies with an unshuffle Hopf algebra. That is, it identifies with the tensor

algebra over S (resp. its completion) equipped with the unshuffle coproduct. The Hopf algebra H is then the enveloping algebra of the free Lie algebra generated by S, which is the central object of the modern theory of free Lie algebras [36].

Dually, a commutative cofree Hopf algebras is a commutative Hopf algebra H that has furthermore a basis indexed by words over a subset X (the set of cogenerators, spanning linearly the vector space of primitive elements) in such a way that the expression of the coproduct in that basis is the deconcatenation coproduct of words. This notion of commutative cofree Hopf algebras is equivalently encoded by the one of commutative B_∞-algebra (a vector space equipped with a family of multilinear operations satisfying ad hoc properties, to be recalled later in the article). When the B_∞ structure is trivial (see Definition 4.4), H identifies with the shuffle Hopf algebra over X, the dual notion to the one of enveloping algebra of the free Lie algebra generated by X, and another fundamental object of the theory of free Lie algebras [36].

The choice of a set S of generators in the free cocommutative case and of a basis of words over a set of cogenerators is in general highly non canonical and there is actually an infinity of possible choices. The most classical example of such phenomena in algebraic combinatorics, in the theory of free Lie algebras and in the structure theory of Hopf algebras is probably provided by change of bases in the Hopf algebra of quasi-symmetric functions or, dually, by change of bases in the Hopf algebra of descents in symmetric groups (see Section 8). We will show later in the present article that there is actually a very close connection between the structure theory of free cocommutative and of commutative cofree Hopf algebras, and the one of descent algebras. This extends the classical connection between the structure theory of graded commutative or cocommutative Hopf algebras and the one of descent algebras.

Various examples of the structures we investigate are provided by familiar and fundamental objects in algebra, combinatorics and topology. Examples of free cocommutative Hopf algebras include tensor algebras equipped with the unshuffle coproduct, the Hopf algebra of descent classes in symmetric groups and the isomorphic Hopf algebra of noncommutative symmetric functions, the enveloping algebras of free preLie algebras, the free noncommutative shuffle (aka dendriform) algebras equipped with the canonical coproduct (the one inherited from the Hopf monadic structure) and, more generally, all enveloping algebras of free Lie algebras. Examples of commutative cofree Hopf algebras include shuffle and quasi-shuffle Hopf algebras over a commutative algebra, Hopf algebra structures constructed on finite topologies, quasi-orders, orders, the Hopf algebra of quasi-symmetric functions... Most of these structures will be discussed below in the article. Details will then be given on their definitions and properties.

Recall that the two notions of free cocommutative Hopf algebras and

commutative cofree Hopf algebras are in perfect duality in the locally finite graded connected case (the two categories are anti-isomorphic; locally finite meaning here that each graded component is finite-dimensional). Most of the above examples correspond to this case. The situation is more complex in the non graded case, of which a typical example are quasi-shuffle Hopf algebras.

The article will accordingly consider largely separately the two cases and be divided into two parts focusing for each on specific ideas. We thus investigate on one side free cocommutative, and commutative cofree graded Hopf algebras H from the point of view of Lie idempotents. It is known in this setting that free cocommutative Hopf algebras are always isomorphic to enveloping algebras of free Lie algebras, that is tensor algebras equipped with the unshuffle coproduct [34, Lemma 22]. The dual result was obtained more recently by C. Bellingeri, E. Ferrucci and N. Tapia[1] in the context of the (Butcher)-Connes-Kreimer Hopf algebra of non planar trees and its applications to the theory of branched rough paths: a graded commutative cofree Hopf algebra is always isomorphic to the shuffle Hopf algebra over the space its primitive elements [2, Remark 3.5].

We recover these results and augment them in several respects. Firstly, we show that *any* family of Lie idempotents (Eulerian, Dynkin, Klyachko, Zassenhaus... — there are infinitely many of them) gives rise to a natural Hopf algebra isomorphism between graded free cocommutative Hopf algebras and unshuffle Hopf algebras. Dually, any such family gives rise to a natural Hopf algebra isomorphism between commutative cofree graded Hopf algebras and shuffle Hopf algebras. The isomorphisms introduced in [34, Lemma 22] and [2, Remark 3.5] correspond to the particular case of the Eulerian family.

On another side, we consider the general (non graded) case. It includes in particular quasi-shuffle Hopf algebras over a commutative algebra A, studied in detail in [26, 15, 14]. Quasi-shuffle Hopf algebras provide a right framework to investigate objects as different as for example Multiple Zeta Values, Rota-Baxter algebras (of weight non 0) or the Hopf algebra of quasi-symmetric functions. The latter is indeed, up to isomorphism, the quasi-shuffle Hopf algebra over the algebra of polynomials in one variable, or in additive notation, over the nonnegative integers. In probability theory, quasi-shuffles are associated to Itô to Stratonovich transformations for continuous semimartingales. There is an large literature on these topics, see e.g. the works quoted above or [30, 8, 6, 7] also for further references.

In order to study free cocommutative and commutative cofree Hopf algebras, we construct an adapted theoretical framework (graded-complete and

[1] We thank them warmly for pointing out to us the relevance of their article on branched rough paths for the theory of commutative cofree Hopf algebras. Our initial project was focusing on the general (non graded) case; their article was a key motivation to extend the scope of the article and systematically study the fine structure phenomena that appear in the graded case.

filtered-graded structures, with an adapted notion of duality between them). This framework accounts for example for the fact that in spite of commutative cofree Hopf algebra not carrying a graded Hopf algebra structure in general (more precisely, the existence of a grading is not part or a direct consequence of their definition), its product has always a nice behavior with respect to the filtration induced by the cofree coalgebra structure. Similar observations hold at the dual level, in spite of the situation being slightly more complex as one has then to consider complete vector spaces.

Building on these definitions, we study free cocommutative Hopf algebras and show that they are (up to completion) enveloping algebras of free Lie algebras. The method is the same as in the graded case excepted for two ingredients: one has to appeal to structure theorems for complete Hopf algebras, and the only idempotent available is the suitable generalisation in this framework of the Eulerian family. We turn then to the dual framework and investigate firstly isomorphisms between a commutative cofree Hopf algebra H and a shuffle Hopf algebra using general Hopf algebra techniques that were first developed to investigate the properties of Hopf algebra endomorphisms. We obtain a parametrisation of such isomorphisms by what we call tangent-to-identity infinitesimal endomorphisms of H. This extends the study that we performed with Jean-Yves Thibon in [15] — in the language of the present article, that article studied natural deformations of the shuffle Hopf algebra and the quasi-shuffle Hopf algebra functors from commutative algebras to commutative cofree Hopf algebras. We essentially drop here the naturality requirement. We use then an idea that was implicitly used in several of our earlier works and was developed systematically in [4], namely the fact that classical structure theorems and key properties of graded connected commutative or cocommutative Hopf algebras still hold under the assumption that the Hopf algebra is unipotent (that is, that its identity map is locally unipotent for the convolution product [4, Def. 4.1.2]). This allows to extend the results in the commutative cofree graded case to the non graded setting.

The article also includes a survey of various classes of Hopf algebras to which these various results apply. Building on [10, 11] and previous constructions in joint works with C. Malvenuto [12, 13], we expand specifically the case of the Hopf algebra of finite topologies, showing how one can in that particular case take advantage of a double bialgebra structure to approach differently the construction of a set of cogenerators.

The article is organized as follows. We recall first in section 2 basic definitions about coalgebras and introduce the notion of graded-complete and filtered-graded coalgebras that will prove useful later, when studying Hoffman-type isomorphisms (between shuffle and quasi-shuffle Hopf algebras). Section 3 considers cofree coalgebras in these categories. Section

4 introduces B_∞-algebras and related notions, whereas section 5 discusses graded B_∞ structures. The next sections introduce fundamental examples: free Lie algebras and their enveloping algebras (section 6), shuffle and quasi-shuffle Hopf algebras (section 7), the Hopf algebra of descents (section 8). Section 9, on finite topologies, illustrates how one can prove that a Hopf algebra can be equipped with a commutative cofree or free cocommutative structure when the existence of such a structure is non trivial. Section 10 investigates the structure of cofree commutative graded Hopf algebras and the dual free cocommutative graded Hopf algebras using duality properties and the theories of free Lie algebras and Lie idempotents. Section 11 studies free cocommutative and cofree commutative Hopf algebras, extending the results of the previous section to the non graded case (under some suitable completion hypotheses). Finally, in section 12, we take advantage of the fine structure on finite topologies to perform explicit computations illustrating the ideas developed in the article, using however a different approach to the generalized Eulerian idempotent relying on the existence of a double bialgebra structure.

2 Filtered-graded coalgebras

Recall first some definitions and properties of coalgebras. Details and proofs can be found in [4, Chap. 2]. All vector spaces are defined over a ground field $\mathbb{K}$ of characteristic 0.

A *graded vector space* is a vector space decomposing as a direct sum $V = \bigoplus_{n\in\mathbb{N}} V_n$. It is *reduced* if $V_0 = 0$. Given V, W two graded vector spaces, a morphism of graded vector spaces from V to W is a morphism of vector spaces $\phi : V \to W$ that respects the graduation ($\phi(V_n) \subset W_n$ for any $n \in \mathbb{N}$). A graded vector space is locally finite-dimensional if all the V_n are finite-dimensional vector spaces. The graded dual of a graded vector space is the graded vector space $V^* = \bigoplus_{n\in\mathbb{N}} V_n^*$, where V_n^* stands for the dual of V_n. It is locally finite-dimensional when V is such. The tensor product of two graded vector spaces is defined by $V \otimes W = \bigoplus_{n\in\mathbb{N}} (V \otimes W)_n$, where $(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q$. The ground field $\mathbb{K}$ identifies with the graded vector space, still written $\mathbb{K}$, with only one non zero component, $\mathbb{K}_0 = \mathbb{K}$. It is the unit of the tensor product ($V \otimes \mathbb{K} = V = V \otimes \mathbb{K}$).

A graded coalgebra is a coalgebra in the tensor category of graded vector spaces. That is, graded coalgebras and other graded structures are defined as usual except for the fact that structure morphisms, for example the coproduct Δ of a graded coalgebra C, have to be morphisms of graded vector spaces, so that

$$\Delta : C_n \to \bigoplus_{p+q=n} C_p \otimes C_q.$$

The counit map from C to $\mathbb{K}$, that we will write ε (or ε_C if we want to emphasize what is the underlying coalgebra) is automatically a null map except in degree 0. A graded coalgebra C is *connected* if $C_0 = \mathbb{K}$; for such a coalgebra, $\Gamma(C) = \{1\}$, where $\Gamma(C)$ stands for the set of group-like elements in C, that is

$$\Gamma(C) := \{c \in C,\ c \neq 0 \text{ and } \Delta(c) = c \otimes c\}.$$

Let now (C, Δ, ε) be a coaugmented coalgebra with coaugmentation $\eta : \mathbb{K} \to C$ (a coaugmentation is a map of coalgebras from the ground field to C, where the ground field is equipped with the identity coproduct: $\mathbb{K} = \mathbb{K} \otimes \mathbb{K}$). In other terms, we fix a group-like element $1 \in C$, that is to say a nonzero element 1 such that $\Delta(1) = 1 \otimes 1$. The coaugmentation is a section of the projection ε to the ground field and one has the decomposition $C = \bar{C} \oplus \mathbb{K}$, where $\bar{C} := \mathrm{Ker}(\varepsilon)$. Notice that a graded connected coalgebra is canonically coaugmented: its coaugmentation is the isomorphism between the ground field and the degree 0 component of the coalgebra, and 1 is the unique group-like element of C. In general, for a coaugmented coalgebra one can define the reduced coproduct from $\bar{C}$ to $\bar{C} \otimes \bar{C}$ by

$$\overline{\Delta}(c) := \Delta(c) - c \otimes 1 - 1 \otimes c;$$

the element c is called primitive if $\overline{\Delta}(c) = 0$, the vector space of primitive elements is denoted $Prim(C)$.

The iterated reduced coproduct from $\bar{C}$ to $\bar{C}^{\otimes n+2}$ is then inductively defined by

$$\bar{\Delta}_{n+2} := (\bar{\Delta} \otimes \mathrm{Id}_{\bar{C}}^{\otimes n}) \circ \bar{\Delta}_{n+1}, \text{ with } \bar{\Delta}_2 := \bar{\Delta}.$$

Let $F_n := \mathrm{Ker}\ (\overline{\Delta}_{n+1}) \subset \bar{C}$. Notice that $\bar{\Delta}_{n+2} = (\bar{\Delta} \otimes \mathrm{Id}_{\bar{C}}^{\otimes n}) \circ \bar{\Delta}_{n+1}$ implies $F_n \subset F_{n+1}$.

Definition 2.1. *The coproduct Δ and the coalgebra C are called conilpotent if $\bar{C} = \bigcup_{n \geqslant 1} F_n$, that is if for every $c \in \bar{C}$ there exists an integer $n \geqslant 2$ such that $\overline{\Delta}_n(c) = 0$.*

Lemma 2.2. *When the coalgebra C is graded and connected, the coproduct is automatically conilpotent.*

Proof. Indeed, when C is graded connected, $\bar{C}_0 = 0$. As $\bar{\Delta}_k$ sends $\bar{C}_n$ to $\bigoplus_{\substack{i_1+\cdots+i_k=n,\\ i_1,\ldots,i_k \geqslant 1}} \bar{C}_{i_1} \otimes \cdots \otimes \bar{C}_{i_k}$, the iterated coproduct $\overline{\Delta}_k$, $k \geqslant 2$, vanishes on C_n for $1 \leqslant n \leqslant k-1$. □

A *filtered* vector space is a vector space V equipped with an increasing filtration by subspaces $V(0) \subset V(1) \subset \cdots \subset V(n) \subset \cdots$, such that $V =$

$\bigcup_{n\in\mathbb{N}} V(n)$. Given V, W two filtered vector spaces, a morphism of filtered vector spaces from V to W is a morphism of vector spaces $\phi : V \to W$ that respects the filtration ($\phi(V(n)) \subset W(n)$ for any $n \in \mathbb{N}$). The tensor product of two filtered vector spaces is defined by requiring $(V \otimes W)(n) := \sum_{i+j=n} V(i) \otimes W(j)$. A graded vector space V is canonically filtered by setting $V(n) := \bigoplus_{p\leqslant n} V_p$. In that case, we say that V is *filtered-graded*. A morphism of filtered-graded vector spaces from V to W is, by definition, a family of morphisms $\phi_0 : V_0 \to W_0$, $\phi_n = \bigoplus_{0<p\leqslant n} \phi_{n,p} : V_n \to \bigoplus_{0<p\leqslant n} V_p$. Notice that, for practical reasons, we deliberately omit the degree 0 component in the image.

A filtered-graded algebra is an algebra A in the category of filtered-graded vector spaces such that furthermore $A_0 = \mathbb{K}$, the ground field. We will be interested later on in Hopf algebras that are filtered-graded as algebras and graded as coalgebras, the typical example being the Hopf algebras of quasi-shuffles over a commutative algebra.

Definition 2.3. *The category* **FgCoalg** *of filtered-graded coalgebras is the category whose objects are graded connected coalgebras and the set of morphisms between two graded connected coalgebras C and D, the set of filtered-graded vector spaces morphisms of coalgebras from C to D.*

That is, a morphism of filtered-graded coalgebras from C to D is a morphism ϕ of coalgebras such that $\phi(C_0) = \mathbb{K} = D_0$ and $\phi(C_n) \subseteq \bigoplus_{0<p\leqslant n} D_p$ for any $n \in \mathbb{N}$.

A standard example of filtered-graded coalgebra morphism is provided by the Hoffman isomorphism between the shuffle and quasi-shuffle Hopf algebras, it will be discussed later on in the article.

A *complete* vector space $\widehat{V}$ is a vector space equipped with a decreasing filtration

$$\widehat{V} = \widehat{V(0)} \supset \widehat{V(1)} \supset \cdots \supset \widehat{V(n)} \supset \ldots$$

such that $\widehat{V} = \lim_{\leftarrow} \widehat{V}/\widehat{V(n)}$.

In the present article, $\widehat{V}$ will always be obtained from a graded vector space V as $\widehat{V} := \prod_{n\in\mathbb{N}} V_n$ and $\widehat{V(i)} := \prod_{n\geqslant i} V_n$. We call such complete vector spaces *graded-complete* vector spaces. A morphism of graded-complete vector spaces from $\widehat{V}$ to $\widehat{W}$ is, by definition, a morphism of complete vector spaces induced by $\phi_0 : V_0 \to W_0$ and $\phi_{n,m} =: V_n \to W_m$ for $m \geqslant n \geqslant 1$.

The complete tensor product of two such spaces is then obtained as $\widehat{V}\widehat{\otimes}\widehat{W} := \prod_{n\in\mathbb{N}} \left(\bigoplus_{p+q=n} V_p \otimes W_q \right) = \prod_{p,q\in\mathbb{N}} V_p \otimes W_q$. A *graded-complete coalgebra* $\widehat{C}$ is a coalgebra in the category of graded-complete vector spaces such that

furthermore $C_0 = \mathbb{K}$. In particular its coproduct Δ is entirely defined by the knowledge of its restrictions

$$\Delta_{n,p,q} : C_n \to C_p \otimes C_q$$

with $p + q \geqslant n \geqslant 1$.

The dual notion to the one of graded-complete coalgebra $\widehat{C}$ is the one of filtered-graded algebra: write $C^* = \bigoplus_{n\in\mathbb{N}} C_n^*$ for the graded dual of the graded vector space C; C^* is equipped with an associative algebra structure by the maps $\mu_{n,m} = \sum_{1\leqslant p\leqslant n+m} \mu_{n,m,p}$ with

$$\mu_{n,m,p} := \Delta^*_{p,n,m} : C_n^* \otimes C_m^* \to C_p^*.$$

Recall, for completeness sake, that a bialgebra B is a unital algebra and a counital coalgebra such that the product and the unit map are morphisms of counital coalgebras. When the coalgebra structure is conilpotent or under suitable completion hypotheses, as it is always the case in the present article, the notions of bialgebra and Hopf algebra identify and we will use the two terminologies indifferently. Denoting π and Δ the product and the coproduct of B, the convolution $f * g$ of two linear endomorphisms f, g of B is defined by $f * g := \pi \circ (f \otimes g) \circ \Delta$. The convolution product defines a unital algebra structure on the vector space of linear endomorphisms of B with unit the composition $\eta \circ \varepsilon$ of the unit and counit maps. See [4] for a systematic treatment.

The quasi-shuffle algebra over a commutative algebra is a filtered-graded algebra and its product map is a filtered-graded morphism of coalgebras (see Section 7. This motivates the following definition.

Definition 2.4. *A Hopf algebra $(H, *, \Delta)$ is called a filtered-graded Hopf algebra if (H, Δ) is a graded coalgebra and the product $* : H \otimes H \to H$ is a filtered-graded coalgebra morphism.*

3 Cofree filtered-graded coalgebras

Let now V be a vector space over $\mathbb{K}$. We denote by $T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$ the tensor gebra[2] of V, and use the word notation for tensors (that is, $v_1 \dots v_n$ will stand for $v_1 \otimes \cdots \otimes v_n$). We will use later a similar notation for the tensor gebra over an alphabet X (the linear span of the set of words — or

[2]We use the terminology of [4] and call gebra a vector space that can be equipped with several algebraic structures — this allows in particular to avoid calling "tensor algebra" the vector space $T(V)$ without equipping it with the algebra structure obtained from the concatenation product of words.

free monoid — over X). The tensor gebra is graded: the *tensor degree* of a word w is its length $\ell(w)$, that is, the number of its letters.

When V is a graded vector space, $T(V)$ is also equipped with the *total degree*: $\deg(v_1 \otimes \cdots \otimes v_n) := |v_1| + \cdots + |v_n|$, when the v_i are homogeneous elements and where $|v_i|$ stands for the degree of v_i in V. To avoid ambiguities, we will write $T^g(V)$ to denote the tensor gebra viewed as a graded vector space when equipped with the total degree.

We equip the tensor gebra with the deconcatenation coproduct, that makes it, together with the canonical projection ε to $\mathbb{K} = V^{\otimes 0}$, a graded (counital, conilpotent, connected, coassociative) coalgebra (for both the tensor and total degrees): for any $v_1, \ldots, v_n \in V$, with $n \geqslant 0$,

$$\Delta(v_1 \ldots v_n) = \sum_{i=0}^{n} v_1 \ldots v_i \otimes v_{i+1} \ldots v_n.$$

We write $T_+(V) := \bigoplus_{n=1}^{\infty} V^{\otimes n}$ for $\overline{T(V)}$. It is equipped with the (coassociative but not counital) reduced deconcatenation coproduct: for any $v_1, \ldots, v_n \in V$, with $n \geqslant 1$,

$$\overline{\Delta}(v_1 \ldots v_n) = \sum_{i=1}^{n-1} v_1 \ldots v_i \otimes v_{i+1} \ldots v_n.$$

The following Lemma further justifies the introduction of the category of filtered-graded coalgebras.

Lemma 3.1. *Let C be graded connected coalgebra and ϕ be a morphism of coalgebras from C to $(T(V), \Delta)$. Then, ϕ is a morphism of filtered graded coalgebras. That is, $\phi(C_n) \subseteq \bigoplus_{k=1}^{n} V^{\otimes k}$ for $n \geqslant 1$.*

Proof. As ϕ is a coalgebra morphism between connected coalgebras, we have, on $\overline{C}$,

$$(\phi \otimes \phi) \circ \overline{\Delta} = \overline{\Delta} \circ \phi,$$

this implies that for any $n \in \mathbb{N}$,

$$\phi^{\otimes n} \circ \overline{\Delta}_n = \overline{\Delta}_n \circ \phi.$$

Therefore, $\phi(\mathrm{Ker}(\overline{\Delta}_{n+1})) \subseteq \mathrm{Ker}(\overline{\Delta}_{n+1})$. It is an easy exercise to show that in $T(V)$, $\mathrm{Ker}(\overline{\Delta}_{n+1}) = \bigoplus_{k=1}^{n} V^{\otimes k}$. Moreover it holds for degree reasons that $C_n \subseteq \mathrm{Ker}(\overline{\Delta}_{n+1})$. We get finally $\phi(C_n) \subseteq \bigoplus_{k=1}^{n} V^{\otimes k}$. □

Definition 3.2 (Cofree coalgebras)**.** *Let V be a reduced graded vector space. A cofree filtered-graded (resp. graded) coalgebra over V is a connected graded*

coalgebra C together with a filtered-graded (resp. graded) vector space map π from $\bar{C}$ to V such that for each connected graded coalgebra D, any filtered-graded (resp. graded) vector space morphism ϕ from $\bar{D}$ to V lifts uniquely to a morphism Φ of coalgebras from D to C in **FgCoalg** *(resp. in the category of graded coalgebras), such that $\pi \circ \Phi_{|\bar{D}} = \phi$. The space V is called the cofreely cogenerating space of C; the map π is called the structure map.*

As usual for cofree objects, any two cofree filtered-graded (resp. cofree graded) coalgebras over V are isomorphic (by a unique isomorphism, see Lemma 3.4 below). This justifies to call (slightly abusively) any cofree filtered-graded (resp. cofree graded) coalgebra over V, "the" cofree filtered-graded (resp. cofree graded) coalgebra over V.

Lemma 3.3. *The tensor gebra $T^g(V)$ over a graded vector space V, equipped with the deconcatenation coproduct Δ, the canonical projection π_V from $T_+(V)$ to V and the total graduation is a cofree filtered-graded (resp. graded) coalgebra over V. We call it the standard cofree filtered-graded (resp. graded) coalgebra over V. With our previous notation, the morphism Φ from D to $T^g(V)$ is obtained as*

$$\varepsilon + \sum_{n \geqslant 1} \phi^{\otimes n} \circ \bar{\Delta}_n.$$

In the formula, it is understood that if $d \in D$ decomposes as $\varepsilon(d) + (d - \varepsilon(d)) \in \mathbb{K} \oplus \bar{D}$,

$$\Phi(d) = \varepsilon(d) + \sum_{n \geqslant 1} \phi^{\otimes n} \circ \bar{\Delta}_n(d - \varepsilon(d)).$$

Proof. Recall that $(T(V), V, \pi_V)$ is a cofree conilpotent coalgebra; that the formula for Φ holds in the cofree conilpotent case (see e.g. [4, Exercise 2.13.3, Remark 2.13.1]) and that graded connected coalgebras are conilpotent. There is thus for any D as in Definition 3.2 a coalgebra map from D to C and it is enough to check that this map is filtered-graded, (resp. graded). This is immediate in both cases due to Formula

$$\Phi = \varepsilon + \sum_{n \geqslant 1} \phi^{\otimes n} \circ \bar{\Delta}_n.$$

since the coproduct is a graded map from $T^g(V)$ to $T^g(V) \otimes T^g(V)$ and since ϕ is a filtered-graded (resp. graded) map from $\overline{D}$ to V. This concludes the proof. Further insights on cofree objects and why some conilpotency assumption is required to have a simple construction thereof can be found e.g. in [4]. □

Remark 3.1. Notice that since $V = Prim(T^g(V))$, the graded vector space of primitive elements of $T^g(V)$, it holds for any cofree filtered-graded (resp. cofree graded, cofree conilpotent) coalgebra C over V that $Prim(C) \cong V$. In

particular one can always choose V to be $Prim(C)$. Fix now a graded basis $(b_i)_{i\in J}$ of $Prim(C)$. The choice of a structure map π can be interpreted as the choice of a basis of C since it induces an isomorphism $C \cong T^g(Prim(C))$, where $T^g(Prim(C))$ has the basis of words $b_{j_1} \dots b_{j_k},\ j_l \in J, l = 1, \dots, k,\ k \in \mathbb{N}$. We will say that the choice of a cofree filtered-graded (resp. cofree graded, cofree conilpotent) coalgebra structure $\pi : C \to Prim(V)$ determines a word *presentation* of C and call the associated basis (resp. graded basis, resp. basis) of C the π-basis.

Remark 3.2. The isomorphism $C \cong T^g(Prim(C))$ is non canonical as it depends heavily on the choice of π. This observation is made more precise in the following Lemma and its proof. A slightly more involved version of the same argument actually allowed us to classify natural endomorphisms and automorphisms of the functor T viewed as a functor from commutative algebras to coalgebras, see [15].

Lemma 3.4. *Let C be a connected graded coalgebra. Cofree filtered-graded (resp. cofree graded) coalgebra structures on C, whenever they exist, are parameterized by the surjections from $\overline{C}$ to $Prim(C)$ that are maps of filtered (resp. graded) vector spaces and restrict to the identity map on $Prim(C)$. All such structures are isomorphic: a cofree filtered-graded (resp. graded) coalgebra structure on a connected graded coalgebra C is thus unique up to isomorphism.*

Proof. Fixing a cofree filtered-graded coalgebra structure on C amounts to fixing an isomorphism of coalgebras between C and $(T^g(V), \pi_V)$, the standard cofree filtered-graded coalgebra over $V := Prim(C)$. We can therefore assume without restriction that C is $T^g(V)$ equipped with the structure map π_V. Any other cofree filtered graded coalgebra structure on C is given by a filtered-graded surjection π from C to V that restricts to the identity map on V. The map π is the structure map of the second cofree structure; by Definition 3.2 it induces a filtered-graded coalgebra endomorphism $\tilde{\pi}$ of $T^g(V)$ obtained as:

$$v_1 \dots v_n \longmapsto \sum_{k<n} \sum_{w_1 \dots w_k = v_1 \dots v_n} \pi(w_1) \dots \pi(w_k). \tag{1}$$

It can be inverted, and the inverse isomorphism $\tilde{\mu}$ such that $\tilde{\mu} \circ \tilde{\pi} = \mathrm{Id}_{T^g(V)}$ is entirely characterized by the identity

$$\pi_V \circ \tilde{\mu} \circ \tilde{\pi} = \pi_V,$$

that is, $\mu(v) = v$ for $v \in V$ and, for $k \geqslant 2$ and $v_1, \dots, v_n \in V$,

$$\mu(v_1 \dots v_n) = -\sum_{k<n} \sum_{w_1 \dots w_k = v_1 \dots v_n} \mu(\pi(w_1) \dots \pi(w_k)). \tag{2}$$

The formula implies, by an induction argument, that μ and $\tilde{\mu}$ are filtered-graded.

The same arguments and formulas apply in the graded case. □

Remark 3.3. When one considers the particular case of natural endomorphisms of T viewed as a functor from commutative algebras to coalgebras, the group of natural filtered-graded automorphisms of T is isomorphic to the group of tangent-to-identity formal power series. In that case, the identity relating π and μ amounts to the calculation of the inverse in the group of tangent-to-identity formal power series [15].

Remark 3.4. The category of filtered-graded coalgebras thus appears to be the natural framework to study Hopf algebras such as quasi-shuffle Hopf algebras. See for example the discussion of natural endomorphisms of quasi-shuffle Hopf algebras in [15, 26] for insights on the role of filtered-graded maps in that context: the surjections that appear in these article should indeed be understood as filtered-graded maps; they map a tensor of order n to a tensor of lower or equal degree.

Let us finally introduce cofree structures for Hopf algebras.

Definition 3.5. *Let $(H, *, \Delta)$ be a conilpotent (resp. graded, resp. filtered-graded) Hopf algebra. A cofree structure on H is the data of a structure map $\pi : \bar{H} \to Prim(H)$ making the triple (H, Δ, π) a cofree conilpotent (resp. graded, resp. filtered-graded) coalgebra over $Prim(H)$. We will say that $(H, *, \Delta, \pi)$ (or simply (H, π) when the underlying Hopf algebra structure is obvious) is a cofree (resp. cofree graded, resp. cofree filtered-graded)Hopf algebra.*

4 B_∞-algebras

The notion of B_∞-algebra was first introduced by Getzler and Jones in [17] for cochain complexes. Their definition extends to other tensor categories, we consider in this section B_∞-algebras in the category of vector spaces. We survey here the fundamental definitions and properties and refer to [9] for further insights and applications.

A B_∞-algebra structure on a vector space V describes in algebraic terms a Hopf algebra structure on the cofree conilpotent coalgebra $(T(V), \Delta, \pi_V)$ (see Proposition 4.7 below). The tensor product of two graded connected coalgebras is a graded connected coalgebra; $T(V)$ is graded by the tensor degree and connected, and $T(V) \otimes T(V)$ is thus a connected graded coalgebra. By Lemma 3.1, the algebra product π, which is a map of coalgebras, necessarily respects the filtrations: $\pi(T_n(V) \otimes T_m(V)) \subset \bigoplus_{k \leqslant n+m} T_k(V)$. In particular, structure results obtained in the previous section for coalgebras in **FgCoalg** apply to the Hopf algebras $T(V)$ associated to B_∞-algebras as described below.

Lemma 4.1. *Let $* : T(V) \otimes T(V) \longrightarrow T(V)$ be a coalgebra map. It is*

entirely characterized by the map $\langle -,-\rangle : T(V)\otimes T(V) \longrightarrow V$ *defined by*

$$\langle v_1 \dots v_k, v_{k+1} \dots v_{k+l}\rangle = \pi_V(v_1 \dots v_k * v_{k+1} \dots v_{k+l}),$$

where $\pi_V : T(V) \longrightarrow V$ *is the canonical surjection.*

Then, $1 * 1 = 1$ *and for any words* $w, w' \in T_+(V)$,

$$w * w' = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1\dots w_k,\\ w'=w'_1\dots w'_k}} \langle w_1, w'_1\rangle \dots \langle w_k, w'_k\rangle. \tag{3}$$

Note that in the sum, the words w_i *or* w'_j *can be empty (in what case* w_i *and* w'_j *stand for 1 in the terms* $\langle w_i, w'_i\rangle$ *or* $\langle w_j, w'_j\rangle$*).*

Moreover,

$$w * 1 = \sum_{k=1}^{\infty} \sum_{w=w_1\dots w_k} \langle w_1, 1\rangle \dots \langle w_k, 1\rangle. \tag{4}$$

$$1 * w' = \sum_{k=1}^{\infty} \sum_{w'=w'_1\dots w'_k} \langle 1, w'_1\rangle \dots \langle 1, w'_k\rangle. \tag{5}$$

Proof. As $*$ is a coalgebra map, it maps $1\otimes 1$, the unique group-like element in $T(V)\otimes T(V)$, to 1, the unique group-like element in $T(V)$. The Lemma follows then from Lemma 3.3: $*$ is entirely characterized by $\langle -,-\rangle := \pi_V \circ *$, and applying the formula expressing $*$ in terms of $\langle -,-\rangle$ yields to Eqs (3,4,5). □

Lemma 4.2. *With the same notation, the product* $*$ *is unital, with unit 1 if, and only if,*

$$\langle -, 1\rangle = \langle 1, -\rangle = \pi_V.$$

Proof. The assertion follows directly from the definitions of $\pi_V, \langle -,-\rangle$ and Eqs (4,5). □

We assume from now on that the product $*$ is unital, with unit 1.

Lemma 4.3. *With the same notation,*

1. *the product* $*$ *is associative if, and only if, for any* $w, w', w'' \in T_+(V)$,

$$\langle w, w' * w''\rangle = \langle w * w', w''\rangle.$$

2. *It is commutative if, and only if, for any* $w, w' \in T_+(V)$,

$$\langle w, w'\rangle = \langle w', w\rangle.$$

Proof. As the product is a map of coalgebra, so are the maps $* \circ (\mathrm{Id} \otimes *)$ and $* \circ (* \otimes \mathrm{Id})$ from $T(V)^{\otimes 3}$ to $T(V)$. They are therefore entirely characterized by the composition with π_V and associativity follows from

$$\pi_V \circ (* \circ (\mathrm{Id} \otimes *)) = \pi_V \circ (* \circ (* \otimes \mathrm{Id})).$$

The second assertion is proved similarly, noticing that the twist map $w \otimes w' \to w' \otimes w$ is a morphism of coalgebras. □

Definition 4.4. *A B_∞-structure on V is a map $\langle -, - \rangle : T(V) \otimes T(V) \longrightarrow V$, such that:*

- *For any word $v_1 \dots v_n \in T(V)$,*

$$\langle 1, v_1 \dots v_n \rangle = \langle v_1 \dots v_n, 1 \rangle = \begin{cases} v_1 \text{ if } n = 1, \\ 0 \text{ otherwise.} \end{cases}$$

- *For any words $w, w', w'' \in T_+(V)$,*

$$\langle w, w' * w'' \rangle = \langle w * w', w'' \rangle,$$

where $*$ is defined by

$$w * w' = \sum_{k=1}^{\infty} \sum_{\substack{w = w_1 \dots w_k, \\ w' = w'_1 \dots w'_k}} \langle w_1, w'_1 \rangle \dots \langle w_k, w'_k \rangle. \tag{6}$$

We shall say that $\langle -, - \rangle$ is commutative if for any $w, w' \in T_+(V)$, $\langle w, w' \rangle = \langle w', w \rangle$. We shall say that $\langle -, - \rangle$ is trivial if furthermore for any $w, w' \in T_+(V)$, $\langle w, w' \rangle = 0$.

Remark 4.1. Equation (6) can be rewritten in this way: for any $v_1, \dots, v_{k+l} \in V$,

$$v_1 \dots v_k * v_{k+1} \dots v_{k+l} = \sum_{n=1}^{k+l} \sum_{\substack{\sigma : [k+l] \twoheadrightarrow [n], \\ \sigma(1) \leqslant \dots \leqslant \sigma(k), \\ \sigma(k+1) \leqslant \dots \leqslant \sigma(k+l)}} \langle v_{\sigma^{-1}(1)} \rangle \dots \langle v_{\sigma^{-1}(n)} \rangle,$$

with the following notation: if $I = \{i_1, \dots, i_q\} \subseteq [n]$, with $i_1 < \dots < i_p \leqslant k < i_{p+1} < \dots < i_q$,

$$v_I = v_{i_1} \dots v_{i_p} \otimes v_{i_{p+1}} \dots v_{i_q}.$$

In particular we get the Lemma:

Lemma 4.5. *The B_∞-algebra structure is trivial if and only if the product $*$ is the shuffle product*[3]*, $* = \sqcup\!\sqcup$, with*

$$v_1 \dots v_k * v_{k+1} \dots v_{k+l} = \sum_{\substack{\sigma:[k+l]\to[k+l],\\ \sigma(1)<\dots<\sigma(k),\\ \sigma(k+1)<\dots<\sigma(k+l)}} v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(n)}.$$

See Section 7 for another classical definition of the shuffle product. In that case, the Hopf algebra $(T(V), \sqcup\!\sqcup, \Delta)$ is called the shuffle Hopf algebra over V.

Definition 4.6. *A B_∞-algebra (resp. commutative) is a vector space V equipped with a B_∞-structure (resp. commutative).*

Proposition 4.7. *Let $P(V)$ be the set of products $*$ on $T(V)$ making the triple $(T(V), *, \Delta)$ a bialgebra and by $B_\infty(V)$ the set of B_∞-algebra structures on V. The following map is a bijection:*

$$\Theta : \left\{ \begin{array}{ccc} P(V) & \longrightarrow & B_\infty(V) \\ * & \longmapsto & \pi_V \circ *. \end{array} \right.$$

Denoting by $P^c(V)$ the set of commutative products on $T(V)$ making the triple $(T(V), *, \Delta)$ a bialgebra and by $B^c_\infty(V)$ of commutative B_∞-algebra structures on V, Θ induces a bijection from $P^c(V)$ to $B^c_\infty(V)$.

Proof. As 1 is the unique group-like of $T(V)$, it is necessarily the unit for the product $*$. By Lemmas 4.1, 4.2 and 4.3, Θ is well-defined. If $* \in P(V)$, then it is a coalgebra morphism from $T(V) \otimes T(V)$ to $T(V)$. By Lemma 3.3, Θ is injective.

Conversely, let $\langle -, - \rangle$ in $B_\infty(V)$. The product $*$ associated to it by (3) is a coalgebra map. It is associative by Lemma 4.3 and has 1 for a unit: $* \in P(V)$, and $\Theta(*) = \langle -, - \rangle$. Thus, Θ is a bijection.

By the last item of Lemma 4.3, $\Theta(P^c(V)) = B^c_\infty(V)$. □

Definition 4.8. *The bialgebra $(T(V), *, \Delta)$ associated to a B_∞-algebra structure $\langle -, - \rangle$ on V is called the B_∞-enveloping algebra of $(V, \langle -, - \rangle)$.*

Remark 4.2. All these notions dualize when V is finite-dimensional, so that $T(V)$ is locally finite. The dual of a B_∞-algebra structure on V is then obtained as a family of linear maps

$$V^* \to (V^*)^{\otimes n} \otimes (V^*)^{\otimes m}$$

[3] Shuffle and quasi-shuffle products can be commutative (as usually the case in Lie theory) or not (as usually in classical algebraic topology where shuffle products appear in relation to cartesian products of simplices). In the present article they will be always commutative, excepted in the example of quasi-shuffle Hopf algebras over an associative algebra (see Section 7) where the quasi-shuffle product is noncommutative when the algebra is noncommutative.

making the completion of $T(V^*)$ a graded-complete coalgebra and a complete Hopf algebra when equipped with the concatenation product. The corresponding notion of B_∞-coalgebra would be obtained by dualizing the B_∞-algebra axioms and would naturally describe complete Hopf algebra structures on algebras of noncommutative formal power series. We will show later on that such Hopf algebras, when cocommutative, are always isomorphic to Hopf algebra structures on algebras of formal power series where the coproduct is the canonical one (the one making the generators of the algebra primitive elements).

5 Graded B_∞-algebras

Let us consider now the notion of B_∞-algebra in the category of graded vector spaces. Most of the classical examples of B_∞-algebras have such a structure, which is by the way closer to the topological notion of B_∞-algebra (that is, the one in the category of chain complexes). We assume therefore in this section that V is a graded vector space. For brevity we do not repeat all definitions in the previous section: they have to be adapted as follows:

- replacing everywhere $T(V)$ (graded by the tensor degree) by $T^g(V)$ (the same vector space, graded by the total degree),
- requiring that *all maps* be maps of graded vector spaces,
- requiring in particular that the product $*$ resp. the structure map $\langle -, - \rangle$ be maps of graded coalgebras resp. of graded vector spaces.

Using the point of view of Definition 4.8, let $H = (T^g(V), *, \Delta)$ be the B_∞-enveloping algebra of $(V, \langle -, - \rangle)$ in the category of graded vector spaces. It is equipped with a graded connected Hopf algebra structure and we say that $T^g(V)$ is then the graded B_∞-enveloping algebra of the graded B_∞-algebra V. For simplicity, we will abusively also say that $(T^g(V), *, \Delta)$ is a graded B_∞-algebra.

Definition 5.1. *When the B_∞ structure is trivial, that is when $\langle w, w' \rangle = 0$ for words both of length greater or equal 1, the product $*$ is the shuffle product $\sqcup\!\sqcup$, $H = (T^g(V), \sqcup\!\sqcup, \Delta)$ is the graded shuffle Hopf algebra over V, and we say that it is a standard graded B_∞-algebra.*

Remark 5.1. A direct inspection of the formulas defining B_∞-enveloping algebras shows that, given a vector space V, $(T(V), *, \Delta)$ is a graded Hopf algebra for the tensor degree on $T(V)$ if and only if the B_∞-algebra structure is trivial (that is, $\langle w, w' \rangle = 0$ for $w, w' \in T_+(V)$ and the product is the shuffle product). Equivalently, if V is graded and concentrated in degree 1 ($V = V_1$), there is a unique graded B_∞-algebra structure on V: the trivial one.

Let now $H = (T^g(V), *, \Delta)$ be a locally finite graded commutative B_∞-enveloping algebra (recall that locally finite means that the graded components H_n are finite-dimensional). The graded dual cocommutative Hopf algebra $H^* = \bigoplus_{n\in\mathbb{N}} H_n^*$ is a free associative algebra generated by the graded dual of V, denoted $V^* := \bigoplus_{n\geqslant 1} V_n^*$. That is, up to a canonical isomorphism, $H^* = T^g(V^*)$, the tensor algebra over V^* equipped with the concatenation product. More generally, if $(H, *, \Delta, \pi)$ is a locally finite cofree graded commutative Hopf algebra, $H^* \cong T^g(Prim(H)^*)$ equipped with the concatenation product. The following definition characterizes such Hopf algebras.

Definition 5.2. *A localy finite graded connected cocommutative Hopf algebra H freely generated as an associative algebra by a graded subspace W is called a free-Lie-type Hopf algebra. When $W \subseteq Prim(H)$ (that is, when H is primitively generated by W), we say that H is a standard free-Lie-type Hopf algebra.*

The reasons for this terminological choice will become clear later on: we will show that a free-Lie-type Hopf algebra is always isomorphic to the enveloping algebra of a free Lie algebra.

Recall already that when H is a standard free-Lie type Hopf algebra, by standard results in the theory of free Lie algebras it is automatically canonically isomorphic to the enveloping algebra of the free Lie algebra generated by W. In that case, the graded dual Hopf algebra is (up to a canonical isomorphism) the shuffle Hopf algebra over W^*, so that standard free Lie-type Hopf algebras and standard graded B_∞-algebras are in duality. On these topics and in particular for details on the duality between enveloping algebras of free Lie algebras (also called unshuffle Hopf algebras later on) and shuffle Hopf algebras, see [36].

6 Free Lie algebras

Let us consider now one of the simplest possible non trivial example of a free cocommutative graded Hopf algebra: the free graded associative algebra $A = \mathbb{Q}\langle x, y\rangle$ on two generators, x of degree 1 and y of degree 2 (so that, for example, the word $xyxy^2$ is of degree 8). It is the enveloping algebra of the free Lie algebra over x and y (using the rewriting trick $[a, b] = ab - ba$ to expand iterated commutators in the free Lie algebra into sums of words). It is then natural to equip A with a standard free-Lie type Hopf algebra structure by requiring x and y to be primitive elements (as A is a free associative algebra, this choice entirely determines the Hopf algebra structure on A).

Consider now the graded dual of A, denoted A^* and write x^* and y^* for the elements dual to x and y in the basis of words. In general, if $y_1 \dots y_n$ is a word in the letters x and y we will write $y_1^* \dots y_n^*$ for the corresponding

element in the dual in the basis of words over x and y. Writing V for the linear span of x^* and y^* with the graduation $|x^*| = 1$, $|y^*| = 2$, A^* identifies to $T^g(V)$ equipped with the deconcatenation product. That is, A^* identifies with the cofree graded coalgebra $T^g(V)$ over V with structure map π_V mapping $y_1^* \dots y_n^*$ to 0 if $n \geqslant 2$ and mapping y_1^* to y_1^*, where $y_i \in \{x, y\}$, $i = 1 \dots n$.

Dualizing the coproduct of A to obtain a product that we denote $\times$, $T^g(V)$ becomes a standard graded B_∞-algebra and identifies as a Hopf algebra to the shuffle Hopf algebra over x^* and y^* (see Remark 4.1 and Section 7). In particular,

$$x^* \times x^* = x^* \sqcup x^* = 2x^*x^*.$$

See [36] for details on these constructions and definitions. The corresponding B_∞-algebra structure on V, obtained through the projection π_V, is trivial.

Consider now the change of variables $t := x$, $z := y + x^2$ (it is convenient to notationally distinguish x and t). One can rewrite $A = \mathbb{Q}\langle t, z\rangle$ but A is not a standard free Lie-type Hopf algebra any more with respect to the linear span of t and z, as z is not primitive: $\overline{\Delta}(z) = 2t \otimes t$. Let us use the same notation as above: if $y_1 \dots y_n$ is a word in the letters t and z we will write $y_1^* \dots y_n^*$ for the corresponding element in the dual basis to the basis of words over t and z. For example, as

$$\begin{aligned} t &= x, & z &= y + x^2, & zt &= yx + x^3, \\ & & t^2 &= x^2, & tz &= xy + x^3, \\ & & & & t^3 &= x^3, \end{aligned}$$

we obtain that

$$\begin{aligned} x^* &= t^*, & y^* &= z^*, & y^*x^* &= z^*t^*, \\ & & x^*x^* &= t^*t^* + z^*, & x^*y^* &= t^*z^*, \\ & & & & x^*x^*x^* &= t^*t^*t^* + t^*z^* + z^*t^*. \end{aligned}$$

Take care that with this notation $x^* = t^*$ and $y^* = z^*$ but $(t^*)^n \neq (x^*)^n$ in general. The coalgebra A^* identifies now to $T(W)$, where W is the linear span of z^* and t^*, equipped with the deconcatenation product (the cofree coalgebra over W with structure map π_W, mapping $y_1^* \dots y_n^*$ to 0 if $n \geqslant 2$ and maps y_1^* to y_1^*, where $y_1, \dots, y_n$ belong now to $\{t, z\}$).

The two graded vector spaces V and W identify, but we distinguish them notationally as the two cofree graded coalgebras $T^g(V)$ and $T^g(W)$ do not. This is the case in particular because the two projections π_V and π_W are different: for example, $\pi_W(x^*) = x^*$, $\pi_W(y^*) = z^*$, $\pi_W(x^*x^*) = z^*$ whereas $\pi_V(x^*) = x^*$, $\pi_V(y^*) = y^*$,$\pi_V(x^*x^*) = 0$. Conversely, $\pi_V(t^*) = x^*$, $\pi_V(z^*) = y^*$, $\pi_V(t^*t^*) = -y^*$. Both π_V and π_W are the null map on all words of degree greater or equal 3.

We also get

$$t^* \times t^* = 2t^*t^* + 2z^*$$

as

$$< t^* \times t^* | tt >=< t^* \otimes t^* | \Delta(tt) >=< t^* \otimes t^* | tt \otimes 1 + 2t \otimes t + 1 \otimes tt >= 2$$

and

$$< t^* \times t^* | z >=< t^* \otimes t^* | \Delta(z) >=< t^* \otimes t^* | z \otimes 1 + 2t \otimes t + 1 \otimes z >= 2.$$

One obtains thus using π_W another graded commutative B_∞-algebra structure $\langle -, - \rangle'$ on W which however is *not* as a Hopf algebra the shuffle algebra over t^* and z^*. We indeed obtain that $\langle t^*, t^* \rangle' = 2z^*$, and $\langle -, - \rangle'$ is 0 on all other pairs of words in t^* and z^*. This illustrates the general idea that a cofree graded commutative Hopf algebra structure on a graded vector space H is actually obtained as a graded connected commutative Hopf algebra structure *plus* the choice of a basis of words making it a cofree coalgebra over the corresponding letters (up to the choice of a basis of $Prim(H)$). There are infinitely many such choices (the basis of $Prim(H)$ being fixed), as our example implies.

The change of basis in A from words in x, y to words in t, z is obtained simply by substituting t for x and $z - tt$ for y (and conversely x for t and $y + xx$ for z). The change of basis in A^* from words in x^*, y^* to words in t^*, z^* is slightly more delicate but follows directly from Lemma 3.3: it is given by

$$y_1^* \dots y_n^* \longmapsto \sum_{k \in \mathbb{N}^*} \pi_W^{\otimes k} \circ \overline{\Delta}_k(y_1^* \dots y_n^*)$$

(with the y_i in $\{x, y\}$) and the inverse map by

$$y_1^* \dots y_n^* \longmapsto \sum_{k \in \mathbb{N}^*} \pi_V^{\otimes k} \circ \overline{\Delta}_k(y_1^* \dots y_n^*)$$

(with the y_i in $\{t, z\}$). For example, using this rule,

$$\begin{aligned} x^*y^*x^*x^*y^* &= \pi_W^{\otimes 4}(x^* \otimes y^* \otimes x^*x^* \otimes y^*) + \pi_W^{\otimes 5}(x^* \otimes y^* \otimes x^* \otimes x^* \otimes y^*) \\ &= t^*z^*z^*z^* + t^*z^*t^*t^*z^*. \end{aligned}$$

7 Shuffle and quasi-shuffle Hopf algebra

Let $\cdot$ be an associative, not necessarily unitary, product on a vector space V. We extend it to a B_∞-structure on V by putting, for any words $w, w' \in T(V)$,

$$\langle w, w' \rangle = \begin{cases} 0 \text{ if } w = w' = 1, \\ w' \text{ if } w = 1 \text{ and } \ell(w') = 1, \\ w \text{ if } \ell(w) = 1 \text{ and } w' = 1, \\ w \cdot w' \text{ if } \ell(w) = \ell(w)' = 1, \\ 0 \text{ otherwise,} \end{cases}$$

where $\ell(w)$ stands for the length of the word w. The associated product on $T(V)$ is called the quasi-shuffle product $\uplus$. It makes $T(V)$ a Hopf algebra called the quasi-shuffle Hopf algebra over V. It is commutative, if, and only if, $\cdot$ is commutative. In the particular case where $\cdot = 0$, we obtain the shuffle product $\sqcup\!\sqcup$ (see also Remark 4.1).

The quasi-shuffle product is classically inductively (and equivalently) defined by the equations $v \uplus 1 = 1 \uplus v = v$ and

$$\begin{aligned} v_1 \dots v_k \uplus v_{k+1} \dots v_{k+l} := v_1(v_2 \dots v_k \uplus v_{k+1} \dots v_{k+l}) \\ + v_{k+1}(v_1 \dots v_k \uplus v_{k+2} \dots v_{k+l}) \\ + (v_1 \cdot v_{k+1})(v_2 \dots v_k \uplus v_{k+2} \dots v_{k+l}), \end{aligned}$$

which restricts to

$$\begin{aligned} v_1 \dots v_k \sqcup\!\sqcup v_{k+1} \dots v_{k+l} := v_1(v_2 \dots v_k \sqcup\!\sqcup v_{k+1} \dots v_{k+l}) \\ + v_{k+1}(v_1 \dots v_k \sqcup\!\sqcup v_{k+2} \dots v_{k+l}), \end{aligned}$$

for the shuffle product. See also e.g. [15] for details on shuffle and quasi-shuffle algebras and their relationships, to be generalized below in the present article.

As already mentioned, in a joint article with J.-Y. Thibon, we investigated and classified more generally natural deformations of the shuffle Hopf algebra structure $(T(A), \sqcup\!\sqcup, \Delta)$ which can be defined on the space of tensors over a commutative algebra A (where natural means functorial) [15]. These deformations, of which the quasi-shuffle Hopf algebra over A is the most important example, are parameterized by formal power series. The logarithmic and exponential series describe the isomorphism between $(T(A), \sqcup\!\sqcup, \Delta)$ and $(T(A), \uplus, \Delta)$. To each such non trivial deformation corresponds a commutative B_∞-algebra structure on A. We refer the reader to our article for definitions and details.

Commutative quasi-shuffle algebras are at the moment the most important example of commutative B_∞-algebras. They can be used to encode certain relations between multi-zeta values and generalizations of them, see [22, 21] for a review on this topic. They play a prominent role in the study of Rota-Baxter algebras [18, 19, 5]. They are used not only in combinatorics and algebra as they have for example applications also in stochastics, where they allow to better understand the equivalence between Itô and Stratonovich integrals, resp. solutions of stochastic differential equations (Itô calculus being encoded by quasi-shuffle algebras and Stratonovich's by shuffle algebras) [6, 7].

Assume now that $V = \bigoplus_{n \geqslant 1} V_n$ is a graded vector space and let $\cdot$ be an associative, commutative and graded product on V. The standard example is the case where V is the semigroup ring of a positively graded commutative semigroup. Direct inspection shows that $(T^g(V), \uplus, \Delta)$ is a cofree graded

commutative Hopf algebra, making V a graded commutative B_∞-algebra. We refer to Hoffman's [20] for details on graded quasi-shuffle algebras and their relationship with graded shuffle algebras. Both structures were used recently in [2] to investigate Itô to Stratonovich transformations in the context of branched rough paths.

8 The gebra of descents

Let us first introduce briefly the context of the following results. Together with section 6 they will serve as a benchmark for our later developments on graded commutative cofree Hopf algebras.

A classical and fundamental theorem by C. Malvenuto in the theory of free Lie algebras, symmetric group representations and symmetric functions asserts that the dual of the Hopf algebra of descents in symmetric groups, whose definition is recalled below, is isomorphic to the Hopf algebra of quasi-symmetric functions [24, 25]. There are several ways to understand this isomorphism, but one of them is particularly important for our later purposes.

The Hopf algebra of descents *Desc* is naturally graded, cocommutative. As an algebra it is a free associative algebra over various sets of integer-graded generators. The exponential map actually allows to move between families of primitive generators to families of group-like generators. When dualizing, choosing a family of primitive generators amounts to considering the dual as equipped with a shuffle Hopf algebra structure, in the sense that the formulas for the product in the corresponding basis are the usual formulas for shuffle products. Choosing group-like generators instead amounts to equip the dual with a specific graded commutative cofree Hopf algebra structure: up to isomorphism, the one of the quasi-shuffle algebra over the semigroup algebra over the positive integers. A change of basis in *Desc* and its graded dual appears therefore as a prototype example for Hoffman's isomorphisms between shuffle and quasi-shuffle Hopf algebras, but there is more to be learned from that example, as we show now.

Let us explain the technical content of these ideas more precisely and recall the definition of the Hopf algebra of descents and some of its key properties, relevant to the present article. The reader is referred to [4, Chap. 5], from which the following definitions and properties are taken, for more details.

Definition 8.1 (Descent sets of permutations)**.** *A permutation σ in the n-th symmetric group S_n is said to have a descent in position $i < n$ if and only if $\sigma(i) > \sigma(i+1)$. The set of descents of a permutation is denoted $desc(\sigma)$:*

$$desc(\sigma) := \{i < n, \sigma(i) > \sigma(i+1)\}.$$

To each subset $S = S' \sqcup \{n\}$ of $[n]$ containing n are associated two elements in the group algebra $\mathbb{Q}[S_n]$:

$$De_{=S} := \sum_{\substack{\sigma \in S_n \\ desc(\sigma) = S'}} \sigma \ , \qquad De_S := \sum_{\substack{\sigma \in S_n \\ desc(\sigma) \subset S'}} \sigma.$$

The De_S (resp. the $De_{=S}$), $S = S' \sqcup \{n\} \subset [n]$ are linearly independent in $\mathbb{Q}[S_n]$. They have the same linear span, written *Desc*.

Let now $X = \{x_1, \dots, x_n, \dots\}$ be a countable alphabet and $(T(X), \cdot, \Delta_{\sqcup\!\sqcup})$ the tensor gebra over X equipped with a graded connected cocommutative Hopf algebra structure by the concatenation product $\cdot$ and the unshuffle coproduct $\Delta_{\sqcup\!\sqcup}$ (the coproduct dual to the shuffle product of words:

$$\Delta_{\sqcup\!\sqcup}(y_1 \dots y_n) := \sum_{I \amalg J = [n]} y_I \otimes y_J$$

where, if $I = \{i_1, \dots, i_k\}, y_I := y_{i_1} \dots y_{i_k}$). Permutations in S_n act on $T_n(X)$ on the right by permutation of the letters of words of length n ($\sigma(y_1 \dots y_n) = y_{\sigma(1)} \dots y_{\sigma(n)}$ where $y_i \in X$, $i \leqslant n$). The convolution product $*$ of linear endomorphisms of $T(X)$, $f * g := \cdot \circ (f \otimes g) \circ \Delta_{\sqcup\!\sqcup}$) induces then a graded algebra structure on the direct sum of linear spans of the symmetric groups $\bigoplus_{n \in \mathbb{N}} \mathbb{Q}[S_n]$ that restricts to a graded algebra structure on *Desc* (elements in S_n being of degree n). This is the standard way to connect the combinatorics of descents with the theory of free Lie algebras [36].

From the action of *Desc* on $T(X)$ (by a process that holds actually more generally for all graded connected cocommutative Hopf algebras), one can derive the existence of a graded cocommutative Hopf algebra structure on *Desc* and one gets the Theorem:

Theorem 8.2. *The descent algebra Desc is a graded cocommutative Hopf algebra, freely generated as a unital associative algebra by any of the following families:*

- *The identity permutations 1_n in the groups S_n, $n \geqslant 1$, that form a group-like family – that is to say, for any $n \in \mathbb{N}$,*

$$\Delta_{\sqcup\!\sqcup}(1_n) = \sum_{k=0}^{n} 1_k \otimes 1_{n-k},$$

- *The Dynkin operators*

$$Dyn_n = \sum_{i=0}^{n-1} (-1)^i De_{=\{1,\dots,i\} \cup \{n\}},$$

which are primitive elements,

- *Solomon's Eulerian idempotents*

$$Sol_n := e_n^1 = \sum_{S\subseteq[n-1]} \frac{(-1)^{|S|}}{n} \binom{n-1}{|S|}^{-1} De_{=S\cup\{n\}},$$

which are primitive elements.

The Theorem implies that the family of Dynkin operators and the family of Eulerian idempotents define a standard free-Lie type Hopf algebra structure on *Desc*: they are primitive elements and freely generate *Desc* as an associative algebra. The graded dual Hopf algebra $Desc^*$ is a shuffle Hopf algebra in the corresponding dual basis of words.

This is a general phenomenon: any family of Lie idempotents (elements in the descent algebra that project the tensor algebra onto the free Lie algebra) would define a unshuffle Hopf algebra structure on *Desc*, and there are infinitely many of them. On Lie idempotents and their generalization to arbitrary connected cocommutative Hopf algebras, see [31, 32, 33].

On the other hand, the family of the identity elements in symmetric groups defines a non standard free-Lie type Hopf algebra structure on *Desc*. Dualizing, one gets a graded commutative B_∞-algebra structure on the space $Prim(Desc^*)$ whose associated Hopf algebra is not a shuffle Hopf algebra over the cogenerating vector space W^*, where W is the linear span of the identity elements in symmetric groups and W^* its graded dual. The graded dual of *Desc* is indeed the quasi-shuffle Hopf algebra over W^* and identifies with the Hopf algebra of quasi-symmetric functions — see [24, 25].

Choosing a family of generators of *Desc* that would not be primitive nor group-like would lead to a graded commutative B_∞-algebra structure on $Prim(Desc^*)$ whose associated Hopf algebra would not be a shuffle Hopf algebra nor a quasi-shuffle Hopf algebra in the corresponding basis. This follows easily from the same references and can be checked directly.

9 The B_∞-algebra structure on finite topologies

Let us detail now a particularly meaningful example: finite topologies (see [13]). The example illustrates how, in a non trivial situation, one can prove that a commutative Hopf algebra can be equipped with a cofree coalgebra structure. This also illustrates on a concrete example the notion of Schur-Weyl categories of bialgebras, about which we refer again to [13].

Notice that the following developments do not require the assumption that structures are graded (the Hopf algebra of finite topologies is canonically graded, but same method would hold,*mutatis mutandis*, for a general conilpotent Hopf algebra).

Let E be a finite set. A topology on E is a set T of subsets of E such that:

- $\varnothing$, $E \in T$.
- If $A, B \in T$, then $A \cup B \in T$ and $A \cap B \in T$.

By Alexandroff's theorem [1], given a finite set E, the set of topologies on E is in one-to-one correspondence with quasi-orders on E, that is to say transitive and reflexive relations on E: given such a relation $\leqslant$ on E, the topology $T_{\leqslant}$ associated to $\leqslant$ is the set of subsets $O \subseteq E$ such that

$$\forall x, y \in E, \qquad x \in O \text{ and } x \leqslant y \Longrightarrow y \in O.$$

Conversely, if T is a topology on a finite set E, it gives rise to a quasi-order on E defined by

$$\forall x, y \in E, \qquad x \leqslant_T y \iff \text{ any } O \in T \text{ containing } x \text{ also contains } y.$$

If $\leqslant$ is a quasi-order on E, we define an equivalence relation on E by

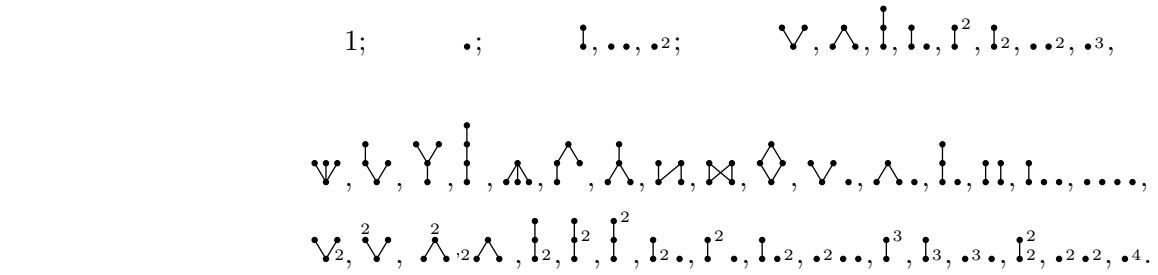
$$x \sim_{\leqslant} y \iff x \leqslant y \text{ and } y \leqslant x.$$

Thus, $E/\sim_{\leqslant}$ inherits an order $\overline{\leqslant}$, defined by

$$\overline{x} \overline{\leqslant} \overline{y} \iff x \leqslant y.$$

We will further on abbreviate $\sim_{\leqslant_T}$ to $\sim_T$.

In the sequel, we shall represent isoclasses of finite topologies by the Hasse graphs of $(E/\sim_T, \overline{\leqslant_T})$, with indices representing the cardinalities of the classes of $\sim_{\leqslant_T}$, when these cardinalities are not equal to 1. Here are finite topologies of cardinality $\leqslant 4$:



By convention, the edges in these graphs are oriented upwards.

The Hopf algebra H_T of quasi-orders, or of finite topologies [12, 13] has for basis the set of (isoclasses) of finite topologies. Its product, that we will write m further on, is given by the disjoint union: if T and T' are two topologies on respective sets E and E', then TT' is a topology on the set $E \sqcup E'$, with

$$TT' = \{O \sqcup O' \mid O \in T,\, O' \in T\}.$$

The Hasse graph of TT' is the disjoint union of the Hasse graphs of T and T'. This product is commutative, and its unit is the unique topology denoted 1 on $\varnothing$. The coproduct Δ is defined on any finite topology T on a set E by

$$\Delta(T) = \sum_{O \in T} T_{|E\setminus O} \otimes T_{|O}.$$

Example 9.1.

$$\Delta(\bullet) = \bullet \otimes 1 + 1 \otimes \bullet,$$
$$\Delta(\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}) = \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \otimes 1 + 1 \otimes \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \bullet \otimes \bullet,$$
$$\Delta(\vee) = \vee \otimes 1 + 1 \otimes \vee + \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \otimes \bullet + \bullet \otimes \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \bullet \otimes \bullet\bullet,$$
$$\Delta(\wedge) = \wedge \otimes 1 + 1 \otimes \wedge + \bullet \otimes \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \bullet \otimes \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \bullet\bullet \otimes \bullet,$$
$$\Delta(\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}) = \begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix} \otimes 1 + 1 \otimes \begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix} + \bullet \otimes \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \otimes \bullet.$$

The Hopf algebra H_T is equipped with an extra structure, the ordinal sum [38, 3]: if T and T' are two topologies on respective finite sets E and E', then $T \downarrow T'$ is a topology on $E \sqcup E'$ defined by

$$T \downarrow T' = T' \cup \{O \sqcup E' \mid O \in T\}.$$

The Hasse graph of $T \downarrow T'$ is obtained by adding an edge from any maximal vertex of the Hasse graph of T to any minimal vertex of the Hasse graph of T'. For example,

$$\bullet \downarrow \bullet = \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}, \quad \bullet \downarrow \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} = \begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}, \quad \bullet \downarrow \bullet\bullet = \vee, \quad \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} \downarrow \bullet = \begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}, \quad \bullet\bullet \downarrow \bullet = \wedge.$$

The product $\downarrow$ is linearly extended to H_T, making it an associative algebra, which unit is again 1. Moreover, by definition of $T \downarrow T'$, for any $x, y \in H_T$,

$$\Delta(x \downarrow y) = (x \otimes 1) \downarrow \Delta(y) + \Delta(x) \downarrow (1 \otimes y) - x \otimes y.$$

Recall that an infinitesimal bialgebra in the sense of [23] is an associative unital algebra with product $\downarrow$ and a coassociative counital coalgebra with coproduct Δ and coaugmentation the unit of $\downarrow$ such that furthermore the previous identity is satisfied. The triple $(H_T, \downarrow, \Delta)$ is thus an infinitesimal bialgebra.

Let us survey some of the properties of these bialgebras with a view towards applications to B_∞ structures. Complementary insights can be found in [13]. The main example of such objects are the tensor gebras $T(V)$, with the concatenation product that we write from now on m_{conc} and the deconcatenation coproduct Δ. In fact, in the conilpotent case, these are the unique examples:

Proposition 9.1. *Let $H = (H, \downarrow, \Delta)$ be an infinitesimal bialgebra. The following map is an injective map of infinitesimal bialgebra:*

$$\Theta : \begin{cases} (T(Prim(H)), m_{conc}, \Delta) & \longrightarrow & (H, \downarrow, \Delta) \\ v_1 \dots v_k & \longmapsto & v_1 \downarrow \dots \downarrow v_k. \end{cases}$$

It is an isomorphism if and only if, the coalgebra (H, Δ) is conilpotent.

Proof. The map Θ is obviously an algebra map. An easy induction on k proves that for any $v_1, \dots, v_k \in Prim(H)$, in H,

$$\Delta(v_1 \downarrow \dots \downarrow v_k) = \sum_{i=0}^{k} v_1 \downarrow \dots \downarrow v_i \otimes v_{i+1} \downarrow \dots \downarrow v_k,$$

so Θ is a coalgebra morphism.

Let us assume that Θ is not injective. Let us consider $w \in \mathrm{Ker}(\Theta)$, non zero. There exists $n \geqslant 1$ such that $w \in \bigoplus_{k=1}^{n} Prim(H)^{\otimes k}$. Let us choose w such that n is minimal. Then the restriction of Θ to $\bigoplus_{k=1}^{n-1} Prim(H)^{\otimes k}$ is injective. Moreover,

$$\begin{aligned} 0 &= \Delta \circ \Theta(w) \\ &= (\Theta \otimes \Theta) \circ \Delta(w) \\ &= \Theta(w) \otimes 1 + 1 \otimes \Theta(w) + (\Theta \otimes \Theta)\overline{\Delta}(w) \\ &= (\Theta \otimes \Theta)\overline{\Delta}(w). \end{aligned}$$

Observing that

$$\overline{\Delta}(w) \in \left(\bigoplus_{k=1}^{n-1} Prim(H)^{\otimes k} \right)^{\otimes 2},$$

we obtain that $\overline{\Delta}(w) = 0$, so $\Delta(w) = w \otimes 1 + 1 \otimes w$ and finally $w \in Prim(H)$. Therefore, $\Theta(w) = w = 0$, which is a contradiction. So Θ is injective.

If Θ is surjective, then the coalgebras $(T(Prim(H)), \Delta)$ and (H, Δ) are isomorphic, so H is conilpotent. Let us assume that H is conilpotent and let us prove that Θ is surjective. Let $x \in Ker\overline{\Delta}_n$ in H, with $n \geqslant 2$, and let us prove that $x \in \mathrm{Im}(\Theta)$ by induction on n.

If $n = 2$, then $x \in Prim(H)$ and $x = \Theta(x)$. Let us assume the result at rank n and let $x \in Ker\overline{\Delta}_{n+1}$. Then, as $\overline{\Delta}$ is coassociative, $\overline{\Delta}_n(x) \in \mathrm{Ker}(\overline{\Delta})^{\otimes n} = Prim(H)^{\otimes n}$. Let us put

$$\overline{\Delta}_n(x) = \sum_{i=1}^{k} v_{i,1} \otimes \dots \otimes v_{i,n}.$$

Then, if $w = \sum_{i=1}^{k} v_{i,1} \dots v_{i,n}$, $\overline{\Delta}_n \circ \Theta(w) = \overline{\Delta}_n(x)$, so the induction hypothesis applies to $x - \Theta(w)$. Therefore, $x - \Theta(w) \in \mathrm{Im}(\Theta)$ and finally $x \in \mathrm{Im}(\Theta)$. □

Proposition 9.2. *Let $(H, \downarrow, \Delta)$ be a conilpotent infinitesimal bialgebra. Then*

$$\bar{H} = \mathrm{Prim}(H) \oplus \bar{H} \downarrow \bar{H},$$

and the projection π on $\mathrm{Prim}(H)$ in this direct sum is given by

$$\pi = \sum_{k=1}^{\infty} (-1)^{k+1} \downarrow^{(k-1)} \circ \overline{\Delta}^{(k-1)}.$$

Proof. By Proposition 9.1, it is enough to prove it for $H = (T(V), m_{conc}, \Delta)$. Let $n \geqslant 1$ and $w \in V^{\otimes n}$.

$$\begin{aligned}
\pi(w) &= \sum_{k \geqslant 1} \sum_{\substack{w = w_1 \dots w_k, \\ w_1, \dots, w_k \neq 1}} (-1)^{k+1} w \\
&= \left(\sum_{k \geqslant 1} \sum_{\substack{w = w_1 \dots w_k, \\ w_1, \dots, w_k \neq 1}} (-1)^{k+1} \right) w \\
&= \left(\sum_{I \subseteq [n-1]} (-1)^{|I|} \right) w \\
&= \delta_{n,1} w.
\end{aligned}$$

Here, I represents the places where the word w is cut, where $i \in [n-1]$ reflects a cut between the letter i and $i+1$ of w. Therefore, π is a projection on $V = \mathrm{Prim}(T(V))$, which vanishes on $T(V)_+ \downarrow T(V)_+ = \bigoplus_{n \geqslant 2} V^{\otimes n}$. □

Remark 9.1. By Takeuchi's formula [39], $\pi + \varepsilon$ is in fact the opposite of the antipode of $(H, \downarrow, \Delta)$, where by antipode is meant the convolution inverse of the identity map for the convolution product on linear endomorphisms induced by the coproduct Δ and the product$\downarrow$. This also follows more abstractly from the implicit definition of π acting on $H = (T(V), m_{conc}, \Delta)$ by the equation $\mathrm{Id} - \varepsilon = \pi * \mathrm{Id}$ or, equivalently, $(-\pi - \varepsilon) * \mathrm{Id} = \varepsilon$.

Therefore, if (H, m, Δ) is a commutative Hopf algebra, with an extra product $\downarrow$ such that $(H, \downarrow, \Delta)$ is an infinitesimal bialgebra, it is cofree as a coalgebra, where one can use the map π as projector onto $Prim(H)$. This is

the case for H_T. For example,

$\pi($$) =$ $,$ $\pi($$) = 0,$ $\pi($$) =$ $-\, 2$$,$

$\pi($$) = 0,$ $\pi($  $) = 0,$ $\pi($$) = 0,$

$\pi($$) =$ $-$ $-$  $+$ $,$

$\pi($$) =$ $- 3$  $- 3$$+ 6$$.$

This projection induces a commutative B_∞ structure on $Prim(H_T)$, given by

$$\langle x_1 \otimes \ldots \otimes x_k, y_1 \otimes \ldots \otimes y_l \rangle = \pi((x_1 \downarrow \ldots \downarrow x_k)(y_1 \downarrow \ldots \downarrow y_l)),$$

where $x_1, \ldots, x_k, y_1, \ldots, y_l \in Prim(H_T)$. For example,

$\langle$$,$$\rangle = \pi($$) =$ $- 2$$,$

$\langle$$\otimes$$,$$\rangle = \langle$$,$$\otimes$$\rangle = \pi($$) =$ $-$ $-$ $+$ $,$

$\langle$$- 2$$,$  $\rangle = \langle$  $,$$- 2$$\rangle = \pi($$- 2$$) =$ $- 2$$-$ $-$ $+ 4$$.$

10 Free cocommutative graded Hopf algebras and their duals

Recall the Poincaré-Birkhoff-Witt (PBW) theorem (we refer again to [4] for details on the materials that follow). Let L be a graded and reduced Lie algebra (reduced meaning that it has no component in degree 0). Its enveloping algebra H is a graded connected cocommutative Hopf algebra and there is a canonical morphism ι from the space $S(L)$ of symmetric tensors over L to H. This map is a coalgebra isomorphism and the decomposition of $S(L)$ according to tensor degrees induces a decomposition of the graded components of the enveloping algebra: $H_n = \bigoplus_{k \leqslant n} H_n \cap \iota(S^k(L))$, where we write $S^k(L)$ for the space of symmetric tensors in $L^{\otimes k}$.

The gebra of tensors $T(X)$ over a set X is, when equipped with the concatenation product and the unshuffle coproduct $\Delta_{\sqcup\!\sqcup}$, the enveloping algebra of the free Lie algebra over X. One of the key properties of the Eulerian idempotents is that they project $T(X)$ to the free Lie algebra over X according to the decomposition of $T(X)$ induced by the Poincaré-Birkhoff-Witt theorem (this is actually how Solomon defined them originally).

This idea was generalized to arbitrary graded connected cocommutative or commutative Hopf algebras H in [29, Th. I,5,6 and Th. I,6,4], see also

[27, 28, 4]. In the cocommutative case, this leads to the definition of generalized Eulerian idempotents that project on the graded components of primitive part of the Hopf algebra in agreement with the Poincaré-Birkhoff-Witt decomposition. We will still write e_n^1 for the generalized Eulerian idempotent, acting on H_n as a projector from H_n to $H_n \cap Prim(H)$. Notice that these generalized idempotents *cannot* be constructed in general as elements of the symmetric group algebras. In particular they are not directly governed by the statistics of descents in symmetric groups — the very reason for the name "Eulerian idempotents" used in the classical case. This is the reason why they should be preferably called "canonical idempotents" — we use below the two terminologies indifferently.

Lemma 10.1. *Given $x \in H_n$, where H is a graded connected cocommutative Hopf algebra, the PBW theorem induces a unique decomposition $x = e_n^1(x) + y$, where $e_n^1(x)$ is a primitive element in H_n and $y \in \bigoplus_{k>1} H_n \cap \iota(S^k(L))$. In particular, it follows from the definition of ι, the PBW isomorphism, that y can be expanded as a sum of products of elements in $\bigoplus_{k<n} H_k$.*

More generally, one can show [4, Th. 5.2.1] that any primitive element ψ_n in the n-th graded component of the descent algebra ($\psi_n \in Prim_n(Desc) := Prim(Desc) \cap Desc_n$) defines a projector from H_n onto $Prim(H) \cap H_n$ provided the coefficient of $1_n \in S_n$ in the expansion of ψ_n, viewed as an element of the group algebra of S_n, is 1 in the basis of permutations. We call a family $(\psi_n)_{n\in\mathbb{N}^*}$ of such projectors a Lie idempotent family. The Dynkin idempotents Dyn_n/n and the Klyachko idempotents provide, together with the Eulerian idempotents, classical examples of such families. There are infinitely many as any convex combination of Lie idempotent families is a Lie idempotent family — this follows from their characterization in terms of the coefficients of $1_n \in S_n$ in their expansion.

It is easy to show using the structure properties of the Hopf algebra of descents that Lemma 10.1 generalizes to these families $(\psi_n)_{n\in\mathbb{N}^*}$ in the following way: given $x \in H_n$, $x = \psi_n(x) + y$, where $\psi_n(x)$ is a primitive element in H_n and y can be expanded as a sum of products of elements in $\bigoplus_{k<n} H_k$.

Theorem 10.2 (Structure theorem for free-Lie-type Hopf algebras)**.** *Let H be a free-Lie-type Hopf algebra with freely generating subspace W. Then, for any Lie idempotent family $(\psi_n)_{n\in\mathbb{N}^*}$, H is a standard free-Lie-type Hopf algebra (an unshuffle Hopf algebra) over the freely generating subspace $W' := \bigoplus_{n\in\mathbb{N}^*} \psi_n(W_n)$. In particular, a free-Lie-type Hopf algebra is always naturally the enveloping algebra of a free Lie algebra over a generating subspace W', and any Lie idempotent family gives rise to such a subspace.*

Proof. Indeed, W freely generates H as a free associative algebra. Let us choose a graded basis $(b_1, \ldots, b_n, \ldots)$, ordered in such a way that $|b_{i+1}| \geqslant |b_i|$. By Lemma 10.1, $b_i = \psi_n(b_i) + r$, where r belongs to the free associative algebra generated by the b_j, $j < i-1$. By a standard triangularity argument, $\psi_n(b_i)$ is freely independent from the b_j, $j < i$ and from the $\psi_n(b_j)$, $j < i$, and the $\psi_n(b_i)$ freely generate H. The Lemma follows. □

The theorem can be recast in categorical terms. Let V a graded vector space together with an isomorphism $\phi : V \to W$. We write ϕ_n for the degree n component of the isomorphism. The data (H, W) of a free-Lie type Hopf algebra are equivalent to the data (H, V, ϕ) of a graded cocommutative Hopf algebra structure on H together with the linear injection (still written) ϕ from V into H that induces an algebra isomorphism $T(V) \cong H$. We call (H, V, ϕ) a presentation of the free-Lie type Hopf algebra (H, W).

Theorem 10.3 (Structure theorem 10.2, categorical formulation)**.** *Let (H, V, ϕ) be a presentation of a free-Lie type Hopf algebra H. Then, for any Lie idempotent family $(\psi_n)_{n\in\mathbb{N}^*}$, $(H, V, \bigoplus_{n\in\mathbb{N}^*} \psi_n \circ \phi_n)$ is a presentation of a standard free-Lie-type Hopf algebra structure on H.*

By duality (hereafter in this section all duals are graded duals and all graded vector spaces are locally finite), we immediately get:

Theorem 10.4 (Structure theorem for cofree graded commutative Hopf algebras)**.** *Let H^* be a cofree graded commutative Hopf algebra with structure map $\phi^* : H^* \to Prim(H^*)$. Then, for any Lie idempotent family $(\psi_n)_{n\in\mathbb{N}^*}$, the structure map $\pi := \sum_{n\in\mathbb{N}^*} \phi_n^* \circ \psi_n^*$ equips H^* with another, isomorphic, cofree coalgebra structure over $Prim(H^*)$. Furthermore, the structure map π induces a Hopf algebra isomorphism with the shuffle Hopf algebra $H^* \cong (T^g(Prim(H^*)), ⧢, \Delta)$.*

The last sentence follows from Theorem 10.2 and the fact that the graded dual Hopf algebra of $(T^g(Prim(H^*)), ⧢, \Delta)$ is the enveloping algebra of the free Lie algebra over the dual of $Prim(H^*)$.

The theorem can be re-expressed in the language of B_∞-algebras.

Theorem 10.5 (Structure theorem for graded commutative B_∞-algebras)**.** *Let V^* be a graded commutative B_∞-algebra and $(T^g(V^*), *, \Delta)$ the associated cofree graded commutative Hopf algebra. Let $(\psi_n)_{n\in\mathbb{N}^*}$ be a Lie idempotent family. Then, the structure map $\sum_{n\in\mathbb{N}^*} \pi_V \circ \psi_n^*$ from $T^g(V^*)$ to V^* induces a Hopf algebra isomorphism $(T^g(V^*), *, \Delta) \cong (T^g(V^*), ⧢, \Delta)$.*

When the Lie idempotent family is the Eulerian family, Theorem 10.2 was obtained in [34] (Lemma 22) and Theorem 10.5 by Bellingeri, Ferrucci and Tapia in [2] (Remark 3.5). This case is important as the Eulerian family is the only family of classical Lie idempotents that generalizes in the non graded case, see the next Section 11.

11 Free cocommutative complete Hopf algebras and their duals

Recall from [35, 16] and [4, Sect. 4.6] that Cartier's structure theorem for graded connected cocommutative Hopf algebras still holds in the complete case. For a complete commutative Hopf algebra $H = \lim\limits_{\leftarrow} H/H(n)$, where $H(n)$ denotes the n-component of the decreasing filtration of H and where it is assumed that $H(0)/H(1) \cong K$, it holds that there is a Hopf algebra isomorphism

$$H \cong \hat{U}(Prim(H)),$$

where $\hat{U}$ denotes the enveloping algebra functor for complete Lie algebras. Moreover, $e^1 = \log^*(\mathrm{Id}_H)$ is well-defined and maps H onto $Prim(H)$ [4, Lemma 4.6.1].

Let us assume furthermore that H is a graded-complete Hopf algebra, that is (by definition, and in the context of the present article) a complete Hopf algebra such that

1. $H = \prod\limits_{n\in\mathbb{N}} H_n$ with $H_0 = \mathbb{K}$, the ground field (so that $H(n) = \prod\limits_{m\geqslant n} H_n$);
2. the associative product on $\prod\limits_{n\in\mathbb{N}} H_n$ is obtained as the completion of a graded algebra structure on $\bigoplus\limits_{n\in\mathbb{N}} H_n$;
3. as a coalgebra, H is graded-complete, that is, the coproduct Δ is obtained from maps:

$$\Delta_n : H_n \to \prod_{p+q\geqslant n} H_p \otimes H_q.$$

In this setting, e^1 maps H_n to $\prod\limits_{p\geqslant n} H_p$.

We say that such a H is free if, as an algebra, $\bigoplus\limits_{n\in\mathbb{N}} H_n$ is a free graded associative algebra over a finite or infinite (but countable) set $S = \{s_1, \dots, s_n, \dots\}$ of graded generators, where we assume that these generators are degree-ordered ($|s_{i+1}| \geqslant |s_i|$) and that there is a finite number of generators in each degree. Thus, $H = K\langle\langle s_1, \dots, s_n, \dots\rangle\rangle$, the algebra of noncommutative formal power series over S. We say that such a H is *standard* if furthermore the generators s_i are primitive elements, so that, in that case, H identifies with the completion of the enveloping algebra of the free Lie algebra over the s_i.

Theorem 11.1. *Let $H = K\langle\langle s_1, \dots, s_n, \dots\rangle\rangle$ be a free cocommutative graded-complete Hopf algebra, then H is canonically isomorphic to the standard free cocommutative graded-complete Hopf algebra $K\langle\langle e^1(s_1), \dots, e^1(s_n), \dots\rangle\rangle$.*

Proof. Indeed, for $s_i \in H_{n_i}$, $e^1(s_i)$ decomposes as $s_i + r_i^{(1)} + r_i^{(2)}$, where $|r_i^{(1)}| = n_i$ and $r_i^{(1)} \in K\langle\langle s_1, \ldots, s_{i-1}\rangle\rangle$ and $r_i^{(2)} \in \prod_{n\in\mathbb{N}} H_{n_i+n+1}$. Therefore, by a standard triangularity argument, the $e^1(s_i)$ are freely independent and generate H as a complete algebra. The Theorem follows as they are primitive elements. □

These results dualize as follows. Let us restate first the characterization of shuffle Hopf algebras (Lemma 4.5).

Lemma 11.2. *Let $(H, *, \Delta, \pi)$ be a cofree commutative Hopf algebra. Then, H is a shuffle algebra in the π basis if and only if π vanishes on $\bar{H} * \bar{H}$, the square of the augmentation ideal of $\bar{H}$.*

In particular, any surjection $\gamma : H \to Prim(H)$ that acts as the identity map on $Prim(H)$ and vanishes on $\bar{H} * \bar{H}$ defines a shuffle algebra structure on H in the γ basis.

Recall (details can be found in [4, Sections 2.10 and 3.3]) that to a commutative Hopf algebra H are classically associated a group and a Lie algebra: the group is the set of algebra maps from H to the ground field $\mathbb{K}$ equipped with the restriction of the convolution product on the algebra $End(H)$ of linear endomorphisms of H. The Lie algebra is the vector space of linear forms on $\bar{H}$ that vanish on $\bar{H} * \bar{H}$ or, equivalently, of linear forms on H that vanish on $\mathbb{K} + \bar{H} * \bar{H}$. These linear forms are usually called infinitesimal characters, their bracket is obtained as the bracket associated to the convolution product. This construction generalizes from linear forms to linear endomorphisms of H. A linear endomorphism of H that vanishes on $\mathbb{K} + \bar{H} * \bar{H}$ is called an infinitesimal endomorphism of H.

Definition 11.3. *An infinitesimal endomorphism ϕ of H is called tangent to identity if and only if its restriction to $Prim(H)$ is the identity map.*

We state the following Corollary of Lemma 11.2 as a Theorem in view of its meaningfulness for the theory.

Theorem 11.4. *Let $(H, *, \Delta, \pi)$ be a cofree commutative Hopf algebra and ϕ be a tangent to identity infinitesimal endomorphism of H. By Lemma 11.2, H is a shuffle Hopf algebra in the $\pi \circ \phi$ basis.*

When restated in the language of B_∞-algebras, the Theorem reads:

Theorem 11.5. *Let $\langle -, - \rangle$ be a commutative B_∞ structure on V with associated Hopf algebra $(T(V), *, \Delta)$. Let ϕ a tangent to identity endomorphism of $T(V)$ and let $\tilde{\omega}$ be the coalgebra automorphism of $(T(V), \Delta)$ induced by*

$\omega := \pi_V \circ \phi$:

$$\tilde{\omega} : \begin{cases} T(V) & \longrightarrow & T(V) \\ 1 & \longmapsto & 1, \\ w \in T_+(V) & \longmapsto & \displaystyle\sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1,\dots,w_k \neq \varnothing}} \omega(w_1) \dots \omega(w_k). \end{cases}$$

Then $\tilde{\omega}$ is a Hopf algebra isomorphism from $(T(V), *, \Delta)$ to $(T(V), \sqcup, \Delta)$.

The calculation of the inverse isomorphism follows from the computation of the inverse of a cofree coalgebra isomorphism in Eq. (2).

Proposition 11.6. *Let notation be as in Theorem 11.5. Let us define inductively ζ (by induction on the length of tensors) by*

$$\zeta : \begin{cases} T_+(V) & \longrightarrow & V \\ v \in V & \longmapsto & v, \\ w \in T_n(V),\ n \geqslant 2 & \longmapsto & \zeta(w) = -\displaystyle\sum_{k=2}^{n} \sum_{\substack{w=w_1 \dots w_k, \\ w_1,\dots,w_k \neq \varnothing}} \tilde{\omega}_*(\zeta(w_1) \dots \zeta(w_k)). \end{cases}$$

Let us then define

$$\tilde{\zeta} : \begin{cases} T(V) & \longrightarrow & T(V) \\ 1 & \longmapsto & 1, \\ w \in T_+(V) & \longmapsto & \displaystyle\sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1,\dots,w_k \neq \varnothing}} \zeta(w_1) \dots \zeta(w_k). \end{cases}$$

Then $\tilde{\zeta}$ is the Hopf algebra isomorphism from to $(T(V), \sqcup, \Delta)$ to $(T(V), *, \Delta)$ inverse to $\tilde{\omega}$ as defined in Theorem 11.5.

In the graded case we saw that any Lie idempotent family defines a shuffle Hopf algebra structure on a cofree graded commutative Hopf algebra. In the non graded case, the key idea to construct a universal tangent to identity endomorphism will be to use the extension of the definition of the Eulerian idempotents from the original case of the tensor Hopf algebra to the case where the Hopf algebra H is commutative and unipotent (being unipotent is a weaker hypothesis than being graded connected; it always holds when the Hopf algebra is conilpotent as a coalgebra). We have already used in previous works on quasi-shuffle algebras this fact that the constructions and proofs of structure results on graded connected cocommutative or commutative Hopf algebras in [29, 27, 28] can be extended to a broader setting as they actually only require the Hopf algebras to be unipotent — this observation was developed systematically in [4, Chap. 4], to which we refer for details and proofs. See in particular [4, Thm 4.4.1].

When these results are applied to a cofree commutative Hopf algebra H, the canonical (or generalized Eulerian) idempotent is thus defined on $\bar{H}$ by

$$e_*(h) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} m_k \circ \bar{\Delta}_k(h),$$

where we write m_k for the iterated product (from $\bar{H}^{\otimes k}$ to H). As in the graded case it projects onto a (canonically constructed) vector subspace $Q(H)$ of H that freely generates H as a commutative algebra. The projection is orthogonal to the square of $\bar{H}$, the augmentation ideal of H and acts as the identity on $Prim(H)$. It is a tangent to identity infinitesimal endomorphism of H.

The reader can find an explicit computation of e_* for quasi-shuffle Hopf algebras in terms of surjections in [26].

When applied to commutative B_∞-algebras, these results imply the Theorem:

Theorem 11.7. *Let $* \in P^c(V)$. The canonical idempotent e_* is defined by $e_*(1) = 0$ and for any non-empty word w of length n,*

$$e_*(w) = \sum_{k=1}^{n} \sum_{\substack{w=w_1 \dots w_k, \\ w_1,\dots,w_k \neq \varnothing}} \frac{(-1)^{k-1}}{k} w_1 * \dots * w_k.$$

Then e_* is a projector, vanishing on $T_+(V) * T_+(V)$, and for any $v \in V$, $e_*(v) = v$: it is a tangent to identity infinitesimal endomorphism of $T(V)$. Moreover, the image of e_* freely generates $(T(V), *)$ as a commutative algebra.

Furthermore, as the product of a B_∞-algebra is a morphism of filtered graded coalgebras,

$$e_*(T_n(V)) \subseteq \bigoplus_{k=1}^{n} T_k(V).$$

Proposition 11.8. *Let $* \in P^c(V)$. We set $\varpi_* := \pi_V \circ e_*$ and call ϖ the canonical commutative B_∞ idempotent. It acts as the identity map on V and sends any non-empty word w to*

$$\varpi_*(w) = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1,\dots,w_k \neq \varnothing}} \frac{(-1)^{k-1}}{k} \langle w_1, w_2 * \dots * w_k \rangle.$$

Proof. The map ϖ_* is indeed an idempotent since π_V is an idempotent and $e_* \circ \pi_V = \pi_V$. The explicit formula follows from the formula for e_* in Theorem 11.7 and the observation that, since $\langle w, w' \rangle = \pi_V(w * w')$,

$$\pi_V(w_1 * w_2 * \dots * w_k) = \pi_V(w_1 * (w_2 * \dots * w_k)) = \langle w_1, w_2 * \dots * w_k \rangle. \quad \square$$

Example 11.1. Let $v_1, v_2, v_3 \in V$.

$$\begin{aligned}
\varpi_*(v_1) &= v_1,\\
\varpi_*(v_1v_2) &= -\frac{1}{2}\langle v_1, v_2\rangle,\\
\varpi_*(v_1v_2v_3) &= -\frac{1}{2}\left(\langle v_1v_2, v_3\rangle + \langle v_1, v_2v_3\rangle\right) + \frac{1}{3}\langle v_1, v_2v_3 + v_3v_2 + \langle v_2, v_3\rangle\rangle.
\end{aligned}$$

Proposition 11.9. *When (H, π) is a cofree commutative Hopf algebra, define similarly the idempotent $\varpi_* := \pi \circ e_*$ and call it the canonical π-idempotent. The Hopf algebra H is then a shuffle Hopf algebra in the ϖ basis.*

Proof. The Proposition follows from the fact that e_* is a tangent to identity infinitesimal endomorphism and from Theorem 11.4. □

In the language of B_∞-algebras, the Proposition reads:

Proposition 11.10. *Let notation be as in Proposition 11.5 but set $\varpi := \pi_V \circ e_*$. Then, $\tilde{\varpi}$ is a Hopf algebra isomorphism from $(T(V), *, \Delta)$ to $(T(V), ⧢, \Delta)$.*

Remark 11.1. In the quasi-shuffle case $* = ⧆$, and the isomorphism and its inverse are known as Hoffman logarithm and exponential [20]. The logarithmic and exponential series also encode the isomorphism as a natural coalgebra automorphism, as we already explained [15]. They are thus given by

$$\varpi_*(v_1 \dots v_n) = \frac{(-1)^{n-1}}{n} v_1 \cdot \dots \cdot v_n.$$

$$\zeta(w) = \frac{1}{n!} v_1 \cdot \dots \cdot v_n,$$

for any $v_1, \dots, v_n \in V$, with $n \geqslant 1$, and where we used the notation of Proposition 11.6 for the inverse.

This can be recovered as follows. Given two non empty words w and w', $\langle w, w'\rangle$ vanishes in the quasi-shuffle case excepted when w and w' are both of length 1. Therefore, using also that the quasi-shuffle product of two words of length p and q is a linear combination of words of length at least $\max(p, q)$, we get

$$\begin{aligned}
\varpi_*(v_1 \dots v_n) &= \frac{(-1)^{n-1}}{n}\langle v_1, v_2 ⧆ \dots ⧆ v_n\rangle.\\
&= \frac{(-1)^{n-1}}{n}\langle v_1, v_2 \cdot \dots \cdot v_n\rangle\\
&= \frac{(-1)^{n-1}}{n} v_1 \cdot v_2 \cdot \dots \cdot v_n.
\end{aligned}$$

One can also check, using for example the identity of coefficients resulting from the formal power series expansion of the identity $\log \circ \exp(x) = x$, that Hoffman's formula for the inverse map

$$\zeta(w) = \frac{1}{n!} v_1 \cdot \ldots \cdot v_n,$$

indeed solves

$$\zeta(w) = -\sum_{k=2}^{n} \sum_{\substack{w=w_1 \ldots w_k, \\ w_1,\ldots,w_k \neq \varnothing}} \frac{(-1)^{k-1}}{k} \zeta(w_1) \cdot \ldots \cdot \zeta(w_k),$$

A similar analysis of the Hoffman isomorphism was performed in [2], in the particular case where the commutative algebra V underlying the construction of the quasi-shuffle Hopf algebra is the algebra of symmetric tensors over a vector space. See also our [15, 14] where more advanced insights on the Hoffman isomorphism and more generally on deformations of shuffle Hopf algebras can be found.

We conclude this section with a remark on the naturality of the constructions presented in the article. Morphisms of commutative B_∞-algebras are defined in the obvious way: given $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ two commutative B_∞-algebras, a linear map f from V to W is a commutative B_∞ morphism if and only if, for any $v_1, \ldots, v_{p+q}$ in V,

$$\langle F(v_1 \cdots v_p), F(v_{p+1} \cdots v_{p+q}) \rangle_W = f(\langle v_1 \otimes \cdots \otimes v_p, v_{p+1} \otimes \cdots \otimes v_{p+q} \rangle_V),$$

where F is defined by $F(v_1 \cdots v_i) := f(v_1) \otimes \cdots \otimes f(v_i)$.

We put $*_V = \Theta^{-1}(\langle -, - \rangle_V)$ and $*_W = \Theta^{-1}(\langle -, - \rangle_W)$ and let the reader check that if f is a commutative B_∞ morphism from V to W, F is a bialgebra morphism from $(T(V), *_V, \Delta)$ to $(T(W), *_W, \Delta)$.

Proposition 11.11. *Let $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ be two commutative B_∞-algebras. Let $f : V \longrightarrow W$ be a morphism of B_∞-algebras. The following diagram is commutative:*

$$\begin{array}{ccc} (T(V), *_V, \Delta) & \xrightarrow{F} & (T(W), *_W, \Delta) \\ {\scriptstyle H_{*_V}}\downarrow & & \downarrow{\scriptstyle H_{*_W}} \\ (T(V), \sqcup, \Delta) & \xrightarrow[F]{} & (T(W), \sqcup, \Delta) \end{array}$$

In other terms, the commutative B_∞/shuffle isomorphism is functorial.

Proof. The Proposition directly follows from the definition of the maps H_*, ϖ and e_*, from the definition of a commutative B_∞ morphism, and the fact that F is a morphism of bialgebras (so that its action commutes in particular with taking products or computing coproducts). □

12 Finite topologies: the double bialgebra structure

In this last section, we take advantage of the fact that the rich combinatorics of finite topologies allows to perform an alternative calculation of the canonical (or generalized Eulerian) idempotents to concretely illustrate the results previously obtained.

The Hopf algebra H_T introduced in section 9 also has a second coproduct δ, defined with the help of the following notions [10, 9]. Let us consider a finite topology T, on a set E, associated to the quasi-order $\leqslant_T$. Let $\sim$ be an equivalence relation on E.

1. $T\mid\sim$ is the topology associated to the quasi-order $\leqslant_{T\mid\sim}$ defined on E by

$$\forall x, y \in E, \qquad x \leqslant_{T\mid\sim} y \Longleftrightarrow x \leqslant_T y \text{ and } x \sim y.$$

2. $T/\sim$ is the topology associated to the quasi-order $\leqslant_{T/\sim}$ defined as the transitive closure of the relation defined on E by

$$\forall x, y \in E, \qquad xRy \Longleftrightarrow x \leqslant_T y \text{ or } x \sim y.$$

3. We shall say that $\sim \in E_c(T)$ if:

   - The connected components of $T\mid\sim$ are the equivalence classes of $\sim$.
   - The relation $\sim_{T/\sim}$ associated to the quasi-order $\leqslant_{T/\sim}$ is $\sim$.

The coproduct δ sends any finite topology T to

$$\delta(T) = \sum_{\sim \in E_c(T)} T/\sim \otimes T\mid\sim .$$

For example,

$$\begin{aligned}
\delta(\bullet) &= \bullet \otimes \bullet,\\
\delta(\bullet\!\bullet) &= \bullet\!\bullet \otimes \bullet + \bullet^{2} \otimes \bullet\!\bullet,\\
\delta(\vee) &= \vee \otimes \bullet\bullet\bullet + 2\,\bullet\!\bullet^{2} \otimes \bullet\!\bullet + \bullet^{3} \otimes \vee,\\
\delta(\bullet\!\bullet\!\bullet) &= \bullet\!\bullet\!\bullet \otimes \bullet\bullet\bullet + \bullet\!\bullet^{2} \otimes \bullet\!\bullet + +\bullet\!\bullet^{2} \otimes \bullet\!\bullet + \bullet^{3} \otimes \bullet\!\bullet\!\bullet,\\
\delta(\wedge) &= \wedge \otimes \bullet\bullet\bullet + 2\,\bullet\!\bullet^{2} \otimes \bullet\!\bullet + \bullet^{3} \otimes \wedge.
\end{aligned}$$

The counit ε_δ of this coproduct sends any finite topology T to 1 if T is discrete (that is to say if $\leqslant_T = \sim_T$) and to 0 otherwise. Then (H_T, m, Δ, δ) is a double bialgebra, that is to say:

1. (H_T, m, δ) is a commutative bialgebra.

2. (H_T, m, Δ) is a commutative bialgebra in the category of right (H_T, m, δ)-comodules with the coaction δ, or in a more detailed version:

   - $\Delta : H_T \longrightarrow H_T \otimes H_T$ is a comodule morphism, that is to say

   $$(\Delta \otimes \mathrm{Id}) \circ \delta = m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta,$$

   where $m_{1,3,24}(h_1 \otimes h_2 \otimes h_3 \otimes h_4) := h_1 \otimes h_3 \otimes m(h_3 \otimes h_4)$.

   - The counit $\varepsilon_\Delta : H_T \longrightarrow \mathbb{K}$ is a comodule morphism, that is to say

   $$\forall x \in H_T, \qquad (\varepsilon_\Delta \otimes \mathrm{Id}) \circ \delta(x) = \varepsilon_\Delta(x)1.$$

For conilpotent double bialgebras, it is possible to obtain the generalized Eulerian idempotent from a single infinitesimal character λ:

Proposition 12.1. *Let (H, m, Δ, δ) be a double bialgebra, such that (H, Δ) is conilpotent. We consider the map $\lambda : H \longrightarrow \mathbb{K}$, defined by $\lambda(1) = 0$ and for any $x \in H_+$,*

$$\lambda(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \varepsilon_\delta^{\otimes k} \circ \overline{\Delta}_k(x).$$

Then the generalized Eulerian idempotent of (H, m, Δ), written e, is given by

$$e = (\lambda \otimes \mathrm{Id}) \circ \delta.$$

Proof. See [11, Section 4]. □

In the case of finite topologies, it is possible to inductively compute this infinitesimal character:

Proposition 12.2. *Let T be a finite topology, associated to the quasi-order $\leqslant_T$ on the set E. We denote by $\min(T)$ the set of classes of $\sim_T$ which are minimal for the order $\overline{\leqslant_T}$. We define $\Upsilon(T) \in \mathbb{Z}[X, X^{-1}]$ by the following:*

$$\Upsilon(T) = \begin{cases} \dfrac{1}{X} \text{ if } T = 1, \\ \displaystyle\sum_{\varnothing \subsetneq I \subseteq \min(T)} X \Upsilon(T_{|E \setminus I}) \text{ otherwise.} \end{cases}$$

Then, for any nonempty finite topology T, $\Upsilon(T) \in \mathbb{Z}[X]$ and

$$\lambda(T) = \int_{-1}^{0} \Upsilon(T)(t) \mathrm{d}t.$$

Proof. Let T be a nonempty finite topology, associated to the quasi-order $\leqslant_T$. Then

$$\begin{aligned}\lambda(T) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{f:E\twoheadrightarrow[k],\\ x\leqslant_T y\Longrightarrow f(x)\leqslant f(y)}} \varepsilon_\delta(T_{|f^{-1}(1)})\dots\varepsilon_\delta(T_{|f^{-1}(k)}) \\ &= \sum_{k=1}^{\infty} \sum_{\substack{f:E\twoheadrightarrow[k],\\ x<_T y\Longrightarrow f(x)<f(y),\\ x\sim_T y\Longrightarrow f(x)=f(y)}} \frac{(-1)^{k-1}}{k}.\end{aligned}$$

We then put, for any finite topology T on a set E,

$$\Upsilon(T) = \sum_{k=0}^{\infty} \sum_{\substack{f:E\twoheadrightarrow[k],\\ x<_T y\Longrightarrow f(x)<f(y),\\ x\sim_T y\Longrightarrow f(x)=f(y)}} X^{k-1}, \tag{7}$$

Note that by convention, $\Upsilon(1) = \dfrac{1}{X}$. Then, for T non empty,

$$\lambda(T) = \int_{-1}^{0} \Upsilon(T)(t)\mathrm{d}t.$$

Let now $f : E \twoheadrightarrow [k]$ be such that for any $x, y \in E$ such that $x <_T y$, then $f(x) < f(y)$ and for any $z, t \in E$ such that $z \sim_T t$, then $f(z) = f(t)$. Then $f^{-1}(1)$ is a nonempty subset of $\min(T)$, and we obtain that

$$\begin{aligned}\Upsilon(T) &= \sum_{\varnothing\subsetneq I\subseteq\min(T)} \sum_{k=1}^{\infty} \sum_{\substack{f:E\setminus I\longrightarrow\{2,\dots,k\},\\ x<_T y\Longrightarrow f(x)<f(y)\\ x\sim_T y\Longrightarrow f(x)=f(y)}} X^{k-2+1} \\ &= \sum_{\varnothing\subsetneq I\subseteq\min(T)} X\Upsilon(P_{|E\setminus I}),\end{aligned}$$

which allows to compute $\Upsilon(T)$ by induction on the number of vertices. □

Example 12.1.

$\Upsilon(\bullet) = 1$, $\Upsilon(\bullet\ \bullet) = 2X + 1$, $\Upsilon(\bullet\ \bullet\ \bullet) = 6X^2 + 6X + 1$,

$\Upsilon(\overset{\bullet}{\bullet}) = X$, $\Upsilon(\vee) = \Upsilon(\wedge) = 2X^2 + X$, $\Upsilon(\begin{smallmatrix}\bullet\\ \bullet\\ \bullet\end{smallmatrix}) = X^2$,

$\Upsilon(\bullet\ \overset{\bullet}{\bullet}) = 3X^2 + 2X$,

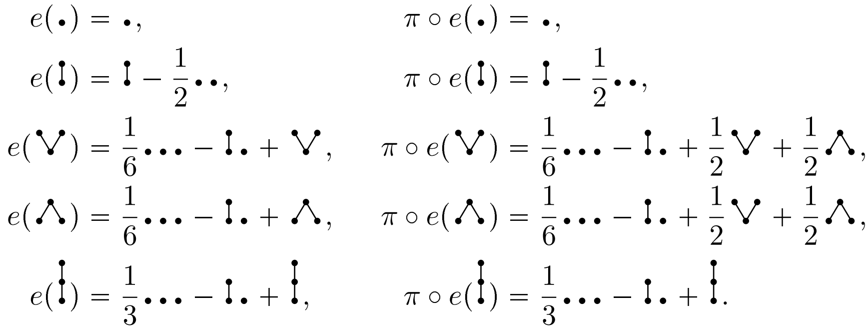
and consequently

$\lambda(\bullet) = 1$, $\lambda(\bullet\ \bullet) = 0$, $\lambda(\bullet\ \bullet\ \bullet) = 0$,

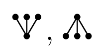
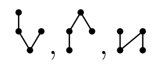
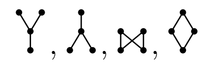
$\lambda(\overset{\bullet}{\bullet}) = -\dfrac{1}{2}$, $\lambda(\vee) = \lambda(\wedge) = \dfrac{1}{6}$, $\lambda(\begin{smallmatrix}\bullet\\ \bullet\\ \bullet\end{smallmatrix}) = -\dfrac{1}{3}$,

$\lambda(\bullet\ \overset{\bullet}{\bullet}) = 0$,

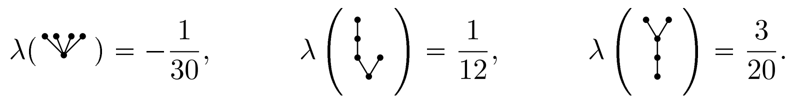
which finally gives, for e and for the canonical π-idempotent:



Here are the values of λ on connected posets of order 4 (it is zero on non connected posets):

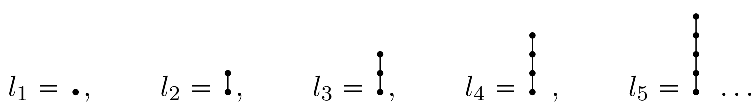
T				
$\lambda(T)$	0	$-\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{4}$

Here are a few examples of order 5:



Let us now give two families of examples.

Proposition 12.3. *1. For any $n \geqslant 1$, the n^{th} ladder is the finite topology associated to the poset $([n], \leqslant)$:*



If $n \geqslant 1$, then $\Upsilon(l_n) = X^{n-1}$ and $\lambda(l_n) = \frac{(-1)^{n+1}}{n}$.

2. For any $n \geqslant 1$, we write

$$\Upsilon(\bullet^n) = \sum_{k=0}^{n-1} s_{n,k} X^k.$$

Then $s_{n,k}$ is the number of surjective maps from $[n]$ to $[k+1]$. In particular, $s_{n,n-1} = n!$ and $s_{n,0} = 1$.

Proof. We use the definition of Υ given by (7). For l_n, indexing the vertices from the root to the leaf, the unique surjective map to be taken in account in the sum defining $\Upsilon(l_n)$ is $\mathrm{Id}_{[n]}$. For $\bullet^n$, all surjective maps have to be taken into account. □

Remark 12.1. The numbers $s_{n,k}$ are related to the Stirling numbers of the second kind $S_2(n,k)$ through the relation

$$s_{n,k} = (k+1)!S_2(n,k+1),$$

see the OEIS [37], Entry A019538 for more details. Here are the first values of $s_{n,k}$:

$n\backslash k$	0	1	2	3	4	5	6
1	1						
2	1	2					
3	1	6	6				
4	1	14	36	24			
5	1	30	150	240	120		
6	1	62	540	1560	1800	720	
7	1	126	1806	8400	16800	15120	5040

Corollary 12.4. *For any $n \geqslant 2$, let us denote by c_n the n^{th} corolla, that is to say the finite topology on $[n]$ given by*

$$\{I \mid I \subseteq [n-1]\} \cup \{[n]\}.$$

Graphically,

$c_2 =$, $c_3 =$  , $c_4 =$, $c_5 =$ ...

Then, for any $n \geqslant 2$,

$$\lambda(c_n) = \sum_{k=0}^{n-2} s_{n-1,k} \frac{(-1)^{k+1}}{k+2}.$$

Proof. As c_n has a unique minimal element,

$$\Upsilon(c_n) = X\Upsilon(\bullet^{n-1}) = \sum_{k=0}^{n-2} s_{n-1,k} X^{k+1}.$$

The results then follows by integration between -1 and 0. □

Example 12.2. This gives

n	2	3	4	5	6	7	8	9	10	11
$\lambda(c_n)$	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$-\frac{617}{6}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Moreover, due to the form of $\delta(l_n)$ and $\delta(c_n)$, we let the reader check that

Corollary 12.5. *For any $n \geqslant 2$,*

$$\begin{aligned} e(l_n) &= \sum_{k=1}^{n} \sum_{\substack{n=i_1+\ldots+i_k,\\ i_1,\ldots,i_k\geqslant 1}} \frac{(-1)^{k+1}}{k} l_{i_1} \ldots l_{i_k} \\ &= \sum_{1i_1+\ldots+ni_n=n} \frac{(-1)^{i_1+\ldots+i_n+1}(i_1+\ldots+i_n-1)!}{i_1!\ldots i_n!} l_1^{i_1}\ldots l_n^{i_n}, \\ e(c_n) &= \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda(c_{n-i}) \bullet^{n-i-1} c_{i+1}, \end{aligned}$$

where by convention $c_1 = \bullet$.

Example 12.3. This gives

$$\begin{aligned} e(l_1) &= l_1, \\ e(l_2) &= l_2 - \frac{1}{2}l_1^2, \\ e(l_3) &= l_3 - l_2 l_1 + \frac{1}{3}l_1^3, \\ e(l_4) &= l_4 + l_3 l_1 + \frac{1}{2}l_2^2 + l_2 l_1^2 - \frac{1}{4}l_1^4, \\ e(l_5) &= l_5 - l_4 l_1 - l_3 l_2 + l_3 l_1^1 + l_2^2 l_1 - l_2 l_1^3 + \frac{1}{5}l_1^5, \\ e(c_2) &= c_2 - \frac{1}{2}c_1^2, \\ e(c_3) &= c_3 - c_2 c_1 + \frac{1}{6}c_1^3, \\ e(c_4) &= c_4 - \frac{3}{2}c_3 c_1 + \frac{1}{2}c_2 c_1^2 + c_4, \\ e(c_5) &= c_5 - 2c_4 c_1 + c_3 c_1^2 - \frac{1}{30}c_1^5. \end{aligned}$$

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