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STICKY DIFFUSIONS ON STAR GRAPHS : CHARACTERIZATION AND ITÔ FORMULA

JULES BERRY AND FAUSTO COLANTONI

ABSTRACT. We investigate continuous diffusions on star graphs with sticky behavior at the vertex. These are Markov processes with continuous paths having a positive occupation time at the vertex. We characterize sticky diffusions as time-changed nonsticky diffusions by adapting the classical technique of Itô and McKean. We prove a form of Itô formula, also known as Freidlin-Sheu formula, for this type of process. As an intermediate step, we also obtain a stochastic differential equation satisfied by the radial component of the process. These results generalize those already known for sticky diffusions on a half-line and skew sticky diffusions on the real line.

Keywords: Diffusion processes, Sticky boundary conditions, Itô formula, Star graphs.

AMS subject classification: 60J60, 60J50, 60J55, 60H10, 05C99.

1. INTRODUCTION

In the unidimensional setting, sticky processes find their roots¹ in the seminal work of Feller [21; 22; 23]. These processes are characterized by a second order term in the boundary condition defining the domain of their generator. For instance, the sticky (reflected) Brownian motion on \mathbb{R}_+ was first obtained by considering the Feller semigroup with generator given by

$$\begin{aligned} Lf(x) &= \frac{1}{2}f''(x) \quad \text{for } x > 0, \\ D(L) &= \{f \in C^2((0, +\infty)) \cap C_0(\mathbb{R}_+) : \eta f''(0^+) = f'(0^+)\}, \end{aligned} \tag{1}$$

where $\eta \geq 0$ is the stickiness parameter. The term “sticky” is motivated by the fact that, when $\eta > 0$, the occupation time of the process at 0 is positive, despite the fact that the set of zeros of the process has empty interior, while it is null for the reflected Brownian motion (which corresponds to the case $\eta = 0$). It was later established by Itô and McKean in [32] that the sticky Brownian motion could be characterized by suitable random time-changes of the reflected Brownian motion. More precisely, let B be a reflected Brownian motion and denote by $\ell_B(t)$ its local time at 0. Define the continuous increasing process $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$V(t) := t + \eta \ell_B(t), \tag{2}$$

consider $V^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ its right inverse, i.e.

$$V^{-1}(t) = \inf \{s > 0 : V(s) > t\},$$

and define the time-changed process $\hat{B}(t) := B(V^{-1}(t))$. It can then be proved that \hat{B} is a Markov process whose semigroup has generator given by (1). From the definition (2), we observe that V is a strictly increasing process that grows faster than t when B is at 0. Therefore, its inverse V^{-1} grows more slowly than t , causing the time-changed process \hat{B} to become “stuck” at 0. This is a direct consequence of the fact that the local time increases only at 0, while it remains constant otherwise. We also note that, when $\eta = 0$, we have $V(t) = t = V^{-1}(t)$, meaning that \hat{B} coincides with B . When $\eta > 0$, outside 0,

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¹See [43] for historical aspects of these matters.

\hat{B} follows the trajectories of B , but with a delay. Recently, it was proved in [19] that the sticky Brownian motion satisfies the stochastic differential equation

$$\begin{cases} dX(t) = \mathbf{1}_{\{X(t)>0\}}dW(t) + \frac{1}{2}d\ell_X(t), \\ \eta\ell_X(t) = \int_0^t \mathbf{1}_{\{X(s)=0\}}ds, \end{cases} \quad (3)$$

for some one-dimensional Brownian motion W , where ℓ_X is the local time of X at 0. More generally, one can consider continuous sticky diffusions on \mathbb{R}_+ as the Markov process with generator given by

$$\begin{aligned} Lf(x) &= \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) \quad \text{for } x > 0, \\ D(L) &= \{f \in C^2((0, +\infty)) \cap C_0(\mathbb{R}_+) : \eta Lf(0^+) = f'(0^+)\}. \end{aligned} \quad (4)$$

In [46], it is established that such diffusions are random time-changes of the reflected diffusion ($\eta = 0$) and satisfy

$$\begin{cases} dX(t) = \sigma(X(t))\mathbf{1}_{\{X(t)>0\}}dW(t) + b(X(t))\mathbf{1}_{\{X(t)>0\}}dt + \frac{1}{2}d\ell_X(t), \\ \eta\ell_X(t) = \int_0^t \mathbf{1}_{\{X(s)=0\}}ds, \end{cases} \quad (5)$$

for some Brownian motion W , where ℓ_X is the local time of X at 0.

Another interesting process is the skew Brownian motion introduced in [48]. Loosely speaking, it is Brownian motion on \mathbb{R} which, upon reaching 0, is reflected in \mathbb{R}_+ with probability p and in \mathbb{R}_- with probability $1 - p$. The generator of this process is given by

$$\begin{aligned} Lf(x) &= \frac{1}{2}f''(x) \quad \text{for } x \neq 0, \\ D(L) &= \left\{ f \in C^2(\mathbb{R} \setminus \{0\}) \cap C_0(\mathbb{R}) : \begin{cases} f''(0^+) = f''(0^-), \\ pf'(0^+) = (1-p)f'(0^-) \end{cases} \right\}. \end{aligned} \quad (6)$$

We refer the reader to [39] and references therein for further information on this class of processes. The version of this process exhibiting a sticky behavior at 0 is then obtained by replacing $D(L)$ in (6) by

$$D(L) = \left\{ f \in C^2(\mathbb{R} \setminus \{0\}) \cap C_0(\mathbb{R}) : \begin{cases} f''(0^+) = f''(0^-), \\ \eta f''(0) = pf'(0^+) - (1-p)f'(0^-) \end{cases} \right\}. \quad (7)$$

This process was recently studied in [46; 47] where it is shown that the process can be obtained as a random time-change of a skew Brownian motion and that it is a weak solution to some stochastic differential equation. In addition, in [47], an explicit expression for the transition kernel of the process is given. Similar results were previously obtained in [30; 19; 5] in the special case $p = 1/2$, in which case it is also called two sided sticky Brownian motion. For more general diffusion processes, considered in [46], the generator takes the form

$$\begin{aligned} Lf(x) &= \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) \quad \text{for } x \neq 0, \\ D(L) &= \left\{ f \in C^2(\mathbb{R} \setminus \{0\}) \cap C_0(\mathbb{R}) : \begin{cases} Lf(0^+) = Lf(0^-), \\ \eta Lf(0) = pf'(0^+) - (1-p)f'(0^-) \end{cases} \right\}. \end{aligned} \quad (8)$$

In [46], it is established that these processes can be constructed from the skew diffusions processes ($\eta = 0$) by using the Itô-McKean technique and satisfy

$$\begin{cases} dX(t) = \sigma(X(t))\mathbf{1}_{\{X(t)\neq 0\}}dW(t) + b(X(t))\mathbf{1}_{\{X(t)\neq 0\}}dt + (2p-1)d\ell_X(t), \\ \eta\ell_X(t) = \int_0^t \mathbf{1}_{\{X(s)=0\}}ds, \end{cases} \quad (9)$$

for some Brownian motion W , where $\ell_X(t)$ is the symmetric local time of X at 0.

Stochastic processes on graphs are an intriguing extension of unidimensional ones, useful for many applications involving complex systems on networks. They were first introduced by Feller's abstract approach in [25]. A fundamental example is certainly the Brownian motion on a star graph, whose paths are constructed in [34; 35]. We can think of it as a stochastic process defined on a number of edges, each of which is a copy of the positive half-line, connected by a single vertex. On each of these edges, the process behaves like a reflecting Brownian motion and randomly switches to another one upon reaching the vertex. In [34; 35], the authors provide a full characterization of the infinitesimal generator of the corresponding Feller semigroup. They also provide a construction of the paths of the process in every case by adapting techniques used for Brownian motions on an interval. In particular the Brownian motion with sticky behavior at the vertex is obtained by using the Itô-McKean random time-change. The construction was then generalized to generic metric graphs, i.e. graphs with more than one vertex, in [36]. Recently, the analysis and construction of Brownian motion with a sticky point, even under non-local dynamic conditions leading to trapping behavior at the vertex, has also been addressed in [8]. A more general case is the Walsh Brownian motion [48; 4], where once the process reaches the vertex, it selects a new edge according to a continuous distribution on $[0, 2\pi)$. An Itô formula for this setup is derived in [29].

Paths of general nondegenerate diffusions on star graphs were studied in [24] under the assumption that the process is nonsticky at the vertex. There, it is established that the radial component of the process is a weak solution to some stochastic differential equation and an Itô formula is obtained. The authors also prove the existence of a process which can be interpreted as the local time of the nonsticky process at the vertex. Our goal is to extend these results to processes exhibiting sticky behavior at the vertex. In addition to [25], the semigroups associated to sticky diffusions on metric graphs have also been analysed in [27] in the special case of pure diffusion processes and a form of singular convergence theorem is also provided. We also mention that, recently, a more general class of nonsticky diffusion processes has been proposed in [40].

The main result of this work is the derivation of an Itô formula for general nondegenerate diffusions on star graphs. As intermediate steps, we also obtain a stochastic differential equation satisfied by radial component of the process and a characterization of sticky diffusions as time-changes of nonsticky ones. On the one hand, our results extend those available in the unidimensional setting [46; 47] by allowing more than one or two edges. On the other hand, we generalize those from [24] by considering sticky processes. This is achieved by making a systematic use of the Itô-McKean random time change in this setting, where we use the local time process obtained in [24] to define the time change. Our choice to focus on diffusions on star graphs was made in order to simplify notations and to avoid obfuscating the exposition with technicalities. However, we expect that our conclusions can be extended to general metric graphs.

As was already mentioned, our main motivation in writing this paper was the derivation of the Itô formula for sticky diffusions. It is used in [7] to prove a verification theorem in the context of an optimal control problem of sticky diffusions on metric graphs. Nevertheless, we believe that our results can be useful in various applications. One example can be found in the recent study of Mean Field Game (MFG) problems on networks. We briefly recall that the theory of MFG was introduced independently in [31] and [37] and aims at studying Nash equilibria of large symmetric dynamic games. General references on this theory are [6; 26; 15; 14]. Adaptation of the theory to metric graphs was first proposed in [12] and further developed in [10; 1; 2; 11]. In all of these references agents are represented by nonsticky diffusions. In contrast, the recent paper [7] mentioned above deals with games where players present a sticky behavior at vertices. MFG of nonsticky diffusions

were recently used in [13] for a model of urban planning. Finally we also mention that another application of nonsticky diffusions to spatial economics can be found in [3]. We believe that considering nonsticky diffusions would be an interesting extension of these models.

The rest of the paper is structured as follows. Our main results are stated in Section 2 together with the necessary notations and definitions. Section 3 contains a summary of the results from [25; 24] that will be useful in our study of sticky diffusions on Γ as well as some technical lemmas on these processes. Section 4 contains the proofs of our main results. Finally Appendix A contains a technical result about a system of ODEs which is used in Section 3.

2. NOTATIONS AND STATEMENT OF MAIN RESULTS

In order to state our main results, we now introduce some notations and definitions. Let N be a positive natural number and let E denote a family of copies of the positive half-line, defined as:

$$E := \bigsqcup_{i=1}^N [0, \infty).$$

For notational simplicity, we represent each point in E by a pair (j, x) , where $j \in \{1, \dots, N\}$ and x represents the Euclidean distance from the origin. Following the approach in [42], we define an equivalence relation on E as follows:

$$(j, x) \sim (i, y) \text{ if and only if } \begin{cases} j = i \text{ and } x = y \\ x = y = 0, \text{ for any } j, i. \end{cases}$$

Then, the star graph is the quotient space $\Gamma := E / \sim$. In this graph, there is a unique vertex $\mathbf{v} = (\cdot, 0)$ that is common to all edges. We emphasize that with this configuration, we do not consider the angles between the edges, allowing it to maintain its abstract nature. Moreover, Γ forms a locally compact Polish space once provided with the following metric:

$$d((j, x), (i, y)) := \begin{cases} |x - y| & \text{if } i = j, \\ x + y & \text{if } i \neq j. \end{cases} \quad (10)$$

Since (Γ, d) is a metric space, a function $f : \Gamma \rightarrow \mathbb{R}$ is considered continuous if it satisfies the standard definition of continuity for functions defined on metric spaces and we denote by $C(\Gamma)$ the space of continuous functions on Γ and $C_0(\Gamma)$ the space of continuous functions vanishing at infinity on each edge. We provide both $C(\Gamma)$ and $C_0(\Gamma)$ with the topology of uniform convergence on compact sets. We can represent a function $f : \Gamma \rightarrow \mathbb{R}$ as

$$f = \bigoplus_{i=1}^N f_i \text{ satisfying } f_i(0) = f_j(0) \text{ for all } i, j \in \{1, \dots, N\},$$

where each $f_i(\cdot) = f(i, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ corresponds to the function restricted to the edge i . With this representation we have

$$C(\Gamma) = \left\{ f = \bigoplus_{i=1}^N f_i \in \bigoplus_{i=1}^N C[0, \infty) \text{ and } f_i(0) = f_j(0), \forall i, j \in \{1, \dots, N\} \right\}.$$

Regarding differentiation, derivatives outside the vertex are viewed as, for $\mathbf{x} = (i, x) \in \Gamma \setminus \{\mathbf{v}\}$

$$f'(\mathbf{x}) = \frac{d}{dx} f_i(x).$$

The same applies to higher order derivatives as well. We then define, for each $k \in \mathbb{N}$,

$$C^k(\Gamma) := \left\{ f = \bigoplus_{i=1}^N f_i \in \bigoplus_{i=1}^N C^k[0, \infty) : f \in C(\Gamma) \right\}.$$

Notice that, in the definition of $C^k(\Gamma)$, only the function is assumed to be continuous at the vertex and the derivatives may be discontinuous. We will also need to consider the following spaces :

$$C_b^k(\Gamma) = \left\{ f \in C^k(\Gamma) : f_i^{(h)} \in C_b[0, \infty) \text{ for all } h \in \{0, \dots, k\} \text{ and } i \in \{1, \dots, N\} \right\}$$

and, for every $k \in \mathbb{N}$,

$$PC^k(\Gamma) = \bigoplus_{i=1}^N C_b^k([0, +\infty)),$$

with the convention that $PC^0(\Gamma) = PC(\Gamma)$. With an abuse of notations, we will say that a function $f: \Gamma \rightarrow \mathbb{R}$ belongs to $PC^k(\Gamma)$ if each $f_i, \dots, f_i^{(k)}$ belongs to $C_b((0, +\infty))$ and can be continuously extended to $[0, +\infty)$. Finally, for $k \geq 0$, we denote by $C_b^{1,k}(\mathbb{R}_+ \times \Gamma)$ the space of function $f: \mathbb{R}_+ \times \Gamma \ni (t, x) \mapsto f(t, x) \in \mathbb{R}$ such that $f, \partial_t f \in C_b(\mathbb{R}_+ \times \Gamma)$ and $\partial_x f_i, \dots, \partial_x^k f_i \in C_b(\mathbb{R}_+ \times \mathbb{R}_+)$ for each $i \in \{1, \dots, N\}$. A comprehensive treatment of functional spaces and equations on the graph is available in [41].

Finally, for $k \geq 0$, we denote by $C_b^{1,k}(\mathbb{R}_+ \times \Gamma)$ the space of function $f: \mathbb{R}_+ \times \Gamma \ni (t, x) \mapsto f(t, x) \in \mathbb{R}$ such that $f, \partial_t f \in C_b(\mathbb{R}_+ \times \Gamma)$ and $\partial_x f_i, \dots, \partial_x^k f_i \in C_b(\mathbb{R}_+ \times \mathbb{R}_+)$ for each $i \in \{1, \dots, N\}$. A comprehensive treatment of functional spaces and equations on the graph is available in [41].

We now turn to defining diffusions on the graph. Let L_i , for $i = 1, \dots, N$, be the linear operators defined on $PC^2(\Gamma)$ by

$$L_i f(x) = \frac{1}{2} \sigma_i^2(x) f_i''(x) + b_i(x) f_i'(x), \quad x > 0, \quad (11)$$

where $\sigma, b \in PC(\Gamma)$ and we assume that exists a constant $\sigma_0 > 0$ such that $\sigma(x) > \sigma_0$ for all $x \in \Gamma$. Observe that, for every $f \in C_b^2(\Gamma)$ with $Lf \in C(\Gamma)$, i.e. such that

$$\lim_{x \rightarrow 0} L_i f_i(x) = \lim_{y \rightarrow 0} L_j f_j(y) \quad \text{for every } i, j \in \{1, \dots, N\}, \quad (12)$$

the operator defined in (11) is an operator on the graph Γ . Indeed, given a point $x = (i, x) \in \Gamma$, we alternatively write

$$Lf(x) = \frac{1}{2} \sigma^2(x) f''(x) + b(x) f'(x), \quad \text{for all } x \in \Gamma, \quad (13)$$

where (12) ensures that the expression (13) makes sense for $x = v$. In order to obtain a process on the graph, we need a transition condition on the vertex, which, in our case, takes the form

$$\eta Lf(v) = \sum_{i=1}^N \rho_i f_i'(0), \quad (14)$$

where $\eta \geq 0$, $\rho_i > 0$ for all $i \in \{1, \dots, N\}$ and $\sum_{i=1}^N \rho_i = 1$. In the end, the generator of a diffusion process on Γ is given by $(L, D(L))$ where L is defined by (13) and

$$D(L) = \left\{ C^2(\Gamma) \cap C_0(\Gamma) : Lf \in C_0(\Gamma) \text{ and } \eta Lf(v) = \sum_{i=1}^N \rho_i f_i'(0) \right\}, \quad (15)$$

The process will be called a *nonsticky diffusion* if $\eta = 0$ and a *sticky diffusion* otherwise. The constat η will called the *stickiness parameter* of the process.

We are now able to state our main results.

Theorem. *Let $X(t) = (i(t), x(t))$ be a sticky diffusion on a star graph Γ with stickiness parameter $\eta > 0$. Then,*

- (i) *there exists a nonsticky diffusion Y on Γ and a strictly increasing continuous function V^{-1} such that X and $Y(V^{-1}(\cdot))$ are indistinguishable. Moreover, there exists an increasing process ℓ_X , adapted to the completion of the natural filtration of X with respect to the collection of probability measures $(\mathbf{P}_x)_{x \in \Gamma}$, such that ℓ_X almost surely increases only when $X = v$ and $V^{-1}(t) = t - \eta \ell_X(t)$;*
- (ii) *up to an extension of the filtered probability space, there exists a one-dimensional Brownian motion W such that, almost surely, we have*

$$\begin{cases} dx(t) = \sigma_{i(s)}(x(t)) \mathbf{1}_{\{x(t) \neq 0\}} dW_t + b_{i(s)}(x(t)) \mathbf{1}_{\{x(t) \neq 0\}} dt + d\ell_X(t), \\ \eta \ell_X(t) = \int_0^t \mathbf{1}_{\{X(s)=v\}} ds; \end{cases} \quad (16)$$

- (iii) *for every $f \in C_b^{1,2}(\mathbb{R}_+ \times \Gamma)$, almost surely, we have the Itô formula*

$$\begin{aligned} f(t, X(t)) &= f(0, X(0)) + \int_0^t (\partial_s f(s, X_s) + Lf(s, X_s)) \mathbf{1}_{\{X(s) \neq v\}} ds \\ &+ \int_0^t \sigma(X_s) \partial_x f(s, X_s) \mathbf{1}_{\{X_s \neq v\}} dW_s + \int_0^t \left(\eta \partial_s f(s, v) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) \right) d\ell_X(s). \end{aligned} \quad (17)$$

Proof. The different statements are proved in Section 4. More precisely (i) is established in Theorem 4.4, (ii) follows from Theorem 4.5 and Proposition 4.2 and (iii) is proved in Theorem 4.7. \square

Notice that one may recover results about sticky and skew sticky diffusions discussed above by considering the spacial cases $N = 1$ and $N = 2$ respectively.

3. DIFFUSION PROCESSES ON Γ

We start by summarizing the conclusions from [25, Theorem 3.1] and [24, Lemma 2.2, Remark 2.5].

Theorem 3.1. *The unbounded linear operator L , defined by (13) with domain given by (15), is the generator of a strongly continuous semigroup on $C_0(\Gamma)$, associated to a conservative Markov process X on Γ having continuous paths. Moreover, writing $X(t) = (i(t), x(t))$, where $x(t)$ is the diffusive part on $[0, \infty)$ generated by L_i and $i(t)$ selects the edges and denoting by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration associated to X , we have the following:*

- (i) *Let $t \geq 0$, and $\tau := \inf\{s > t : x(s) = 0\}$. Then x coincides with a one dimensional diffusion on $[0, \infty)$, with generator $L_{i(t)}$, on $[t, \tau)$.*
- (ii) *If $\eta = 0$, then we have*

$$\int_0^t \mathbf{1}_{\{x(s)=0\}} ds = 0 \quad \text{for all } t \geq 0,$$

and there exists a one-dimensional Brownian motion W and a continuous increasing process ℓ_X , both adapted to $(\mathcal{F}_t)_{t \geq 0}$, such that

$$dx(t) = \sigma_{i(t)}(x(t)) dW_t + b_{i(t)}(x(t)) dt + d\ell_X(t), \quad (18)$$

and the process ℓ_X increases only when $x(t) = 0$. Furthermore, we have

$$\left(\sum_{i=1}^N \frac{2\rho_i}{\sigma_i^2(0)} \right) \ell_X(t) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_0^t \mathbf{1}_{\{x(s) \leq \delta\}} ds \quad \text{in expectation.}$$

To complete the notation, we denote by \mathbf{E}_x the expected value with respect to \mathbf{P}_x , where x is the starting point. We continue with a simple observation.

Proposition 3.2. *The vertex \mathbf{v} is not a trap for the Markov process generated by (13) and (15), for a finite $\eta \geq 0$.*

Proof. Arguing by contradiction, assume that \mathbf{v} is a trap for the Markov process X generated by (13)-(15). Denote by $(P_t)_{t \geq 0}$ the strongly continuous semigroup of bounded linear operators on $C_0(\Gamma)$ associated to X . Then $P_t g(\mathbf{v}) = \mathbf{E}_\mathbf{v}[g(X(t))] = g(\mathbf{v})$ for all $g \in C_0(\Gamma)$ and $t \geq 0$. Let $u \in D(L)$ be such that $Lu(\mathbf{v}) = 1$, such a function may be obtained by extending the one obtain in Proposition A.1 with $f = 1$. Then, using Kolmogorov's forward equation, we have

$$0 = \frac{d}{dt} P_t u(\mathbf{v}) = P_t (Lu)(\mathbf{v}) = 1,$$

a contradiction. □

The following lemma provides an asymptotic estimate on the mean first passage time of the process started at \mathbf{v} at the boundary of the ball $B(\mathbf{v}, \delta)$ as δ tends to zero.

Lemma 3.3. *Let $X(t) = (i(t), x(t))$ be a Markov process with generator given by (13) and (15) for $\eta \geq 0$ and, for every $\delta > 0$, set*

$$T_X^\delta := \inf\{t \geq 0 : x(t) \geq \delta\}. \quad (19)$$

Then $|\mathbf{E}_\mathbf{v}[T_X^\delta] - \eta\delta| = O(\delta^2)$ when $\delta \downarrow 0$.

Proof. For each $\delta > 0$ and $1 \leq i \leq N$, consider $u^\delta \in D(L)$ such that the restriction of u^δ to $B(\mathbf{v}, \delta)$ satisfies the system of ODEs

$$\begin{cases} \frac{1}{2}\sigma_i^2(x)u_i^{\delta''}(x) + b_i(x)u_i^{\delta'}(x) = -1 & \text{for all } x \in (0, \delta), 1 \leq i \leq N, \\ \eta Lu(\mathbf{v}) - \sum_{i=1}^N \rho_i u_i^{\delta'}(0) = 0, \\ u_i^\delta(\delta) = 0 & \text{for all } i \in \{1, \dots, N\}, \end{cases} \quad (20)$$

and

$$\sup_{0 < d(x, \mathbf{v}) < \delta} |u^{\delta''}(x)| = O(1) \quad \text{as } \delta \downarrow 0. \quad (21)$$

Such a function can be obtained by extending the function obtained in the postponed Proposition A.1, with $f_i(x) = -1$ for all $0 \leq x < \delta$. Then, using Dynkin's formula, we have

$$u_i^\delta(0) = \mathbf{E}_\mathbf{v} \left[u^\delta(0) - u^\delta(X(T_X^\delta)) \right] = \mathbf{E}_\mathbf{v} \left[- \int_0^{T_X^\delta} Lu^\delta(X(s)) ds \right] = \mathbf{E}_\mathbf{v} \left[T_X^\delta \right].$$

On the other hand, using Taylor's formula, there exists $(\theta_1, \dots, \theta_N) \in (0, \delta)^N$ such that

$$u_i^\delta(0) = \sum_{i=1}^N \rho_i \left(u_i^\delta(0) - u_i^\delta(\delta) \right) = - \sum_{i=1}^N \rho_i \left(u_i^{\delta'}(0)\delta + u_i^{\delta''}(\theta_i) \frac{\delta^2}{2} \right) = \eta\delta - \sum_{i=1}^N \rho_i u_i^{\delta''}(\theta_i) \frac{\delta^2}{2}.$$

We conclude that

$$\left| \mathbf{E}_\mathbf{v}[T_X^\delta] - \eta\delta \right| \leq \sup_{0 < d(x, \mathbf{v}) < \delta} |u^{\delta''}(x)| \delta^2.$$

The conclusion then follows from (21). □

The following result was established in [24, Corollary 2.4] in the case $\eta = 0$. We provide here a generalization of this result the case to $\eta \geq 0$ with a slightly different proof.

Lemma 3.4. *Let $X(t) = (i(t), x(t))$ be a Markov process with generator given by (13) and (15) for $\eta \geq 0$ and, for all $\delta > 0$, define T_X^δ according to (19). Then*

$$\rho_j = \lim_{\delta \downarrow 0} \mathbf{P}_v \left(i(T_X^\delta) = j \right) \quad \text{for all } 1 \leq i \leq N,$$

where $(\rho_j)_{1 \leq j \leq N}$ are the constants appearing in (15).

Proof. Let $f \in D(L)$ and set $\tilde{\rho}_j^\delta := \mathbf{P}_v (i(T_X^\delta) = j)$ for all $1 \leq j \leq N$ and $\delta > 0$. Using Taylor's formula, we have

$$\mathbf{E}_v \left[f(X(T_X^\delta)) - f(v) \right] = \sum_{j=1}^N \tilde{\rho}_j^\delta (f_j(\delta) - f_j(0)) = \sum_{j=1}^N \tilde{\rho}_j^\delta (f_j'(0)\delta + o(\delta)). \quad (22)$$

On the other hand, from Dynkin's formula, we also have

$$\begin{aligned} \mathbf{E}_v \left[\delta^{-1} \left(f(X(T_X^\delta)) - f(v) \right) \right] &= \mathbf{E}_v \left[\frac{1}{\delta} \int_0^{T_X^\delta} Lf(X(s)) ds \right] \\ &= \frac{\mathbf{E}_v [T_X^\delta]}{\delta} \left(Lf(v) + \mathbf{E}_v \left[\frac{1}{\mathbf{E}_v [T_X^\delta]} \int_0^{T_X^\delta} Lf(X(s)) - Lf(v) ds \right] \right), \end{aligned}$$

and notice that

$$\left| \mathbf{E}_v \left[\frac{1}{\mathbf{E}_v [T_X^\delta]} \int_0^{T_X^\delta} Lf(X(s)) - Lf(v) ds \right] \right| \leq \sup_{x,y \in B(v,\delta)} |Lf(x) - Lf(y)| \xrightarrow{\delta \rightarrow 0} 0.$$

Let $(\delta_n)_{n \in \mathbb{N}}$ be any sequence in $(0, \infty)$ converging to 0 and such that $\tilde{\rho}_j^{\delta_n}$ converges to some $\tilde{\rho}_j$ for each j . Then, passing to the limit $n \rightarrow \infty$ and using the fact that $|\mathbf{E}_v [T_X^\delta] - \eta\delta| = o(\delta)$ (see Lemma 3.3), we obtain that

$$\eta Lf(v) = \sum_{j=1}^N \tilde{\rho}_j f_j'(0).$$

Since the latter holds for every $f \in D(L)$, we conclude that $\tilde{\rho}_j = \rho_j$. \square

Lemma 3.5. *Let $X(t) = (i(t), x(t))$ be a Markov process with generator given by (13) and (15) for $\eta \geq 0$ and, for every $\delta > 0$, define the stopping time T_X^δ by (19). Then*

$$\left| \mathbf{E}_v \left[\int_0^{T_X^\delta} \mathbf{1}_{\{X(s)=v\}} ds \right] - \eta\delta \right| = O(\delta^2) \quad \text{when } \delta \downarrow 0.$$

Proof. For each $\delta > 0$, consider a sequence $f^n \in \bigoplus_{i=1}^N C([0, \delta])$ satisfying

$$0 \leq f_i^n \leq 1 \quad \text{and} \quad f^n(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \delta/3, \\ 0 & \text{for } 2\delta/3 \leq x \leq \delta, \end{cases}$$

for all $1 \leq i \leq N$, and, as in the proof of Lemma 3.3, a sequence of functions $u^n \in D(L)$ such that the restriction of u^n to $B(v, \delta)$ satisfies

$$\begin{cases} \frac{1}{2}\sigma_i^2(x)u_i^{n''}(x) + b_i(x)u_i^{n'}(x) = -f^n(x) & \text{for all } x \in (0, \delta), 1 \leq i \leq N, \\ \eta Lu^n(v) - \sum_{i=1}^N \rho_i u_i^{n'}(0) = 0, \\ u_i^n(\delta) = 0 & \text{for all } i \in \{1, \dots, N\}. \end{cases} \quad (23)$$

From Dynkin's formula, we have

$$\begin{aligned} u^n(\mathbf{v}) &= \mathbf{E}_{\mathbf{v}} \left[u^n(\mathbf{v}) - u^\delta(X(T_X^\delta)) \right] = \mathbf{E}_{\mathbf{v}} \left[- \int_0^{T_X^\delta} Lu^n(X(s)) ds \right] \\ &= \mathbf{E}_{\mathbf{v}} \left[\int_0^{T_X^\delta} f^n(X(s)) ds \right] \end{aligned}$$

On the other hand, using Taylor's formula, there exists $(\theta_1^n, \dots, \theta_N^n) \in (0, \delta)^N$ such that

$$\begin{aligned} u^n(0) &= \sum_{i=1}^N \rho_i (u_i^n(0) - u_i^n(\delta)) = - \sum_{i=1}^N \rho_i \left(u_i^{n'}(0)\delta + u_i^{n''}(\theta_i^n) \frac{\delta^2}{2} \right) \\ &= -\eta Lu^n(\mathbf{v})\delta - \sum_{i=1}^N \rho_i u_i^{n''}(\theta_i^n) \frac{\delta^2}{2} \\ &= \eta\delta - \sum_{i=1}^N \rho_i u_i^{n''}(\theta_i^n) \frac{\delta^2}{2} \end{aligned}$$

We conclude that

$$\left| \mathbf{E}_{\mathbf{v}} \left[\int_0^{T_X^\delta} f^n(X(s)) ds \right] - \eta\delta \right| \leq \sup_{0 < d(\mathbf{x}, \mathbf{v}) < \delta} |u^{n''}(\mathbf{x})| \frac{\delta^2}{2}. \quad (24)$$

Finally, noticing that f^n converges point-wise to $\mathbf{1}_{\{\mathbf{v}\}}$ and using Proposition A.1 to bound the right-hand side of (24), we deduce from Lebesgue's dominated convergence theorem that, for every $\delta_0 > 0$, there is a constant $C_0 > 0$ such that

$$\left| \mathbf{E}_{\mathbf{v}} \left[\int_0^{T_X^\delta} \mathbf{1}_{\{X(s)=\mathbf{v}\}} ds \right] - \eta\delta \right| \leq C_0 \delta^2.$$

for all $0 < \delta \leq \delta_0$. This concludes the proof. \square

4. MAIN RESULTS

4.1. Construction of trajectories of sticky diffusions. As mentioned in the introduction, sticky conditions are characterized by a slowdown which, in probabilistic terms, is given by a time-change of the process involving the local time. Our first result concerns the extension of this idea to metric graphs for a general diffusion.

Let $Y(t) = (j(t), y(t))$ be the nonsticky process generated by (13) and (15) for $\eta = 0$, and denote by $(\mathcal{G}_t)_{t \geq 0}$ the natural filtration associated to Y . As for the one dimensional case, we define the time-changed process

$$X(t) = Y(V^{-1}(t)), \quad (25)$$

where V^{-1} is the right inverse of V , defined by

$$V^{-1}(t) = \inf\{s \geq 0 : V(s) > t\}, \quad t > 0 \quad (26)$$

and

$$V(t) = t + \eta \ell_Y(t). \quad (27)$$

Since V is continuous, we know from [45, Proposition 4.6 p.43] that each $V^{-1}(t)$ is a stopping time for the natural filtration of ℓ_Y . In particular it is a stopping time for \mathcal{G}_t because ℓ_Y is adapted to \mathcal{G}_t . Moreover, since V is also strictly increasing, it is known that the process V^{-1} is also strictly increasing and continuous and satisfies $V^{-1}(V(t)) = t$

almost surely. Consequently, it is a time change in the sense of [45, Definition 1.2 p.180]. In particular the process X is adapted to the time-changed filtration

$$\mathcal{F}_t := \mathcal{G}_{V^{-1}(t)} := \{A \in \mathcal{G}_\infty : A \cap \{V^{-1}(t) \leq s\} \in \mathcal{G}_s \text{ for all } s \geq 0\}.$$

It is known that X is a Markov process for the filtration \mathcal{F}_t (see [45, Theorem 2.18 p.417] for this result in the case of a Brownian motion and [18, Theorem 10.10] for the general case). We will prove that this process is a version of the sticky diffusion on Γ , with stickiness parameter η . This is achieved by a characterization of the generator of the process X .

Proposition 4.1. *Let $Y(t) = (j(t), y(t))$ be the nonsticky process generated by (13) and (15) for $\eta = 0$ and define the time-changed X according to (25). Then, the semigroup induced by X on $C_0(\Gamma)$ is generated by (13) and (15), where η is the same constant appearing in (27). In particular X is a version of the Markov process given by (13) and (15). Moreover $V^{-1}(t) = t - \eta \ell_X(t)$, where $\ell_X(t) := \ell_Y(V^{-1}(t))$ and ℓ_Y is the process appearing in Theorem 3.1-(ii).*

Proof. Let $f \in D(L^\eta) \cap C_b^2(\Gamma)$, with $\eta > 0$, where L^η denotes the generator of the semigroup induced by X on $C_0(\Gamma)$ and $D(L^\eta)$ is its domain. One may check that $L^\eta f(x) = Lf(x)$ for every $x \neq v$ (see Theorem 4.4 below for a similar argument). We therefore turn to the characterization of the junction condition.

Fix $\delta > 0$, we recall the definition of the stopping time

$$T_X^\delta := \inf\{t \geq 0 : x(t) \geq \delta\}.$$

From the Dynkin's formula we know

$$\mathbf{E}_v[f(X_{T_X^\delta})] - f(v) = \mathbf{E}_v \left[\int_0^{T_X^\delta} Lf(X(s)) ds. \right] \quad (28)$$

Setting $\rho_j^\delta := \mathbf{P}_v(i(T_X^\delta) = j)$ for each $1 \leq j \leq N$, we have

$$\mathbf{E}_v[f(X_{T_X^\delta})] - f(v) = \sum_{k=1}^n \rho_k^\delta f_k(\delta) - f(v) = \sum_{k=1}^n \rho_k^\delta (f_k(\delta) - f_k(0)). \quad (29)$$

So that

$$\mathbf{E}_v \left[\int_0^{T_X^\delta} Lf(X(s)) ds \right] = \sum_{k=1}^n \rho_k^\delta (f_k(\delta) - f_k(0)). \quad (30)$$

We introduce the analogous hitting time for the nonsticky diffusion Y

$$T_Y^\delta := \inf\{t \geq 0 : y(t) \geq \delta\}.$$

Our claim is that $T_X^\delta = V(T_Y^\delta)$, where V is the process defined through (27). Since V^{-1} is its inverse and it is a continuous process, trivially we have $V^{-1}(V(T_Y^\delta)) = T_Y^\delta$ and $Y(V^{-1}(V(T_Y^\delta))) = \delta$. Because $X(t) = Y(V^{-1}(t))$, we obtain the inequality $T_X^\delta \leq V(T_Y^\delta)$, which we do not know a priori to be an equality as it might not be the infimum. If we assume that the last inequality is strict, then by composing with V^{-1} we have

$$V^{-1}(T_X^\delta) < V^{-1}(V(T_Y^\delta)) = T_Y^\delta,$$

since V^{-1} is strictly increasing. But $Y(V^{-1}(T_X^\delta)) = X(T_X^\delta) = \delta$, in contradiction with the definition of hitting time T_Y^δ . We conclude that

$$T_X^\delta = V(T_Y^\delta) = T_Y^\delta + \eta \ell_Y(T_Y^\delta).$$

In expectation, we get

$$\mathbf{E}_v[T_X^\delta] = \mathbf{E}_v[T_Y^\delta] + \eta \mathbf{E}_v[\ell_Y(T_Y^\delta)].$$

Using (18), Lemma 3.3 and the fact that b is bounded, we have

$$\mathbf{E}_v[\ell_Y(T_Y^\delta)] = \mathbf{E}_v[y(T_Y^\delta)] + \mathbf{E}_v \left[\int_0^{T_Y^\delta} b_{j(s)}(y(s)) ds \right] = \delta + O(\delta^2) \quad \text{as } \delta \downarrow 0.$$

Moreover, also from Lemma 3.3, we have $\mathbf{E}_v[T_Y^\delta] = O(\delta^2)$. From (30), we deduce

$$\frac{\mathbf{E}_v [T_X^\delta]}{\delta} \left(Lf(v) + \mathbf{E}_v \left[\frac{1}{\mathbf{E}_v [T_X^\delta]} \int_0^{T_X^\delta} Lf(X(s)) - Lf(v) ds \right] \right) = \sum_{k=1}^n \rho_k^\delta \frac{f_k(\delta) - f_k(0)}{\delta}.$$

Using the fact that

$$\left| \mathbf{E}_v \left[\frac{1}{\mathbf{E}_v [T_X^\delta]} \int_0^{T_X^\delta} Lf(X(s)) - Lf(v) ds \right] \right| \leq \sup_{x,y \in B(v,\delta)} |Lf(x) - Lf(y)| \xrightarrow{\delta \rightarrow 0} 0,$$

together with Lemmas 3.3 and 3.4, we conclude by passing to the limit $\delta \downarrow 0$ that

$$\eta Lf(v) = \sum_{k=1}^n \rho_k f'_k(0).$$

This proves that $D(L^\eta) \cap C_b^2(\Gamma) \subset D(L)$. It then follows that the semigroups induced by $(L, D(L))$ and $(L^\eta, D(L^\eta))$ coincide.

Finally, we define

$$\ell_X(t) = \ell_Y(V^{-1}(t)) \quad \text{a.s. for all } t \geq 0. \quad (31)$$

Moreover, notice that

$$t = V(V^{-1}(t)) = V^{-1}(t) + \eta \ell_Y(V^{-1}(t)) = V^{-1}(t) + \eta \ell_X(t),$$

and therefore

$$V^{-1}(t) = t - \eta \ell_X(t). \quad (32)$$

□

From Theorem 4.1 we are able, similarly to [46, Theorem 3.1], to characterize the occupation time of the sticky diffusion process at v .

Proposition 4.2. *For every bounded and measurable function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ the time-changed process X satisfies*

$$\int_0^t f(s) \mathbf{1}_{\{X_s=v\}} ds = \eta \int_0^t f(s) d\ell_X(s) \quad \text{almost surely for every } t > 0, \quad (33)$$

where the integral in the right-hand side must be understood in the Lebesgue-Stieltjes sense. In particular, the occupation time of X at v is given by

$$\int_0^t \mathbf{1}_{\{X_s=v\}} ds = \eta \ell_X(t) \quad \text{almost surely for every } t > 0. \quad (34)$$

Proof. We start by proving (34). From Theorem 3.1-(ii), we know that, since the diffusion Y has $\eta = 0$ in (15),

$$\int_0^t \mathbf{1}_{\{Y(s)=v\}} ds = 0 \quad \text{almost surely for all } t > 0.$$

Then,

$$\int_0^{V^{-1}(t)} \mathbf{1}_{\{Y(s)=v\}} ds = 0 \quad \text{almost surely for all } t > 0,$$

and since $V^{-1}(t)$ is an increasing function (in particular it has finite variation) and using formula (32), we rewrite the integral as the Lebesgue-Stieltjes integral

$$\begin{aligned} 0 &= \int_0^{V^{-1}(t)} \mathbf{1}_{\{Y(s)=v\}} ds = \int_0^t \mathbf{1}_{\{X(s)=v\}} dV^{-1}(s) = \int_0^t \mathbf{1}_{\{X(s)=v\}} d(t - \eta\ell_X(t)) \\ &= \int_0^t \mathbf{1}_{\{X(s)=v\}} dt - \eta \int_0^t \mathbf{1}_{\{X(s)=v\}} d\ell_X(t) \\ &= \int_0^t \mathbf{1}_{\{X(s)=v\}} dt - \eta\ell_X(t), \end{aligned} \tag{35}$$

where in the last step we used the fact that the random measure $d\ell_X$ is almost surely supported on the set $\{t \geq 0 : X(t) = v\}$. We now turn to the proof of (33). Notice first that the conclusion is a direct consequence of (34) if f is a simple function. The general case then follows from the dominated convergence theorem, after approximation of f by simple functions. \square

Remark 4.3. The diffusion X we are considering is a Markov process, as guaranteed by [25, Theorem 3.1]. However, if we want to introduce memory to the process and make it non-Markovian, we can use recent developments provided by the Fractional Boundary Value Problem, as described in [8, Section 4] and [17]. A complete treatment, on star graphs, is also available in [9, Section 4].

Let us denote by H a subordinator characterized by

$$\mathbf{E}_0[e^{-\lambda H(t)}] = e^{-t\Phi(\lambda)}, \quad \lambda > 0,$$

where Φ is a Bernstein function

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda z}) \Pi^\Phi(dz), \quad \lambda > 0,$$

and Π^Φ is a Lévy measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge z) \Pi^\Phi(dz) < \infty$. For us, $\bar{\Pi}^\Phi(z) = \Pi^\Phi(z, \infty)$ is the tail of the Lévy measure. We also introduce the process

$$V_H(t) = t + H(\eta\ell_Y(t))$$

and V_H^{-1} is its right inverse. The most significant difference will be this dependence on H in the time change.

Let $M > 0$, $w \geq 0$ and \mathcal{M}_w be the set of (piecewise) continuous function on $[0, \infty)$ of exponential order w such that $|\varphi(t)| \leq Me^{wt}$. Let $\varphi \in \mathcal{M}_w \cap C[0, \infty)$ with $\varphi' \in \mathcal{M}_w$. Then, we define the Caputo-Dzherbashian-type derivative as

$$\mathfrak{D}_t^\Phi \varphi(t) := \int_0^t \varphi'(s) \bar{\Pi}^\Phi(t-s) ds \tag{36}$$

which is a convolution type operator. Then, by proceeding as in [8, Theorem 22], we have that $Y(V_H^{-1}(t))$ is related to the non-local dynamic condition condition

$$\eta \mathfrak{D}_t^\Phi u(t, v) = \sum_{i=1}^n \rho_i \frac{d}{dx} u_i(t, x)|_{x=0}.$$

In this case, the dynamics is no longer Markovian, since in the time change appears the inverse of the subordinator H , which is non-Markov. Then, it spends a non-exponential time in the vertex. Just as the process X has its roots in the skew sticky Brownian motion, the process $Y(V_H^{-1})$ originates from the non-local skew sticky Brownian motion, described in [16].

4.2. Characterization of sticky diffusions. We now prove that any sticky process has a modification that is a time-changed nonsticky one.

Theorem 4.4. *Let $X(t) = (i(t), x(t))$ be the process generated by (13) and (15) with $\eta > 0$. Then there exists a nonsticky diffusion \tilde{Y} on Γ and a strictly increasing continuous function V^{-1} such that X and $\tilde{Y}(V^{-1}(\cdot))$ are indistinguishable. Moreover, there exists an increasing process ℓ_X , adapted to the completion of the natural filtration of X with respect to the collection of probability measures $(\mathbf{P}_x)_{x \in \Gamma}$, such that ℓ_X almost surely increases only when $X = \mathbf{v}$, that $\eta \ell_X(t) = \int_0^t \mathbf{1}_{\{X(s)=\mathbf{v}\}} ds$ and $\ell_X(t) = \ell_{\tilde{Y}}(V^{-1}(t))$ almost surely, where $\ell_{\tilde{Y}}$ is the process appearing in Theorem 3.1-(ii).*

Proof. The argument is reminiscent of part 4 of the proof of [19, Theorem 5]. Set

$$R(t) := \int_0^t \mathbf{1}_{\{X(s) \neq \mathbf{v}\}} ds = t - \int_0^t \mathbf{1}_{\{X(s)=\mathbf{v}\}} ds,$$

and notice that R is a.s. continuous and nondecreasing. Furthermore, since the vertex \mathbf{v} is not a trap when η is finite, we deduce that $\lim_{t \rightarrow \infty} R(t) = \infty$ almost surely. We may therefore consider the right inverse

$$R^{-1}(t) := \inf \{s > 0 : R(s) > t\}$$

which is strictly increasing and right-continuous. We claim that R cannot be constant on some interval $[a, b] \subset \mathbb{R}_+$ with $b > a$. Indeed, if it is the case, we have $\int_a^b \mathbf{1}_{\{X(s) \neq \mathbf{v}\}} ds = 0$ and therefore the set $\{t \in [a, b] : X(t) \neq \mathbf{v}\}$ is negligible. Since X almost surely has continuous paths, this implies that $X(t) = \mathbf{v}$ a.s. on $[a, b]$, a contradiction. This implies that R^{-1} is in fact continuous and that $R^{-1}(R(t)) = t$ almost surely. Moreover, since R is \mathcal{F}_t -adapted, where \mathcal{F}_t is the natural filtration of X , we have that each $R^{-1}(t)$ is a \mathcal{F}_t -stopping time. It follows that the process R^{-1} defines a time-change in the sense of [45, Definition 1.2 p.180]. We define the time-changed process

$$\tilde{Y}(t) := X(R^{-1}(t)) \quad t \geq 0,$$

which is a Markov process for the filtration $\tilde{\mathcal{G}}_t := \mathcal{F}_{R^{-1}(t)}$ (see [45, Theorem 2.18 p. 417] for a proof of this fact in the case of a Brownian motion and [18, Theorem 10.10] for the general case). Let now $(\tilde{L}, D(\tilde{L}))$ be the generator of the semigroup induced by \tilde{Y} on $C_0(\Gamma)$. We claim that the semigroups generated by $(\tilde{L}, D(\tilde{L}))$ and $(L, D(L))$, for $\eta = 0$, coincide. Let $\bar{x} > 0$ and set $\bar{x} = (i, \bar{x})$ for some $i \in \{1, \dots, N\}$. Let $f \in D(\tilde{L}) \cap C_b^2(\Gamma)$ and let $\psi \in C^\infty(\mathbb{R}_+)$ be such that

$$0 \leq \psi \leq 1 \quad \text{and} \quad \psi(x) = \begin{cases} 0 & \text{for } x \leq \bar{x}/3, \\ 1 & \text{for } x \geq 2\bar{x}/3, \end{cases}$$

and define $g \in C_b^2(\Gamma)$ by setting $g_i(x) = f_i(x)\psi(x)$. Notice that $g \in D(L)$. Set also $\tau := \inf\{t \geq 0 : y(t) \leq 2\bar{x}/3\}$ and notice that $R(t) = t$ on $[0, \tau]$ if X is started at \bar{x} . Then, using Dynkin's formula, we have

$$\begin{aligned} \mathbf{E}_{\bar{x}}[f(\tilde{Y}(t \wedge \tau))] - f(\bar{x}) &= \mathbf{E}_{\bar{x}}[g(X(R^{-1}(t \wedge \tau))) - g(\bar{x})] = \mathbf{E}_{\bar{x}}[g(X(t \wedge \tau))] - g(\bar{x}) \\ &= \mathbf{E}_{\bar{x}} \left[\int_0^{t \wedge \tau} Lg(X(s)) ds \right]. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{L}f(\bar{x}) &= \lim_{t \rightarrow 0^+} t^{-1} \left(\mathbf{E}_{\bar{x}}[f(\tilde{Y}(t \wedge \tau))] - f(\bar{x}) \right) \\ &= \lim_{t \rightarrow 0^+} t^{-1} \mathbf{E}_{\bar{x}} \left[\int_0^{t \wedge \tau} Lg(s) ds \right] = Lg(\bar{x}) = Lf(\bar{x}), \end{aligned}$$

so that $\tilde{L}f(x) = Lf(x)$ for all $x \neq \mathbf{v}$ and $f \in D(\tilde{L}) \cap C_b^2(\Gamma)$.

We now turn to the characterization of $D(\tilde{L})$. Let $f \in \tilde{D} \cap C_b^2(\Gamma)$ and consider, for all $\delta > 0$, the stopping times

$$T_X^\delta := \inf \{t \geq 0 : x(t) \geq \delta\},$$

and $T_{\tilde{Y}}^\delta$, defined analogously. Following the argument in Proposition 4.1, one may prove that $T_{\tilde{Y}}^\delta = R(T_X^\delta)$. Using Taylor's formula, we have

$$\begin{aligned} \mathbf{E}_\nu \left[f(\tilde{Y}(T_{\tilde{Y}}^\delta)) - f(\nu) \right] &= \mathbf{E}_\nu \left[f(X(T_X^\delta)) - f(\nu) \right] = \sum_{j=1}^N \rho_j^\delta (f_j(\delta) - f_j(0)) \\ &= \sum_{j=1}^N \rho_j^\delta (f_j'(0)\delta + O(\delta^2)), \end{aligned}$$

where $\rho_j^\delta = \mathbf{P}_\nu(i(T_X^\delta) = j)$. On the other hand, Dynkin's formula yields

$$\begin{aligned} \left| \mathbf{E}_\nu \left[f(\tilde{Y}(T_{\tilde{Y}}^\delta)) - f(\nu) \right] \right| &= \left| \mathbf{E}_\nu \left[\int_0^{T_{\tilde{Y}}^\delta} Lf(\tilde{Y}(s)) ds \right] \right| = \left| \mathbf{E}_\nu \left[\int_0^{R(T_X^\delta)} Lf(X(R^{-1}(s))) ds \right] \right| \\ &\leq C \mathbf{E}_\nu \left[R(T_X^\delta) \right] = C \mathbf{E}_\nu \left[T_X^\delta \right] - \mathbf{E}_\nu \left[\int_0^{T_X^\delta} \mathbf{1}_{\{X(s)=\nu\}} ds \right] \\ &= O(\delta^2), \end{aligned}$$

where we used Lemmas 3.3 and 3.5 to deduce that last line. Dividing by δ and sending $\delta \downarrow 0$, using Lemma 3.4, we deduce that

$$\sum_{j=1}^N \rho_j f_j'(0) = 0,$$

which proves that $D(\tilde{L}) \cap C_b^2(\Gamma) \subset D(L)$. We conclude that \tilde{Y} is a nonsticky diffusion on Γ . According to Proposition 4.1 we can define a sticky process $\tilde{X}(t) = \tilde{Y}(V^{-1}(t)) = X(R^{-1}(V^{-1}(t)))$, with stickiness parameter η . We claim that $R^{-1}(V^{-1}(t)) = t$ \mathbf{P}_ν -almost surely. Since

$$t = R^{-1} \circ V^{-1} \circ V \circ R(t) \quad \text{almost surely,}$$

it is enough to prove that $V(R(t)) = t$ \mathbf{P}_ν -almost surely. We a.s. have

$$\begin{aligned} \frac{1}{\delta} \int_0^{R(t)} \mathbf{1}_{\{y(s) \leq \delta\}} ds &= \frac{1}{\delta} \int_0^t \mathbf{1}_{\{x(s) \leq \delta\}} dR(s) = \frac{1}{\delta} \int_0^t \mathbf{1}_{\{x(s) \leq \delta\}} ds - \frac{1}{\delta} \int_0^t \mathbf{1}_{\{x(s)=0\}} ds \\ &= \frac{1}{\delta} \int_0^t \mathbf{1}_{\{0 < x(s) \leq \delta\}} ds \end{aligned}$$

so that, using Theorem 3.1, we have

$$\ell_Y(R(t)) = \lim_{\delta \rightarrow 0} \frac{\Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < x(s) \leq \delta\}} ds \quad \text{in expectation,} \quad (37)$$

where

$$\Lambda := \left(\sum_{i=1}^N \frac{2\rho_i}{\sigma_i(0)^2} \right)^{-1}.$$

Recall that

$$V(R(t)) = R(t) + \eta \ell_Y(R(t)) = t - \int_0^t \mathbf{1}_{\{x(s)=0\}} ds + \ell_Y(R(t)).$$

Therefore, we obtain the claim if we are able to prove that $\int_0^t \mathbf{1}_{\{x(s)=0\}} ds = \eta \hat{\ell}_X(t)$ \mathbf{P}_v -almost surely, where

$$\hat{\ell}_X(t) := \lim_{\delta \rightarrow 0} \frac{\Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < x(s) \leq \delta\}} ds = \ell_Y(R(t)).$$

Recall that we have defined $\ell_{\tilde{X}}(t) := \ell_Y(V^{-1}(t))$ and that

$$V^{-1}(t) = t - \eta \ell_{\tilde{X}}(t) = t - \int_0^t \mathbf{1}_{\{\tilde{x}(s)=0\}} ds, \quad (38)$$

where we have used Proposition 4.2. Noticing also that, using (38),

$$\int_0^{V^{-1}(t)} \mathbf{1}_{\{y(s) \leq \delta\}} ds = \int_0^t \mathbf{1}_{\{\tilde{x}(s) \leq \delta\}} dV^{-1}(s) = \int_0^t \mathbf{1}_{\{0 < \tilde{x}(s) \leq \delta\}} ds,$$

we deduce from Theorem 3.1 that

$$\ell_{\tilde{X}}(t) = \lim_{\delta \rightarrow 0} \frac{\Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < \tilde{x}(s) \leq \delta\}} ds \quad \text{in expectation.} \quad (39)$$

Using Proposition 4.2 and the fact that X and \tilde{X} have the same transition function, we deduce that

$$\begin{aligned} & \mathbf{E}_x \left[\left| \int_0^t \mathbf{1}_{\{x(s)=0\}} ds - \frac{\eta \Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < x(s) \leq \delta\}} ds \right| \right] \\ &= \mathbf{E}_x \left[\left| \int_0^t \mathbf{1}_{\{\tilde{x}(s)=0\}} ds - \frac{\eta \Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < \tilde{x}(s) \leq \delta\}} ds \right| \right] \\ &= \eta \mathbf{E}_x \left[\left| \ell_{\tilde{X}}(t) - \frac{\Lambda}{\delta} \int_0^t \mathbf{1}_{\{0 < \tilde{x}(s) \leq \delta\}} ds \right| \right] \xrightarrow{\delta \rightarrow 0} 0. \end{aligned}$$

The claim is proved. In particular, it follows that $X(t) = \tilde{X}(t)$ \mathbf{P}_x -almost surely for every $t \geq 0$. Since these processes are continuous, they are in fact indistinguishable. We now check that \tilde{X} is adapted to $(\hat{\mathcal{F}}_t^X)_{t \geq 0}$, the completion of the natural filtration $(\mathcal{F}_t^X)_{t \geq 0}$ of X with respect to $(\mathbf{P}_x)_{x \in \Gamma}$. Indeed, there is a \mathbf{P}_x -negligible set N such that $X(t, \omega) = \tilde{X}(t, \omega)$ for every $\omega \notin N$. Then, for $E \in \mathcal{F}_t^X$ we have

$$\{\omega : \tilde{X}(t, \omega) \in E\} = (\{\omega : X(t, \omega) \in E\} \cap N^c) \cup (\{\omega : \tilde{X}(t, \omega) \in E\} \cap N) \in \hat{\mathcal{F}}_t^X.$$

Moreover, from (39), we deduce that $\ell_{\tilde{X}}$ is also adapted to $(\mathcal{F}_t^X)_{t \geq 0}$. This concludes the proof if we set $\ell_X = \ell_{\tilde{X}}$. □

4.3. Stochastic differential equation. Applying Theorem 3.1-(ii) to the nonsticky process $Y(t) = (j(t), y(t))$ we have that

$$dy(t) = \sigma_{j(s)}(y(t)) dW_t + b_{j(s)}(y(t)) dt + d\ell_Y(t), \quad (40)$$

where W is a one-dimensional Brownian motion measurable with adapted to \mathcal{G}_t , the natural filtration generated by Y . On the other hand, an SDE representation for sticky process was obtained in [46, Theorem 3.2] in the particular case $N = 2$. The point of the next result is to extend this representation to the case $N > 2$.

Theorem 4.5. *Up to an extension of the filtered probability space, there exists a one dimensional Brownian motion \tilde{W} , such that the sticky process $X(t) = (x(t), i(t))$ satisfies*

$$dx(t) = \sigma_{i(s)}(x(t)) \mathbf{1}_{\{x(t) \neq 0\}} d\tilde{W}_t + b_{i(s)}(x(t)) \mathbf{1}_{\{x(t) \neq 0\}} dt + d\ell_X(t), \quad (41)$$

where ℓ_X is the process obtained in Theorem 4.4.

Proof. The only difference with [46, Theorem 3.2], is that now the existence of the SDE for the diffusion y is given by (40). For the rest, we can proceed with the same steps. Using Theorem 4.4, we may assume that, almost surely, $X(t) = \tilde{Y}(V^{-1}(t))$ and $\ell_X(t) = \ell_{\tilde{Y}}(V^{-1}(t))$ for some nonsticky diffusion $\tilde{Y}(\cdot) = (j(\cdot), \tilde{y}(\cdot))$. From (32) and (34), we have

$$V^{-1}(t) = t - \eta \ell_X(t) = \int_0^t \mathbf{1}_{\{X(s) \neq v\}} ds$$

Using the fact that $x(t) = \tilde{y}(V^{-1}(t))$, we have

$$\int_0^t \mathbf{1}_{\{X(s) \neq v\}} ds = \int_0^t \mathbf{1}_{\{x(s) \neq 0\}} ds.$$

By applying the time change $V^{-1}(t)$, we get

$$dx(t) = \sigma(x(t))dS(t) + b(x(t))\mathbf{1}_{\{x(t) \neq 0\}} dt + d\ell_X(t)$$

where $S(t) = W \left(\int_0^t \mathbf{1}_{\{x(s) \neq 0\}} ds \right)$. We define

$$\widetilde{W}(t) := W \left(\int_0^t \mathbf{1}_{\{x(s) \neq 0\}} ds \right) + W_2 \left(\int_0^t \mathbf{1}_{\{x(s) = 0\}} ds \right),$$

where W_2 is a Brownian motion independent of W (which is known to exist up to extension of the probability space, see [33, Remark 4.1 p.169]). This concludes the proof since $dS(t) = \mathbf{1}_{\{x(t) \neq 0\}} d\widetilde{W}(t)$. \square

Let us now provide a result, consistent with the well-known convergence of the upcrossing number to the local time, which we will later use in the proof of Itô's formula.

Corollary 4.6. *Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be bounded and continuous and let $X(t) = (i(t), x(t))$ be a sticky process on Γ , started at v . Define, for $\delta > 0$, the collections of stopping times $(\tau_n^\delta)_{n \in \mathbb{N}}$ and $(\theta_n^\delta)_{n \in \mathbb{N}}$ by $\tau_0^\delta = \theta_0^\delta = 0$ and*

$$\begin{aligned} \theta_n^\delta &= \inf \left\{ t \geq \tau_{n-1}^\delta : x(t) = \delta \right\}, \\ \tau_n^\delta &= \inf \left\{ t \geq \theta_{n-1}^\delta : x(t) = 0 \right\}, \end{aligned} \tag{42}$$

for $n \geq 1$. Then, for every $t \geq 0$,

$$\lim_{\delta \rightarrow 0} \delta \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} = \int_0^t f(s) d\ell_X(s) \quad \text{in probability.}$$

In particular

$$\lim_{\delta \rightarrow 0} \delta \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} = \ell_X(t) \quad \text{in probability.}$$

Proof. For this proof, we rely on the one for the convergence in probability of the number of upcrossings to the local time; see for example [38, Proposition 9.11, page 248]. Notice first that, for every $n \in \mathbb{N}$, we have, from Theorem 4.5,

$$\begin{aligned} R_n &:= f(\theta_{n+1}^\delta)(x(\theta_{n+1}^\delta \wedge t) - x(\tau_n^\delta \wedge t)) = f(\theta_{n+1}^\delta) \int_{\tau_n^\delta \wedge t}^{\theta_{n+1}^\delta \wedge t} \sigma_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} d\widetilde{W}(s) \\ &\quad + f(\theta_{n+1}^\delta) \int_{\tau_n^\delta \wedge t}^{\theta_{n+1}^\delta \wedge t} b_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} ds + f(\theta_{n+1}^\delta)(\ell_X(\theta_{n+1}^\delta \wedge t) - \ell_X(\tau_n^\delta \wedge t)). \end{aligned}$$

We sum over $n \in \mathbb{N}$

$$\begin{aligned} \sum_{n \in \mathbb{N}} R_n &= \int_0^t \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{(\tau_n^\delta, \theta_{n+1}^\delta]}(s) \sigma_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} d\widetilde{W}(s) \\ &\quad + \int_0^t \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{(\tau_n^\delta, \theta_{n+1}^\delta]}(s) b_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} ds \\ &\quad + \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) (\ell_X(\theta_{n+1}^\delta \wedge t) - \ell_X(\tau_n^\delta \wedge t)) \end{aligned}$$

where we have used that there are only finitely many values of n such that $\theta_{n+1}^\delta \leq t$. Since b and f are bounded and

$$0 \leq \mathbf{1}_{(\tau_n^\delta, \theta_{n+1}^\delta]}(s) \mathbf{1}_{\{x(s) > 0\}} \leq \mathbf{1}_{\{0 < x(s) \leq \delta\}} \xrightarrow{\delta \rightarrow 0} 0 \quad \text{almost surely,}$$

we have

$$\int_0^t \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{(\tau_n^\delta, \theta_{n+1}^\delta]}(s) b_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} ds \xrightarrow{\delta \downarrow 0} 0 \quad \text{almost surely.}$$

Similarly, using [38, Proposition 5.8, page 111], we get

$$\int_0^t \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{(\tau_n^\delta, \theta_{n+1}^\delta]}(s) \sigma_{i(s)}(x(s)) \mathbf{1}_{\{x(s) > 0\}} d\widetilde{W}(s) \xrightarrow{\delta \downarrow 0} 0 \quad \text{in probability,}$$

From the properties of Riemann-Stieltjes integrals and the fact that ℓ_X a.s. does not increase between θ_{n+1}^δ and τ_{n+1}^δ , we also have

$$\lim_{\delta \rightarrow 0} \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) (\ell_X(\theta_{n+1}^\delta \wedge t) - \ell_X(\tau_n^\delta \wedge t)) = \int_0^t f(s) d\ell_X(s) \quad \text{almost surely.}$$

We conclude the proof by observing that

$$\sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) (x(\theta_{n+1}^\delta \wedge t) - x(\tau_n^\delta \wedge t)) = \delta \sum_{n \in \mathbb{N}} f(\theta_{n+1}^\delta) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} + O(\delta),$$

Indeed, $x(\theta_{n+1}^\delta \wedge t) - x(\tau_n^\delta \wedge t) = \delta$ when $\theta_{n+1}^\delta \leq t$, and when $\theta_{n+1}^\delta > t$, we have $x(t) - x(\tau_n^\delta \wedge t) \leq \delta$. Hence, the claim holds. \square

4.4. Freidlin-Sheu-Itô formula. The goal of this section is to extend the Itô type formula proved in [24, Lemma 2.3] for the nonsticky diffusion to the case of sticky diffusions. Note that the special case $N = 2$ was considered in [46, Lemma 3.5].

Theorem 4.7. *Let $X(t) = (i(t), x(t))$ be the process generated by (13) and (15) with $\eta > 0$, started at $X(0) = \mathbf{x} \in \Gamma$, and let $f \in C_b^{1,2}(\mathbb{R}_+ \times \Gamma)$. Then,*

$$\begin{aligned} f(t, X(t)) &= f(0, \mathbf{x}) + \int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) \neq \mathbf{v}\}} ds \\ &\quad + \int_0^t \sigma(X(s)) \partial_x f(s, X(s)) \mathbf{1}_{\{X(s) \neq \mathbf{v}\}} d\widetilde{W}(s) + \int_0^t \left(\eta \partial_s f(s, \mathbf{v}) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) \right) d\ell_X(s), \end{aligned} \tag{43}$$

where \widetilde{W} is the Brownian motion obtained in Theorem 4.5. In particular, if $Lf(t, \cdot) \in C(\Gamma)$ for all $t \geq 0$, we have

$$\begin{aligned} f(t, X(t)) &= f(0, \mathbf{x}) + \int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) ds \\ &+ \int_0^t \sigma(X(s)) \partial_x f(s, X(s)) \mathbf{1}_{\{X_s \neq \mathbf{v}\}} d\widetilde{W}(s) + \int_0^t \left(-\eta Lf(s, \mathbf{v}) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) \right) d\ell_X(s). \end{aligned} \quad (44)$$

Proof. We follow the proof of the nonsticky case (see [24; 28]).

Without loss of generality, we may assume that $\mathbf{x} = (i, 0)$. Fix $\delta > 0$. We recursively define the following sequence of stopping times : $\tau_0^\delta = \theta_0^\delta = 0$ and, for every integer $n \geq 1$,

$$\begin{aligned} \theta_n^\delta &= \inf \left\{ t \geq \tau_{n-1}^\delta : x(t) = \delta \right\}, \\ \tau_n^\delta &= \inf \left\{ t \geq \theta_{n-1}^\delta : x(t) = 0 \right\}. \end{aligned} \quad (45)$$

Since $x(\cdot)$ is a diffusion process on \mathbb{R}_+ , reflected at 0, it is in particular a semimartingale so that we know that its number of upcrossings from 0 to δ up to time t is almost surely finite. We then decompose

$$\begin{aligned} f(t, X(t)) - f(0, \mathbf{x}) &= \sum_{n=0}^{+\infty} f(\theta_{n+1}^\delta \wedge t, X(\theta_{n+1}^\delta \wedge t)) - f(\theta_n^\delta \wedge t, X(\theta_n^\delta \wedge t)) \\ &= Q_1^\delta + Q_2^\delta + Q_3^\delta, \end{aligned}$$

where

$$\begin{aligned} Q_1^\delta &:= \sum_{n \geq 0} \left(f(\theta_{n+1}^\delta \wedge t, X(\theta_{n+1}^\delta \wedge t)) - f(\tau_n^\delta \wedge t, X(\tau_n^\delta \wedge t)) \right) \\ &- \sum_{j=1}^N \sum_{n \geq 0} \delta \partial_x f_j(\theta_{n+1}^\delta, 0) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t, i(\theta_{n+1}^\delta) = j\}}, \end{aligned} \quad (46)$$

$$Q_2^\delta := \sum_{j=1}^N \sum_{n \geq 0} \delta \partial_x f_j(\theta_{n+1}^\delta, 0) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t, i(\theta_{n+1}^\delta) = j\}}, \quad (47)$$

and

$$Q_3^\delta := \sum_{n \geq 0} f(\tau_n^\delta \wedge t, X(\tau_n^\delta \wedge t)) - f(\theta_n^\delta \wedge t, X(\theta_n^\delta \wedge t)). \quad (48)$$

We also write $Q_1^\delta = Q_{(1,1)}^\delta + Q_{(1,2)}^\delta$, where

$$Q_{(1,1)}^\delta := \sum_{j=1}^N \sum_{n \geq 0} \left(f_j(\theta_{n+1}^\delta, x(\theta_{n+1}^\delta)) - f_j(\tau_n^\delta, x(\tau_n^\delta)) - \delta \partial_x f_j(\theta_{n+1}^\delta, 0) \right) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t, i(\theta_{n+1}^\delta) = j\}} \quad (49)$$

and

$$Q_{(1,2)}^\delta := \sum_{j=1}^N \sum_{n \geq 0} \left(f_j(\theta_{n+1}^\delta \wedge t, x(\theta_{n+1}^\delta \wedge t)) - f_j(\tau_n^\delta \wedge t, x(\tau_n^\delta \wedge t)) \right) \mathbf{1}_{\{\theta_{n+1}^\delta > t, i(\theta_{n+1}^\delta) = j\}}. \quad (50)$$

Step 1: $Q_1^\delta \xrightarrow{\delta \rightarrow 0} \eta \int_0^t \partial_s f(s, \mathbf{v}) \ell_X(ds)$ in probability.

From the definition of θ_n, τ_n and the continuity of X we know that, for every $n \geq 0$ and $j \in \{1, \dots, N\}$,

$$f_j(\theta_{n+1}^\delta, x(\theta_{n+1}^\delta)) = f(\theta_{n+1}^\delta, \delta) \quad \text{and} \quad f_j(\tau_n^\delta, x(\tau_n^\delta)) = f(\tau_n^\delta, 0) \quad \text{almost surely.}$$

It then follows that, on each edge j ,

$$\begin{aligned}
 & f_j(\theta_{n+1}^\delta, x(\theta_{n+1}^\delta)) - f_j(\tau_n^\delta, x(\tau_n^\delta)) - \delta \partial_x f_j(\theta_{n+1}^\delta, 0) \\
 &= f_j(\theta_{n+1}^\delta, \delta) - f_j(\tau_n^\delta, 0) - \delta \partial_x f_j(\theta_{n+1}^\delta, 0) \\
 &= f_j(\theta_{n+1}^\delta, \delta) - f_j(\theta_{n+1}^\delta, 0) - \delta \partial_x f_j(\theta_{n+1}^\delta, 0) + f_j(\theta_{n+1}^\delta, 0) - f_j(\tau_n^\delta, 0) \\
 &= \delta \int_0^\delta \partial_z f_j(\theta_{n+1}^\delta, z) - \partial_x f_j(\theta_{n+1}^\delta, 0) dz + \int_{\tau_n^\delta}^{\theta_{n+1}^\delta} \partial_s f_j(s, 0) ds.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & \left| Q_{(1,1)}^\delta - \left[\sum_{j=1}^N \sum_{n \geq 0} \int_{\tau_n^\delta}^{\theta_{n+1}^\delta} \partial_s f_j(s, 0) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t, i(\theta_{n+1}^\delta) = j\}} ds \right] \right| \\
 &= \left| Q_{(1,1)}^\delta - \sum_{n \geq 0} \int_0^{\theta_{n+1}^\delta} \partial_s f(s, 0) \mathbf{1}_{\{x(s) \leq \delta, \theta_{n+2}^\delta > t\}} ds \right| \\
 &\leq N \delta^2 \|\partial_x^2 f\|_\infty \sum_{n \geq 0} \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}}.
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 & \left| Q_{(1,2)}^\delta - \sum_{j=1}^N \sum_{n \geq 0} \int_{\tau_n^\delta}^t \partial_s f_j(s, 0) \mathbf{1}_{\{\theta_{n+1}^\delta > t, i(\theta_{n+1}^\delta) = j\}} ds \right| \\
 &= \left| \sum_{j=1}^N \sum_{n \geq 0} \left(f_j(t, x(t)) - f_j(\tau_n^\delta, x(\tau_n^\delta)) - \int_{\tau_n^\delta}^t \partial_s f_j(s, 0) ds \right) \mathbf{1}_{\{\theta_{n+1}^\delta > t, i(\theta_{n+1}^\delta) = j\}} \right| \\
 &\leq \sum_{j=1}^N \sum_{n \geq 0} \left| (f_j(t, x(t)) - f_j(\tau_n^\delta, x(\tau_n^\delta)) - \int_{\tau_n^\delta}^t \partial_s f_j(s, 0) ds) \mathbf{1}_{\{\theta_{n+1}^\delta > t, i(\theta_{n+1}^\delta) = j\}} \right| \\
 &\leq \delta \|\partial_x f\|_\infty.
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 & \left| Q_1^\delta - \sum_{n \geq 0} \int_{\tau_n^\delta \wedge t}^{\theta_{n+1}^\delta \wedge t} \partial_s f(s, 0) \mathbf{1}_{\{x(s) \leq \delta\}} ds \right| \\
 &\leq \left| Q_{(1,1)}^\delta - \sum_{j=1}^N \sum_{n \geq 0} \int_{\tau_n^\delta}^{\theta_{n+1}^\delta} \partial_s f_j(s, 0) \mathbf{1}_{\{\theta_{n+1}^\delta \leq t, i(\theta_{n+1}^\delta) = j\}} ds \right| \\
 &\quad + \left| Q_{(1,2)}^\delta - \sum_{j=1}^N \sum_{n \geq 0} \int_{\tau_n^\delta}^t \partial_s f_j(s, 0) \mathbf{1}_{\{\theta_{n+1}^\delta > t, i(\theta_{n+1}^\delta) = j\}} ds \right| \\
 &\leq N \delta^2 \|\partial_x^2 f\|_\infty \sum_{n \geq 0} \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} + \delta \|\partial_x f\|_\infty.
 \end{aligned}$$

From Corollary 4.6, we recall that $\delta \sum_{n \geq 0} \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}}$ goes, for $\delta \downarrow 0$, in probability to $\ell_X(t)$. Then, we have

$$N \delta^2 \|\partial_x^2 f\|_\infty \sum_{n \geq 0} \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} \xrightarrow{\delta \rightarrow 0} 0 \quad \text{in probability}$$

and

$$\delta \|\partial_x f\|_\infty \xrightarrow{\delta \rightarrow 0} 0 \quad \text{a.s.}$$

Therefore,

$$Q_1^\delta - \sum_{n \geq 0} \int_{\tau_n^\delta \wedge t}^{\theta_{n+1}^\delta \wedge t} \partial_s f(s, 0) \mathbf{1}_{\{X(s) \leq \delta\}} ds \xrightarrow{\delta \rightarrow 0} 0 \quad \text{in probability}$$

But, from Lebesgue's dominated convergence theorem, we get

$$\lim_{\delta \rightarrow 0} \sum_{n \geq 0} \int_{\tau_n^\delta \wedge t}^{\theta_{n+1}^\delta \wedge t} \partial_s f(s, 0) \mathbf{1}_{\{X(s) \leq \delta\}} ds = \int_0^t \partial_s f(s, 0) \mathbf{1}_{\{X(s) = v\}} ds,$$

so that, using Proposition 4.2,

$$\int_0^t \partial_s f(s, 0) \mathbf{1}_{\{X(s) = v\}} ds = \eta \int_0^t \partial_s f(s, 0) \ell_X(ds).$$

Then, our first claim holds true.

Step 2: $Q_2^\delta \xrightarrow{\delta \rightarrow 0} \int_0^t \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) d\ell_X(s)$ in probability.
First, we write

$$Q_2^\delta = \sum_{n \geq 0} \delta \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} \sum_{j=1}^N \partial_x f_j(\theta_{n+1}^\delta, 0) \mathbf{1}_{\{i(\theta_{n+1}^\delta) = j\}}.$$

We observe that from Lemma 3.4 we have

$$\lim_{\delta \rightarrow 0} \mathbf{E}_v \left[\mathbf{1}_{\{i(\theta_{n+1}^\delta) = j\}} \right] = \lim_{\delta \rightarrow 0} \mathbf{P}_v(i(\theta_{n+1}^\delta) = j) = \rho_j. \quad (51)$$

From Corollary 4.6, we obtain

$$\sum_{n \geq 0} \delta \mathbf{1}_{\{\theta_{n+1}^\delta \leq t\}} \partial_x f_j(\theta_{n+1}^\delta, 0) \xrightarrow{\delta \rightarrow 0} \int_0^t \partial_x f_j(s, 0) d\ell_X(s) \quad \text{in probability.}$$

Now, including the choice of the edge (51), we conclude

$$Q_2^\delta \xrightarrow{\delta \rightarrow 0} \int_0^t \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) d\ell_X(s) \quad \text{in probability.}$$

Step 3:

$$\begin{aligned} \lim_{\delta \rightarrow 0} Q_3^\delta &= f(0, x) + \int_0^t (\partial_t f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) \neq v\}} \\ &\quad + \int_0^t \sigma(X(s)) \partial_x f(s, X(s)) \mathbf{1}_{\{X(s) \neq v\}} d\widetilde{W}(s) \end{aligned}$$

in probability.

This follows from the standard one-dimension Itô formula and the dominated convergence theorem for stochastic integrals.

Step 4: Conclusion.

It should be noted that

$$f(t, X(t)) - f(0, x) = Q_1^\delta + Q_2^\delta + Q_3^\delta,$$

by using steps 1 to 3, we get

$$\begin{aligned} \lim_{\delta \rightarrow 0} Q_1^\delta + Q_2^\delta + Q_3^\delta &= \int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) \neq v\}} ds \\ &+ \int_0^t \sigma(X_s) \partial_x f(s, X(s)) \mathbf{1}_{\{X(s) \neq v\}} d\widetilde{W}(s) + \int_0^t \left(\eta \partial_s f(s, 0) + \sum_{i=1}^N \tilde{\rho}_i \partial_x f_i(s, 0) \right) d\ell_X(s). \end{aligned}$$

We observe

$$\begin{aligned} &\int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) \neq v\}} ds = \\ &\int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) ds - \int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) = v\}} ds, \end{aligned}$$

but, from Proposition 4.2, we have

$$\int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) \mathbf{1}_{\{X(s) = v\}} ds = \eta \int_0^t (\partial_s f(s, v) + Lf(s, v)) d\ell_X(s).$$

Then, we conclude

$$\begin{aligned} f(t, X(t)) &= f(0, x) + \int_0^t (\partial_s f(s, X(s)) + Lf(s, X(s))) ds \\ &+ \int_0^t \sigma(X(s)) \partial_x f(s, X(s)) \mathbf{1}_{\{X(s) \neq v\}} d\widetilde{W}(s) + \int_0^t \left(-\eta Lf(s, v) + \sum_{i=1}^N \rho_i \partial_x f_i(s, 0) \right) d\ell_X(s). \end{aligned}$$

□

APPENDIX A. A BOUNDARY VALUE PROBLEM

We use here the same notations and assumptions as those from Section 2.

Proposition A.1. *For every $\delta > 0$ and $f \in \bigoplus_{i=1}^N C_b([0, \delta])$, with $f_i(0) = f_j(0)$ for all $1 \leq i, j \leq N$, there is a unique $u \in \bigoplus_{i=1}^N C^2([0, \delta])$ such that*

$$\begin{cases} L_i u_i(x) = f_i(x) & \text{for all } x \in (0, \delta), 1 \leq i \leq N, \\ \eta Lu(v) - \sum_{i=1}^N \rho_i u_i'(0) = 0, & \\ u_i(\delta) = 0 & \text{for all } i \in \{1, \dots, N\}, \\ u_i(0) = u_j(0) & \text{for all } i, j \in \{1, \dots, N\}, \end{cases} \quad (52)$$

given by

$$u_i(x) = \frac{\kappa_i \alpha_i(x)}{\alpha_i(0)} + \beta_i(x) \quad \text{for all } 0 \leq x \leq \delta,$$

where

$$\begin{aligned} \kappa_i &= \left(\eta f(v) - \sum_{i=1}^N \frac{\rho_i (\beta_1(0) - \beta_i(0))}{\alpha_i(0)} \right) \left(\sum_{i=1}^N \frac{\rho_i}{\alpha_i(0)} \right)^{-1} + \beta_1(0) - \beta_i(0) \\ \alpha_i(x) &= \int_x^\delta \exp \left(- \int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz \right) dy \\ \beta_i(x) &= \int_x^\delta \left(\int_0^y \frac{2f(z)}{\sigma_i^2(z)} \exp \left(\int_0^z \frac{2b_i(s)}{\sigma_i^2(s)} ds \right) dz \right) \exp \left(- \int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz \right) dy \end{aligned}$$

for all $1 \leq i \leq N$. Furthermore, we have

$$\max_{1 \leq i \leq N} \max_{x \in [0, \delta]} |u_i''(x)| = O(1) \quad \text{as } \delta \downarrow 0,$$

where the hidden constant only depends on f through $\|f\|_\infty$.

Proof. We divide the proof into three steps.

1. *Proof of existence.* Using the positivity of σ^2 we first rewrite the problem in the form

$$u_i''(x) + \frac{2b_i(x)}{\sigma_i^2(x)} u_i'(x) = \frac{2f(x)}{\sigma_i^2(x)} \quad \text{for all } x \in (0, \delta).$$

This implies that

$$u_i'(x) = \left(A_i + \int_0^x \frac{2f(y)}{\sigma_i^2(y)} \exp\left(\int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz\right) dy \right) \exp\left(-\int_0^x \frac{2b_i(y)}{\sigma_i^2(y)} dy\right) \quad \text{for } x \in (0, \delta), \quad (53)$$

were the parameters A_i are constants. In particular we have $u_i'(0) = \lim_{x \downarrow 0} u_i'(x) = A_i$. It follows that

$$u_i(x) = \int_x^\delta \left(A_i + \int_0^y \frac{2f(z)}{\sigma_i^2(z)} \exp\left(\int_0^z \frac{2b_i(s)}{\sigma_i^2(s)} ds\right) dz \right) \exp\left(-\int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz\right) dy \quad \text{for } x \in (0, \delta).$$

Notice in particular that $u_i(\delta) = 0$. Setting

$$\begin{aligned} \alpha_i(x) &= \int_x^\delta \exp\left(-\int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz\right) dy \\ \beta_i(x) &= \int_x^\delta \left(\int_0^y \frac{2f(z)}{\sigma_i^2(z)} \exp\left(\int_0^z \frac{2b_i(s)}{\sigma_i^2(s)} ds\right) dz \right) \exp\left(-\int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz\right) dy \end{aligned}$$

we get $u_i(x) = A_i \alpha_i(x) + \beta_i(x)$. The continuity of u then implies $A_i \alpha_i(0) + \beta_i(0) = A_j \alpha_j(0) + \beta_j(0)$ for all $1 \leq i, j \leq N$. Since $\alpha_i(0) > 0$ we obtain

$$A_i = \frac{A_1 \alpha_1(0) + \beta_1(0) - \beta_i(0)}{\alpha_i(0)} \quad \text{for all } 1 \leq i \leq N.$$

The second condition in (52) then reads

$$\eta f(\mathbf{v}) = \eta Lu(\mathbf{v}) = \sum_{i=1}^N \rho_i u_i'(0) = \sum_{i=1}^N \rho_i A_i = A_1 \alpha_1(0) \left(\sum_{i=1}^N \frac{\rho_i}{\alpha_i(0)} \right) + \sum_{i=1}^N \frac{\rho_i (\beta_1(0) - \beta_i(0))}{\alpha_i(0)}$$

so that

$$A_1 = \left(\eta f(\mathbf{v}) - \sum_{i=1}^N \frac{\rho_i (\beta_1(0) - \beta_i(0))}{\alpha_i(0)} \right) \left(\sum_{i=1}^N \frac{\rho_i \alpha_1(0)}{\alpha_i(0)} \right)^{-1}.$$

2. *Proof of uniqueness.* Clearly it is enough to prove that $v = 0$ is the unique element in $\bigoplus_{i=1}^N C_b([0, \delta])$ satisfying

$$\begin{cases} \frac{1}{2} \sigma_i^2(x) v_i''(x) + b_i(x) v_i'(x) = 0 & \text{for all } x \in (0, \delta), 1 \leq i \leq N, \\ \sum_{i=1}^N \rho_i v_i'(0) = 0, \\ u_i(0) = u_j(0) & \text{for all } i, j \in \{1, \dots, N\}, \\ v_i(\delta) = 0 & \text{for all } i \in \{1, \dots, N\}. \end{cases} \quad (54)$$

Applying the strong maximum principle [20, Theorem 3 p.349] to u_i we deduce that it cannot achieve its maximum (resp. minimum) on $(0, \delta)$ unless it is constant. In the case where u_i is constant we deduce that $u_i = 0$ from the Dirichlet boundary condition. We therefore proceed by assuming that the u_i is not all constant. If the maximum (resp. minimum) of u is attained at 0, then, Hopf's lemma [20, Lemma p.347], or the simpler result [44, Theorem 2 p.4], implies that $u_i'(0) < 0$ (resp. $u_i'(0) > 0$) for all $1 \leq i \leq N$, which contradicts the Kirchhoff conditions. We conclude that the maximum (resp. minimum) must be $u_i(\delta) = 0$ for some $1 \leq i \leq N$, so that $u \leq 0$ (resp. $u \geq 0$). This proves uniqueness.

3. *Proof of estimate.* From (53) and uniqueness of solutions we obtain

$$u_i''(x) = \left(\frac{2f(x)}{\sigma_i^2(x)} - \frac{2b_i(x)}{\sigma_i^2(x)} \left(\int_0^x \frac{2f(y)}{\sigma_i^2(y)} \exp \left(\int_0^y \frac{2b_i(z)}{\sigma_i^2(z)} dz \right) dy \right) \right) \exp \left(- \int_0^x \frac{2b_i(y)}{\sigma_i^2(y)} dy \right) - \frac{2A_i b_i(x)}{\sigma_i^2(x)} \exp \left(- \int_0^x \frac{2b_i(y)}{\sigma_i^2(y)} dy \right)$$

so that

$$|u_i''(x)| \leq \frac{2(\|f\|_\infty + A_i \|b\|_\infty)}{\sigma_0^2} + \frac{4\|b\|_\infty \|f\|_\infty \delta}{\sigma_0^4} \exp \left(\frac{2\delta \|b\|_\infty}{\sigma_0^2} \right).$$

Notice also that

$$\delta \exp \left(- \frac{2\delta \|b\|_\infty}{\sigma_0^2} \right) \leq \alpha_i(0) \leq \delta$$

and

$$0 < \beta_i(0) \leq \frac{2\delta^2}{\sigma_0^2} \exp \left(\frac{2\delta \|b\|_\infty}{\sigma_0^2} \right).$$

So that $A_i = O(1)$ as $\delta \downarrow 0$, with hidden constant depending on f through $\|f\|_\infty$. The conclusion follows. \square

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