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An ill-posed mechanical problem with friction

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Abstract – Many models involve the Coulomb's law in order to describe dynamical properties of friction phenomena. In order to generalize this Coulomb's law and to deal with its correct mathematical expression, we study a nonlinear equation where we take into account a maximal monotone graph. In the particular case of Coulomb's law, existence and uniqueness are proved. But in the general case, only existence persists. A counter-example to uniqueness is given.

friction / Coulomb's law / maximal monotone graph

1. Introduction

For twenty years, many studies have been devoted to nonlinear oscillations. Either they can deal with smooth nonlinearities due to the geometry (oscillations of a pendulum or large flexural displacements of a beam), or to constitutive laws (nonlinear elasticity, etc.), or to active forces (aeroelastic dry and lift forces depending on velocity and displacement of a structure). They can also investigate non-smooth non-linearities that modelize for example impacts (Whiston, 1987) or constitutive laws (Capecchi and Vestroni, 1995), or friction phenomena (Jean and Pratt, 1985).

In this study, we only deal with mechanical models including dry friction. We consider two distinct classes of models. A first class involves all the mechanical models with Coulomb's friction, and a second one involves all the other models: this last class contains all the models with a friction force which is not a constant during the dynamical phase.

We consider the one-dimensional motion of a material point on a plane support moving at velocity $v(t)$; this point is submitted to a normal force directed downwards, F_n , which creates a longitudinal friction force. Moreover, the material point is submitted to given dynamics. The mathematical formulation of this problem is

$$\ddot{x}(t) = F(t, x(t), \dot{x}(t)) - \mu_D F_n \operatorname{sgn}(\dot{x}(t) - v(t)), \quad \text{if } \dot{x}(t) - v(t) \neq 0, \quad (1)$$

$$\ddot{x}(t) = F(t, x(t), \dot{x}(t)) + F_l, \quad \text{with } F_l \in [-\mu_S F_n, \mu_S F_n], \quad \text{if } \dot{x}(t) - v(t) = 0, \quad (2)$$

where μ_D and μ_S are the dynamical and static friction coefficients; $\dot{x}(t)$ denotes velocity relative to the support, and F describe the longitudinal dynamics. We assume that F is a Lipschitz continuous function with respect to its last arguments (x, \dot{x}) .

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This model had been studied for a long time because it is very simple, especially in the case of a spring-mass system with a passive external solicitation. In such a case, F is written as

$$F(t, \alpha, \beta) = -\alpha\beta - \omega^2\alpha \quad (3)$$

with specific (per unit mass) viscous damping a and eigenfrequency of the system ω .

Den Hartog (1956) studied the Coulomb's model with a linear spring and viscous damping. This article focused on the instability induced by friction. Other authors deal with friction coefficients μ_D and μ_S which are constants (Awrejcewicz and Delfs, 1990), which are increasing functions of time (Caudet, 1964; Baumberger, et al. 1990) or which are functions of the relative velocity (Dowell and Schwarz, 1983; Popp, 1989; Rabinowicz, 1996). If μ_D is a constant equal to μ_D , a proof of existence and uniqueness of a solution has been given in (Monteiro Marques, 1994). This paper by Monteiro Marques provides two sufficient conditions for uniqueness.

This article is organised as follows. In Section 2, we present a simple model where μ_D is a constant and μ_S is an increasing function of time (this case includes Coulomb's friction). In Section 3, we give an example of non-uniqueness, and in Section 4 we give a numerical scheme which has the same non-uniqueness properties as the continuous problem. Finally we draw conclusions.

2. Introduction of the model

Let us consider again Eqs (1) and (2). Let us choose $F_n = 1$. We intend to study a solid with abscissa $x(t)$, moving on a plane rigid support, with velocity $v(t)$, submitted to an external smooth force $F(t, x(t), \dot{x}(t))$ and to friction $\phi(t)$; $\dot{x}(t)$ denotes the velocity relative to the referential. Thus this model can be expressed as

$$\ddot{x}(t) = F(t, x(t), \dot{x}(t)) + \phi(t), \quad (4)$$

$$x(0) = x_0, \quad (5)$$

$$\dot{x}(0) = \dot{x}_0. \quad (6)$$

Now let us describe the friction term: our model of friction takes into account the points of view of Caudet (1964) and Baumberger et al. (1990). This model is based upon the motion of last blocking time. Let w be a continuous function on $[0, T]$ and let

$$b(t, w) = \begin{cases} t, & \text{if } w(t) \neq 0, \\ \sup\{s \in [0, t[\text{ such that } w(s) \neq 0\}, & \text{if } w(t) = 0. \end{cases} \quad (7)$$

This definition assumes that if $w(0) = 0$, a non positive $b(0, w)$ is given. We consider the different phases: static ones ($\dot{x}(t) - v(t) = 0$) and dynamic ones ($\dot{x}(t) - v(t) \neq 0$).

We introduce two coefficients: a dynamic friction coefficient μ_D and a static friction coefficient $\mu_S(t)$. The coefficient μ_D is a strictly positive constant and $\mu_S(t)$ is an increasing positive function, bounded on every static phase. So we have

$$\begin{cases} \phi(t) = -\mu_D \operatorname{sgn}(\dot{x}(t) - v(t)), & \text{if } \dot{x}(t) - v(t) \neq 0, \\ \phi(t) \in [-\mu_S(t), \mu_S(t)], & \text{if } \dot{x}(t) - v(t) = 0, \end{cases} \quad (8)$$

where

$$\mu_S(t) = h(t - b(t, \dot{x} - v)), \quad (9)$$

where h is an increasing bounded positive function from $]0, +\infty[$ to $]0, +\infty[$. We can simplify (8) and (9) by setting

$$h(0) = \mu_D \quad (10)$$

with $\mu_D \leq \mu_S$, and we consider the maximal monotone graph σ defined by

$$\sigma(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \\ [-1, 1], & \text{if } x = 0. \end{cases} \quad (11)$$

So (8) and (9) are equivalent to

$$\phi(t) \in -\sigma(\dot{x}(t) - v(t))h(t - b(t, \dot{x} - v)). \quad (12)$$

Thus, for given F, v, b, σ, h, x_0 , and \dot{x}_0 , we seek functions x and ϕ from $[0, T]$ to \mathbb{R} , such that Eqs (4) and (12) are verified with initial conditions (5) and (6). Let us observe that ϕ can be eliminated from (4) and (12)

$$\ddot{x}(t) + \sigma(\dot{x}(t) - v(t))h(t - b(t, \dot{x} - v)) \ni F(t, x(t), \dot{x}(t)). \quad (13)$$

If we assume that h is a positive constant $\mu = \mu_D$, (12) is equivalent to

$$\begin{cases} \phi(t) = -\mu \operatorname{sgn}(\dot{x}(t) - v(t)), & \text{if } \dot{x}(t) - v(t) \neq 0, \\ \phi(t) \in [-\mu, \mu], & \text{if } \dot{x}(t) - v(t) = 0 \end{cases} \quad (14)$$

and we find again the particular case of Coulomb's model. If velocity \dot{x} is denoted by y , Eqs (5), (6) and (13) are equivalent to

$$\dot{y}(t) + \sigma(y(t) - v(t))h(t - b(t, y - v)) \ni F\left(t, x_0 + \int_0^t y(s) ds, y(t)\right), \quad (15)$$

where y verifies the initial condition

$$y(0) = \dot{x}_0. \quad (16)$$

3. Existence and uniqueness results: ill-posed model

F is continuous with respect to all its arguments and Lipschitz continuous with respect to its last two arguments x and \dot{x} ; then there exists a solution x to (5), (6) and (13). In the particular case of Coulomb's friction, x is unique. In the general case, uniqueness is not true. Moreover, one can exhibit an example with non-uniqueness for the problem (15), (16). We assume now that

$$1. \quad y(0) = 0, \quad v = 0, \quad (17)$$

$$2. \quad h \text{ is increasing, continuous and strictly positive on } \mathbb{R}_+, \quad (18)$$

$$3. \quad \text{Let also } F \text{ be defined by } F(t, a, b) = h(t). \quad (19)$$

A family of distinct functions $\{y_u\}_{u \in [0, T]}$ is defined as follows

$$\forall t \in [0, u], \quad y_u(t) = 0, \quad \text{and } \forall t \in [u, T], \quad y_u(t) = \int_u^t h(s) ds - h(0)(t - u). \quad (20)$$

We verify readily that y_u solves (15), (16). We also notice that $y_u \neq y_v$ when $u \neq v$. Thus, the solution to Eqs (15) and (16) is not unique.

By using the same idea, can show the sensitivity of the solutions to data. Let us set

$$\forall t \in [0, T], \quad y_0(t) = \int_0^t h(s) ds - h(0)t, \quad \text{and} \quad y_T \equiv 0 \quad (21)$$

and for every $\varepsilon \in \mathbb{R}$

$$f_\varepsilon = h + \varepsilon. \quad (22)$$

Consider functions $\{Y_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ such that for every $\varepsilon \in \mathbb{R}$ and for every $t \in [0, T]$

$$Y_\varepsilon(t) = \begin{cases} y_0(t) + \varepsilon t, & \text{if } \varepsilon > 0 \\ 0, & \text{if } \varepsilon \leq 0. \end{cases} \quad (23)$$

For every $\varepsilon \in [-2h(0), +\infty[$, we have

$$\dot{Y}_\varepsilon + \sigma(Y_\varepsilon(t))h(t - b(t, Y_\varepsilon)) \ni f_\varepsilon, \quad \text{and} \quad Y_\varepsilon(0) = 0. \quad (24)$$

But y_0 and y_T are two distinct solutions to (15), (16) and

$$\lim_{\varepsilon \rightarrow 0^+} Y_\varepsilon = y_0, \quad \lim_{\varepsilon \rightarrow 0^-} Y_\varepsilon = y_T, \quad \lim_{\varepsilon \rightarrow 0} f_\varepsilon = h, \quad \text{in } C^0([0, T]). \quad (25)$$

Thus the solutions are not continuous with respect to the data.

4. Numerical scheme

Let us consider the counter-example to uniqueness of Section 3 defined by Eqs (15) and (20) and assumptions (17), (18) and (19). Its solutions can be approximated by a numerical scheme similar to the implicit Euler scheme for first-order differential equations. Let $\delta \in [0, T]$, let N be the integer part of T/δ and $t_n = n\delta$, for every $n \in \{0, \dots, N\}$. Define continuous and piecewise linear functions z_δ on $[0, T]$ by their values at the nodes t_n

$$\forall n \in \{0, \dots, N\}, \quad Z_n = z_\delta(t_n). \quad (26)$$

Let $Z_0 = y_0$ and assume that the Z_n 's satisfy the following relation

$$\forall n \in \{0, \dots, N-1\}, \quad \frac{Z_{n+1} - Z_n}{\delta} + \sigma(Z_{n+1})h(t_{n+1} - b(t_{n+1}, z_\delta)) \ni h(t_{n+1}). \quad (27)$$

According to Brezis (1973), if A is a maximal monotone graph, for every $\lambda > 0$, $I + \lambda A$ is invertible; i.e., for every $y \in \mathbb{R}$, the equation

$$x + \lambda A(x) \ni y \quad (28)$$

admits a unique solution, denoted

$$x = (I + \lambda A)^{-1}(y). \quad (29)$$

In the case of the Coulomb's friction, we have $h = \mu > 0$ and according to (27), (28) and (29) Z_{n+1} is defined by

$$\forall n \in \{0, \dots, N-1\}, \quad Z_{n+1} = (I + \mu\delta\sigma)^{-1}(\delta h(t_{n+1}) + Z_n). \quad (30)$$

Let us show that with assumptions (17), (18), (19), scheme (27) has several solutions.

Let us assume for $n \in \{0, \dots, N-1\}$, $Z_0 = \dots = Z_n = 0$.

(1) If $Z_{n+1} = 0$, we have according to Eq. (7)

$$b(t_{n+1}, z_\delta) = 0 \quad \text{and} \quad h(t_{n+1} - b(t_{n+1}, z_\delta)) = h(t_{n+1}) \quad (31)$$

and Eq. (27) is verified.

(2) If $Z_{n+1} > 0$, we have according to Eq. (7)

$$b(t_{n+1}, z_\delta) = t_{n+1} \quad \text{and} \quad h(t_{n+1} - b(t_{n+1}, z_\delta)) = h(0) \quad (32)$$

and Eqs (27), (28), (29) imply

$$Z_{n+1} = (I + h(0)\delta\sigma)^{-1}(\delta h(t_{n+1})). \quad (33)$$

Because σ is strictly increasing, one can verify that the value of Z_{n+1} given by (33) is strictly positive. Then in both cases, Z_{n+1} is a solution of (27). This numerical scheme has therefore N solutions.

5. Conclusion

We have generalized Coulomb's law into a law which involves the last blocking time. We have proved existence for this model, and given a counter-example to uniqueness. Therefore the model under study is defective, and other models which possess better properties should be considered. It would be an interesting perspective to mathematically analyze models or which μ_S could depend on the relative abscissa, or the velocity of the support.

Otherwise, in order to justify non-uniqueness of solutions to problem (15) in the general case, we can note that this problem can be written in the form

$$\dot{y}(t) + A_{t,y}(y(t)) \ni F\left(t, x_0 + \int_0^t y(s) ds, y(t)\right), \quad (34)$$

where at each given time t , the graph $u \rightarrow A_{t,y}(u)$ is maximal monotone one. But, the friction is no longer maximal monotone when t is varying in $[0, T]$. This remark can explain a posteriori the lack of uniqueness for the studied problem.

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