



Nonsmooth Dynamical Systems: an overview

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Nonsmooth Dynamical Systems: an overview

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Abstract. This lecture constitutes a short introduction to the subject of the stability (in the sense of Lyapounov) of evolution variational inequalities and non-smooth dynamical systems. After recalling some basic concepts and results in Convex and Set-Valued Analysis and in the field of Differential Inclusions (including sweeping processes), we present some basic existence results and an Invariance principle for a class of first order evolution variational inequalities. Using this approach, stability and asymptotic properties of important classes of second order dynamic systems with dry friction are studied. Using Brouwer's topological degree, necessary conditions for asymptotic stability of evolution variational inequalities are also studied.

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1 Introduction

In recent year, the theory of stability (in the sense of Lyapounov) of stationary solutions of dynamical systems has been considerably developed. It is well known that this field is of major importance in both applied mathematics and engineering. With the emergence of many engineering disciplines, it is not surprising that the unilateral dynamical system has played a central role in the understanding of mechanical processes. The mathematical formulation of the unilateral dynamical system involved inequality constraints and necessarily contains natural non-smoothness. The non-smoothness could originate from the discontinuous control term, or from the environment (non-smooth impact), or from the dry friction. It is well-known that dry friction generates instabilities and consequently influences the performance and the behavior of mechanical systems. It seems that the formalism of evolution variational inequalities represents a large class of unilateral dynamical systems [8], [25], [26] [16], [10] . Due to the lack of smoothness, classical mathematical methods (see e.g. [23], [30]) are applicable only to a limited amount and require naturally extensions for both analytical and numerical methods. Recently, new analytical tools have been developed for the study of evolution variational inequalities [10], [15], [16].

In [16], the authors has developed a Lyapunov approach to study the stability of stationary solutions of first order evolution variational inequalities in Hilbert spaces. This approach was efficient for giving sufficient conditions of stability for the problem in the form of a variational inequality. Some results of stability, asymptotic stability and unstability for general evolution variational inequalities are provided. The method of Lyapounov function [30], was extended to the case of variational inclusions. An important particular case arises for constrained problems on a closed, convex subset K , where the arguments involve the indicator function of K . Some geometric conditions in terms of tangent, normal and recession cones are used. The case where K is a cone is of special interest for applications.

Recently, in [2] the authors has developed a LaSalle's invariance theory applicable to a general class of first order non-linear evolution variational inequalities. This approach was applied to the study of the stability and the asymptotic properties of second order dynamical systems involving friction forces. Equally important, is the study of the attractivity properties of the set of stationary solutions which correspond in general to a stationary mode where the friction elements are sticking.

The lecture is organized as follows. In section 2, we recall some definitions and backgrounds of convex analysis and degree theory which will be used later. In section 3, we will give an existence result for first order evolution variational inequalities. In section 4, we recall some functional framework for studying the stability of first order evolution variational inequalities. In section 5, we will show how to apply the Lyapounov approach to linear variational inequalities in finite dimensional spaces. Section 6 is devoted to the development of an invariance principle for evolution variational inequalities. In section 7, we show how this approach can be used to study the stability and asymptotic properties of important classes of second order dynamic systems with friction. In section 8, we give a necessary condition for asymptotic stability of evolution variational inequalities.

2 Basic Convex Analysis

We recall several definitions and results from Convex Analysis. The general setting is that of a real Hilbert space H , with the associated scalar product, denoted by $\langle \cdot, \cdot \rangle$, or simply by a dot $\cdot$, and with the associated norm $\| \cdot \|$. We shall often consider only the finite dimensional setting, more precisely, $H = \mathbb{R}^n$ with the usual Euclidean scalar product $\langle x, y \rangle = x \cdot y = x^T y$.

2.1. Convex sets and cones

A set $C \subset H$ is *convex* if it contains all the convex combinations $\lambda x + (1 - \lambda)y$ with $0 \leq \lambda \leq 1$, for any pair $x, y \in C$. In other words,

$$\lambda C + (1 - \lambda)C \subset C, \quad \forall 0 \leq \lambda \leq 1.$$

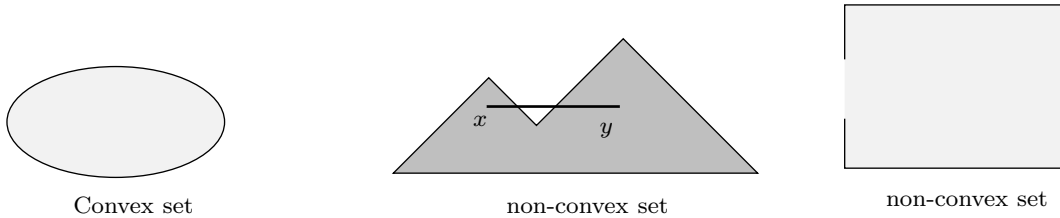


Figure 1: Example of convex and non-convex sets.

A *cone* is a set which contains all λx , for $\lambda \geq 0$ and x in the set. Then a closed convex set K is a cone of H iff $K + K \subset K$.

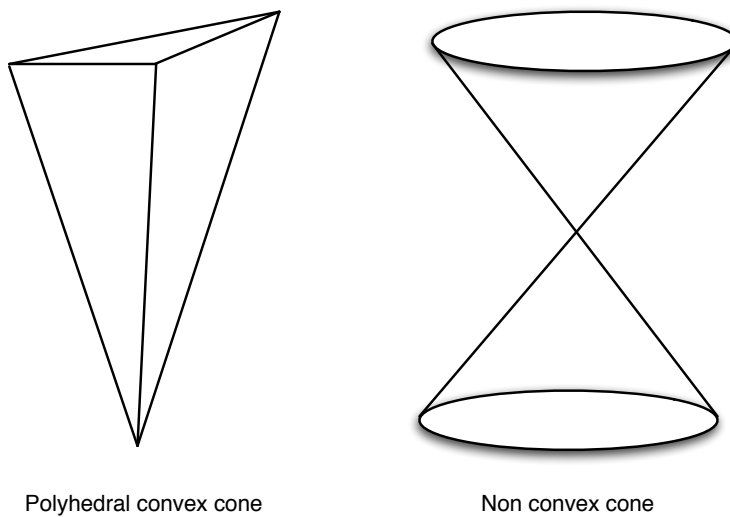


Figure 2: Example of cones.

The *polar cone* (or negative polar cone [4]) of K is defined by:

$$K^\circ \text{ (or } K^-) = \{x^* \in H : \langle x^*, x \rangle \leq 0, \forall x \in K\}.$$

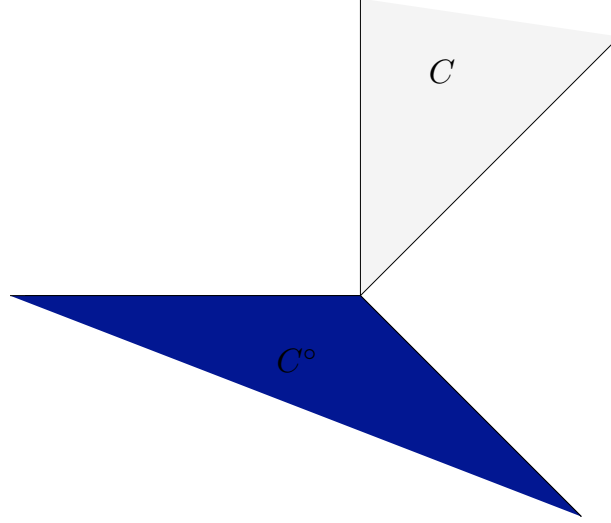


Figure 3: A cone C and its negative polar C° .

A related definition is that of the (positive) *dual cone*, found in other authors, e.g. [11], where the sign of the inequality is reversed:

$$K^* = \{x^* \in H : \langle x^*, x \rangle \geq 0, \forall x \in K\}.$$

For $\bar{x} \in C$, we define the (outward) *normal cone* at $\bar{x}$ by

$$N_C(\bar{x}) = \{p \in \mathbb{R}^n : \langle p, x - \bar{x} \rangle \leq 0, \forall x \in C\}.$$

If $\bar{x}$ belongs to the interior of C , then $N_C(\bar{x}) = \{0\}$.

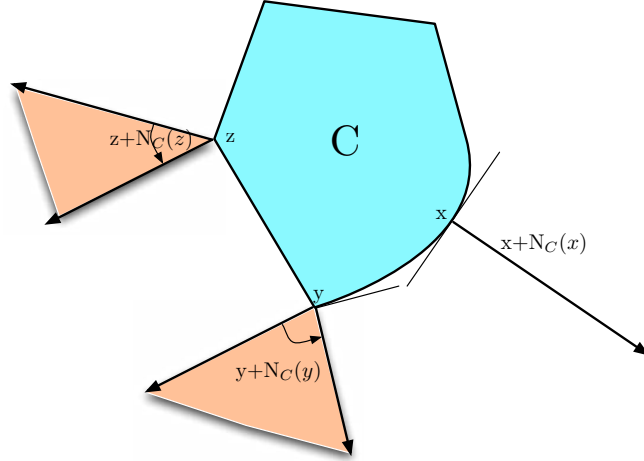


Figure 4: Normal cone to a convex set.

We can also define the *tangent cone* of a subset $C \subset \mathbb{R}^n$ at a point $x \in C$, which we denote $T_C(x)$, by the following formula:

$$y \in T_C(x) \iff \exists (x_k) \in C \text{ with } x_k \rightarrow x, \text{ and } \exists \alpha_k \in \mathbb{R}_+ \text{ such that } \alpha_k \rightarrow 0 \text{ and } \frac{x_k - x}{\alpha_k} \rightarrow y.$$

Furthermore, if C is convex, then

$$z \in \left(T_C(x)\right)^\circ \iff \langle z, y - x \rangle \leq 0, \forall y \in C,$$

i.e. the polar of the tangent cone is the normal cone: $\left(T_C(x)\right)^\circ = N_C(x)$.

Let K be a closed convex subset of H . Then the *recession cone* of H is the closed convex cone

$$K_\infty := \bigcap_{t>0} \left[\frac{K - x_0}{t} \right],$$

where x_0 is arbitrarily chosen in K (indeed, it is independent of the choice of x_0). Equivalently, this amounts to say that d belongs to K_∞ if and only if $x_0 + \lambda d \in K$, for all $\lambda \geq 0$ and $x_0 \in K$. Since K is closed and convex, the translate $K - x_0$ is also closed and convex. $\frac{K - x_0}{t}$ as a homothetic of a closed and convex set is also closed and convex. Finally K_∞ is closed (convex) as an intersection of a closed (convex) sets. Furthermore, K_∞ is a cone which means that if $d \in K_\infty$, $\lambda d \in K_\infty, \forall \lambda > 0$. Hence, K_∞ is a closed and convex cone in H . If K is a closed convex cone, then clearly $K_\infty = K$.

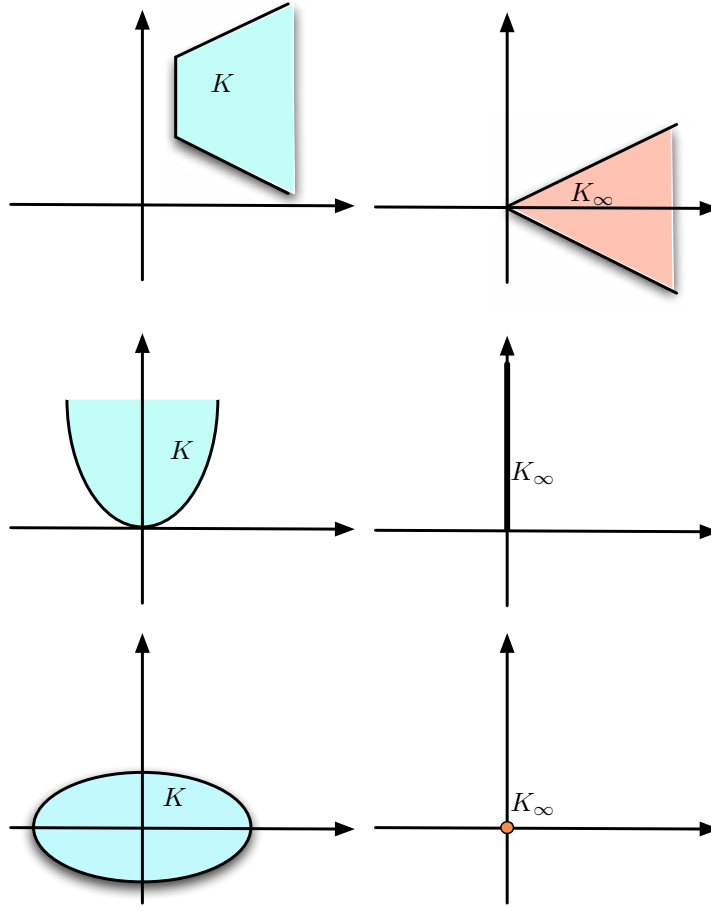


Figure 5: Three subsets and their recession cones.

In finite dimensional spaces, we have the following property:

Proposition 2.1 *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set. Then the following are equivalent:*

1. $C_\infty = \{0\}$
2. C is bounded.

PROOF:

(2) $\Rightarrow$ (1) :

Suppose there exists $d \neq 0$ such that $d \in C_\infty$. Hence, for every $n \in \mathbb{N}$ and $x_0 \in C$, we have: $x_0 + nd \in C$. We have

$$\|x_0 + nd - x_0\| = n\|d\| \rightarrow \infty.$$

Therefore,

$$\text{Diam } C = \sup_{x,y \in C} \|x - y\| \geq n\|d\|.$$

Hence, $\text{Diam } C = \infty$ and C is unbounded.

(1) $\Rightarrow$ (2) :

Suppose C is unbounded, then there exists a sequence (x_n) in C such that $\|x_n\| \rightarrow \infty$. So for some n_0 , $\|x_n\| > 0, \forall n > n_0$. Let $d_n = x_n/\|x_n\|$. Note that $\|d_n\| = 1$, for all $n > n_0$. Hence there is a subsequence (d_{n_k}) which converges to d with $\|d\| = 1$. Now for a fixed $x_0 \in C$, and $t > 0$, let

$$\begin{aligned} y_{nk} &= \left(1 - \frac{t}{\|x_{nk}\|}\right) x_0 + t d_{nk} \\ &= \left(1 - \frac{t}{\|x_{nk}\|}\right) x_0 + \frac{t}{\|x_{nk}\|} x_{nk}. \end{aligned}$$

For large k , $t/\|x_{nk}\| < 1$, so $y_{nk} \in C$ since it is a convex combination of elements in C and C is convex. Also, since $(y_{nk}) \rightarrow x_0 + td$ and C is closed, then $x_0 + td \in C$. Hence there exists a $d \neq 0$ such that $x_0 + td \in C$, for all $t > 0$, and so $C_\infty \neq \{0\}$, which completes the proof.

Remark 2.1 In infinite dimensional spaces, there exist unbounded convex sets such that $C_\infty = \{0\}$, e.g.

$$C = \{x \in l^2(\mathbb{N}^*) : |\langle x, e_n \rangle| \leq n, \forall n \in \mathbb{N}^*\}.$$

2.2. Convex functions

Let $\varphi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be an arbitrary function.

Definition 2.1 A function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *convex* iff $\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y)$, for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^n$.

We say that φ is *proper* if its effective domain

$$\text{Dom } \varphi := \{x \in H \mid \varphi(x) < +\infty\}$$

is nonempty.

Let C be a subset of $\mathbb{R}^n$. We write $I_C(x)$ for the *indicator function* of C , i.e.,

$$I_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}$$

Then C is a convex set iff I_C is a convex function.

Proposition 2.2 Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of class C^1 , where $U \subset \mathbb{R}^n$ is an open set and let C be a convex subset of U . The following are equivalent

1. f is convex on C ;
2. $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ for all $x, y \in C$.

Proposition 2.3 Let $f \in C^2(U; \mathbb{R})$ with $U \subset \mathbb{R}^n$ is an open convex set. The following are equivalent

1. f is convex on U ;
2. The Hessian matrix $\nabla^2 f(x)$ is positive semi-definite for every $x \in U$, i.e.

$$\langle \nabla^2 f(x)h, h \rangle \geq 0, \quad \forall h \in \mathbb{R}^n.$$

Note that if $\nabla^2 f(x)$ is positive definite for every $x \in U$, then f is strictly convex on U .

We now recall some general continuity concepts.

Definition 2.2 (1) $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi-continuous (lsc)* at x_0 if for all $a < \varphi(x_0)$ there exists a ball $B(x_0, r)$ such that for all $x \in B(x_0, r)$, we have: $a < \varphi(x)$. φ is said to be *lower semi-continuous* if and only if it is lower semi-continuous at each point.

(2) φ is said to be *upper semi-continuous (usc)* at x_0 if $-\varphi$ is lsc at x_0 . φ is *upper semi-continuous* if and only if it is upper semi-continuous at each point.

Remark 2.2 φ is continuous if and only if it is both lower semi-continuous and upper semi-continuous.

For $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, we set

$$\text{Epi}(\varphi) = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \varphi(x) \leq \lambda\},$$

the *epigraph* of φ . Given $\lambda \in \mathbb{R}$, the *lower λ -level* subset of φ is

$$\text{Lev}_\lambda(\varphi) = \{x \in \mathbb{R}^n : \varphi(x) \leq \lambda\},$$

and $\text{argmin}(\varphi) = \{\bar{x} \in \mathbb{R}^n : \varphi(\bar{x}) = \inf_{x \in \mathbb{R}^n} \varphi(x)\}$.

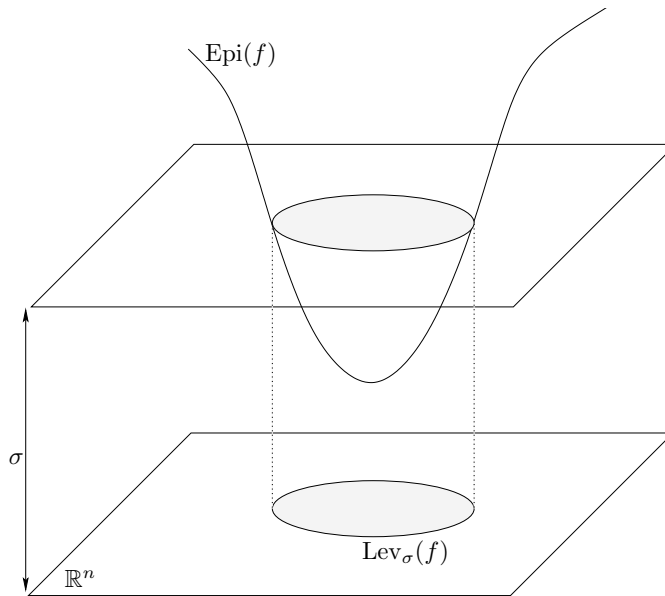


Figure 6: $\text{Epi}(f)$ and $\text{Lev}_\sigma(f)$.

Proposition 2.4 Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. Then the following are equivalent:

1. φ is lower semi-continuous;
2. the epigraph $\text{Epi}(\varphi)$ is closed;
3. For all λ , the sublevel $\text{Lev}_\lambda(\varphi)$ is closed;
4. For all $x \in \mathbb{R}^n$, $\varphi(x) \leq \liminf_{y \rightarrow x} \varphi(y) = \sup_{r>0} \inf \{\varphi(y) : y \in B(x, r)\}$;
5. If $x_n \rightarrow x$ then $\varphi(x) \leq \liminf_{n \rightarrow \infty} \varphi(x_n)$.

We set $\Gamma_0(H)$ the set of all convex, lower semicontinuous and proper functions.

Definition 2.3 Let $\varphi \in \Gamma_0(H)$. A point $p \in H$ is called a *subgradient* of φ at the point $u \in \text{Dom } \varphi$ iff

$$\varphi(v) \geq \varphi(u) + \langle p, v - u \rangle \quad \forall v \in H.$$

We denote

$$\partial\varphi(u) = \{p \in H : \varphi(v) \geq \varphi(u) + \langle p, v - u \rangle \quad \forall v \in H\}$$

and we say that $\partial\varphi(u)$ is the *subdifferential* of φ at u .

This means that elements of $\partial\varphi$ are slopes of the hyperplanes supporting the epigraph of φ at $(u, \varphi(u))$.

For instance, it can be observed that

$$\partial I_C(\bar{x}) = N_C(\bar{x}).$$

We have also the following characterization of the subdifferential in terms of the normal cone to the epigraph. Let $f \in \Gamma_0(\mathbb{R}^n)$ and $x \in \text{Dom}(f)$, then

$$p \in \partial f(x) \iff (p, -1) \in N_{\text{Epi}(f)}(x, f(x)).$$

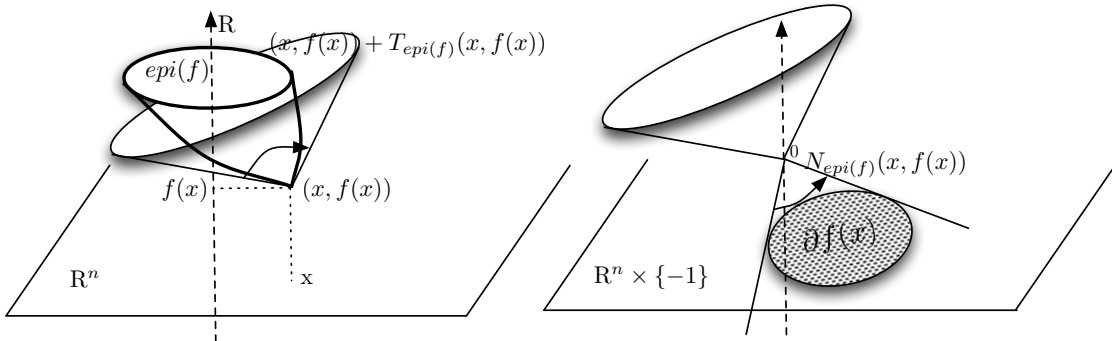


Figure 7: Illustration of the subdifferential by the normal cone.

Example 2.1 Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$. Then

$$\partial f(x) = \begin{cases} 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \|x\|_2$ (Euclidean norm). Then

$$\partial f(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } x \neq 0 \\ B(0, 1) & \text{if } x = 0 \end{cases}$$

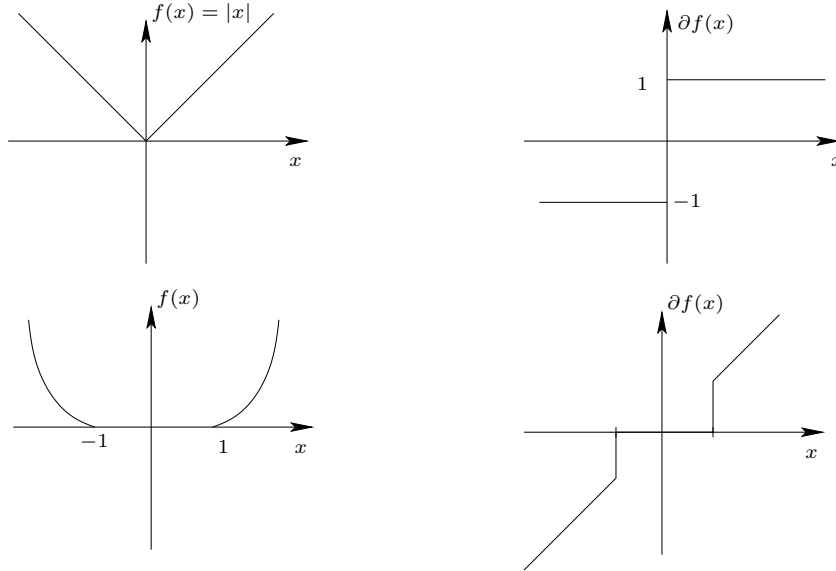


Figure 8: Example of function and their subdifferential.

Example 2.2 If $C = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ where $g \in \Gamma_0(\mathbb{R}^n)$ and $\bar{x} \in \mathbb{R}^n$ such that $0 \notin \partial g(\bar{x})$ then

$$N_C(\bar{x}) = \begin{cases} 0 & \text{if } g(\bar{x}) < 0 \\ \emptyset & \text{if } g(\bar{x}) > 0 \\ \mathbb{R}_+ \partial g(\bar{x}) & \text{if } g(\bar{x}) = 0. \end{cases}$$

Here $\mathbb{R}_+ \partial g(\bar{x}) = \{\lambda p : \lambda > 0 \text{ and } p \in \partial g(\bar{x})\}$.

Theorem 2.1 *Let $f \in \Gamma_0(\mathbb{R}^n)$. Then $\partial f(\bar{x})$ is a closed convex set, possibly empty. But if $x \in \text{Int}(\text{Dom } f)$, then $\partial f(\bar{x}) \neq \emptyset$. In particular if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then for all $\bar{x} \in \mathbb{R}^n$, $\partial f(\bar{x})$ is a nonempty, convex and compact set of $\mathbb{R}^n$.*

A bit of calculus now:

Theorem 2.2 *Let $f, g \in \Gamma_0(\mathbb{R}^n)$ and suppose that f is continuous at a point $x_0 \in \text{Dom } g$. Then $\forall x \in \text{Dom } (f + g)$, we have*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Proposition 2.5 (Chain Rule) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function and let $A \in \mathbb{R}^{n \times m}$ be a matrix. We define the $F : \mathbb{R}^m \rightarrow \mathbb{R}$, $x \mapsto f(Ax)$. Then*

$$\partial F(x) = A^T \partial f(Ax) = \{A^T p : p \in \partial f(Ax)\}.$$

We turn our attention to optimization problems which we translate into the language of convex analysis. For the *unconstrained minimum problem*, we have:

$$\begin{aligned} f(\bar{x}) = \inf_{x \in \mathbb{R}^n} f(x) &\iff \bar{x} \in \text{Argmin } f \\ &\iff 0 \in \partial f(\bar{x}) \\ &\iff \nabla f(\bar{x}) = 0 \quad (\text{if } f \text{ is } C^1). \end{aligned}$$

For $f \in \Gamma_0(\mathbb{R}^n)$ and C a closed convex subset of $\mathbb{R}^n$, we consider the *constrained optimization problem*

$$\min_{x \in C} f(x) = \min_{x \in \mathbb{R}^n} (f + I_C)(x), \tag{1}$$

It is clear that

$$\bar{x} \text{ is a solution of (1)} \iff 0 \in \partial(f + I_C)(\bar{x}).$$

If f is continuous at a point of C , then

$$0 \in \partial f(\bar{x}) + \partial I_C(\bar{x}). \tag{2}$$

If $f \in C^1(\mathbb{R}^n; \mathbb{R})$, then (2) becomes

$$-\nabla f(\bar{x}) \in N_C(\bar{x}). \tag{3}$$

The classic references on convex analysis are [27] and [29].

3 Maximal monotone operators and variational inequalities

We recall some definitions and some results about maximal monotone operators theory drawn from [7] (see also [6]). Let H be a real Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For a multivalued operator $T : H \rightrightarrows H$, we denote by:

$$D(T) := \{u \in H \mid T(u) \neq \emptyset\},$$

the *domain* of T ,

$$R(T) := \bigcup_{u \in H} T(u),$$

the *range* of T ,

$$\text{Graph } (T) := \{(u, u^*) \in H \times H \mid u \in D(T) \text{ and } u^* \in T(u)\},$$

the *graph* of T . Throughout the paper we identify operators with their graphs.

We recall that T is *monotone* if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$ we have

$$\langle v^* - u^*, v - u \rangle \geq 0.$$

T is *maximal monotone* if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

We say that a single-valued mapping A is *hemicontinuous* (following [6], p. 26) if, for all $x, y \in H$:

$$A((1 - t)x + ty) \rightarrow A(x), \text{ as } t \rightarrow 0.$$

A continuous map is therefore hemicontinuous. It can be shown that, if $A : D(A) = H \rightarrow H$ is monotone and hemicontinuous, then A is maximal monotone (see [6], Proposition 2.4). Also, if A is monotone and hemicontinuous and B is maximal monotone, then $A + B$ is maximal monotone ([6], p. 37).

From *Minty's theorem*, we know that:

$$T \text{ is maximal monotone} \iff R(T + I) = H,$$

where I stands for the identity mapping.

T^{-1} is the operator defined by

$$v \in T^{-1}(u) \iff u \in T(v).$$

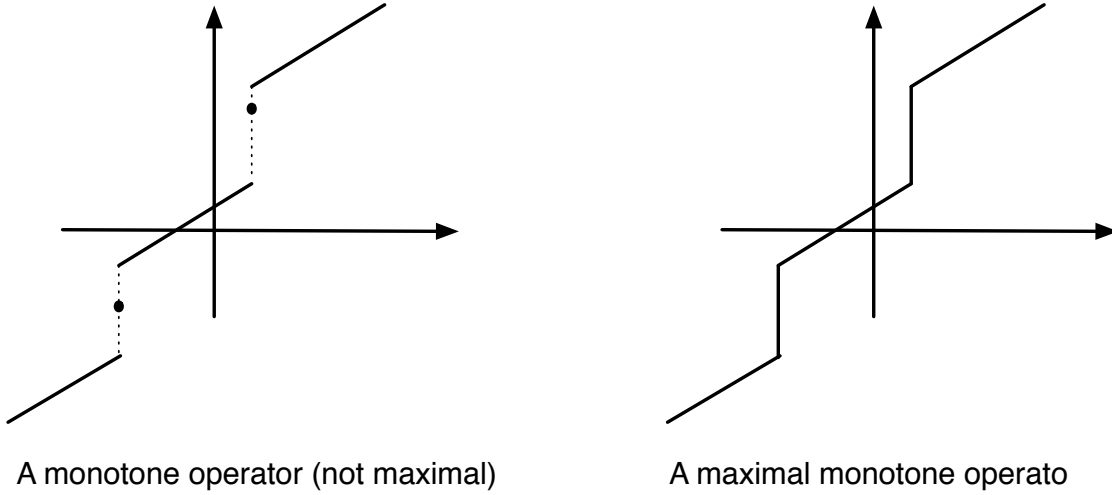


Figure 9: Example of monotone operators.

When T is maximal monotone, the operator $(I + T)^{-1}$ (where I is the identity operator in H) is defined in the whole space H , it is single-valued and Lipschitz-continuous; indeed, it is non-expansive:

$$\|(I + T)^{-1}(x) - (I + T)^{-1}(y)\| \leq \|x - y\|.$$

Moreover, the operators $(\lambda I + T)^{-1}$ or $(I + \lambda T)^{-1}$ with $\lambda > 0$ are similarly well-defined, single-valued and Lipschitz-continuous.

Let $y \in \mathbb{R}^n$ be given. We consider the following *variational inequality problem*: find $x \in \mathbb{R}^n$ such that

$$\langle x - y, v - x \rangle + \varphi(v) - \varphi(x) \geq 0, \forall v \in \mathbb{R}^n. \quad (4)$$

This is equivalent to $y - x \in \partial\varphi(x)$, that is, $y \in (I + \partial\varphi)(x)$, i.e. $x \in (I + \partial\varphi)^{-1}(y)$. Since the subdifferential of a proper l.s.c. convex function is a maximal monotone operator, we may apply the above theory.

Thus, Problem (4) has a unique solution, that we denote by

$$P_\varphi(y) = (I + \partial\varphi)^{-1}(y).$$

The operator $P_\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n; y \mapsto P_\varphi(y)$ is well-defined on the whole of $\mathbb{R}^n$ and

$$P_\varphi(\mathbb{R}^n) \subset D(\partial\varphi). \quad (5)$$

For instance, if $\varphi \equiv I_K$, where K is a nonempty closed convex set and I_K denotes the indicator function of K , then

$$P_\varphi \equiv P_{I_K} \equiv P_K$$

where P_K denotes the projection operator onto K , which is defined by the formula:

$$\|x - P_K x\| = \min_{w \in K} \|x - w\|.$$

We may also consider the set-valued operator $\mathcal{A}_\varphi : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ defined by

$$\mathcal{A}_\varphi(x) = \{f \in \mathbb{R}^n : \langle x - f, v - x \rangle + \varphi(v) - \varphi(x) \geq 0, \forall v \in \mathbb{R}^n\}. \quad (6)$$

We see that $f \in \mathcal{A}_\varphi(x) \iff f - x \in \partial\varphi(x)$ so that

$$\mathcal{A}_\varphi^{-1}(f) = P_\varphi(f). \quad (7)$$

It is also easy to see that

$$(t\mathcal{A}_\varphi)^{-1}(tf) = P_\varphi(f), \forall t > 0. \quad (8)$$

Note that

$$\mathcal{A}_\varphi(x) = \partial\left\{\frac{\|\cdot\|^2}{2} + \varphi(\cdot)\right\}(x), \forall x \in \mathbb{R}^n,$$

so that $\mathcal{A}_\varphi$ is a maximal monotone operator. It results that for any $t > 0$, the operator $(id_{\mathbb{R}^n} + t\mathcal{A}_\varphi)^{-1}$ is a well defined single-valued operator.

Given the definition of P_φ , we know from the general theory that

Proposition 3.1 *The operator P_φ is nonexpansive, i.e.*

$$\|P_\varphi v - P_\varphi y\| \leq \|v - y\|, \forall v, y \in \mathbb{R}^n.$$

Proof: Let $v, y \in \mathbb{R}^n$ be given. We set $x := P_\varphi(v)$ and $x^* := P_\varphi(y)$. We have

$$\langle x - v, w - x \rangle + \varphi(w) - \varphi(x) \geq 0, \forall w \in \mathbb{R}^n \quad (9)$$

and

$$\langle x^* - y, w - x^* \rangle + \varphi(w) - \varphi(x^*) \geq 0, \forall w \in \mathbb{R}^n. \quad (10)$$

Setting $w := x^*$ in (9) and $w := x$ in (10), we obtain the relations $\langle x - v, x - x^* \rangle - \varphi(x^*) + \varphi(x) \leq 0$ and $-\langle x^* - y, x - x^* \rangle - \varphi(x) + \varphi(x^*) \leq 0$, from which we deduce that $\|x - x^*\|^2 \leq \|x - x^*\| \|v - y\|$. It follows that

$$\|P_\varphi(v) - P_\varphi(y)\| \leq \|v - y\|.$$

■

We may consider a more general *inequality problem*. Given a continuous mapping $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find $\bar{x} \in \mathbb{R}^n$ such that

$$\langle \Phi(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \forall v \in \mathbb{R}^n. \quad (11)$$

In other words,

$$-\Phi(\bar{x}) \in \partial\varphi(\bar{x}).$$

Having defined the operator P_φ , it is now clear that problem (11) is equivalent to a nonlinear equation: find $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} - P_\varphi(\bar{x} - \Phi(\bar{x})) = 0. \quad (12)$$

This may be handled by fixed point theory, since $\bar{x}$ is a fixed point of the operator $x \mapsto P_\varphi(x - \Phi(x))$, or by degree theory.

4 Linear complementarity problems

The basic *Linear Complementarity Problem*, LCP for short, is presented in [11] (pp. 1-32) as follows. Consider $H = \mathbb{R}^n$ with the standard scalar product denoted by a dot and the order relation $x \geq y$ iff $x_i \geq y_i$ for all $1 \leq i \leq n$. This is the order associated to the cone $K = R_+^n$, since $x \geq y$ iff $x - y \in K$. Then the LCP problem denoted by $LCP(q, M)$ requires that, given $q \in R^n$ and a $n \times n$ matrix M we find z satisfying

$$z \geq 0, \quad q + Mz \geq 0, \quad z \cdot (q + Mz) = 0. \quad (13)$$

This is often written as

$$0 \leq z \perp q + Mz \geq 0.$$

It is easy to verify that this is equivalent to finding z such that

$$-(q + Mz) \in N_K(z). \quad (14)$$

More generally, we may consider the *Nonlinear Complementarity Problem* (NCP) associated in a similar manner to a function f from R^n into itself: find z such that

$$z \geq 0, \quad f(z) \geq 0, \quad z \cdot f(z) = 0. \quad (15)$$

This is equivalent to finding z such that

$$-f(z) \in N_K(z), \quad (16)$$

where again $K = R_+^n$. Since $N_K(z) = \partial I_K(z)$, this is nothing but a special case of (11).

The NCP may be generalized as follows. Let K be a cone in R^n , with dual cone K^* , and f be given as above. Then the *complementarity problem over the cone K* , denoted $CP(K, f)$, consists in finding z such that

$$z \in K, \quad f(z) \in K^*, \quad z \cdot f(z) = 0. \quad (17)$$

If $K = R_+^n$, then it is self-dual, i. e. $K^* = K$, and thus CP reduces to NCP.

An associated problem is the following *variational inequality* $VI(K, f)$: find z such that

$$z \in K, \quad f(z) \cdot (y - z) \geq 0, \quad \forall y \in K. \quad (18)$$

It can easily be shown that the problems are equivalent (cf. [11], Prop. 1.5.2):

Theorem 4.1 *Consider a cone $K \subset R^n$, a map $f : R^n \rightarrow R^n$ and $z \in R^n$. Then z solves $CP(K, f)$ if and only if it solves $VI(K, f)$.*

Again, $VI(K, f)$ is equivalent to (16) - now with a general cone K - and so it is a special case of (11).

It is worth mentioning that the standard existence result for the LCP is the one due to Samuelson, Thrall and Wesler, which says that (cf. [11], p. 148):

Theorem 4.2 *The LCP $0 \leq z \perp (q + Mz) \geq 0$ has a unique solution for all q if and only if M is a P-matrix, i.e. all its principal minors are positive.*

Notice that if M is symmetric, then M is a P-matrix if and only if it is positive definite. In this situation, the LCP is equivalent to a constrained minimum problem.

5 Existence results for first order evolution variational inequalities

In the mathematical literature, there are a lot of existence theorems for evolution variational inequalities. Here we choose to use a variant of Kato's theorem [17] which is a powerful tool for handling nonlinear evolution problems.

Theorem 5.1 *Let H be a real Hilbert space and let $A : D(A) \subset H \rightrightarrows H$ be a maximal monotone operator. Let $t_0 \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $u_0 \in D(A)$ be given and suppose that $f : [t_0, +\infty) \rightarrow H$ satisfies*

$$f \in C^0([t_0, +\infty); H), \quad \frac{df}{dt} \in L^1_{\text{loc}}(t_0, +\infty; H).$$

Then there exists a unique $u \in C^0([t_0, +\infty); H)$ satisfying

$$\begin{aligned} \frac{du}{dt} &\in L^\infty_{\text{loc}}(t_0, +\infty; H); \\ u &\text{ is right-differentiable on } [t_0, +\infty); \\ u(t) &\in D(A), \quad t \geq t_0; \\ u(t_0) &= u_0; \\ -\frac{du}{dt}(t) + \sigma u(t) + f(t) &\in Au(t), \quad \text{a.e. } t \geq t_0. \end{aligned} \tag{19}$$

Using this Theorem, we get the following existence result for evolution variational inequalities.

Corollary 5.1 [16] *Let H be a real Hilbert space and let $\varphi \in \Gamma_0(H)$. Let $A : H \rightarrow H$ be a hemicontinuous operator such that for some $w_1 \geq 0$, $A + w_1 I$ is monotone. Let $B : H \rightarrow H$ be an operator such that*

$$\|Bu - Bv\| \leq w_2 \|u - v\|, \quad \forall u, v \in H$$

for some $w_2 > 0$. Let $t_0 \in \mathbb{R}$ and $u_0 \in D(\partial\varphi)$ be given and suppose that $f : [t_0, +\infty) \rightarrow H$ satisfies

$$f \in C^0([t_0, +\infty); H), \quad \frac{df}{dt} \in L^1_{\text{loc}}(t_0, +\infty; H).$$

Then there exists a unique $u \in C^0([t_0, +\infty); H)$ such that

$$\frac{du}{dt} \in L^\infty_{\text{loc}}(t_0, +\infty; H); \tag{20}$$

$$u \text{ is right-differentiable on } [t_0, +\infty); \tag{21}$$

$$u(t) \in D(\partial\varphi), \quad t \geq t_0; \tag{22}$$

$$u(t_0) = u_0; \tag{23}$$

$$\left\langle \frac{du}{dt}(t) + Au(t) + Bu(t) - f(t), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geq 0, \quad \forall v \in H, \quad \text{a.e. } t \geq t_0. \tag{24}$$

In other words, we solve:

$$-\frac{du}{dt}(t) + f(t) \in \partial\varphi(u(t)) + Au(t) + Bu(t). \quad (25)$$

Proof. Let w_1 and w_2 be positive numbers. Then (25) is equivalent to

$$-\frac{du}{dt}(t) + (w_1 + w_2)u(t) + f(t) \in \partial\varphi(u(t)) + (A + w_1)u(t) + (B + w_2)u(t)$$

or

$$-\frac{du}{dt}(t) + (w_1 + w_2)u(t) + f(t) \in \partial\varphi(u(t)) + T_1u(t) + T_2u(t),$$

if we set $T_1 = A + w_1I$ and $T_2 = B + w_2I$.

It is clear that T_1 is monotone and hemicontinuous. The operator T_2 is Lipschitz continuous and thus hemicontinuous. In addition, it is seen that

$$\begin{aligned} \langle T_2x - T_2y, x - y \rangle &= \langle Bx - By, x - y \rangle + w_2\|x - y\|^2 \\ &\geq -\|Bx - By\| \|x - y\| + w_2\|x - y\|^2 \\ &\geq -w_2\|x - y\|^2 + w_2\|x - y\|^2 = 0, \quad \forall x, y \in H \end{aligned}$$

thus T_2 is monotone.

Hence, $T = T_1 + T_2$ is monotone and hemicontinuous and thus maximal monotone. We now apply Theorem 5.1 with $A + T$ (which is also maximal monotone) and with $\sigma = w_1 + w_2$ to conclude that a unique map $u \in C^0([t_0, +\infty); H)$ can be found to fulfill (20)-(24).

6 Degree theory

We mention briefly some preliminaries on topological degree theory.

Let $D \subset \mathbb{R}^n$ be an open and bounded set. Let $f \in C^1(D; \mathbb{R}^n) \cap C^0(\bar{D}; \mathbb{R}^n)$ be given.

Set $Z_f(D) := \{x \in \bar{D} : f(x) = 0\}$ (the zeros of f in D) and $C_f(D) := \{x \in D : \det J_f(x) = 0\}$ (the critical points of f in D) where $J_f(x)$ denotes the Jacobian matrix of f at x defined by $(J_f)_{ij} := \frac{\partial f_i}{\partial x_j}(x)$, $i, j \in \{1, \dots, n\}$. If $Z_f(D) \cap C_f(D) = \emptyset$ and $0 \notin f(\partial D)$, then $Z_f(D)$ is a finite set and the *Brouwer topological degree* of f with respect to D and 0 is well-defined by the formula

$$\deg(f, D, 0) = \sum_{x \in Z_f(D)} \text{sgn}(\det J_f(x)),$$

where $\text{sgn}(t) = 1$ for $t > 0$ and $\text{sgn}(t) = -1$ for $t < 0$. More generally, if $f : \bar{D} \rightarrow \mathbb{R}^n$ is continuous and $0 \notin f(\partial D)$ then the Brouwer topological degree of f with respect to D and 0 is well-defined (see e.g. [20]) and denoted by $\deg(f, D, 0)$.

Let us now recall some useful properties of the topological degree.

1. If $0 \notin f(\partial D)$ and $\deg(f, D, 0) \neq 0$ then there exists $x \in D$ such that $f(x) = 0$.
2. Let $\varphi : [0, 1] \times \bar{D} \rightarrow \mathbb{R}^n; (\lambda, x) \mapsto \varphi(\lambda, x)$, be continuous such that, for each $\lambda \in [0, 1]$, one has $0 \notin \varphi(\lambda, \partial D)$. Then the map $\lambda \mapsto \deg(\varphi(\lambda, \cdot), D, 0)$ is constant on $[0, 1]$.
3. If $0 \in D$ then

$$\deg(id_{\mathbb{R}^n}, D, 0) = 1.$$

4. If $0 \notin f(\partial D)$ and $\alpha > 0$ then

$$\deg(\alpha f, D, 0) = \deg(f, D, 0)$$

and

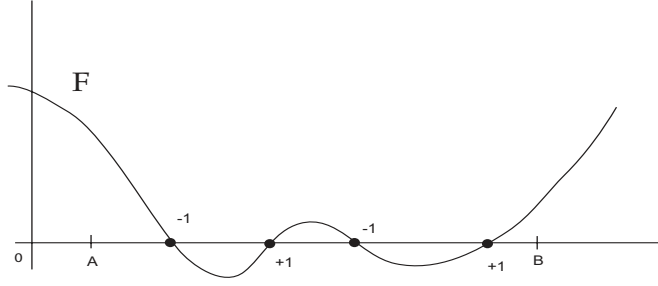
$$\deg(-\alpha f, D, 0) = (-1)^n \deg(f, D, 0).$$

5. Let $D' \subset D$ be an open set such that $Z_f(D) \subset D'$. Then

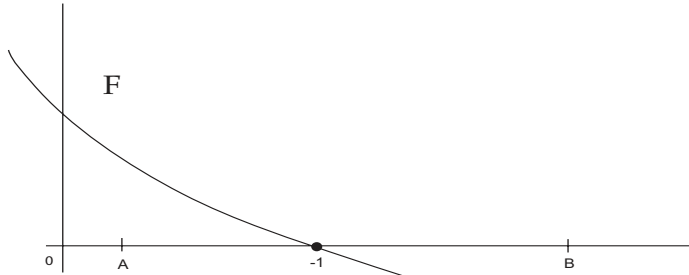
$$\deg(f, D, 0) = \deg(f, D', 0).$$

6. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. If $0 \in D$ then $\deg(A, D, 0) = \text{sgn}(\det A)$.
7. Let $f : [-r, +r] \rightarrow \mathbb{R}$ be a continuous function. Suppose that $f(-r) \neq 0$ and $f(+r) \neq 0$. Then

$$\begin{aligned} \deg(f,]-r, +r[, 0) &= 0 \quad \text{if } f(-r)f(+r) > 0, \\ \deg(f,]-r, +r[, 0) &= +1 \quad \text{if } f(-r) < 0 \text{ and } f(+r) > 0, \\ \deg(f,]-r, +r[, 0) &= -1 \quad \text{if } f(-r) > 0 \text{ and } f(+r) < 0. \end{aligned}$$



$$\deg(F,]A, B[, 0) = -1 + 1 - 1 + 1 = 0$$



$$\deg(F,]A, B[, 0) = -1$$

Figure 10: Degree of F .

From this theory, *Brouwer fixed point theorem* may be deduced:

Theorem 6.1 *Let $K \subset R^n$ be a non-empty closed convex set and $\phi : K \rightarrow R^n$ be a continuous map such that $\phi(K) \subset K$. Then ϕ has a fixed point in K : there exists $\bar{x} \in K$ such that $\phi(\bar{x}) = \bar{x}$.*

It is worth noting that the degree theory can be extended to infinite dimensional spaces (*Leray-Schauder degree*) and to multivalued (set-valued) maps. For the latter, we may refer the reader to [20] (section 7.3) or to [12].

7 Set-valued Analysis

A few preliminaries on set-valued or multivalued analysis are given, which may be needed in the sequel.

For a multifunction, i.e. a set-valued mapping

$$F : X \rightrightarrows Y, x \mapsto F(x) \subset Y$$

there are two possible inverse images: if $C \subset Y$, we denote

$$F^-(C) = F^{-1}(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\},$$

$$F_-(C) = \{x \in X \mid F(x) \subset C\}.$$

When X and Y are topological spaces, this leads to different continuity notions.

F is *upper semi-continuous (USC)* if $F_-(V)$ is open, for every open set $V \subset Y$; F is *lower semi-continuous (LSC)* if $F^-(V)$ is open, for every open set $V \subset Y$.

This is equivalent (respectively) to requiring that, for any $x \in X$, F be *USC at x* , that is, for any open set W containing $F(x)$, there exists a neighbourhood U of x such that $F(U) \subset W$ (i.e. $F(z) \subset W$, $\forall z \in U$); or that F be *LSC at x* , i.e., that for every $y \in F(x)$ and every neighbourhood V of y , there exists a neighbourhood U of x such that $F(z) \cap V \neq \emptyset$, for all $z \in U$.

For a multifunction F between metric spaces X and Y , it can be shown that if F is USC with closed values then its graph is closed in $X \times Y$. Moreover, if Y is compact and F has a closed graph, then F is USC. Therefore, *USC* is closely related to the *closed graph* assumption, which in the case of metric spaces is stated as:

$$(x_n \rightarrow x, y_n \in F(x_n) (\forall n), y_n \rightarrow y) \Rightarrow y \in F(x),$$

while F is LSC iff

$$(x_n \rightarrow x, y \in F(x)) \Rightarrow \exists y_n \in F(x_n) (\forall n) : y_n \rightarrow y.$$

F is *continuous* if it is both USC and LSC. There exist also "metric" versions of these definitions, which use the so-called *Hausdorff distance* or Hausdorff metric h or d_H between subsets A and B of a metric space, say (X, d) :

$$h(A, B) = \max(e(A, B), e(B, A)), \text{ where } e(A, B) = \sup_{a \in A} d(a, B)$$

and $d(a, B) = \inf_{b \in B} d(a, b)$.

We say that a set-valued function F with values in a metric space is *Hausdorff continuous* at t iff $s \rightarrow t$ implies $h(F(s), F(t)) \rightarrow 0$. If F is Hausdorff-continuous at each t in the domain, we say that it is Hausdorff-continuous or *H-continuous*, for short.

When the domain is also a metric space, an important class of H-continuous multifunctions is that of *Lipschitz-continuous* ones, which satisfy for some k :

$$h(F(x), F(y)) \leq k d(x, y), \forall x, y.$$

When F has compact values, F is H-continuous iff F is continuous.

Another important topic is the existence of *selections* of a multifunction F , that is, of functions f such that $f(x) \in F(x)$ for every x . *Michael's theorem* ensures that LSC multifunctions on a metric space X with non-empty closed convex values in a Banach space Y do have continuous selections $f : X \rightarrow Y$. If instead F is USC, then approximate continuous selections exist: for any $\varepsilon > 0$, there is a continuous f_ε such that, for all x , $d((x, f_\varepsilon(x)), \text{Graph}(F)) \leq \varepsilon$.

The approximate continuous selections can be used in combination with fixed point theory or degree theory to obtain fixed point theorems and a degree theory for set-valued functions. For instance, we have the following *Tikhonov-Kakutani-Ky Fan Fixed Point Theorem* (cf. [4], p. 85):

Theorem 7.1 *Let K be a non-empty compact convex set of a Banach space X and F be a USC multifunction defined on K with non-empty compact convex values contained in K . Then F has at least one fixed point, i.e. there is $\bar{x} \in K$ such that*

$$\bar{x} \in F(\bar{x}).$$

Further topics of general interest include:

- measurability and existence of measurable selections,
- the integral of set-valued functions,
- convergence results.

The latter will appear in the study of differential inclusions. References for the subjects in this section include [4] and [5].

8 Differential inclusions a la Filippov

Differential inclusions are relations involving derivatives of functions and set-valued functions, such as, in a normalized form:

$$\frac{dx}{dt}(t) \in F(t, x(t)).$$

We may consider examples:

1) If F has a continuous selection $f(t, x) \in F(t, x)$ for all (t, x) , then any solution of the ODE $dx/dt(t) = f(t, x(t))$ is a solution of the differential inclusion.

2) The evolution problems in Section 4 are also differential inclusions, associated to monotone operators. They are a class of their own and are dealt with specific techniques. A special type of such evolution problems, called the sweeping process, will be the object of the last section.

Here, we shall be concerned with a standard existence result and the so-called Filippov solutions. To escape "triviality" (example 1 above), it is clear that we have to consider USC multifunctions, if we are dealing with closed convex-valued righthand sides. Then:

Theorem 8.1 *Let $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an USC multifunction with non-empty closed convex values $F(t, x) \subset \mathbb{R}^d$. Assume moreover that F satisfies some growth condition:*

$$F(t, x) \subset (a + b\|x\|) B_1, \quad \forall (t, x) \in [0, t] \times \mathbb{R}^d,$$

where B_1 is the unit ball. Let $x_0 \in \mathbb{R}^d$. Then there exists a Lipschitz-continuous function $x : [0, T] \rightarrow \mathbb{R}^d$ such that

$$x(0) = x_0, \quad \frac{dx}{dt}(t) \in F(t, x(t)),$$

for almost every $t \in [0, T]$.

Remarks - 1) The roles of convexity, USC and the weak convergence of derivatives in the proof will deserve more attention (in the oral presentation).

2) The growth condition may be relaxed (cf. [4],[12]).

The so-called *viability problems*, where it is additionally required that the solution remains in a given closed set D , or that $x(t) \in D(t)$ for all t , may also be solved, if a certain tangency condition is satisfied. Let us mention just one result ([12]):

Theorem 8.2 *Let $F : [0, T] \times R^d \rightarrow R^d$ be as in the previous theorem. Let D be a non-empty closed (convex) subset of R^d and assume that*

$$F(t, x) \cap T_D(x) \neq \emptyset, \quad \forall x \in D, \forall t.$$

Let $x_0 \in D$. Then there exists a Lipschitz-continuous function $x : [0, T] \rightarrow R^d$ such that

$$x(0) = x_0, \quad x(t) \in D, \quad \forall t, \quad \frac{dx}{dt}(t) \in F(t, x(t)),$$

for almost every $t \in [0, T]$.

A very important class of differential inclusions arises from the consideration of *differential equations with discontinuous righthand sides*, i.e. $dx/dt(t) \in f(t, x(t))$, where $f(t, x)$ is discontinuous with respect to the state variable x . For instance, in many applications, we have to consider piecewise linear equations, where different linear righthand sides are considered in regions separated by hyperplanes.

For simplicity, let us consider the autonomous case $f = f(x)$. If we try to solve the ODE by a discretization scheme, what do we expect the derivative of the limit function x to be, say at time t , when f is discontinuous at $x(t)$? This question leads to the consideration of an associated set-valued function and an associated differential inclusion, linked to the name of Filippov ([13]; cf. also [4], p. 101-103).

Theorem 8.3 *If f is locally bounded (bounded in a neighbourhood of each point) then the set-valued function defined by*

$$F(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} f(x + \varepsilon B_1)$$

is USC with compact convex values. Moreover, for all x , $f(x) \in F(x)$ and if f is continuous at x , then $F(x) = \{f(x)\}$.

Therefore, every solution of the ODE

$$\frac{dx}{dt}(t) = f(x(t))$$

is a solution of the differential inclusion

$$\frac{dx}{dt}(t) \in F(x(t)),$$

while if f is continuous at $x(t)$ and the differential inclusion is satisfied at that t , then the ODE is also satisfied. We may therefore investigate instead the solutions of the differential inclusion, which exist (under suitable growth conditions for f , hence for F), according to the above theory.

It may be convenient to discard information coming from sets of measure zero, say a line in the plane. For that purpose, it is useful to consider another set-valued map.

Theorem 8.4 *If f is measurable and locally bounded, then the multifunction defined by*

$$F_0(x) = \bigcap_{\varepsilon > 0} \bigcap_{N: \text{meas}(N)=0} \overline{\text{co}} f((x + \varepsilon B_1) \setminus N)$$

is USC with compact convex values. Moreover $f(x) \in F_0(x)$, for almost every x , and if f is continuous at x , then $F_0(x) = \{f(x)\}$.

Similar relations hold between the ODE and the differential inclusion

$$\frac{dx}{dt}(t) \in F_0(x(t)).$$

Solutions of the differential inclusions are called *Filippov solutions* and the theory is thoroughly presented in [13]. In the lecture, a few more points may be highlighted, if the schedule permits.

9 Sweeping processes

Sweeping processes (S.P.) are differential inclusions associated to convex sets. The basic form is the following:

$$-\frac{dx}{dt}(t) \in N_{C(t)}(x(t)),$$

where the righthand side is the outward normal cone to a closed convex set $C(t)$, which depends on t . By choosing a convenient set, the above formalism also encompasses inclusions such as

$$-\frac{dx}{dt}(t) \in N_{C(t)}(x(t)) + f(t, x(t)).$$

Notice that the normal cone is an unbounded set-valued function, so that we would need existence results for "unbounded" differential inclusions. Moreover, the S.P. implicitly requires that

$$x(t) \in C(t), \quad \forall t,$$

so that it may also be considered as a viability problem. In addition, the normal cone is the subdifferential of an indicator function, therefore it is a monotone operator; hence, the S.P. is a special evolution equation where the monotone operators have time-dependent domains (the convex sets $C(t)$, to be precise). It is not surprising that the study of the S.P. has common features with the previous theories, but it also leads to new developments.

The S.P. was introduced in the early seventies by J. J. Moreau and it was extensively studied by him and then mainly by the Montpellier school (C. Castaing, M. Valadier, L. Thibault and

many others, including the present author). A basic reference is the paper [28]. In [18], it can be found an introduction to the topic and to some applications to quasi-statics and to dynamics.

In the present lecture, the basics of the topic will be presented.

Theorem 9.1 (Moreau) *If $C : t \mapsto C(t)$ is a Lipschitz-continuous multifunction with non-empty closed convex values in R^d (or in a Hilbert space H), and if $x_0 \in C(0)$, then there exists a unique Lipschitz-continuous function x such that $x(0) = x_0$, $x(t) \in C(t)$ for all t and*

$$-\frac{dx}{dt}(t) \in N_{C(t)}(x(t)), \text{ for a.e. } t.$$

The proof uses a numerical scheme called the *catching-up algorithm*, where the new approximate value of x is obtained by the nearest point projection into the updated set:

$$x_{n,i+1} = \text{proj}(x_{n,i}, C(t_{n,i+1})).$$

The existence theory may be extended to less smooth cases. It may be assumed that C has bounded variation, in the sense that for some nondecreasing function v :

$$h(C(s), C(t)) \leq v(t) - v(s), \quad \forall s \leq t.$$

Then, a solution of the S.P.

$$-du \in N_{C(t)}(u(t))$$

is expected to be a function u with values in R^d having bounded variation, and therefore a differential or Stieltjes measure du . If u' is a (vector-valued) density function such that $du = u'd\mu$ ($d\mu$ being a positive measure), then it is required that

$$-u'(t) \in N_{C(t)}(u(t)),$$

for $d\mu$ almost every t . This formulation is not ambiguous since the r.h.s. is a cone.

Theorem 9.2 (Moreau) *Let $C : t \mapsto C(t)$ be a multifunction with non-empty closed convex values in R^d (or in a Hilbert space H) having bounded variation, which is right-continuous (in the sense of Hausdorff distance) i.e. v above is right-continuous. Let $u_0 \in C(0)$. Then there exists a unique right-continuous function u with bounded variation such that $u(0) = u_0$, $u(t) \in C(t)$ for all t and*

$$-du \in N_{C(t)}(u(t))$$

in the above sense.

The catching-up algorithm is again used. There exist also b. v. solutions for sweeping processes defined by LSC moving closed convex sets with nonempty interior.

In the field of the dynamics with unilateral contact, problems may be formulated as differential inclusions in terms of velocities, which are only expected to be functions of bounded variation. Also, cones are present, either the cone of admissible velocities at contact or the friction cone. Therefore, the ideas which were helpful to deal with the S.P. are still fruitful in the new setting. Algorithms which resemble the catching-up algorithm are successfully used in current practice, while the "theoretical crew" tries to follow that up with convergence and existence proofs, usually several years behind.

Two references (among others) for the reader interested in applications are [8] and [9].

10 A Lyapunov Approach for Evolution Variational Inequalities

Let us first specify the general mathematical framework for stability theory and formulate the following assumptions (H) :

(H₁) H is a real Hilbert space

(H₂) $\varphi \in \Gamma_0(H)$ such that $0 \in D(\partial\varphi)$

(H₃) $A : H \rightarrow H$ is a hemicontinuous operator such that for $w_1 \geq 0$, $A + w_1 I$ is monotone

(H₄) $B : H \rightarrow H$ is an operator such that for some $w_2 > 0$,

$$\|Bu - Bv\| \leq w_2 \|u - v\|, \forall u, v \in H$$

(H₅) $f \in C^0([t_0, +\infty); H)$, $\frac{df}{dt} \in L^1_{\text{loc}}(t_0, +\infty; H)$

(H₆) $(A(0) + B(0) - f(t), v) + \varphi(v) - \varphi(0) \geq 0, \forall v \in H, \forall t \geq t_0$.

Condition (H₆) is equivalent to

$$f(t) - (A(0) + B(0)) \in \partial\varphi(0), \forall t \geq t_0.$$

From Corollary 5.1, we deduce that for each $u_0 \in D(\partial\varphi)$ problem $P(t_0, u_0)$: (24) has a unique solution $t \rightarrow u(t; t_0, u_0)$ ($t \geq t_0$) with $u \in C^0([t_0, +\infty); H)$, $\frac{du}{dt} \in L^\infty_{\text{loc}}(t_0, +\infty; H)$, and u right-differentiable on $[t_0, +\infty)$.

Moreover, conditions $0 \in D(\partial\varphi)$ and (H₆) ensure that

$$u(t; t_0, 0) = 0, \quad t \geq t_0,$$

i.e. the trivial solution 0 is the unique solution of problem $P(t_0, 0)$. This solution is called stationary solution because for the unilateral system modeled by (3.1) the trajectory remains in the same position 0 for all times $t \geq t_0$.

Note that if $\varphi = I_K$, where K is a closed convex subset of H such that $0 \in K$ and I_K denotes the indicator function of K , then problem $P(t_0, u_0)$ ($t_0 \in \mathbb{R}, u_0 \in K$) reads as

$$\left(\frac{du}{dt}(t) + Au(t) + Bu(t) - f(t), v - u(t)\right) \geq 0, \forall v \in K, \text{ a.e. } t \geq t_0 \quad (26)$$

$$u(t) \in K, \quad t \geq t_0 \quad (27)$$

$$u(t_0) = u_0. \quad (28)$$

This last model appears frequently in the applications. Note that in this case (H₂) is satisfied and (H₆) takes the form

$$(f(t) - (A(0) + B(0)), v) \leq 0, \forall v \in K, \forall t \geq t_0, \quad (29)$$

which can be expressed as follows

$$f(t) - (A(0) + B(0)) \in N_K(0), \quad \forall t \geq t_0,$$

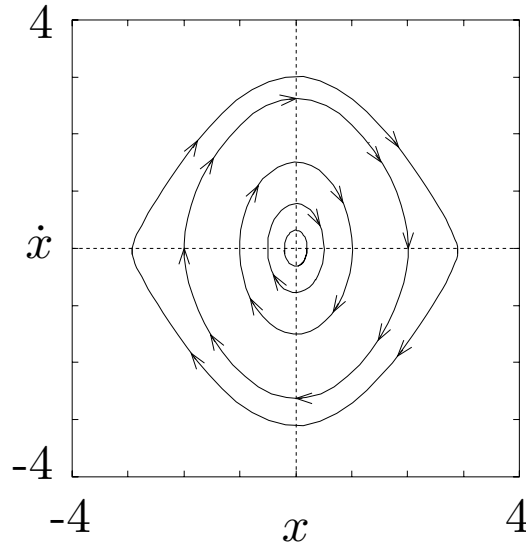
with the normal cone

$$N_K(x) = \{w \in H : (w, y - x) \leq 0, \quad \forall y \in K\}, \quad x \in K.$$

We may now define the stability of the trivial solution in the setting of evolution variational inequalities (for the corresponding definitions in the case of ordinary differential equations see, e.g., [10]). The stationary solution 0 is called stable if small perturbations of the initial condition $u(t_0) = 0$ lead to solutions which remain in the neighborhood of 0 for all $t \geq t_0$, precisely:

Definition 10.1 *The solution 0 is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that for any $u_0 \in D(\partial\varphi)$ with $\|u_0\| \leq \eta$ the solution $u(\cdot; t_0, u_0)$ of problem $P(t_0, u_0)$ satisfies*

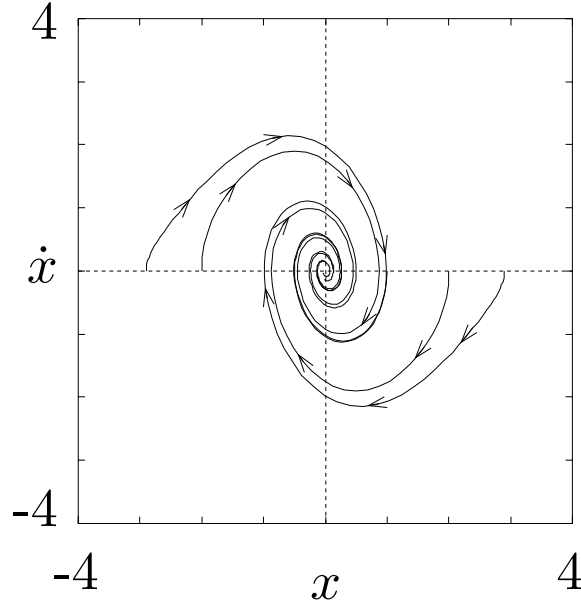
$$\|u(t; t_0, u_0)\| \leq \varepsilon, \quad \forall t \geq t_0.$$



If in addition the trajectories of the perturbed solutions are attracted by 0 then we say that the stationary solution is asymptotically stable, precisely:

Definition 10.2 *We say that the solution 0 is asymptotically stable if it is stable and there exists $\delta > 0$ such that for any $u_0 \in D(\partial\varphi)$ with $\|u_0\| \leq \delta$ the solution $u(\cdot; t_0, u_0)$ of problem $P(t_0, u_0)$ fulfills*

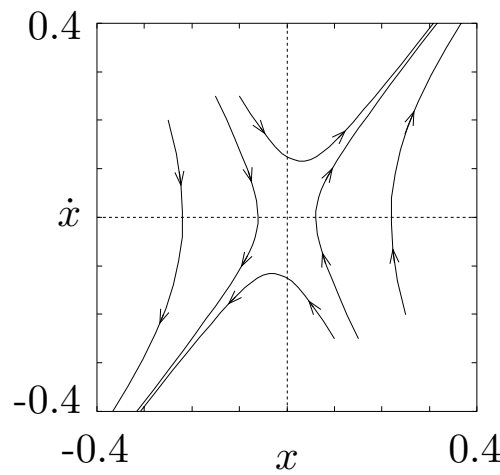
$$\lim_{t \rightarrow +\infty} \|u(t; t_0, u_0)\| = 0.$$



The notion of unstability is given below.

Definition 10.3 We say that the solution 0 is unstable if it is not stable (see Definition 3.1), i.e. there exists $\varepsilon > 0$ such that for any $\eta > 0$, one may find $u_0 \in D(\partial\varphi)$ with $\|u_0\| \leq \eta$ and $\bar{t} \geq t_0$ such that the solution $u(\cdot; t_0, u_0)$ of problem $P(t_0, u_0)$ verifies

$$\|u(\bar{t}; t_0, u_0)\| > \varepsilon.$$



Theorem 10.1 (Stability) Assume that conditions (H) hold and there exist $\sigma > 0$ and $V \in C^1(H; \mathbb{R})$ such that

(1)

$$V(x) \geq a(\|x\|), \quad x \in D(\partial\varphi), \quad \|x\| \leq \sigma,$$

with $a : [0, \sigma] \rightarrow \mathbb{R}$ satisfying $a(t) > 0, \forall t \in (0, \sigma)$;

(2) $V(0) = 0$;

(3) $(Ax + Bx - f(t), V'(x)) + (w, V'(x)) \geq 0, \quad x \in D(\partial\varphi), \quad \|x\| \leq \sigma, \quad w \in \partial\varphi(x), \quad t \geq t_0.$

Then the (trivial) solution of problem $P(t_0, 0)$ is stable.

Theorem 10.2 (Asymptotic Stability) Assume that conditions (H) hold and there exist $\sigma > 0, \lambda > 0$ and $V \in C^1(H; \mathbb{R})$ such that

(1)

$$V(x) \geq a(\|x\|), \quad x \in D(\partial\varphi), \quad \|x\| \leq \sigma,$$

with $a : [0, \sigma] \rightarrow \mathbb{R}$ satisfying $a(t) \geq ct^\tau, \forall t \in [0, \sigma]$, for some constants $c > 0, \tau > 0$;

(2) $V(0) = 0$;

(3) $(Ax + Bx - f(t), V'(x)) + (w, V'(x)) \geq \lambda V(x), \quad x \in D(\partial\varphi), \quad \|x\| \leq \sigma, \quad w \in \partial\varphi(x), \quad t \geq t_0.$

Then the (trivial) solution of $P(t_0, 0)$ is asymptotically stable.

Let us recall that the tangent cone $T_K(x)$ of a subset K of H at $x \in K$ is defined by

$$T_K(x) = N_K(x)^\circ = \{w \in H : (w, v) \leq 0, \forall v \in N_K(x)\}.$$

Corollary 10.1 Assume condition (H_1) and let K be a closed convex subset of H with $0 \in K$. Assume that conditions (H_3) - (H_5) and (29) hold. Suppose that there exists $\sigma > 0$ and $V \in C^1(H; \mathbb{R})$ such that

(1)

$$V(x) \geq a(\|x\|), \quad x \in K, \quad \|x\| \leq \sigma,$$

with $a : [0, \sigma] \rightarrow \mathbb{R}$ satisfying $a(t) > 0, \forall t \in (0, \sigma)$;

(2) $V(0) = 0$;

(3) $-V'(x) \in T_K(x), \quad x \in \partial K, \quad \|x\| \leq \sigma$;

(4) $(Ax + Bx - f(t), V'(x)) \geq 0, \quad x \in K, \quad \|x\| \leq \sigma, \quad t \geq t_0.$

Then the (trivial) solution of (26)-(28) with $u_0 = 0$ is stable. In particular, the conclusion holds true if (3) is replaced by

(3)' $x - V'(x) \in K, \quad x \in \partial K, \quad \|x\| \leq \sigma.$

Proof. Let us check that condition $(3)'$ implies (3) . Assume that $(3)'$ is satisfied. Then for an x as in (3) and every $v \in N_K(x)$ we have

$$(v, y - x) \leq 0, \forall y \in K.$$

Therefore, taking $y = x - V'(x)$ (cf. $(3)'$), one obtains

$$(v, -V'(x)) \leq 0, \forall v \in N_K(x),$$

which means that $-V'(x) \in T_K(x)$, thus (3) is verified.

We apply Theorem 10.1 with $\varphi = I_K$, i.e. φ is equal to the indicator function of K . Then (H_2) is clearly verified and (29) is equivalent to (H_6) . Since (H_1) , $(H_3) - (H_5)$ as well as (1) , (2) have been admitted, it remains to verify condition (3) in Theorem 10.1. Let $x \in K$ with $\|x\| \leq \sigma$. If $x \in \text{int}\{K\}$ then $\partial I_K(x) = \{0\}$. If $x \in \partial K$ then $\partial I_K(x) = N_K(x)$. Thus if $x \in \partial K$ and $w \in \partial I_K(x)$ then, by (3) , we obtain

$$(-V'(x), w) \leq 0.$$

It results that

$$(w, V'(x)) \geq 0, \forall x \in K, \|x\| \leq \sigma, w \in \partial I_K(x).$$

Therefore (3) in Theorem 10.1 is satisfied. The conclusion follows from Theorem 10.1. $\square$

Example 3.1. Let us consider problem (26)-(28) with $u_0 = 0$ where $H = \mathbb{R}^2$, $K = \mathbb{R}_+ \times \mathbb{R}_+$, $A \equiv 0$, $\Phi \equiv 0$, $f \equiv 0$ and

$$B(x_1, x_2) = (x_2, -\sin(x_1)), (x_1, x_2) \in \mathbb{R}^2.$$

It is clear that assumptions (H_1) , $(H_3) - (H_5)$ and (29) in Corollary 10.1 are satisfied. We choose

$$V(x_1, x_2) = \frac{x_2^2}{2} - \cos(x_1) + 1, \forall (x_1, x_2) \in \mathbb{R}^2.$$

We have for $|x_1|$ small enough that

$$1 - \cos(x_1) \geq \frac{x_1^2}{4}.$$

Thus, there exists $\sigma > 0$ such that

$$V(x_1, x_2) \geq \frac{x_1^2}{4} + \frac{x_2^2}{4}, \|x\| \leq \sigma,$$

which establishes (1) in Corollary 10.1 taking $a(t) = \frac{t^2}{4}$ for $0 \leq t \leq \sigma$. Moreover, (2) in Corollary 10.1 is obviously satisfied. Since

$$V'(x_1, x_2) = (\sin(x_1), x_2)$$

it follows that

$$(B(x_1, x_2), V'(x_1, x_2)) = 0, (x_1, x_2) \in \mathbb{R}^2,$$

i.e. (4) in Corollary 10.1 holds. Using

$$(x_1, x_2) - V'(x_1, x_2) = (x_1 - \sin x_1, 0),$$

it turns out for σ above that

$$x = (x_1, x_2) \in \partial K, \|x\| \leq \sigma \Rightarrow (x_1, x_2) - V'(x_1, x_2) \in K,$$

i.e. (3)' in Corollary 10.1 holds true. All assumptions of Corollary 10.1 are satisfied and we may conclude to the stability of the trivial solution of problem (26)-(28) with $u_0 = 0$.

Theorem 10.3 (Unstability) Assume that conditions (H) hold. Suppose in addition that 0 is a cluster point of $D(\partial\varphi)$. If there exist $V \in C^1(H, \mathbb{R})$ and $\alpha > 0$ such that

(1)

$$V(x) \leq b(\|x\|), x \in D(\partial\varphi),$$

with $b : [0, +\infty) \rightarrow \mathbb{R}$ satisfying $b(t) \leq kt^s, \forall t \geq 0$, for some constants $k > 0, s > 0$;

(2) $V(x) > 0, x \in D(\partial\varphi), x \neq 0$ near 0;

(3) $(Ax + Bx - f(t), V'(x)) + (w, V'(x)) \leq -\alpha V(x), x \in D(\partial\varphi), w \in \partial\varphi(x), t \geq t_0$,

then the (trivial) solution of $P(t_0, 0)$ is unstable.

Corollary 10.2 Assume condition (H₁) and let K be a closed convex set in H with $0 \in K$ and $K \setminus \{0\} \neq \emptyset$. Suppose that conditions (H₃) - (H₅) and (29) hold. If there exist $V \in C^1(H; \mathbb{R})$ and $\alpha > 0$ such that

(1)

$$V(x) \leq b(\|x\|), x \in K,$$

with $b : [0, +\infty) \rightarrow \mathbb{R}$ satisfying $b(t) \leq kt^s, \forall t \geq 0$, for some constants $k > 0, s > 0$;

(2) $V(x) > 0, x \in K, x \neq 0$ near 0;

(3) $V'(x) \in K_\infty, x \in \partial K$;

(4) $(Ax + Bx - f(t), V'(x)) \leq -\alpha V(x), x \in K, t \geq t_0$,

then the (trivial) solution of (26)-(28) with $u_0 = 0$ is unstable.

11 Linear Variational Inequalities in $\mathbb{R}^n$

Let $A \in \mathbb{R}^{n \times n}$ be a real square matrix of order n and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function such that $0 \in \partial\varphi(0)$. Note that A satisfies condition (H_3) with $w_1 \geq \sup_{\|x\|=1} (-Ax, x)$.

For $(t_0, u_0) \in \mathbb{R} \times D(\partial\varphi)$, we consider the problem: Find $u : [t_0, +\infty) \rightarrow \mathbb{R}^n$ such that

$$\left(\frac{du}{dt}(t) + Au(t), v - u(t)\right) + \varphi(v) - \varphi(u(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0 \quad (30)$$

$$u(t) \in D(\partial\varphi), \quad t \geq t_0 \quad (31)$$

$$u(t_0) = u_0. \quad (32)$$

Here $(\cdot, \cdot)$ denotes the euclidean scalar product in $\mathbb{R}^n$. It is worth to point out that problem (30)-(32) has a unique solution u in the sense of (20)-(24). We denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the spectral radius of A , respectively.

A particular case of interest is obtained for

$$\varphi(x) := I_K(x),$$

where K is a closed convex set in $\mathbb{R}^n$ such that $0 \in K$ and I_K denotes the indicator function of K . In this case problem (30)-(32) for $(t_0, u_0) \in \mathbb{R} \times K$ reads: Find $u : [t_0, +\infty) \rightarrow \mathbb{R}^n$ such that

$$\left(\frac{du}{dt}(t) + Au(t), v - u(t)\right) \geq 0, \quad \forall v \in K, \text{ a.e. } t \geq t_0 \quad (5.4)$$

$$u(t) \in K, \quad t \geq t_0 \quad (5.5)$$

$$u(t_0) = u_0. \quad (5.6)$$

Let us recall that a matrix $A \in \mathbb{R}^{n \times n}$ is said to be positive stable if each eigenvalue of A has positive real part.

Theorem 11.1 (Lyapunov) *Let $A \in \mathbb{R}^{n \times n}$ be given. Then A is positive stable if and only if there exists a positive definite matrix $G \in \mathbb{R}^{n \times n}$ such that $GA + A^T G$ is positive definite.*

Remark 11.1 *Recall that if A is a nonsingular M-matrix, i.e.*

$$A = \alpha I - P, \quad P \geq 0, \quad \alpha > \rho(P),$$

then there is a positive diagonal matrix D such that $DA + A^T D$ is positive definite.

We first state a direct consequence of the results in Section 4.

Theorem 11.2 *Let $\varphi \in \Gamma_0(\mathbb{R}^n)$ such that $0 \in \partial\varphi(0)$.*

(i) *If A is positive semi-definite, i.e.*

$$(Ax, x) \geq 0, \quad x \in \mathbb{R}^n,$$

then the (trivial) solution of (30)-(32) with $u_0 = 0$ is stable.

(ii) *If A is positive definite, i.e.*

$$(Ax, x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0,$$

then the (trivial) solution of (30)-(32) with $u_0 = 0$ is asymptotically stable.

12 The Invariance Principle

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous operator such that for some $\omega \geq 0$, $F + \omega I$ is monotone. Here I denotes the identity mapping on $\mathbb{R}^n$. For $(t_0, x_0) \in \mathbb{R} \times D(\partial\varphi)$, we consider the problem $P(t_0, x_0)$: Find a function $t \mapsto x(t)$ ($t \geq t_0$) with $x \in C^0([t_0, +\infty); \mathbb{R}^n)$, $\frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n)$ and such that:

$$\begin{cases} x(t) \in D(\partial\varphi), \quad t \geq t_0, \\ \left\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \right\rangle + \varphi(v) - \varphi(x(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0, \\ x(t_0) = x_0. \end{cases} \quad (33)$$

Here $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. The corresponding norm is denoted by $\|\cdot\|$. It follows from standard convex analysis that (33) can be rewritten equivalently as the differential inclusion

$$\frac{dx}{dt}(t) + F(x(t)) \in -\partial\varphi(x(t)). \quad (34)$$

Remark 12.1 *Note that if $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $k > 0$ then F is continuous and $F + kI$ is monotone.*

In this section, we will assume that the assumptions of Corollary 5.1 are always satisfied in order to have the existence and the uniqueness of a solution $x(t; t_0, x_0)$ for problem (33). We prove below that for $t \geq t_0$ fixed, the application $x(t; t_0, \cdot)$ is uniformly continuous on $D(\partial\varphi)$. This property will be used later in Section 3. Let us first recall some Gronwall inequality that is used in our next result (see e.g. Lemma 4.1 in [32]).

Lemma 12.1 *Let $T > 0$ be given and let $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ with $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. Let the absolutely continuous function $w : [t_0, t_0 + T] \rightarrow \mathbb{R}_+$ satisfy*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \text{ a.e. } t \in [t_0, t_0 + T],$$

where $0 \leq \alpha < 1$. Then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau)d\tau} b(s)ds, \forall t \in [t_0, t_0 + T].$$

■

Theorem 12.1 *Let $\tau \geq t_0$ be fixed. The application*

$$x(\tau; t_0, \cdot) : x_0 \mapsto x(\tau; t_0, x_0)$$

is uniformly continuous on $D(\partial\varphi)$.

Proof: Let $\tau \geq t_0$ be fixed. Let $\varepsilon > 0$ be given and set

$$\delta := \frac{\varepsilon}{\sqrt{e^{2\omega(\tau-t_0)}}}.$$

We claim that if $x_0, x_0^* \in D(\partial\varphi)$, $\|x_0 - x_0^*\| \leq \delta$ then $\|x(\tau; t_0, x_0) - x(\tau; t_0, x_0^*)\| \leq \varepsilon$. Indeed, let us set $x(t) := x(t; t_0, x_0)$ and $x^*(t) := x(t; t_0, x_0^*)$. We know that

$$\left\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \right\rangle + \varphi(v) - \varphi(x(t)) \geq 0, \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0 \quad (35)$$

and

$$\left\langle \frac{dx^*}{dt}(t) + F(x^*(t)), z - x^*(t) \right\rangle + \varphi(z) - \varphi(x^*(t)) \geq 0, \forall z \in \mathbb{R}^n, \text{ a.e. } t \geq t_0. \quad (36)$$

Setting $v = x^*(t)$ in (35) and $z = x(t)$ in (36), we obtain the relations:

$$-\left\langle \frac{dx}{dt}(t) + F(x(t)), x^*(t) - x(t) \right\rangle + \varphi(x(t)) - \varphi(x^*(t)) \leq 0, \text{ a.e. } t \geq t_0 \quad (37)$$

and

$$\left\langle \frac{dx^*}{dt}(t) + F(x^*(t)), x^*(t) - x(t) \right\rangle + \varphi(x^*(t)) - \varphi(x(t)) \leq 0, \text{ a.e. } t \geq t_0. \quad (38)$$

It results that

$$\begin{aligned} \left\langle \frac{d(x^* - x)}{dt}(t), x^*(t) - x(t) \right\rangle &\leq \langle \omega x^*(t) - \omega x(t), x^*(t) - x(t) \rangle \\ -\langle [F + \omega I](x^*(t)) - [F + \omega I](x(t)), x^*(t) - x(t) \rangle &\leq \omega \|x^*(t) - x(t)\|^2, \text{ a.e. } t \geq t_0. \end{aligned}$$

Recalling that $x \in C^0([t_0, +\infty); \mathbb{R}^n)$ and $\frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n)$, we may write

$$\frac{d}{dt} \|x^\star(t) - x(t)\|^2 \leq 2\omega \|x^\star(t) - x(t)\|^2, \text{ a.e. } t \geq t_0. \quad (39)$$

We may apply Lemma 12.1 with $T > \tau - t_0$, $\alpha = 0$, $b(\cdot) = 0$, $a(\cdot) = 2\omega$ and $w(\cdot) = \|x^\star(\cdot) - x(\cdot)\|^2$ to get

$$\|x^\star(t) - x(t)\|^2 \leq \|x_0^\star - x_0\|^2 e^{2\omega(t-t_0)}, \forall t \in [t_0, t_0 + T]. \quad (40)$$

It follows that

$$\|x^\star(\tau) - x(\tau)\| \leq \delta \sqrt{e^{2\omega(\tau-t_0)}} = \varepsilon.$$

■

Suppose now in addition that

$$0 \in D(\partial\varphi), \quad F(0) \in -\partial\varphi(0). \quad (41)$$

Then

$$x(t; t_0, 0) = 0, \forall t \geq t_0,$$

i.e. the trivial stationary solution 0 is the unique solution of problem $P(t_0, 0)$.

Let us here denote by $\mathcal{S}(F, \varphi)$ the set of stationary solutions of (33), that is

$$\mathcal{S}(F, \varphi) := \{z \in D(\partial\varphi) : \langle F(z), v - z \rangle + \varphi(v) - \varphi(z) \geq 0, \forall v \in \mathbb{R}^n\}.$$

Condition (41) ensures that $0 \in \mathcal{S}(F, \varphi)$.

Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ be given. We set

$$E(F, \varphi, V) := \{x \in D(\partial\varphi) : \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = 0\}. \quad (42)$$

For $x_0 \in D(\partial\varphi)$, we denote by $\gamma(x_0)$ the *orbit*

$$\gamma(x_0) := \{x(\tau; t_0, x_0); \tau \geq t_0\}$$

and by $\Lambda(x_0)$ the *limit set*

$$\Lambda(x_0) := \{z \in \mathbb{R}^n : \exists \{\tau_i\} \subset [t_0, +\infty); \tau_i \rightarrow +\infty \text{ and } x(\tau_i; t_0, x_0) \rightarrow z\}.$$

We say that a set $\mathcal{D} \subset D(\partial\varphi)$ is *invariant* provided that

$$x_0 \in \mathcal{D} \Rightarrow \gamma(x_0) \subset \mathcal{D}.$$

Here we denote by $d(s, \mathcal{M})$ the distance from a point $s \in \mathbb{R}^n$ to a set $\mathcal{M} \subset \mathbb{R}^n$, that is $d(s, \mathcal{M}) := \inf_{m \in \mathcal{M}} \|s - m\|$.

Remark 12.2 Let $x_0 \in D(\partial\varphi)$ be given.

i) It is clear that

$$\gamma(x_0) \subset D(\partial\varphi), \Lambda(x_0) \subset \overline{D(\partial\varphi)}.$$

ii) It is easy to check that

$$\Lambda(x_0) \subset \overline{\gamma(x_0)}.$$

iii) If $\gamma(x_0)$ is bounded then $\Lambda(x_0) \neq \emptyset$.

Indeed, if $\gamma(x_0)$ is bounded then we may find a sequence $x(\tau_i; t_0, x_0)$ ($\tau_i \geq t_0$) such that $x(\tau_i; t_0, x_0) \rightarrow z \in \mathbb{R}^n$. It results that $z \in \Lambda(x_0)$.

iv) If $\gamma(x_0)$ is bounded then

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \Lambda(x_0)) = 0.$$

Indeed, if we suppose the contrary then we can find $\varepsilon > 0$ and $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \rightarrow +\infty$ and $d(x(\tau_i; t_0, x_0), \Lambda(x_0)) \geq \varepsilon$. The sequence $x(\tau_i; t_0, x_0)$ is bounded and along a subsequence, we may suppose that $x(\tau_i; t_0, x_0) \rightarrow x^*$. Thus $x^* \in \Lambda(x_0)$. On the other hand we get the contradiction $d(x^*, \Lambda(x_0)) \geq \varepsilon$.

v) The set of stationary solutions $\mathcal{S}(F, \varphi)$ is invariant. Indeed, if $x_0 \in \mathcal{S}(F, \varphi)$ then $x(\tau; t_0, x_0) = x_0, \forall \tau \geq t_0$, and thus $\gamma(x_0) = \{x_0\} \subset \mathcal{S}(F, \varphi)$.

Thanks to Theorem 12.1, we can prove that the set $\Lambda(x_0) \cap D(\partial\varphi)$ is invariant by using standard topological arguments (see e.g. [30]).

Theorem 12.2 Let $x_0 \in D(\partial\varphi)$ be given. The set $\Lambda(x_0) \cap D(\partial\varphi)$ is invariant.

Proof: Let $z \in \Lambda(x_0) \cap D(\partial\varphi)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \rightarrow +\infty$ and $x(\tau_i; t_0, x_0) \rightarrow z$. Let $\tau \geq t_0$ be given. Using Theorem 12.1, we obtain $x(\tau; t_0, z) = \lim_{i \rightarrow \infty} x(\tau; t_0, x(\tau_i; t_0, x_0))$. Then remarking that $x(\tau; t_0, x(\tau_i; t_0, x_0)) = x(\tau - t_0 + \tau_i; t_0, x_0)$, we get $x(\tau; t_0, z) = \lim_{i \rightarrow \infty} x(\tau - t_0 + \tau_i; t_0, x_0)$. Thus setting $w_i := \tau - t_0 + \tau_i$, we see that $w_i \geq t_0$, $w_i \rightarrow +\infty$ and $x(w_i; t_0, x_0) \rightarrow x(\tau; t_0, z)$. It results that $x(\tau; t_0, z) \in \Lambda(x_0) \cap D(\partial\varphi)$. ■

Our goal is now to prove an extension of the LaSalle Invariance Principle applicable to the first order evolution variational inequality given in (33).

Lemma 12.2 Let Ψ be a compact subset of $\mathbb{R}^n$. We assume that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that

$$(1) \quad \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad x \in D(\partial\varphi) \cap \Psi.$$

Let $x_0 \in D(\partial\varphi)$ be given. If $\gamma(x_0) \subset \Psi$ then there exists a constant $k \in \mathbb{R}$ such that

$$V(x) = k, \forall x \in \Lambda(x_0).$$

Proof: Let $T > 0$ be given. We define the mapping $V^* : [t_0; +\infty) \rightarrow \mathbb{R}$ by the formula

$$V^*(t) := V(x(t; t_0, x_0)).$$

The function $x(\cdot) \equiv x(\cdot; t_0, x_0)$ is absolutely continuous on $[t_0, t_0 + T]$ and thus V^* is a.e. strongly differentiable on $[t_0, t_0 + T]$. We have

$$\frac{dV^*}{dt}(t) = \langle V'(x(t)), \frac{dx}{dt}(t) \rangle, \text{ a.e. } t \in [t_0, t_0 + T].$$

We know (by hypothesis) that

$$x(t) \in D(\partial\varphi) \cap \Psi, \forall t \geq t_0,$$

and

$$\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle + \varphi(v) - \varphi(x(t)) \geq 0, \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0. \quad (43)$$

Setting $v = x(t) - V'(x(t))$ in (43), we obtain

$$\langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq -\varphi(x(t)) + \varphi(x(t) - V'(x(t))) - \langle F(x(t)), V'(x(t)) \rangle, \text{ a.e. } t \geq t_0.$$

and thus using assumption 1, we obtain

$$\langle \frac{dx}{dt}(t), V'(x(t)) \rangle \leq 0, \text{ a.e. } t \geq t_0. \quad (44)$$

Thus

$$\frac{dV^*}{dt}(t) \leq 0, \text{ a.e. } t \in [t_0, t_0 + T].$$

We know that $x \in C^0([t_0, t_0 + T]; \mathbb{R}^n)$, $\frac{dx}{dt} \in L^\infty(t_0, t_0 + T; \mathbb{R}^n)$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$. It follows that $V^* \in W^{1,1}(t_0, t_0 + T; \mathbb{R})$ and applying e.g. Lemma 3.1 in [16], we obtain that V^* is decreasing on $[t_0, t_0 + T]$. The real T has been chosen arbitrary and thus V^* is decreasing on $[t_0, +\infty)$. Moreover V^* is bounded from below on $[t_0, +\infty)$ since $\gamma(x_0) \subset \Psi$ and V is continuous on the compact set Ψ . It results that

$$\lim_{\tau \rightarrow +\infty} V(x(\tau; t_0, x_0)) = k,$$

for some $k \in \mathbb{R}$.

let $y \in \Lambda(x_0)$ be given. There exists $\{\tau_i\} \subset [t_0, +\infty)$ such that $\tau_i \rightarrow +\infty$ and $x(\tau_i; t_0, x_0) \rightarrow y$. by continuity

$$\lim_{i \rightarrow +\infty} V(x(\tau_i; t_0, x_0)) = V(y).$$

Therefore $V(y) = k$. Here y has been chosen arbitrary in $\Lambda(x_0)$ and thus

$$V(y) = k, \forall y \in \Lambda(x_0).$$

■

Lemma 12.3 *We assume that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

$$(1) \quad \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad x \in D(\partial\varphi).$$

Let $a \in \mathbb{R}$ be given and set

$$\bar{\Psi} := \{x \in \mathbb{R}^n : V(x) \leq a\}.$$

The set $D(\partial\varphi) \cap \bar{\Psi}$ is invariant

Proof: Let $x_0 \in D(\partial\varphi) \cap \bar{\Psi}$ be given. Then $x_0 \in D(\partial\varphi)$ and $V(x_0) \leq a$. If $\tau \geq t_0$ then $x(\tau; t_0, x_0) \in D(\partial\varphi)$ and as in the proof of Lemma 12.2, we check that $V(x(\cdot; t_0, x_0))$ is decreasing on $[t_0, +\infty)$. Thus

$$V(x(\tau; t_0, x_0)) \leq V(x(t_0; t_0, x_0)) = V(x_0) \leq a.$$

It results that

$$\gamma(x_0) \subset D(\partial\varphi) \cap \bar{\Psi}.$$

■

Theorem 12.3 (Invariance Theorem) *Let $\Psi \subset \mathbb{R}^n$ be a compact set and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ a function such that*

- (1) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ is lower semicontinuous on $D(\partial\varphi) \cap \Psi$;
- (2) $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad x \in D(\partial\varphi) \cap \Psi$;
- (3) $D(\partial\varphi)$ is closed.

We set

$$E_\Psi(F, \varphi, V) := E(F, \varphi, V) \cap \Psi$$

and we denote by $\mathcal{M}$ the largest invariant subset of $E_\Psi(F, \varphi, V)$. Then for each $x_0 \in D(\partial\varphi)$ such that $\gamma(x_0) \subset \Psi$, we have

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

Proof: Here $\gamma(x_0)$ is bounded and thus (see Remark 12.2 iii) and iv)) $\Lambda(x_0)$ is nonempty and

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \Lambda(x_0)) = 0.$$

Let us now check that $\Lambda(x_0) \subset E_\Psi(F, \varphi, V)$. We first note that

$$\Lambda(x_0) \subset \overline{\gamma(x_0)} \subset \overline{D(\partial\varphi) \cap \bar{\Psi}} = D(\partial\varphi) \cap \Psi.$$

From Lemma 12.2, there exists $k \in \mathbb{R}$ such that $V(x) = k, \forall x \in \Lambda(x_0)$. Let $z \in \Lambda(x_0)$ be given. Using Theorem 12.2, we see that $x(t; t_0, z) \in \Lambda(x_0), \forall t \geq t_0$ and thus

$$V(x(t; t_0, z)) = k, \forall t \geq t_0.$$

It results that

$$\frac{d}{dt}V(x(t; t_0, z)) = 0, \text{ a.e. } t \geq t_0. \quad (45)$$

Setting $x(\cdot) \equiv x(\cdot; t_0, z)$, we check as in the proof of Lemma 12.2 that

$$\langle V'(x(t)), \frac{dx}{dt}(t) \rangle \leq -\langle F(x(t)), V'(x(t)) \rangle \quad (46)$$

$$-\varphi(x(t)) + \varphi(x(t) - V'(x(t))), \text{ a.e. } t \geq t_0. \quad (47)$$

From (45) and (47) we deduce that

$$\langle F(x(t)), V'(x(t)) \rangle + \varphi(x(t)) - \varphi(x(t) - V'(x(t))) \leq 0, \text{ a.e. } t \geq t_0.$$

Using assumption 1, we see that the mapping

$$t \mapsto \langle F(x(t; t_0, z)), V'(x(t; t_0, z)) \rangle + \varphi(x(t; t_0, z)) - \varphi(x(t; t_0, z) - V'(x(t; t_0, z)))$$

is lower semicontinuous on $[t_0, +\infty)$ and thus taking the $\liminf$ as $t \rightarrow t_0$, we obtain

$$\langle F(z), V'(z) \rangle + \varphi(z) - \varphi(z - V'(z)) \leq 0.$$

This together with assumption 2 ensure that $z \in E_\Psi(F, \varphi, V)$. Finally $\Lambda(x_0) \subset \mathcal{M}$ since $\Lambda(x_0) \subset E_\Psi(F, \varphi, V)$ and $\Lambda(x_0)$ is invariant (see Theorem 12.2). The conclusion follows. ■

Remark 12.3 *Note that the conditions of Theorem 12.3 ensure that*

$$\mathcal{S}(F, \varphi) \cap \Psi \subset \mathcal{M}.$$

We have, $\mathcal{S}(F, \varphi) \cap \Psi \subset E_\Psi(F, \varphi, V)$ and $\mathcal{S}(F, \varphi) \cap \Psi$ is invariant.

Corollary 12.1 *Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ be a function such that*

- (1) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ is lower semicontinuous on $D(\partial\varphi)$;
- (2) $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, x \in D(\partial\varphi)$;
- (3) $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty, x \in D(\partial\varphi)$;
- (4) $D(\partial\varphi)$ is closed.

Let $\mathcal{M}$ be the largest invariant subset of $E(F, \varphi, V)$. Then for each $x_0 \in D(\partial\varphi)$, the orbit $\gamma(x_0)$ is bounded and

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

Proof: Let $x_0 \in D(\partial\varphi)$ be given. We set $\bar{\Psi} := \{x \in \mathbb{R}^n : V(x) \leq V(x_0)\}$ and $\Psi = \bar{\Psi} \cap D(\partial\varphi)$. The set $\bar{\Psi}$ is closed. Assumption 3 and 4 ensure that $D(\partial\varphi) \cap \bar{\Psi}$ is bounded and closed. Thus Ψ is compact. Lemma 12.3 ensures that Ψ is invariant. Here $x_0 \in \Psi$ and thus $\gamma(x_0) \subset \Psi$. It results that $\gamma(x_0)$ is bounded. Moreover, from Theorem 12.3, we obtain

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \mathcal{M}_*) = 0,$$

where $\mathcal{M}_*$ is the largest invariant subset of $E_\Psi(F, \varphi, V)$. It is clear that $\mathcal{M}_* \subset \mathcal{M}$ and the conclusion follows. $\blacksquare$

Corollary 12.2 *Suppose that condition (41) holds. Suppose that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1)

$$V(x) \geq a(\|x\|), \quad x \in D(\partial\varphi),$$

with $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying $a(0) = 0$, a strictly increasing on $\mathbb{R}_+$;

(2) $V(0) = 0$;

(3) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ is lower semicontinuous on $D(\partial\varphi)$;

(4) $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0$, $x \in D(\partial\varphi)$;

(5) $D(\partial\varphi)$ is closed;

(6) $E(F, \varphi, V) = \{0\}$.

Then the trivial solution of (33) is (a) the unique stationary solution of (33), (b) asymptotically stable, (c) globally attractive, i.e. for each $x_0 \in D(\partial\varphi)$, $\lim_{t \rightarrow +\infty} \|x(t; t_0, x_0)\| = 0$.

Proof: Assertion (a) and the stability are let to the reader.

Moreover, we may apply Corollary 12.1 with $\mathcal{M} = \{0\}$ (since $E(F, \varphi, V) = \{0\}$) to obtain that for any $x_0 \in D(\partial\varphi)$ the limit

$$\lim_{\tau \rightarrow +\infty} x(\tau; t_0, x_0) = 0$$

holds. Assertions (b) and (c) follow. $\blacksquare$

Corollary 12.3 *Suppose that condition (41) hold. Suppose that there exists $\sigma > 0$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1)

$$V(x) \geq a(\|x\|), \quad x \in D(\partial\varphi) \cap B_\sigma,$$

with $a : [0, \sigma] \rightarrow \mathbb{R}$ satisfying $a(t) > 0, \forall t \in (0, \sigma)$;

(2) $V(0) = 0$;

(3) $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$ is lower semicontinuous on $D(\partial\varphi) \cap B_\sigma$;

(4) $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad x \in D(\partial\varphi) \cap B_\sigma$;

(5) $D(\partial\varphi)$ is closed;

(6) $E(F, \varphi, V) \cap B_\sigma = \{0\}$.

Then the trivial solution of (33) is (a) isolated in $\mathcal{S}(F, \varphi)$, (b) asymptotically stable.

Proof: Assertion (a) and the stability are let to the reader. The stability ensures the existence of $\delta > 0$ such that if $x_0 \in D(\partial\varphi) \cap B_\delta$ then

$$\gamma(x_0) \subset B_\sigma.$$

Applying Theorem 12.3 with $\Psi = B_\sigma$, we obtain for $x_0 \in D(\partial\varphi) \cap B_\delta$ that

$$\lim_{t \rightarrow +\infty} d(x(t; t_0, x_0), \mathcal{M}) = 0,$$

where $\mathcal{M}$ is the largest invariant subset of $E_\Psi(F, \varphi, V)$. It is clear that assumption 6 yields $\mathcal{M} = \{0\}$. The attractivity and assertion (b) follow. ■

Corollary 12.4 Suppose that condition (41) hold. Assume that $D(\partial\varphi)$ is closed and suppose that there exists $\sigma > 0$ such that

$$\langle F(x), x \rangle + \varphi(x) - \varphi(0) > 0, \quad x \in D(\partial\varphi) \cap B_\sigma, \quad x \neq 0.$$

Then the trivial stationary solution of (33) is (a) isolated in $\mathcal{S}(F, \varphi)$ and (b) asymptotically stable.

Proof: This follows from Corollary 12.3 that we may apply with $V \in C^1(\mathbb{R}^n; \mathbb{R})$ defined by $V(x) = \frac{1}{2}\|x\|^2, x \in \mathbb{R}^n$. ■

13 Second Order Dynamic Systems with friction

In this Section, we deal with the following class of second order dynamic systems:

Let $\Phi : \mathbb{R}^l \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. Let $M, C, K \in \mathbb{R}^{m \times m}$, $H_1 \in \mathbb{R}^{m \times l}$ and $H_2 \in \mathbb{R}^{l \times m}$ be given matrices. For $(t_0, q_0, \dot{q}_0) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ with $H_2 \dot{q}_0 \in D(\partial\Phi)$, we consider the problem $P(t_0, q_0, \dot{q}_0)$: Find a function $t \mapsto q(t)$ ($t \geq t_0$) with $q \in C^1([t_0, +\infty); \mathbb{R}^m)$, and such that:

$$\frac{d^2 q}{dt^2} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^m), \quad (48)$$

$$\frac{dq}{dt} \text{ is right-differentiable on } [t_0, +\infty), \quad (49)$$

$$q(t_0) = q_0, \quad (50)$$

$$\frac{dq}{dt}(t_0) = \dot{q}_0, \quad (51)$$

$$H_2 \frac{dq}{dt}(t) \in D(\partial\Phi), \quad t \geq t_0, \quad (52)$$

$$M \frac{d^2 q}{dt^2}(t) + C \frac{dq}{dt}(t) + Kq(t) \in -H_1 \partial\Phi(H_2 \frac{dq}{dt}(t)), \quad \text{a.e. } t \geq t_0. \quad (53)$$

The model in (86) can be used in Mechanics to describe the motion of various systems having frictional contact. For such problems, m is the number of degrees of freedom of the system, M is the mass matrix of the system, C is the viscous damping matrix of the system and K is the stiffness matrix. The term $H_1 \partial\Phi(H_2 \cdot)$ is used to model the unilaterality of the contact induced by friction forces.

The euclidean scalar product in $\mathbb{R}^m$ is denoted by $\langle \cdot, \cdot \rangle_m$ and the corresponding norm by $\|\cdot\|_m$. The subordinate matrix norm is also denoted by $\|\cdot\|_m$. In this Section, we also use the notations I_m and $0_{p \times q}$ to denote the $m \times m$ identity matrix and the $p \times q$ null matrix respectively.

Theorem 13.1 (*Existence and Uniqueness*) Suppose that the following assumptions are satisfied:

- (1) M is nonsingular;
- (2) there exists a matrix $R \in \mathbb{R}^{m \times m}$, symmetric and nonsingular such that :

$$R^{-2} H_2^T = M^{-1} H_1;$$

- (3) there exists $y_0 = H_2 R^{-1} x_0$, ($x_0 \in \mathbb{R}^m$) at which Φ is finite and continuous.

Let $t_0 \in \mathbb{R}$, $q_0, \dot{q}_0 \in \mathbb{R}^m$ with $H_2 \dot{q}_0 \in D(\partial\Phi)$. Then there exists a unique $q \in C^1([t_0, +\infty); \mathbb{R}^m)$ satisfying conditions (84)-(86).

Proof: Let us here for a function f use the notations $\ddot{f} = \frac{d^2 f}{dt^2}$ and $\dot{f} = \frac{df}{dt}$. We first remark that (86), i.e.

$$M\ddot{q} + C\dot{q} + Kq \in -H_1\partial\Phi(H_2\dot{q})$$

is equivalent to

$$\ddot{q} + M^{-1}C\dot{q} + M^{-1}Kq \in -M^{-1}H_1\partial\Phi(H_2\dot{q}).$$

Hence,

$$R\ddot{q} + RM^{-1}CR^{-1}R\dot{q} + RM^{-1}KR^{-1}Rq \in -RM^{-1}H_1\partial\Phi(H_2R^{-1}R\dot{q}). \quad (54)$$

Setting $z = Rq$ in (54), we get

$$\ddot{z} + RM^{-1}CR^{-1}\dot{z} + RM^{-1}KR^{-1}z \in -RM^{-1}H_1\partial\Phi(H_2R^{-1}\dot{z}).$$

Using now assumption 2, we obtain

$$\ddot{z} + RM^{-1}CR^{-1}\dot{z} + RM^{-1}KR^{-1}z \in -R^{-1}H_2^T\partial\Phi(H_2R^{-1}\dot{z}). \quad (55)$$

Let us here define the function $\chi : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ by the formula

$$\chi(w) = (\Phi \circ H_2R^{-1})(w), \forall w \in \mathbb{R}^m.$$

It is clear that χ is convex and lower semicontinuous. Moreover, thanks to assumption 3, χ is proper and we have (see e.g. Proposition ??)

$$\partial\chi(w) = R^{-1}H_2^T\partial\Phi(H_2R^{-1}w), \forall w \in \mathbb{R}^m.$$

Thus (55) reduces to

$$\ddot{z} + RM^{-1}CR^{-1}\dot{z} + RM^{-1}KR^{-1}z \in -\partial\chi(\dot{z}). \quad (56)$$

We note also that (50), (51) and (52) can be written here respectively in term of the variable z as $z(t_0) = Rq_0$, $\dot{z}(t_0) = R\dot{q}_0$ and $\dot{z}(t) \in D(\partial\chi), \forall t \geq t_0$. Moreover, $R\dot{q}_0 \in D(\partial\chi)$ since $H_2\dot{q}_0 \in D(\partial\Phi)$.

Let us now set

$$x_1 := z, \quad x_2 := \dot{z}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (57)$$

It is clear that (56) is equivalent to the following first order system :

$$\begin{cases} \dot{x}_1 - x_2 = 0 \\ \dot{x}_2 + RM^{-1}CR^{-1}x_2 + RM^{-1}KR^{-1}x_1 \in -\partial\chi(x_2). \end{cases}$$

It results that problem $P(t_0, q_0, \dot{q}_0)$ can be written as follows:

$$\begin{cases} \dot{x} + Ax \in -\partial\varphi(x) \\ x(t_0) = x_0 \end{cases}$$

where the matrix $A \in \mathbb{R}^{n \times n}$ ($n = 2m$) is defined by

$$A = \begin{pmatrix} 0_{m \times m} & -I_m \\ RM^{-1}KR^{-1} & RM^{-1}CR^{-1} \end{pmatrix}, \quad (58)$$

the vector $x_0 \in \mathbb{R}^n$ is given by

$$x_0 = \begin{pmatrix} Rq_0 \\ R\dot{q}_0 \end{pmatrix}, \quad (59)$$

and the proper, convex and lower semicontinuous function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(x) := \chi(x_2). \quad (60)$$

The result is thus a direct consequence of Corollary 5.1 (with $F(\cdot) = A(\cdot)$). Indeed, A is Lipschitz continuous (see Remark 12.1). ■

Suppose that the assumptions of Theorem 13.1 hold and let us now denote by $q(\cdot; t_0, q_0, \dot{q}_0)$ the unique solution of Problem $P(t_0, q_0, \dot{q}_0)$.

The set $\mathcal{W}$ of stationary solutions of (52)-(86) is given by

$$\mathcal{W} = \{\bar{q} \in \mathbb{R}^m : K\bar{q} \in -H_1\partial\Phi(0)\}.$$

We suppose that

$$0 \in D(\partial\Phi). \quad (61)$$

Remark 13.1 i) If $0 \in \partial\Phi(0)$ then it is clear that $0 \in \mathcal{W}$.

ii) If $0 \in D(\partial\Phi)$ and K is nonsingular then $\mathcal{W} = -K^{-1}H_1\partial\Phi(0)$.

iii) If $\partial\Phi(0) = \{0\}$ then $\mathcal{W} = \ker K$.

iv) If $\Phi'(0)$ exists and K is nonsingular then the trivial stationary solution of (52)-(86) is the unique stationary solution of (52)-(86). Indeed, here we have $\mathcal{W} = \{-K^{-1}H_1\Phi'(0)\}$.

We consider the stability of a stationary solution with respect to the "generalized coordinates" $q_1, \dots, q_m$ and the "generalized velocities" $\frac{dq_1}{dt}, \dots, \frac{dq_m}{dt}$. More precisely, we say that a stationary solution $\bar{q} \in \mathcal{W}$ is *stable* provided that for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $q_0 \in \mathbb{R}^m, \dot{q}_0 \in \mathbb{R}^m, H_2\dot{q}_0 \in D(\partial\Phi)$ with $\sqrt{\|q_0 - \bar{q}\|_m^2 + \|\dot{q}_0\|_m^2} \leq \eta$ the solution $q(\cdot; t_0, q_0, \dot{q}_0)$ of Problem $P(t_0, q_0, \dot{q}_0)$ satisfies $\sqrt{\|q(t; t_0, q_0, \dot{q}_0) - \bar{q}\|_m^2 + \|\frac{dq}{dt}(t; t_0, q_0, \dot{q}_0)\|_m^2} \leq \varepsilon, \forall t \geq t_0$.

If there exists a $\delta > 0$ such that for any $q_0 \in \mathbb{R}^m, \dot{q}_0 \in \mathbb{R}^m, H_2\dot{q}_0 \in D(\partial\Phi)$ with $\sqrt{\|q_0 - \bar{q}\|_m^2 + \|\dot{q}_0\|_m^2} \leq \delta$ the solution $q(\cdot; t_0, q_0, \dot{q}_0)$ of Problem $P(t_0, q_0, \dot{q}_0)$ satisfies the limits

$$\lim_{t \rightarrow +\infty} \|q(t; t_0, q_0, \dot{q}_0) - \bar{q}\|_m = 0 \quad (62)$$

and

$$\lim_{t \rightarrow +\infty} \left\| \frac{dq}{dt}(t; t_0, q_0, \dot{q}_0) \right\|_m = 0, \quad (63)$$

then we say that the stationary solution $\bar{q}$ is *attractive*. If the limits in (62) and (63) hold for any $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ then we say that the stationary solution $\bar{q}$ is *globally attractive*. Finally, a stable and attractive stationary solution is said *asymptotically stable*.

Theorem 13.2 (Stability) *Suppose that the assumptions of Theorem 13.1 together with condition (61) hold. Suppose in addition that*

- (1) $RM^{-1}CR^{-1}$ is positive semidefinite;
- (2) $RM^{-1}KR^{-1}$ is symmetric and positive definite.

Then $\mathcal{W} \neq \emptyset$ and any stationary solution $\bar{q} \in \mathcal{W}$ of (52)-(86) is stable.

Proof: Condition (61) ensures that $\partial\Phi(0) \neq \emptyset$ and assumption 2 entails that K is nonsingular. Thus $\mathcal{W} = -K^{-1}H_1\partial\Phi(0) \neq \emptyset$.

Let $\bar{q} \in \mathcal{W}$ be given. Setting $Q := q - \bar{q}$, we see that the question of stability of $\bar{q}$ reduces to the one of the trivial stationary solution of the system:

$$M\ddot{Q} + C\dot{Q} + KQ + K\bar{q} \in -H_1\partial\Phi(H_2\dot{Q}). \quad (64)$$

Setting $x_1 := RQ$, $x_2 := R\dot{Q}$ and $x := (x_1 \ x_2)^T$, we check as in the proof of Theorem 13.1 that the system in (64) can be written as follows:

$$\dot{x} + F(x) \in -\partial\varphi(x)$$

where

$$\begin{aligned} F(x) &= Ax + \bar{F}, \\ A &= \begin{pmatrix} 0_{m \times m} & -I_m \\ RM^{-1}KR^{-1} & RM^{-1}CR^{-1} \end{pmatrix}, \\ \bar{F} &= \begin{pmatrix} 0_{m \times 1} \\ RM^{-1}K\bar{q} \end{pmatrix}, \\ \varphi(x) &= \chi(x_2) \ (:= \Phi \circ H_2R^{-1}(x_2)), \end{aligned}$$

and

$$\partial\varphi(x) = \begin{pmatrix} 0_{m \times 1} \\ R^{-1}H_2^T\partial\Phi(H_2R^{-1}x_2) \end{pmatrix}.$$

The mapping $F(\cdot)$ is Lipschitz continuous. Moreover, condition (41) holds since $\bar{q} \in \mathcal{W} \Leftrightarrow K\bar{q} \in -H_1\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -RM^{-1}H_1\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -R^{-1}H_2^T\partial\Phi(0) \Leftrightarrow RM^{-1}K\bar{q} \in -\partial\chi(0) \Leftrightarrow \bar{F} \in -\partial\varphi(0)$.

Let $V \in C^1(\mathbb{R}^n; \mathbb{R})$ ($n = 2m$) be given by

$$V(x) = \frac{1}{2} \langle RM^{-1}KR^{-1}x_1, x_1 \rangle_m + \frac{1}{2} \|x_2\|_m^2.$$

We have

$$V'(x) = \begin{pmatrix} RM^{-1}KR^{-1}x_1 \\ x_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} \langle Ax, V'(x) \rangle + \langle \bar{F}, V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) &= \langle RM^{-1}CR^{-1}x_2, x_2 \rangle_m \\ &+ \langle RM^{-1}K\bar{q}, x_2 \rangle_m + \chi(x_2) - \chi(0). \end{aligned}$$

Assumption 1 yields

$$\langle RM^{-1}CR^{-1}x_2, x_2 \rangle_m \geq 0. \quad (65)$$

Moreover $\bar{q} \in \mathcal{W}$ and thus $RM^{-1}K\bar{q} \in -\partial\chi(0)$. It results that

$$\langle RM^{-1}K\bar{q}, x_2 \rangle + \chi(x_2) - \chi(0) \geq 0. \quad (66)$$

The conclusion is thus a consequence of Theorem 3 in [2]. $\blacksquare$

It is easy to see from the proof of Theorem 13.2 that the following variant can also be stated.

Theorem 13.3 *Suppose that the assumptions of Theorem 13.1 together with condition (61) hold. Let $\bar{q} \in \mathcal{W}$ be a stationary solution of (52)-(86). Suppose that*

- (1) $\langle RM^{-1}CR^{-1}z + RM^{-1}K\bar{q}, z \rangle_m + \Phi(H_2R^{-1}z) - \Phi(0) \geq 0, z \in \mathbb{R}^m;$
- (2) $RM^{-1}KR^{-1}$ is symmetric and positive definite.

Then $\bar{q}$ is stable.

It follows from Remark 13.1 that an equilibrium point $\bar{q}$ is in general not isolated in $\mathcal{W}$. The concept of attractivity is for such case not really appropriated. It is then worthwhile to verify if the trajectories of the perturbed solutions are attracted by $\mathcal{W}$.

Theorem 13.4 (Attractivity of $\mathcal{W}$) *Suppose that the assumptions of Theorem 13.1 together with condition (61) hold. Suppose also that*

- (1) $RM^{-1}KR^{-1}$ is symmetric and positive definite;
- (2) $\langle RM^{-1}CR^{-1}z, z \rangle_m + \Phi(H_2R^{-1}z) - \Phi(0) > 0, z \in \mathbb{R}^m \setminus \{0\};$
- (3) $D(\partial\Phi)$ is closed.

Then (a) for any $q_0 \in \mathbb{R}^m, \dot{q}_0 \in \mathbb{R}^m, H_2\dot{q}_0 \in D(\partial\Phi)$, the orbit

$$\Omega(q_0, \dot{q}_0) := \{(q(\tau; t_0, q_0, \dot{q}_0) \frac{dq}{dt}(\tau; t_0, q_0, \dot{q}_0))^T; \tau \geq t_0\}$$

is bounded and (b) the following asymptotic properties hold:

$$\lim_{\tau \rightarrow +\infty} d(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}) = 0 \text{ and } \lim_{\tau \rightarrow +\infty} \frac{dq}{dt}(\tau; t_0, q_0, \dot{q}_0) = 0.$$

Proof: From the proof of Theorem 13.1, we know that the study of our problem reduces to the one of the first order system

$$\dot{x} + Ax \in -\partial\varphi(x),$$

where A is defined in (58) and φ is given by (60).

Let us first check that all assumptions of Corollary 12.1 are satisfied with $V \in C^1(\mathbb{R}^n; \mathbb{R})$ ($n = 2m$) defined as in the proof of Theorem 13.2, i.e.

$$V(x) = \frac{1}{2} \langle RM^{-1}KR^{-1}x_1, x_1 \rangle_m + \frac{1}{2} \|x_2\|_m^2.$$

We have $\varphi(x) - \varphi(x - V'(x)) = \chi(x_2) - \chi(0)$ and the application $x \mapsto \varphi(x) - \varphi(x - V'(x))$ is thus lower semicontinuous. It results that hypothesis 1 of Corollary 12.1 is satisfied.

We have

$$\langle Ax, V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = \langle RM^{-1}CR^{-1}x_2, x_2 \rangle_m + \chi(x_2) - \chi(0).$$

Assumption 2 ensure that hypothesis 2 of Corollary 12.1 holds.

It is clear that hypothesis 3 of Corollary 12.1 is satisfied. Finally hypothesis 4 of Corollary 12.1 follows from assumption 3 which ensures that $D(\partial\varphi) = \mathbb{R}^m \times D(\partial(\Phi \circ H_2R^{-1}))$ is closed.

Here, we have

$$E(A, \varphi, V) = \{x \in D(\partial\varphi) : \langle RM^{-1}CR^{-1}x_2, x_2 \rangle_m + \chi(x_2) - \chi(0) = 0\}.$$

Using assumption 2, we get

$$E(A, \varphi, V) = \{(x_1, 0); x_1 \in \mathbb{R}^m\}. \quad (67)$$

Corollary 12.1 ensures that for any $x_0 \in D(\partial\varphi)$, the orbit $\gamma(x_0)$ is bounded. If $q_0 \in \mathbb{R}^m$, $\dot{q}_0 \in \mathbb{R}^m$, $H_2\dot{q}_0 \in D(\partial\Phi)$ then $R\dot{q}_0 \in D(\partial\chi)$. It results that the conclusion of Corollary 12.1 with $x_0 = (Rq_0 \ R\dot{q}_0)^T$ means that the set $\Omega(q_0, \dot{q}_0)$ is bounded. This gives part (a) of our result.

Corollary 12.1 ensures also that

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0,$$

where $\mathcal{M}$ is the largest invariant subset of $E(A, \varphi, V)$. We have $S(A, \varphi) \subset E(A, \varphi, V)$. From Remark 12.2 (v), we know also that $S(A, \varphi)$ is invariant. Thus $S(A, \varphi)$ is an invariant subset of $E(A, \varphi, V)$. We prove now that $S(A, \varphi)$ is the largest invariant subset of $E(A, \varphi, V)$.

Since, $S(A, \varphi) \subset E(A, \varphi, V)$, by (67) we have

$$S(A, \varphi) = \{(x_1, 0) : \langle RM^{-1}KR^{-1}x_1, h \rangle_m + \chi(h) - \chi(0) \geq 0, \forall h \in \mathbb{R}^m\}.$$

Let us set

$$\mathcal{N} := \{x_1 \in \mathbb{R}^m : RM^{-1}KR^{-1}x_1 \in -\partial\chi(0)\}.$$

Then, we may write

$$S(A, \varphi) = \mathcal{N} \times \{0\}.$$

Let $\mathcal{D}$ be any invariant subset of $E(A, \varphi, V)$ and let $z \in \mathcal{D}$ be given. The function $x(\cdot; t_0, z)$ satisfies

$$\left\langle \frac{dx_1}{dt}(t; t_0, z) - x_2(t; t_0, z), v_1 - x_1(t; t_0, z) \right\rangle_m \geq 0, \forall v_1 \in \mathbb{R}^m, \text{ a.e. } t \geq t_0, \quad (68)$$

and

$$\begin{aligned} & \left\langle \frac{dx_2}{dt}(t; t_0, z) + RM^{-1}KR^{-1}x_1(t; t_0, z) + RM^{-1}CR^{-1}x_2(t; t_0, z), v_2 - x_2(t; t_0, z) \right\rangle_m \\ & + \chi(v_2) - \chi(x_2(t; t_0, z)) \geq 0, \forall v_2 \in \mathbb{R}^m, \text{ a.e. } t \geq t_0. \end{aligned} \quad (69)$$

However, $\gamma(z) \subset \mathcal{D} \subset E(A, \varphi, V)$ and thus $x_2(t; t_0, z) = 0, \forall t \geq t_0$. Thus (68) reduces to $\frac{dx_1}{dt}(t; t_0, z) = 0, \text{ a.e. } t \geq t_0$ from which we deduce that $x_1(\cdot; t_0, z) = z_1, \forall t \geq t_0$. Then (69) yields

$$\langle RM^{-1}KR^{-1}z_1, v_2 \rangle_m + \chi(v_2) - \chi(0) \geq 0, \forall v_2 \in \mathbb{R}^m.$$

Thus

$$z = (z_1, z_2) \in \mathcal{N} \times \{0\}.$$

It results that $\mathcal{D} \subset S(A, \varphi)$ and $S(A, \varphi)$ is well the largest invariant subset of $E(A, \varphi, V)$.

Thus

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), S(A, \varphi)) = 0.$$

This implies that

$$\lim_{\tau \rightarrow +\infty} d(x_1(\tau; t_0, x_0), \mathcal{N}) = 0 \quad (70)$$

and

$$\lim_{\tau \rightarrow +\infty} x_2(\tau; t_0, x_0) = 0. \quad (71)$$

Recall that in terms of the vector $q = R^{-1}x_1$ of "generalized coordinates" and the vector $\dot{q} = R^{-1}x_2$ of "generalized velocities" we have $RM^{-1}KR^{-1}x_1 \in -\partial\chi(0) \Leftrightarrow Kq \in -MR^{-2}H_2^T\partial\Phi(0) = -H_1\partial\Phi(0)$. Thus the limit in (70) reads

$$\lim_{\tau \rightarrow +\infty} d(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}) = 0. \quad (72)$$

On the other hand, the limit in (71) gives

$$\lim_{\tau \rightarrow +\infty} \dot{q}(\tau; t_0, q_0, \dot{q}_0) = 0. \quad (73)$$

Part (b) of our result is thus proved. ■

Remark 13.2 i) Note that if $0 \in \partial\Phi(0)$ then assumption 2 in Theorem 13.4 is satisfied provided that either $RM^{-1}CR^{-1}$ is positive definite or $RM^{-1}CR^{-1}$ is positive semidefinite and $\{z \in \mathbb{R}^m : \Phi(H_2R^{-1}z) = \Phi(0)\} = \{0\}$.

ii) If $H_2^T = H_1$ and M is symmetric and positive definite then the matrix $R = M^{\frac{1}{2}}$ satisfies assumption 2 of Theorem 13.1. Then conditions 1 and 2 in Theorem 13.2 hold if and only if C is positive semidefinite and K is symmetric and positive definite. Indeed, here $\langle RM^{-1}CR^{-1}., . \rangle_m = \langle CM^{-\frac{1}{2}}, M^{-\frac{1}{2}}. \rangle_m$ and $\langle RM^{-1}KR^{-1}., . \rangle_m = \langle KM^{-\frac{1}{2}}, M^{-\frac{1}{2}}. \rangle_m$.

iii) The conditions discussed in Remark 13.2 ii) are usually satisfied as soon as concrete applications in Mechanics are considered.

iv) Assumption 1 in Theorem 13.4 implies that K is nonsingular. Hence $\mathcal{W} = -K^{-1}H_1\partial\Phi(0)$.

v) Suppose that the assumptions of Theorem 13.4 hold. Suppose in addition that $\partial\Phi(0) = \{0\}$. Then $\mathcal{W} = \{0\}$ and thus the trivial solution of (52)-(86) is (a) the unique stationary solution of (52)-(86), (b) stable and (c) globally attractive. In particular, the results in (b) and (c) ensure that the trivial solution of (52)-(86) is asymptotically stable

14 Examples in Unilateral Mechanics

Example 1. The model of Figure 11 consists of a mass $m > 0$ restrained by a spring with stiffness constant $k > 0$ and a damper with viscous damping coefficient $c > 0$. The motion of the mass has frictional contact. A Coulomb model is assumed for the friction force f , i.e.

$$f \in -\partial\Phi(\dot{q}),$$

with

$$\Phi(x) = \gamma|x|,$$

where $\gamma > 0$ denotes the coefficient of friction.

The motion of the system is described by the model:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) \in -\partial\Phi(\dot{u}(t)). \quad (74)$$

Here $q = (u)$, $M = (m)$, $K = (k)$, $C = (c)$, $H_1 = (1)$, $H_2 = (1)$, $D(\partial\Phi) = \mathbb{R}$ and $\partial\Phi(0) = [-\gamma, +\gamma]$. Setting $R = (\sqrt{m})$, we see that both assumptions of Theorem 13.1 hold. The set of stationary solutions is here given by

$$\mathcal{W} = \left[-\frac{\gamma}{k}, +\frac{\gamma}{k}\right].$$

This set defines a steady zone due to friction.

It is also easy to check that both assumptions of Theorems 13.2 and 13.4 are satisfied. It results that **each stationary solution $\bar{q} \in \mathcal{W}$ is stable**. Moreover,

$$\lim_{\tau \rightarrow +\infty} d(u(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}) = 0.$$

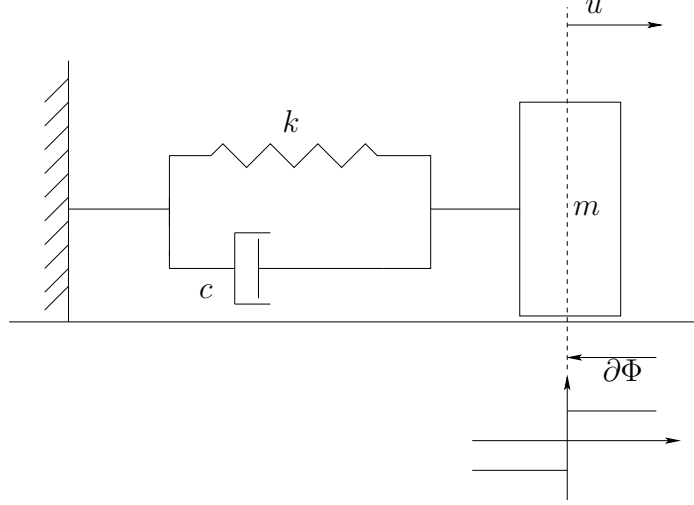


Figure 11: Example 1

and

$$\lim_{\tau \rightarrow +\infty} \dot{u}(\tau; t_0, q_0, \dot{q}_0) = 0.$$

Some numerical results ($m = 1, k = 1, c = 0.2, \gamma = 1$) are given in Figures 2, 3 and 4 so as to illustrate and support this last theoretical result.

Example 2. We consider the model given in Figure 5. A mass $m > 0$ is restrained by a vertical spring with stiffness constant $k_V > 0$ in parallel with a damper with coefficient of viscous damping $c_V > 0$ and some inclined device formed by a spring with stiffness constant $k_I > 0$ in parallel with a nonlinear damper whose characteristic (feedback force-speed) is described by a monotone set-valued graph $\partial\Phi$ as the one depicted in Figure 6. The angle of inclination is denoted by $\theta \in (0, \frac{\pi}{2})$. The horizontal and vertical displacement of the mass m are respectively denoted by u_N and u_T .

The model describing the motion of this system is of the form given in (86) with

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} k_I \sin^2 \theta & -k_I \sin \theta \cos \theta \\ -k_I \sin \theta \cos \theta & k_V + k_I \cos^2 \theta \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & c_V \end{pmatrix},$$

$$H_1 = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \quad H_2 = H_1^T, \quad q = \begin{pmatrix} u_T \\ u_N \end{pmatrix}$$

and with $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ as depicted in Figure 14.

Here $D(\partial\Phi) = \mathbb{R}$, $\partial\Phi(0) = \{0\}$ and $\Phi(x) > 0, \forall x \neq 0$.

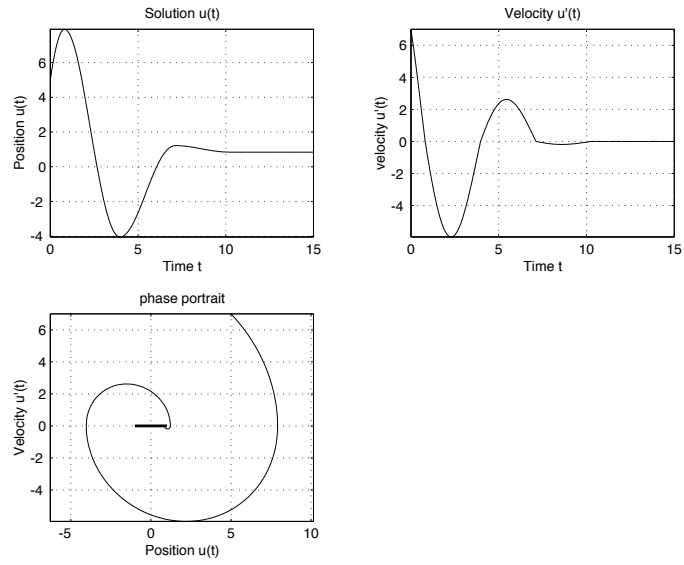


Figure 12: Example 1

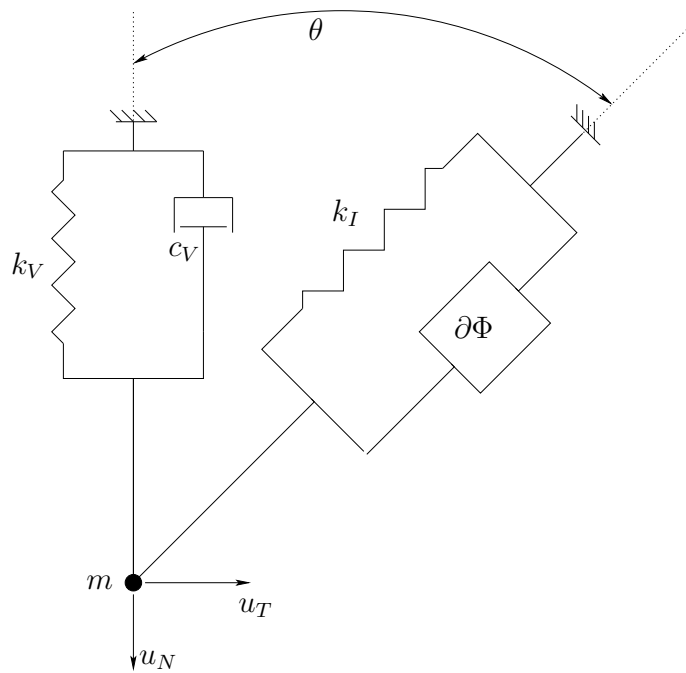


Figure 13: Example 2

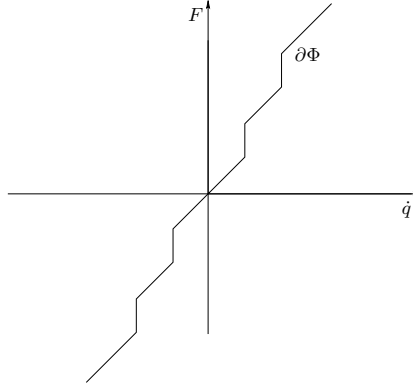


Figure 14: monotone law

It is clear that all the assumptions of Theorem 13.1 hold with

$$R = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix}.$$

The set of stationary solutions reduces here to $\{0\}$ since K is nonsingular and $\partial\Phi(0) = \{0\}$. Thus the trivial stationary solution is here the unique stationary solution. We see that $RM^{-1}CR^{-1}$ is positive semidefinite and $RM^{-1}KR^{-1}$ is symmetric and positive definite.

We may apply Theorem 13.2 and conclude that the trivial stationary solution is stable.

Let us now check that Theorem 13.4 can also be applied. It remains to verify that Assumption 2 in Theorem 13.4 holds. We have

$$\langle RM^{-1}CR^{-1}z, z \rangle_2 + \Phi(H_2R^{-1}z) - \Phi(0) = \frac{c_V}{m} |z_2|^2 + \Phi\left(\frac{1}{\sqrt{m}} \sin(\theta)z_1 - \frac{1}{\sqrt{m}} \cos(\theta)z_2\right).$$

It is thus clear that $\langle RM^{-1}CR^{-1}z, z \rangle_2 + \Phi(H_2R^{-1}z) - \Phi(0) \geq 0$. Suppose now that $\langle RM^{-1}CR^{-1}z, z \rangle_2 + \Phi(H_2R^{-1}z) - \Phi(0) = 0$. Then $|z_2|^2 = 0$ and $\Phi\left(\frac{1}{\sqrt{m}} \sin(\theta)z_1 - \frac{1}{\sqrt{m}} \cos(\theta)z_2\right) = 0$. This yields $z_2 = 0$ and next $z_1 = 0$. Assumption 2 of Theorem 13.4 is thus satisfied.

Theorem 13.4 ensures that the trivial stationary solution is globally attractive.

In conclusion, the trivial stationary solution is (a) **the unique stationary solution**, (b) **stable** and (c) **globally attractive**. Properties (b) and (c) entail that the trivial stationary solution is **asymptotically stable**.

A numerical simulation is given in Figure 7.

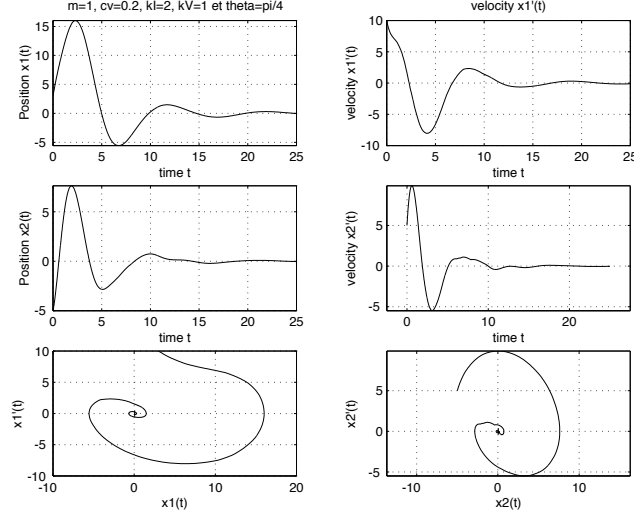


Figure 15: Example 2

15 Necessary conditions for asymptotic stability of evolutionary variational inequalities

This section is based on the article [15] and its aim is to state necessary conditions of asymptotic stability for a class of unilateral dynamical systems. More precisely, we consider the problem: Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonlinear continuous operator. Let $x_0 \in \mathbb{R}^n$ be given. We consider the **problem** $P(x_0)$: Find a unique continuous mapping $t \mapsto u(t)$ such that

$$\frac{du}{dt} \in L_{\text{loc}}^\infty(0, +\infty; \mathbb{R}^n); \quad (75)$$

$$u(t) \in D(\partial\varphi), \forall t \geq 0; \quad (76)$$

$$\left\langle \frac{du}{dt}(t) + F(u(t)), v - u(t) \right\rangle + \varphi(v) - \varphi(u(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq 0, \quad (77)$$

and

$$u(0) = x_0. \quad (78)$$

The variational inequality in (77) can also be formulated as the set-valued differential equation:

$$\frac{du}{dt}(t) + F(u(t)) \in -\partial\varphi(u(t)), \text{ a.e. } t \geq 0, \quad (79)$$

where $\partial\varphi$ denotes the subdifferential of φ . It is assumed that $F(0) \in -\partial\varphi(0)$ i.e. the origin 0 of the system is a trivial stationary solution of (79).

If $\varphi \equiv 0$ then (79) reduces to a standard system of differential equations and it is well known (see e.g. [21]) in this case that if 0 is an isolated zero of F and is asymptotically stable, then there exists $\rho_0 > 0$ such that

$$\deg(F, B_\rho, 0) = 1, \forall \rho \in (0, \rho_0],$$

where $\deg(F, B_\rho, 0)$ denotes the Brouwer degree of F with respect to the open ball $B_\rho := \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and 0.

Our goal in this paper is to generalize this famous result to the model in (78)-(79). More precisely, let us define the mapping Λ as

$$\Lambda(x) := x - P_\varphi(x - F(x))$$

where $P_\varphi := (id_{\mathbb{R}^n} + \partial\varphi)^{-1}$ and $id_{\mathbb{R}^n}$ denotes the identity mapping on $\mathbb{R}^n$. We will prove that if 0 is an isolated zero of Λ and is asymptotically stable, then there exists $\rho_0 > 0$ such that

$$\deg(\Lambda, B_\rho, 0) = 1, \forall \rho \in (0, \rho_0].$$

We suppose that:

- (h_1) $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous;
- (h_2) $D(\partial\varphi)$ is closed;
- (h_3) $0 \in D(\partial\varphi)$;
- (h_4) There exists a neighborhood $\mathcal{N}$ of 0 and a constant $C_1 > 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| \leq C_1 \|x_1 - x_2\|, \forall x_1, x_2 \in \mathcal{N} \cap \text{dom}\{\varphi\};$$

Condition (h_2) ensure that $D(\partial\varphi) = \text{dom}\{\varphi\}$. Thus $D(\partial\varphi)$ is also convex. The projection operator $P_{D(\partial\varphi)}$ onto $D(\partial\varphi)$ is well-defined. Recall that

$$P_{D(\partial\varphi)}^2 = P_{D(\partial\varphi)},$$

$$\|P_{D(\partial\varphi)}x - P_{D(\partial\varphi)}y\| \leq \|x - y\|, \forall x, y \in \mathbb{R}^n,$$

and

$$\langle P_{D(\partial\varphi)}x - x, w - P_{D(\partial\varphi)}x \rangle \geq 0, \forall w \in D(\partial\varphi).$$

We suppose also that

- (h_5) $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous operator such that for some $\bar{\omega} \in \mathbb{R}$, $F + \bar{\omega}I$ is monotone;
- (h_6) F is locally Lipschitz at 0.

Finally, we suppose that

(h_7) 0 is an isolated stationary solution of (77), i.e.

$$0 = P_\varphi(-F(0)) \quad (80)$$

and there is a neighborhood $\mathcal{V}$ of 0 such that

$$x \neq P_\varphi(x - F(x)), \forall x \in \mathcal{V} \setminus \{0\}. \quad (81)$$

Note that the relation in (80) can also be written as

$$\langle F(0), w \rangle + \varphi(w) - \varphi(0) \geq 0, \forall w \in \mathbb{R}^n.$$

Let us now define the continuous mapping $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\Lambda(x) := x - P_\varphi(x - F(x)). \quad (82)$$

If (h_7) is satisfied then for $r > 0$ small enough, $\deg(\Lambda(\cdot), B_r, 0)$ is well defined and constant. We set

$$\zeta(\Lambda, 0) := \deg(\Lambda(\cdot), B_r, 0) \text{ for } r > 0 \text{ small enough.} \quad (83)$$

Remark 15.1 If $n = 1$ then $\zeta(\Lambda, 0)$ can be computed by using property 7 (see Section 2) of degree. More precisely,

$$\zeta(\Lambda, 0) = 0 \text{ if } \Lambda(-r)\Lambda(+r) > 0, \text{ for } r > 0 \text{ small enough,}$$

$$\zeta(\Lambda, 0) = +1 \text{ if } \Lambda(-r) < 0 \text{ and } \Lambda(+r) > 0, \text{ for } r > 0 \text{ small enough,}$$

$$\zeta(\Lambda, 0) = -1 \text{ if } \Lambda(-r) > 0 \text{ and } \Lambda(+r) < 0, \text{ for } r > 0 \text{ small enough.}$$

We have the following result (for a proof we refer to [15]).

Theorem 15.1 Suppose that assumptions (h_1) – (h_7) are satisfied. If the trivial stationary solution of (77) is asymptotically stable then

$$\zeta(\Lambda, 0) = 1.$$

Or equivalently, if $\zeta(\Lambda, 0) \neq 1$ then the trivial stationary solution of (77) is not asymptotically stable.

Example 15.1 i) Let F and φ be defined by $F(x) = 2x$ and $\varphi(x) = I_{\mathbb{R}_+}$. Here

$$\Lambda(x) = x - \max\{0, -x\}.$$

For $r > 0$, we have $\Lambda(-r) = -2r < 0$ and $\Lambda(r) = r > 0$ and thus $\zeta(\Lambda, 0) = +1$. Hence, the necessary condition of asymptotic stability $\zeta(\Lambda, 0) = 1$ is satisfied.

ii) Let F and φ be defined by $F(x) = -2x$ and $\varphi(x) = I_{\mathbb{R}_+}$. Here

$$\Lambda(x) = x - \max\{0, 3x\}.$$

For $r > 0$, we have $\Lambda(-r) = -r < 0$ and $\Lambda(r) = -2r < 0$ and thus $\zeta(\Lambda, 0) = 0$. Hence, the trivial stationary solution is not asymptotically stable.

Theorem 15.2 Suppose that $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and $\text{dom}\{\varphi\} = \mathbb{R}^n$. Suppose that assumptions $(h_5) - (h_7)$ are satisfied. Suppose in addition that there exists $\sigma > 0$ and a nonsingular and symmetric matrix $A \in \mathbb{R}^{n \times n}$ such that

$$\langle F(x), Ax \rangle + \varphi(x) - \varphi(x - Ax) > 0, \quad x \in \overline{B}_\sigma \setminus \{0\}.$$

If $\det A < 0$ then the trivial stationary solution of (77) is unstable.

Example 15.2 Let F and φ be defined by

$$F(x) = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\varphi(x) = d |x_2|,$$

with $a < 0, b > 0, c \in \mathbb{R}$ and $d > 0$. Setting

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that $\langle F(x), Ax \rangle + \varphi(x) - \varphi(x - Ax) = -ax_1^2 + bx_2^2 + d |x_2| > 0, \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Here $\det A = -1$ and thus the trivial stationary solution of (77) is unstable.

Second Order Dynamical Systems in Mechanics

Let us here deal with the following class of second order dynamical systems:

Let $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a convex function with $\text{dom}(\Phi) = \mathbb{R}^m$. Let $M, C, K \in \mathbb{R}^{m \times m}$ be given matrices. We consider the problem: Find a function $t \mapsto q(t)$ ($t \geq 0$) with $q \in C^1([0, +\infty); \mathbb{R}^m)$, and such that:

$$\frac{d^2 q}{dt^2} \in L_{\text{loc}}^\infty(0, +\infty; \mathbb{R}^m), \quad (84)$$

$$\frac{dq}{dt} \text{ is right-differentiable on } [0, +\infty), \quad (85)$$

$$M \frac{d^2 q}{dt^2}(t) + C \frac{dq}{dt}(t) + Kq(t) \in -\partial\Phi\left(\frac{dq}{dt}(t)\right), \text{ a.e. } t \geq 0. \quad (86)$$

The model in (86) can be used in Mechanics to describe the motion of various systems having frictional contact. For such problems, m is the number of degrees of freedom of the system, M is the mass matrix of the system, C is the viscous damping matrix of the system and K is the stiffness matrix. The term $\partial\Phi(\cdot)$ is used to model the unilaterality of the contact induced by friction forces.

Let us first assume that:

(H_1) M is symmetric and positive definite;

(H_2) K is symmetric and nonsingular;

(H_3) $x^T Cx + \Phi(x) - \Phi(0) > 0, \forall x \in \mathbb{R}^m, x \neq 0$;

(H_4) $0 \in \partial\Phi(0)$.

Let

$$\sigma(M, K) := \{\lambda \in \mathbb{C} : \det(\lambda M + K) = 0\} = \{\lambda_1, \dots, \lambda_m\}.$$

Using assumptions (H_1) and (H_2), we may assert that (see e.g. [?]):

$$\sigma(M, K) \subset \mathbb{R} \setminus \{0\},$$

and there exists a nonsingular matrix $R \in \mathbb{R}^{m \times m}$ such that

$$R^T M R = I \quad \text{and} \quad R^T K R = K_0,$$

where I denotes the $m \times m$ identity matrix and K_0 is the diagonal matrix defined by $(K_0)_{ii} = \lambda_i$, ($i \in \{1, \dots, m\}$).

Setting

$$q := Rz,$$

we may rewrite (86) as follows:

$$\frac{d^2 z}{dt^2}(t) + R^T C R \frac{dz}{dt}(t) + K_0 z(t) \in -R^T \partial\Phi(R \frac{dz}{dt}(t)), \quad \text{a.e. } t \geq 0. \quad (87)$$

Let us here define the convex function $\chi : \mathbb{R}^m \rightarrow \mathbb{R}$ by the formula

$$\chi(w) = (\Phi \circ R)(w), \quad \forall w \in \mathbb{R}^m.$$

Then (see Theorem 23.9 in [29]),

$$\partial\chi(w) = R^T \partial\Phi(Rw), \quad \forall w \in \mathbb{R}^m.$$

Let us now set

$$x_1 := z, \quad x_2 := \frac{dz}{dt}, \quad x = (x_1 \ x_2)^T, \quad \dot{x}_1 = \frac{dx_1}{dt}, \quad \dot{x}_2 = \frac{dx_2}{dt}. \quad (88)$$

It is clear that (87) is equivalent to the following first order system :

$$\begin{cases} \dot{x}_1 - x_2 = 0 \\ \dot{x}_2 + R^T C R x_2 + K_0 x_1 \in -\partial\chi(x_2). \end{cases} \quad (89)$$

It results that our problem can be written as in (77) with $n = 2m$ and where F and φ are defined by

$$F(x) = \begin{pmatrix} 0 & -I \\ K_0 & R^T C R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\varphi(x) = \chi(x_2).$$

Let V be defined by

$$V(x) = \frac{1}{2} \langle Ax, x \rangle$$

where

$$A = \begin{pmatrix} K_0 & 0 \\ 0 & I \end{pmatrix}.$$

We see that

$$V'(x) = Ax \neq 0, \forall x \in \mathbb{R}^n, x \neq 0, \quad (90)$$

$$\deg(V', B_r, 0) = \text{sgn}(\det A) = \text{sgn}(\det K_0), \forall r > 0, \quad (91)$$

and

$$\begin{aligned} \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) &= x_2^T R^T C R x_2 + \\ &+ \chi(x_2) - \chi(0) \geq 0, \forall x \in \mathbb{R}^n. \end{aligned} \quad (92)$$

Here

$$\mathcal{S}(F, \varphi) = \{(x_1, 0) : K_0 x_1 \in -\partial\chi(0)\}$$

and

$$\begin{aligned} \mathcal{E}(F, \varphi, V) &= \{x \in \mathbb{R}^n : x_2^T R^T C R x_2 + \chi(x_2) - \chi(0) = 0\} = \\ &= \{(x_1, 0) : x_1 \in \mathbb{R}^m\}. \end{aligned}$$

Moreover, it can be proved that the largest invariant subset $\mathcal{M}(F, \varphi, V)$ of $\mathcal{E}(F, \varphi, V)$ coincides with the set of stationary solutions of (77) (see the proof of Theorem 9 in [2]), that is:

$$\mathcal{M}(F, \varphi, V) = \mathcal{S}(F, \varphi).$$

Our theory can be applied provided that the trivial stationary solution of (89) is isolated in $\mathcal{S}(F, \varphi)$. Let us so now assume in place of (H_4) that:

$$(H_4)' \partial\Phi(0) = \{0\}.$$

Then $\mathcal{S}(F, \varphi) = \{0\}$. So, if assumptions $(H_1) - (H_3), (H_4)'$ are satisfied, then properties (90)-(92) hold, $\mathcal{M}(F, \varphi, V) = \{0\}$ and we may apply Theorem ?? to conclude that if

$$\det K_0 < 0$$

then the trivial stationary solution of (89) is unstable.

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