

# Well-posedness of a non local ocean-atmosphere coupling model

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*S. Thery (2023). Well-posedness of a non local ocean-atmosphere coupling model: study of a 1d ekman boundary layer problem with non local kpp-type turbulent viscosity profile*

# Application of ocean-atmosphere coupling

- Various physical phenomena are governed by the ocean-atmosphere (OA) interaction
- OA models have originally been constructed separately, by two distinct communities.  
⇒ mathematics coherence of such coupling ?

## Modelisation

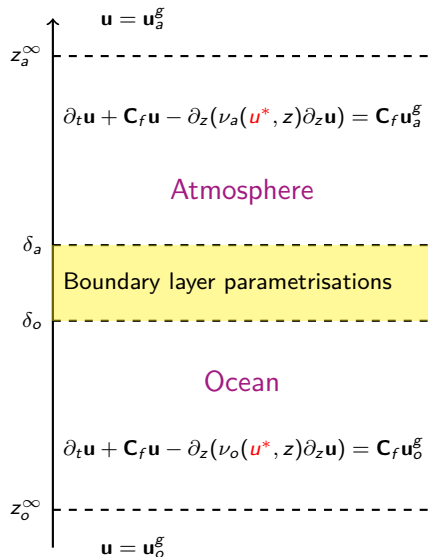
- Translate the realistic OA model in mathematical terms
- Construct a model that take into account account the specificities brought by the numerical models

Work by : Eric Blayo (UGA), Florian Lemarié (Inria-UGA), Charles Pelletier (ECMWF)

## Mathematical study of this model

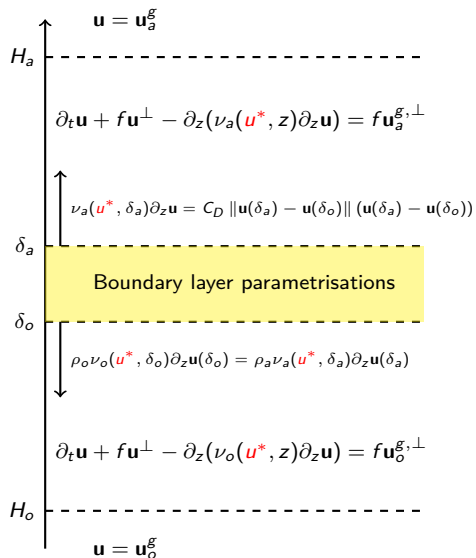
Conditions for the well-posedness (existence and unicity of solutions)

# A simplified OA model



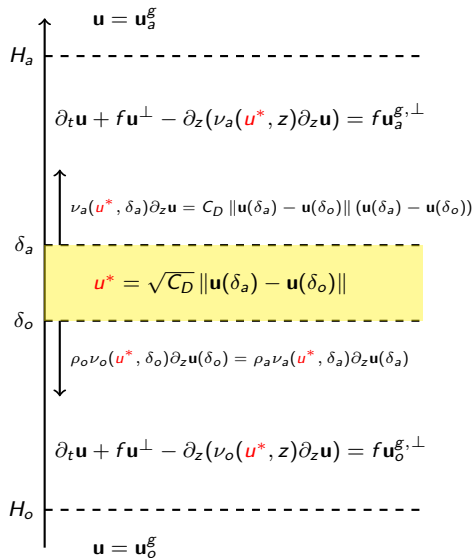
- Navier-stokes + **Simplification hypotheses** : (hydrostatic, ect...)
- 1D vertical,  $\mathbf{u}$  horizontal wind/current  
 $\mathbf{C}_f$  Coriolis force  
 $\mathbf{u}^g$  source term
- **Boussinesq hypothesis** :  
 subgrid-scale parametrisations  
 $\Rightarrow \nu$  turbulent viscosity
- **Boundaries layer parametrisation**:  
 Interface is a "buffer-zone" where solutions are parameterized

# Our local OA model



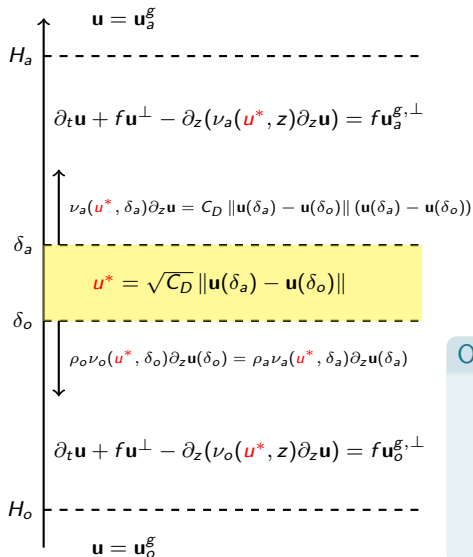
- Friction law at the interface
- Viscosity parameterized by  $u^*$  and  $z$

# Our non-local OA model



- Friction law at the interface
- Viscosity parameterized by  $u^*$  and  $z$
- Boundary layer parametrisation: computation of  $u^*$  (Pelletier et al. 2021)
- $u^*$  wears the non-locality

# Our non-local OA model



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## Our goal

- Study the well-posedness of this problem
- Application to OA framework (specific viscosity and order of magnitude)

- 1 Ocean-atmosphere coupling model
  - Context
  - A non-local OA model
- 2 Well-posedness of the non local problem
  - A fixed point problem
  - The stationary problem
  - The non stationary problem
- 3 Application to OA order of magnitude
  - Specific viscosity profile
  - A necessary and sufficient criteria for stationary problem
- 4 Conclusion

## Existing work and our approach

- **Ocean-Atmosphere coupling model** : *Lions and al. (1993)*
  - ▶ coupling of primitive equations (local problem)
  - ▶ not taking into account real numerical scheme in realistic model
- **Fluid dynamic community** : *Bernardi and al. (2002)*
  - ▶ coupling of two turbulent fluids
  - ▶ stationary and non local problem
  - ▶ very close problem with other kind of viscosity
  - ▶ well-posedness depending on the viscosity profile and its variations.

### Our approach:

- Use fixed point method on our simplified problem (viscosity parametrized) → well-posedness criteria
- Extend the method to non-stationary problem



# A fixed point formulation

$$\left\{ \begin{array}{l} \partial_t \mathbf{u}_\alpha + \mathbf{C}_f \mathbf{u}_\alpha - \partial_z (\nu_\alpha(\mathbf{u}^*, z) \partial_z \mathbf{u}) = \mathbf{C}_f \mathbf{u}_\alpha^g \quad \text{on } (\delta_\alpha, z_\alpha^\infty) \times ]0, T[ \\ \mathbf{u}_\alpha(z_\alpha^\infty) = \mathbf{u}_\alpha^g|_{z_\alpha^\infty} \\ \mathbf{u}_\alpha(z, t=0) = \mathbf{u}_\alpha^0(z) \\ \nu_o \partial_z \mathbf{u}_o(\delta_o) = \lambda^2 \nu_a \partial_z \mathbf{u}_a(\delta_a) \\ \nu_a \partial_z \mathbf{u}_a(\delta_a) = C_D \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| (\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)) \end{array} \right.$$

$$\mathbf{u}^* = \sqrt{C_D} \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\|$$

## Strategy for mathematical studying

Write as a fixed point problem :

Show  $P : u^* \rightarrow \sqrt{C_D} \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\|$  is a contracting mapping.

## Our point of view

Sufficient well-posedness criteria on viscosity profile

## First step : study the local problem

Suppose  $u^*$  is given, the stationary local problem is :

$$\left\{ \begin{array}{l} \mathbf{C}_f \mathbf{u}_\alpha - \partial_z (\nu_\alpha(u^*, z) \partial_z \mathbf{u}) = \mathbf{C}_f \mathbf{u}_\alpha^g \quad \text{on } (\delta_\alpha, z_\alpha^\infty) \\ \mathbf{u}_\alpha(z_\alpha^\infty) = \mathbf{u}_\alpha^g|_{z_\alpha^\infty} \\ \nu_o \partial_z \mathbf{u}_o(\delta_o) = \lambda^2 \nu_a \partial_z \mathbf{u}_a(\delta_a) \\ \nu_a \partial_z \mathbf{u}_a(\delta_a) = C_D \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| (\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)) \end{array} \right.$$

### Property: Well-posedness of the stationary local problem

Weak formulation of the stationary local problem is well-posed in  $H^1(\Omega)$ , with  $\Omega := \Omega_o \cup \Omega_a$

**Proof:** Using Galerkin method.

Apriori estimate on  $\|\cdot\|_{\text{OA}} \approx \|\cdot\|_{L^2(\Omega_o)} + \|\cdot\|_{L^2(\Omega_a)}$

## Fixed point problem

For a given  $\mathbf{u}^g$  with "good" regularity, there exist  $u_{\max}^*$  such that

$$P : \begin{cases} [0, u_{\max}^*] & \rightarrow [0, u_{\max}^*] \\ u^* & \rightarrow \sqrt{C_D} \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| \\ & \mathbf{u} \text{ sol. local pb with } \nu(u^*) \end{cases}$$

### Sufficient well-posedness criteria

$P$  is contractant if  $\forall u^*, v^* \in [0, u_{\max}^*]$  :

$$\max_{z \in \Omega_\alpha} \left| \frac{\nu_\alpha(z, u^*) - \nu_\alpha(z, v^*)}{\sqrt{\nu_\alpha(z, u^*) \nu_\alpha(z, v^*)}} \right| \leq C (\mathcal{M}^e)^{-1} |u^* - v^*|$$

with  $\mathcal{M}^e$  an upper bound of  $\|\sqrt{\nu} \partial_z \mathbf{u}\|_{\text{OA}}$  for all  $\mathbf{u}$  solution of the local problem.

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with  $\mathcal{M}^e$  an upper bound of  $\|\sqrt{\nu} \partial_z \mathbf{u}\|_{\text{OA}}$  for all  $\mathbf{u}$  solution of the local problem. Another more restrictive condition can be derived as

$$\|\partial_{u^*} \nu_\alpha\|_{\mathcal{L}^\infty(\Omega, [0, u_{\max}^*])} \leq C \min(\nu_\alpha) (\mathcal{M}^e)^{-1} \quad C \text{ cst obtain via trace thm}$$

## Same criteria for the non stationary problem

For a given  $\mathbf{u}^g$  with "good" regularity, there exist  $u_{\max}^*$  such that

$$P : \begin{cases} \mathcal{V}^* & \rightarrow \mathcal{V}^* \\ u^*(t) & \rightarrow \sqrt{C_D} \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| (t) \\ & \mathbf{u} \text{ sol. local pb with } \nu(u^*) \end{cases}$$

with  $\mathcal{V}^* := \{u^* \in C^1([0, T]), 0 \leq u^*(t) \leq u_{\max}^*\}$

### Property: Well-posedness criteria

$P$  is contractant if  $\forall u^*, v^* \in [0, u_{\max}^*]$  :

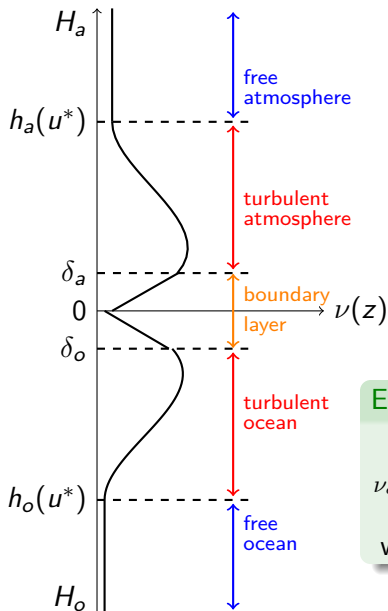
$$\max_{z \in \Omega_\alpha} \left| \frac{\nu_\alpha(z, u^*) - \nu_\alpha(z, v^*)}{\sqrt{\nu_\alpha(z, u^*) \nu_\alpha(z, v^*)}} \right| \leq C(\mathcal{M})^{-1} |u^* - v^*|$$

with  $\mathcal{M}$  an upper bound of  $\sup_{t \in [0, T]} \|\sqrt{\nu} \partial_z \mathbf{u}\|_{O_A}$  for all  $\mathbf{u}$  solution of the local problem.

**Justification of the choice of space  $\mathcal{V}^*$  :**  $\mathcal{M}$  exist if  $\|\partial_t \nu\| \in \mathcal{L}^\infty(\Omega \times ]0, T[)$

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## KPP viscosity



- **Turbulent part** : influenced by  $u^*$
- **Free part** :  $\nu_\alpha = \nu_\alpha^m$
- Turbulence layer thickness :  $h_\alpha(u^*) = c_\alpha u^*$
- Coherence with interface boundary parametrisation :  $\nu(\delta_\alpha) \approx \kappa u^* |\delta_\alpha|$

Example *O'Brien, (1970)*

$$\nu_\alpha(u^*, z) = \kappa u^* |z| \left(1 - \frac{z}{h_\alpha}\right)^2 H\left(1 - \frac{z}{h_\alpha}\right) + \nu_\alpha^m$$

with  $H$  heaviside function

## Application to OA order of magnitude

Example with stationary state and  $\mathbf{u}_\alpha^g$  constant :  $\mathcal{M}^e = \frac{C_D}{2} \|\mathbf{u}_a^g - \mathbf{u}_o^g\|^{3/2}$

- Criteria of well-posedness :

$$\|\partial_{u^*} \nu_\alpha\|_{\mathcal{L}^\infty(\Omega, [0, u_{\max}^*])} \leq 2.5 \times 10^{-5} \|\mathbf{u}_a^g - \mathbf{u}_o^g\|^{3/2}$$

- OA order of magnitude  $\|\mathbf{u}_a^g - \mathbf{u}_o^g\| \approx 10 \text{ms}^{-1}$  give very small upper bound for  $\|\partial_{u^*} \nu_\alpha\|$
- KPP O'Brien viscosity :  $h_\alpha(u^*) \leq 2.5 \times 10^{-5} \|\mathbf{u}_a^g - \mathbf{u}_o^g\|^{3/2}$  that is possible if  $\|\mathbf{u}_a^g - \mathbf{u}_o^g\| \leq 10^{-4} \text{ms}^{-1}$

### Cause of non unicity ?

- Due to the too large bounding in the criterion?
- Due to the profile of  $\nu$  and OA order of magnitude ?



# A necessary and sufficient criteria for stationary problem

## Property

We we can solve the ODE :

$$\mathbf{C}_f \mathbf{u}_\alpha - \partial_z (\nu_\alpha (u^*, z) \partial_z \mathbf{u}) = \mathbf{C}_f \mathbf{u}_\alpha^g \quad \text{on } (\delta_\alpha, z_\alpha^\infty)$$

$$\mathbf{u}_\alpha (z_\alpha^\infty) = \mathbf{u}_\alpha^g |_{z_\alpha^\infty}$$

$$\nu_o \partial_z \mathbf{u}_o(\delta_o) = \lambda^2 \nu_a \partial_z \mathbf{u}_a(\delta_a)$$

$$\nu_a \partial_z \mathbf{u}_a(\delta_a) = C_D \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\| (\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o))$$

$$u^* = \sqrt{C_D} \|\mathbf{u}(\delta_a) - \mathbf{u}(\delta_o)\|$$

We can write

$$\|\mathbf{u}_a^g - \mathbf{u}_o^g\| = F(u^*)$$

## Sufficient and necessary well-posedness criteria

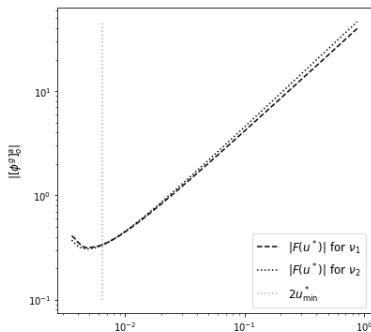
If  $F$  in continuous and injective on  $I^*$  then non-local problem is well-posed on  $u^* \in I^*$

# No unicity of solution in OA order of magnitude

## Exemple

KPP viscosity profile approximate by  $P^2(z)$  polynomial.

- $I^* := [u_{\min}^*, u_{\max}^*]$  depending of  $z_\alpha^\infty$  and  $\delta_\alpha$ .
- $F$  is combination of Legendre polynomial (complicated formula)
- OA order of magnitude : we can prove that we have a inflexion point when  $u^* \approx 2u_{\min}^*$



# Conclusion

- **Wellposedness criteria :**

- ▶ Problem re-write as a fixed point formulation
- ▶ Sufficient well-posedness condition depending on viscosity profile and its variation
- ▶ Variation  $\|\partial_{u^*}\nu\|$  small compare to  $\min(\nu)$

- **Non unicity of solution in OA frame work**

- ▶ Sufficient well-posedness criteria non verify in the OA order of magnitude
- ▶ Necessary and sufficient condition for stationary state with KPP viscosities  
→ non unicity of solution for small value of  $u^*$

- **Improvement**

- ▶ Use alternative boundary conditions
- ▶ More realistic boundary layer parametrisation →  $C_D$  depending on  $u^*$
- ▶ Find another method than the fixed point formulation