

Cloaking by a thin plate in water waves

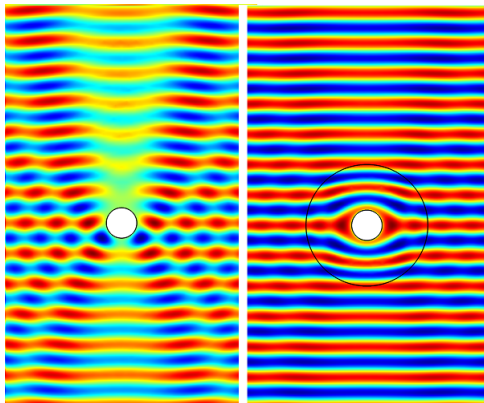
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WAVES July 4th 2024

Joint work with Malte Peter (University of Augsburg), Luke Bennetts (University of Adelaide) and Sébastien Guenneau (Imperial College London)

Cloaking principle :

Create a invisibility cloak to conceal an object

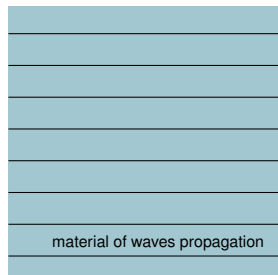


source : Wikipedia, Metamaterial cloaking

Invariant form for a change of coordinate

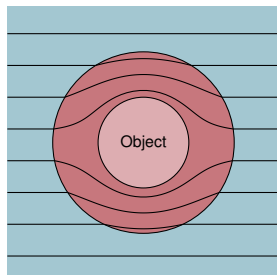
Example with conductivity equation

space e , (x_1, x_2)



change of coordinate

space E , (X_1, X_2)



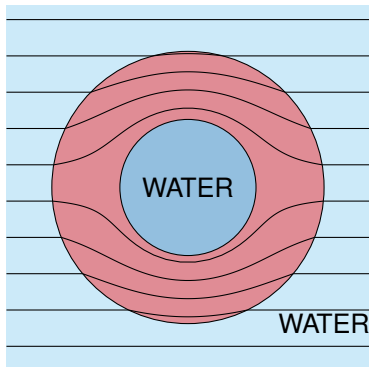
$$\nabla_x \sigma(x) \nabla_x v = f(x)$$

isotropic material, σ is scalar

$$\nabla_X \underbrace{J\sigma(x)J^T|J|^{-1}}_{\Sigma(X)} \nabla_X V = f(x)|J|^{-1}$$

anisotropic material, Σ is matrix

Cloaking in water waves by a thin plate



Difference with electromagnetism:

- Coupling of two different physical domains
- Water wave and elastic plate equations are not shape invariant.

Two different goals:

- "Protected" area → change of coordinate in the plate
- "Invisibility" cloak → other methods

Cloaking in water waves

Related works

- Shallow water by designed water depth
Porter and al 2004; Zareei and Alam 2015

Cloaking in water waves

Related works

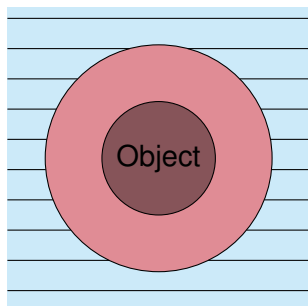
- Shallow water by designed water depth

Porter and al 2004; Zareei and Alam 2015

- On the "invisibility cloak" :

Iida and al. 2023; Farhat and al 2020; Newman 2014

Numerical optimization to minimize the scattering of an object



⇒ No guarantee the object will be "protected"

⇒ We want explicit material property

Cloaking in thin plate

Related works

- Change of coordinate in a isotropic plate

Zareei and Alam 2017, Colquitt 2014

- ▶ Kirchhoff-Love plate equation is **not shape invariant**
- ▶ We have to add an external force (prestress) to have a perfect cloaking

Cloaking in thin plate

Related works

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Zareei and Alam 2017, Colquitt 2014
 - ▶ Kirchhoff-Love plate equation is **not shape invariant**
 - ▶ We have to add an external force (prestress) to have a perfect cloaking

Our approach

- Take a general anisotropic plate equation
- Apply to the coupled "water-wave-plate" model

1 Introduction

- Cloaking principle and existing work
- Existing work and our approach

2 Cloaking in the plate

- Change of coordinates in an anisotropic plate
- How to choose the plate parameters and transformation

3 Cloaking of water waves by thin plate

- Change of coordinate in water domain
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Anisotropic plate : Kirchhoff-Love equation

General equation for thin isotropic plate :

$$D\Delta^2 w + \rho_0 d \partial_t^2 w = pg \quad \text{in } \mathbf{P} \subset \mathbb{R}^2$$

+ boundary conditions on $\partial\mathbf{P}$

with w the vertical displacement

Anisotropic plate : Kirchhoff-Love equation

General equation for thin anisotropic plate :

$$\begin{aligned} (D_{ijkl} w_{,kl})_{,ij} + \rho_0 d \partial_t^2 w &= p g && \text{in } \mathbf{P} \subset \mathbb{R}^2 \\ + \text{boundary conditions} &&& \text{on } \partial \mathbf{P} \end{aligned}$$

with w vertical plate displacement, and $i, j, k, l \in \{1, 2\}$.

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with w vertical plate displacement, and $i, j, k, l \in \{1, 2\}$.

Variational formulation from Lagrangian :

$$\delta L_p(w)(\tilde{w}) = - \int_{\mathbf{P}} \tilde{w}_{,ij} D_{ijkl} w_{,kl} + \int_{\mathbf{P}} \rho_0 d \partial_t w \partial_t \tilde{w} - \int_{\mathbf{P}} p g \tilde{w}$$

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We suppose symmetries $D_{ijkl} = D_{jikl} = D_{ijlk}$.

$$\tilde{w}_{,ij} D_{ijkl} w_{,kl} = \begin{pmatrix} \tilde{w}_{,11} \\ \tilde{w}_{,22} \\ \sqrt{2} \tilde{w}_{,12} \end{pmatrix}^T \begin{pmatrix} D_{1111} & D_{1122} & \sqrt{2} D_{1112} \\ D_{1122} & D_{2222} & \sqrt{2} D_{2221} \\ \sqrt{2} D_{1112} & \sqrt{2} D_{2221} & 2 D_{1212} \end{pmatrix} \begin{pmatrix} w_{,11} \\ w_{,22} \\ \sqrt{2} w_{,12} \end{pmatrix}$$

Matrix representation

- Isotropic plate

$$\mathbf{D}^{iso} = \begin{pmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & (1 - \nu)D \end{pmatrix}$$

with $D = \frac{Eh^3}{12(1 - \nu)}$, E the Young's modulus and ν the Poisson ratio.

- General anisotropic plate : 6 degree of freedom

$$\mathbf{D} = \begin{pmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{1122} & D_{2222} & D_{2221} \\ D_{1112} & D_{2221} & D_{1212} \end{pmatrix}$$

Change of coordinates in an anisotropic plate

For a change of coordinates $(X_1, X_2) = F(x_1, x_2)$:

$$\int_e \tilde{w}_{,ij} D_{ijkl} w_{,kl} dv = \int_E \tilde{W}_{,IJ} D_{IJKL} W_{,KL} + \tilde{W}_{,IJ} b_{IJK} W_{,K} + \tilde{W}_{,I} b_{KLI} W_{,KL} + \delta W_{,I} c_{IK} W_{,K} dV$$

Change of coordinates in an anisotropic plate

For a change of coordinates $(X_1, X_2) = F(x_1, x_2)$:

$$(D_{IJKL}W_{,KL})_{,IJ} \quad -N_{IJ}W_{,IJ} + S_I W_{,I} \quad + \rho_0 h \partial_t^2 w = pg$$

anisotropic plate prestress force

With N_{IJ} , S_I depending on b_{IJK} , c_I .

Change of coordinates in an anisotropic plate

For a change of coordinates $(X_1, X_2) = F(x_1, x_2)$:

$$\underbrace{(D_{IJKL}W_{,KL})_{,IJ}}_{\text{anisotropic plate}} \quad \underbrace{-N_{IJ}W_{,IJ} + S_I W_{,I}}_{\text{prestress force}} + \rho_0 h \partial_t^2 w = pg$$

With N_{IJ} , S_I depending on b_{IJK} , c_I .

Issues

- Ignoring **prestress force** provide an inefficient cloaking.
- **Prestress force** is null for linear transformation *Pomot and al. 2019*

⇒ **How to choose D_{ijkl} and F to get an efficient cloaking**

Work on linear forms

Define $B_F : \mathbb{R}^6 \rightarrow \mathbb{R}^9$

$$B_F((D_{ijkl})_{i,j,k,l \in \{1,2\}}) = (b_{IJK}, c_{IJ})_{I,J,K \in \{1,2\}}.$$

Work on linear forms

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$$B_F((D_{ijkl})_{i,j,k,l \in \{1,2\}}) = (b_{IJK}, c_{IJ})_{I,J,K \in \{1,2\}}.$$

- $B_F(\mathbb{R}^6) = \{0\}$ for linear transformation or s.t $F_{21} = F_{12} = 0$

Work on linear forms

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- $B_F(\mathbb{R}^6) = \{0\}$ for linear transformation or s.t $F_{21} = F_{12} = 0$
- $\ker(B_F) \neq 0$:

$$\text{in } e \quad c(x_1, x_2) \begin{pmatrix} \Gamma_1^2 & \Gamma_1 \Gamma_2 & \Gamma_1 \Gamma_3 \\ \Gamma_1 \Gamma_2 & \Gamma_2^2 & \Gamma_2 \Gamma_3 \\ \Gamma_1 \Gamma_3 & \Gamma_2 \Gamma_3 & \Gamma_3^2 \end{pmatrix} \quad \forall c \in \mathbf{C}^2(\mathbf{P})$$

\Rightarrow

$$\text{in } E \quad \begin{pmatrix} G_1^2 & G_1 G_2 & G_1 G_3 \\ G_2 G_2 & G_1^2 & G_2 G_3 \\ G_1 G_3 & G_2 G_3 & G_3^2 \end{pmatrix} \quad +\text{NO prestress force}$$

with Γ_i, G_i depending on Hessian of F .

How to choose plate parameters

For general non linear transformations

- Isotropic plate is transformed to anisotropic plate + prestress force
- Equation is shape invariant for specific family of anisotropic plate

Issues:

- Difficulty in predicting wave behaviour in a completely anisotropic material
- Choose a good couple transformation-plate parameters

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Change of coordinate in water

Equation in the water domain

$$\nabla^2 \Phi = 0 \quad \mathbb{R}^2 \times]-h, 0[$$

$$\partial_z \Phi = 0 \quad z = -h$$

$$\partial_z \Phi = \partial_t \eta \quad z = 0$$

$$\partial_t \Phi + g\eta = p/\rho_w \quad z = 0$$

with Φ potential and η water elevation.

Change of coordinate in water

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with Φ potential and η water elevation.

For an horizontal change of variable in the water domain

$$\nabla_{X,Y} \left(\frac{JJ^T}{|J|} \nabla_{X,Y} \Phi \right) + \partial_z^2 \Phi = 0 \quad \mathbb{R}^2 \times]d, 0[$$

$$\partial_z \Phi = 0 \quad z = d$$

$$\partial_z \Phi = \partial_t \eta \quad z = 0$$

$$\partial_t \Phi + g\eta = P/\rho_w \quad z = 0$$

Change of coordinate in water

Equation in the water domain

$$\begin{aligned}\nabla_{x,y} (h \nabla_{x,y} \Phi) &= -\partial_t \eta && \text{in } \mathbb{R}^2 \\ \partial_t \Phi + g\eta &= p/\rho_w && \text{in } \mathbb{R}^2\end{aligned}$$

For an horizontal change of variable in the water domain

$$\begin{aligned}\nabla_{X,Y} (H \nabla_{X,Y} \Phi) &= -\partial_t \eta && \text{in } \mathbb{R}^2 \\ \partial_t \Phi + g\eta &= P/\rho_w && \text{in } \mathbb{R}^2\end{aligned}$$

- ⇒ Consider shallow water regime
- ⇒ To be shape invariant, water thickness have to be anisotropic
- ⇒ **Plate immersion** is anisotropic

Variational formulation for system plate+water

In shallow water regime, :

$$\begin{aligned} S_{p+w}(\phi, \eta)(\psi, \xi) = & \int_{t_0}^{t_1} \int_{\Omega} -\rho_w \psi \partial_t \eta + \rho_w \nabla \psi h \nabla \Phi + \rho_w \partial_t \Phi \xi + \rho_w g \eta \xi \\ & + \xi_{,ij} D_{ijkl} \eta_{,kl} - \rho_0 d \partial_t \xi \partial_t \eta \quad dx_1 dx_2 \end{aligned}$$

with h water thickness,

After horizontal change of coordinate:

$$\begin{aligned} S_{p+w}(\phi, \eta)(\psi, \xi) = & \int_{t_0}^{t_1} \int_{\tilde{\Omega}} -\tilde{\rho}_w \psi \partial_t \eta + \tilde{\rho}_w \nabla \psi H \nabla_h \Phi + \tilde{\rho}_w \partial_t \Phi \xi + \tilde{\rho}_w g \eta \xi \\ & + \xi_{,ij} D_{ijkl} \eta_{,kl} - \tilde{\rho}_0 \tilde{\rho} \partial_t \xi \partial_t \eta \quad dX_1 dX_2 \end{aligned}$$

- Anisotropic stiffness matrix
- Anisotropic plate immersion

An analytical time-harmonic solution

Using a particular couple plate parameters/tranformation :

$$\text{in } e \quad c(x_1, x_2) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \forall c \in \mathbf{C}^2(\mathbf{P})$$

$$\Rightarrow \quad F : r \rightarrow c_1 r + \frac{c_2}{r} \quad \forall c_i \in \mathbb{R}$$

$$\text{in } E \quad \left(c_1 + \frac{c_2}{r^2}\right) c(x_1, x_2) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad r = \sqrt{x_1^2 + x_2^2}$$

- Plate with hole or object in initial space
- Analytical solution using eigen matching method

Conclusion

Cloaking with a floating plate brings **several difficulties**:

- Coupling of physical domain: plate/water wave interaction
⇒ non "invisibility"
- Resulting plate is anisotropic on the stiffness and the thickness

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Solution: Proposed tools help to find a family of anisotropic plate with "good" property to create a protected area

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Next step

- Homogenisation technique to create such a plate
- What's happening inside the "hole" or on the object

From electromagnetism

First reference work

- Theoretical : method using a change of coordinate, *Pendry and al. 2006*; *Leonhardt 2006*
- Experimental : using metamaterial *Schrig and al. 2006*
⇒ Homogeneisation

Cloaking principle applied to many other waves since