On reconstruction from imaginary part for radiation solutions in two dimensions

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A.V. Nair, R.G. Novikov

Abstract

We consider a radiation solution ψ for the Helmholtz equation in an exterior region in $\mathbb{R}^2$. We show that ψ in the exterior region is uniquely determined by its imaginary part $Im(\psi)$ on an interval of a line L lying in the exterior region. This result has holographic prototype in the recent work [Nair, Novikov, arXiv:2408.08326]. Some other curves for measurements instead of the lines L are also considered. Applications to the Gelfand-Krein-Levitan inverse problem and passive imaging are also indicated.

Keywords: Two-dimensional Helmholtz equation, radiation solutions, Gelfand-Krein-Levitan inverse problem, passive imaging, holographic type uniqueness
AMS Subject classification: 35J05, 35J08, 35P25, 35R30

1 Introduction

We consider the two-dimensional Helmholtz equation

$$\Delta\psi(x) + \kappa^2\psi(x) = 0, \quad x \in \mathcal{U}, \quad \kappa > 0, \tag{1}$$

where Δ is the Laplacian in x, and $\mathcal{U}$ is a region (open connected set) in $\mathbb{R}^2$ consisting of all points outside a closed bounded regular curve S (for example as in [13]). For equation (1) we consider the radiation solution ψ such that: ψ is of class C^2 and satisfies the Sommerfeld's radiation condition

$$\sqrt{|x|}\big(\frac{\partial}{\partial|x|} - i\kappa\big)\psi(x) \to 0 \ \ as \ \ |x| \to +\infty, \tag{2}$$

uniformly in $x/|x|$.

Let

$$L = L_{x_0,\theta} = \{x \in \mathbb{R}^2 : \ \ x = x(s) = x_0 + s\theta, \ \ -\infty < s < +\infty\},$$

$$L^+ = L^+_{x_1,\theta} = \{x \in \mathbb{R}^2 : x = x(s) = x_1 + s\theta, \ \ 0 < s < +\infty\},$$

$$L^- = L^-_{x_2,\theta} = \{x \in \mathbb{R}^2 : \ \ x = x(s) = x_2 - s\theta, \ \ 0 < s < +\infty\}, \tag{3}$$

where $x_0, x_1, x_2 \in \mathbb{R}^2$, $\theta \in \mathbb{S}^1$ and $x_1, x_2 \in L$.
The present work combines results in [16], and methods of [25] (the three-dimensional case), to obtain analogical results to [25] in the two-dimensional case. In particular, we show that for any straight line $L \subset \mathcal{U}$, any complex-valued radiation solution ψ on $\mathcal{U}$ is uniquely determined by the $Im(\psi)$ on an arbitrary non-empty interval Λ of L; see Theorem 2 (which is a corollary of Theorem 1) and Corollary 1 in Section 2. We also consider other curves for measurements instead of the lines L; see Example 1 and Theorem 3 in Section 2.

Our studies are motivated by the Gelfand-Krein-Levitan problem and passive imaging, see, e.g., [2], [3], [5], [8], [11], [12], [14], [15], [25], [26] and references therein. In particular, in this respect, we continue studies of [25], where the three-dimensional case is considered.

The Gelfand-Krein-Levitan problem (in its fixed energy version in dimension $d = 2$) consists in determining the potential v in the Schrödinger equation

$$-\Delta\psi(x) + v(x)\psi(x) = \kappa^2\psi(x) + \delta(x-y), \quad x, y \in \mathbb{R}^2, \quad \kappa > 0, \tag{4}$$

from the imaginary part of its radiation solutions $\psi = R_v^+(x, y, \kappa)$ (that is, satisfying (2)) for one κ and all x, y on some part of the boundary of a domain containing the support of v. Here, δ is the Dirac delta function. In this problem, $\mathrm{Im}(R_v^+)$ is related to the spectral measure of the Schrödinger operator $H = -\Delta + v$. For more information about the Gelfand-Krein-Levitan problem and its relevance to passive imaging, see [2], [5], [25] and references therein.

Theorems 1, 2, Corollary 1, and Theorem 3 mentioned above admit the same applications to the Gelfand-Krein-Levitan inverse problem and passive imaging in two dimensions as their prototypes in [25] in three dimensions, see subsection 2.2 for more details.

In the present work, we use the Karp expansion (9) below for the radiation solutions ψ of equation (1) instead of the Atkinson-Wilcox expansion, used in [25], for radiation solutions of the Helmholtz equation in three dimensions. In addition, we use very recent results on the Karp expansion obtained in [16].

The main results of the present work are presented in more detail and proved in Section 2, Subsection 3.2, and Section 4. In our proofs, we proceed from the results recalled in Section 3.

2 Main results

2.1 Determination of radiation solutions ψ from $\mathbf{Im}(\psi)$

Our key result is as follows.

Theorem 1. *Let ψ be a radiation solution of equation (1) as in (3). Let L, L^+ and L^- be as given in (3) such that $L^+ = L^+_{x_1,\theta} \subset \mathcal{U}$ and $L^- = L^-_{x_2,\theta} \subset \mathcal{U}$, $x_1, x_2 \in L$, where $\mathcal{U}$ is the region in (1). Then ψ on $L^+ \cup L^-$ is uniquely determined by $Im(\psi)$ on $\Lambda^+ \cup \Lambda^-$, where Λ^+ and Λ^- are arbitrary non-empty intervals of L^+ and L^-, respectively.*

As a corollary, we also get the following result.

Theorem 2. *Let ψ be a radiation solution of equation (1) as in (3). Let L be as given in (3) such that $L = L_{x_0,\theta} \subset \mathcal{U}$, where $\mathcal{U}$ is the region in (1). Then ψ on L is uniquely determined by $Im(\psi)$ on Λ, where Λ is an arbitrary non-empty interval of L.*

Theorem 1 is proved in Section 4 using the Karp expansion of [13] for the radiation solutions of equation (1), results of [16], and methods of [25]. Similar to [25], we use a two-point approximation for ψ in terms of $Im(\psi)$; see Proposition 1 in Section 3.

Note that ψ and $Im(\psi)$ are real-analytic on $\mathcal{U}$, and therefore on $L^+ \cup L^-$ in Theorem 1 and on L in Theorem 2. Because of this analyticity, Theorem 1 reduces to the case when $\Lambda^+ \cup \Lambda^- = L^+ \cup L^-$ and Theorem 2 reduces to the case when $\Lambda = L$.

Theorem 2 is proved as follows (for example). We assume that $\Lambda = L$. Then we simply consider $L^+ \subset L$ and $L^- \subset L$ such that $L^+ \cap L^- \neq \emptyset$. Then we may apply Theorem 1.

Corollary 1. *Under the assumptions of Theorem 2, $Im(\psi)$ on L uniquely determines ψ in the entire region $\mathcal{U}$.*

Corollary 1 follows from Theorem 2, formula (14) recalled in Section 3, and analyticity of ψ in $\mathcal{U}$.

Theorems 1 and 2, and Corollary 1 have holographic prototypes in [16].

In connection with other curves of measurements instead of the lines L our results are as follows.

The results in Theorems 1 and 2, don't hold for some other curves in place of L. An example is as follows. Let

$$\mathbb{S}_r = \{x \in \mathbb{R}^2 : |x| = r\}, \quad r > 0. \tag{5}$$

Example 1. *Let $\psi(x) = G^+(x,\kappa) = \frac{i}{4}H_0(\kappa|x|)$, where H_0 is the Hankel function of first kind of order zero. Then ψ is a non-zero radiation solution of equation (1) if $\{0\} \subset \mathbb{R}^2 \setminus \overline{\mathcal{U}}$, but $Im(\psi) \equiv 0$ on the circles $\mathbb{S}_r$ for $r = \frac{c_j}{\kappa}$, $j \in \mathbb{N}$ where c_j are the positive real roots of the Bessel function $J_0 = Re(H_0)$.*

The fact that there are infinite positive real roots of J_0 and they tend to positive infinity can be found in [1]. Nevertheless, as in the three-dimensional scenario, the uniqueness results of Theorem 2 and Corollary 1, remain valid for many interesting curves instead of the lines L.

Suppose that

$$\begin{gathered} \mathcal{D} \text{ is an open bounded domain in } \mathbb{R}^2, \\ \Gamma = \partial\mathcal{D} \text{ is real analytic and connected.} \end{gathered} \tag{6}$$

Theorem 3. *Let ψ be a radiation solution of equation (1) as in (2). Let Γ be a curve as above, where κ is not a Dirichlet eigenvalue for $\mathcal{D}$, and $\overline{\mathcal{D}} = \mathcal{D} \cup \Gamma \subset \mathcal{U}$. Then ψ on $\mathcal{U}$ is uniquely determined by $Im(\psi)$ on Λ where Λ is an arbitrary non-empty open interval of Γ.*

The proof of Theorem 3 is similar to that of Theorem 4 in [25]. In particular, this proof uses real analytic continuations, solving the Dirichlet problem for the Helmholtz equation in $\mathcal{D}$, and Corollary 1 in place of its three-dimensional prototype in [25].

2.2 Applications to Gelfand-Krein-Levitan problem

The aforementioned results on recovering a radiation solution ψ from $\mathrm{Im}(\psi)$ give a reduction of the Gelfand-Krein-Levitan problem (of inverse spectral theory and passive imaging in dimension $d = 2$) to the inverse scattering problem of finding v in (4) from boundary values of $\psi = R_v^+$. This reduction and known results imply, in particular, that v in (4) is uniquely determined by $\mathrm{Im}(R_v^+(x, y, \kappa))$ for one κ and all x, y on an arbitrary open interval Λ of L, where *supp* $v \subset \Omega$, L is a line in $\mathbb{R}^2 \setminus \overline{\Omega}$, Ω is an open bounded connected domain in $\mathbb{R}^2$, $\overline{\Omega}$ is the closure of Ω; see Theorem 4 below.

Theorem 4. *Let $v \in L^\infty(\mathbb{R}^2)$, supp $v \subset \Omega$, and $L \subset \mathbb{R}^2 \setminus \overline{\Omega}$, where Ω is an open bounded connected domain in $\mathbb{R}^2$, $\overline{\Omega}$ is the closure of Ω, and L is a straight line. Let $R_v^+(x, y, \kappa)$ be the outgoing Green function for equation (4). Then v is uniquely determined by $Im(R_v^+(x, y, \kappa))$ for one κ and all x, y on Λ, where Λ is an arbitrary non-empty open interval of L.*

Theorem 5. *Let $v \in L^\infty(\mathbb{R}^2)$, supp $v \subset \mathbb{R}^2 \setminus \overline{\mathcal{U}}$, where $\mathcal{U}$ is as in (1), and property (17) hold for fixed $\kappa > 0$. Let Γ be a curve as in (6), where κ is not a Dirichlet eigenvalue for $\mathcal{D}$, and $\overline{\mathcal{D}} = \mathcal{D} \cup \Gamma \subset \mathcal{U}$. Then v is uniquely determined by $Im(R_v^+(x, y, \kappa))$ on $\Lambda \times \Lambda$, where Λ is an arbitrary non-empty open interval of Γ.*

In Theorems 4 and 5, we do not assume that v is real-valued, but we assume that (17) formulated in Subsection 3.4 holds.
Theorems 4 and 5 can be proved in a similar way to Theorems 3 and 5 in [25], respectively. These proofs use Theorems 1,2, and 3 in place of their three-dimensional prototypes in [25], some known results on two-dimensional direct scattering in place of similar results on three-dimensional direct scattering used in [25], and also the two-dimensional global uniqueness result on recovering from the scattering amplitude A in (21) at fixed κ (see Corollary 5.1 in [6] in addition to results of [17], [18]). These proofs follow from these indications in a straightforward way.

3 Preliminaries

Let (r, ϕ) denote the polar coordinates of a point x and let $\theta = \frac{x}{|x|}$ be its direction. So $-\theta$ will be the direction of $-x$ whose polar coordinates will be $(r, \phi + \pi)$.
Let

$$B_\rho = \{x \in \mathbb{R}^2 \mid |x| < \rho\}, \quad \rho > 0. \tag{7}$$

We use the following asymptotic expansion for radiation solution ψ of equation (1):

$$\psi(x) \sim \sqrt{\frac{2}{\pi\kappa|x|}} e^{i(\kappa|x| - \frac{\pi}{4})} \sum_{j=0}^{\infty} \frac{f_j(\phi)}{|x|^j} \quad \text{for} \quad x \in \mathbb{R}^2, \quad |x| \to \infty. \tag{8}$$

Expansion (8) looks similar to the convergent Atkinson-Wilcox expansion in three dimensions (in [4]). However, the series in (8) diverges in general and in particular for $\psi(x) = H_0(\kappa|x|)$ (see [13] for more details).

3.1 The Karp Expansion

Due to [13], we have the following Karp expansion:

Suppose that ψ is a radiation solution of equation (1), and $\mathbb{R}^2 \setminus B_\rho \subset \mathcal{U}$.

$$\psi(x) = H_0(\kappa|x|) \sum_{j=0}^{\infty} \frac{F_j(\phi)}{|x|^j} + H_1(\kappa|x|) \sum_{j=0}^{\infty} \frac{G_j(\phi)}{|x|^j}, \tag{9}$$

$$\text{for} \quad x \in \mathbb{R}^2 \setminus B_\rho,$$

where H_0 and H_1 are the Hankel functions of the first kind of order zero and one respectively. The series converges absolutely and uniformly for $|x| \geqq \rho_1 > \rho$ (for any such ρ_1).

The important result is that $F_j(\phi), G_j(\phi),\ j = 0, 1, 2, ...n$, in (9) are uniquely determined from $f_j(\phi), f_j(\phi + \pi),\ j = 0, 1, 2, ...n$, in (8) via recurrent formulas; (see Theorem III of [13] for $n = 0$, Theorem 3 of [16] for the general case).

3.2 A two-point approximation for ψ

Let

$$I(x) = \sqrt{|x|} Im(\psi(x)), \quad x \in \mathcal{U}, \tag{10}$$

where ψ is a radiation solution of equation (1).

We have that

$$2iI(x) = \sqrt{\frac{2}{\pi\kappa}}(e^{i(\kappa|x|-\frac{\pi}{4})} f_0(\phi) - e^{-i(\kappa|x|-\frac{\pi}{4})}\overline{f_0(\phi)}) + O(\frac{1}{|x|}),\ \text{as } |x| \to \infty, \tag{11}$$

uniformly in $\theta = x/|x|$, where f_0 is the leading coefficient in (8).

Proposition 1. *Let ψ be a radiation solution of equation (1). Then*

$$f_0(\phi) = \frac{1}{sin(\kappa\tau)}\sqrt{\frac{\pi\kappa}{2}}\big(I(x)e^{-i(\kappa|y|-\frac{\pi}{4})} - I(y)e^{-i(\kappa|x|-\frac{\pi}{4})} + O(\frac{1}{|x|})\big),\ \textit{as } |x| \to \infty, \tag{12}$$

$$x, y \in L^+_{x_0,\theta},\ x_0 = 0,\ y = x + \tau\theta,\ \theta \in \mathbb{S},\ \tau > 0,$$

uniformly in θ, where f_0 is the leading coefficient in (8), I is defined in (10), L^+ is the ray defined by (3), and $sin(\kappa\tau) \neq 0$ for fixed τ.

Formula (12) is a two-point approximation for f_0 and together with (8) also gives a two-point approximation for ψ in terms of I. For phaseless inverse scattering and holography formulas of such a type go back to [19] (see also [23], [22], [21], [16], [25]).

Remark 1. *If an arbitrary function I on $L_{0,\theta}$ satisfies (11), then formula (12) holds, for fixed $\theta \in \mathbb{S}$.*

We obtain (12) from the system of equations for $f_0, \overline{f_0}$:

$$\sqrt{\frac{2}{\pi\kappa}}(e^{i(\kappa|x|-\frac{\pi}{4})} f_0(\phi) - e^{-i(\kappa|x|-\frac{\pi}{4})}\overline{f_0(\phi)}) = 2iI(x) + O(|x|^{-1}), \tag{13}$$

$$\sqrt{\frac{2}{\pi\kappa}}(e^{i(\kappa|y|-\frac{\pi}{4})} f_0(\phi) - e^{-i(\kappa|y|-\frac{\pi}{4})}\overline{f_0(\phi)}) = 2iI(y) + O(|y|^{-1}),$$

where x, y are as in (12). In particular, we use that $|y| = |x| + \tau$. In turn, (13) follows from (11).

3.3 A Green type formula

The following formula holds:

$$\psi(x) = 2\int_L \frac{\partial G^+(x-y,\kappa)}{\partial \nu_y}\psi(y)dy, \quad x \in V_L, \tag{14}$$
$$G^+(x,\kappa) = \frac{i}{4}H_0(\kappa|x|), \quad x \in \mathbb{R}^2,$$

where ψ is a radiation solution of equation (1), L and V_L are line and open half-plane in $\mathcal{U}$, where L is the boundary of V_L, ν is the outward normal to L relative to V_L; see, for example, formula 5.84 in [7].

3.4 Some facts of direct scattering

The radiation solution $R_v^+ = R_v^+(.,y,\kappa)$ for equation (4) satisfies the integral equation

$$R_v^+(x,y,\kappa) = -G^+(x-y,\kappa) + \int_\Omega G^+(x-y,\kappa)v(z)R_v^+(z,y,\kappa)dz, \tag{15}$$

where $x, y \in \mathbb{R}^2$, G^+ is given in (14).
We consider equations (4) and (15) assuming for simplicity that

$$v \text{ is complex-valued, } v \in L^\infty(\Omega), \ v \equiv 0 \text{ on } \mathbb{R}^2 \setminus \overline{\Omega}, \tag{16}$$

and

$$\text{equation (15) is uniquely solvable for } R_v^+(.,y,\kappa) \in L^2(\Omega). \tag{17}$$

It is known that if v satisfies (16) and is real-valued (or $\mathrm{Im}(v) \le 0$) then (17) is fulfilled automatically; see, for example, [9], at least in three dimensions.

Note that

$$R_v^+(x,y,\kappa) = R_v^+(y,x,\kappa), \quad x,y \in \mathbb{R}^2; \tag{18}$$

We also consider the scattering wave functions ψ^+ for the homogeneous equation (4) (i.e without δ):

$$\psi^+ = \psi^+(x.k) = e^{ikx} + \psi_{sc}^+(x,k), \quad x,k \in \mathbb{R}^2, \ |k| = \kappa, \tag{19}$$

where $\psi_{sc}^+(x,k)$ satisfies the radiation condition (2) at fixed k.

The following formulas hold:

$$R_v^+(x,y,\kappa) = -\frac{1}{2}\sqrt{\frac{1}{2\pi\kappa|x|}}e^{i(\kappa|x|+\frac{\pi}{4})}\psi^+(y.-\kappa\frac{x}{|x|}) + O\big(\frac{1}{|x|^{3/2}}\big), \ \text{ as } |x| \to \infty, \tag{20}$$

at fixed y;

$$\psi^+_{sc}(x,k) = \frac{e^{i\kappa|x|}}{|x|^{1/2}} A(k, \kappa\frac{x}{|x|}) + O\big(\frac{1}{|x|^{3/2}}\big), \quad \text{as } |x| \to \infty, \tag{21}$$

at fixed k, where A arising in (21) is the scattering amplitude for the homogeneous equation (4).
In connection with the aforementioned facts concerning R^+_v and ψ^+, see, e.g., [10].

4 Proof of Theorem 1

4.1 Case $L^+_{x_1,\theta} \cup L^-_{x_2,\theta} \subseteq L_{0,\theta}$

In a similar way to [25] (where the three-dimensional case is considered), we prove that $\mathrm{Im}(\psi)$ on $L^+_{x_1,\theta} \cup L^-_{x_2,\theta} \subseteq L_{0,\theta}$ defined as in (3) where 0 denotes the origin in $\mathbb{R}^2$, uniquely determines $f_j(\phi)$ in (8) $\forall j \in \mathbb{N}$. The proof is given below. Then, using formulas in [13] and [16] as recalled in Subsection 3.1, all $F_j(\phi), G_j(\phi)$, in (9) are uniquely determined by $f_j(\phi), f_j(\phi+\pi)$. The rest follows from the convergence of the series in (9) and analyticity of ψ and $\mathrm{Im}(\psi)$ on $L^+ \subseteq L^+_{0,\theta}$.

The determination of $f_0(\phi)$ from $\mathrm{Im}(\psi)$ on $L^+_{x_1,\theta}$ follows from (12).

Suppose that $f_0(\phi), f_1(\phi), f_2(\phi), ...f_n(\phi)$ are determined then the determination of $f_{n+1}(\phi)$ from $\mathrm{Im}(\psi)$ on $L^+_{x_1,\theta}$ is as follows.

Let

$$\psi_n(x) = \sqrt{\frac{2}{\pi\kappa|x|}} e^{i(\kappa|x|-\frac{\pi}{4})} \sum_{j=0}^{n} \frac{f_j(\phi)}{|x|^j}, \quad \text{where } \theta = \frac{x}{|x|} = (cos(\phi), sin(\phi)), \tag{22}$$

$$I_n(x) = \sqrt{|x|} Im(\psi(x)), \tag{23}$$

$$J_n(x) = |x|^{n+1}(I(x) - I_n(x)), \tag{24}$$

where x is as in (9), $I(x)$ is defined by (10).

We have that:

$$2iI(x) = 2I_n(x) + \sqrt{\frac{2}{\pi\kappa}} \frac{e^{i(\kappa|x|-\frac{\pi}{4})}}{|x|^{n+1}} f_{n+1}(\phi) - \sqrt{\frac{2}{\pi\kappa}} \frac{e^{-i(\kappa|x|-\frac{\pi}{4})}}{|x|^{n+1}} \overline{f_{n+1}(\phi)} \tag{25}$$
$$+ O\big(\frac{1}{|x|^{n+2}}\big)$$

$$2iJ_n(x) = \sqrt{\frac{2}{\pi\kappa}} (e^{i(\kappa|x|-\frac{\pi}{4})} f_{n+1}(\phi) - e^{-i(\kappa|x|-\frac{\pi}{4})} \overline{f_{n+1}(\phi)}) + O\big(\frac{1}{|x|}\big), \tag{26}$$

as $|x| \to +\infty$, uniformly in $\theta = x/|x|$, where I is defined by (10).

Due to (26) and Remark 1, as $|x| \to +\infty$ we get:

$$f_{n+1}(\phi) = \frac{1}{sin(\kappa\tau)}\sqrt{\frac{\pi\kappa}{2}}\big(J_n(x)e^{-i(\kappa|y|-\frac{\pi}{4})} - J_n(y)e^{-i(\kappa|x|-\frac{\pi}{4})} + O(\frac{1}{|x|})\big), \quad (27)$$

$$x, y \in L^+ \subseteq L^+_{0,\theta}, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^1, \quad \tau > 0,$$

uniformly in θ, assuming that $sin(\kappa\tau) \neq 0$ for fixed τ (where the parameter τ can always be fixed in such a way for fixed $\kappa > 0$).

Formulas (10), (22)-(24) and (27) determine $f_{n+1}(\phi)$, give the step of induction for finding all $f_j(\phi)$ from $\mathrm{Im}(\psi)$ on $L^+_{x_1,\theta}$. The determination of $f_j(\phi + \pi)$ from $\mathrm{Im}(\psi)$ on $L^-_{x_2,\theta}$ is completely similar.

This completes the proof of Theorem 1 for the case $L^+_{x_1,\theta} \cup L^-_{x_2,\theta} \subseteq L_{0,\theta}$.

4.2 General Case

In fact, in a similar way with [21], [25], and [16], the general case reduces to the case of Subsection 4.1 by the change of variables

$$x' = x - q \quad (28)$$

$$\text{for some fixed} \quad q \in \mathbb{R}^2 \quad \text{such that} \quad L^+ \subseteq L^+_{q,\theta}, \quad L^- \subseteq L^-_{q,\theta}.$$

In the new variables x', where $x' \in \mathcal{U}' = \mathcal{U} - q$ we have that:

$$L^+ \subseteq L^+_{q,\theta} = L^+_{0,\theta} \qquad L^- \subseteq L^-_{q,\theta} = L^-_{0,\theta}. \quad (29)$$

In addition,

$$\psi(x') = \sqrt{\frac{2}{\pi\kappa|x'|}}e^{i(\kappa|x'|-\frac{\pi}{4})}\sum_{j=0}^{\infty}\frac{f'_j(\phi)}{|x'|^j} \quad \text{for} \quad x' \in \mathbb{R}^2 \setminus B_{r'}, \quad (30)$$

for some new f'_j, where r' is such that $\mathbb{R}^2 \setminus B_{r'} \subset \mathcal{U}'$.
We also have

$$\psi(x') = H_0(\kappa|x'|)\sum_{j=0}^{\infty}\frac{F'_j(\phi)}{|x'|^j} + H_1(\kappa|x'|)\sum_{j=0}^{\infty}\frac{G'_j(\phi)}{|x'|^j}, \quad (31)$$

where F'_j and G'_j are appropriate new coefficients (in accordance with subsection 6.2 of [16]), and the series converges absolutely and uniformly in $|x| > r'$.

In view of (30), (31), we complete the proof of Theorem 1 by repeating the proof of Subsection 4.1.

Remark 2. *Our proof of Theorem 1 has holographic prototypes in [16]. Additional formulas for finding f_j can also be obtained using approaches of [20], [23], [24].*

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