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L. Bertini, D. Gabrielli, Claudio Landim

METASTABLE Γ-EXPANSION OF FINITE STATE MARKOV CHAINS LEVEL TWO LARGE DEVIATIONS RATE FUNCTIONS.

L. BERTINI, D. GABRIELLI, AND C. LANDIM

Abstract. We examine two analytical characterisation of the metastable behavior of a Markov chain. The first one expressed in terms of its transition probabilities, and the second one in terms of its large deviations rate functional.

Consider a sequence of continuous-time Markov chains $(X_t^{(n)} : t \geq 0)$ evolving on a fixed finite state space V. Under a hypothesis on the jump rates, we prove the existence of times-scales $\theta_n^{(p)}$ and probability measures with disjoint supports $\pi_j^{(p)}$, $j \in S_p$, $1 \leq p \leq \mathfrak{q}$, such that (a) $\theta_n^{(1)} \to \infty$, $\theta_n^{(k+1)}/\theta_n^{(k)} \to \infty$, (b) for all p, $x \in V$, $t > 0$, starting from x, the distribution of $X^{(n)}_{t\theta_n^{(p)}}$ converges, as $n \to \infty$, to a convex combination of the probability measures $\pi_j^{(p)}$. The weights of the convex combination naturally depend on x and t.

Let $\mathcal{I}_n$ be the level two large deviations rate functional for $X_t^{(n)}$, as $t \to \infty$. Under the same hypothesis on the jump rates and assuming, furthermore, that the process is reversible, we prove that $\mathcal{I}_n$ can be written as $\mathcal{I}_n = \mathcal{I}^{(0)} + \sum_{1 \leq p \leq \mathfrak{q}} (1/\theta_n^{(p)})\, \mathcal{I}^{(p)}$ for some rate functionals $\mathcal{I}^{(p)}$ which take finite values only at convex combinations of the measures $\pi_j^{(p)}$: $\mathcal{I}^{(p)}(\mu) < \infty$ if, and only if, $\mu = \sum_{j \in S_p} \omega_j\, \pi_j^{(p)}$ for some probability measure ω in S_p.

1. Introduction

The metastable behavior of continuous-time Markov chains has attracted some interest in recent years. We refer to the monographs [52, 13, 32, 37] for the latest developments. In this article, we propose to investigate the Markov chains metastable behaviour from an analytical perpective, by showing that the Markov chains semigroup and large deviations rate function encode the metastable properties of the process. The main results explain how to extract from these functionals the metastable time-scales, states and wells.

To tackle this problem we consider a sequence of continuous-time Markov chains $(X_t^{(n)} : t \geq 0)$ evolving on a finite state space V. Under a natural hypothesis on the jump rates of these chains, stated in equation (2.4) below, we prove the existence of

(a) time-scales $\theta_n^{(1)}, \dots, \theta_n^{(\mathfrak{q})}$ such that, as $n \to \infty$, $\theta_n^{(1)} \to \infty$, $\theta_n^{(p+1)}/\theta_n^{(p)} \to \infty$ for $1 \leq p < \mathfrak{q}$;

(b) and metastable states $\pi_1^{(p)}, \dots, \pi_{\mathfrak{n}_p}^{(p)}$, $1 \leq p \leq \mathfrak{q}$.

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The parameter p is called the level and indicates the depth of the wells or, equivalently, the time-scale at which a metastable behaviour is observed. The metastable states $\pi_j^{(p)}$ are probability measures on V. It will be shown that, for each fixed level p, the support of the measures $\pi_1^{(p)}, \dots, \pi_{\mathfrak{n}_p}^{(p)}$ are disjoint. They represent the wells among which the process $X_t^{(n)}$ evolves in the time-scale $\theta_n^{(p)}$. The number of metastable set decreases as the time-scales increase: $\mathfrak{n}_{p+1} < \mathfrak{n}_p$. A metastable state at level $p+1$ is a convex combination of metastable states at level p: for each $1 \le p < \mathfrak{q}$ and $1 \le m \le \mathfrak{n}_{p+1}$, $\pi_m^{(p+1)} = \sum_j \omega_j^{(m)} \pi_j^{(p)}$ for some probability measure $\omega^{(m)}$ on $\{1, \dots, \mathfrak{n}_p\}$.

The first main result of this article, Theorem 3.1.(b), states that for all $t > 0$, $x \in V$, the distribution of $X^{(n)}_{t\theta_n^{(p)}}$ starting from x converges to a convex combination of the measures $\pi_j^{(p)}$, $1 \le j \le \mathfrak{n}_p$. More precisely, denote by $p_t^{(n)}(x,y)$ the transition probabilities of the Markov chain $X_t^{(n)}$. Then, for each $1 \le p \le \mathfrak{q}$, $t > 0$, $x \in V$, there exists a probability measure $\omega_{t,x}^{(p)}(\,\cdot\,)$ on $\{1, \dots, \mathfrak{n}_p\}$ such that

$$\lim_{n\to\infty} p^{(n)}_{t\theta_n^{(p)}}(x, \,\cdot\,) \;=\; \sum_{j=1}^{\mathfrak{n}_p} \omega_{t,x}^{(p)}(j)\, \pi_j^{(p)}(\,\cdot\,)\,. \tag{1.1}$$

The weights $\omega_{t,x}^{(p)}(j)$ of this convex combination naturally depend on x and t, and are obtained by a recursion procedure.

Theorem 3.1 also characerises the asymptotic behavior of the transition probabilities at all intermediate time-scales β_n. Fix $0 \le p \le \mathfrak{q}$, set $\theta_n^{(0)} = 1$, $\theta_n^{(\mathfrak{q}+1)} = +\infty$, and consider a sequence β_n such that $\beta_n/\theta_n^{(p)} \to \infty$, $\beta_n/\theta_n^{(p+1)} \to 0$. Theorem 3.1 provides a formula for the limit of $p^{(n)}_{\beta_n}(x, \,\cdot\,)$ as $n \to \infty$. It corresponds to the limit obtained in (1.1) by letting $t \to \infty$ after $n \to \infty$.

Freidlin and Koralov [20], after [3] and [43], examined sequences of Markov chains on finite state spaces under the same hypothesis (2.4) assumed below and taken from [3, 43]. Their main results describes the asymptotic behavior of the transition probabilities at the intermediate time-scales β_n introduced above. These results demonstrate the interest of the theory developed in [1, 5, 35, 51, 37], which permits to investigate the asymptotic behavior of the Markov chain exactly at the metastable time-scale, and not just before or after it.

We turn to the large deviations. Denote by $\mathcal{I}_n$ the level two large deviations rate functional of the Markov chain $X_t^{(n)}$, as $t \to \infty$ [56]. Under the hypothesis of reversibility, the second main result of this article provides a Γ-expansion of the functional $\mathcal{I}_n$ as

$$\mathcal{I}_n \;=\; \mathcal{I}^{(0)} \;+\; \sum_{p=1}^{\mathfrak{q}} \frac{1}{\theta_n^{(p)}}\, \mathcal{I}^{(p)}\,. \tag{1.2}$$

This expansion has to be understood in the sense that $\mathcal{I}_n$, $\theta_n^{(p)}\, \mathcal{I}_n$, $1 \le p \le \mathfrak{q}$, Γ-converge to $\mathcal{I}^{(0)}$, $\mathcal{I}^{(p)}$, respectively. The rate functionals $\mathcal{I}^{(p)}$ take finite values only at convex combinations of the metastable states $\pi_j^{(p)}$: $\mathcal{I}^{(p)}(\mu) < \infty$ if, and only if, $\mu = \sum_{j\in S_p} \omega_j\, \pi_j^{(p)}$ for some probability measure ω in S_p.

Therefore, both the semigroup and the level two large deviations rate functionals encode all characteristics of the metastable behaviour of a Markov chain. They provide the time-scales, the metastable states and the wells. In particular, it becomes a natural problem to prove such an expansion in other contexts.

We believe that the inductive approach presented here provides a general method to derive these results, as well as the metastable behavior in the classical sense [1], of Markov chains with wells of different depths, even if the state space is not fixed, as assumed here. To be applied, one needs (a) to show that the process quickly reaches one of the wells (the initial step of the induction procedure) and (b) to compute the capacities (2.7) and the asymptotic jump rates (2.10).

More precisely, inspecting the proof of Theorem 3.3 reveals that it essentially relies on the convergences of the generator of the trace process on the wells (more exactly on the convergence of the average rates $r_n^{(p)}(i,j)$ introduced in (2.9) below). Since this convergence has been obtained in many different contexts, by following the strategy proposed here it should be possible to derive the metastable Γ-expansion of the large deviations level two rate function for dynamics in which the state space is not fixed.

This includes random walks in potential fields [39, 41], condensing zero-range models [2, 31, 54], inclusion processes [25, 17, 9, 27, 28], or statistical mechanical models in which the volume grows as the temperature decreases. For example, the Curie-Weiss model in random environment [11, 10], the Blume-Capel model [34], the Potts model [40, 30, 44], or the Kawasaki dynamics for the Ising model [24].

In particular, it should be possible to apply this approach to non-reversible diffusions in potential fields,[55, 12, 38, 42, 53, 45, 46], extending Di Gesù and Mariani [23], who prove the Γ-expansion in the reversible case in which there is only one well at each different depth.

2. The model

Let $G = (V, E)$ be a finite directed graph, where V represents the finite set of vertices, and E the set of directed edges. Denote by $(X_t^{(n)} : t \geq 0)$, $n \geq 1$, a sequence of V-valued, irreducible continuous-time Markov chains, whose jump rates are represented by $R_n(x,y)$. We assume that $R_n(x,y) > 0$ for all $(x,y) \in E$ and $n \geq 1$. The generator reads as

$$(\mathcal{L}_n f)(x) \;=\; \sum_{y\colon (x,y)\in E} R_n(x,y)\,\{\, f(y) \,-\, f(x)\,\}\,.$$

Denote by $\lambda_n(x)$, $x \in V$, the holding rates of the Markov chain $X_t^{(n)}$ and by $p_n(x,y)$, $x, y \in V$, the jump probabilities, so that $R_n(x,y) = \lambda_n(x)\, p_n(x,y)$.

Let π_n stand for the unique stationary state. The so-called Matrix tree Theorem [21, Lemma 6.3.1] provides a representation of the measure π_n in terms of arborescences of the graph (V, E).

Denote by $D(\mathbb{R}_+, W)$, W a finite set, the space of right-continuous functions $\mathfrak{x} : \mathbb{R}_+ \to W$ with left-limits endowed with the Skorohod topology and the associated Borel σ-algebra. Let $\mathbf{P}_x = \mathbf{P}_x^n$, $x \in V$, be the probability measure on the path space $D(\mathbb{R}_+, V)$ induced by the Markov chain $X_t^{(n)}$ starting from x. Expectation with respect to $\mathbf{P}_x$ is represented by $\mathbf{E}_x$.

Denote by $p_t^{(n)}(x,y)$ the transition probability of the Markov chain $X_t^{(n)}$:

$$p_t^{(n)}(x,y) \,:=\, \mathbf{P}_x^n\big[\, X_t \,=\, y\,\big]\,, \quad x\,,\, y\in V\,,\;\; t\,>\,0\,.$$

Since the chain is irreducible and π_n its stationary state, by the ergodic theorem for finite state-spaces Markov chains,

$$\lim_{t\to\infty} p_t^{(n)}(x,y) \;=\; \pi_n(y) \;\text{ for all } x,\, y\in V\,.$$

Longer time-scales. Assume that $\lim_n R_n(x,y)$ exists for all $(x,y)\in E$, and denote by $\mathbb{R}_0(x,y)\in[0,\infty)$ its limit:

$$\mathbb{R}_0(x,y) \,:=\, \lim_n R_n(x,y)\,, \quad (x,y)\,\in\, E\,. \tag{2.1}$$

Let $\mathbb{E}_0$ be the set of edges whose asymptotic rate is positive: $\mathbb{E}_0 \,:=\, \{\,(x,y)\,\in\, E\,:\,\mathbb{R}_0(x,y)\,>\,0\,\}$, and assume that $\mathbb{E}_0\neq\varnothing$. The jump rates $\mathbb{R}_0(x,y)$ induce a continuous-time Markov chain on V, denoted by $(\mathbb{X}_t : t\geq 0)$, which, of course, may be reducible. Denote by $\mathbb{L}^{(0)}$ its generator.

Denote by $\mathcal{V}_1,\dots,\mathcal{V}_{\mathfrak{n}}$, $\mathfrak{n}\geq 1$, the closed irreducible classes of $\mathbb{X}_t$, and let

$$S \,:=\, \{1,\dots,\mathfrak{n}\}\,, \quad \mathcal{V} \,:=\, \bigcup_{j\in S}\mathcal{V}_j\,, \quad \Delta \,:=\, V\,\setminus\,\mathcal{V}\,. \tag{2.2}$$

The set Δ may be empty and some of the sets $\mathcal{V}_j$ may be singletons.

Let $\mathbb{Q}_x$ be the probability measure on $D(\mathbb{R}_+,V)$ induced by the Markov chain $\mathbb{X}_t$ starting from x.

For two sequences of positive real numbers $(\alpha_n : n\geq 1)$, $(\beta_n : n\geq 1)$, $\alpha_n\prec\beta_n$ or $\beta_n\succ\alpha_n$ means that $\lim_{n\to\infty}\alpha_n/\beta_n = 0$. Similarly, $\alpha_n\preceq\beta_n$ or $\beta_n\succeq\alpha_n$ indicates that either $\alpha_n\prec\beta_n$ or α_n/β_n converges to a positive real number $a\in(0,\infty)$.

Let

$$\gamma_n \;:=\; \max_{(x,y)\in E\setminus\mathbb{E}_0} R_n(x,y)$$

so that $\gamma_n\prec 1$. Choose a sequence β_n such that $1\prec\beta_n\prec\gamma_n^{-1}$. Couple $X_t^{(n)}$ and $\mathbb{X}_t$ making them jump as much as possible together. Denote by $\widehat{\mathbf{P}}_x$ the coupling measure. Since $\beta_n\prec\gamma_n^{-1}$, for all $x\in V$

$$\lim_{n\to\infty}\widehat{\mathbf{P}}_x\,\big[\,\mathbf{X}_t^{(n)} \,=\, \mathbb{X}_t\,,0\leq t\leq\beta_n\,\big] \;=\; 1\,.$$

In particular, for all $x,\, y\in V$

$$\lim_{n\to\infty} p_{\beta_n}^{(n)}(x,y) \;=\; \lim_{n\to\infty}\widehat{\mathbf{P}}_x\,\big[\,\mathbb{X}_{\beta_n} \,=\, y\,\big] \;=\; \sum_{j\in S}\mathfrak{a}^{(0)}(x,j)\,\pi_j^\sharp(y)\,,$$

where $\pi_j^\sharp$, $1\leq j\leq\mathfrak{n}$, represents the stationary states of the Markov chain $\mathbb{X}$ restricted to $\mathcal{V}_j$ and $\mathfrak{a}^{(0)}(x,j)$ the probability that the chain $\mathbb{X}_t$ starting from x is absorbed by the closed recurrent class $\mathcal{V}_j$:

$$\mathfrak{a}^{(0)}(x,j) \,:=\, \lim_{t\to\infty}\mathbb{Q}_x\,\big[\,\mathbb{X}_t\,\in\,\mathcal{V}_j\,\big]\,. \tag{2.3}$$

In the first part of this article, we investigate the asymptotic behaviour of $p_{\beta_n}^{(n)}(x,y)$ in different time-scales β_n. The definition of the time-scales and the description of the asymptotic behaviour is based on a construction of a tree [3, 43] presented after the statement of the main hypothesis of the article.

The main assumption. Two sequences of positive real numbers $(\alpha_n : n \geq 1)$, $(\beta_n : n \geq 1)$ are said to be *comparable* if $\alpha_n \prec \beta_n$, $\beta_n \prec \alpha_n$ or $\alpha_n/\beta_n \to a \in (0, \infty)$. This condition excludes the possibility that the sequence α_n/β_n oscillates between two finite values and does not converge.

A set of sequences $(\alpha_n^{\mathfrak{u}} : n \geq 1)$, $\mathfrak{u} \in \mathfrak{R}$, of positive real numbers, indexed by some finite set $\mathfrak{R}$, is said to be comparable if for all $\mathfrak{u}$, $\mathfrak{v} \in \mathfrak{R}$ the sequence $(\alpha_n^{\mathfrak{u}} : n \geq 1)$, $(\alpha_n^{\mathfrak{v}} : n \geq 1)$ are comparable.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and let Σ_m, $m \geq 1$, be the set of functions $k : E \to \mathbb{Z}_+$ such that $\sum_{(x,y)\in E} k(x, y) = m$. We assume throughout this article that for every $m \geq 1$ the set of sequences

$$\Big(\prod_{(x,y)\in E} R_n(x, y)^{k(x,y)} : n \geq 1 \Big) , \quad k \in \Sigma_m , \tag{2.4}$$

is comparable.

Remark 2.1. *This hypothesis on the jump rates is taken from* [3] *and* [43]*. It also appears in* [20]*, what supports the assertions that this condition is natural in the context of metastability.*

As observed in [3]*, assumption* (2.4) *is fulfilled by all statistical mechanics models which evolve on a fixed state space and whose metastable behaviour has been derived. This includes the Ising model* [49, 50, 7, 15]*, the Potts model with or without a small external field* [48, 29]*, the Blume-Capel model* [18, 33]*, and conservative Kawasaki dynamics* [14, 22, 26, 6]*.*

A rooted tree. In this subsection, we present the construction, proposed in [3, 43], of a rooted tree which describes all different metastable behaviours of the Markov chain $X_t^{(n)}$. This construction plays a fundamental role in the statement of the main theorems of this article. The reader will find at the end of this section a simple example which may help to understand the construction.

The tree satisfies the following conditions:

(a) Each vertex of the tree represents a subset of V;
(b) Each generation forms a partition of V;
(c) The children of each vertex form a partition of the parent.
(d) The generation $p + 1$ is strictly coarser than the generation p.

The tree is constructed by induction starting from the leaves to the root. It corresponds to a deterministic coalescence process. Denote by $\mathfrak{q}$ the number of steps in the recursive construction of the tree. At each level $1 \leq p \leq \mathfrak{q}$, the procedure generates a partition $\{\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}, \Delta_p\}$, a time-scale $\theta_n^{(p)}$ and a $\{1, \dots, \mathfrak{n}_p\}$-valued continuous-time Markov chains $\mathbb{X}_t^{(p)}$ which describes the evolution of the chain $X_{t\theta_n^{(p)}}^{(n)}$ among the subsets $\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}$, called hereafter *wells*.

The leaves are the sets $\mathcal{V}_1, \dots, \mathcal{V}_{\mathfrak{n}}, \Delta$ introduced in (2.2). We proceed by induction. Let $S_1 = S$, $\mathfrak{n}_1 = \mathfrak{n}$, $\mathcal{V}_j^{(1)} = \mathcal{V}_j$, $j \in S_1$, $\Delta_1 = \Delta$, and assume that the recursion has produced the sets $\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}, \Delta_p$ for some $p \geq 1$, which forms a partition of V.

Denote by $H_{\mathcal{A}}$, $H_{\mathcal{A}}^+$, $\mathcal{A} \subset V$, the hitting and return time of $\mathcal{A}$:

$$H_{\mathcal{A}} := \inf \big\{ t > 0 : X_t^{(n)} \in \mathcal{A} \big\} , \quad H_{\mathcal{A}}^+ := \inf \big\{ t > \tau_1 : X_t^{(n)} \in \mathcal{A} \big\} , \tag{2.5}$$

where τ_1 represents the time of the first jump of the chain $X_t^{(n)}$: $\tau_1 = \inf\{t > 0 : X_t^{(n)} \neq X_0^{(n)}\}$.

For two non-empty, disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of V, denote by $\mathrm{cap}_n(\mathcal{A},\mathcal{B})$ the capacity between $\mathcal{A}$ and $\mathcal{B}$:

$$\mathrm{cap}_n(\mathcal{A},\mathcal{B}) \;:=\; \sum_{x\in\mathcal{A}} \pi_n(x)\,\lambda_n(x)\,\mathbf{P}_x^n\big[\,H_{\mathcal{B}} < H_{\mathcal{A}}^+\,\big]\;. \tag{2.6}$$

Set $S_p = \{1,\dots,\mathfrak{n}_p\}$, and let $\theta_n^{(p)}$ be defined by

$$\frac{1}{\theta_n^{(p)}} \;:=\; \sum_{i\in S_p} \frac{\mathrm{cap}_n(\mathcal{V}_i^{(p)},\breve{\mathcal{V}}_i^{(p)})}{\pi_n(\mathcal{V}_i^{(p)})}\;, \quad \text{where } \breve{\mathcal{V}}_i^{(p)} \;:=\; \bigcup_{j\in S_p\setminus\{i\}} \mathcal{V}_j^{(p)}\;. \tag{2.7}$$

The ratio $\pi_n(\mathcal{V}_i^{(p)})/\mathrm{cap}_n(\mathcal{V}_i^{(p)},\breve{\mathcal{V}}_i^{(p)})$ represents the time it takes for the chain $X_t^{(n)}$, starting from a point in $\mathcal{V}_i^{(p)}$ to reach the set $\breve{\mathcal{V}}_i^{(p)}$. Therefore, $\theta_n^{(p)}$ corresponds to the smallest time needed to observe such a jump.

Recall from (A.1) the definition of the trace of a Markov chain. Denote by $\{Y_t^{n,p} : t\ge 0\}$ the trace of $\{X_t^{(n)} : t\ge 0\}$ on $\mathcal{V}^{(p)}$, and by $R_n^{(p)} : \mathcal{V}^{(p)}\times\mathcal{V}^{(p)} \to \mathbb{R}_+$ its jump rates. By equation (2.5) in [32],

$$R_n^{(p)}(x,y) \;=\; \lambda_n(x)\,\mathbf{P}_x^n\big[H_y = H_{\mathcal{V}^{(p)}}^+\big]\;,\quad x\,,\ y\in\mathcal{V}^{(p)}\,,\ x\neq y\;. \tag{2.8}$$

Denote by $r_n^{(p)}(i,j)$ the mean rate at which the trace process jumps from $\mathcal{V}_i^{(p)}$ to $\mathcal{V}_j^{(p)}$:

$$r_n^{(p)}(i,j) \;:=\; \frac{1}{\pi_n(\mathcal{V}_i^{(p)})} \sum_{x\in\mathcal{V}_i^{(p)}} \pi_n(x) \sum_{y\in\mathcal{V}_j^{(p)}} R_n^{(p)}(x,y)\;. \tag{2.9}$$

Under the assumption (2.4), [43] proved that the sequences $\theta_n^{(p)}\, r_n^{(p)}(i,j)$ converge for all $i\neq j\in S_p$. Denote the limits by $r^{(p)}(i,j)$:

$$r^{(p)}(i,j) \;:=\; \lim_{n\to\infty} \theta_n^{(p)}\, r_n^{(p)}(i,j) \;\in\; \mathbb{R}_+\;. \tag{2.10}$$

Denote by $(\mathbb{X}_t^{(p)} : t\ge 0)$ the S_p-valued continuous-time Markov chain induced by the jump rates $r^{(p)}(j,k)$, and by $\mathbb{L}^{(p)}$ its generator. Let $\Phi_p : \mathcal{V}^{(p)} \to S_p$ be the projection which sends the points in $\mathcal{V}_j^{(p)}$ to j:

$$\Phi_p \;:=\; \sum_{k\in S_p} k\,\chi_{\mathcal{V}_k^{(p)}}\;.$$

In this formula and below, $\chi_{\mathcal{A}}$ stands for the indicator function of the set $\mathcal{A}$.

Next theorem is the main result in [43].

Theorem 2.2. *Assume that condition* (2.4) *is in force. Then, for each* $1\le p\le \mathfrak{q}$, $j\in S_p$, $x\in\mathcal{V}_j^{(p)}$, *under the measure* $\mathbf{P}_x^n$, *the sequence of* S_p*-valued, hidden Markov processes* $\Phi_p(X^{(n)}_{t\theta_n^{(p)}})$ *converges weakly in the Skorohod topology to* $\mathbb{X}_t^{(p)}$. *Moreover, the time spent in* Δ_p *is negligible in the sense that for all* $t>0$,

$$\lim_{n\to\infty} \max_{y\in\mathcal{V}^{(p)}} \mathbf{E}_x^n\Big[\int_0^t \chi_{\Delta_p}(X^{(n)}_{s\theta_n^{(p)}})\,ds\Big] \;=\; 0\;.$$

The process $\mathbb{X}_t^{(p)}$ describes therefore how the chain $X_t^{(n)}$ evolves among the wells $\mathcal{V}_j^{(p)}$ in the time-scale $\theta_n^{(p)}$. Let $p_t^{(p)}(i,j)$ be the transition probabilities:

$$p_t^{(p)}(i,j) \;=\; \mathbb{Q}_i^{(p)}[\, X_t = j\,]\,, \quad t \,\geq\, 0\,, i\,,\, j \in S_p\,, \tag{2.11}$$

where $\mathbb{Q}_i^{(p)}$ stands for the probability measure on the path space $D(\mathbb{R}_+, S_p)$ induced by the Markov chain $\mathbb{X}_t^{(p)}$ starting from i.

By [43, Theorem 2.7], there exists $j,\, k \in S$ such that $r^{(p)}(j,k) > 0$. Actually, by the proof of this result

$$\sum_{k\neq j} r^{(p)}(j,k) > 0 \text{ for all } j \in S_p \text{ such that } \quad \lim_{n\to\infty} \theta_n^{(p)}\, \frac{\mathrm{cap}_n(\mathcal{V}_j^{(p)}, \breve{\mathcal{V}}_j^{(p)})}{\pi_n(\mathcal{V}_j^{(p)})} > 0\,. \tag{2.12}$$

Denote by $\mathfrak{R}_1^{(p)}, \dots, \mathfrak{R}_{\mathfrak{n}_{p+1}}^{(p)}$ the recurrent classes of the S_p-valued chain $\mathbb{X}_t^{(p)}$, and by $\mathfrak{T}_p$ the transient states. Let $\mathfrak{R}^{(p)} = \cup_j \mathfrak{R}_j^{(p)}$, and observe that $\{\mathfrak{R}_1^{(p)}, \dots, \mathfrak{R}_{\mathfrak{n}_{p+1}}^{(p)}, \mathfrak{T}_p\}$ forms a partition of the set S_p. This partition of S_p induces a new partition of the set V. Let

$$\mathcal{V}_m^{(p+1)} \;:=\; \bigcup_{j\in\mathfrak{R}_m^{(p)}} \mathcal{V}_j^{(p)}\,, \quad \mathcal{T}^{(p+1)} \;:=\; \bigcup_{j\in\mathfrak{T}_p} \mathcal{V}_j^{(p)}\,, \quad m \in S_{p+1} \;:=\; \{1, \dots, \mathfrak{n}_{p+1}\}\,,$$

so that $V \;=\; \Delta_{p+1} \,\cup\, \mathcal{V}^{(p+1)}$, where

$$\mathcal{V}^{(p+1)} \;=\; \bigcup_{m\in S_{p+1}} \mathcal{V}_m^{(p+1)}\,, \quad \Delta_{p+1} \;:=\; \Delta_p \,\cup\, \mathcal{T}^{(p+1)}\,. \tag{2.13}$$

The subsets $\mathcal{V}_1^{(p+1)}, \dots, \mathcal{V}_{\mathfrak{n}_{p+1}}^{(p+1)}, \Delta_{p+1}$ of V are the result of the recursive procedure. We claim that conditions (a)–(d) hold at step $p+1$ if they are fulfilled up to step p in the induction argument.

The sets $\mathcal{V}_1^{(p+1)}, \dots, \mathcal{V}_{\mathfrak{n}_{p+1}}^{(p+1)}$, Δ_{p+1} constitute a partition of V because the sets $\mathfrak{R}_1^{(p)}, \dots, \mathfrak{R}_{\mathfrak{n}_{p+1}}^{(p)}$, $\mathfrak{T}_p$ form a partition of S_p, and the sets $\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}$, Δ_p one of V. Conditions (a)–(c) are therefore satisfied.

To show that the partition obtained at step $p+1$ is strictly coarser than $\{\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}, \Delta_p\}$, observe that, by (2.12), $r^{(p)}(j,k) > 0$ for some $k \neq j \in S_p$. Hence, either j is a transient state for the process $\mathbb{X}_t^{(p)}$ or the closed recurrent class which contains j also contains k. In the first case $\Delta_p \subsetneq \Delta_{p+1}$, and in the second one there exists $m \in S_{p+1}$ such that $\mathcal{V}_j^{(p)} \cup \mathcal{V}_k^{(p)} \subset \mathcal{V}_m^{(p+1)}$. Therefore, the new partition $\{\mathcal{V}_1^{(p+1)}, \dots, \mathcal{V}_{\mathfrak{n}_{p+1}}^{(p+1)}, \Delta_{p+1}\}$ of V satisfies the conditions (d).

The construction terminates when the S_p-valued Markov chain $\mathbb{X}_t^{(p)}$ has only one recurrent class so that $\mathfrak{n}_{p+1} = 1$. In this situation, the partition at step $p+1$ is $\mathcal{V}_1^{(p+1)}$, Δ_{p+1}.

This completes the construction of the rooted tree. Recall that we denote by $\mathfrak{q}$ the number of steps of the scheme. As claimed at the beginning of the procedure, for each $1 \leq p \leq \mathfrak{q}$, we generated a time-scale $\theta_n^{(p)}$, a partition $\mathcal{P}_p = \{\mathcal{V}_1^{(p)}, \dots, \mathcal{V}_{\mathfrak{n}_p}^{(p)}, \Delta_p\}$, where $\mathcal{P}_1 = \{\mathcal{V}_1, \dots, \mathcal{V}_{\mathfrak{n}}, \Delta\}$, $\mathcal{P}_{\mathfrak{q}+1} = \{\mathcal{V}_1^{(\mathfrak{q}+1)}, \Delta_{\mathfrak{q}+1}\}$, and a S_p-valued continuous-time Markov chain $\mathbb{X}_t^{(p)}$.

Furthermore, by construction,

$$\Delta_p \;\subset\; \Delta_{p+1}\,, \quad 1 \,\leq\, p \,\leq\, \mathfrak{q}\,, \tag{2.14}$$

by [43, Assertion 8.B],

$$\theta_n^{(p)} \prec \theta_n^{(p+1)}\,, \quad 1 \le p < \mathfrak{q}\,, \tag{2.15}$$

and by [43, Assertion 8.A] or equation (8.2) of this article,

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})} \text{ exists and belongs to } (0,1] \tag{2.16}$$

for all $1 \le p \le \mathfrak{q}+1$, $j \in S_p$, $x \in \mathcal{V}_j^{(p)}$.

The partitions $\mathcal{P}_1, \dots, \mathcal{P}_{\mathfrak{q}+1}$ form a rooted tree whose root (0-th generation) is V, first generation is $\{\mathcal{V}_1^{(\mathfrak{q}+1)}, \Delta_{\mathfrak{q}+1}\}$ and last ($(\mathfrak{q}+1)$-th) generation is $\{\mathcal{V}_1, \dots, \mathcal{V}_{\mathfrak{n}}, \Delta\}$. Note that the set $\mathcal{V}^{(p+1)}$ corresponds to the set of recurrent points for the chain $\mathbb{X}_t^{(p)}$. In contrast, the points in Δ_{p+1} are either transient for this chain or negligible in the sense that the chain $X_t^{(n)}$ remains a negligible amount of time on the set Δ_p in the time-scale $\theta_n^{(p)}$ (cf. [3, 43]).

Example. We conclude this section with an example to help the reader understanding the tree's construction. Let $V = \{0, \dots, 29\}$, and consider the energy $\mathbb{H} : V \to \{0, \dots, 4\}$ given in Figure 1. Note that $\mathbb{H}(k+1) - \mathbb{H}(k) = \pm 1$ for $0 \le k < 29$. The energy $\mathbb{H}$ has 9 local minima, represented in Figure 1 by $x_1, \dots x_9$.

Consider the V-valued continuous-time Markov chain $X_t^{(n)}$ whose jump rates are given by $R_n(k,j) = 0$ if $j \neq k \pm 1$ and $R_n(k, k\pm 1) = \exp\{-n\,[\mathbb{H}(k\pm 1) - \mathbb{H}(k)]^+\}$, where $a^+ = \max\{a, 0\}$. Hence if $\mathbb{H}(k\pm 1) - \mathbb{H}(k) = -1$ the chain jumps from k to $k \pm 1$ at rate 1, while if $\mathbb{H}(k \pm 1) - \mathbb{H}(k) = +1$ it jumps from k to $k \pm 1$ at rate e^{-n}. More simply, observing the energy landscape presented in Figure 1, the chain jumps "downwards" at rate 1 and jumps "upwards" at rate e^{-n}.

It is easy to check that the stationary state, denoted by π_n, is given by $\pi_n(k) = (1/Z_n)\exp\{-n\,\mathbb{H}(k)\}$, where Z_n is a normalising constant, and that π_n satisfies the detailed balance conditions. In particular, and since the downward jump rates are equal to 1, $c_n(j,k) := \pi_n(j)\,R_n(j,k) = \pi_n(j) \wedge \pi_n(k)$. It follows from this identity and Lemma 7.3 below that the capacities introduced in (2.6) are easy to estimate in this example.

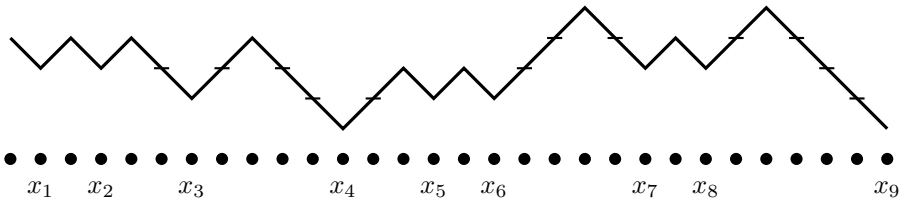

FIGURE 1. The energy landscape of the Markov chain $X_t^{(n)}$.

Consider the tree construction presented at the beginning of this section.

Step 1: the leaves. In the first step we determine the leaves of the tree, which correspond to the closed irreducible classes of the chain $\mathbb{X}_t$. In this example, the closed irreducible classes are the local minima of the energy $\mathbb{H}$ so that $\mathfrak{n} = 9$, $\mathcal{V}_j = \{x_j\}$, $1 \le j \le 9$, $\Delta = V \setminus \{x_1, \dots, x_9\}$, and the leaves are the sets Δ and $\mathcal{V}_j$, $1 \le j \le 9$.

Denote by $\mathfrak{q}+1$, $\mathfrak{q}\geq 0$, to total number of generations of the tree. The exact value of $\mathfrak{q}+1$ will only be known at the end of the construction.

Step 2: the generation $\mathfrak{q}$. The second step consists in determining the smallest transition time between a well $\mathcal{V}_j$ to a well $\mathcal{V}_k$. This is the smallest time-scale it takes for the process $X_t^{(n)}$ starting from $\mathcal{V}_j$ to hit $\mathcal{V}_k$. In the above example this time-scale is $\theta_n^{(1)}=e^n$. In this time scale, the trace of $X_t^{(n)}$ on $\mathcal{V}=\cup_j\mathcal{V}_j$ evolves as a Markov chain and converges, as $n\to\infty$, to a $\mathcal{V}$-valued Markov chain, represented by $\mathbb{X}_t^{(1)}$. The states x_1 and x_2 are transient states for $\mathbb{X}_t^{(1)}$ and absorbed at the recurrent state x_3. Similarly, the states x_5 and x_6 are transient states for $\mathbb{X}_t^{(1)}$ and are absorbed by x_4. The states x_7, x_8 form a closed irreducible class of $\mathbb{X}_t^{(1)}$, as well as the point x_9.

Therefore, $\mathfrak{T}_1=\{1,2,5,6\}$, $\mathfrak{R}_1^{(1)}=\{3\}$, $\mathfrak{R}_2^{(1)}=\{4\}$, $\mathfrak{R}_3^{(1)}=\{7,8\}$, $\mathfrak{R}_4^{(1)}=\{9\}$, so that $\mathcal{V}_1^{(2)}=\{x_3\}$, $\mathcal{V}_2^{(2)}=\{x_4\}$, $\mathcal{V}_3^{(2)}=\{x_7,x_8\}$, $\mathcal{V}_4^{(2)}=\{x_9\}$, $\mathcal{T}_2=\{x_1,x_2,x_5,x_6\}$. Moreover, the generation $\mathfrak{q}$ of the tree has 5 elements: $\Delta_2=\Delta\cup\mathcal{T}_2$, and $\mathcal{V}_j^{(2)}$, $1\leq j\leq 4$.

Step 3: the generation $\mathfrak{q}-1$. At this point, we need to determine the smallest transition time between the wells $\mathcal{V}_1^{(2)}$, $\mathcal{V}_2^{(2)}$, $\mathcal{V}_3^{(2)}$ and $\mathcal{V}_4^{(2)}$. In this example the smallest transition time is $\theta_n^{(2)}=e^{2n}$.

Let $\mathcal{V}^{(2)}=\cup_{1\leq j\leq 4}\mathcal{V}_j^{(2)}$, and denote by $Y_t^{n,2}$ the trace of the process $X_t^{(n)}$ on $\mathcal{V}^{(2)}$. Consider the projection $\Phi_2:\mathcal{V}^{(2)}\to S_2=\{1,2,3,4\}$ which sends the points in $\mathcal{V}_j^{(2)}$ to j. Note that Φ_2 is not a bijection. In consequence the process $\Phi_2(Y_t^{n,2})$ is not a Markov chain. It is however possible to prove (cf. [1]) that the process $\Phi_2(Y^{n,2}_{t\theta_n^{(2)}})$ converges to a S_2-valued Markov chain, denoted by $\mathbb{X}_t^{(2)}$.

The states 1 and 3, which corresponds to the sets $\mathcal{V}_1^{(2)}$ and $\mathcal{V}_3^{(2)}$, respectively, are transient for the chain $\mathbb{X}_t^{(2)}$, while the states 2 and 4, which corresponds to the sets $\mathcal{V}_2^{(2)}$ and $\mathcal{V}_4^{(2)}$, respectively, form closed irreducible classes. The state 1 is absorbed at 2, while the state 3 may be absorbed at 2 or 4.

Thus, in this example, $\mathfrak{T}_2=\{1,3\}$, $\mathfrak{R}_1^{(2)}=\{2\}$, $\mathfrak{R}_2^{(2)}=\{4\}$, so that $\mathcal{V}_1^{(3)}=\{x_4\}$, $\mathcal{V}_2^{(3)}=\{x_9\}$, $\mathcal{T}_3=\{x_3,x_7,x_8\}$. The generation $\mathfrak{q}-1$ of the tree has 3 elements: $\Delta_3=\Delta_2\cup\mathcal{T}_3$, and $\mathcal{V}_j^{(3)}$, $j=1,2$.

Step 4: the generation $\mathfrak{q}-2$. We need now to determine the smallest transition time between the wells $\mathcal{V}_1^{(3)}$ and $\mathcal{V}_2^{(3)}$. In this example it is $\theta_n^{(3)}=e^{3n}$.

Let $\mathcal{V}^{(3)}=\mathcal{V}_1^{(3)}\cup\mathcal{V}_2^{(3)}$, and denote by $Y_t^{n,3}$ the trace of the process $X_t^{(n)}$ on $\mathcal{V}^{(3)}$. It is however possible to prove (cf. [1]) that the process $Y^{n,3}_{t\theta_n^{(3)}}$ converges to a $\{1,2\}$-valued Markov chain, denoted by $\mathbb{X}_t^{(3)}$.

The states $\{1,2\}$ form a irreducible class for $\mathbb{X}_t^{(3)}$. Hence $\mathfrak{T}_3$ is empty and $\mathfrak{R}^{(3)}=\mathfrak{R}_1^{(3)}=\{1,2\}$, so that $\mathcal{V}_1^{(4)}=\{x_4,x_9\}$, $\mathcal{T}_4=\varnothing$. The generation $\mathfrak{q}-2$ of the tree has 2 elements: $\Delta_4=\Delta_3$, and $\mathcal{V}_1^{(4)}$.

As there is only one closed irreducible class, the construction is completed and the value of $\mathfrak{q}$ is revealed. The partition $\{\Delta_4\,,\,\mathcal{V}_1^{(4)}\}$ of V corresponds to the first generation. Since, by construction, it is also the $(\mathfrak{q}-2)$-th generation, we deduce that $\mathfrak{q}=3$ and that the tree has $\mathfrak{q}+1=4$ generations. To get a rooted tree, we declare that the root, which corresponds to the zeroth generation, is the set

$V = \Delta_4 \cup \mathcal{V}_1^{(4)}$. The tree associated to the example presented in Figure 1 is depicted in Figure 2.

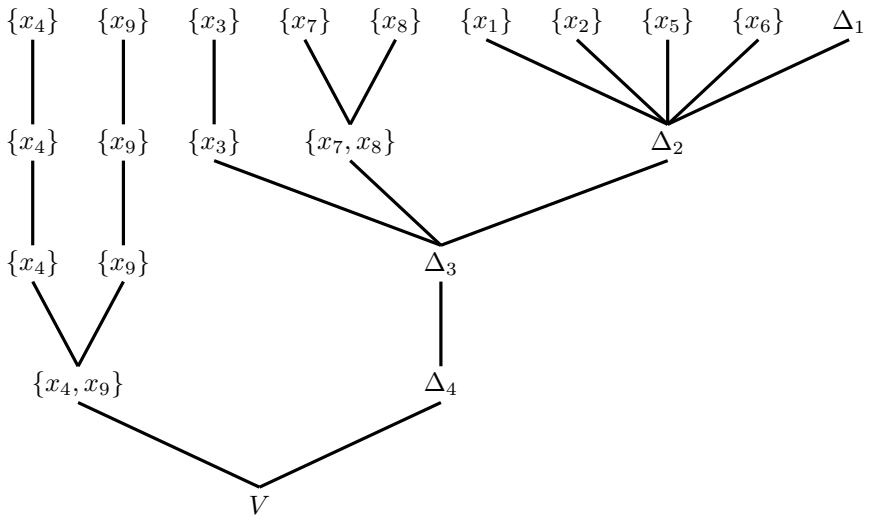

FIGURE 2. The tree or coalescence process generated by the Markov chain $X_t^{(n)}$.

3. The main results

In this section, we enunciate the main results of the article. The statements require a further layer in the tree construction presented in the previous section. At each step $1 \le p \le \mathfrak{q}+1$, we introduce a set of probability measures $\pi_j^{(p)}$, $j \in S_p$, on V. The construction of these measures is carried out below by induction. In Proposition 3.2, however, we characterise the measure $\pi_j^{(p)}$ as the limit of the stationary state π_n conditioned to $\mathcal{V}_j^{(p)}$. In particular,

$$\text{the support of } \pi_j^{(p)} \text{ is the set } \mathcal{V}_j^{(p)} \,. \tag{3.1}$$

Moreover, in Theorem 3.1.(b) we show that for all $t > 0$, $x \in V$, the distribution of $X^{(n)}_{t\theta_n^{(p)}}$ starting from x converges to a convex combination of the measures $\pi_j^{(p)}$, $j \in S_p$. The weights of this convex combination depend on x and t. This result asserts, therefore, that the measures $\pi_j^{(p)}$ are the metastable states of the process $X_t^{(n)}$ observed on the time-scale $\theta_n^{(p)}$.

We proceed by induction. Let $\pi_j^{(1)}$, $j \in S_1$, be the probability measure on $\mathcal{V}_j^{(1)}$ given by $\pi_j^{(1)} = \pi_j^\sharp$, where, recall, $\pi_j^\sharp$ represents the stationary states of the Markov chain $\mathbb{X}_t$ restricted to the closed irreducible set $\mathcal{V}_j^{(1)} = \mathcal{V}_j$. Clearly, condition (3.1) is fulfilled.

Fix $1 \le p \le \mathfrak{q}$, and assume that the probability measures $\pi_j^{(p)}$, $j \in S_p$, has been defined and satisfy condition (3.1). Denote by $M_m^{(p)}(\cdot)$, $m \in S_{p+1}$, the stationary state of the Markov chain $\mathbb{X}_t^{(p)}$ restricted to $\mathfrak{R}_m^{(p)}$. The measure $M_m^{(p)}$ is understood

as a measure on $S_p = \{1, \dots, \mathfrak{n}_p\}$ which vanishes on the complement of $\mathfrak{R}_m^{(p)}$. Let $\pi_m^{(p+1)}$ be the probability measure on $\mathcal{V}_m^{(p)}$ given by

$$\pi_m^{(p+1)}(x) \;:=\; \sum_{j \in \mathfrak{R}_m^{(p)}} M_m^{(p)}(j)\, \pi_j^{(p)}(x)\;, \quad x \in V\;. \tag{3.2}$$

Clearly, condition (3.1) is in force. Moreover, $\pi_m^{(p+1)}$ is a convex combination of the measures $\pi_j^{(p)}$, $j \in \mathfrak{R}_m^{(p)}$. A fortiori, for each $1 \le p \le \mathfrak{q}+1$, $m \in S_p$, $\pi_m^{(p)}$ is a convex combination of the measures $\pi_j^\sharp$, $j \in S$.

We further add absorption probabilities at each step. Let $\mathfrak{a}^{(0)}(x,j)$, $x \in V$, $j \in S_1$, be the probability that the Markov chain $\mathbb{X}_t$ starting from x is absorbed at the closed irreducible set $\mathcal{V}_j^{(1)}$:

$$\mathfrak{a}^{(0)}(x,j) \;:=\; \lim_{t\to\infty} \mathbb{Q}_x\big[\, \mathbb{X}_t \,\in\, \mathcal{V}_j \,\big]\;. \tag{3.3}$$

Note that $a^{(0)}(x,\,\cdot\,)$ is a probability measure on S_1 for each $x \in V$.

Fix $1 \le p \le \mathfrak{q}$ and assume that $\mathfrak{a}^{(p-1)}(x,j)$ has been defined. Let $\mathfrak{A}^{(p)}(j,m)$, $j \in S_p$, $m \in S_{p+1}$, be the probability that the chain $\mathbb{X}_t^{(p)}$ starting from j has been absorbed at the closed irreducible set $\mathfrak{R}_m^{(p)}$:

$$\mathfrak{A}^{(p)}(j,m) \;:=\; \lim_{t\to\infty} \sum_{k \in \mathfrak{R}_m^{(p)}} p_t^{(p)}(j,k)\;, \quad j \in S_p\;,\;\; m \in S_{p+1}\;. \tag{3.4}$$

For $x \in V$, $m \in S_{p+1}$, let

$$\mathfrak{a}^{(p)}(x,m) \;:=\; \sum_{j \in S_p} \mathfrak{a}^{(p-1)}(x,j)\, \mathfrak{A}^{(p)}(j,m)\;. \tag{3.5}$$

Since $\mathfrak{A}^{(p)}(j,\,\cdot\,)$ is a probability measure on S_{p+1}, it is easy to show by induction that $a^{(p)}(x,\,\cdot\,)$ is a probability measure on S_{p+1} for each $x \in V$, $1 \le p \le \mathfrak{q}$.

Let $\theta_n^{(0)} = 1$, $\theta_n^{(\mathfrak{q}+1)} = +\infty$ for all $n \ge 1$. The first main result of the article reads as follows. It provides a complete description of the ergodic behavior of the Markov chain $X_t^{(n)}$.

Theorem 3.1. *Assume that condition* (2.4) *is in force. Then,*

(a) *For each* $1 \le p \le \mathfrak{q}+1$, *sequence* $(\beta_n : n \ge 1)$ *such that* $\theta_n^{(p-1)} \prec \beta_n \prec \theta_n^{(p)}$, *and* $x \in V$,

$$\lim_{n\to\infty} p_{\beta_n}^{(n)}(x,\,\cdot\,) \;=\; \Pi_{p-1}(x,\,\cdot\,) \;:=\; \sum_{j \in S_p} \mathfrak{a}^{(p-1)}(x,j)\, \pi_j^{(p)}(\,\cdot\,)\;. \tag{3.6}$$

(b) *For each* $1 \le p \le \mathfrak{q}$, $t > 0$, $x \in V$,

$$\lim_{n\to\infty} p_{t\theta_n^{(p)}}^{(n)}(x,\,\cdot\,) \;=\; \sum_{j \in S_p} \omega_t^{(p)}(x,j)\, \pi_j^{(p)}(\,\cdot\,)\;, \tag{3.7}$$

where

$$\omega_t^{(p)}(x,j) \;=\; \sum_{k \in S_p} \mathfrak{a}^{(p-1)}(x,k)\, p_t^{(p)}(k,j)\;.$$

(c) *For all* $1 \le p \le \mathfrak{q}$, $j \in S_p$, $x \in V$,

$$\lim_{t\to 0} \lim_{n\to\infty} p_{t\theta_n^{(p)}}^{(n)}(x,\,\cdot\,) \;=\; \sum_{j \in S_p} \mathfrak{a}^{(p-1)}(x,j)\, \pi_j^{(p)}(\,\cdot\,)$$

(d) *For all* $1 \le p \le \mathfrak{q}$, $1 \le j \le \mathfrak{n}_p$, $x \in V$,

$$\lim_{t\to\infty} \lim_{n\to\infty} p^{(n)}_{t\theta^{(p)}_n}(x,\,\cdot\,) \;=\; \sum_{m\in S_{p+1}} \mathfrak{a}^{(p)}(x,m)\, \pi^{(p+1)}_m(\,\cdot\,)\,.$$

Moreover,

$$\lim_{n\to\infty} \pi_n(\Delta_{\mathfrak{q}+1}) \;=\; 0\,, \quad \lim_{n\to\infty} \pi_n(x) \;\; \textit{exists and belongs to } (0,1] \tag{3.8}$$

for all $x \in \mathcal{V}^{(\mathfrak{q}+1)}$.

Note that the right-hand side of (c) and (d) coincide with the one obtained in (a). These assertion state that at the time-scale $\theta^{(p)}_n$ a smooth transition between two different regimes is observed.

Part (b) of this theorem states that, starting from x, the distribution of the process at time $t\theta^{(p)}_n$ is close to a convex combination of the measures $\pi^{(p)}_k$, $k \in S_p$. The weight of the measure $\pi^{(p)}_k$ is given by the probability that the process is initially attracted to a well $\mathcal{V}^{(p)}_j$ times the probability that the dynamics among the wells drives the process from the well $\mathcal{V}_j$ to the well $\mathcal{V}_k$ in the "macroscopic" time intervall $[0,t]$.

The next result provides a formula for the measures $\pi^{(p)}_j$ and for the absorbing probabilities $\mathfrak{a}^{(p-1)}(x,j)$. Recall that for each $x \in V$, $\mathfrak{a}^{(p-1)}(x,\,\cdot\,)$ is a probability measure on S_p.

Proposition 3.2. *Fix* $1 \le p \le \mathfrak{q}+1$, $j \in S_p$. *For all* $z \in \mathcal{V}^{(p)}_j$,

$$\lim_{n\to\infty} \frac{\pi_n(z)}{\pi_n(\mathcal{V}^{(p)}_j)} \;=\; \pi^{(p)}_j(z)\,.$$

If $x \in \mathcal{V}^{(p)}_j$, *then* $\mathfrak{a}^{(p-1)}(x,k) = \delta_{j,k}$, $k \in S_p$. *On the other hand, if* $x \notin \mathcal{V}^{(p)}$, *then*

$$\mathfrak{a}^{(p-1)}(x,j) \;=\; \lim_{n\to\infty} \mathbf{P}^n_x\big[\, H_{\mathcal{V}^{(p)}_j} \,<\, H_{\breve{\mathcal{V}}^{(p)}_j}\,\big]\,.$$

Large deviations rate function expansion. We assume from now on that the dynamics is reversible: $\pi_n(x)\, R_n(x,y) = \pi_n(y)\, R_n(y,x)$ for all $(x,y) \in E$. For a probability measure ν on a finite space W and two functions f, $g : W \to \mathbb{R}$, let

$$\langle\, f\,,\, g\,\rangle_\nu \;=\; \sum_{x\in W} f(x)\, g(x)\, \nu(x)\,.$$

By [56], for each fixed $n \ge 1$, the occupation time distribution of the chain X^n_t, defined by

$$\frac{1}{t}\int_0^t \delta_{X^n_s}\, ds\,,$$

satisfies a large deviations principle as $t \to \infty$, the so-called level 2 LDP. In this formula, δ_x, $x \in V$, represents the Dirac measure concentrated at x, so that $t^{-1}\int_0^t \delta_{X^n_s}\, ds$ is a random element of $\mathcal{P}(V)$, the space of probability measures on V. Denote by $\mathcal{I}_n : \mathcal{P}(V) \to [0,\infty]$ the level two large deviations rate function:

$$\mathcal{I}_n(\mu) \;=\; -\inf_u \sum_{x\in V} \frac{(\mathcal{L}_n u)(x)}{u(x)}\, \mu(x)\,, \tag{3.9}$$

where the infimum is performed over all functions $u : V \to (0, \infty)$. Since we assumed reversibility and $\pi_n(x) > 0$ for all $x \in V$, for all measures $\mu \in \mathcal{P}(V)$, by [19, Theorem 5],

$$\mathcal{I}_n(\mu) \;=\; \langle\, \sqrt{f_n}\,, (-\mathcal{L}_n)\sqrt{f_n}\,\rangle_{\pi_n}\,, \tag{3.10}$$

where $f_n(x) = \mu(x)/\pi_n(x)$.

The second main result of this article provides an expansion of the rate function $\mathcal{I}_n$. Recall that we denote by $\mathbb{L}^{(0)}$ the generator of the Markov chain $\mathbb{X}_t$ introduced right after (2.1). Let $\mathcal{I}^{(0)} : \mathcal{P}(V) \to \mathbb{R}_+$ be given by

$$\mathcal{I}^{(0)}(\mu) \;=\; -\inf_{u>0} \sum_{x\in V} \mu(x) \frac{(\mathbb{L}^{(0)}u)(x)}{u(x)}\,, \tag{3.11}$$

where the supremum is carried over all functions $u : V \to (0, \infty)$. Theorem 3.3 below states that the sequence of rate functions $\mathcal{I}_n$ Γ-converges to $\mathcal{I}^{(0)}$. In (8.8), we show that $\mathcal{I}^{(0)}(\mu) = 0$ if and only if there exists a probability measure ω on S_1 such that

$$\mu \;=\; \sum_{j\in S_1} \omega_j\, \pi_j^{(1)}\,. \tag{3.12}$$

For such measures μ, it is natural to consider the limit $\beta_n \mathcal{I}_n(\mu)$ for some sequence $\beta_n \to \infty$.

Fix $1 \le p \le \mathfrak{q}$. Denote by $\mathcal{P}(S_p)$ the set of probability measures on S_p. Let $\mathcal{I}^{(p)} \colon \mathcal{P}(V) \to [0, +\infty]$ be the functional given by

$$\mathcal{I}^{(p)}(\mu) \;:=\; \begin{cases} -\inf_{\mathbf{h}} \sum_{j\in S_p} \omega_j\, \dfrac{\mathbb{L}^{(p)}\mathbf{h}}{\mathbf{h}} & \text{if } \mu = \sum_{j\in S_p} \omega_j\, \pi_j^{(p)} \text{ and } \omega \in \mathcal{P}(S_p)\,, \\ +\infty \quad \text{otherwise}\,. \end{cases} \tag{3.13}$$

In this formula, the infimum is carried over all functions $\mathbf{h} : S_p \to (0, \infty)$. We prove in (8.12) that

$$\mathcal{I}^{(p+1)}(\mu) \;<\; \infty \quad \text{if and only if} \quad \mathcal{I}^{(p)}(\mu) \;=\; 0\,.$$

By (3.12), this assertion holds also for $p = 0$.

Recall the definition of Γ-convergence. We refer to [16] for an overview on this subject. Fix a Polish space $\mathcal{X}$ and a sequence $(U_n : n \in \mathbb{N})$ of functionals on $\mathcal{X}$, $U_n \colon \mathcal{X} \to [0, +\infty]$. The sequence U_n Γ-*converges* to the functional $U \colon \mathcal{X} \to [0, +\infty]$ if and only if the two following conditions are met:

(i) Γ-*liminf.* The functional U is a Γ-liminf for the sequence U_n: For each $x \in \mathcal{X}$ and each sequence $x_n \to x$, we have that $\liminf_n U_n(x_n) \ge U(x)$.

(ii) Γ-*limsup.* The functional U is a Γ-limsup for the sequence U_n: For each $x \in \mathcal{X}$ there exists a sequence $x_n \to x$ such that $\limsup_n U_n(x_n) \le U(x)$.

Theorem 3.3. *The functional $\mathcal{I}_n$ Γ-converges to $\mathcal{I}^{(0)}$. Moreover, for each $1 \le p \le \mathfrak{q}$, the functional $\theta_n^{(p)}\, \mathcal{I}_n$ Γ-converges to $\mathcal{I}^{(p)}$.*

This theorem provides an expansion of the large deviations rate function $\mathcal{I}_n$ which can be written as

$$\mathcal{I}_n \;=\; \mathcal{I}^{(0)} \;+\; \sum_{p=1}^{\mathfrak{q}} \frac{1}{\theta_n^{(p)}}\, \mathcal{I}^{(p)}\,. \tag{3.14}$$

Therefore, the rate function $\mathcal{I}_n$ encodes all the characteristics of the metastable behavior of the chain $X_t^{(n)}$. The time-scales $\theta_n^{(p)}$ appear as the weights of the expansion, and the meta-stable states $\pi_j^{(p)}$, $j \in S_p$, generate the space where the rate functional $\mathcal{I}^{(p)}(\mu)$ is finite. Indeed, by (8.9), $\mathcal{I}^{(p)}(\mu)$ is finite if and only if μ is a convex combination of the measures $\pi_j^{(p)}$, $j \in S_p$.

Next result is a simple consequence of the level two large deviations principle and the Γ-convergence stated in Theorem 3.3 (cf. Corollary 4.3 in [47]).

Corollary 3.4. *Fix $0 \le p \le \mathfrak{q}$ and recall that $\theta_n^{(0)} = 1$. For every $x \in V$, closed subset F and open subset G of $\mathcal{P}(V)$,*

$$\limsup_{n\to\infty} \limsup_{t\to\infty} \frac{\theta_n^{(p)}}{t} \log \mathbf{P}_x^n \Big[\frac{1}{t} \int_0^t \delta_{X_s^n}\, ds \in F \Big] \le - \inf_{\mu\in F} \mathcal{I}^{(p)}(\mu)\,,$$

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \frac{\theta_n^{(p)}}{t} \log \mathbf{P}_x^n \Big[\frac{1}{t} \int_0^t \delta_{X_s^n}\, ds \in G \Big] \ge - \inf_{\mu\in G} \mathcal{I}^{(p)}(\mu)\,.$$

Theorem 3.3 extends to the context of continuous-time Markov chains evolving on finite state-spaces a result by Di Gesù and Mariani [23] proved for reversible diffusions with a single valley at each different depth.

Remark 3.5. *Theorem 3.3 should hold for nonreversible dynamics. Reversibility is assumed here only to compute the Γ-limsup through formula* (3.10)*. It should also be possible to obtain a metastable Γ-expansion for the level 2.5 large deviations rate function derived in* [8].

Remark 3.6. *The proof of Theorems 3.1 and 3.3 do not require the full strength of assumption* (2.4)*, but only the ability to compute some capacities, the limit of the ratio of some measures and of mean jump rates. Stating, however, the minimal conditions would require much work.*

4. The first time-scale

In this section, we prove conditions (a) and (b) of Theorem 3.1 for $p = 1$. Throughout the article, we adopt the following notation, $O(\varepsilon)$ represents a term whose absolute value is bounded by $C_0\,\varepsilon$ for some constant C_0 independent of n and ε. Similarly, $o_n(1)$ represents a term which vanishes as $n \to \infty$.

Recall that we denote by $(\mathbb{X}_t : t \ge 0)$ the V-valued continuous-time Markov chain with jump rates $\mathbb{R}_0(x,y)$, and by $\mathbb{Q}_x$ the probability measure on $D(\mathbb{R}_+, V)$ induced by the chain $\mathbb{X}_t$ with jump rates $\mathbb{R}_0$ starting from x. For x, $y \in S$, let

$$\omega(x,y) := \lim_{t\to\infty} \mathbb{Q}_x\big[\mathbb{X}_t = y\big]\,. \tag{4.1}$$

Clearly,

$$\omega(x,y) = 0\,,\quad y \in \Delta \quad\text{and}\quad \omega(x,y) = \mathfrak{a}^{(0)}(x,j)\,\pi_j^\sharp(y)\,,\quad y \in \mathcal{V}_j\,, \tag{4.2}$$

where $\mathfrak{a}^{(0)}(x,j)$ has been introduced in (3.3).

Denote by $\mathcal{W}_j$, $j \in S$, the set of points in V which may end in the set $\mathcal{V}_j$:

$$\mathcal{W}_j := \big\{ x \in V : \mathfrak{a}^{(0)}(x,j) > 0 \big\}\,. \tag{4.3}$$

Note that $V = \cup_j \mathcal{W}_j$. Let $\mathcal{B}_j$ be the set of points attracted to $\mathcal{V}_j$:

$$\mathcal{B}_j := \big\{ x \in V : \mathfrak{a}^{(0)}(x,j) = 1 \big\}\,.$$

Clearly, $\mathcal{V}_j \subset \mathcal{B}_j \subset \mathcal{W}_j$, and $\mathcal{B}_j = \mathcal{W}_j \setminus (\cup_{k\neq j}\mathcal{W}_k) = V \setminus (\cup_{k\neq j}\mathcal{W}_k)$. In other words, $\mathcal{B}_j^c = \cup_{k\neq j}\mathcal{W}_k$. Moreover, as $\mathfrak{a}^{(0)}(x,j) = 0$ for $x \in \cup_{k\neq j}\mathcal{B}_k$ and $\mathfrak{a}^{(0)}(x,j) = 1$ for $x \in \mathcal{B}_j$,

$$\omega(x,y) \;=\; 0\,, \quad \omega(x,z) \;=\; \pi^\sharp_j(z)\,, \quad x\,,\, z \in \mathcal{V}_j\,,\;\; y \in \mathcal{V}_k\,,\;\; k \neq j\,. \tag{4.4}$$

The first result describes the asymptotic behavior of $p_t^{(n)}(x,y)$ in the slowest time-scale, $t = O(1)$.

Lemma 4.1. *For every $\varepsilon > 0$, there exists T_ε such that*

$$\limsup_{n\to\infty} \big|\, \mathbf{P}^n_x[\, X_{T_\varepsilon} = y\,] \,-\, \omega(x,y)\,\big| \;\le\; \varepsilon \quad \textit{for all } \; x\,,\, y \in V\,,$$

where $\omega(x,y)$ has been introduced in (4.1).

Proof. Fix $\varepsilon > 0$. By the ergodic theorem, there exists $T_\varepsilon < \infty$ such that

$$\big|\, \mathbb{Q}_x[\, \mathbb{X}_{T_\varepsilon} = y\,] \,-\, \omega(x,y)\,\big| \;\le\; \varepsilon \tag{4.5}$$

for all x, $y \in V$.

Couple $X_t^{(n)}$ and $\mathbb{X}_t$ making them jump together as much as possible. Denote by $\mathbb{P}_x^{(n)}$ the measure on $D(\mathbb{R}_+, V\times V)$ induced by the basic coupling starting from (x,x). By (2.1), for all $T > 0$,

$$\lim_{n\to\infty} \mathbb{P}_z^{(n)}\big[\, \mathbb{X}_t \,=\, X_t^{(n)}\,,\;\; 0 \le t \le T\,\big] \;=\; 1\,. \tag{4.6}$$

The assertion of the lemma follows from (4.5) and (4.6) with $T = T_\varepsilon$. □

Recall the definition of the sets $\mathcal{V}_j$, $1 \le j \le \mathfrak{n}$, introduced in (2.2). The chain $\mathbb{X}_t$ has only one closed irreducible class if, and only if, $\mathfrak{n} = 1$.

Corollary 4.2. *Assume that $\mathfrak{n} = 1$, Then, $\lim_{n\to\infty} p^{(n)}_{\beta_n}(x,y) = \pi^\sharp(y)$ for all x, $y \in V$, $\beta_n \succ 1$.*

Proof. Fix $\varepsilon > 0$, and let T_ε be the constant given by Lemma 4.1. By the Markov property,

$$p^{(n)}_{\beta_n}(x,y) \;=\; \sum_{z\in V} p^{(n)}_{\beta_n - T_\varepsilon}(x,z)\, p^{(n)}_{T_\varepsilon}(z,y)\,.$$

By Lemma 4.1 and (4.2), since $\mathfrak{a}^{(0)}(y,1) = 1$ for all $y \in V$, the right-hand side is equal to

$$\sum_{z\in V} p^{(n)}_{\beta_n - T_\varepsilon}(x,z)\, \pi^\sharp(y) \;+\; O(\varepsilon) \;+\; o_n(1) \;=\; \pi^\sharp(y) \;+\; O(\varepsilon) \;+\; o_n(1)\,,$$

which completes the proof of the corollary. □

Corollary 4.2 shows that the asymptotic behavior of the transition probability $p_t^{(n)}$ is trivial if $\mathfrak{n} = 1$, that is if the Markov chain $\mathbb{X}_t$ has a unique closed irreducible class. Assume that $\mathfrak{n} \ge 2$.

The time-scale $\theta_n^{(1)}$. Recall the definition of $\mathfrak{n}_1$, S_1, and the sets $\mathcal{V}_j^{(1)}$, $j \in S_1$, Δ_1, introduced just above (2.5). Let $\theta_n = \theta_n^{(1)}$ be given by (2.7) with $p = 1$.

Recall from [43, Section 2.3] the definition of the sequence α_n. In the present context, by (2.1), the sequence α_n converges to a positive real number. By Assertions 7.B and equation (7.4) in [43], $\theta_n \succ 1$. The next result is the first assertion of Theorem 3.1.

Proposition 4.3. *Let $(\beta_n : n \geq 1)$ be a sequence such that $1 \prec \beta_n \prec \theta_n$. Then, (3.6) holds for all x, $y \in V$.*

Recall that we call the sets $\mathcal{V}_j$ wells. A time scale $\beta_n \prec \theta_n$ is not long enough to allow the process to jump from a well to another. This is the content of the next two results. Lemma 4.4 states that starting from a well $\mathcal{V}_j$ the process does not visit another well (the set $\breve{\mathcal{V}}_j$ introduced in (2.7)) in a time-scale β_n such that $\beta_n \prec \theta_n$. Corollary 4.5 extends this result asserting that the points that might end up in another well (the set $\cup_{k\neq j}\mathcal{W}_k$) are also not visited in this time-scale.

Lemma 4.4. *Let $(\beta_n : n \geq 1)$ be a sequence such that $\beta_n \prec \theta_n$. Then, for all $j \in S_1$, $x \in \mathcal{V}_j$,*

$$\lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\breve{\mathcal{V}}_j} < \beta_n \,\big] = 0 \,.$$

Proof. Fix $j \in S$, $x \in \mathcal{V}_j$. By Lemma A.4 and (2.16), the probability appearing in the statement of the lemma is bounded by $C_0\, \beta_n \operatorname{cap}_n(\{x\}\,, \breve{\mathcal{V}}_j)/\pi_n(\mathcal{V}_j)$ for some finite constant C_0, independent of n and whose value may change from line to line. By equation (B2) in [32], this expression is bounded by $C_0\, \beta_n \operatorname{cap}_n(\mathcal{V}_j\,, \breve{\mathcal{V}}_j)/\pi_n(\mathcal{V}_j)$. By the definition (2.7) of θ_n, this expression is less than or equal to $C_0\, \beta_n \,/\, \theta_n$. This concludes the proof of the lemma. □

Corollary 4.5. *Let $(\beta_n : n \geq 1)$ be an increasing sequence such that $\beta_n \prec \theta_n$. Then, for all $j \in S$, $x \in \mathcal{V}_j$,*

$$\lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathcal{B}_j^c} < \beta_n \,\big] = 0 \,.$$

Proof. Assume first that $\beta_n \succ 1$. Fix $j \in S$ and $x \in \mathcal{V}_j$ and keep in mind that $\mathcal{B}_j^c = \cup_{k\neq j}\mathcal{W}_k$.

We proceed by contradiction. Suppose the assertion does not hold. In this case, there exists $\delta > 0$, $k \neq j$, $z \in \mathcal{W}_k$ and a subsequence n', still denoted by n, such that $\mathbf{P}_x^n[\, H_z < \beta_n \,] > \delta$ for all n. By the strong Markov property and this bound,

$$\mathbf{P}_x^n \big[\, H_{\breve{\mathcal{V}}_j} < 2\,\beta_n \,\big] \geq \mathbf{P}_x^n[\, H_z < \beta_n \,]\, \mathbf{P}_z^n \big[\, H_{\breve{\mathcal{V}}_j} < \beta_n \,\big] \geq \delta\, \mathbf{P}_z^n \big[\, H_{\breve{\mathcal{V}}_j} < \beta_n \,\big] \,.$$

Since $z \in \mathcal{W}_k$, there exists $\delta' > 0$ and $T_0 < \infty$, such that $\mathbb{Q}_z \big[\, H_{\mathcal{V}_k} < T_0 \,\big] > \delta'$. By (4.6) this estimate extends to $X_t^{(n)}$: $\mathbf{P}_z^n \big[\, H_{\mathcal{V}_k} < T_0 \,\big] > \delta'/2$ for all n sufficiently large.

Combining the previous estimates yields that $\mathbf{P}_x^n[\, H_{\breve{\mathcal{V}}_j} < 2\,\beta_n \,] \geq \delta\,\delta'/2$ for all n sufficiently large because $\beta_n \to \infty$. This result contradicts the assertion of Lemma 4.4 and completes the proof of the corollary in the case $\beta_n \succ 1$.

If the sequence β_n is bounded, the result follows from the coupling (4.6) because $\mathbb{Q}_x[\, H_{\mathcal{W}_k} < \infty \,] \leq \mathbb{Q}_x[\, H_{\mathcal{V}_j^c} < \infty \,] = 0$ for all $x \in \mathcal{V}_j$, $k \neq j$. □

Proof of Proposition 4.3. Fix x, $y \in V$, $\varepsilon > 0$, and recall the definition of $\omega(x,y)$ introduced in (4.1). Since $\mathcal{V}$ represents the set of recurrent points of the chain $\mathbb{X}_t$, there exists $T_\varepsilon > 0$ such that

$$\mathbb{Q}_w\big[\, \mathbb{X}_T \,=\, \mathcal{V} \,\big] \,\geq\, 1 - \varepsilon\,, \quad \big|\, \mathbb{Q}_w\big[\, \mathbb{X}_T \,=\, z \,\big] \,-\, \omega(w,z) \,\big| \,\leq\, \varepsilon \tag{4.7}$$

for all w, $z \in V$, $T \geq T_\varepsilon$.

Assume first that $y \in \Delta$. By the Markov property,

$$\mathbf{P}^n_x\big[\, X_{\beta_n} \,=\, y \,\big] \,=\, \sum_{z\in V} \mathbf{P}^n_x\big[\, X_{\beta_n - T_\varepsilon} \,=\, z \,\big]\, \mathbf{P}^n_z\big[\, X_{T_\varepsilon} \,=\, y \,\big]\,.$$

By (4.6), (4.7) and (4.2), the right-hand side is bounded by $o_n(1) + \varepsilon$, which proves (3.6) for $y \in \Delta$.

Assume that $y \in \mathcal{V}_k$ for some $k \in S_1$. By the Markov property,

$$\mathbf{P}^n_x\big[\, X_{\beta_n} \,=\, y \,\big] \,=\, \sum_{z\in V} \mathbf{P}^n_x\big[\, X_{T_\varepsilon} \,=\, z \,\big]\, \mathbf{P}^n_z\big[\, X_{\beta_n - T_\varepsilon} \,=\, y \,\big]\,.$$

By (4.6), (4.7) and (4.2), the right-hand side is equal to

$$\sum_{j\in S} \sum_{z\in\mathcal{V}_j} \mathfrak{a}^{(0)}(x,j)\, \pi^\sharp_j(z)\, \mathbf{P}^n_z\big[\, X_{\beta_n - T_\varepsilon} \,=\, y \,\big] \,+\, o_n(1) \,+\, O(\varepsilon)\,.$$

Since $\beta_n \prec \theta_n$, by Corollary 4.5, we may add inside the probability the event $\{H_{\mathcal{B}_j^c} \geq \beta_n\}$. The previous sum is thus equal to

$$\sum_{j\in S} \sum_{z\in\mathcal{V}_j} \mathfrak{a}^{(0)}(x,j)\, \pi^\sharp_j(z)\, \mathbf{P}^n_z\big[\, X_{\beta_n - T_\varepsilon} \,=\, y\,,\, H_{\mathcal{B}_j^c} \,\geq\, \beta_n \,\big] \,+\, o_n(1) \,+\, O(\varepsilon)\,.$$

As y belongs to $\mathcal{V}_k$ and $\mathcal{V}_k \cap \mathcal{B}_j = \varnothing$ if $j \neq k$, this sum is equal to

$$\sum_{z\in\mathcal{V}_k} \mathfrak{a}^{(0)}(x,k)\, \pi^\sharp_k(z)\, \mathbf{P}^n_z\big[\, X_{\beta_n - T_\varepsilon} \,=\, y\,,\, H_{\mathcal{B}_k^c} \,\geq\, \beta_n \,\big] \,+\, o_n(1) \,+\, O(\varepsilon)\,.$$

In view of the presence of the event $\{H_{\mathcal{B}_k^c} \geq \beta_n\}$, the previous probability is equal to

$$\sum_{w\in\mathcal{B}_k} \mathbf{P}^n_z\big[\, X_{\beta_n - T_\varepsilon} \,=\, y\,,\, X_{\beta_n - 2T_\varepsilon} \,=\, w\,,\, H_{\mathcal{B}_k^c} \,\geq\, \beta_n \,\big]$$

By Corollary 4.5, we may remove the event $\{H_{\mathcal{B}_k^c} \geq \beta_n\}$ at a cost $o_n(1)$ and apply the Markov property to conclude that the previous sum is equal to

$$\sum_{w\in\mathcal{B}_k} \mathbf{P}^n_z\big[\, X_{\beta_n - 2T_\varepsilon} \,=\, w \,\big]\, \mathbf{P}^n_w\big[\, X_{T_\varepsilon} \,=\, y \,\big] \,+\, o_n(1)\,.$$

By (4.6), (4.7) and (4.2), this expression is equal to

$$\sum_{w\in\mathcal{B}_k} \mathbf{P}^n_z\big[\, X_{\beta_n - 2T_\varepsilon} \,=\, w \,\big]\, \mathfrak{a}^{(0)}(w,k)\, \pi^\sharp_k(y) \,+\, o_n(1) \,+\, O(\varepsilon)\,.$$

Since w belongs to $\mathcal{B}_k$, $\mathfrak{a}^{(0)}(w,k) = 1$ and the previous expression is equal to

$$\pi^\sharp_k(y)\, \mathbf{P}^n_z\big[\, X_{\beta_n - 2T_\varepsilon} \,\in\, \mathcal{B}_k \,\big] \,+\, o_n(1) \,+\, O(\varepsilon)\,.$$

Since z belongs to $\mathcal{V}_k$ and $\{X_{\beta_n - 2T_\varepsilon} \notin \mathcal{B}_k\} \subset \{H_{\mathcal{B}_k^c} \leq \beta_n\}$, by Corollary 4.5, the expression in the previous displayed equation is equal to $\pi^\sharp_k(y) + o_n(1) + O(\varepsilon)$.

Combining the previous estimates yields that

$$\begin{aligned}\mathbf{P}_x^n\big[\, X_{\beta_n} \,=\, y\,\big] \;&=\; \sum_{z\in\mathcal{V}_k} \mathfrak{a}^{(0)}(x,k)\,\pi_k^\sharp(z)\pi_k^\sharp(y) \,+\, o_n(1) \,+\, O(\varepsilon)\\ &=\; \mathfrak{a}^{(0)}(x,k)\,\pi_k^\sharp(y) \,+\, o_n(1) \,+\, O(\varepsilon)\;,\end{aligned}$$

as claimed. □

The time-scale $t\,\theta_n$. We turn to the proof of Theorem 3.1.(b) for $p=1$.

Proposition 4.6. *Assertion* (3.7) *holds for $p=1$ and all $t>0$, $x\in V$.*

The proof of this result relies on the following lemma.

Lemma 4.7. *Recall the definition of the set Δ introduced in* (2.2)*. Then,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\max_{j\in S}\max_{x\in\mathcal{V}_j}\sup_{2\delta\le s\le 3\delta}\mathbf{P}_x^n\big[\, X_{s\theta_n}\in\Delta\,\big] \;=\; 0\;.$$

Proof. Fix $\varepsilon>0$ and let T_ε be the constant given by Lemma 4.1. By the Markov property, the probability appearing in the statement of the lemma is bounded by

$$\max_{y\in V}\mathbf{P}_y^n\big[\, X_{T_\varepsilon}\in\Delta\,\big]$$

By (4.2) and Lemma 4.1, this expression is bounded by $\varepsilon+o_n(1)$, which proves the lemma. □

By [35, Proposition 2.1], [43, Theorem 2.7] and Lemma 4.7, for every $t>0$, j, $k\in S_1$, $x\in\mathcal{V}_j$,

$$\lim_{n\to\infty}\mathbf{P}_x^n\big[\, X_{t\theta_n}\,\in\,\mathcal{V}_k\,\big] \;=\; p_t^{(1)}(j,k)\;, \tag{4.8}$$

where the transition probability $p_t^{(1)}$ has been introduced in (2.11).

Proof of Proposition 4.6. Suppose that $y\in\Delta$ and fix $t>0$, $\varepsilon>0$. In this case, by the Markov property

$$\mathbf{P}_x^n\big[\, X_{t\theta_n}\,=\,y\,\big] \;=\; \sum_{z\in V}\mathbf{P}_x^n\big[\, X_{t\theta_n-T_\varepsilon}\,=\,z\,\big]\,\mathbf{P}_z^n\big[\, X_{T_\varepsilon}\,=\,y\,\big]\;,$$

where T_ε is given by Lemma 4.1. By this lemma, the second probability on the right hand side is bounded by $\omega(z,y)+\varepsilon+o_n(1)$. By (4.2), as $y\in\Delta$, $\omega(z,y)=0$ so that

$$\lim_{n\to\infty}\mathbf{P}_x^n\big[\, X_{t\theta_n}\,=\,y\,\big] \;=\; 0\;,$$

as claimed.

Suppose that $y\in\mathcal{V}_m$ for some $m\in S_1$ and fix $t>0$, $\varepsilon>0$. By the Markov property

$$\mathbf{P}_x^n\big[\, X_{t\theta_n}\,=\,y\,\big] \;=\; \sum_{z,z'\in V}\mathbf{P}_x^n\big[\, X_{T_\varepsilon}\,=\,z\,\big]\,\mathbf{P}_z^n\big[\, X_{t\theta_n-2T_\varepsilon}\,=\,z'\,\big]\,\mathbf{P}_{z'}^n\big[\, X_{T_\varepsilon}\,=\,y\,\big]\;,$$

where T_ε is given by Lemma 4.1. By this lemma and (4.2), which asserts that $\omega(x',y')=0$ if $y'\in\Delta$, this expression is equal to

$$\sum_{z'\in V}\sum_{j\in S_1}\sum_{z\in\mathcal{V}_j}\omega(x,z)\,\mathbf{P}_z^n\big[\, X_{t\theta_n-2T_\varepsilon}\,=\,z'\,\big]\,\omega(z',y) \,+\, \varepsilon \,+\, o_n(1)\;.$$

The first part of the proof permits to restrict the first sum to $z' \in \mathcal{V}$. Since $y \in \mathcal{V}_m$, by (4.4), we may further restrict the sum to $z' \in \mathcal{V}_m$, and then replace $\omega(z', y)$ by $\pi_m^{(1)}(y)$. Hence the previous sum is equal to

$$\pi_m^{(1)}(y) \sum_{j \in S_1} \sum_{z \in \mathcal{V}_j} \omega(x,z)\, \mathbf{P}_z^n \big[\, X_{t\theta_n - 2T_\varepsilon} \,\in\, \mathcal{V}_m \,\big] \;+\; \varepsilon \;+\; o_n(1)\,,$$

where we summed over $z' \in \mathcal{V}_m$. By (4.8), as $n \to \infty$, this expression converges to

$$\pi_m^{(1)}(y) \sum_{j \in S_1} \sum_{z \in \mathcal{V}_j} \omega(x,z)\, p_t^{(1)}(j,m) \;+\; \varepsilon \;=\; \sum_{j \in S_1} \mathfrak{a}^{(0)}(x,j)\, p_t^{(1)}(j,m)\, \pi_m^{(1)}(y) \;+\; \varepsilon\,,$$

as claimed. □

5. Longer time-scales

In this section, we complete the proof of Theorem 3.1. We first derive some properties of the weights $\mathfrak{a}^{(p)}$ needed in the argument. Recall that $\mathfrak{q}$ represents the number of time-scales or steps in the construction of the rooted tree in Section 2. Moreover, the chain $\mathbb{X}_t^{(\mathfrak{q})}$ has only one closed irreducible class.

Next result states that a point in the closed irreducible class $\mathcal{V}_\ell^{(p+1)}$ is not absorbed at $\mathcal{V}_m^{(p+1)}$ for $m \neq \ell$.

Lemma 5.1. *For all $0 \le p < \mathfrak{q}$, $\ell \in S_{p+1}$, $x \in \mathcal{V}_\ell^{(p+1)}$,*

$$\mathfrak{a}^{(p)}(x,m) \;=\; 0 \quad \textit{for all} \quad m \,\in\, S_{p+1} \setminus \{\ell\}\,. \tag{5.1}$$

Proof. The proof is by induction in p. For $p = 0$, by definition (3.3) of $\mathfrak{a}^{(0)}$, for all $\ell \in S_1$, $x \in \mathcal{V}_\ell$, $m \in S_1 \setminus \{\ell\}$,

$$\mathfrak{a}^{(0)}(x,m) \;=\; \lim_{t \to \infty} \mathbb{Q}_x \big[\, \mathbb{X}_t \,\in\, \mathcal{V}_m \,\big] \;=\; 0$$

because the sets $\mathcal{V}_k$ are the closed irreducible classes of the chain $\mathbb{X}_t$.

Assume that (5.1) holds for $0 \le p \le r-1$. Fix $\ell \in S_{r+1}$, $x \in \mathcal{V}_\ell^{(r+1)}$, $m \in S_{r+1} \setminus \{\ell\}$. By definition of $\mathfrak{a}^{(r)}(x,m)$,

$$\mathfrak{a}^{(r)}(x,m) \;:=\; \sum_{j \in S_r} \mathfrak{a}^{(r-1)}(x,j)\, \mathfrak{A}^{(r)}(j,m)\,.$$

We may restrict the sum to $j \in \mathfrak{R}_m^{(r)}$. Indeed, since $S_r \setminus \mathfrak{R}_m^{(r)} = \cup_{k \in S_{r+1} \setminus \{m\}} \mathfrak{R}_k^{(r)}$ and since the sets $\mathfrak{R}_k^{(r)}$, $k \in S_{r+1}$, are the closed irreducible classes of the chain $\mathbb{X}_t^{(r)}$, $\mathfrak{A}^{(r)}(j,m) = 0$ for $j \in S_r \setminus \mathfrak{R}_m^{(r)}$. Hence,

$$\mathfrak{a}^{(r)}(x,m) \;:=\; \sum_{j \in \mathfrak{R}_m^{(r)}} \mathfrak{a}^{(r-1)}(x,j)\, \mathfrak{A}^{(r)}(j,m)\,.$$

On the other hand, as $x \in \mathcal{V}_\ell^{(r+1)} = \cup_{i \in \mathfrak{R}_\ell^{(r)}} \mathcal{V}_i^{(r)}$ and $\mathfrak{R}_\ell^{(r)} \cap \mathfrak{R}_m^{(r)} = \varnothing$ because $\ell \neq m$, x belongs to some $\mathcal{V}_i^{(r)}$ with $i \notin \mathfrak{R}_m^{(r)}$. Thus, by the induction assumption $\mathfrak{a}^{(r-1)}(x,j) = 0$ for all $j \in \mathfrak{R}_m^{(r)}$, which yields that $\mathfrak{a}^{(r)}(x,m) = 0$, as claimed. □

The previous result is stated for $p < \mathfrak{q}$ because $\mathbb{X}_t^{(\mathfrak{q})}$ has only one irreducible class which makes $S_{\mathfrak{q}+1}$ a singleton.

It has been noted, just before the statement of Theorem 3.1, that $\mathfrak{a}^{(p)}(x, \cdot)$ is a probability measure on S_{p+1} for all $x \in V$. Therefore, by the previous lemma, for all $1 \le p < \mathfrak{q}$, $\ell \in S_{p+1}$, $x \in \mathcal{V}_\ell^{(p+1)}$,

$$\mathfrak{a}^{(p)}(x,\ell) \;=\; 1 \quad \text{so that} \quad \Pi_p(x, \cdot\,) \;=\; \pi_\ell^{(p+1)}(\,\cdot\,)\,, \tag{5.2}$$

where $\Pi_p(x, \cdot\,)$ has been introduced in (3.6). In particular, under these conditions on ℓ and x,

$$\Pi_p(x,y) \;=\; 0 \tag{5.3}$$

for all $y \in \mathcal{V}_m^{(p+1)}$, $m \in S_{p+1} \setminus \{\ell\}$.

This identity can be extended. Since the support of the measure $\pi_m^{(p+1)}(\,\cdot\,)$ is the set $\mathcal{V}_m^{(p+1)}$, $m \in S_{p+1}$, and $\cup_m \mathcal{V}_m^{(p+1)} = \mathcal{V}^{(p+1)}$,

$$\Pi_p(x,y) \;=\; 0 \quad \text{for all} \quad x \in V\,,\; y \in \big(\mathcal{V}^{(p+1)}\big)^c \;=\; \Delta_{p+1}\,. \tag{5.4}$$

Induction hypotheses: Assume that we proved for some $1 \le p < \mathfrak{q}$ that for all $t > 0$, x, $y \in V$,

$$\lim_{n\to\infty} p^{(n)}_{t\,\theta_n^{(p)}}(x,y) \;=\; \sum_{k\in S_p} \omega_t^{(p)}(x,k)\,\pi_k^{(p)}(y)\,, \tag{5.5}$$

where $\omega_t^{(p)}$, $\pi_k^{(p)}$ are as in the statement of Theorem 3.1. This assertion for $p = 1$ is the content of Proposition 4.6.

The time scale $t\theta_n^{(p)}$, as $t \to \infty$. Recall the definition of $\mathfrak{A}^{(p)}(j,m)$, $m \in S_{p+1}$, $j \in S_p$, introduced in (3.4). With this notation, for every j, $k \in S_p$,

$$\lim_{t\to\infty} p_t^{(p)}(j,k) \;=\; \sum_{m\in S_{p+1}} \mathfrak{A}^{(p)}(j,m)\, M_m^{(p)}(k)\,, \tag{5.6}$$

where, recall, $M_m^{(p)}(\cdot)$, $m \in S_{p+1}$, the stationary state of the Markov chain $\mathbb{X}_t^{(p)}$ restricted to $\mathfrak{R}_m^{(p)}$. In particular, $\lim_{t\to\infty} p_t^{(p)}(j,k) = 0$ for every $k \in \mathfrak{T}_p$.

By the induction assumption (5.5), the definition of $\omega_t^{(p)}$, (5.6) and the fact that the support of the measure $M_m^{(p)}$ is the set $\mathfrak{R}_m^{(p)}$, for all $x \in V$,

$$\lim_{t\to\infty}\lim_{n\to\infty} p^{(n)}_{t\,\theta_n^{(p)}}(x,\,\cdot\,) \;=\; \sum_{j\in S_p}\sum_{m\in S_{p+1}}\sum_{k\in\mathfrak{R}_m^{(p)}} \mathfrak{a}^{(p-1)}(x,j)\,\mathfrak{A}^{(p)}(j,m)\,M_m^{(p)}(k)\,\pi_k^{(p)}(\,\cdot\,)\,.$$

By definition of the measures $\pi_m^{(p+1)}$ and by the one of $a^{(p)}(x,m)$, given in (3.5), this expression is equal to

$$\sum_{j\in S_p}\sum_{m\in S_{p+1}} \mathfrak{a}^{(p-1)}(x,j)\,\mathfrak{A}^{(p)}(j,m)\,\pi_m^{(p+1)}(\,\cdot\,) \;=\; \sum_{m\in S_{p+1}} \mathfrak{a}^{(p)}(x,m)\,\pi_m^{(p+1)}(\,\cdot\,)\,.$$

Hence, we proved that for all $x \in V$,

$$\lim_{t\to\infty}\lim_{n\to\infty} p^{(n)}_{t\,\theta_n^{(p)}}(x,\,\cdot\,) \;=\; \sum_{m\in S_{p+1}} \mathfrak{a}^{(p)}(x,m)\,\pi_m^{(p+1)}(\,\cdot\,) \;=\; \Pi_p(x,\,\cdot\,)\,. \tag{5.7}$$

The argument above shows that Theorem 3.1.(d) follows from Theorem 3.1.(b). Assertion (c) of this theorem follows from the fact that $p_t^{(p)}(j,k)$ converges to $\delta_{j,k}$ as $t \to 0$.

The time scale $\theta_n^{(p)} \prec \beta_n \prec \theta_n^{(p+1)}$. By (5.7) and (5.4),

$$\lim_{t\to\infty} \lim_{n\to\infty} \max_{x\in V} \mathbf{P}_x^n \big[X_{t\theta_n^{(p)}} \notin \mathcal{V}^{(p+1)} \big] \;=\; 0\,. \tag{5.8}$$

In particular,

$$\lim_{t\to\infty} \limsup_{n\to\infty} \max_{x\in V} \mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}^{(p+1)}} > t\theta_n^{(p)} \big] \;=\; 0\,. \tag{5.9}$$

Suppose that S_{p+1} is a singleton. In other words, that the chain $\mathbb{X}_t^{(p)}$ has a unique closed irreducible class. In this case $p = \mathfrak{q}$ and $\theta_n^{(p+1)} = +\infty$ for all $n \ge 1$. If S_{p+1} is not a singleton, recall from (2.7) the definition of $\theta_n^{(p+1)}$. As stated in (2.15), by [43, Assertion 8.B], $\theta_n^{(p)} \prec \theta_n^{(p+1)}$.

Lemma 5.2. *Let $(\beta_n : n \ge 1)$ be a sequence such that $\theta_n^{(p)} \prec \beta_n \prec \theta_n^{(p+1)}$. Then, for all $m \in S_{p+1}$,*

$$\lim_{n\to\infty} \max_{x\in\mathcal{V}_m^{(p+1)}} \mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}_m^{(p+1)}} < \beta_n \big] \;=\; 0\,,$$

where $\breve{\mathcal{V}}_m^{(p+1)}$ has been introduced in (2.7).

Proof. Fix $m \in S_{p+1}$, $x \in \mathcal{V}_m^{(p+1)}$. By Lemma A.4 and (2.16), the probability appearing above is bounded by $C_0\,\beta_n\,\mathrm{cap}_n(\{x\}\,,\,\breve{\mathcal{V}}_m^{(p+1)})/\pi_n(\mathcal{V}_m^{(p+1)})$ for some finite constant C_0 independent of n. By equation (B2) in [32], this expression is bounded by $C_0\,\beta_n\,\mathrm{cap}_n(\mathcal{V}_m^{(p+1)}\,,\,\breve{\mathcal{V}}_m^{(p+1)})/\pi_n(\mathcal{V}_m^{(p+1)})$. By the definition (2.7) of $\theta_n^{(p+1)}$, this expression is less than or equal to $C_0\,\beta_n\,/\,\theta_n^{(p+1)}$. This concludes the proof of the lemma. □

Let $\Delta_{p+1,m}$, $m \in S_{p+1}$, be the set of points in Δ_{p+1} which may be absorbed by a set $\mathcal{V}_\ell^{(p+1)}$, $\ell \ne m$, in the time-scale $\theta_n^{(p)}$:

$$\Delta_{p+1,m} \;:=\; \Big\{ x \in \Delta_{p+1} : \sum_{\ell\in S_{p+1}\setminus\{m\}} \mathfrak{a}^{(p)}(x,\ell) > 0 \Big\}\,.$$

Corollary 5.3. *Let $(\beta_n : n \ge 1)$ be a sequence such that $\theta_n^{(p)} \prec \beta_n \prec \theta_n^{(p+1)}$. Then, for all $m \in S_{p+1}$,*

$$\lim_{n\to\infty} \max_{x\in\mathcal{V}_m^{(p+1)}} \mathbf{P}_x^n \big[H_{\Delta_{p+1,m}} < \beta_n \big] \;=\; 0\,.$$

Proof. Suppose the assertion is not true. Then, there exists $\delta > 0$, $x \in \mathcal{V}_m^{(p+1)}$ and a subsequence n', still denoted by n, such that

$$\mathbf{P}_x^n \big[H_{\Delta_{p+1,m}} < \beta_n \big] \;\ge\; \delta$$

for all n sufficiently large.

Fix $t > 0$ to be chosen later. Denote by $\vartheta_s : D(\mathbb{R}_+, V) \to D(\mathbb{R}_+, V)$, $s \ge 0$, the semigroup of translations of a trajectory: $(\vartheta_s\mathfrak{x})(r) = \mathfrak{x}(r+s)$, $r \ge 0$. By the strong Markov property,

$$\begin{aligned}
\mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}_m^{(p+1)}} < \beta_n \,+\, t\,\theta_n^{(p)} \big] \;&\ge\; \mathbf{P}_x^n \big[H_{\Delta_{p+1,m}} < \beta_n\,,\, H_{\breve{\mathcal{V}}_m^{(p+1)}} \circ \vartheta_{H_{\Delta_{p+1,m}}} < t\,\theta_n^{(p)} \big] \\
&\ge\; \mathbf{P}_x^n \big[H_{\Delta_{p+1,m}} < \beta_n \big] \min_{z\in\Delta_{p+1,m}} \mathbf{P}_z^n \big[H_{\breve{\mathcal{V}}_m^{(p+1)}} < t\,\theta_n^{(p)} \big] \\
&\ge\; \mathbf{P}_x^n \big[H_{\Delta_{p+1,m}} < \beta_n \big] \min_{z\in\Delta_{p+1,m}} \mathbf{P}_z^n \big[X_{t\,\theta_n^{(p)}} \in \breve{\mathcal{V}}_m^{(p+1)} \big]\,.
\end{aligned}$$

By the first part of the proof, the first term is bounded below by δ for n sufficiently large. By Theorem 3.1.(d), proved in the previous subsection for p, for each $z \in \Delta_{p+1,m}$, the second probability converges, as $n \to \infty$ and then $t \to \infty$, to

$$\sum_{\ell \in S_{p+1} \setminus \{m\}} \mathfrak{a}^{(p)}(z,\ell) \,.$$

By definition of $\Delta_{p+1,m}$, this term is strictly positive for each $z \in \Delta_{p+1,m}$. Therefore, there exist $\delta' > 0$ and $t_0 < \infty$ such that

$$\liminf_{n\to\infty} \min_{z\in\Delta_{p+1,m}} \mathbf{P}_z^n \big[X_{t_0\,\theta_n^{(p)}} \in \breve{\mathcal{V}}_m^{(p+1)} \big] \;\geq\; \delta' \,.$$

Putting together the previous estimates yields that

$$\liminf_{n\to\infty} \mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}_m^{(p+1)}} < \beta_n \,+\, t_0\,\theta_n^{(p)} \big] \;>\; 0 \,,$$

in contradiction with the statement of Lemma 5.2. This completes the proof of the corollary. □

For $m \in S_{p+1}$, let

$$\mathcal{U}_m^{(p+1)} \;:=\; \Big\{ x \in V : \mathfrak{a}^{(p)}(x,m) = 1 \Big\} \,.$$

By (5.2) and the definition of the set $\Delta_{p+1,m}$, introduced just before the statement of Corollary 5.3, the set $\mathcal{U}_m^{(p+1)}$ is equal to $\mathcal{V}_m^{(p+1)} \cup [\, \Delta_{p+1} \setminus \Delta_{p+1,m} \,]$. Thus, $(\mathcal{U}_m^{(p+1)})^c = \breve{\mathcal{V}}_m^{(p+1)} \cup \Delta_{p+1,m}$.

Proposition 5.4. *Let $\theta_n^{(p)} \prec \beta_n \prec \theta_n^{(p+1)}$. Then, for all $x \in V$,*

$$\lim_{n\to\infty} p_{\beta_n}^{(n)}(x,\,\cdot\,) \;=\; \Pi_p(x,\,\cdot\,) \,,$$

where $\Pi_p(x,\,\cdot\,)$ has been introduced in (3.6).

Proof. Fix $\varepsilon > 0$. By (5.7), there exists t_ε such that

$$\Big| \lim_{n\to\infty} p_{t\,\theta_n^{(p)}}^{(n)}(x,y) \,-\, \Pi_p(x,y) \Big| \;<\; \varepsilon \tag{5.10}$$

for all x, $y \in V$, $t > t_\varepsilon$.

Fix $t > t_\varepsilon$. By the Markov property,

$$p_{\beta_n}^{(n)}(x,y) \;=\; \sum_{z\in V} p_{t\,\theta_n^{(p)}}^{(n)}(x,z)\, p_{\beta_n - t\,\theta_n^{(p)}}^{(n)}(z,y) \,.$$

By (5.10) and (5.4), this expression is equal to

$$\sum_{z\in\mathcal{V}^{(p+1)}} \Pi_p(x,z)\, \mathbf{P}_z^n \big[X_{\beta_n - t\,\theta_n^{(p)}} = y \big] \;+\; o_n(1) \;+\; O(\varepsilon) \,. \tag{5.11}$$

Fix $s > 0$, and rewrite the sum appearing in (5.11) as

$$\sum_{m\in S_{p+1}} \sum_{z\in\mathcal{V}_m^{(p+1)}} \sum_{w\in V} \Pi_p(x,z)\, \mathbf{P}_z^n \big[X_{\beta_n - t\,\theta_n^{(p)}} = y \,,\, X_{\beta_n - (t+s)\,\theta_n^{(p)}} = w \big] \,.$$

We have shown just above the statement of the proposition that $(\mathcal{U}_m^{(p+1)})^c = \breve{\mathcal{V}}_m^{(p+1)} \cup \Delta_{p+1,m}$. Hence, by Lemma 5.2 and Corollary 5.3, we may restrict the

third sum to $w \in \mathcal{U}_m^{(p+1)}$ by paying a price of order $o_n(1)$. Apply the Markov property to rewrite the resulting expression as

$$\sum_{m\in S_{p+1}} \sum_{z\in\mathcal{V}_m^{(p+1)}} \sum_{w\in\mathcal{U}_m^{(p+1)}} \Pi_p(x,z)\, \mathbf{P}_z^n\big[\, X_{\beta_n-(t+s)\,\theta_n^{(p)}} = w\,\big]\, \mathbf{P}_w^n\big[\, X_{s\,\theta_n^{(p)}} = y\,\big]\ .$$

By (5.7) the last probability converges, as $n\to\infty$, and then $s\to\infty$, to $\Pi_p(w,y)$. By definition of Π_p and the one of $\mathcal{U}_m^{(p+1)}$, since $w\in\mathcal{U}_m^{(p+1)}$ and $\mathfrak{a}^{(p)}(x,\,\cdot\,)$ is a probability measure on S_{p+1}, $\Pi_p(w,y) = \pi_m^{(p+1)}(y)$. This expression does not depend on w. By Lemma 5.2 and Corollary 5.3, the previous sum is thus equal to

$$\sum_{m\in S_{p+1}} \sum_{z\in\mathcal{V}_m^{(p+1)}} \Pi_p(x,z)\, \pi_m^{(p+1)}(y) \;+\; o_n(1)\ .$$

By the definition (3.6) of Π_p, this expression is equal to

$$\sum_{\ell\in S_{p+1}} \mathfrak{a}^{(p)}(x,\ell)\, \pi_\ell^{(p+1)}(y)\ ,$$

as claimed. □

The time scale $\theta_n^{(p+1)}$. If S_{p+1} is a singleton, $p=\mathfrak{q}$, $\theta_n^{(p+1)} = +\infty$ for all n and the proof of Theorem 3.1 ends at the previous step where we considered the time-scale $\theta_n^{(p)} \prec \beta_n \prec \theta_n^{(p+1)} \equiv +\infty$.

Assume that S_{p+1} is not a singleton. The next result completes the recursive argument and the proof of Theorem 3.1. It states that the induction hypothesis (5.5) holds at level $p+1$ if it holds at level p.

Proposition 5.5. *For all $t>0$, x, $y\in V$,*

$$\lim_{n\to\infty} p^{(n)}_{t\,\theta_n^{(p+1)}}(x,y) \;=\; \sum_{m\in S_{p+1}} \omega_t^{(p+1)}(x,m)\, \pi_m^{(p+1)}(y)\ .$$

The proof of this result is based on Lemma 5.6 below.

Lemma 5.6. *Recall the definition of the set Δ_{p+1} introduced in* (2.13)*. Then,*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \max_{m\in S_{p+1}} \max_{x\in\mathcal{V}_m^{(p+1)}} \sup_{2\delta\le s\le 3\delta} \mathbf{P}_x^n\big[\, X_{s\theta_n^{(p+1)}} \in \Delta_{p+1}\,\big] \;=\; 0\ .$$

Proof. Fix $\delta>0$, $\varepsilon>0$. By (5.8), there exists $t_\varepsilon<\infty$

$$\lim_{n\to\infty} \max_{x\in V} \mathbf{P}_x^n\big[\, X_{t\theta_n^{(p)}} \notin \mathcal{V}^{(p+1)}\,\big] \;\le\; \varepsilon \tag{5.12}$$

for all $t\ge t_\varepsilon$. By the Markov property, since $\theta_n^{(p)} \prec \theta_n^{(p+1)}$, the probability appearing in the statement of the lemma is bounded by

$$\max_{y\in V} \mathbf{P}_y^n\big[\, X_{t_\varepsilon\theta_n^{(p)}} \in \Delta_{p+1}\,\big]$$

for all $x\in V$, $s\in[2\delta,3\delta]$. By (5.12), this expression is bounded by $\varepsilon + o_n(1)$, which proves the lemma. □

By [35, Proposition 2.1], [43, Theorem 2.7] and Lemma 5.6 for every $t>0$, ℓ, $m\in S_{p+1}$, $x\in\mathcal{V}_\ell^{(p+1)}$,

$$\lim_{n\to\infty} \mathbf{P}_x^n\big[\, X_{t\theta_n^{(p+1)}} \,\in\, \mathcal{V}_m^{(p+1)}\,\big] \;=\; p_t^{(p+1)}(\ell,m)\ , \tag{5.13}$$

where, recall, $p_t^{(p+1)}(\ell, m)$ is the transition probability of the S_{p+1}-valued Markov chain $\mathbb{X}_t^{(p+1)}$.

Proof of Proposition 5.5. Suppose that $y \in \Delta_{p+1}$ and fix $t > 0$, $\varepsilon > 0$. Recall the definition of t_ε introduced in (5.12). By the Markov property,

$$\begin{aligned}\mathbf{P}_x^n \big[X_{t\theta_n^{(p+1)}} = y \big] &= \sum_{z \in V} \mathbf{P}_x^n \big[X_{t\theta_n^{(p+1)} - t_\varepsilon \theta_n^{(p)}} = z \big] \, \mathbf{P}_z^n \big[X_{t_\varepsilon \theta_n^{(p)}} = y \big] \\ &\le \max_{z \in V} \mathbf{P}_z^n \big[X_{t_\varepsilon \theta_n^{(p)}} = y \big] \, .\end{aligned}$$

By (5.12), this maximum is bounded by $\varepsilon + o_n(1)$, so that

$$\lim_{n \to \infty} \mathbf{P}_x^n \big[X_{t\theta_n^{(p+1)}} = y \big] = 0 \, ,$$

as claimed.

Suppose that $y \in \mathcal{V}_m^{(p+1)}$ for some $m \in S_{p+1}$ and fix $t > 0$, $\varepsilon > 0$. Recall the definition of Π_p, introduced in (3.6). Choose t_ε large enough for

$$\max_{z, z' \in V} \Big| \lim_{n \to \infty} p_{t\, \theta_n^{(p)}}^{(n)}(z, z') - \Pi_p(z, z') \Big| \le \varepsilon \tag{5.14}$$

for all $t \ge t_\varepsilon$.

By the Markov property, as $\theta_n^{(p)} \prec \theta_n^{(p+1)}$,

$$\begin{aligned}&\mathbf{P}_x^n \big[X_{t\theta_n^{(p+1)}} = y \big] \\ &\quad = \sum_{z, z' \in V} \mathbf{P}_x^n \big[X_{t_\varepsilon\, \theta_n^{(p)}} = z \big] \, \mathbf{P}_z^n \big[X_{t\theta_n^{(p+1)} - 2t_\varepsilon\, \theta_n^{(p)}} = z' \big] \, \mathbf{P}_{z'}^n \big[X_{t_\varepsilon\, \theta_n^{(p)}} = y \big] \, .\end{aligned}$$

By (5.14) and (5.4), this expression is equal to

$$\sum_{z' \in V} \sum_{\ell \in S_{p+1}} \sum_{z \in \mathcal{V}_\ell^{(p+1)}} \Pi_p(x, z) \, \mathbf{P}_z^n \big[X_{t\theta_n^{(p+1)} - 2t_\varepsilon\, \theta_n^{(p)}} = z' \big] \, \Pi_p(z', y) + O(\varepsilon) + o_n(1) \, .$$

The first part of the proof permits to restrict the first sum to $z' \in \mathcal{V}^{(p+1)}$. Since $y \in \mathcal{V}_m^{(p+1)}$, by (5.3) we may further restrict the sum to $z' \in \mathcal{V}_m^{(p+1)}$. At this point, by (5.2), we may replace $\Pi_p(z', y)$ by $\pi_m^{(p+1)}(y)$. Hence, the previous sum is equal to

$$\pi_m^{(p+1)}(y) \sum_{\ell \in S_{p+1}} \sum_{z \in \mathcal{V}_\ell^{(p+1)}} \Pi_p(x, z) \, \mathbf{P}_z^n \big[X_{t\theta_n^{(p+1)} - 2t_\varepsilon\, \theta_n^{(p)}} \in \mathcal{V}_m^{(p+1)} \big] + O(\varepsilon) + o_n(1) \, ,$$

where we summed over $z' \in \mathcal{V}_m^{(p+1)}$. By (5.13), as $n \to \infty$, this expression converges to

$$\sum_{\ell \in S_{p+1}} \sum_{z \in \mathcal{V}_\ell^{(p+1)}} \Pi_p(x, z) \, p_t^{(p+1)}(\ell, m) \, \pi_m^{(p+1)}(y) + O(\varepsilon) \, .$$

By the definition (3.6) of Π_p and since the measure $\pi_k^{(p+1)}(\,\cdot\,)$, $k \in S_{p+1}$, is supported on $\mathcal{V}_k^{(p+1)}$, the previous expression is equal to

$$\sum_{\ell \in S_{p+1}} \mathfrak{a}^{(p)}(x, \ell) \, p_t^{(p+1)}(\ell, m) \, \pi_m^{(p+1)}(y) + O(\varepsilon) \, ,$$

as claimed. □

Proof of (3.8)**.** Recall that $\theta_n^{(\mathfrak{q}+1)} \equiv +\infty$, and fix a sequence β_n such that $\theta_n^{(\mathfrak{q})} \prec \beta_n \prec \theta_n^{(\mathfrak{q}+1)}$. Since π_n is the stationary state,

$$\pi_n(\Delta_{\mathfrak{q}+1}) \;=\; \sum_{x\in V} \pi_n(x)\, \mathbf{P}_x^n\big[\, X_{\beta_n} \in \Delta_{\mathfrak{q}+1}\,\big] \;\le\; \max_{x\in V} \mathbf{P}_x^n\big[\, X_{\beta_n} \in \Delta_{\mathfrak{q}+1}\,\big]\,.$$

By the tree construction, $S_{\mathfrak{q}+1}$ is a singleton and there is only one measure at step $\mathfrak{q}+1$, the measure $\pi_1^{(\mathfrak{q}+1)}$ which is concentrated on $\mathcal{V}_1^{(\mathfrak{q}+1)} = \mathcal{V}^{(\mathfrak{q}+1)}$. Since $\pi_1^{(\mathfrak{q}+1)}(\Delta_{\mathfrak{q}+1}) = 0$, by (3.6), and the previous displayed equation,

$$\limsup_{n\to\infty} \pi_n(\Delta_{\mathfrak{q}+1}) \;\le\; \pi_1^{(\mathfrak{q}+1)}(\Delta_{\mathfrak{q}+1}) = 0\,.$$

It follows from the previous estimate that $\lim_{n\to\infty} \pi_n(\mathcal{V}^{(\mathfrak{q}+1)}) = 1$. Hence, by (2.16), for all $x \in \mathcal{V}^{(\mathfrak{q}+1)}$,

$$\lim_{n\to\infty} \pi_n(x) \;\text{ exists and belongs to } (0,1]\,.$$

6. Proof of Proposition 3.2

The proof is divided in several lemmata. We start with the asymptotic behavior of the stationary states π_n.

Lemma 6.1. *For all $j \in S_1$, $x \in \mathcal{V}_j$,*

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j)} \;=\; \pi^\sharp(x) \;>\; 0\,.$$

Proof. Fix $j \in S_1$. By (2.16), the limit $\pi_n(x)/\pi_n(\mathcal{V}_j)$ exists for all $x \in \mathcal{V}_j$ and is strictly positive. It remains to show that it is equal to $\pi^\sharp(x)$. Denote the limit by $m(x)$. Since π_n is a stationary state, for all $x \in \mathcal{V}_j$,

$$\sum_{y\in V} \pi_n(x)\, R_n(x,y) \;=\; \sum_{y\in V} \pi_n(y)\, R_n(y,x) \;\ge\; \sum_{y\in \mathcal{V}_j} \pi_n(y)\, R_n(y,x)\,.$$

As $\mathcal{V}_j$ is a closed irreducible class for the chain $\mathbb{X}_t$, dividing by $\pi_n(\mathcal{V}_j)$ and passing to the limit yields that

$$\sum_{y\in \mathcal{V}_j} m(x)\, \mathbb{R}_0(x,y) \;\ge\; \sum_{y\in \mathcal{V}_j} m(y)\, \mathbb{R}_0(y,x)\,.$$

Summing over $x \in \mathcal{V}_j$ shows that this inequality must be an identity for all $x \in \mathcal{V}_j$. Therefore, m is a stationary state for the chain $\mathbb{X}_t$ on $\mathcal{V}_j$ what implies that $m = \pi^\sharp$, as claimed. □

Lemma 6.2. *Fix $1 \le p \le \mathfrak{q}$. For all $m \in S_{p+1}$, $j \in \mathfrak{R}_m^{(p)}$,*

$$\lim_{n\to\infty} \frac{\pi_n(\mathcal{V}_j^{(p)})}{\pi_n(\mathcal{V}_m^{(p+1)})} \;=\; M_m^{(p)}(j)\,.$$

Proof. Fix $1 \le p \le \mathfrak{q}$ and $m \in S_{p+1}$. Consider the sequence of measures on $\mathfrak{R}_m^{(p)}$ defined by $m_n(j) = \pi_n(\mathcal{V}_j^{(p)})/\pi_n(\mathcal{V}_m^{(p+1)})$. By (2.16), it converges to a limiting measure, denoted by $m(j)$.

By [1, Proposition 6.3], $\pi_n(\,\cdot\,)/\pi_n(\mathcal{V}^{(p)})$ is the stationary state of the chain $Y_t^{n,p}$, the trace of $X_t^{(n)}$ on $\mathcal{V}^{(p)}$. Hence, for all $j \in \mathfrak{R}_m^{(p)}$, $x \in \mathcal{V}_j^{(p)}$,

$$\sum_{y\in\mathcal{V}^{(p)}} \pi_n(x)\, R_n^{(p)}(x,y) \;=\; \sum_{y\in\mathcal{V}^{(p)}} \pi_n(y)\, R_n^{(p)}(y,x) \;\ge\; \sum_{k\in\mathfrak{R}_m^{(p)}} \sum_{y\in\mathcal{V}_k^{(p)}} \pi_n(y)\, R_n^{(p)}(y,x)\,.$$

Sum over all $x \in \mathcal{V}_j^{(p)}$ to get that

$$\sum_{k\in S_p} \sum_{x\in\mathcal{V}_j^{(p)}} \pi_n(x)\, R_n^{(p)}(x,\mathcal{V}_k^{(p)}) \;\ge\; \sum_{k\in\mathfrak{R}_m^{(p)}} \sum_{y\in\mathcal{V}_k^{(p)}} \pi_n(y)\, R_n^{(p)}(y,\mathcal{V}_j^{(p)})\,,$$

where $R_n^{(p)}(z,\mathcal{V}_\ell^{(p)}) = \sum_{w\in\mathcal{V}_\ell^{(p)}} R_n^{(p)}(z,w)$. Remove on both sides of this inequality the case $k=j$. By (2.9), this new expression divided by $\pi_n(\mathcal{V}_m^{(p+1)})$ is equal to

$$\frac{\pi_n(\mathcal{V}_j^{(p)})}{\pi_n(\mathcal{V}_m^{(p+1)})} \sum_{k\in S_p\setminus\{j\}} r_n^{(p)}(j,k) \;\ge\; \sum_{k\in\mathfrak{R}_m^{(p)}\setminus\{j\}} \frac{\pi_n(\mathcal{V}_k^{(p)})}{\pi_n(\mathcal{V}_m^{(p+1)})}\, r_n^{(p)}(k,j)\,.$$

By the assumption on the measure m_n and by (2.10), as $n\to\infty$, this expression multiplied by $\theta_n^{(p)}$ on both sides converges to

$$m(j) \sum_{k\in S_p\setminus\{j\}} r^{(p)}(j,k) \;\ge\; \sum_{k\in\mathfrak{R}_m^{(p)}\setminus\{j\}} m(k)\, r^{(p)}(k,j)\,.$$

Since $\mathfrak{R}_m^{(p)}$ is a closed irreducible class for the chain $\mathbb{X}_t^{(p)}$, $r^{(p)}(j,k) = 0$ for all $k\notin\mathfrak{R}_m^{(p)}$, and the first sum can be restricted to this later set. Summing over j yields that this inequality must be an identity for all j. Therefore, m is a stationary state for the Markov chain $\mathbb{X}_t^{(p)}$ restricted to $\mathfrak{R}_m^{(p)}$. By ergodicity, $m = M_m^{(p)}$, as claimed. □

Corollary 6.3. *Fix $1\le p\le \mathfrak{q}+1$. For all $j\in S_p$, $x\in\mathcal{V}_j^{(p)}$,*

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})} \;=\; \pi_j^{(p)}(x)\,.$$

Proof. The proof is performed by induction. Lemma 6.1 covers the case $p=1$. Assume that this corollary has been proven for all $1\le p<p_0$, where $p_0\le\mathfrak{q}+1$. Fix $j\in S_{p_0}$ and $x\in\mathcal{V}_j^{(p_0)}$. By construction of $\mathcal{V}_j^{(p_0)}$, there exists $k\in S_{p_0-1}$ such that $x\in\mathcal{V}_k^{(p_0-1)}\subset\mathcal{V}_j^{(p_0)}$. We can write

$$\frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p_0)})} \;=\; \frac{\pi_n(x)}{\pi_n(\mathcal{V}_k^{(p_0-1)})}\, \frac{\pi_n(\mathcal{V}_k^{(p_0-1)})}{\pi_n(\mathcal{V}_j^{(p_0)})}\,.$$

By Lemma 6.2 and the induction assumption, as $n\to\infty$, this expression converges to

$$\pi_k^{(p_0-1)}(x)\, M_j^{(p_0-1)}(k)\,.$$

By (3.2), this expression is equal to $\pi_j^{(p_0)}(x)$ as claimed. □

We turn to the absorbing probabilities. We first consider the case where the state belongs to the valley.

Lemma 6.4. *For all $1\le p\le\mathfrak{q}+1$, $j\in S_p$ and $x\in\mathcal{V}_j^{(p)}$, $\mathfrak{a}^{(p-1)}(x,j)=1$.*

Proof. The proof is by induction on p. Fix $j \in S_1$ and $x \in \mathcal{V}_j$. By (3.3), $\mathfrak{a}^{(0)}(x,j) = 1$ because $\mathcal{V}_j$ is a closed irreducible class for $\mathbb{X}_t$ and x belongs to $\mathcal{V}_j$.

Suppose that the results has been proved for $p-1$. This means that if $j \in S_p$ and $x \in \mathcal{V}_j^{(p)}$, then $\mathfrak{a}^{(p-1)}(x,j) = 1$. As $\mathfrak{a}^{(p-1)}(x\,,\,\cdot\,)$ is a probability measure on S_p, $\mathfrak{a}^{(p-1)}(x,k) = 0$ for all $k \in S_p \setminus \{j\}$.

Fix $m \in S_{p+1}$ and $x \in \mathcal{V}_m^{(p+1)}$. As $\mathcal{V}_m^{(p+1)} = \cup_{j \in \mathfrak{R}_m^{(p)}} \mathcal{V}_j^{(p)}$, $x \in \mathcal{V}_j^{(p)}$ for some $j \in \mathfrak{R}_m^{(p)}$. By (3.5), and since, by the induction hypothesis, $\mathfrak{a}^{(p-1)}(x,k) = \delta_{j,k}$,

$$\mathfrak{a}^{(p)}(x,m) \;=\; \sum_{k \in S_p} \mathfrak{a}^{(p-1)}(x,k)\, \mathfrak{A}^{(p)}(k,m) \;=\; \mathfrak{A}^{(p)}(j,m)\;.$$

As $j \in \mathfrak{R}_m^{(p)}$ and $\mathfrak{R}_m^{(p)}$ is a closed irreducible class for $\mathbb{X}_t^{(p)}$, by the definition (3.4) of $\mathfrak{A}^{(p)}$, $\mathfrak{A}^{(p)}(j,m) = 1$, which completes the proof of the lemma. □

It follows from this lemma and from (3.6) that for all $1 \le p \le \mathfrak{q}+1$, $j \in S_p$, $x \in \mathcal{V}_j^{(p)}$ and sequences β_n such that $\theta_n^{(p-1)} \prec \beta_n \prec \theta_n^{(p)}$

$$\lim_{n\to\infty} p_{\beta_n}^{(n)}(x,\,\cdot\,) \;=\; \pi_j^{(p)}(\,\cdot\,)\;. \tag{6.1}$$

Lemma 6.4 provides a formula for $a^{(p-1)}(x\,,\,\cdot\,)$ when $x \in \mathcal{V}^{(p)}$. Lemma 6.5 completes the characterisation of $a^{(p-1)}(x\,,\,\cdot\,)$. The proof of this result relies on the following bound.

We claim that for all $a > 0$, $1 \le p \le \mathfrak{q}+1$, $x \notin \mathcal{V}^{(p)}$ and sequence β_n such that $\theta_n^{(p-1)} \prec \beta_n \prec \theta_n^{(p)}$,

$$\lim_{n\to\infty} \max_{x \in V} \mathbf{P}_x^n\big[\, H_{\mathcal{V}^{(p)}} \,>\, a\,\beta_n \,\big] \;=\; 0\;. \tag{6.2}$$

If $x \in \mathcal{V}^{(p)}$, there is nothing to prove. Fix $x \notin \mathcal{V}^{(p)}$ and observe that $\{H_{\mathcal{V}^{(p)}} > a\,\beta_n\} \subset \int_{[0,a\beta_n]} \chi_{\Delta_p}(X_s^n)\,ds \ge a\,\beta_n$. Hence, by Chebyshev inequality,

$$\mathbf{P}_x^n\big[\, H_{\mathcal{V}^{(p)}} \,>\, a\,\beta_n \,\big] \;\le\; \mathbf{P}_x^n\Big[\, \int_0^{a\beta_n} \chi_{\Delta_p}(X_s^n)\,ds \ge a\,\beta_n \,\Big] \;\le\; \frac{1}{a}\int_0^a \mathbf{E}_x^n\big[\, \chi_{\Delta_p}(X_{s\beta_n}^n)\,ds \,\big]\;.$$

The last term can be written as

$$\sum_{z \in \Delta_p} \frac{1}{a}\int_0^a p_{s\beta_n}^{(n)}(x,z)\,ds\;.$$

For each fixed $0 < s < a$ the sequence $s\beta_n$ satisfies the hypotheses of Theorem 3.1.(a). Hence, since $\pi_j^{(p)}(\Delta_p) = 0$ for all $j \in S_p$, $p_{s\beta_n}^{(n)}(x,z) \to 0$. Therefore, by the dominated convergence theorem, the previous expression vanishes, which proves claim (6.2).

Lemma 6.5. *For all $1 \le p \le \mathfrak{q}+1$, $j \in S_p$, $x \in V$,*

$$\mathfrak{a}^{(p-1)}(x,j) \;=\; \lim_{n\to\infty} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} \,<\, H_{\check{\mathcal{V}}_j^{(p)}} \,\big]\;.$$

Proof. Fix $1 \le p \le \mathfrak{q}+1$ and $j \in S_p$. If $x \in \mathcal{V}^{(p)}$, this result follows from Lemma 6.4. Assume that $x \notin \mathcal{V}^{(p)}$ and fix a sequence β_n such that $\theta_n^{(p-1)} \prec \beta_n \prec \theta_n^{(p)}$. On the one hand, by (3.6),

$$\lim_{n\to\infty} \sum_{y \in \mathcal{V}_j^{(p)}} p_{\beta_n}^{(n)}(x,y) \;=\; \mathfrak{a}^{(p-1)}(x,j)\;.$$

On the other hand,

$$\sum_{y\in\mathcal{V}_j^{(p)}} p_{\beta_n}^{(n)}(x,y) \;=\; \mathbf{P}_x^n\big[\, X_{\beta_n} \,\in\, \mathcal{V}_j^{(p)}\,\big] \;=\; \sum_{k\in S_p} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_k^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,,\, X_{\beta_n} \,\in\, \mathcal{V}_j^{(p)}\,\big]\,.$$

Fix $k\in S_p$ and $0<\varepsilon<1$. By (6.2), the previous probability for the fixed k is equal to

$$\mathbf{P}_x^n\big[\, H_{\mathcal{V}^{(p)}} \,<\, \varepsilon\,\beta_n\,,\, H_{\mathcal{V}_k^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,,\, X_{\beta_n} \,\in\, \mathcal{V}_j^{(p)}\,\big] \;+\; o_n(1)\,.$$

By the strong Markov property at $H_{\mathcal{V}^{(p)}}$, the previous probability is equal to

$$\mathbf{E}_x^n\Big[\, H_{\mathcal{V}^{(p)}} \,<\, \varepsilon\,\beta_n\,,\, H_{\mathcal{V}_k^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,,\, \mathbf{P}^n_{X(H_{\mathcal{V}^{(p)}})}\big[\, X_{\beta_n - H_{\mathcal{V}^{(p)}}} \,\in\, \mathcal{V}_j^{(p)}\,\big]\,\Big]\,.$$

In this formula, one computes the probability $\mathbf{P}^n_{X(H_{\mathcal{V}^{(p)}})}[\, X_{\beta_n - t} \,\in\, \mathcal{V}_j^{(p)}\,]$ and then replace t by $H_{\mathcal{V}^{(p)}}$. After the proof of this lemma, we show that for all $z\in V$

$$\sup_{t\le\varepsilon\beta_n} \mathbf{P}_z^n\big[\, X_{\beta_n - t} \,\in\, \mathcal{V}_j^{(p)}\,\big] \;\le\; \max_{y\in\mathcal{V}_j^{(p)}} \mathbf{P}_y^n\big[\, H_{\check{\mathcal{V}}_j^{(p)}} < \varepsilon\,\beta_n\,\big] \;+\; \mathbf{P}_z^n\big[\, X_{\beta_n} \,\in\, \mathcal{V}_j^{(p)}\cup\Delta_p\,\big]\,. \tag{6.3}$$

By (6.4),

$$\lim_{n\to\infty} \max_{y\in\mathcal{V}_j^{(p)}} \mathbf{P}_y^n\big[\, H_{\check{\mathcal{V}}_j^{(p)}} < \varepsilon\,\beta_n\,\big] \;=\; 0\,.$$

Therefore, up to this point, we proved that

$$\mathfrak{a}^{(p-1)}(x,j) \;\le\; \sum_{k\in S_p} \liminf_{n\to\infty} \mathbf{E}_x^n\Big[\, H_{\mathcal{V}^{(p)}} \,<\, \varepsilon\,\beta_n\,,\, H_{\mathcal{V}_k^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,,\, \mathbf{P}^n_{X(H_{\mathcal{V}^{(p)}})}\big[\, X_{\beta_n} \,\in\, \mathcal{V}_j^{(p)}\cup\Delta_p\,\big]\,\Big]\,.$$

By (3.6) and Lemma 6.4, if $k\neq j$ the previous expectation vanishes as $n\to\infty$. If $k=j$ by the same reasons, the probability inside the expectation converges to 1 as $n\to\infty$. Hence,

$$\mathfrak{a}^{(p-1)}(x,j) \;\le\; \liminf_{n\to\infty} \mathbf{P}_x^n\big[\, H_{\mathcal{V}^{(p)}} \,<\, \varepsilon\,\beta_n\,,\, H_{\mathcal{V}_j^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,\big]\,.$$

Therefore, by (6.2), for all $j\in S_p$,

$$\mathfrak{a}^{(p-1)}(x,j) \;\le\; \liminf_{n\to\infty} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,\big]\,.$$

The previous inequality implies that equality holds for all $j\in S_p$. Indeed, assume that strict inequality holds for some $j\in S_p$. Then, as $\mathfrak{a}^{(p-1)}(x\,,\,\cdot\,)$ is a probability measure on S_p,

$$\begin{aligned} 1 \;&=\; \sum_{j\in S_p} \mathfrak{a}^{(p-1)}(x,j) \;<\; \sum_{j\in S_p} \liminf_{n\to\infty} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,\big] \\ &\le\; \liminf_{n\to\infty} \sum_{j\in S_p} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} \,=\, H_{\mathcal{V}^{(p)}}\,\big] \;=\; 1\,, \end{aligned}$$

which is a contradiction. □

We turn to the proof of (6.3). Inserting the event $\{X_{\beta_n} \in \mathcal{V}_j^{(p)} \cup \Delta_p\}$ and its complement inside the probability appearing on the left-hand side of (6.3) yields that this probability is bounded by

$$\begin{aligned} &\mathbf{P}_z^n \big[X_{\beta_n - t} \in \mathcal{V}_j^{(p)}, X_{\beta_n} \notin \mathcal{V}_j^{(p)} \cup \Delta_p \big] + \mathbf{P}_z^n \big[X_{\beta_n} \in \mathcal{V}_j^{(p)} \cup \Delta_p \big] \\ &\quad \le \max_{y \in \mathcal{V}_j^{(p)}} \mathbf{P}_y^n \big[X_t \notin \mathcal{V}_j^{(p)} \cup \Delta_p \big] + \mathbf{P}_z^n \big[X_{\beta_n} \in \mathcal{V}_j^{(p)} \cup \Delta_p \big] , \end{aligned}$$

where we used the Markov property to estimate the first by the second line. As $t \le \varepsilon \beta_n$, this expression is clearly bounded by

$$\max_{y \in \mathcal{V}_j^{(p)}} \mathbf{P}_y^n \big[H_{\breve{\mathcal{V}}_j^{(p)}} < \varepsilon \beta_n \big] + \mathbf{P}_z^n \big[X_{\beta_n} \in \mathcal{V}_j^{(p)} \cup \Delta_p \big] ,$$

as claimed in (6.3).

To complete the proof of Lemma 6.5, it remains to show that for all $1 \le p \le \mathfrak{q}$, $j \in S_p$,

$$\lim_{a \to 0} \limsup_{n \to \infty} \max_{x \in \mathcal{V}_j^{(p)}} \mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}_j^{(p)}} < a \, \theta_n^{(p)} \big] = 0 . \tag{6.4}$$

Fix $1 \le p \le \mathfrak{q}$, $j \in S_p$, $x \in \mathcal{V}_j^{(p)}$. Recall that $Y^{n,p}$ represents the trace of the process X_t^n on $\mathcal{V}^{(p)}$, and that $\Phi_p : \mathcal{V}^{(p)} \to S_p$ stands for the projection which sends $x \in \mathcal{V}_j^{(p)}$ to j. By [43, Theorems 2.1 and 2.12], under $\mathbf{P}_x^n$, the process $\Phi_p(Y^{n,p}_{t\theta_n^{(p)}})$ converges weakly in the Skorohod topology to $\mathbb{X}_t^{(p)}$. In particular,

$$\lim_{a \to 0} \limsup_{n \to \infty} \mathbf{P}_x^n \big[H_{\breve{\mathcal{V}}_j^{(p)}}(Y^{n,p}) < a \, \theta_n^{(p)} \big] = 0 .$$

In this formula, $H_{\breve{\mathcal{V}}_j^{(p)}}(Y^{n,p})$ stands for the hitting time of $\breve{\mathcal{V}}_j^{(p)}$ for the process $Y_t^{n,p}$. Since $H_{\breve{\mathcal{V}}_j^{(p)}}(Y^{n,p}) \le H_{\breve{\mathcal{V}}_j^{(p)}}$, assertion (6.4) follows from this last result.

We complete this section with a consequence of Lemma 6.5. Recall from (2.11) that $\mathbb{Q}_k^{(p)}$ stands for the measure on $D(\mathbb{R}_+, S_p)$ induced by the process $\mathbb{X}_t^{(p)}$ starting from $k \in S_p$.

Lemma 6.6. *For all $2 \le p \le \mathfrak{q}$, $i \in S_{p-1}$ and $x \in \mathcal{V}_i^{(p-1)}$,*

$$\mathfrak{a}^{(p-1)}(x, j) = \mathbb{Q}_i^{(p-1)} \big[H_{\mathfrak{R}_j^{(p)}} < H_{\breve{\mathfrak{R}}_j^{(p)}} \big] , \quad j \in S_p ,$$

where $\breve{\mathfrak{R}}_j^{(p)} = \cup_{k \in S_p \setminus \{j\}} \mathfrak{R}_k^{(p)}$.

Proof. Recall that $Y_t^{n,p-1}$ represents the trace of $X_t^{(n)}$ on $\mathcal{V}^{(p-1)}$. By [3, Theorem 2.1], under the measure $\mathbf{P}_x^n$ the process $\mathbb{X}_t^{n,p-1} := \Phi_{p-1}(Y^{n,p-1}_{t\theta_n^{(p-1)}})$ converges weakly in the Skorohod topology to the S_{p-1}-valued process $\mathbb{X}_t^{(p-1)}$ introduced below (2.10).

Clearly, under the measure $\mathbf{P}_x^n$,

$$\big\{ H_{\mathcal{V}_j^{(p)}}(X^n) < H_{\breve{\mathcal{V}}_j^{(p)}}(X^n) \big\} = \big\{ H_{\mathcal{V}_j^{(p)}}(Y^{n,p-1}) < H_{\breve{\mathcal{V}}_j^{(p)}}(Y^{n,p-1}) \big\} .$$

This identity asserts that the process $X^{(n)}$ hits the set $\mathcal{V}_j^{(p)}$ before the set $\breve{\mathcal{V}}_j^{(p)}$ if and only if this happens to the trace process $Y^{n,p-1}$. By projecting the process $Y^{n,p-1}$ with Φ_{p-1}, the last event becomes

$$\big\{ H_{\mathfrak{R}_j^{(p)}}(\mathbb{X}^{n,p-1}) < H_{\breve{\mathfrak{R}}_j^{(p)}}(\mathbb{X}^{n,p-1}) \big\} ,$$

Therefore, by Lemma 6.5, for $2 \le p \le \mathfrak{q}+1$, $j \in S_p$

$$\begin{aligned}\mathfrak{a}^{(p-1)}(x,j) &= \lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathcal{V}_j^{(p)}} < H_{\check{\mathcal{V}}_j^{(p)}} \,\big] \\ &= \lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathfrak{R}_j^{(p)}}(\mathbb{X}^{n,p-1}) < H_{\check{\mathfrak{R}}_j^{(p)}}(\mathbb{X}^{n,p-1}) \,\big]\,.\end{aligned}$$

As $\mathbb{X}^{n,p-1}$ converges weakly in the Skorohod topology to $\mathbb{X}^{(p-1)}$,

$$\lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathfrak{R}_j^{(p)}}(\mathbb{X}^{n,p-1}) < H_{\check{\mathfrak{R}}_j^{(p)}}(\mathbb{X}^{n,p-1}) \,\big] = \mathbb{Q}_i^{(p-1)} \big[\, H_{\mathfrak{R}_j^{(p)}} < H_{\check{\mathfrak{R}}_j^{(p)}} \,\big]\,,$$

as claimed. □

7. Preliminary estimates

In this section, we present some estimates needed in the proof of Theorem 3.3. We assume throughout it that the process is reversible. We start with some estimates on the stationary state, now assumed to be reversible.

Fix $x \in \Delta$. As x is a transient state for the chain $\mathbb{X}_t$, it is eventually absorbed by a closed irreducible class $\mathcal{V}_k$, $k \in S_1$. Fix $j \in S_1$ such that $\mathfrak{a}^{(0)}(x,j) > 0$, where $\mathfrak{a}^{(0)}(x,j)$ has been introduced in (2.3). We claim that

$$\pi_n(x) \prec \pi_n(\mathcal{V}_j)\,. \tag{7.1}$$

Indeed, as $\mathfrak{a}^{(0)}(x,j) > 0$, there exists a sequence $x = x_0, \dots, x_\ell$ of elements of V such that $\mathbb{R}_0(x_i, x_{i+1}) > 0$, $x_i \in \Delta$, $0 \le i < \ell$, $x_\ell \in \mathcal{V}_j$. By reversibility,

$$\frac{\pi_n(x_i)}{\pi_n(x_{i+1})} = \frac{R_n(x_{i+1}, x_i)}{R_n(x_i, x_{i+1})}\,.$$

Since $R_n(x_i, x_{i+1}) \to \mathbb{R}_0(x_i, x_{i+1}) > 0$, by (2.1), $\pi_n(x_i) \preceq \pi_n(x_{i+1})$. As $x_{\ell-1} \in \Delta$, $x_\ell \in \mathcal{V}_j$, $R_n(x_\ell, x_{\ell-1}) \to \mathbb{R}_0(x_\ell, x_{\ell-1}) = 0$, so that $\pi_n(x_{\ell-1}) \prec \pi_n(x_\ell)$, which proves claim (7.1).

Next result extends this estimate

Lemma 7.1. *Fix $2 \le p \le \mathfrak{q}$, $j \in S_p$, $x \in \mathcal{V}^{(p-1)} \setminus \mathcal{V}^{(p)}$. If $\mathfrak{a}^{(p-1)}(x,j) > 0$, then, $\pi_n(x) \prec \pi_n(\mathcal{V}_j^{(p)})$.*

Proof. The proof is similar to the one presented to derive (7.1). Suppose that $x \in \mathcal{V}_i^{(p-1)} \setminus \mathcal{V}^{(p)}$ for $i \in S_{p-1}$. As x does not belong to $\mathcal{V}^{(p)}$, $i \in \mathfrak{T}_{p-1}$.

As $\mathfrak{a}^{(p-1)}(x,j) > 0$, by Lemma 6.6, there exists a sequence $i = i_0, \dots, i_\ell$ of elements of S_{p-1} such that $r^{(p-1)}(i_a, i_{a+1}) > 0$, $i_a \in \mathfrak{T}_{p-1}$, $0 \le a < \ell$, $i_\ell \in \mathfrak{R}_j^{(p-1)}$. By reversibility, (2.8) and (2.9),

$$\frac{\pi_n(\mathcal{V}_{i_a}^{(p-1)})}{\pi_n(\mathcal{V}_{i_{a+1}}^{(p-1)})} = \frac{r_n^{(p-1)}(i_{a+1}, i_a)}{r_n^{(p-1)}(i_a, i_{a+1})}\,. \tag{7.2}$$

Since $\theta_n^{(p-1)}\, r_n^{(p-1)}(i_a, i_{a+1}) \to r^{(p-1)}(i_a, i_{a+1}) > 0$, by (2.10), $\pi_n(\mathcal{V}_{i_a}^{(p-1)}) \preceq \pi_n(\mathcal{V}_{i_{a+1}}^{(p-1)})$. As $i_{\ell-1} \in \mathfrak{T}_{p-1}$, $i_\ell \in \mathfrak{R}_j^{(p-1)}$, $\theta_n^{(p-1)}\, r_n^{(p)}(i_\ell, i_{\ell-1}) \to r^{(p)}(i_\ell, i_{\ell-1}) = 0$, so that $\pi_n(\mathcal{V}_{i_{\ell-1}}^{(p-1)}) \prec \pi_n(\mathcal{V}_{i_\ell}^{(p-1)})$. Since $i_\ell \in \mathfrak{R}_j^{(p-1)}$, $\mathcal{V}_{i_\ell}^{(p-1)} \subset \mathcal{V}_j^{(p)}$, and the lemma is proved. □

Corollary 7.2. *Fix $2 \le p \le \mathfrak{q}$, $j \in S_p$, $x \in V \setminus \mathcal{V}^{(p)}$. If $\mathfrak{a}^{(p-1)}(x,j) > 0$, then, $\pi_n(x) \prec \pi_n(\mathcal{V}_j^{(p)})$.*

Proof. Fix $x \in V \setminus \mathcal{V}^{(p)}$ and let $r(x)$ be the element r of $\{1, \dots, p\}$ such that $x \in \mathcal{V}^{(r-1)} \setminus \mathcal{V}^{(r)}$, where $\mathcal{V}^{(0)} = V$. The proof is by induction on $r(x)$.

If $r(x) = p$, the assertion corresponds to the one of Lemma 7.1. Suppose that the corollary has been proved for $y \in \mathcal{V}^{(r-1)} \setminus \mathcal{V}^{(r)}$ and all $r \in \{s+1, \dots, p\}$, and fix $x \in \mathcal{V}^{(s-1)} \setminus \mathcal{V}^{(s)}$. By the strong Markov property at time $H_{\mathcal{V}^{(s)}}$,

$$\begin{aligned} \mathfrak{a}^{(p-1)}(x,j) \;&=\; \lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathcal{V}_j^{(p)}} \,<\, H_{\check{\mathcal{V}}_j^{(p)}} \,\big] \\ &= \lim_{n\to\infty} \sum_{k\in S_s} \sum_{z\in\mathcal{V}_k^{(s)}} \mathbf{P}_x^n \big[\, H_{\mathcal{V}_k^{(s)}} \,=\, H_{\mathcal{V}^{(s)}} \,,\, X(H_{\mathcal{V}^{(s)}}) \,=\, z \,\big]\, \mathbf{P}_z^n \big[\, H_{\mathcal{V}_j^{(p)}} \,<\, H_{\check{\mathcal{V}}_j^{(p)}} \,\big] \,. \end{aligned}$$

The sum can be restricted to elements $k \in S_s$ and $z \in \mathcal{V}_k^{(s)}$ such that $\mathfrak{a}^{(s-1)}(x,k) > 0$, $\mathfrak{a}^{(p-1)}(z,j) > 0$. By Lemma 7.1, $\pi_n(x) \prec \pi_n(\mathcal{V}_k^{(s)})$ and by the induction assumption, $\pi_n(z) \preceq \pi_n(\mathcal{V}_j^{(p)})$. The previous estimate may not be strict as it might happen that z belongs to $\mathcal{V}_j^{(p)}$. By (2.16), $\pi_n(\mathcal{V}_k^{(s)}) \sim \pi_n(z)$ so that $\pi_n(x) \prec \pi_n(\mathcal{V}_j^{(p)})$, as claimed. □

Potential theory. We turn to estimates involving the capacity. Recall the definition of comparable sequences introduced just before the main hypothesis (2.4). Let $c_n \colon E \to \mathbb{R}_+$ be given by $c_n(x,y) := \pi_n(x) R_n(x,y)$ and note that c_n is symmetric. It follows from (2.4) (cf. equation (2.5) in [3]) that the sequences $c_n(x,y)$ are comparable. A self-avoiding path γ from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{A}, \mathcal{B} \subset V$, $\mathcal{A} \cap \mathcal{B} = \varnothing$, is a sequence of sites $(x_0, x_1, \dots, x_m)$ such that $x_0 \in \mathcal{A}$, $x_m \in \mathcal{B}$, $x_i \neq x_j$, $i \neq j$, $R_n(x_i, x_{i+1}) > 0$, $0 \le i < m$. Denote by $\Gamma_{\mathcal{A},\mathcal{B}}$ the set of self-avoiding paths from $\mathcal{A}$ to $\mathcal{B}$ and let

$$c_n(\mathcal{A},\mathcal{B}) \;:=\; \max_{\gamma\in\Gamma_{\mathcal{A},\mathcal{B}}} c_n(\gamma) \,, \quad c_n(\gamma) \;:=\; \min_{0\le i<m} c_n(x_i, x_{i+1}) \,.$$

Note that there might be more than one optimal path and that $c_n(\{x\},\{y\}) \ge c_n(x,y)$, with possibly a strict inequality. Next result is [3, Lemma 4.1].

Lemma 7.3. *There exists a positive and finite constant C_1 such that*

$$C_1^{-1} \;\le\; \frac{\mathrm{cap}_n(\mathcal{A},\mathcal{B})}{c_n(\mathcal{A},\mathcal{B})} \;\le\; C_1$$

for all $n \ge$ and non-empty, disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of V.

Fix two disjoint, non-empty subsets $\mathcal{A}$, $\mathcal{B}$ of V, and let $h_{\mathcal{A},\mathcal{B}}$ be the equilibrium potential between $\mathcal{A}$ and $\mathcal{B}$:

$$h_{\mathcal{A},\mathcal{B}}(x) \;:=\; \mathbf{P}_x^n[H_{\mathcal{A}} < H_{\mathcal{B}}] \,, \quad x \in V \,.$$

Denote by $D_n(f)$ the Dirichlet form of a function $f : V \to \mathbb{R}$:

$$D_n(f) \;:=\; \langle\, f \,,\, (-\mathcal{L}_n) f \,\rangle_{\pi_n} \,.$$

It is well known [32, equation (B.7)], that

$$\mathrm{cap}_n(\mathcal{A}\,,\,\mathcal{B}) \;=\; D_n(h_{\mathcal{A},\mathcal{B}}) \,.$$

Lemma 7.4. *There exists a finite constant C_0, independent of n, such that*

$$h_{\mathcal{A},\mathcal{B}}(x)^2 \;\le\; C_0 \, \frac{\mathrm{cap}_n(\mathcal{A}\,,\,\mathcal{B})}{\mathrm{cap}_n(\{x\}\,,\,\mathcal{B})}$$

for all $x \notin \mathcal{A} \cup \mathcal{B}$.

Proof. Let $h = _{\mathcal{A},\mathcal{B}}$, and let $\gamma = (x = x_0, \dots, x_m)$ be a self-avoiding path between x and $\mathcal{B}$. Hence $R_n(x_i, x_{i+1}) > 0$, $x_i \notin \mathcal{B}$, $0 \le i < m$ and $x_m \in \mathcal{B}$. As $x_m \in \mathcal{B}$, $h(x_m) = 0$ so that

$$h(x)^2 \;=\; (\, h(x_0) - h(x_m)\,)^2 \;\le\; \sum_{i=0}^{m-1} c_n(x_i, x_{i+1})\, [\, h(x_{i+1} - h(x_i)\,]^2 \sum_{i=0}^{m-1} \frac{1}{c_n(x_i, x_{i+1})}\,.$$

As the path is self-avoiding, this quantity is bounded by

$$|E|\, D_n(h) \max_{0\le i<m} \frac{1}{c_n(x_i, x_{i+1})} \;=\; |E|\, \mathrm{cap}_n(\mathcal{A}\,,\,\mathcal{B}) \max_{0\le i<m} \frac{1}{c_n(x_i, x_{i+1})}\,.$$

Minimising over all possible paths γ from x to $\mathcal{B}$ yields that

$$h_{\mathcal{A},\mathcal{B}}(x)^2 \;\le\; |E|\, \mathrm{cap}_n(\mathcal{A}\,,\,\mathcal{B})\, \frac{1}{\max_\gamma \min_{0\le i<m} c_n(x_i, x_{i+1})}\,.$$

The assertion of the lemma follows from Lemma 7.3. □

Lemma 7.5. *Fix $1 \le p \le \mathfrak{q}$, and suppose that $r^{(p)}(j,k) > 0$ for some j, $k \in S_p$. Then,*

$$\liminf_{n\to\infty} \frac{\theta_n^{(p)}}{\pi_n(\mathcal{V}_j^{(p)})}\, \mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{V}_k^{(p)}) \;>\; 0\,.$$

We do not exclude the possibility that this lim sup *is $+\infty$.*

Proof. We argue by contradiction, proving that if the lim sup vanishes than $r^{(p)}(j,k) = 0$, but we first derive a consequence of the positivity of $r^{(p)}(j,k)$.

Fix $x \in \mathcal{V}_j^{(p)}$. The main result in [3] states that under the measure $\mathbf{P}_x^n$, the process $\mathbb{X}_t^{n,p} = \Phi_p(Y^{n,p}_{t\theta_n^{(p)}})$ converges weakly in the Skorohod topology to the S_p-valued process $\mathbb{X}_t^{(p)}$. Hence, if $r^{(p)}(j,k) > 0$, for every $a > 0$,

$$\liminf_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathcal{V}_k^{(p)}}(Y^{n,p}) \,<\, a\,\theta_n^{(p)} \,\big] \;\ge\; \mathbb{Q}_j^{(p)} \big[\, H_k \,<\, a \,\big] \;>\; 0\,. \tag{7.3}$$

Denote by $Y_t^{n,j,k}$ the trace of X_t^n on $\mathcal{V}_j^{(p)} \cup \mathcal{V}_k^{(p)}$. By [1, Theorem 2.6] (for the process $Y_t^{n,j,k}$ and with $\mathcal{B} = \mathcal{W} = \mathcal{V}_j^{(p)}$, $\mathcal{B}^c = \mathcal{V}_k^{(p)}$) and [3, Theorem 7.1] (Condition T4 ensures that the hypothesis (2.14) of [1, Theorem 2.6] is in force), under $\mathbf{P}_x^n$, the random variable $H_{\mathcal{V}_k^{(p)}}(Y^{n,j,k})/\theta_n^{j,k}$ converges in distribution to a mean-one exponencial random variable. In this formula,

$$\theta_n^{j,k} \;=\; \frac{\pi_n^{j,k}(\mathcal{V}_j^{(p)})}{\mathrm{cap}_n^{j,k}(\mathcal{V}_j^{(p)}\,,\,\mathcal{V}_k^{(p)})}\,, \quad \pi_n^{j,k}(\mathcal{V}_j^{(p)}) \;=\; \frac{\pi_n(\mathcal{V}_j^{(p)})}{\pi_n(\mathcal{V}_j^{(p)} \cup \mathcal{V}_j^{(p)})}\,,$$

and $\mathrm{cap}_n^{j,k}$ stands for the capacity with respect to the trace process $Y_t^{n,j,k}$. By [1, Lemma 6.9], $\mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{V}_k^{(p)}) = \pi_n(\mathcal{V}_j^{(p)} \cup \mathcal{V}_j^{(p)})\, \mathrm{cap}_n^{j,k}(\mathcal{V}_j^{(p)}\,,\,\mathcal{V}_k^{(p)})$, so that

$$\theta_n^{j,k} \;=\; \frac{\pi_n(\mathcal{V}_j^{(p)})}{\mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{V}_k^{(p)})}\,.$$

Suppose by contradiction that the limit appearing in the statement of the lemma vanishes, so that $\theta_n^{(p)}/\theta_n^{j,k} \to 0$ and $\mathbf{P}_x^n[H_{\mathcal{V}_k^{(p)}}(Y^{n,j,k}) < a\theta_n^{(p)}] \to 0$ for all $a > 0$.

Hence, as $H_{\mathcal{V}_k^{(p)}}(Y^{n,j,k}) \le H_{\mathcal{V}_k^{(p)}}(Y^{n,p})$,

$$\lim_{n\to\infty} \mathbf{P}_x^n \big[\, H_{\mathcal{V}_k^{(p)}}(Y^{n,p}) \,<\, a\,\theta_n^{(p)} \,\big] \;=\; 0\,.$$

This contradicts (7.3), and therefore one must have that $r^{(p)}(j,k) = 0$, completing the proof of the lemma by contradiction. □

Fix $1 \le p \le \mathfrak{q}$, $j \in \mathfrak{T}_p$. Let A_j be the recurrent points of the chain $\mathbb{X}_t^{(p)}$ which can be hit before any other recurrent point when the chain starts from j. More precisely, $\ell \in A_j$ if, and only if, $\ell \in \mathfrak{R}^{(p)}$ and there exists a path $j_0 = j, j_1, \dots, j_m = \ell$ such that $r^{(p)}(j_a, j_{a+1}) > 0$, $j_a \in \mathfrak{T}_p$, $0 \le a < m$. Let $\mathcal{A}_j = \cup_{\ell\in A_j} \mathcal{V}_\ell^{(p)}$.

Lemma 7.6. *Fix $1 \le p \le \mathfrak{q}$, $x \in \mathcal{V}_j^{(p)}$, $j \in \mathfrak{T}_p$. Then,*

$$\liminf_{n\to\infty} \frac{\theta_n^{(p)}}{\pi_n(x)}\, \mathrm{cap}_n(\{x\}\,,\,\mathcal{A}_j) \;>\; 0\,.$$

Proof. As $x \in \mathcal{V}_j^{(p)}$, $j \in \mathfrak{T}_p$, there exists a path $j_0 = j, j_1, \dots, j_m$ such that $r^{(p)}(j_a, j_{a+1}) > 0$, $j_a \in \mathfrak{T}_p$, $0 \le a < m$, $j_m \in A_j$. Moreover, for $0 \le a < m$, by (7.2), with p instead of $p-1$, $\pi_n(\mathcal{V}_{j_a}^{(p)}) \preceq \pi_n(\mathcal{V}_{j_{a+1}}^{(p)})$, and, by Lemma 7.5,

$$\liminf_{n\to\infty} \frac{\theta_n^{(p)}}{\pi_n(\mathcal{V}_{j_a}^{(p)})}\, \mathrm{cap}_n(\mathcal{V}_{j_a}^{(p)}\,,\,\mathcal{V}_{j_{a+1}}^{(p)}) \;>\; 0\,.$$

This limit is finite because this capacity is bounded by the one obtained by replacing $\mathcal{V}_{j_{a+1}}^{(p)}$ by $\breve{\mathcal{V}}_{j_a}^{(p)}$, and the limit for this later one is finite in view of (2.10).

By the previous displayed equation and Lemma 7.3, the exist a positive constant c_0 and self-avoiding paths γ_a from $\mathcal{V}_{j_a}^{(p)}$ to $\mathcal{V}_{j_{a+1}}^{(p)}$ such that $c_n(\gamma_a) \ge c_0\, \pi_n(\mathcal{V}_{j_a}^{(p)})/\theta_n^{(p)} \ge c_0\, \pi_n(\mathcal{V}_j^{(p)})/\theta_n^{(p)}$, $0 \le a < m$.

Denote by y_a, $0 \le a < m$, the starting points of the paths γ_a, and by x_{a+1} its ending point. Let $x_0 = x$. Hence x_a, y_a belongs to the same well $\mathcal{V}_{j_a}^{(p)}$. By Property (T4) in [3, Theorem 7.1] and Lemma 7.3, there exist self-avoiding paths γ_a' from x_a to y_a such that $c_n(\gamma_a') \ge c_0 \pi_n(\mathcal{V}_{j_a}^{(p)})/\theta_n^{(p-1)} \ge c_0 \pi_n(\mathcal{V}_j^{(p)})/\theta_n^{(p)}$, where the value of the constant c_0 may change from line to line.

By concatenating the paths γ_a, γ_a', we obtain a path γ from x to $\mathcal{A}_j$ such that $c_n(\gamma) \ge c_0 \pi_n(\mathcal{V}_j^{(p)})/\theta_n^{(p)}$. If it is not self-avoiding, we may shorten it improving the lower on $c_n(\gamma)$ and keeping it as a path from x to $\mathcal{A}_j$. At this point, the assertion of the lemma follows from Lemma 7.3 and (2.16). □

Fix $x \in \Delta$. Let $\mathcal{A}_x$ be the recurrent points of the chain $\mathbb{X}_t$ which can be hit before any other recurrent point when the chain starts from x. More precisely, $y \in \mathcal{A}_x$ if, and only if, $y \in \mathcal{V}$ and there exists a path $x_0 = x, x_1, \dots, x_m = y$ such that $\mathbb{R}_0(x_a, x_{a+1}) > 0$, $x_a \in \Delta$, $0 \le a < m$.

Lemma 7.7. *Fix $x \in \Delta$. Then,*

$$\liminf_{n\to\infty} \frac{1}{\pi_n(x)}\, \mathrm{cap}_n(\{x\}\,,\,\mathcal{A}_x) \;>\; 0\,.$$

Proof. By definition of the path from x to $\mathcal{A}_x$, $\mathbb{R}_0(x_a, x_{a+1}) > 0$ for all $0 \le a < m$. Hence $\pi_n(x_a) \preceq \pi_n(x_{a+1})$ and $c_n(x_a, x_{a+1}) = \pi_n(x_a)\, R_n(x_a, x_{a+1}) \succeq \pi_n(x_a) \succeq$

$\pi_n(x)$. This proves that $c_n(\gamma) \succeq \pi_n(x)$ and completes the proof of the lemma in view of Lemma 7.3. □

Lemma 7.8. *Fix $1 \le p \le \mathfrak{q}$. Then, for all for $x \notin \mathcal{V}^{(p)}$, $j \in S_p$,*

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})}\, \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} = H_{\mathcal{V}^{(p)}}\,\big]^2 \;=\; 0\,.$$

Proof. If $\pi_n(x)/\pi_n(\mathcal{V}_j^{(p)}) \to 0$, the conclusion is straightforward. If $\pi_n(\mathcal{V}_j^{(p)}) \sim \pi_n(x)$, by Corollary 7.2, $\mathfrak{a}^{(p-1)}(x,j) = 0$, so that, by Lemma 6.5,

$$\lim_{n\to\infty} \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} = H_{\mathcal{V}^{(p)}}\,\big] \;=\; 0\,,$$

and the assertion of the lemma follows.

Assume that $\pi_n(\mathcal{V}_j^{(p)}) \prec \pi_n(x)$, and suppose that $x \in \mathcal{V}$. Let $1 \le r < p$ such that $x \in \mathcal{V}_k^{(r)}$ for some $k \in \mathfrak{T}_r$. Such r exists and is smaller than p because $x \notin \mathcal{V}^{(p)}$.

Recall the definition of the sets A_k, $\mathcal{A}_k$ introduced just before Lemma 7.6. Add the index r to recall that $k \in \mathfrak{T}_r$ and write $A_{r,k}$, $\mathcal{A}_{r,k}$ instead of A_k, $\mathcal{A}_k$, respectively. By definition, $\mathcal{A}_{r,k} \subset \mathcal{V}^{(r+1)}$.

By the tree construction, since $r < p$, there exists $B \subset S_{r+1}$, such that $\mathcal{V}_j^{(p)} = \cup_{i\in B}\mathcal{V}_i^{(r+1)}$. By Lemma 7.1, $\pi_n(x) \prec \pi_n(\mathcal{V}_\ell^{(r)})$, $\ell \in A_{r,k}$. Thus, as $\pi_n(\mathcal{V}_j^{(p)}) \prec \pi_n(x)$, $\pi_n(\mathcal{V}_i^{(r+1)}) \prec \pi_n(\mathcal{V}_\ell^{(r)})$ for all $i \in B$, $\ell \in A_{r,k}$. Hence, since by (2.16), all elements of the same valley have measures of the same order, $\mathcal{A}_{r,k} \cap \mathcal{V}_j^{(p)} = \varnothing$.

The proof is by induction on r. We first prove it for $r = p-1$. In the sequel, we show that if it holds for all $r \in \{r_0+1, \dots, p-1\}$, then it holds for r_0 also. First, assume that $r = p-1$ (and keep the index r of $\mathcal{A}_{r,k}$ as r, though $r = p-1$). In this case, since $\mathcal{A}_{r,k} \subset \mathcal{V}^{(p)}$ and $\mathcal{A}_{r,k} \cap \mathcal{V}_j^{(p)} = \varnothing$, we have that $\mathcal{A}_{r,k} \subset \breve{\mathcal{V}}_j^{(p)}$. Therefore,

$$\mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\breve{\mathcal{V}}_j^{(p)}}\,\big] \;\le\; \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\mathcal{A}_{r,k}}\,\big]\,.$$

By Lemma 7.4,

$$\frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})}\, \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\mathcal{A}_{r,k}}\,\big]^2 \;\le\; C_0\, \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})}\, \frac{\mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{A}_{r,k})}{\mathrm{cap}_n(\{x\}\,,\,\mathcal{A}_{r,k})} \tag{7.4}$$

for some finite constant C_0. By equation (B.2) in [32],

$$\mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{A}_{r,k}) \;\le\; \mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\breve{\mathcal{V}}_j^{(p)})\,.$$

By (2.7), this expression is bounded by $C_0\,\pi_n(\mathcal{V}_j^{(p)})/\theta_n^{(p)}$ for some finite constant C_0 whose value may change from line to line.

On the other hand, by Lemma 7.6, as $r = p-1$,

$$\mathrm{cap}_n(\{x\}\,,\,\mathcal{A}_{r,k}) \;\ge\; c_0\,\pi_n(x)/\theta_n^{(p-1)} \tag{7.5}$$

for some positive constant c_0. Putting together the two previous estimates, we obtain that the expression in (7.4) vanishes as $n \to \infty$. This completes the proof of the lemma in the case $r = p-1$.

We turn to the induction argument. Fix $r < p$ and assume that the result holds for $r+1, \dots, p-1$. Recall the notation introduced at the beginning of the proof

and write

$$\mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} = H_{\mathcal{V}^{(p)}} \,\big] \;\le\; \mathbf{P}_x^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\mathcal{A}_{r,k}} \,\big] \;+\; \mathbf{P}_x^n\big[\, H_{\mathcal{A}_{r,k}} < H_{\mathcal{V}_j^{(p)}} < H_{\check{\mathcal{V}}_j^{(p)}} \,\big]\,. \tag{7.6}$$

We estimate separately the square of each term on the right-hand side.

The argument for the first term is similar to the one presented for $r = p-1$. By Lemma 7.4, (7.4) holds for some finite constant C_0. By equations (B.1) and (B.2) in [32],

$$\mathrm{cap}_n(\mathcal{V}_j^{(p)}\,,\,\mathcal{A}_{r,k}) \;\le\; \sum_{i\in B} \mathrm{cap}_n(\mathcal{V}_i^{(r+1)}\,,\,\mathcal{A}_{r,k}) \;\le\; \sum_{i\in B} \mathrm{cap}_n(\mathcal{V}_i^{(r+1)}\,,\,\check{\mathcal{V}}_i^{(r+1)})\,.$$

By (2.7), this expression is bounded by $\sum_{i\in B} \pi_n(\mathcal{V}_i^{(r+1)})/\theta_n^{(r+1)} = \pi_n(\mathcal{V}_j^{(p)})/\theta_n^{(r+1)}$. On the other hand, by Lemma 7.6, (7.5) is in force with $\theta_n^{(r)}$ in place of $\theta_n^{(p-1)}$ and some positive constant c_0. Putting together the two previous estimates, we obtain that the expression in (7.4) vanishes as $n\to\infty$.

We turn to the second term in (7.6). By the strong Markov property, it is bounded by

$$\max_{z\in\mathcal{A}_{r,k}} \mathbf{P}_z^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\check{\mathcal{V}}_j^{(p)}} \,\big]\,.$$

To complete the proof, it remains to show that for all $z\in\mathcal{A}_{r,k}$,

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{V}_j^{(p)})}\, \mathbf{P}_z^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\check{\mathcal{V}}_j^{(p)}} \,\big]^2 \;=\; 0\,.$$

Since $\pi_n(x) \prec \pi_n(z)$, it is enough to show that

$$\lim_{n\to\infty} \frac{\pi_n(z)}{\pi_n(\mathcal{V}_j^{(p)})}\, \mathbf{P}_z^n\big[\, H_{\mathcal{V}_j^{(p)}} < H_{\check{\mathcal{V}}_j^{(p)}} \,\big]^2 \;=\; 0\,. \tag{7.7}$$

This follows from the induction hypothesis. Indeed, as $z\in\mathcal{V}^{(r+1)}$, either z belongs to $\mathcal{V}^{(p)}$ or z belongs to some $\mathcal{V}_\ell^{(s)}$, $r<s<p$, for some $\ell\in\mathfrak{T}_s$. In the first case, the probability vanishes because $z\in\check{\mathcal{V}}_j^{(p)}$ (as $z\in\mathcal{V}^{(p)}\cap\mathcal{A}_{r,k}$ and $\mathcal{A}_{r,k}\cap\mathcal{V}_j^{(p)}=\varnothing$, $z\in\check{\mathcal{V}}_j^{(p)}$). In the second case, (7.7) holds by the induction hypothesis.

It remains to consider the case where $\pi_n(\mathcal{V}_j^{(p)}) \prec \pi_n(x)$ and $x\in\Delta$. We repeat the induction argument. Write (7.6) with $\mathcal{A}_x$ instead of $\mathcal{A}_{r,k}$. We estimate the first term on the right-hand side of (7.6) as before, applying Lemma 7.7 instead of Lemma 7.6. The second term is also bounded as before. At the end of the argument one needs to estimate (7.7) for $z\in\mathcal{V}$, $\pi_n(\mathcal{V}_j^{(p)}) \prec \pi_n(z)$. This has been done in the first part of the proof. □

8. Proof of Theorem 3.3

We assume in this section that the dynamics is reversible: $\pi_n(x)\,R_n(x,y) = \pi_n(y)\,R_n(y,x)$ for all $(x,y)\in E$.

Elementary properties of π_n. The proof of Theorem 3.3 requires some preparation. We first introduce the transient equivalent classes of the chain $X_t^{(n)}$. We say that y is equivalent to x, $y\sim x$ if $y=x$ or if there exists a sequence $x=x_0,\dots,x_\ell=y$, $y_0=y,\dots,y_m=x$ such that $\mathbb{R}_0(x_i,x_{i+1})>0$, $\mathbb{R}_0(y_j,y_{j+1})>0$ for all $0\le i<\ell$, $0\le j<m$.

This relation divides the set V into equivalent classes. Clearly the sets $\mathcal{V}_j$ are equivalent classes, but there might be others. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_{\mathfrak{m}}$ the equivalent classes which *have more than one element* and are not one of the sets $\mathcal{V}_j$, $1 \le j \le \mathfrak{n}$. Note that the sets $\mathcal{C}_1, \dots, \mathcal{C}_{\mathfrak{m}}, \mathcal{V}_1, \dots, \mathcal{V}_{\mathfrak{n}}$ may not exhaust V: the set V may contain elements which do not belong to one of the $\mathcal{C}_k$'s nor to one of the $\mathcal{V}_j$'s.

The first assertion extends (2.16) to the sets $\mathcal{C}_k$. We claim that if x, y belong to the same class $\mathcal{C}_k$, then

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(y)} \;=\; a \in (0,\infty)\;. \tag{8.1}$$

Indeed, by definition, there exists a sequence $x = x_0, \dots, x_\ell = y$, such that $\mathbb{R}_0(x_i, x_{i+1}) > 0$, for all $0 \le i < \ell$. By reversibility,

$$\frac{\pi_n(x)}{\pi_n(y)} \;=\; \frac{R_n(x_\ell, x_{\ell-1}) \dots R_n(x_1, x_0)}{R_n(x_0, x_1) \dots R_n(x_{\ell-1}, x_\ell)}\;.$$

By hypothesis, the denominator converges to a positive real number. On the other hand, by (2.1), the numerator converges to a non-negative real number. This proves that $\pi_n(x)/\pi_n(y) \to a \in [0,\infty)$. Inverting the roles of x and y we conclude that $a \in (0,\infty)$, as claimed in (8.1).

Fix an oriented edge $(x,y) \in E$ whose endpoints belong to the same equivalent class $\mathcal{V}_j^{(1)}$ or $\mathcal{C}_k$, $j \in S_1$, $1 \le k \le \mathfrak{m}$. We claim that

$$\mathbb{R}_0(x,y) > 0 \text{ if and only if } \mathbb{R}_0(y,x) > 0\;. \tag{8.2}$$

Indeed, assume that $\mathbb{R}_0(x,y) > 0$. Since $\pi_n(x)\, R_n(x,y) = \pi_n(y)\, R_n(y,x)$, by (2.16) and (8.1), $\lim_{n\to\infty} R_n(y,x) = \mathbb{R}_0(x,y) \lim_{n\to\infty} \pi_n(x)/\pi_n(y) > 0$.

Denote by $\mathbb{L}_j^{(0)}$, $\mathbb{L}_{T,k}^{(0)}$, $j \in S_1$, $1 \le k \le \mathfrak{m}$, the generators associated to the rates $\mathbb{R}_0$ restricted to the equivalent classes $\mathcal{V}_j^{(1)}$, $\mathcal{C}_k$, respectively. This means that we set to 0 all jumps from $\mathcal{C}_k$ to its complement. Denote by ν_k the stationary state of the Markov chain associated to the generator $\mathbb{L}_{T,k}^{(0)}$.

We claim that for all $1 \le k \le \mathfrak{m}$,

$$\lim_{n\to\infty} \frac{\pi_n(x)}{\pi_n(\mathcal{C}_k)} \;=\; \nu_k(x) \quad \text{for all } x \in \mathcal{C}_k \text{ and that } \nu_k \text{ is reversible}\;. \tag{8.3}$$

This result extends Lemma 6.1 to the transient sets $\mathcal{C}_k$. To establish (8.3), let $m \in \mathcal{P}(\mathcal{C}_k)$ be the limit of the sequence of measures $\pi_n(\,\cdot\,)/\pi_n(\mathcal{C}_k)$. This limit exists by (8.1). By reversibility, for all x, $y \in \mathcal{C}_k$,

$$\frac{\pi_n(x)}{\pi_n(\mathcal{C}_k)}\, R_n(x,y) \;=\; \frac{\pi_n(y)}{\pi_n(\mathcal{C}_k)}\, R_n(y,x)\;.$$

Passing to the limit yields that m satisfies the detailed balance conditions with respect to $\mathbb{R}_0$. Hence m is stationary (actually, reversible), and, by uniqueness, $m = \nu_k$. This proves that the sequence of measures $\pi_n(\,\cdot\,)/\pi_n(\mathcal{C}_k)$ converges to ν_k and that ν_k is reversible.

The same statement yields that $\pi_j^{(1)}$ is a reversible measure for the chain $\mathbb{X}_t$ restricted to $\mathcal{V}_j$, $j \in S_1$.

The functionals $\mathcal{I}^{(p)}$. The first result of this section provides an alternative formula for the functional $\mathcal{I}^{(0)}$ introduced in (3.11). Its proof relies on the construction of a directed graph without directed loops. The equivalence classes of the chain $\mathbb{X}_t$ form the set of vertices of this directed graph. Denote them by $\mathcal{Q}_1, \dots, \mathcal{Q}_\ell$. The sets $\mathcal{V}_j$ and $\mathcal{C}_k$ belongs to this set and are vertices of the graph. In other words, for each $j \in S_1$, there exists $1 \le a \le \ell$ such that $\mathcal{V}_j = \mathcal{Q}_a$. A similar statement holds for the sets $\mathcal{C}_k$.

Draw a directed arrow from $\mathcal{Q}_a$ to $\mathcal{Q}_b$ if there exists $x \in \mathcal{Q}_a$ and $y \in \mathcal{Q}_b$ such that $\mathbb{R}_0(x,y) > 0$. Denote the set of directed edges by $\mathbb{A}$ and the graph by $\mathbb{G} = (\mathbf{Q}, \mathbb{A})$, where $\mathbf{Q}$ is the set $\{\mathcal{Q}_1, \dots, \mathcal{Q}_\ell\}$ of vertices.

A path in the graph $\mathbb{G} = (\mathbf{Q}, \mathbb{A})$ is a sequence vertices $(\mathcal{Q}_{a_j} : 0 \le j \le m)$, such that there is a directed arrow from $\mathcal{Q}_{a_j}$ to $\mathcal{Q}_{a_{j+1}}$ for $0 \le j < m$.

This directed graph has no directed loops because the existence of a directed loop would contradict the definition of the sets $\mathcal{Q}_a$ as equivalent classes. (Mind that undirected loops might exist). On the other hand, since the sets $\mathcal{V}_j$ are closed irreducible classes, these sets are not the tail of a directed edge in the graph. Finally, fix an equivalent class $\mathcal{Q}_a$ which is not a set $\mathcal{V}_j$. Hence, the elements of $\mathcal{Q}_a$ are transient for the chain $\mathbb{X}_t$. In particular, there is a path $(\mathcal{Q}_a = \mathcal{Q}_{a_0}, \dots, \mathcal{Q}_{a_m})$ such that $\mathcal{Q}_{a_j}$ is not a closed irreducible class for $0 \le j < m$, and $\mathcal{Q}_{a_m}$ is one.

Fix an equivalent class $\mathcal{Q}_a$ which is not a set $\mathcal{V}_j$. Denote by $\mathbf{D}(\mathcal{Q}_a)$ the length of the longest path from $\mathcal{Q}_a$ to a closed irreducible class. The function $\mathbf{D}$ is well defined because (a) the set of vertices is finite, (b) there is at least a path, (c) there are no directed loops in the graph.

Fix a, b such that there is a directed arrow from $\mathcal{Q}_a$ to $\mathcal{Q}_b$. Then,

$$\mathbf{D}(\mathcal{Q}_a) \;\ge\; \mathbf{D}(\mathcal{Q}_b) \;+\; 1\,. \tag{8.4}$$

Indeed, it is enough to consider the longest path from $\mathcal{Q}_b$ to the irreducible classes. $\mathcal{Q}_a$ does not belong to the path because there are no directed loops. By adding $\mathcal{Q}_a$ at the beginning of the path from $\mathcal{Q}_b$ to the irreducible classes, we obtain a path from $\mathcal{Q}_a$ to the irreducible classes of length $\mathbf{D}(\mathcal{Q}_b) + 1$, proving (8.4).

We may lift the function $\mathbf{D}$ to V by setting $\mathbf{D}(x) = \mathbf{D}(\mathcal{Q}_a)$ for all $x \in \mathcal{Q}_a$.

Let $\mathcal{J}^{(0)} \colon \mathcal{P}(V) \to [0, +\infty]$ be the functional defined by

$$\begin{aligned} \mathcal{J}^{(0)}(\mu) \;=\; & \sum_{j \in S_1} \big\langle \sqrt{f_j}\,, (-\mathbb{L}_j^{(0)}) \sqrt{f_j} \,\big\rangle_{\pi_j^{(1)}} \;+\; \sum_{k=1}^{\mathfrak{m}} \big\langle \sqrt{g_k}\,, (-\mathbb{L}_{T,k}^{(0)}) \sqrt{g_k} \,\big\rangle_{\nu_k} \\ & + \sum_{k=1}^{\mathfrak{m}} \sum_{x \in \mathcal{C}_k} \sum_{y \notin \mathcal{C}_k} \mu(x)\, \mathbb{R}_0(x,y) + \sum_{x \notin \mathcal{C} \cup \mathcal{V}} \sum_{y \in V} \mu(x)\, \mathbb{R}_0(x,y)\,. \end{aligned} \tag{8.5}$$

In this formula, $\mathcal{C} = \cup_k \mathcal{C}_k$, $f_j(x) = \mu(x)/\pi_j^{(1)}(x)$, $g_k(z) = \mu(z)/\nu_k(z)$, $x \in \mathcal{V}_j^{(1)}$, $z \in \mathcal{C}_k$.

Lemma 8.1. *For every $\mu \in \mathcal{P}(V)$, $\mathcal{I}^{(0)}(\mu) = \mathcal{J}^{(0)}(\mu)$.*

Proof. Fix $\mu \in \mathcal{P}(V)$. We first prove that $\mathcal{J}^{(0)}(\mu) \le \mathcal{I}^{(0)}(\mu)$. By definition of the generator $\mathbb{L}^{(0)}$,

$$\mathcal{I}^{(0)}(\mu) \;=\; \sup_{u>0} - \sum_{(x,y) \in \mathbb{E}_0} \frac{\mu(x)}{u(x)}\, \mathbb{R}_0(x,y)\, [\, u(y) - u(x)\,]\,, \tag{8.6}$$

where the sum is performed over all directed edges of $\mathbb{E}_0$.

Fix $\ell \geq 1$, and define $u_\ell : V \to (0, \infty)$ by

$$u_\ell(x) \,=\, \ell^{\mathbf{D}(x)} \, \sqrt{\frac{\mu(x)+\varepsilon}{\pi_j^{(1)}(x)}} \,, \quad u_\ell(y) \,=\, \ell^{\mathbf{D}(y)} \, \sqrt{\frac{\mu(y)+\varepsilon}{\nu_k(y)}} \,, \quad u_\ell(z) \,=\, \ell^{\mathbf{D}(z)} \,,$$

for $x \in \mathcal{V}_j$, $j \in S_1$, $y \in \mathcal{C}_k$, $1 \leq k \leq \mathfrak{m}$, and $z \notin \mathcal{V} \cup \mathcal{C}$. Here, $\varepsilon = 1/\ell$ and guarantees that u is positive. By definition of $\mathcal{I}^{(0)}$,

$$\mathcal{I}^{(0)}(\mu) \,\geq\, \limsup_{\ell \to \infty} - \sum_{(x,y) \in \mathbb{E}_0} \frac{\mu(x)}{u_\ell(x)} \, \mathbb{R}_0(x,y) \, [\, u_\ell(y) - u_\ell(x) \,] \,. \tag{8.7}$$

We examine the asymptotic behavior of the right-hand of (8.7). Fix $(x, y) \in \mathbb{E}_0$, and suppose, first, that x, $y \in \mathcal{V}_j$ for some $j \in S_1$. In this case, the factors $\ell^{\mathbf{D}}$ cancel, and, as $\ell \to \infty$, the corresponding term in (8.7) converges to

$$\pi_j^{(1)}(x) \, \sqrt{f_j(x)} \, \mathbb{R}_0(x,y) \, \big[\, \sqrt{f_j(y)} \,-\, \sqrt{f_j(x)} \, \big]$$

where $f_j(x) = \mu(x)/\pi_j^{(1)}(x)$. Therefore, the contributions to the right-hand side of (8.7), of the sum over the edges $(x, y) \in \mathbb{E}_0$ such that x, $y \in \mathcal{V}_j$ is

$$\big\langle \sqrt{f_j} \,,\, (-\mathbb{L}_j^{(0)}) \sqrt{f_j} \, \big\rangle_{\pi_j^{(1)}} \,.$$

The same argument yields that the contributions to the right-hand side of (8.7), of the sum over the edges $(x, y) \in \mathbb{E}_0$ such that x, $y \in \mathcal{C}_k$ for some $1 \leq k \leq \mathfrak{m}$, is

$$\big\langle \sqrt{g_k} \,,\, (-\mathbb{L}_{T,k}^{(0)}) \sqrt{g_k} \, \big\rangle_{\nu_k} \,,$$

where $g_k(x) = \mu(x)/\nu_k(x)$.

Up to this point we considered all edges $(x, y) \in \mathbb{E}_0$ whose head and tail belong to the same equivalent class $\mathcal{V}_j$ or $\mathcal{C}_k$. Assume now that this is not the case, and consider the term

$$-\, \frac{\mu(x)}{u_\ell(x)} \, \mathbb{R}_0(x,y) \, [\, u_\ell(y) - u_\ell(x) \,] \,=\, \mu(x) \, \mathbb{R}_0(x,y) \,-\, \frac{\mu(x)}{u_\ell(x)} \, \mathbb{R}_0(x,y) \, u_\ell(y) \,.$$

By definition, and since the measures $\pi_j^{(1)}$, ν_k are strictly positive, $u_\ell(y) \leq C_0 \, \ell^{\mathbf{D}(y)}$, $\mu(x)/u_\ell(x) \leq C_0 \, \ell^{-\mathbf{D}(x)}$ for some finite constant C_0 independent of x, y and ℓ. The absolute value of the second term is thus bounded above by $C_0 \, \ell^{\mathbf{D}(y) - \mathbf{D}(x)}$. Since there is an edge from x to y, by (8.4), $\mathbf{D}(x) \geq \mathbf{D}(y) + 1$, which proves that the second term of the previous displayed equation vanishes as $\ell \to \infty$.

Fix an edge $(x, y) \in \mathbb{E}_0$ whose head and tail do not belong to the same equivalent class $\mathcal{V}_j$ or $\mathcal{C}_k$. Since $\mathcal{V}_j$ is a closed irreducible class, $x \notin \mathcal{V}$. Suppose that $x \in \mathcal{C}_k$. Hence, $y \notin \mathcal{C}_k$ because they do not belong to the same class. These are the terms which respond for the third sum in (8.5). the terms in which $x \notin \mathcal{C}$ respond for the fourth sum in (8.5), completing the proof that $\mathcal{I}^{(0)}(\mu) \geq \mathcal{J}^{(0)}(\mu)$.

We turn to the reverse inequality, $\mathcal{J}^{(0)}(\mu) \geq \mathcal{I}^{(0)}(\mu)$. By (8.6),

$$\mathcal{I}^{(0)}(\mu) \,\leq\, \sum_{j \in S_1} \mathcal{I}^{(0)}_{\mathcal{V}_j}(\mu) \,+\, \sum_{k=1}^{\mathfrak{m}} \mathcal{I}^{(0)}_{\mathcal{C}_k}(\mu) \,+\, \mathcal{I}^{(0)}_{\mathcal{R}}(\mu) \,,$$

where $\mathcal{I}^{(0)}_{\mathcal{V}_j}(\mu)$ is given by formula (8.6) when the sum is performed over the directed edges (x, y) whose head and tail belong to $\mathcal{V}_j$. $\mathcal{I}^{(0)}_{\mathcal{C}_k}(\mu)$ is defined similarly, while $\mathcal{I}^{(0)}_{\mathcal{R}}(\mu)$ contains the remaining edges.

By [56, Theorem 5],

$$\mathcal{I}^{(0)}_{\mathcal{V}_j}(\mu) \;=\; \big\langle \sqrt{f_j}\,,\, (-\mathbb{L}^{(0)}_j)\sqrt{f}\,\big\rangle_{\pi^{(1)}_j}\,,$$

where $f_j(x) = \mu(x)/\pi^{(1)}_j(x)$, $x \in \mathcal{V}_j$. An analogous result holds for $\mathcal{I}^{(0)}_{\mathcal{C}_k}(\mu)$. These two terms correspond to the first two terms in (8.5).

We turn to $\mathcal{I}^{(0)}_{\mathcal{R}}(\mu)$, which can be written as

$$\mathcal{I}^{(0)}_{\mathcal{R}}(\mu) \;=\; \sum_{(x,y)} \mu(x)\,\mathbb{R}_0(x,y) \;+\; \sup_{u>0} - \sum_{(x,y)} \frac{\mu(x)}{u(x)}\,\mathbb{R}_0(x,y)\,u(y)\,.$$

where the sums are performed over directed edges whose head and tail belong to different equivalent classes. Since the second term is negative,

$$\mathcal{I}^{(0)}_{\mathcal{R}}(\mu) \;\le\; \sum_{(x,y)} \mu(x)\,\mathbb{R}_0(x,y)\,.$$

We have seen in the first part of the proof that this sum can be written as the third and fourth terms in $\mathcal{J}^{(0)}(\mu)$, completing the proof of the lemma. □

Note that for each $1 \le k \le \mathfrak{m}$, there exists at least on $x \in \mathcal{C}_k$ such that $\mathbb{R}_0(x,y) > 0$ for some $y \notin \mathcal{C}_k$. On the other hand, as the Markov chain associated to $\mathbb{L}^{(0)}_{T,k}$ is ergodic, $\big\langle \sqrt{g}\,,\, (-\mathbb{L}^{(0)}_{T,k})\sqrt{g}\,\big\rangle_{\nu_k} = 0$ entails that g is constant. Therefore, $\mathcal{I}^{(0)}(\mu) = 0$ if and only if there exists a probability measure ω on S_1 such that

$$\mu \;=\; \sum_{j\in S_1} \omega_j\,\pi^{(1)}_j\,. \tag{8.8}$$

Fix $1 \le p \le \mathfrak{q}$, and let $\mathcal{J}^{(p)}\colon \mathcal{P}(V) \to [0,+\infty]$ be the functional defined as follows. If $\mu = \sum_{j\in S_p} \omega_j\,\pi^{(p)}_j$ for some probability measure ω in S_p, $\omega \in \mathcal{P}(S_p)$,

$$\mathcal{J}^{(p)}(\mu) \;:=\; \sum_{m\in S_{p+1}} \big\langle \sqrt{f_m}\,,\, (-\mathbb{L}^{(p)}_m)\sqrt{f_m}\,\big\rangle_{M^{(p)}_m} \;+\; \sum_{j\in\mathfrak{T}_p}\sum_{k\in S_p} \omega_j\,r^{(p)}(j,k)\,. \tag{8.9}$$

In this formula, $\mathbb{L}^{(p)}_m$ stands for the generator associated to Markov chain $\mathbb{X}^{(p)}_t$ restricted to the closed irreducible set $\mathfrak{R}^{(p)}_m$ and $f_m(j) = \omega_j/M^{(p)}_m(j)$, $j \in \mathfrak{R}^{(p)}_m$. To complete the definition of $\mathcal{J}^{(p)}$, set

$$\mathcal{J}^{(p)}(\mu) \;:=\; +\infty \quad \text{if } \mu \text{ is not a convex combination of } \pi^{(p)}_j,\ j \in S_p\,. \tag{8.10}$$

Recall from (3.13) the definition of the functional $\mathcal{I}^{(p)}\colon \mathcal{P}(V) \to [0,+\infty]$. The proof of Lemma 8.1 yields that

Lemma 8.2. *For all $1 \le p \le \mathfrak{q}$, $\mu \in \mathcal{P}(V)$, $\mathcal{I}^{(p)}(\mu) = \mathcal{J}^{(p)}(\mu)$.*

Note that, by (8.9) and (3.2), $\mathcal{I}^{(p)}(\mu) = 0$ if and only if there exists a probability measure $\widehat{\omega}$ in S_{p+1} such that

$$\mu \;=\; \sum_{m\in S_{p+1}} \widehat{\omega}_m \sum_{j\in\mathfrak{R}^{(p)}_m} M^{(p)}_m(j)\,\pi^{(p)}_j \;=\; \sum_{m\in S_{p+1}} \widehat{\omega}_m\,\pi^{(p+1)}_m\,. \tag{8.11}$$

On the other hand, if μ is not of this form, by (8.10), $\mathcal{I}^{(p+1)}(\mu)$ is set to be equal to $+\infty$. Hence, the functional $\mathcal{I}^{(p+1)}$ is finite only at the 0-level set of $\mathcal{I}^{(p)}$. Furthermore, since the right-hand side of (8.9) is always finite,

$$\mathcal{I}^{(p+1)}(\mu) \;<\; \infty \quad \text{if and only if} \quad \mathcal{I}^{(p)}(\mu) \;=\; 0\,. \tag{8.12}$$

By (8.8), this assertion holds also for $p = 0$.

The Γ-convergence. We turn to the proof of Theorem 3.3. We proceed by induction. We first show that $\mathfrak{I}_n$ Γ-converges to the functional $\mathfrak{I}^{(0)}$. Then, we observe that, according to (8.8), the 0-level set of $\mathfrak{I}^{(0)}$ corresponds to the convex combinations of the measures $\pi_j^{(1)}$, $j \in S_1$. In the sequel, we prove that $\theta_n^{(1)}\,\mathfrak{I}_n$ Γ-converges to $\mathfrak{I}^{(1)}$. Clearly, by definition, $\mathfrak{I}^{(1)}(\mu) = +\infty$ if μ is not a convex combinations of the measures $\pi_j^{(1)}$, $j \in S_1$, while $\mathfrak{I}^{(1)}(\mu) < +\infty$ if it is. By (8.11), the 0-level set of $\mathfrak{I}^{(1)}$ consists of the convex combinations of the measures $\pi_j^{(2)}$, $j \in S_2$.

At this point, we iterate the procedure by examining the behavior of $\theta_n^{(2)}\,\mathfrak{I}_n$, and so on until proving that $\theta_n^{(\mathfrak{q})}\,\mathfrak{I}_n$ Γ-converges to $\mathfrak{I}^{(\mathfrak{q})}$. The 0-level set of this functional is the singleton formed by the measure $\pi^{(\mathfrak{q}+1)}$. As the level set is a singleton, the iterative procedures ends. Note that this approach produced the state $\pi^{(\mathfrak{q}+1)}$ which is is the limit of the stationary measures π_n: $\pi^{(\mathfrak{q}+1)}(x) = \lim_{n\to\infty} \pi_n(x)$, $x \in V$.

We turn to the proof that $\mathfrak{I}_n$ Γ-converges to $\mathfrak{I}^{(0)}$.

Proposition 8.3. *The functional $\mathfrak{I}_n$ Γ-converges to $\mathfrak{I}^{(0)}$.*

Proof. We start with the $\Gamma - \limsup$. Fix $\mu \in \mathcal{P}(V)$ and consider the sequence μ_n constant equal to μ. By (3.10),

$$\mathfrak{I}_n(\mu) \;=\; \frac{1}{2} \sum_{(x,y)\in E} \pi_n(x)\, R_n(x,y) \Big\{ \sqrt{\frac{\mu(y)}{\pi_n(y)}} \;-\; \sqrt{\frac{\mu(x)}{\pi_n(x)}} \Big\}^2 \,,$$

Fix an edge $(x, y) \in E$. We examine the asymptotic behavior of

$$\pi_n(x)\, R_n(x,y) \Big\{ \sqrt{\frac{\mu(y)}{\pi_n(y)}} \;-\; \sqrt{\frac{\mu(x)}{\pi_n(x)}} \Big\}^2 \,. \tag{8.13}$$

By reversibility, this term is symmetric in x, y.

There are three types of edges. Assume first that $R_n(x, y) \to 0$ and $R_n(y, x) \to 0$. By [43, Lemma 3.1], either $\pi_n(x)/\pi_n(y)$ converges to a nonnegative real number or so does $\pi_n(y)/\pi_n(x)$. Assume, without loss of generality because (8.13) is symmetric, that $\pi_n(y)/\pi_n(x) \to a \in [0, \infty)$. In this case, (8.13) is equal to

$$R_n(y,x) \Big\{ \sqrt{\frac{\mu(x)\,\pi_n(y)}{\pi_n(x)}} \;-\; \sqrt{\mu(y)} \Big\}^2 \,,$$

which vanishes as $n \to \infty$.

Assume that $R_n(x, y) \not\to 0$ and $R_n(y, x) \to 0$. Hence, $\mathbb{R}_0(x, y) > 0$, $\mathbb{R}_0(y, x) = 0$. In particular, as the set $\mathcal{V}_j$ are closed irreducible classes, $x \notin \mathcal{V}$ (if $x \in \mathcal{V}_j$ and $\mathbb{R}_0(x, y) > 0$, then $y \in \mathcal{V}_j$ because it is a closed irreducible class. Hence, by (8.2), $\mathbb{R}_0(y, x) > 0$, which is a contradiction). Two possibilities remain, either $x \in \mathcal{C}_k$ for some k or $x \notin \mathcal{C}$.

By reversibility, $\pi_n(x)/\pi_n(y) \to 0$. Hence, (8.13) is equal to

$$R_n(x,y) \Big\{ \sqrt{\frac{\mu(y)\,\pi_n(x)}{\pi_n(y)}} \;-\; \sqrt{\mu(x)} \Big\}^2 \,,$$

which converges to $\mu(x)\,\mathbb{R}_0(x, y)$. If $x \in \mathcal{C}_k$, by (8.2), $y \notin \mathcal{C}_k$. These are the pairs which appear in the third term on the right-hand side of (8.5). If $x \notin \mathcal{C}$ these pairs are responsible for the fourth term on the right-hand side of (8.5).

Finally, assume that $R_n(x,y) \not\to 0$ and $R_n(y,x) \not\to 0$. This means that x and y belong to the same equivalence class, say $\mathcal{V}_j$ or $\mathcal{C}_k$. Assume that x and $y \in \mathcal{V}_j$. The argument is identical if we replace $\mathcal{V}_j$ by $\mathcal{C}_k$. Replace $\pi_n(x)$, $\pi_n(y)$ by $\pi_n(x)/\pi_n(\mathcal{V}_j)$, $\pi_n(y)/\pi_n(\mathcal{V}_j)$, respectively. By Lemma 6.1, $\pi_n(x)/\pi_n(\mathcal{V}_j)$ converges to $\pi^{\natural}_j(x) = \pi^{(1)}_j(x) > 0$. Hence, (8.13) converges to

$$\pi^{(1)}_j(x)\, \mathbb{R}_0(x,y) \Big\{ \sqrt{\frac{\mu(y)}{\pi^{(1)}_j(y)}} \,-\, \sqrt{\frac{\mu(x)}{\pi^{(1)}_j(x)}} \Big\}^2 .$$

Putting together the previous estimates yields that $\mathfrak{I}_n(\mu) \to \mathcal{J}^{(0)}(\mu)$. To complete the proof of the $\Gamma - \limsup$, it remains to recall the statement of Lemma 8.1.

We turn to the $\Gamma - \liminf$. Fix $\mu \in \mathcal{P}(V)$ and a sequence of probability measures μ_n in $\mathcal{P}(\mathcal{V})$ converging to μ. By definition of $\mathfrak{I}_n$,

$$\mathfrak{I}_n(\mu_n) \,\ge\, -\int_V \frac{\mathcal{L}_n u}{u}\, d\mu_n \,=\, -\sum_{x\in V} \frac{\mu_n(x)}{u(x)} \sum_{y\in V} R_n(x,y)\, [\, u(y) - u(x)\,]$$

for all $u : V \to (0,\infty)$. As $\mu_n \to \mu$ and $R_n \to \mathbb{R}_0$, this expression converges to

$$-\sum_{(x,y)\in \mathbb{E}_0} \frac{\mu(x)}{u(x)}\, \mathbb{R}_0(x,y)\, [\, u(y) - u(x)\,] .$$

Therefore,

$$\liminf_{n\to\infty} \mathfrak{I}_n(\mu_n) \,\ge\, \sup_{u>0} -\sum_{(x,y)\in \mathbb{E}_0} \frac{\mu(x)}{u(x)}\, \mathbb{R}_0(x,y)\, [\, u(y) - u(x)\,] \,=\, \mathfrak{I}^{(0)}(\mu) ,$$

which completes the proof of the lemma. □

Recall from (8.9) the definition of the functionals $\mathfrak{I}^{(p)}$, $1 \le p \le \mathfrak{q}$.

Proposition 8.4. *Fix $1 \le p \le \mathfrak{q}$. The functional $\theta_n^{(p)}\, \mathfrak{I}_n$ Γ-converges to $\mathfrak{I}^{(p)}$.*

Proof. We start with the $\Gamma - \limsup$. Fix $\mu \in \mathcal{P}(V)$. If μ is not a convex combinations of the measures $\pi^{(p)}_j$, $j \in S_p$, there is nothing to prove. Assume, therefore, that $\mu = \sum_{j\in S_p} \omega_j\, \pi^{(p)}_j$ for some weights ω_j.

Let $f_n : \mathcal{V}^{(p)} \to \mathbb{R}_+$ be the function given by $f_n = \sum_{j\in S_p} \omega_j(n)\, \chi_{\mathcal{V}^{(p)}_j}$, where $\omega_j(n) = \omega_j/\pi_n(\mathcal{V}^{(p)}_j)$. To extend this function to V, solve the Poisson equation (A.2) with $\mathcal{L} = \mathcal{L}_n$, $V_0 = \mathcal{V}^{(p)}$, $g = \sqrt{f_n}$. Denote by h_n the solution of the equation. Let $\mu_n = \alpha_n\, h_n^2\, \pi_n$, where α_n is a normalizing constant which turns μ_n into a probability measure.

We claim that $\alpha_n \to 1$ and $\mu_n \to \mu$. By definition,

$$\alpha_n^{-1} \,=\, \sum_{x\notin \mathcal{V}^{(p)}} h_n(x)^2\, \pi_n(x) \,+\, \sum_{j\in S_p} \sum_{x\in \mathcal{V}^{(p)}_j} f_n(x)\, \pi_n(x) .$$

By definition of h_n, for $x \notin \mathcal{V}^{(p)}$,

$$\begin{aligned} h_n(x)^2\, \pi_n(x) \,&=\, \Big\{ \sum_{j\in S_p} \sqrt{\omega_j(n)}\, \mathbf{P}^n_x \big[\, H_{\mathcal{V}^{(p)}_j} = H_{\mathcal{V}^{(p)}} \,\big] \Big\}^2 \pi_n(x) \\ &\le\, C_0 \sum_{j\in S_p} \frac{\omega_j}{\pi_n(\mathcal{V}^{(p)}_j)}\, \mathbf{P}^n_x \big[\, H_{\mathcal{V}^{(p)}_j} = H_{\mathcal{V}^{(p)}} \,\big]^2 \pi_n(x) , \end{aligned}$$

where the constant C_0 bounds the cardinality of V. By Lemma 7.8, this expression vanishes as $n \to \infty$. By definition of f_n, the second term of the penultimate displayed equation is equal to

$$\sum_{j\in S_p} \sum_{x\in\mathcal{V}_j^{(p)}} \frac{\omega_j}{\pi_n(\mathcal{V}_j^{(p)})}\, \pi_n(x) \;=\; 1\,,$$

which proves that $\alpha_n \to 1$.

The previous argument shows that $\mu_n(x) = \alpha_n\, h_n(x)^2\, \pi_n(x) \to 0 = \mu(x)$ if $x \notin \mathcal{V}^{(p)}$. If $x \in \mathcal{V}_j^{(p)}$, $\mu_n(x) = \alpha_n\, f_n(x)\, \pi_n(x) = \alpha_n\, \omega_j\, \pi_n(x)/\pi_n(\mathcal{V}_j^{(p)})$. Since $\alpha_n \to 1$, by Corollary 6.3, the previous expression converges to $\omega_j\, \pi_j^{(p)}(x) = \mu(x)$.

To complete the proof of the $\Gamma-$lim sup, it remains to show that $\limsup_n \theta_n^{(p)}\, \mathfrak{I}_n(\mu_n) \le \mathfrak{I}^{(p)}(\mu)$. By (3.10) and the definition of μ_n,

$$\mathfrak{I}_n(\mu_n) \;=\; \alpha_n\, \langle\, h_n\,,\, (-\mathcal{L}_n)\, h_n\,\rangle_{\pi_n}\,.$$

By Corollary A.2 and the definition of h_n, the right-hand side is equal to

$$\begin{aligned}
&\alpha_n\, \pi_n(\mathcal{V}^{(p)})\, \langle \sqrt{f_n}\,, (-\mathcal{L}_n^{(p)})\sqrt{f_n}\rangle_{\pi_n^{(p)}} \\
&\quad = \; -\,\alpha_n \sum_{x,y\in\mathcal{V}^{(p)}} \pi_n(x)\, R_n^{(p)}(x,y)\, \sqrt{f_n(x)}\, \big\{\, \sqrt{f_n(y)} \,-\, \sqrt{f_n(x)}\,\big\}\,,
\end{aligned}$$

where $\mathcal{L}_n^{(p)}$ stands for the generator of the trace process $Y_t^{n,p}$ introduced in (2.8), and $\pi_n^{(p)}$ for the measure π_n conditioned to $\mathcal{V}^{(p)}$. Since f_n is constant and equal to $\omega_j(n)$ on each set $\mathcal{V}_j^{(p)}$, the previous expression is equal to

$$\begin{aligned}
&-\,\alpha_n \sum_{j\in S_p} \sqrt{\omega_j(n)} \sum_{k\in S_p\setminus\{j\}} \big\{\, \sqrt{\omega_k(n)} \,-\, \sqrt{\omega_j(n)}\,\big\} \sum_{x\in\mathcal{V}_j^{(p)}} \pi_n(x) \sum_{y\in\mathcal{V}_k^{(p)}} R_n^{(p)}(x,y) \\
&\quad = \; -\,\alpha_n \sum_{j\in S_p} \sqrt{\omega_j(n)} \sum_{k\in S_p\setminus\{j\}} \big\{\, \sqrt{\omega_k(n)} \,-\, \sqrt{\omega_j(n)}\,\big\} \pi_n(\mathcal{V}_j^{(p)})\, r_n^{(p)}(j,k)\,,
\end{aligned}$$

where $r_n^{(p)}(j,k)$ is defined in (2.9). Up to this point, we proved that

$$\theta_n^{(p)}\, \mathfrak{I}_n(\mu_n) \;=\; \frac{\alpha_n}{2}\, \theta_n^{(p)} \sum_{j,k\in S_p} \pi_n(\mathcal{V}_j^{(p)})\, r_n^{(p)}(j,k) \Big\{\, \sqrt{\omega_k(n)} \,-\, \sqrt{\omega_j(n)}\,\Big\}^2\,,$$

where we used that $\pi_n(\mathcal{V}_j^{(p)})\, r_n^{(p)}(j,k) = \pi_n(\mathcal{V}_k^{(p)})\, r_n^{(p)}(k,j)$, an identity which follows from the reversibility assumption.

Recall that $\alpha_n \to 1$. In view of the definition of $\omega_i(n)$, it remains to examine the asymptotic behavior of

$$\pi_n(\mathcal{V}_j^{(p)})\, \theta_n^{(p)}\, r_n^{(p)}(j,k) \Big\{\, \sqrt{\frac{\omega_k}{\pi_n(\mathcal{V}_k^{(p)})}} \,-\, \sqrt{\frac{\omega_j}{\pi_n(\mathcal{V}_j^{(p)})}}\,\Big\}^2\,. \tag{8.14}$$

As in the proof of Proposition 8.3, we divide the pairs (j,k) in three types. Assume first that $\theta_n^{(p)}\, r_n^{(p)}(j,k) \to 0$ and $\theta_n^{(p)}\, r_n^{(p)}(k,j) \to 0$. By [43, Lemma 3.1], and (2.16), either $\pi_n(\mathcal{V}_j^{(p)})/\pi_n(\mathcal{V}_k^{(p)})$ converges to a nonnegative real number or so does $\pi_n(\mathcal{V}_k^{(p)})/\pi_n(\mathcal{V}_j^{(p)})$. Assume that $\pi_n(\mathcal{V}_k^{(p)})/\pi_n(\mathcal{V}_j^{(p)}) \to a \in [0,\infty)$. In this case,

by reversibility, (8.14) is equal to

$$\theta_n^{(p)}\, r_n^{(p)}(k,j)\, \Big\{ \sqrt{\omega_k} \,-\, \sqrt{\omega_j\, \frac{\pi_n(\mathcal{V}_k^{(p)})}{\pi_n(\mathcal{V}_j^{(p)})}} \Big\}^2 \,, \tag{8.15}$$

which vanishes as $n \to \infty$.

Next, suppose that $\theta_n^{(p)}\, r_n^{(p)}(j,k) \to 0$ and $\theta_n^{(p)}\, r_n^{(p)}(k,j) \to r^{(p)}(k,j) > 0$, where $r^{(p)}(k,j)$ has been introduced in (2.10). In particular, k is a transient state of the chain $\mathbb{X}_t^{(p)}$. By reversibility, $\pi_n(\mathcal{V}_k^{(p)})/\pi_n(\mathcal{V}_j^{(p)}) = r_n^{(p)}(j,k)/r_n^{(p)}(k,j) \to 0$. Hence, (8.14), which is equal to (8.15), converges to

$$r^{(p)}(k,j)\, \omega_k \,.$$

Finally, suppose that $\theta_n^{(p)}\, r_n^{(p)}(j,k) \to r^{(p)}(j,k) > 0$ and $\theta_n^{(p)}\, r_n^{(p)}(k,j) \to r^{(p)}(k,j) > 0$. This means that j and k belong to some closed irreducible class $\mathfrak{R}_m^{(p)}$ of the chain $\mathbb{X}_t^{(p)}$. By Lemma 6.2, the expression (8.14) converges to

$$M_m^{(p)}(j)\, r^{(p)}(k,j)\, \Big\{ \sqrt{\frac{\omega_k}{M_m^{(p)}(k)}} \,-\, \sqrt{\frac{\omega_j}{M_m^{(p)}(j)}} \Big\}^2 \,.$$

Combining the previous estimates yields that $\theta_n^{(p)} \mathcal{I}_n(\mu_n)$ converges to $\mathcal{J}^{(p)}(\mu)$, which completes the proof of the $\Gamma - \limsup$ in view of Lemma 8.2.

We turn to the $\Gamma - \liminf$ where we use an induction argument. Fix $1 \le p \le \mathfrak{q}$ and assume that the Γ-convergence of $\theta_n^{(p-1)}\, \mathcal{I}_n$ to $\mathcal{I}^{(p-1)}$ has been proved. Fix a probability measure μ on V and a sequence μ_n converging to μ.

Suppose that $\mathcal{I}^{(p-1)}(\mu) > 0$. In this case, since $\theta_n^{(p-1)}\, \mathcal{I}_n$ Γ-converges to $\mathcal{I}^{(p-1)}$ and $\theta_n^{(p)}/\theta_n^{(p-1)} \to \infty$,

$$\liminf_{n\to\infty} \theta_n^{(p)}\, \mathcal{I}_n(\mu_n) \;=\; \liminf_{n\to\infty} \frac{\theta_n^{(p)}}{\theta_n^{(p-1)}}\, \theta_n^{(p-1)}\, \mathcal{I}_n(\mu_n) \;\ge\; \mathcal{I}^{(p-1)}(\mu)\, \lim_{n\to\infty} \frac{\theta_n^{(p)}}{\theta_n^{(p-1)}} \;=\; \infty \,.$$

On the other hand, by (8.12), $\mathcal{I}^{(p)}(\mu) = \infty$. $\mathcal{I}^{(p-1)}(\mu) = \infty$. This proves the $\Gamma - \liminf$ convergence for measures μ such that $\mathcal{I}^{(p-1)}(\mu) > 0$.

Assume that $\mathcal{I}^{(p-1)}(\mu) = 0$. By (8.10), there exists a probability measure ω on S_p such that $\mu = \sum_{j\in S_p} \omega_j\, \pi_j^{(p)}$. By (3.9),

$$\mathcal{I}_n(\mu_n) \;\ge\; -\int_V \frac{\mathcal{L}_n u}{u}\, d\mu_n$$

for all $u : V \to (0,\infty)$.

Fix a function $h : \mathcal{V}^{(p)} \to (0,\infty)$ which is constant on each $\mathcal{V}_j^{(p)}$, $j \in S_p$: $h = \sum_{j\in S_p} \mathbf{h}(j)\, \chi_{\mathcal{V}_j^{(p)}}$. Let $u_n \colon V \to \mathbb{R}$ be the solution of the Poisson equation (A.2) with $\mathcal{L} = \mathcal{L}_n$, $V_0 = \mathcal{V}^{(p)}$ and $g = h$. By the representation (A.3), it is clear that $u_n(x) \in (0,\infty)$ for all $x \in V$.

Since u_n is harmonic on $V \setminus \mathcal{V}^{(p)}$ and $u_n = h$ on $\mathcal{V}^{(p)}$, by Lemma A.1, the right-hand side of the previous displayed equation with $u = u_n$ is equal to

$$-\int_{\mathcal{V}^{(p)}} \frac{\mathcal{L}_n u_n}{u_n}\, d\mu_n \;=\; -\int_{\mathcal{V}^{(p)}} \frac{\mathcal{L}_n u_n}{h}\, d\mu_n \;=\; -\int_{\mathcal{V}^{(p)}} \frac{\mathcal{L}_n^{(p)} h}{h}\, d\mu_n \,.$$

Here, as in the first part of the proof, $\mathcal{L}_n^{(p)}$ stands for the generator of the trace process $Y_t^{n,p}$ introduced in (2.8).

Since h is constant on each set $\mathcal{V}_j^{(p)}$ (and equal to $\mathbf{h}(j)$), the last integral is equal to

$$-\sum_{j,k\in S_p} \frac{[\,\mathbf{h}(k)-\mathbf{h}(j)\,]}{\mathbf{h}(j)} \sum_{x\in\mathcal{V}_j^{(p)}} \pi_n(x)\,\frac{\mu_n(x)}{\pi_n(x)}\,R_n^{(p)}(x,\mathcal{V}_k^{(p)})\;,$$

where $R_n^{(p)}(x,\mathcal{V}_k^{(p)}) = \sum_{y\in\mathcal{V}_k^{(p)}} R_n^{(p)}(x,y)$. By Proposition 3.2, $\pi_n(x)/\pi_n(\mathcal{V}_j^{(p)}) \to \pi_j^{(p)}(x)$ for all $x\in\mathcal{V}_j^{(p)}$. Thus, since $\mu_n\to\mu=\sum_{j\in S_p}\omega_j\,\pi_j^{(p)}$,

$$\lim_{n\to\infty} \pi_n(\mathcal{V}_j^{(p)})\,\frac{\mu_n(x)}{\pi_n(x)} \;=\; \omega_j \quad \text{for all } x\in\mathcal{V}_j^{(p)}\;.$$

Therefore, by (2.10), as $n\to\infty$, the penultimate expression multiplied by $\theta_n^{(p)}$ converges to

$$-\sum_{j\in S_p}\omega_j\,\frac{1}{\mathbf{h}(j)}\sum_{k\in S_p} r^{(p)}(j,k)\,[\,\mathbf{h}(k)-\mathbf{h}(j)\,] \;=\; -\sum_{j\in S_p}\omega_j\,\frac{\mathbb{L}^{(p)}\mathbf{h}}{\mathbf{h}}\;.$$

Summarising, we proved that

$$\liminf_{n\to\infty}\theta_n^{(p)}\,\mathfrak{I}_n(\mu_n) \;\ge\; \sup_{\mathbf{h}} -\sum_{j\in S_p}\omega_j\,\frac{\mathbb{L}^{(p)}\mathbf{h}}{\mathbf{h}}\;,$$

where the supremum is carried over all functions $\mathbf{h}: S_p\to(0,\infty)$. By (3.13), the right-hand side is precisely $\mathfrak{I}^{(p)}(\mu)$, which completes the proof of the $\Gamma-\liminf$. □

Appendix A. Potential theory

We present in this section some results on potential theory used in the article. We do not assume reversibility. We keep the same notation of the article, removing the index n. In particular, X_t is a V-valued, continuous-time irreducible Markov process whose jump rates are represented by $R(x,y)$. Denote by $(\mathcal{F}_t : t\ge 0)$ the canonical filtration induced by the chain X_t. Hence, $\mathcal{F}_t$ is the σ-algebra generated by the variables X_s, $0\le s\le t$.

We first recall for the reader's convenience the definition of the trace of a process on a subset.

Trace process. Fix a non-empty subset W of V. Denote by $T^W(t)$ the total time the process X_t spends in W in the time-interval $[0,t]$:

$$T^W(t) \;=\; \int_0^t \chi_W(X_s)\,ds\;,$$

where, recall, χ_W represents the indicator function of the set W. Denote by $S^W(t)$ the generalized inverse of $T^W(t)$:

$$S^W(t) \;=\; \sup\{\,s\ge 0\,:\,T^W(s)\le t\,\}\;.$$

The trace of X_t on W, denoted by $(X_t^W : t\ge 0)$, is defined by

$$X_t^W \;=\; X_{S^W(t)}\;;\quad t\ge 0\;. \tag{A.1}$$

By Propositions 6.1 and 6.3 in [1], the trace process is an irreducible, W-valued continuous-time Markov chain, obtained by turning off the clock when the process

X_t visits the set W^c, that is, by deleting all excursions to W^c. For this reason, it is called the trace process of X_t on W.

Denote by $\mathcal{L}_W$, R_W, λ_W, p_W and π_W its generator, jump rates, holding times, transition matrix and stationary state, respectively. The measure π_W is obtained by conditioning π to W: $\pi_W(x) = \pi(x)/\pi(W)$.

Let $\mathbf{P}_x^W$, $x \in W$, be the probability measure on the path space $D(\mathbb{R}_+, W)$ induced by the Markov chain X_t^W starting from x. Expectation with respect to $\mathbf{P}_x^W$ is represented by $\mathbf{E}_x^W$.

Poisson equation. Fix a non-empty proper subset V_0 of V and a function $g : V_0 \to \mathbb{R}$. Let f be the solution of the Poisson equation

$$\begin{cases} \mathcal{L} f \,=\, 0\,, & V \setminus V_0\,, \\ f \,=\, g\,, & V_0\,. \end{cases} \tag{A.2}$$

Recall from (2.5) the definition of the hitting and return times to a subset $\mathcal{A}$ of V. By the strong Markov property, the solution of the Poisson equation can be represented as

$$f(x) \,=\, \mathbf{E}_x[\, g(X_{H_{V_0}})\,]\,, \quad x \in V\,. \tag{A.3}$$

Fix $V_0 \subset W \subset V$ and denote by f_W the solution of the Poisson equation

$$\begin{cases} \mathcal{L}_W f \,=\, 0\,, & W \setminus V_0\,, \\ f \,=\, g\,, & V_0\,. \end{cases} \tag{A.4}$$

Mind that W may be equal to V_0.

Starting from $x \notin V_0$, the processes X_t and X_t^W hit the set V_0 at the same point: $X^W_{H_{V_0}(X^W)} = X_{H_{V_0}}$, $\mathbf{P}_x$ a.s. In this formula and below, $H_{V_0}(X^W)$, $H^+_{V_0}(X^W)$ stand for hitting and return time to V_0 for the process X^W. By the representation (A.3) and the previous observation, for $x \in W$

$$f_W(x) \,=\, \mathbf{E}_x^W[\, g(X_{H_{V_0}})\,] \,=\, \mathbf{E}_x[\, g(X^W_{H_{V_0}(X^W)})\,] \,=\, \mathbf{E}_x[\, g(X_{H_{V_0}})\,] \,=\, f(x)\,. \tag{A.5}$$

Lemma A.1. *Fix $V_0 \subset W \subset V$. Denote by f, f_W the solutions of the Poisson equations* (A.2), (A.4), *respectively. Then,*

$$(\mathcal{L} f)(x) \,=\, (\mathcal{L}_W f_W)(x)\,, \quad x \in V_0\,.$$

Proof. Fix $x \in V_0$. The left-hand side of the identity appearing in the statement of the lemma can be written as

$$\lambda(x) \sum_{y \in V} p(x,y)\,[\, f(y) \,-\, f(x)\,]\,.$$

Without loss of generality, assume that $p(z,z) = 0$ for all $z \in V$ (if this is not the case, one redefines the holding time $\lambda(z)$ for the identity to hold). By (A.3), $f(y) = \mathbf{E}_y[\, g(X_{H_{V_0}})\,]$ for all $y \in V$, and by the strong Markov property $\mathbf{E}_x[\, g(X_{H^+_{V_0}})\,] = \sum_{y \in V} p(x,y)\, f(y)$. Hence, the previous sum can be written as

$$\lambda(x)\, \big\{\, \mathbf{E}_x[\, g(X_{H^+_{V_0}})\,] \,-\, f(x)\,\big\}\,. \tag{A.6}$$

Recall that we denote by X_t^W the trace of the process X_t on W. We consider two cases. If $H_W^+ < H_x^+$ then the process X_t and X_t^W return to V_0 at the same point $X_{H^+_{V_0}}$ (to prove this assertion, consider separately the two situations $\{H^+_{V_0} < H^+_x\}$

and $\{H^+_{V_0} = H^+_x\}$). Thus, if $H^+_W < H^+_x$ we may replace in (A.6) $g(X_{H^+_{V_0}})$ by $g(X^W_{H^+_{V_0}(X^W)})$.

If $H^+_W = H^+_x$, the process X_t returns to V_0 (and also to W) at x. In contrast, in the time interval $[0, H^+_W]$ the trace process on W remains at x, and X^W_t may return to V_0 at a point $y \neq x$. In particular, X_t and X^W_t may return to V at different points. In this case, since $H^+_{V_0} = H^+_x$, we have

$$\mathbf{E}_x\big[\, g(X_{H^+_{V_0}})\, \chi(H^+_x = H^+_W)\,\big] \;=\; g(x)\, \mathbf{P}_x\big[\, H^+_x = H^+_W\,\big]\,.$$

Up to this point, we proved that for $x \in V_0$,

$$\mathbf{E}_x\big[\, g(X_{H^+_{V_0}})\,\big] \;=\; \mathbf{E}_x\big[\, g(X^W_{H^+_{V_0}(X^W)})\, \chi_{A^c}\,\big] \;+\; g(x)\, \mathbf{P}_x[\, A\,]\,,$$

where A is the event $\{H^+_x = H^+_W\}$. Write χ_{A^c} as $1 - \chi_A$. On the event A, $H^+_{V_0}(X^W) = H^+_{V_0} + H^+_{V_0}(X^W) \circ \vartheta_{H^+_{V_0}}$. Hence, conditioning on $\mathcal{F}_{H^+_{V_0}}$, since A is $\mathcal{F}_{H^+_{V_0}}$-measurable and $X(H^+_{V_0}) = x$ on the event A, by the strong Markov property,

$$\mathbf{E}_x\big[\, g(X^W_{H^+_{V_0}(X^W)})\, \chi_{A}\,\big] \;=\; \mathbf{P}_x[\, A\,]\, \mathbf{E}_x\big[\, g(X^W_{H^+_{V_0}(X^W)})\,\big]\,.$$

Therefore, for $x \in V_0$,

$$\mathbf{E}_x\big[\, g(X_{H^+_{V_0}})\,\big] \,-\, g(x) \;=\; \mathbf{P}_x[\, H^+_W < H^+_x\,]\, \Big\{\, \mathbf{E}_x\big[\, g(X^W_{H^+_{V_0}(X^W)})\,\big] \,-\, g(x)\,\Big\}\,. \tag{A.7}$$

By equation (6.9) in [1], $\lambda(x)\, \mathbf{P}_x[\, H^+_W < H^+_x\,] = \lambda_W(x)$. Therefore, (A.6) is equal to

$$\lambda_W(x)\, \big\{\, \mathbf{E}^W_x[\, g(X_{H^+_{V_0}})\,] \,-\, f(x)\,\big\}\,.$$

By the strong Markov property and (A.5), this expression is equal to

$$\begin{aligned}
&\lambda_W(x) \sum_{y\in W} p_W(x,y)\, \big\{\, \mathbf{E}^W_y[\, g(X_{H_{V_0}})\,] \,-\, f(x)\,\big\} \\
&\quad =\; \lambda_W(x) \sum_{y\in W} p_W(x,y)\, \big\{\, f_W(y) \,-\, f_W(x)\,\big\} \;=\; (\mathcal{L}_W f_W)(x)\,,
\end{aligned}$$

as claimed. □

Denote by $D(f)$ the Dirichlet form of a function $f : V \to \mathbb{R}$:

$$D(f) \;:=\; \langle\, f\,,\, (-\mathcal{L})f\,\rangle_\pi\,.$$

Corollary A.2. *Fix $V_0 \subset W \subset V$. Denote by f, f_W the solutions of the Poisson equations* (A.2), (A.4), *respectively. Then,*

$$D(f) \;=\; \pi(W)\, \langle\, f_W\,,\, (-\mathcal{L}_W) f_W\,\rangle_{\pi_W}\,. \tag{A.8}$$

Proof. By definition of the Dirichlet form,

$$D(f) \;=\; \langle\, f\,,\, (-\mathcal{L})f\,\rangle_\pi \;=\; -\sum_{x\in V} \pi(x)\, f(x)\, (\mathcal{L}f)(x)\,.$$

Since f is harmonic on V_0^c, the sum can be restricted to V_0. Hence, the previous expression is equal to

$$-\sum_{x\in V_0} \pi(x)\, f(x)\, (\mathcal{L}f)(x)\,.$$

By (A.5) and Lemma A.1, this sum is equal to

$$-\sum_{x\in V_0}\pi(x)\,f_W(x)\,(\mathcal{L}_W f_W)(x)\;.$$

Since f_W is $\mathcal{L}_W$-harmonic on $W\setminus V_0$, we may extend the sum to W. To complete the proof, it remains to recall that $\pi_W(\,\cdot\,)=\pi(\,\cdot\,)/\pi(W)$. □

The same proof yields the following result.

Corollary A.3. *Fix $V_0\subset V$, $g:V_0\to\mathbb{R}$, and let u be the solution of* (A.2)*. Then,*

$$\int_V \frac{\mathcal{L}\,u}{u}\,d\mu \;=\; \int_{V_0}\frac{\mathcal{L}_{V_0}\,g}{g}\,d\mu$$

for all probability measures μ on V.

Proof. Since u is harmonic on $V\setminus V_0$, we may restrict the integral to V_0. By Lemma A.1, on V_0 we may replace $\mathcal{L}u$ by $\mathcal{L}_{V_0}u_{V_0}$, where u_{V_0} is the solution of (A.4) with $W=V_0$. However, as $W=V_0$, the solution of (A.4) is $u_{V_0}=g$. Hence, $\mathcal{L}_{V_0}u_{V_0}=\mathcal{L}_{V_0}g$. As $u=g$ on V_0, the proof is complete. □

We turn to an estimate of hitting times. Denote by π_A, $A\subset V$, the stationary measure π conditioned to A

$$\pi_A(x)\;=\;\frac{\pi(x)}{\pi(A)}\;,\quad x\in A\;.$$

Next result is [36, Proposition 8.4]. It holds for non-reversible dynamics. The assertion in the case where A is a singleton follows from the proofs of [5, Corollary 4.2] and [36, Proposition 8.4].

Lemma A.4. *Let A, B be two nonempty disjoint subsets of E. Then, for every probability measure ν concentrated on the set A and $\varrho>0$*

$$\mathbf{P}_\nu\big[\,H_B\;\le\;\varrho\,\big]^2\;\le\;e^2\,E_{\pi_A}\Big[\,\Big(\frac{\nu}{\pi_A}\Big)^2\,\Big]\,\frac{\mathrm{cap}(A,\,B)}{\pi(A)}\,\varrho\;.$$

If A is a singleton, $A=\{x\}$, then for every $\varrho>0$

$$\mathbf{P}_x\big[\,H_B\;\le\;\varrho\,\big]\;\le\;e\,\frac{\mathrm{cap}(\{x\},\,B)}{\pi(x)}\,\varrho\;.$$

This result helps in showing that the left-hand side vanishes asymptotically if $[\,\mathrm{cap}_n(\{x\},\,B)/\pi_n(x)\,]\,\varrho_n\to 0$.

Remark A.5. *For two sets A, B satisfying the hypotheses of Lemma A.4, let $\nu_{A,B}$ be the equilibrium measure on A:*

$$\nu_{A,B}(x)\;=\;\frac{1}{\mathrm{cap}(A\,,\,B)}\,\pi(x)\,\lambda(x)\,\mathbf{P}_x[\,H_B<H_A^+\,]\;,\quad x\in A\;.$$

By Chebychev inequality and [4, Proposition A.2]*,*

$$\mathbf{P}_{\nu_{A,B}}\big[\,H_B\;\ge\;\varrho\,\big]\;\le\;\frac{1}{\varrho}\,E_{\nu_{A,B}}\big[\,H_B\,\big]\;=\;\frac{E_\pi[h^*_{A,B}]}{\varrho\,\mathrm{cap}(A,\,B)}\;,$$

where $h^*_{A,B}$ stands for the equilibrium potential of the time-reversed process (sometimes called the adjoint process): $h^*_{A,B}(y)=\mathbf{P}^*_y[H_A<H_B]$, and $\mathbf{P}^*$ stands for the

distribution of the continuous-time Markov chain with jump rates $R^*(x,y)$ *given by* $R^*(x,y) = \pi(y)\, R(y,x)/\pi(x)$. *In many cases,* $E_{\nu_{A,B}}[\, H_B\,] \,=\, [1+o(1)]\,\pi(A)$ *so that*

$$\mathbf{P}_{\nu_{A,B}}\big[\, H_B \;\geq\; \varrho\,\big] \;\leq\; [1+o(1)]\,\frac{\pi(A)}{\varrho\,\mathrm{cap}(A,\,B)}\;.$$

This inequality demonstrates that the bound in Lemma A.4 is sharp whenever $E_{\nu_{A,B}}[\, H_B\,] \,=\, [1+o(1)]\,\pi(A)$.

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Lorenzo Bertini
Dipartimento di Matematica, Università di Roma 'La Sapienza'
P.le Aldo Moro 2, 00185 Roma, Italy
Email address: bertini@mat.uniroma1.it

Davide Gabrielli
DISIM, Università dell'Aquila
67100 Coppito, L'Aquila, Italy
Email address: davide.gabrielli@univaq.it

Claudio Landim
IMPA
Estrada Dona Castorina 110,
J. Botanico, 22460 Rio de Janeiro, Brazil
and
CNRS UMR 6085, Université de Rouen,
Avenue de l'Université, BP.12, Technopôle du Madrillet,
F76801 Saint-Étienne-du-Rouvray, France.
Email address: landim@impa.br