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Stability analysis of stochastic optimal control: the linear discounted quadratic case

Mathieu Granzotto, Romain Postoyan, Dragan Nešić, and Andrew R. Teel

Abstract—We analyze the stability properties of stochastic linear systems in closed-loop with an optimal policy that minimizes a discounted quadratic cost in expectation. In particular, the linear system is perturbed by both additive and multiplicative stochastic disturbances. We provide conditions under which mean-square boundedness, mean-square stability and recurrence properties hold for the closed-loop system. We distinguish two cases, when these properties are verified for any value of the discount factor sufficiently close to 1, or when they hold for a fixed value of the discount factor in which case tighter conditions are derived as illustrated in an example. The analysis exploits properties of the optimal value function, as well as a detectability property of the system with respect to the stage cost, to construct a Lyapunov function for the stochastic linear quadratic regulator problem.

I. INTRODUCTION

The linear quadratic regulator provides an analytical, systematic way to produce an optimal controller that minimizes a quadratic cost for a linear control system. Its theory, which culminated in the famous work of [16] in the 1960's, is now well established in the control literature [3]. In this work, we concentrate on linear discrete-time systems and quadratic costs that are discounted, i.e., the quadratic costs are weighted by an exponentially decreasing term γ^k , where $\gamma \in (0, 1)$ is the discount factor and k is the time step. Such costs are ubiquitous in the dynamic programming literature [4], and play a major role in reinforcement learning, see, e.g., [5,22,25,27,30]. In these settings, the systems are typically modeled as stochastic processes [4,27]. Our objective is to unravel the link between optimality and stability in this context, namely for stochastic linear discrete-time systems whose inputs minimize a discounted quadratic cost in expectation. The case of deterministic systems with discounted costs is addressed in, e.g., [24], and the related results for stochastic systems concentrate on undiscounted costs, see, e.g., [8–10,15,20,23,31]. The discounted stochastic linear quadratic case is an open problem as far as we are aware, which we will now address.

There exist various definitions of stability for stochastic processes [11], and we propose conditions to guarantee mean-square boundedness, e.g., [18], and recurrence, e.g., [12,19,26,28]. Mean-square boundedness implies that the expected value of the norm of the state squared along the stochastic solutions decreases exponentially down to a constant; when the constant is zero we recover the well-known mean-square stability, e.g., [11]. A strongly globally recurrent set, on the other hand, is a set where any stochastic solution of the system is guaranteed to enter an infinite number of times with probability one. This notion guarantees that the closed-loop system always drives its state back into the recurrent set despite potentially large perturbations. Other stochastic counterparts of the deterministic

notions of attractivity or forward invariance as found in [17,29] require that we stay in given sets almost surely, which is too strong of a requirement in the presence of additive noise, hence the motivation in using the weaker notion of recurrence.

To guarantee these stability properties, we assume: (i) mild conditions on the random inputs; (ii) that the discounted optimal control problem is feasible and is given by a quadratic-like function; (iii) that the stochastic system satisfies a detectability property with respect to the quadratic stage cost. These assumptions are consistent with the deterministic case as seen in [1,16,24]. We first establish mean-square properties and determine strongly globally recurrent sets for the linear system in closed-loop with the optimal policy for any value of the discount factor sufficiently close to 1. Afterwards, we also establish these desired stability properties for a given value of the discount factor, which allows us to relax the required conditions. These results supplement each other, as the former provide us with a range of discount factors such that stability hold, while the latter refines these properties when a discount factor has been fixed. Estimates of the exponential decay and ultimate bound for mean-square properties, and estimates of the recurrent set are provided. These highlight that the discount factor has to be suitably selected to allow for stability to hold. The analysis involves a construction of a Lyapunov function that combines the optimal value function with a Lyapunov-like detectability function.

The rest of the paper is organized as follows. We formalize the optimal control problem in Section II. The problem statement is given in Section III, and the main assumptions are provided in Section IV. In Sections V and VI, we present stability results, for any values of γ sufficiently close to one and then for a given $\gamma \in (0, 1)$, respectively. An illustrative example is given in Section VII. Section VIII concludes the paper and provides future work perspectives. Finally, the proofs and some auxiliary data of the illustrative example are given in the appendices to avoid breaking the flow of exposition.

Notation: Let \mathbb{R} be the set of real numbers, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\bar{\mathbb{R}} := [-\infty, \infty]$, $\bar{\mathbb{R}}_{\geq 0} := [0, \infty]$, \mathbb{Z} be the set of integers, $\mathbb{Z}_{>0} := \{1, 2, \dots\}$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$, $\bar{\mathbb{Z}}_{>0} := \mathbb{Z}_{>0} \cup \{\infty\}$, $\bar{\mathbb{Z}}_{\geq 0} := \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We denote the Euclidean norm by $|\cdot|$. The notation (x, y) stands for $[x^\top, y^\top]^\top$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $n, m \in \mathbb{Z}_{>0}$. The identity matrix is denoted by \mathbb{I}_n with $n \in \mathbb{Z}_{>0}$. The identity map from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ is denoted by $\mathbb{I}_{\geq 0}$, and the zero map from $\mathbb{R}_{\geq 0}$ to $\{0\}$ by $\mathbf{0}$. We write $A \succeq 0$ when real matrix A is symmetric and positive semi-definite, and $A \succ 0$ when it is symmetric and positive definite. We write $A \succeq B$ when $A - B \succeq 0$, and $A \succ B$ when $A - B \succ 0$, for any symmetric matrices A, B . The Kronecker product is denoted by \otimes . The smallest and the biggest eigenvalue of a real, symmetric matrix A is denoted $\underline{\lambda}(A)$ and $\bar{\lambda}(A)$, respectively. The Borel algebra on topological space \mathcal{A} is denoted $\mathcal{B}(\mathcal{A})$. Letting (Ω, \mathcal{F}) be a measurable space, like $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with $n \in \mathbb{Z}_{>0}$, a mapping $M : \Omega \rightrightarrows \mathbb{R}^n$ is \mathcal{F} -measurable (simply measurable when \mathcal{F} is clear from the context) if, for each open set $\mathcal{O} \subset \mathbb{R}^n$, $M^{-1}(\mathcal{O}) := \{\omega \in \Omega : M(\omega) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{F}$. Letting $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $n \in \mathbb{Z}_{>0}$, we say v is a random variable when v is a measurable function from Ω to \mathbb{R}^n , that is, $v^{-1}(F) := \{\omega \in \Omega : v(\omega) \in F\} \in \mathcal{F}$ for each

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$F \in \mathbf{B}(\mathbb{R}^n)$. The expected value of a random variable \mathbf{v} over the entire sample space Ω is denoted by $\mathbb{E}[\mathbf{v}]$. Moreover, $\mathbf{a} \sim \mathcal{N}_{n_a}(0, \Sigma)$ denotes a random variable \mathbf{a} with a multivariate normal distribution of dimension $n_a \in \mathbb{Z}_{>0}$ with zero mean and covariance matrix $\Sigma \in \mathbb{R}^{n_a \times n_a}$, while $\mathbf{m} \sim \mathcal{B}(p)$ denotes a random variable \mathbf{m} with a Bernoulli distribution, i.e., $\mathbb{P}(\mathbf{m} = 0) = (1 - p)$ and $\mathbb{P}(\mathbf{m} = 1) = p$. Given sets $\mathcal{A} \subseteq \mathcal{X}$, the characteristic function of \mathcal{A} is $\mathbb{1}_{\mathcal{A}}(x) := 1$ when $x \in \mathcal{A}$ and $\mathbb{1}_{\mathcal{A}}(x) := 0$ for $x \in \mathcal{X} \setminus \mathcal{A}$. For a sequence $\mathbf{u} = (u(0), u(1), \dots)$ of d elements, with $d \in \mathbb{Z}_{\geq 0}$ and $u(i) \in \mathbb{R}^n$ for $i \in \{1, \dots, d\}$, $n \in \mathbb{Z}_{>0}$, we use $\mathbf{u}|_k$ to denote the first k elements of \mathbf{u} where $k \in \mathbb{Z}_{\geq 0}$, i.e., $\mathbf{u}|_k = (u(0), \dots, u(k-1))$ and $\mathbf{u}|_0 = \emptyset$ by convention.

II. OPTIMAL CONTROL PROBLEM

We consider discrete-time linear systems given by

$$x^+ = (A + M(m))x + (B + N(m))u + La \quad (1)$$

where the state is $x \in \mathbb{R}^{n_x}$, the control input is $u \in \mathbb{R}^{n_u}$, and the stochastic disturbances are $a \in \mathbb{R}^{n_a}$ and $m \in \mathbb{R}^{n_m}$, with $n_x, n_u, n_a, n_m \in \mathbb{Z}_{>0}$. Matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $L \in \mathbb{R}^{n_x \times n_a}$ and continuous matrix-valued maps¹ $M : \mathbb{R}^{n_m} \rightarrow \mathbb{R}^{n_x \times n_x}$, $N : \mathbb{R}^{n_m} \rightarrow \mathbb{R}^{n_x \times n_u}$ are assumed to be known.

System (1) encompasses three notable classes of stochastic linear systems: (i) systems with multiplicative noise when $N(m) \neq 0$ or $M(m) \neq 0$ for some $m \in \mathbb{R}^{n_m}$ [20, Section 2]; (ii) systems with additive noise when $L \neq 0$ [8, Chapter 6.2]; (iii) systems with the so-called random system matrices when $A = 0$ and $B = 0$ [4, Chapter 4.1].

In the following, we concentrate on inputs to system (1) that are generated by policies, i.e., $u = h(x)$ with $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$. This is justified by the fact that, in the considered stochastic setting, minimizing a cost in expectation over policies may produce smaller costs than minimizing over any inputs, see [4, p. 4]. Given $x \in \mathbb{R}^{n_x}$, we denote for brevity $f(x, h, v) := (A + M(m))x + (B + N(m))h(x) + La$ with $v := (a, m) \in \mathbb{R}^{n_v}$, $n_v := n_a + n_m$, and $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$. Moreover, for a given policy h and a fixed sequence of perturbations $\bar{\mathbf{v}}_\infty := (\bar{v}(0), \bar{v}(1), \dots)$, we define $\phi^h(k+1, x, \bar{\mathbf{v}}_\infty|_{k+1}) := f(\phi^h(k, x, \bar{\mathbf{v}}_\infty|_k), h, \bar{v}(k))$ and $\phi^h(0, x, \cdot) := x$.

We investigate the scenario where the policy produces control inputs that minimize a discounted quadratic cost along solutions to (1). We consider for this purpose the cost function defined by, for any initial state x , policy h and sequence $\bar{\mathbf{v}}_\infty$,

$$J_\gamma(x, h, \bar{\mathbf{v}}_\infty) := \sum_{k=0}^{\infty} \gamma^k \left| C\phi^h(k, x, \bar{\mathbf{v}}_\infty|_k) + Dh(\phi^h(k, x, \bar{\mathbf{v}}_\infty|_k)) \right|^2 \quad (2)$$

where $\gamma \in (0, 1)$ is a discount factor, and $C \in \mathbb{R}^{n_\ell \times n_x}$ and $D \in \mathbb{R}^{n_\ell \times n_u}$ are known weighting matrices with $n_\ell \in \mathbb{Z}_{>0}$. Function J_γ is the cost incurred by system (1), policy h and perturbation $\bar{\mathbf{v}}_\infty$. We denote, for the sake of convenience, $\ell(x, u) := |Cx + Du|^2$ for any $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$. We emphasize that cost (2) is equivalent to the classical quadratic cost with cross-terms as $\ell(x, u) = x^\top Qx + u^\top Ru + 2x^\top Zu$ with $Q = C^\top C$, $R = D^\top D$ and $Z = C^\top D$, and verifies by construction the well-known condition [1, Chapter 3.4] $Q - ZR^{-1}Z^\top \succeq 0$ when $R \succ 0$. We are now ready to formalize the stochastic optimal control problem.

Given $x \in \mathbb{R}^{n_x}$, let $\mathbf{v}_\infty := (v_0, v_1, \dots)$ be a sequence of independent, identically distributed (i.i.d.) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, \mathbf{v}_∞ maps from outcomes $\omega \in \Omega$ to infinite sequences of perturbations in \mathbb{R}^{n_v} , hence

¹In the sense that each element of $M(m)$ and $N(m)$ are functions of m .

$\phi^h(k, x, \mathbf{v}_\infty|_k)$ for $k \in \mathbb{Z}_{\geq 0}$ is also a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ when h is a continuous function. We focus on the case where the policy applied to system (1) minimizes cost J_γ in (2) in expectation with perturbation inputs given by the sequence of random variables \mathbf{v}_∞ . In particular, we consider continuous optimal policies, which lead to the *stochastic optimal value function*

$$V_\gamma^*(x) := \inf_{h \in \mathcal{H}} \mathbb{E}[J_\gamma(x, h, \mathbf{v}_\infty)], \quad (3)$$

where we minimize across all continuous policies $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$, denoted in (3) by $\mathcal{H} := \mathcal{C}^0(\mathbb{R}^{n_x}, \mathbb{R}^{n_u})$. We will formally assume in Section IV that there exists one and only one function $h_\gamma^* \in \mathcal{H}$ such that $V_\gamma^*(x) = \mathbb{E}[J_\gamma(x, h_\gamma^*, \mathbf{v}_\infty)]$, so that $V_\gamma^*(x) = \min_{h \in \mathcal{H}} \mathbb{E}[J_\gamma(x, h, \mathbf{v}_\infty)]$.

III. PROBLEM STATEMENT

In this section, we introduce the different stability notions we analyze and we summarize our objectives.

A. Stochastic stability notions

Our aim is to analyse the stability properties of the system given by, for any $x \in \mathbb{R}^{n_x}$,

$$x^+ = (A + M(m))x + (B + N(m))h_\gamma^*(x) + La, \quad (4)$$

where h_γ^* is defined after (3).

We formalize the desired stochastic stability notions for system (4) in the following. To reduce notational burden, from now on we will denote the *random solution* of system (1) in closed loop with h_γ^* initialized at $x \in \mathbb{R}^{n_x}$ by ϕ^* , i.e., given time step $i \in \mathbb{Z}_{\geq 0}$, $\phi_i^* : \Omega \rightarrow \mathbb{R}^{n_x}$ with $\phi_i^*(\omega) := \phi^{h_\gamma^*}(i, x, \mathbf{v}_\infty(\omega)|_i)$ for any time step $i \in \mathbb{Z}_{>0}$ and outcome $\omega \in \Omega$ and $\phi_0^*(\cdot) := x$. We are ready to present our stochastic stability notions. First, we define mean-square boundedness/stability properties as in [7,20].

Definition 1: System (4) is *mean-square bounded* when there exist $g_\gamma \in [1, \infty)$, $\lambda_\gamma \in [0, 1)$ and $G_\gamma \in \mathbb{R}_{\geq 0}$, such that, for any $x \in \mathbb{R}^{n_x}$ and $i \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{E}[\phi_i^{*\top} \phi_i^*] \leq g_\gamma \lambda_\gamma^i |x|^2 + G_\gamma. \quad (5)$$

Moreover, when $G_\gamma = 0$, system (4) is *mean-square stable*. \square

The next stability notion we define is recurrence, which is slightly adapted from [26] to our setting.

Definition 2: An open bounded set $\mathcal{O} \subset \mathbb{R}^{n_x}$ is said to be *strongly globally recurrent* for system (4) if for every $x \in \mathbb{R}^{n_x}$, $\mathbb{E} \left[\prod_{i \in \mathbb{Z}_{\geq 0}} \mathbb{1}_{\mathbb{R}^{n_x} \setminus \mathcal{O}}(\phi_i^*) \right] = 0$ or, equivalently, that $\mathbb{P} \left(\bigcup_{i \in \mathbb{Z}_{\geq 0}} \{\phi_i^*\} \cap \mathcal{O} \neq \emptyset \right) = 1$. \square

According to Definition 2, strong global recurrence is the property that, for any $x \in \mathbb{R}^{n_x}$, ϕ^* has an element in set \mathcal{O} with probability 1. Note that the above indeed ensures that ϕ^* enters \mathcal{O} infinitely many times with probability one, as for any realization $\phi_i^*(\omega) = \bar{x} \notin \mathcal{O}$ for a given time step $i \in \mathbb{Z}_{\geq 0}$ and outcome $\omega \in \Omega$, Definition 2 implies that the stochastic solution of system (4) with initial state \bar{x} will also enter \mathcal{O} in the future with probability one.

Both of Definitions 1 and 2 are important in their own right, as they provide valuable information in spite of the presence of stochastic perturbations: Definition 1 informs us on the decay of the second moment of the state norm (up to a constant); Definition 2 provides us with a stochastic but weaker counterpart of the notions of attractivity and forward invariance for deterministic systems. We illustrate the difference between these stability notions in an example.

Example 1: Consider the following scalar system

$$x^+ = mx + a, \quad (6)$$

where $x, a, m \in \mathbb{R}$. We assume that the random variables \mathbf{a} and \mathbf{m} are independent and such that $\mathbb{E}[\mathbf{a}] = 0$, $\mathbb{E}[\mathbf{a}^2] = \sigma_a$, $\mathbb{E}[\mathbf{m}] = \bar{m}$, $\mathbb{E}[\mathbf{m}^2] = \sigma_m$ with $\bar{m} \in \mathbb{R}$ and $\sigma_m \in \mathbb{R}_{\geq 0}$. We also assume that $\mathbb{P}(\mathbf{m}x + \mathbf{a} = 0 | x \neq 0) = p$ with $p > 0$ for all $x \in \mathbb{R}$, which for instance occurs when $\mathbb{P}(\mathbf{a} = 0) = p_1$ and $\mathbb{P}(\mathbf{m} = 0) = p_2$ where $p_1 p_2 = p$ and $\mathbb{P}(\mathbf{m} = r) = 0$ for each $r \neq 0$, e.g., $\mathbf{a} \sim \mathcal{B}(1 - p_1)\mathcal{N}_1(0, \sigma_a/1 - p_1)$ and $\mathbf{m} \sim \mathcal{B}(1 - p_2)\mathcal{N}_1(\bar{m}/1 - p_2, \sqrt{\sigma_m^2 - \bar{m}^2}/1 - p_2)$ with $\sigma_m > \bar{m}$. We verify that $\mathbb{E}[(\mathbf{m}x + \mathbf{a})^2] = \mathbb{E}[\mathbf{m}^2]x^2 + \mathbb{E}[\mathbf{a}^2] + 2\mathbb{E}[\mathbf{m}]\mathbb{E}[\mathbf{a}]x = \sigma_m x^2 + \sigma_a$ for any $x \in \mathbb{R}$. This implies by the law of iterated expectation that, for any $i \in \mathbb{Z}_{>0}$, $\mathbb{E}[\phi_i^{*2}] = \sigma_m^i x^2 + \sigma_a \sum_{k=0}^{i-1} \sigma_m^k = \sigma_m^i x^2 + \sigma_a \frac{1 - \sigma_m^i}{1 - \sigma_m}$. Furthermore, we calculate, for any $x \neq 0$, and $i \in \mathbb{Z}_{\geq 0}$, that $\mathbb{P}(\phi_{i+1}^* = 0 | \phi_k^* \neq 0, k \in \{0, \dots, i\}) = \mathbb{P}(\mathbf{m}x + \mathbf{a} = 0 | x \neq 0) = p$ and moreover $\mathbb{P}(\phi_k^* \neq 0, k \in \{0, \dots, i\}) = (1 - p)^i$, hence, $\mathbb{P}\left(\bigcup_{i \in \mathbb{Z}_{\geq 0}} \{\phi_i^* \neq 0\} \cap \{0\} \neq \emptyset\right) = \sum_{i=0}^{\infty} \mathbb{P}(\phi_{i+1}^* = 0 | \phi_k^* \neq 0, k \in \{0, \dots, i\}) \mathbb{P}(\phi_k^* \neq 0, k \in \{0, \dots, i\}) = \sum_{i=0}^{\infty} p(1 - p)^i = 1$. We thus conclude that system (6) is

- mean-square stable if and only if $\sigma_m \in [0, 1)$ and $\sigma_a = 0$;
- mean-square bounded if and only if $\sigma_m \in [0, 1)$;
- strongly globally recurrent relative to any open set containing the origin for any $\bar{m} \in \mathbb{R}$ and $\sigma_m \in \mathbb{R}_{\geq 0}$. \square

This scalar example illustrates the differences between Definitions 1 and 2. Nevertheless the conditions we provide in the sequel allow to conclude recurrence and mean-square properties simultaneously.

B. Objectives

Our objective is to provide conditions under which system (4) is mean-square bounded, mean-square stable and strongly globally recurrent for some set, respectively. Like in the deterministic case, the discount factor γ in (2) plays a key role for the satisfaction of these properties as illustrated by the next example.

Example 2: Consider the scalar system $x^+ = 2x + u + a$ with $a \sim 1 - 2\mathcal{B}(1/2)$ and the discounted quadratic cost (2) with $C = (1 \ 0)^\top$ and $D = (0 \ 1)^\top$. The optimal solution is given by the feedback law $h_\gamma^*(x) = K_\gamma^* x$ with $K_\gamma^* = -2/(1 + 2/(5\gamma - 1 + \sqrt{(5\gamma - 1)^2 + 4\gamma}))$, see [8, Chapter 6.2]. The closed-loop system is mean-square bounded if and only if $2 + K_\gamma^* \in (-1, 1)$, which is equivalent to $\gamma \in (1/3, 1)$. Moreover, there exists a strongly recurrent bounded open set containing the origin if and only if $\gamma \in (1/3, 1)$. Hence, in both cases γ needs to be sufficiently close to 1 for this example to exhibit the desired stability properties. \square

We first establish mean-square and recurrence properties for any γ sufficiently close to 1, see Section V. For this purpose, we rely on assumptions that do not involve γ . Afterwards, we establish mean-square bounded, mean-square stable and strong global recurrence properties for a given value of $\gamma \in (0, 1)$ only, see Section VI. In this case, less conservative conditions are required to obtain the desired results.

To derive our desired properties, we require the next set of assumptions on system (1) and cost (2).

IV. STANDING ASSUMPTIONS

A. Stochastic disturbances

We make the next assumption on the stochastic disturbance $v = (a, m)$.

Standing Assumption 1 (SA1): The following holds for any $i \in \mathbb{Z}_{\geq 0}$.

- $\mathbf{v}_i = (\mathbf{a}_i, \mathbf{m}_i) : \Omega \rightarrow \mathbb{R}^{n_a} \times \mathbb{R}^{n_m}$ is such that \mathbf{a}_i and \mathbf{m}_i are mutually independent.
- $\mathbb{E}[\mathbf{a}_i]$, $\mathbb{E}[M(\mathbf{m}_i)]$, $\mathbb{E}[N(\mathbf{m}_i)]$ are zero.

- $\mathbb{E}[\mathbf{a}_i \otimes \mathbf{a}_i]$, $\mathbb{E}[M(\mathbf{m}_i) \otimes M(\mathbf{m}_i)]$, $\mathbb{E}[N(\mathbf{m}_i) \otimes N(\mathbf{m}_i)]$ and $\mathbb{E}[N(\mathbf{m}_i) \otimes M(\mathbf{m}_i)]$ are finite element-wise. \square

Item (i) of SA1 states that the multiplicative and the additive noises at every time step are also mutually independent. In view of the i.i.d. property of \mathbf{v}_∞ , we can omit the subscript of $\mathbf{v}_i = (\mathbf{a}_i, \mathbf{m}_i)$ and write $\mathbf{v} = (a, m)$ in the following for the sake of convenience. The property that $\mathbb{E}[\mathbf{a}] = 0$ in item (ii) of SA1 is usual in the literature [8] for linear systems affected by additive noise. On the other hand, the assumption that M and N have zero mean is without loss of generality when $\mathbb{E}[M(\mathbf{m})]$ and $\mathbb{E}[N(\mathbf{m})]$ are non-zero and known, as we can always rewrite system (1) as $x^+ = ((A + \mathbb{E}[M(\mathbf{m})]) + (M(\mathbf{m}) - \mathbb{E}[M(\mathbf{m})]))x + ((B + \mathbb{E}[N(\mathbf{m})]) + (N(\mathbf{m}) - \mathbb{E}[N(\mathbf{m})]))u + La$, as done in an example in Section VII. Items (i)-(ii) of SA1 imply that, $\mathbb{E}[\mathbf{a}^\top T M(\mathbf{m})] = \mathbb{E}[\mathbf{a}^\top] T \mathbb{E}[M(\mathbf{m})] = 0$ and $\mathbb{E}[\mathbf{a}^\top T N(\mathbf{m})] = \mathbb{E}[\mathbf{a}^\top] T' \mathbb{E}[N(\mathbf{m})] = 0$ for any matrices $T \in \mathbb{R}^{n_a \times n_x}$, $T' \in \mathbb{R}^{n_a \times n_x}$, which will be important in the sequel. Finally, item (iii) of SA1 is made to guarantee that $\mathbb{E}[M(\mathbf{m})^\top P M(\mathbf{m})]$ and $\mathbb{E}[N(\mathbf{m})^\top P' N(\mathbf{m})]$ are finite for any matrices $P \in \mathbb{R}^{n_x \times n_x}$, $P' \in \mathbb{R}^{n_x \times n_u}$.

B. Optimal value function and policy

Next, we assume the existence of a unique solution to (3).

Standing Assumption 2 (SA2): For any $\gamma \in (0, 1)$, $V_\gamma^*(x)$ in (3) is finite for all $x \in \mathbb{R}^{n_x}$ and is given by

$$V_\gamma^*(x) = x^\top P_\gamma x + \frac{\gamma}{1 - \gamma} \mathbb{E}[\mathbf{a}^\top L^\top P_\gamma L \mathbf{a}] = \mathbb{E}[J(x, h_\gamma^*, \mathbf{v}_\infty)] \quad (7)$$

with $h_\gamma^*(x) = -K_\gamma^* x$, where $P_\gamma \succeq 0$ satisfies

$$P_\gamma = C^\top C + \gamma(A^\top P_\gamma A + \mathbb{E}[M(\mathbf{m})^\top P_\gamma M(\mathbf{m})]) - (D^\top C + \gamma(B^\top P_\gamma A + \mathbb{E}[N(\mathbf{m})^\top P_\gamma M(\mathbf{m})]))^\top K_\gamma^* \quad (8)$$

and $K_\gamma^* := (D^\top D + \gamma(B^\top P_\gamma B + \mathbb{E}[N(\mathbf{m})^\top P_\gamma N(\mathbf{m})]))^{-1} (D^\top C + \gamma(B^\top P_\gamma A + \mathbb{E}[N(\mathbf{m})^\top P_\gamma M(\mathbf{m})]))$. \square

SA2 implies that the stochastic optimal control problem is feasible, and its solution is related to the generalized algebraic Riccati equation (8). Equation (8) also corresponds to the Riccati equation for the deterministic case when no stochastic component is present, i.e., when L , $M(\mathbf{m})$, $N(\mathbf{m})$ are zero for all $\mathbf{m} \in \mathbb{R}^{n_m}$. Conditions for the feasibility of (8) in SA2 can be found for several special cases of (1), in, e.g., [8, 20], by considering the time-varying change of coordinates $z(k) = \sqrt{\gamma^k} x(k)$. That is, applying the cited result to system $z^+ = \sqrt{\gamma}((A + M(\mathbf{m}))z + (B + N(\mathbf{m}))u)$ and $\tilde{\phi}^h := \sqrt{\gamma^k} \phi^h$ given policy h to cost $J_1(x, h, \bar{\mathbf{v}}_\infty) := \sum_{k=0}^{\infty} \ell(\tilde{\phi}^h(k, x, \bar{\mathbf{v}}_\infty | k), h(\tilde{\phi}^h(k, x, \bar{\mathbf{v}}_\infty | k)))$.

Under SA2, the closed-loop system (4) is equivalent to, for any $x \in \mathbb{R}^{n_x}$,

$$x^+ = \mathcal{A}_\gamma^*(m)x + La, \quad (9)$$

where $\mathcal{A}_\gamma^*(m) := A + M(\mathbf{m}) - (B + N(\mathbf{m}))K_\gamma^*$ for all $\mathbf{m} \in \mathbb{R}^{n_m}$. Moreover, the so-called Bellman equation [6, Proposition 9.8] implies

$$V_\gamma^*(x) = \mathbb{E}[\ell(x, -K_\gamma^* x) + \gamma V_\gamma^*(\mathcal{A}_\gamma^*(m)x + La)]. \quad (10)$$

This equation plays a key role in the analysis presented in Sections V and VI.

Remark 1: It is important to note that P_γ verifying (8) in SA2 may not exist for certain choices of matrices and multiplicative noise that is sufficiently large. This is known as the uncertainty threshold principle in the random matrices literature, see, e.g., [2, 32] and [4, Volume 1]. \square

Remark 2: Unless we note otherwise, we assume in the following that both P_γ and K_γ are known. In general, we may resort to

dynamic programming methods like value iteration [4] to obtain approximations of P_γ and K_γ . This is how we proceed for the example in Section VII. \square

C. Detectability

We make the next assumption on the open-loop system and the incurred stage cost, which is related to the Lyapunov characterization of detectability employed for optimal control in the deterministic setting, see, e.g., [13,14,24].

Standing Assumption 3 (SA3): There exist $S \succeq 0$ and $\tilde{S} \succ 0$ such that, for all $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$,

$$\mathbb{E}[(\hat{x}^+)^T S \hat{x}^+] \leq x^T S x - x^T \tilde{S} x + \ell(x, u), \quad (11)$$

where $\hat{x}^+ := (A + M(\mathbf{m}))x + (B + N(\mathbf{m}))u$ is a random variable that depends on the multiplicative noise \mathbf{m} and ℓ is the stage cost defined after (2). \square

SA3 is a detectability property of system (1) with respect to stage cost ℓ in presence of multiplicative noise and in spite of the additive noise. SA3 requires that the Lyapunov-like function $W(x) := x^T S x$ decreases in expectation along the solution of the stochastic system (1) for $L = 0$ when the stage cost is sufficiently small. SA3 has a trivial solution $S = 0$ when $C^T C \succ 0$ and $D^T D \succ 0$. When no multiplicative noise is present, $C = (\sqrt{Q} \ 0)^T$ and $D = (0 \ \sqrt{R})^T$, where \sqrt{Q} denotes a positive semi-definite matrix such that $\sqrt{Q}\sqrt{Q}^T = Q$ for any $Q \succeq 0$, pair (A, \sqrt{Q}) detectable implies SA3 holds. In particular cases, SA3 can be verified by LMI methods as seen in Section VII. Note that verifying (11) without the expectation for some values of m is not enough to verify SA3 and one has to take into consideration all values of m in the support of \mathbf{m} when solving (11) for S and \tilde{S} .

We are ready to present the main stability results.

V. RESULTS FOR ANY γ SUFFICIENTLY CLOSE TO 1

In this section, we show the existence of a $\gamma^* \in [0, 1)$ such that the stochastic stability properties defined in Section III hold for any $\gamma \in (\gamma^*, 1)$. The proofs of this section are given in Appendix I to avoid breaking the flow of exposition.

A. Stabilizability assumption

Because we are aiming at stability guarantees that hold for any $\gamma \in (\gamma^*, 1)$ we need the next assumption.

Assumption 1: There exists $P_1 \succeq 0$ that verifies the generalized Riccati equation related to the stochastic optimal control problem for undiscounted costs, i.e., $\gamma = 1$, without additive noise, $L = 0$, i.e.,

$$P_1 = C^T C + A^T P_1 A + \mathbb{E}[M(\mathbf{m})^T P_1 M(\mathbf{m})] - (D^T C + B^T P_1 A + \mathbb{E}[N(\mathbf{m})^T P_1 M(\mathbf{m})])^T K_1^* \quad (12)$$

with $K_1^* := (D^T D + B^T P_1 B + \mathbb{E}[N(\mathbf{m})^T P_1 N(\mathbf{m})])^{-1} (D^T C + B^T P_1 A + \mathbb{E}[N(\mathbf{m})^T P_1 M(\mathbf{m})])$. \square

Assumption 1 is related to the stabilizability of the linear system in presence of multiplicative disturbances, as seen in, e.g., [8,20]. Note that $\gamma = 1$ is not covered by SA2 when L is different from 0, as SA2 implies that V_γ^* tends to infinity when $\gamma \rightarrow 1$ due to the constant term $\frac{\gamma}{1-\gamma} \mathbb{E}[\mathbf{a}^T L^T P_\gamma L \mathbf{a}]$ that comes from the additive noise. Assumption 1 will no longer be required in Section VI when considering a given value of $\gamma \in (0, 1)$. The next proposition states that $P_\gamma \preceq P_1$; its proof is given in Appendix I.

Proposition 1: When Assumption 1 holds, $P_\gamma \preceq P_1$. \square

B. Mean-square properties

We show next the mean-square boundedness and stability guarantees for system (4).

Theorem 1: Suppose Assumption 1 holds and let $\gamma^* \in (0, 1)$ be such that $\tilde{S} \succeq (1 - \gamma^*)P_1$. For any $\gamma \in (\gamma^*, 1)$, system (4) is mean-square bounded. In particular, (5) holds with $g_\gamma = \tilde{\lambda}(P_1 + \frac{S}{\gamma})/\lambda(\tilde{S})$, $\lambda_\gamma = \alpha_\gamma$ and $G_\gamma = \mathbb{E}[\mathbf{a}^T L^T (P_\gamma + \frac{S}{\gamma}) L \mathbf{a}]/(1 - \alpha_\gamma)$, where $\alpha_\gamma = 1 - \frac{\gamma - \gamma^*}{\gamma(1 - \gamma^*)} a_Y$ with sufficiently small² $a_Y \in (0, 1)$ such that $a_Y(P_1 + \frac{S}{\gamma}) \preceq \tilde{S}$. Hence, when $\mathbb{E}[\mathbf{a}^T L^T L \mathbf{a}] = 0$, system (4) is mean-square stable. \square

Theorem 1 shows that mean-square boundedness properties hold provided that the discount factor is sufficiently close to 1. When no additive noise is present, $\mathbb{E}[\mathbf{a}^T L^T L \mathbf{a}] = 0$ and mean-square stability is guaranteed. It is always possible to find γ^* sufficiently close to one such that $\tilde{S} \succeq (1 - \gamma^*)P_1$ given $\tilde{S} \succ 0$, which holds by SA3. Interestingly, the condition $\tilde{S} \succeq (1 - \gamma^*)P_1$ allows to derive stability properties in presence of the discount factor without explicitly calculating P_γ . Moreover, this condition and the bounds on λ_γ and G_γ show the interplay between the optimal value function, the detectability properties in SA3 and the discount factor.

We now concentrate on strong global recurrence properties.

C. Strong global recurrence

We provide a strongly globally recurrence property of a given set for system (4), provided that γ is sufficiently close to 1.

Theorem 2: Suppose Assumption 1 holds and let $\gamma^* \in (0, 1)$ be such that $\tilde{S} \succeq (1 - \gamma^*)P_1$ as in Theorem 1. Moreover, let $\gamma \in (\gamma^*, 1)$ and

$$\mathcal{K}_\gamma := \left\{ x \in \mathbb{R}^{n_x} \mid x^T \tilde{S} x \leq \frac{1 - \gamma^*}{\gamma - \gamma^*} \mathbb{E} \left[\mathbf{a}^T L^T (\gamma P_1 + S) L \mathbf{a} \right] \right\}, \quad (13)$$

with P_1 , S and \tilde{S} from Assumption 1 and SA3. Then, any bounded open set \mathcal{O}_γ such that $\mathcal{O}_\gamma \supset \mathcal{K}_\gamma$ is strongly globally recurrent for system (4). \square

Theorem 2 establishes a recurrence property for set \mathcal{O}_γ for system (4), provided that the discount factor γ is larger than γ^* and smaller than one. The recurrent set contains an ellipsoid centered at the origin, whose axes are the eigenvectors of \tilde{S} . We note that $\mathcal{K}_{\gamma'} \subset \mathcal{K}_\gamma$ for any $\gamma' > \gamma$, hence the strongly globally recurrent set “shrinks” as γ increases to 1.

Theorem 2 also allows for the next observation.

Corollary 1: Suppose Assumption 1 holds and $\mathbb{E}[\mathbf{a}^T L^T L \mathbf{a}] = 0$. For any $\gamma \in (\gamma^*, 1)$ with γ^* selected as in Theorem 1, any bounded open set containing the origin is strongly globally recurrent for system (4). \square

Corollary 1 is a stronger result than Theorem 2 when no additive noise is present, e.g., when $L = 0$, and is similar to the second part of Theorem 1. While in Theorem 2 the recurrent set depends on the interplay of the multiplicative and additive noise, Corollary 1 shows that, ultimately, \mathcal{O}_γ is arbitrarily small when no additive noise is present and γ is sufficiently close to 1.

The conditions above allow us to derive the desired stability properties for stochastic linear quadratic regulator in great generality, yet the approach taken leads to some conservatism in the choice of γ^* (in particular due to SA3) as we will see in Section VII. In the next section, we provide tailored conditions for the desired stability properties to hold for a given value of $\gamma \in (0, 1)$.

²We can always take $a_Y = \lambda(\tilde{S})/\tilde{\lambda}(P_1 + \frac{S}{\gamma^*})$, however larger values for a_Y may exist and are desirable.

VI. RESULTS FOR GIVEN $\gamma \in (0, 1)$

We now provide conditions such that the stochastic stability properties in Definitions 1 and 2 hold for system (4) for a given value of $\gamma \in (0, 1)$. The proofs of this section are omitted to avoid repetition, as the following results are shown by following the steps of the proofs of Section V presented in Appendix I by replacing S , \tilde{S} with S_γ and \tilde{S}_γ , respectively.

A. Detectability assumption

We make the next assumption on the closed-loop system and the incurred stage cost, which relaxes SA3 as explained below.

Assumption 2: Given $\gamma \in (0, 1)$, there exist $S_\gamma \succeq 0$ and $\tilde{S}_\gamma \succ 0$ such that

$$\mathbb{E}[x^\top \mathcal{A}_\gamma^*(\mathbf{m})^\top S_\gamma \mathcal{A}_\gamma^*(\mathbf{m}) x] \leq x^\top S_\gamma x - x^\top \tilde{S}_\gamma x + \ell(x, -K_\gamma^* x), \quad (14)$$

where $\mathcal{A}_\gamma^*(\mathbf{m})$ comes from (9). \square

While SA3 states a detectability of system (1) with respect to $\ell(x, u)$ for any input $u \in \mathbb{R}^{n_u}$, Assumption 2 requires this property to hold only for $u = -K_\gamma^* x$ and allows S and \tilde{S} in SA3 to depend on γ thereby relaxing SA3.

B. Mean-square properties

We assert next the mean-square stability guarantee given Assumption 2.

Theorem 3: Suppose Assumption 2 holds for a given $\gamma \in (0, 1)$ and let $\varpi_\gamma \in (0, 1)$ be such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$. If $\varpi_\gamma < \gamma$, system (4) is mean-square bounded. In particular, (5) holds with $g_\gamma = \bar{\lambda}(P_\gamma + \frac{S_\gamma}{\gamma}) / \Delta(\tilde{S}_\gamma)$, $\lambda_\gamma = \alpha_\gamma$ and $G_\gamma = \mathbb{E}[\mathbf{a}^\top L^\top (P_\gamma + \frac{S_\gamma}{\gamma}) L \mathbf{a}] / (1 - \alpha_\gamma)$, where $\alpha_\gamma = 1 - \frac{\gamma - \varpi_\gamma}{\gamma(1 - \varpi_\gamma)} a_\gamma$ with sufficiently small³ $a_\gamma \in (0, 1)$ such that $a_\gamma (P_\gamma + \frac{S_\gamma}{\gamma}) \preceq \tilde{S}_\gamma$. Hence, when $\mathbb{E}[\mathbf{a}^\top L^\top L \mathbf{a}] = 0$, system (4) is mean-square stable. \square

Theorem 3 relaxes Theorem 1 to a fixed $\gamma \in (0, 1)$, therefore showing that mean-square properties hold provided that we find \tilde{S}_γ and $\varpi_\gamma < \gamma$ such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$. These conditions are the main difference with Theorem 3, where ϖ_γ plays the role of γ^* but the inequality it must satisfy, namely $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$, involves γ -dependent matrices contrary to γ^* in Theorem 1, which may help to consider smaller values of γ . In addition, because the proof of Theorem 3 relies on a Lyapunov-like function with γ -dependent matrices, less conservative estimates of g_γ , λ_γ and G_γ in (5) are given by Theorem 3 compared to Theorem 1. These differences are illustrated in the example given in Section VII.

We now move to strong global recurrence properties.

C. Strong global recurrence

We show next the existence of a strongly globally recurrent set for system (4).

Theorem 4: Suppose Assumption 2 holds for a given $\gamma \in (0, 1)$ and let $\varpi_\gamma \in (0, 1)$ be such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$ as in Theorem 3. Moreover, let

$$\mathcal{K}_{\varpi_\gamma} := \left\{ x \in \mathbb{R}^{n_x} \mid x^\top \tilde{S}_\gamma x \leq \frac{1 - \varpi_\gamma}{\gamma - \varpi_\gamma} \mathbb{E}[\mathbf{a}^\top L^\top (\gamma P_\gamma + S_\gamma) L \mathbf{a}] \right\} \quad (15)$$

with P_γ , S_γ and \tilde{S}_γ coming from SA2 and Assumption 2. If $\varpi_\gamma < \gamma$, then any bounded open set \mathcal{O}_γ such that $\mathcal{O}_\gamma \supset \mathcal{K}_{\varpi_\gamma}$ is strongly globally recurrent for system (4). \square

We can make similar observations on how Theorem 4 relaxes the conditions of Theorem 2 as above. We also derive the next corollary.

³Similar to Theorem 1, we can always take $a_\gamma = \Delta(\tilde{S}_\gamma) / \bar{\lambda}(P_\gamma + \frac{S_\gamma}{\gamma})$.

Corollary 2: Suppose Assumption 2 holds for a given $\gamma \in (0, 1)$ and let $\varpi_\gamma \in (0, 1)$ such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$ as in Theorem 3. When $\varpi_\gamma < \gamma$ and $\mathbb{E}[\mathbf{a}^\top L^\top L \mathbf{a}] = 0$, any bounded open set containing the origin is strongly globally recurrent for system (4). \square

We now illustrate the presented stochastic stability tools in an illustrative example.

VII. ILLUSTRATIVE EXAMPLE

A. System, cost and assumptions

We consider the model of a linearized cart pole system [21], which we (exactly) discretize by employing a zero-order hold with sampling period $T = 0.1$. The state is $x = (s, \dot{s}, \psi, \dot{\psi}) \in \mathbb{R}^4$, where s is the position of the cart and ψ is the angle of the pole. Moreover, we consider the presence of additive noise and that actuation fails intermittently with probability $p \in [0, 1]$. The system is thus given by

$$x^+ = Ax + (\tilde{B} + \tilde{N}(m))u + a \quad (16)$$

where $\mathbf{a} \sim \mathcal{N}_4(0, \Sigma)$, $\mathbf{m} \sim \mathcal{B}(p)$, $\tilde{N}(0) = 0$ and $\tilde{N}(1) = -\tilde{B}$. The numerical values including the system matrices are given in Appendix II to avoid breaking the flow of exposition. By the Bernoulli distribution of \mathbf{m} , $\mathbb{P}(\tilde{N}(m) = -\tilde{B}) = p$ and $\mathbb{P}(\tilde{N}(m) = 0) = 1 - p$. As $\mathbb{E}[\tilde{N}(m)] \neq 0$, we consider instead $B := \tilde{B} + \mathbb{E}[\tilde{N}(m)] = \tilde{B} - p\tilde{B} = (1 - p)\tilde{B}$ and $N(m) = \tilde{N}(m) - \mathbb{E}[\tilde{N}(m)]$. In particular, $N(0) = 0 - (-p\tilde{B}) = p\tilde{B}$ and $N(1) = -\tilde{B} - (-p\tilde{B}) = -(1 - p)\tilde{B}$. In this way, and considering that \mathbf{a} and \mathbf{m} are mutually independent random variables with finite (co)variance, SA1 holds for $x^+ = Ax + (B + N(m))u + a$.

For the cost, we choose C and D such that $C^\top C = \mathbb{I}_4$, $D^\top D = 1$ and $C^\top D = 0$. We have derived both P_γ and P_1 from SA2 and Assumption 1 using a value iteration procedure. We observed that finite P_γ does not exist with $p > 0.32$, which we attribute to the uncertainty threshold principle [32]. Therefore, we fix $p = 0.10$, and numerically conclude that P_γ verifies (8) from SA2 and (12) from Assumption 1, for $\gamma \in (0, 1)$ and $\gamma = 1$, respectively.

To verify SA3, we calculate S and \tilde{S} by employing an LMI procedure given by Lemma 1 in Appendix III with $A_0 := A_1 := A$, $B_0 := \tilde{B}$, $B_1 := 0$, $p_0 := (1 - p)$ and $p_1 := p$.

We are ready to apply the results of Sections V and VI.

B. Stability guarantees

Given the obtained \tilde{S} and P_1 , we find $\gamma^* = 0.993$ such that $\tilde{S} \succeq (1 - \gamma^*)P_1$. Hence, Theorems 1 and 2 hold. For the following, we fix the discount factor to $\gamma = 0.9965$ and the covariance matrix of the additive noise as $\Sigma = 10^{-2}\mathbb{I}_4$. The quantitative estimates of mean-square boundedness from Theorem 1 are given in Table I. Similarly, we estimate the strongly globally recurrent set from Theorem 2 as any open bounded set $\mathcal{O}_\gamma \subset \mathbb{R}^4$ such that

$$\mathcal{R} := \left\{ x \in \mathbb{R}^{n_x} \mid x^\top \tilde{S} x \leq 62 \right\} \subset \mathcal{O}_\gamma. \quad (17)$$

To study the conservatism of the above estimates, we apply the results of Section VI for $\gamma = 0.9965$. We verify Assumption 2 again by an LMI procedure, based on (27) in Appendix III. We find that $\varpi_\gamma = 0$ is such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$, thus Theorems 3 and 4 hold as $\gamma = 0.9965 > 0$. The estimated constants related to mean-square boundedness are also given in Table I. Compared to ones given by Theorem 1, by means of Theorem 3 we have obtained a smaller guaranteed overshoot g_γ , a faster decay rate λ_γ and a smaller ultimate bound G_γ . Similarly, we estimate the strongly globally recurrent set from Theorem 4 as any open bounded set $\mathcal{O}_\gamma \subset \mathbb{R}^4$ such that

$$\mathcal{R}_\gamma := \left\{ x \in \mathbb{R}^{n_x} \mid x^\top \tilde{S}_\gamma x \leq 284 \right\} \subset \mathcal{O}_\gamma, \quad (18)$$

	g_γ	λ_γ	G_γ
Theorem 1	$3 \cdot 10^5$	0.9998	$1.5 \cdot 10^8$
Theorem 3	$3 \cdot 10^3$	0.9858	$1 \cdot 10^3$

TABLE I
CONSTANTS IN (5)

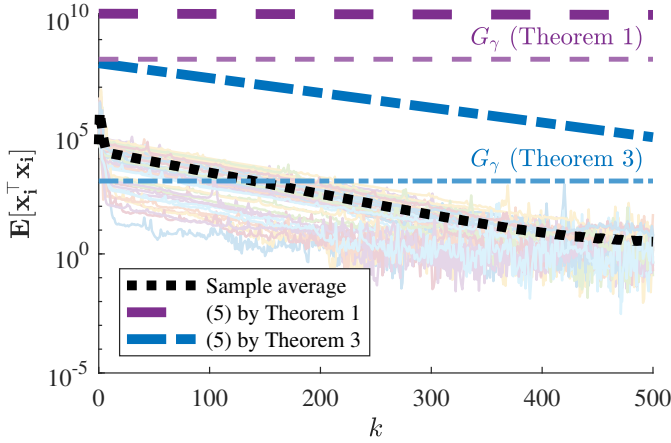


Fig. 1. The evolution of the mean of the state norm squared of (16) in closed-loop with K_γ^* for $\gamma = 0.9965$, given by 10^6 arbitrary realizations, and its theoretical upper-bound (5) given by Theorems 1 and 3.

which is strictly included in \mathcal{R} , as we have $\tilde{S}_\gamma/284 \succ \tilde{S}/62$. We further illustrate the results by means of simulations.

In Figure 1, we simulate 500 time steps over 10^6 realizations and we present the arithmetic mean of the sample. The initial state of each realization (which a subset is plotted in faded colors in Figure 1) were arbitrarily chosen to verify $x^T x = 200^2$. We also plot the upper-bounds in (5) guaranteed by Theorems 1 and 3. As expected, we observe tighter bounds when employing Theorem 3, with the decay rate given by Theorem 3 relatively close to the decay rate of the estimated mean of the sample.

In Figure 2 we show the evolution over 200 time steps of the position and angle of the cart pole system against the ellipsoid projection of the recurrent set for these state variables. We observe that the two given realizations with distinct initial states always (re-)enter these projected sets given in (17) and (18), despite the presence of stochastic perturbations consistently with Theorems 2 and 4.

In Figures 3 and 4, we illustrate the stronger guarantees when no additive noise is present. We do so by repeating the above experiments while setting $L = 0$ and increasing the number of time steps. We see that the (mean-square) state approaches the origin exponentially in the presence of multiplicative noise. This is consistent with mean-square stability and recurrence properties predicted by Theorems 1 and 3 and Corollaries 1 and 2.

Finally, we apply the results of Section VI and investigate whether stability properties can be ensured for some $\gamma \in (0, 0.993)$. For the given example, the sufficient condition for stability properties, namely $\gamma > \varpi_\gamma$ with ϖ_γ such that $\tilde{S}_\gamma \succeq (1 - \varpi_\gamma)P_\gamma$ in Theorems 3 and 4, is verified for all tested values of γ larger than 0.586. Therefore, we see that the results of Section VI indeed allows to certify stability properties for $\gamma < 0.993$. Interestingly, simulation results suggest that mean-square stability of the cart-pole system does not hold when $\gamma \in (0, 0.586)$.

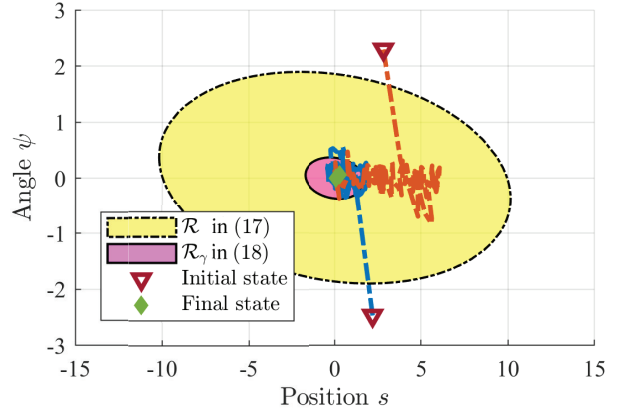


Fig. 2. Two arbitrary realizations of (16) in closed-loop with K_γ^* for $\gamma = 0.9965$ projected in the $s \times \psi$ plane (re-)entering \mathcal{R} and \mathcal{R}_γ .

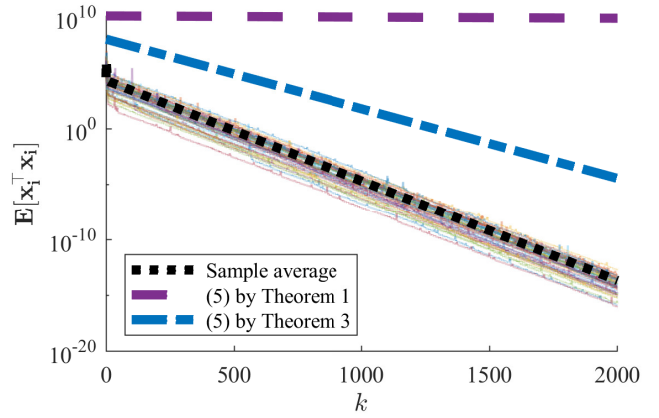


Fig. 3. The evolution of the mean of the state norm squared of (16) without additive noise ($L = 0$) in closed-loop with K_γ^* for $\gamma = 0.9965$, and its theoretical upper-bound converging to zero.

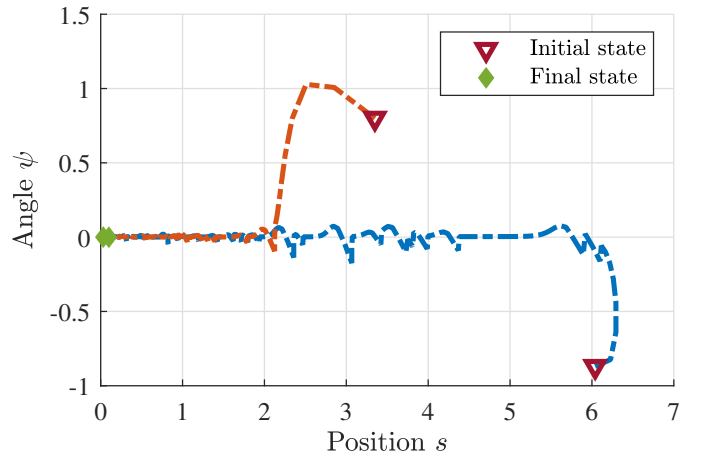


Fig. 4. Two arbitrary realizations of (16) without additive noise ($L = 0$) in closed-loop with K_γ^* for $\gamma = 0.9965$ projected in the $s \times \psi$ plane approaching the origin.

VIII. CONCLUSIONS

We have provided general conditions to ensure stability properties for the stochastic linear quadratic regulator problem with discounted costs.

In particular and under our assumptions, we have derived that it is desirable to select the discount factor sufficiently close to one for stability purposes. Several questions could be investigated in future work based on the presented results. For example, it would be interesting to formally study if such property holds when only an approximation of the controller $u = -K_{\gamma}^*x$ is available (by, e.g., dynamic programming methods as done in the example) or for general nonlinear systems and stage costs.

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APPENDIX I PROOFS OF SECTION V

A. Sketch of proof of Proposition 1

It suffices to consider the optimal value function in expectation like in (3) for system $\hat{x}^+ = (A + M(m))\hat{x} + (B + N(m))u$, which is system (1) with $L = 0$, and cost (2). We have $V_{\gamma} = x^{\top} P_{\gamma} x$ for $\gamma \in (0, 1]$. It then follows $P_{\gamma} \preceq P_1$ from $V_{\gamma}^*(x) \leq V_{\gamma'}^*(x)$ for any $\gamma \leq \gamma'$.

B. Preliminaries of the proofs of Theorems 1 and 2

Let $x \in \mathbb{R}^{n_x}$ and $\gamma \in (\gamma^*, 1)$ with γ^* selected as in Theorem 2. The main steps of the proof are to show, first, that $P_{\gamma} + \frac{S}{\gamma}$ is positive definite, second, that $\mathbb{E}[Y_{\gamma}(x^+)] \leq (1 - \lambda)Y_{\gamma}(x) + \mathbb{E}[a^{\top} L^{\top} (P_{\gamma} + S_{\gamma}) L a]$ for some $\lambda \in (0, 1)$ with $Y_{\gamma}(x) := x^{\top} (P_{\gamma} + \frac{S}{\gamma}) x$. By the law of iterated expectations, these two properties allows to derive the conclusions of Theorem 1, and by invoking [26, Theorem 1], conclude that \mathcal{O}_{γ} given in Theorem 2 is strongly globally recurrent for (4).

We now show $P_{\gamma} + \frac{S}{\gamma}$ is positive definite. In view of (8), (11) with $u = -K_{\gamma}^*x$, $P_{\gamma} + \frac{S}{\gamma} \succeq 0$ and $\gamma \in (0, 1)$, we calculate $Y_{\gamma}(x) = x^{\top} (P_{\gamma} + \frac{S}{\gamma}) x \geq \ell(x, -K_{\gamma}^*x) + x^{\top} S x \geq \ell(x, -K_{\gamma}^*x) + x^{\top} \tilde{S} x - \ell(x, -K_{\gamma}^*x)$, thus

$$Y_{\gamma}(x) \geq x^{\top} \tilde{S} x. \quad (19)$$

Hence $P_{\gamma} + \frac{S}{\gamma} \succ 0$. Moreover, there exists $a_Y \in (0, 1)$, independent of γ , such that $a_Y (P_{\gamma} + \frac{S}{\gamma}) \preceq \tilde{S}$. Indeed, consider $a_Y := \lambda(\tilde{S}) / \bar{\lambda}(P_1 + \frac{S}{\gamma^*})$. It follows that $\lambda(\tilde{S}) \mathbb{I}_{n_x} \preceq \tilde{S} \preceq P_{\gamma} + \frac{S}{\gamma} \preceq \bar{\lambda}(P_1 + \frac{S}{\gamma^*}) \mathbb{I}_{n_x} \preceq \frac{\bar{\lambda}(P_1 + \frac{S}{\gamma^*})}{\lambda(\tilde{S})} \tilde{S}$, hence $\tilde{S} \succeq \frac{\lambda(\tilde{S})}{\bar{\lambda}(P_1 + \frac{S}{\gamma^*})} (P_{\gamma} + \frac{S}{\gamma}) = a_Y (P_{\gamma} + \frac{S}{\gamma})$.

We now evaluate, for any $x \in \mathbb{R}^{n_x}$, $\mathbb{E}[Y_{\gamma}(x^+)] = \mathbb{E}[(x^+)^{\top} P_{\gamma} x^+] + \mathbb{E}[(x^+)^{\top} \frac{S}{\gamma} x^+]$. On the one hand, from (10), $\gamma \mathbb{E}[V_{\gamma}^*(x^+)] = -\ell(x, -K_{\gamma}^*x) + (1 - \gamma)V_{\gamma}^*(x) + \gamma V_{\gamma}^*(x)$, and by recalling $V_{\gamma}^*(x) = x^{\top} P_{\gamma} x + c_{\gamma}$ with $c := \frac{\gamma}{1-\gamma} \mathbb{E}[a^{\top} L^{\top} P_{\gamma} L a]$

in view of (7), $\gamma \mathbb{E}[(\mathbf{x}^+)^\top P_\gamma \mathbf{x}^+] + \gamma c_\gamma = -\ell(x, -K_\gamma^* x) + (1 - \gamma)x^\top P_\gamma x + (1 - \gamma)c_\gamma + \gamma x^\top P_\gamma x + \gamma c_\gamma$. Therefore,

$$\mathbb{E}[\mathbf{x}^{+\top} P_\gamma \mathbf{x}^+] = \frac{1}{\gamma} \left(-\ell(x, -K_\gamma^* x) + (1 - \gamma)x^\top P_\gamma x + \gamma \mathbb{E}[\mathbf{a}^\top L^\top P_\gamma L \mathbf{a}] \right) + x^\top P_\gamma x. \quad (20)$$

On the other hand, from SA3, we have

$$\mathbb{E}[\mathbf{x}^{+\top} S \mathbf{x}^+] = \mathbb{E}[\mathcal{A}_\gamma^*(\mathbf{m})^\top S \mathcal{A}_\gamma^*(\mathbf{m})] + \mathbb{E}[\mathbf{a}^\top L^\top S L \mathbf{a}] + \mathbb{E}[2\mathcal{A}_\gamma^*(\mathbf{m})^\top S L \mathbf{a}], \quad (21)$$

where \mathcal{A}_γ^* comes from (9). Note that $\mathbb{E}[2\mathcal{A}_\gamma^*(\mathbf{m})^\top S L \mathbf{a}] = 2\mathbb{E}[\mathcal{A}_\gamma^*(\mathbf{m})^\top S L \mathbb{E}[\mathbf{a}]] = 0$ in view of items (i) and (ii) of SA1.

Again in view of (20), (21), we derive $\mathbb{E}[Y_\gamma(\mathbf{x}^+)] \leq \frac{1}{\gamma} \left(-\ell(x, -K_\gamma^* x) + (1 - \gamma)x^\top P_\gamma x + \gamma \mathbb{E}[\mathbf{a}^\top L^\top P_\gamma L \mathbf{a}] \right) + x^\top P_\gamma x + \frac{1}{\gamma} \left(\ell(x, -K_\gamma^* x) + x^\top S x - x^\top \tilde{S} x + \mathbb{E}[\mathbf{a}^\top L^\top S L \mathbf{a}] \right) = x^\top P_\gamma x + x^\top \frac{S}{\gamma} x + \frac{1}{\gamma} \left(-x^\top \tilde{S} x + (1 - \gamma)x^\top P_\gamma x + \gamma \mathbb{E}[\mathbf{a}^\top L^\top (\gamma P_\gamma + S) L \mathbf{a}] \right)$, and thus

$$\begin{aligned} \mathbb{E}[Y_\gamma(\mathbf{x}^+)] &= Y_\gamma(x) + \frac{1}{\gamma} \left(-x^\top \tilde{S} x + (1 - \gamma)x^\top P_\gamma x \right. \\ &\quad \left. + \gamma \mathbb{E}[\mathbf{a}^\top L^\top (\gamma P_\gamma + S) L \mathbf{a}] \right). \end{aligned} \quad (22)$$

By Proposition 1 and as $(1 - \gamma^*)P_1 \preceq \tilde{S}$, we obtain $(1 - \gamma)P_\gamma \preceq \frac{1 - \gamma}{1 - \gamma^*} \tilde{S}$. Hence $(1 - \gamma)P_\gamma - \tilde{S} \preceq -\frac{\gamma - \gamma^*}{1 - \gamma^*} \tilde{S}$, and, in view of (22),

$$\begin{aligned} \mathbb{E}[Y_\gamma(\mathbf{x}^+)] &\leq Y_\gamma(x) - \frac{\gamma - \gamma^*}{\gamma(1 - \gamma^*)} x^\top \tilde{S} x \\ &\quad + \mathbb{E} \left[\mathbf{a}^\top L^\top \left(P_1 + \frac{S}{\gamma} \right) L \mathbf{a} \right]. \end{aligned} \quad (23)$$

Given the above and a_Y such that $a_Y \left(P_\gamma + \frac{S}{\gamma} \right) \preceq \tilde{S}$ holds,

$$\mathbb{E}[Y_\gamma(\mathbf{x}^+)] \leq \alpha_\gamma Y_\gamma(x) + \mathbb{E}[\mathbf{a}^\top L^\top (P_\gamma + \frac{S}{\gamma}) L \mathbf{a}] \quad (24)$$

where $\alpha_\gamma := 1 - \frac{\gamma - \gamma^*}{\gamma(1 - \gamma^*)} a_Y$. Note that α_γ necessarily is between 0 and 1, since $\frac{\gamma - \gamma^*}{\gamma(1 - \gamma^*)} a_Y < 1$ for all $\gamma \in (\gamma^*, 1)$.

We are ready to prove the statements of Section V.

C. Proof of Theorem 1

By the law of iterated expectations, we obtain from (24) that, for $i \in \mathbb{Z}_{>0}$,

$$\begin{aligned} \mathbb{E}[Y_\gamma(\phi_i^*)] &\leq \alpha_\gamma \mathbb{E}[Y_\gamma(\phi_{i-1}^*)] + \mathbb{E} \left[\mathbf{a}^\top L^\top \left(P_\gamma + \frac{S}{\gamma} \right) L \mathbf{a} \right] \\ &\leq \alpha_\gamma^i Y_\gamma(x) + \sum_{k=0}^{i-1} \alpha_\gamma^k \mathbb{E} \left[\mathbf{a}^\top L^\top \left(P_\gamma + \frac{S}{\gamma} \right) L \mathbf{a} \right]. \end{aligned} \quad (25)$$

We are allowed to do so as $Y_\gamma(\phi_i^*)$ is \mathcal{F}_i -measurable for each $i \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ where \mathcal{F}_i is the natural filtration associated to the sequence of random variables \mathbf{v}_∞ , hence the law of iterated expectations applies. Note that $Y_\gamma(\phi_i^*)$ is indeed \mathcal{F}_i -measurable as Y_γ is continuous by construction and ϕ_i^* is \mathcal{F}_i measurable for the same reasons. Therefore, in view of (19) and (25), we conclude the first part of Theorem 1 holds. Indeed, we obtain $g_\gamma \in [1, \bar{\lambda}(P_1 + \frac{S}{\gamma^*})/\Delta(\tilde{S})]$, $\lambda_\gamma \in [0, \alpha_\gamma]$ and $G_\gamma \in [0, \mathbb{E}[\mathbf{a}^\top L^\top (P_\gamma + \frac{S}{\gamma}) L \mathbf{a}]/(1 - \alpha_\gamma)]$, as $\frac{1}{1 - \alpha_\gamma} \geq \sum_{i=0}^{k-1} \alpha_\gamma^i$ for any $i \in \mathbb{Z}_{\geq 0}$. Moreover, for the second part of Theorem 1, it simply follows from $\mathbb{E}[\mathbf{a}^\top L^\top (P_\gamma + \frac{S}{\gamma}) L \mathbf{a}] \leq \bar{\lambda}(P_1 + \frac{S}{\gamma^*}) \mathbb{E}[\mathbf{a}^\top L^\top L \mathbf{a}] = 0$, hence $G_\gamma = 0$.

D. Proof of Theorem 2

Given \mathcal{O}_γ as in Theorem 2, it follows in view of (23) that

$$\mathbb{E}[Y_\gamma(\mathbf{x}^+)] \leq Y_\gamma(x) - \varrho_\gamma(x) \quad (26)$$

for $\varrho_\gamma(x) := \frac{\gamma - \gamma^*}{\gamma(1 - \gamma^*)} x^\top \tilde{S} x - \mathbb{E} \left[\mathbf{a}^\top L^\top \left(P_1 + \frac{S}{\gamma} \right) L \mathbf{a} \right]$, with $\varrho_\gamma(x)$ strictly positive for $x \in \mathbb{R}^{n_x} \setminus \mathcal{O}_\gamma$. Hence, Y_γ is a sufficient recurrence-Lyapunov function relative to \mathcal{O}_γ for (4), see [26, Eq. (4)]. The proof of Theorem 4 is concluded by invoking [26, Theorem 1].

E. Proof of Corollary 1

From (23), it is clear that, when $\mathbb{E}[\mathbf{a}^\top L^\top L \mathbf{a}] = 0$, $\varrho_\gamma(x)$ given in (26) is strictly positive for any $x \neq 0$. Hence, for any open \mathcal{O}_γ containing the origin, Y_γ is a sufficient recurrence-Lyapunov function relative to \mathcal{O}_γ for (4) and recurrence is established by invoking [26, Theorem 1].

APPENDIX II

NUMERICAL VALUES FOR SECTION VII

$$A := \begin{bmatrix} 1 & 0.0283 & -0.0047 & -0.0002 \\ 0 & 0.0244 & -0.0670 & -0.0047 \\ 0 & 0.2453 & 1.1821 & 0.1061 \\ 0 & 3.5334 & 3.6549 & 1.1821 \end{bmatrix}$$

$$\tilde{B} := [0.0114 \quad 0.1553 \quad -0.0391 \quad -0.5625]^\top$$

$$P_\gamma = \begin{bmatrix} 75.32 & 29.06 & 54.86 & 9.53 \\ 29.06 & 261.99 & 540.28 & 92.47 \\ 54.86 & 540.28 & 1140.03 & 191.95 \\ 9.53 & 92.47 & 191.95 & 33.78 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 87.66 & 33.90 & 63.98 & 11.11 \\ 33.90 & 265.06 & 546.29 & 93.51 \\ 63.98 & 546.29 & 1151.98 & 193.99 \\ 11.11 & 93.51 & 193.99 & 34.13 \end{bmatrix}$$

$$S = \begin{bmatrix} 39.72 & -19.09 & 41.02 & -7.33 \\ -19.09 & 215.28 & -457.60 & 78.65 \\ 41.02 & -457.60 & 1261.09 & -215.57 \\ -7.33 & 78.65 & -215.57 & 36.85 \end{bmatrix}$$

$$\tilde{S} = \begin{bmatrix} 0.62 & 0.19 & 0.65 & 0.04 \\ 0.19 & 2.37 & 1.64 & 1.05 \\ 0.65 & 1.64 & 17.86 & -0.38 \\ 0.04 & 1.05 & -0.38 & 0.55 \end{bmatrix}$$

$$S_\gamma = \begin{bmatrix} 6574.11 & 2494.90 & 4633.39 & 832.60 \\ 2494.90 & 4808.83 & 5857.55 & 1709.10 \\ 4633.39 & 5857.55 & 14794.83 & 2075.15 \\ 832.60 & 1709.10 & 2075.15 & 659.54 \end{bmatrix}$$

$$\tilde{S}_\gamma = \begin{bmatrix} 98.70 & 38.62 & 68.35 & 21.78 \\ 38.62 & 696.16 & 593.61 & 292.47 \\ 68.35 & 593.61 & 2133.07 & 220.15 \\ 21.78 & 292.47 & 220.15 & 144.61 \end{bmatrix}$$

APPENDIX III

DETECTABILITY FOR DISCRETE MULTIPLICATIVE NOISE

Given system (1) where \mathbf{m} is a random variable with countable support, i.e., $A + M(i) =: A_i$, $B + N(i) =: B_i$ and $\mathbb{P}(m = i) = p_i$ for all $i \in \{0, \dots, m_{\max}\}$, for some $m_{\max} \geq 0$ and $p_i \geq 0$ such that $\sum_{k=0}^{m_{\max}} p_k = 1$, the following lemma holds.

Lemma 1: Consider system (1) with discrete multiplicative noise as above. If there exist $S \succeq 0$ and $\tilde{S} \succ 0$ such that

$$\sum_{k=0}^{m_{\max}} p_k \begin{bmatrix} A_k^\top S A_k & A_k^\top S B_k \\ B_k^\top S A_k & B_k^\top S B_k \end{bmatrix} \preceq \begin{bmatrix} S - \tilde{S} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D \end{bmatrix} \quad (27)$$

then SA3 holds. \square

Proof: For any $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$, $\mathbb{E}[(\mathbf{x}^+)^\top S \mathbf{x}^+]$, where $\mathbf{x}^+ = (A + M(\mathbf{m}))x + (B + N(\mathbf{m}))u$, is equal to $[x^\top u^\top] Z [x^\top u^\top]^\top$ where $Z := \mathbb{E} \left[\begin{bmatrix} (A + M(\mathbf{m}))^\top \\ (B + N(\mathbf{m}))^\top \end{bmatrix} S \begin{bmatrix} (A + M(\mathbf{m})) & (B + N(\mathbf{m})) \end{bmatrix} \right] = \sum_{k=0}^{m_{\max}} p_k \begin{bmatrix} A_k^\top \\ B_k^\top \end{bmatrix} S \begin{bmatrix} A_k & B_k \end{bmatrix} = \sum_{k=0}^{m_{\max}} p_k \begin{bmatrix} A_k^\top S A_k & A_k^\top S B_k \\ B_k^\top S A_k & B_k^\top S B_k \end{bmatrix}$. In view of (27), $[x^\top u^\top] Z [x^\top u^\top]^\top \leq x^\top (S - \tilde{S}) x + \ell(x, u)$, therefore SA3 holds. \blacksquare