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Low-power high-gain homogeneous feedback control for chain of integrators with disturbances

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Abstract: We present a new high-gain feedback control design ensuring finite-time input-to-state stabilization of chains of integrators of dimension n subject to additive disturbances. The proposed high-gain homogeneous controller involves only control gains of power limited to 2 regardless of the dimension of the chains of integrators in contrast to the classical high-gain controllers involving gains with powers up to $n + 1$ which grows with the growth of the chain of integrators. The stability analysis of the closed-loop system is achieved by means of classical Lyapunov-based tools and homogeneity-based concepts. This result can be seen as the dual design of low-power high-gain observers recently extended to the homogeneous context.

Keywords: High-gain control, homogeneous feedback, finite-time stability, ISS, chain of integrators, disturbances.

1. INTRODUCTION

In this article, we study the stabilization problem for class of chain of integrators with external disturbances, i.e.:

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + \omega_i(t), & \forall i \in \{0, \dots, n-2\}, \\ \dot{x}_{n-1}(t) &= u(t) + \omega_{n-1}(t), \end{aligned} \quad (1)$$

where $t \geq 0$ is the time variable, $x(t) = [x_0(t), \dots, x_{n-1}(t)]^\top \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is a nonlinear control input, and $\omega(t) = [\omega_0(t), \dots, \omega_{n-1}(t)]^\top \in \mathbb{R}^n$ is a vector of extra inputs.

Among the different techniques adopted for robust stabilization and observation of nonlinear systems in chain of integrators form, high-gain methods have been shown to be advantageous in many control/estimation scenarios in the literature due to their simplicity and good performance in noise-free settings (see, e.g., [Isidori, 1995, Teel and Praly, 1995, Lin, 2009, Praly and Khalil, 2014] and references therein). In fact, by increasing the high-gain parameter, one can increase the convergence rate of the controlled/observer system's states towards the equilibrium. However, increasing the high-gain parameter means also increasing the transient state peaking phenomenon (referred to in short as the peaking phenomenon) especially since the high-gain parameter is usually powered up to the system's dimension. Due to this phenomenon, the high-gain observers/controllers are less desirable in numerical implementation whenever the dimension of the plant is large. For this reason, a different high-gain design methodology, referred to as “low-power high-gain”, has been proposed in [Astolfi and Marconi, 2015] for

nonlinear systems in the chain of integrator form. The proposed “low-power high-gain” observer design does not only preserve all the main positive features of classical high-gain observers, but also elevates the issues related to the transient state peaking by having the power of the high-gain parameter raised just up to the order 2, regardless of the dimension of the observed system, with the (mild) drawback of doubling the dimension of the observer. The peculiarity of such a paradigm consists in a construction of interconnected 2nd-order high-gain observers. Notably, a similar step-by-step procedure have been also proposed in the context of sliding-mode observers in [Barbot and Floquet, 2010]. Recently, in order to cope with non-Lipschitz nonlinearities and guarantee finite-time convergence, two direct extensions have been proposed: [Andrieu et al., 2021] directly extending the so-called super-twisting methodology [Barbot and Floquet, 2010, Moreno and Osorio, 2012] to chain of integrator of any order, and [Moreno, 2021] combining the low-power high-gain design techniques with homogeneity techniques in the framework of Levant's differentiator (see, e.g., [Andrieu et al., 2008, Levant, 2005]).

Notably, it has been remarked in [Astolfi et al., 2017] that the same procedure followed in the context of high-gain observers [Astolfi and Marconi, 2015] can be also employed in the context of high-gain feedback stabilization. As a consequence, one can extend the classical high-gain state-feedback methodology [Isidori, 1995, Teel and Praly, 1995] developed for the stabilization of chain of integrators of order n , in order to build a high-gain dynamic-feedback controller containing a high-gain parameter which is raised up to the power of 2 regardless of the dimension of n . We highlight that standard high-gain feedback has been

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developed also in the homogeneous context for finite-time stabilization, see, e.g. [Levant, 2003, 2005, Andrieu et al., 2008].

The main objective of this article is to combine the series of works [Astolfi and Marconi, 2015, Astolfi et al., 2017, Moreno, 2021, Levant, 2003] in order to develop a new low-power high-gain methodology for finite-time stabilization of the disturbance-free system (1) and input-to-state stabilization in the presence of external disturbances. In contrast to the existing results, the proposed (dynamical) control feedback features high-gain parameters of quadratic form at most and ensures that the closed-loop system is homogeneous of negative degree.

The rest of the article is organized as follows. In Section 2, we briefly revisit the concept of weighted homogeneity and its role in classical homogeneous high-gain feedback control design for (1). Then, we present our novel low-power version of the classical high-gain homogeneous feedback controller along with our main stability-related result. In section 3, we provide a detailed proof of our main results. In Section 4, we present a numerical example to illustrate these results. Finally, conclusions and perspectives are given in Section 5.

Notation. The function $x \mapsto [x]^p = |x|^p \text{sign}(x)$ is the signed power $p \geq 0$ of $x \in \mathbb{R}$.

2. MAIN RESULT

2.1 Highlights on Homogeneous High-Gain State Feedback

Before we delve into our low-power high-gain strategy, let us recall first the well-known concept of weighted homogeneity and its application to classical high-gain state feedback stabilization of systems of the form (1). For this, define a vector of positive weights $r = [r_0, \dots, r_{n-1}]$ ($r_i > 0, \forall i \in \{0, \dots, n-1\}$), and denote by the dilation matrix the following matrix:

$$\Lambda_r(\lambda) = \text{diag}[\lambda^{r_0}, \dots, \lambda^{r_{n-1}}], \quad \forall \lambda > 0.$$

and by the homogeneous norm $\|\cdot\|_r$ the following map:

$$z \mapsto \|z\|_r := \left(\sum_{i=0}^n |z_i|^{\rho/r_i} \right)^{1/\rho}, \quad \rho \geq \max_{1 \leq j \leq n} r_j.$$

Using these two notations, a function $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be homogeneous of degree $d \in \mathbb{R}$ iff $\phi(\Lambda_r(\lambda)x) = \lambda^d \phi(x)$ is satisfied for every $\lambda > 0$. Moreover, a vector field $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is homogeneous of degree d iff $f(\Lambda_r(\lambda)x) = \lambda^d \Lambda_r(\lambda)f(x)$ is satisfied for every $\lambda > 0$.

Notice that $\|\cdot\|_r$ is homogeneous of degree $d = 1$ and satisfies, $\forall x \in \mathbb{R}^n$, the following inequality:

$$\left(\min_{\{z: \|z\|=1\}} \|z\|_r \right) \|x\| \leq \|x\|_r \leq \left(\max_{\{z: \|z\|=1\}} \|z\|_r \right) \|x\|. \quad (2)$$

For system (1) subject to a control law of the form:

$$u(t) := -\phi(x(t); \kappa) = -\kappa^{n+1} k_n \left[\sum_{i=0}^{n-1} k_i [\kappa^{-(i+1)} x_i(t)]^{\frac{\alpha_i}{r_i}} \right]^{\frac{r_n}{\alpha_i}}, \quad (3)$$

the closed loop system (equivalently its vector field), with $\omega = 0$, can be rendered homogeneous of any degree $\delta \in (-1, 0)$ with weights r_0, \dots, r_{n-1} given by:

$$r_i = 1 - (n-1-i)\delta, \quad \forall i \in \{0, \dots, n\}, \quad (4)$$

and with appropriately selected positive real gains $k_i > 0, \forall i \in \{0, \dots, n\}$. The parameter $\alpha \geq r_0$ is a tuning parameter used to select a different member of the family of controllers, and the gain $\kappa > 0$ is a scalar high-gain used to accelerate the convergence of the closed-loop system.

Consequently, by adopting the arguments in Huang et al. [2005], it can be proved that:

Proposition 1. For any homogeneity degree $\delta \in (-1, 0)$, there exist gains $k_0 > 0, \dots, k_n > 0$, such that for every $\kappa > 0$ the origin of (1)-(3) is globally finite-time stable (FTS) when $\omega = 0$ and input-to-state stable (ISS) w.r.t to ω when $\omega \neq 0$.

Remark 1. Note that the (linear) change of coordinates $z_i = \kappa^{-(i+1)} x_i, \forall i \in \{0, \dots, n-1\}$, transforms (1) with controller (3) into:

$$\dot{z}_i(t) = \kappa z_{i+1}(t) + \frac{1}{\kappa^{i+1}} \omega_i(t), \quad \forall i \in \{0, \dots, n-2\},$$

$$\dot{z}_{n-1}(t) = -\kappa \phi(z(t); \kappa = 1) + \frac{1}{\kappa^n} \omega_{n-1}(t).$$

From this, it is clear that the scaled gains lead to an acceleration (for $\kappa \gg 1$) of the convergence, assuming that the plant (1) with the control law (3) (with $\kappa = 1$) is asymptotically stable (AS).

2.2 Homogeneous High-Gain State Feedback with Limited Gain Power

Similarly to [Astolfi et al., 2017, Moreno, 2021], it is possible to stabilize system (1) by means of a homogeneous dynamic control law of dimension $n-2$, in which the power of $\kappa_i, \forall i \in \{1, \dots, n-1\}$ is limited to 2, in contrast to the classical control law (3), that has a power $n+1$.

To this end, select any homogeneity degree $\delta \in (-1, 0)$ and let $\{\bar{k}_i, \underline{k}_i\}_{i \in \{0, \dots, n-1\}}$ be a family of sufficiently large positive gains. We define next the homogeneous dynamic controller as follows:

$$\begin{aligned} \eta_0(t) &= x_0(t), \\ \dot{\eta}_i(t) &= x_{i+1}(t) + v_i(t), \quad \forall i \in \{1, \dots, n-2\}, \\ \eta_{n-1}(t) &= 0, \\ u(t) &= -v_{n-1}(t), \end{aligned} \quad (5)$$

with inputs $v_i(t), \forall i \in \{1, \dots, n-1\}$, selected as:

$$\begin{aligned} v_i(t) &:= \phi_i(x(t); \kappa_i), \\ &= \kappa_i^2 \underline{k}_i \left[[\eta_{i-1}(t)]^{\frac{\alpha_i}{r_i}} + \bar{k}_i [\kappa_i^{-1} (x_i(t) - \eta_i(t))]^{\frac{\alpha_i}{r_i}} \right]^{\frac{r_{v_i}}{\alpha_i}}, \end{aligned} \quad (6)$$

where, $\forall i \in \{1, \dots, n-1\}$, the vector of homogeneity weights $[r_i, \bar{r}_i, r_{v_i}]$ and the gains κ_i are selected respectively as:

$$r_i = 1 - (n-i)\delta, \quad \bar{r}_i = r_i + \delta, \quad r_{v_i} = \bar{r}_i + \delta, \quad (7)$$

and

$$\kappa_i := \kappa_0 \bar{\kappa}_i > 0, \quad (8)$$

where κ_0 is any positive gain and $\bar{\kappa}_i$ is a family of sufficiently large positive gains. The parameters $\alpha_i \geq r_i, \forall i \in \{1, \dots, n-1\}$, are tuning parameters. Finally, we have the following result.

Theorem 1. Consider the feedback law (5)-(6) and let the parameters selected as in (7), (8). Then, system (1) in

closed loop with the homogeneous dynamic control law (5)-(6) is homogeneous of degree δ , the origin is Globally **FTS** for any $\kappa_0 > 0$ when $\omega = 0$ and **ISS** w.r.t to ω when $\omega \neq 0$. In particular, increasing the gain κ_0 increases the convergence rate.

Remark 2. Note that when $\delta = 0$ we recover the linear case, already considered in [Astolfi et al., 2017]. However, conditions of Theorem 1 are more restrictive for the linear case than those in [Astolfi et al., 2017] due to the technical proof which relies on homogeneous Lyapunov functions of order 2. Moreover, when $\delta = -1$, it is still possible to recover the same results of Theorem 1 when we have particularly $|\omega_{n-1}| \leq D$. However, since in this case the controller is discontinuous we need to adapt the proof rigorously using Filippov techniques for differential inclusions. For this short conference version, we prefer excluding this case.

3. PROOF OF THE MAIN RESULT

The proof of Theorem 1 is divided into three parts: First, we study the stability of the decoupled undisturbed homogeneous closed-loop system with high-gain coefficients set to 1. In this situation, the closed-loop system is composed of n decoupled homogeneous double chain of integrators that can be proven to be stable using a homogeneous Lyapunov function. Next, we include the effect of the high-gain coefficients and the coupling terms and prove that the closed-loop system keeps the stability and homogeneity properties. Finally, using the homogeneity properties of the closed-loop system, we prove that when external disturbances are considered, we recover input-to-state stabilization.

Let us start by rearranging the closed-loop system (1) with (5), we recover the following cascade system:

$$\begin{aligned} \dot{x}_0(t) &= x_1(t) + \omega_0(t), \\ \dot{x}_i(t) &= x_{i+1}(t) + \omega_i(t), \quad \forall i \in \{1, \dots, n-2\}, \\ \dot{\eta}_i(t) &= x_{i+1}(t) + v_i(t), \quad \forall i \in \{1, \dots, n-2\}, \\ \dot{x}_{n-1}(t) &= -v_{n-1}(t) + \omega_{n-1}(t), \end{aligned} \quad (9)$$

on which we use the following change of coordinates $[x, \eta] \mapsto \xi$:

$$\xi_i(t) = [\xi_{i,1}(t), \xi_{i,2}(t)]^T := [\eta_{i-1}(t), x_i(t) - \eta_i(t)]^T, \quad (10)$$

$\forall i \in \{1, \dots, n-1\}, \forall t \geq 0$. This results in the following system:

$$\begin{aligned} \dot{\xi}_{1,1}(t) &= \xi_{1,2}(t) + \xi_{2,1}(t) + \omega_0(t), \\ \dot{\xi}_{1,2}(t) &= -v_1(t) + \omega_1(t), \\ \dot{\xi}_{i,1}(t) &= \xi_{i,2}(t) + \xi_{i+1,1}(t) + v_{i-1}(t), \\ \dot{\xi}_{i,2}(t) &= -v_i(t) + \omega_i(t), \quad \forall i \in \{2, \dots, n-1\}, \end{aligned} \quad (11)$$

with $\xi_{n,1} = \eta_{n-1} = 0$ due to (5). Note that the control laws $v_i(t)$ given in (6) can be also expressed in the ξ_i -coordinates as follows:

$$\begin{aligned} v_i(t) &:= \phi_i(\xi_i(t); \kappa_i), \\ &= \kappa_i^2 \underline{k}_i \left[|\xi_{i,1}(t)|^{\frac{\alpha_i}{r_i}} + \bar{k}_i [\kappa_i^{-1} \xi_{i,2}(t)]^{\frac{\alpha_i}{\bar{r}_i}} \right]^{\frac{r_i v_i}{\alpha_i}}. \end{aligned} \quad (12)$$

3.1 Stability analysis of decoupled undisturbed subsystems with $\kappa_i = 1$:

Let us consider the i -th subsystem of (11) with $\kappa_i = 1$ and in the absence of the vector input ω and the coupling

terms $\xi_{i+1,1}$ and v_{i-1} , i.e.:

$$\begin{aligned} \dot{\xi}_{i,1}(t) &= \xi_{i,2}(t), \\ \dot{\xi}_{i,2}(t) &= -v_i(t). \end{aligned} \quad (13)$$

Let us consider the following Lyapunov function candidate:

$$\begin{aligned} V_i(\xi_i) &= \left(\gamma_i \frac{r_i}{m} + \frac{m - \bar{r}_i}{m} \right) |\xi_{i,1}|^{\frac{m}{r_i}} + \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} |\xi_{i,1}|^{\frac{m - \bar{r}_i}{r_i}} |\xi_{i,2}| \\ &\quad + \frac{\bar{r}_i}{m} \hat{k}_i^{\frac{m}{m - \bar{r}_i}} |\xi_{i,2}|^{\frac{m}{\bar{r}_i}}, \end{aligned} \quad (14)$$

where $\{\hat{k}_i\}_{i \in \{1, \dots, n-1\}}$ is a family of sufficiently large positive gains, $\gamma_i > 0$ is a positive tuning parameter and m is the degree of homogeneity of V_i taken to satisfy $m \geq r_i + \bar{r}_i$ in order to guarantee the differentiability of V_i . Notice that V_i is positive-definite. In fact, using Young's inequality given in Lemma 1 on the second term of V_i with $p = \frac{m}{m - \bar{r}_i}$, leads to:

$$\begin{aligned} -|\xi_{i,1}|^{\frac{m - \bar{r}_i}{r_i}} \left(\hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} |\xi_{i,2}| \right) &\leq |\xi_{i,1}|^{\frac{m - \bar{r}_i}{r_i}} \left| \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} |\xi_{i,2}| \right|, \\ &\leq \left(1 - \frac{\bar{r}_i}{m} \right) |\xi_{i,1}|^{\frac{m}{r_i}} \\ &\quad + \frac{\bar{r}_i}{m} \hat{k}_i^{\frac{m}{m - \bar{r}_i}} |\xi_{i,2}|^{\frac{m}{\bar{r}_i}}, \\ &\leq \left(\gamma_i \frac{r_i}{m} + \left(\frac{m - \bar{r}_i}{m} \right) \right) |\xi_{i,1}|^{\frac{m}{r_i}} \\ &\quad + \frac{\bar{r}_i}{m} \hat{k}_i^{\frac{m}{m - \bar{r}_i}} |\xi_{i,2}|^{\frac{m}{\bar{r}_i}}, \end{aligned} \quad (15)$$

This shows that V_i stays always nonnegative. Moreover, by computing $\frac{\partial V_i(\xi_i)}{\partial \xi_{i,1}}$ and $\frac{\partial V_i(\xi_i)}{\partial \xi_{i,2}}$ the partial derivatives of V_i with respect to $\xi_{i,1}$ and $\xi_{i,2}$ respectively:

$$\begin{aligned} \frac{\partial V_i(\xi_i)}{\partial \xi_{i,1}} &= \left(\gamma_i + \left(\frac{m - \bar{r}_i}{r_i} \right) \right) |\xi_{i,1}|^{\frac{m - r_i}{r_i}} \\ &\quad + \left(\frac{m - \bar{r}_i}{r_i} \right) \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} |\xi_{i,1}|^{\frac{m - r_i - \bar{r}_i}{r_i}} |\xi_{i,2}|, \end{aligned} \quad (16)$$

and

$$\frac{\partial V_i(\xi_i)}{\partial \xi_{i,2}} = \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} \left(|\xi_{i,1}|^{\frac{m - \bar{r}_i}{r_i}} + \hat{k}_i |\xi_{i,2}|^{\frac{m - \bar{r}_i}{\bar{r}_i}} \right), \quad (17)$$

it is easy to check that $[\bar{\xi}_{i,1}, \bar{\xi}_{i,2}] = [0, 0]$ is the only critical point of V_i . Moreover, since V_i is always nonnegative and since $V_i(0) = 0$, the origin is the unique point where V_i achieves its minimal value that is equal to 0. Consequently, V_i is positive-definite. Furthermore, computing the time derivative of V_i along the trajectories of (13) and replacing by the expression of $\phi_i(\xi; \kappa_i)$, give us:

$$\begin{aligned} \dot{V}_i(\xi_i) &= \left(\gamma_i + \left(\frac{m - \bar{r}_i}{r_i} \right) \right) |\xi_{i,1}|^{\frac{m - r_i}{r_i}} \xi_{i,2} \\ &\quad + \left(\frac{m - \bar{r}_i}{r_i} \right) \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} |\xi_{i,1}|^{\frac{m - r_i - \bar{r}_i}{r_i}} |\xi_{i,2}|^2 \\ &\quad - \underline{k}_i \hat{k}_i^{\frac{\bar{r}_i}{m - \bar{r}_i}} \left(|\xi_{i,1}|^{\frac{m - \bar{r}_i}{r_i}} + \hat{k}_i |\xi_{i,2}|^{\frac{m - \bar{r}_i}{\bar{r}_i}} \right) \\ &\quad \times \left[|\xi_{i,1}|^{\frac{\alpha_i}{r_i}} + \bar{k}_i |\xi_{i,2}|^{\frac{\alpha_i}{\bar{r}_i}} \right]^{\frac{r_i v_i}{\alpha_i}}. \end{aligned} \quad (18)$$

By taking $\hat{k}_i = \bar{k}_i^{\frac{m - \bar{r}_i}{\alpha_i}}$ in (18), we can ensure that the last term of (18) is negative for any $m \geq r_i + \bar{r}_i$. To prove this, it suffices to study the following cases:

- (1) both of $\xi_{i,1}$ and $\xi_{i,2}$ are either positive or negative,
- (2) they have different signs, and $|\xi_{i,1}|^{\frac{1}{r_i}} \geq \bar{k}_i^{\frac{1}{\alpha_i}} |\xi_{i,2}|^{\frac{1}{\bar{r}_i}}$,

(3) they have different signs, and $|\xi_{i,1}|^{\frac{1}{r_i}} \leq \bar{k}_i^{\frac{1}{\alpha_i}} |\xi_{i,2}|^{\frac{1}{\bar{r}_i}}$.

Now seeing that the last term of (18) vanishes on the set:

$$\mathcal{S}_i = \left\{ \left[\xi_{i,1} \right]^{\frac{m-\bar{r}_i}{r_i}} + \hat{k}_i \left[\xi_{i,2} \right]^{\frac{m-\bar{r}_i}{\bar{r}_i}} = 0 \right\}, \quad (19)$$

and using Lemma 2, we can render \dot{V}_i negative-definite. More precisely, evaluating $\dot{V}_i(\xi_i)$ on the set \mathcal{S}_i as follows:

$$\dot{V}_i(\xi_i) \Big|_{\mathcal{S}_i} = -\hat{k}_i^{\frac{-\bar{r}_i}{m-\bar{r}_i}} \gamma_i |\xi_{i,1}|^{\frac{m-r_i+\bar{r}_i}{r_i}} < 0, \quad (20)$$

can be used to conclude that

$$\dot{V}_i(\xi_i) = -W_i(\xi_i) \quad (21)$$

can be rendered negative-definite by selecting \underline{k}_i sufficiently large.

3.2 Scaling of the system and stability analysis of the undisturbed system (11):

Using the following change of variables:

$$z_i := [z_{i,1}, z_{i,2}]^T = [\xi_{i,1}, \kappa_i^{-1} \xi_{i,2}]^T, \quad (22)$$

on system (11), we obtain:

$$\begin{aligned} \dot{z}_{1,1}(t) &= \kappa_1 z_{1,2}(t) + z_{2,1}(t) + \omega_0(t), \\ \dot{z}_{1,2}(t) &= -\kappa_1 \tilde{v}_1(z_1) + \frac{1}{\kappa_1} \omega_1(t), \\ \dot{z}_{i,1}(t) &= \kappa_i z_{i,2}(t) + z_{i+1,1}(t) + \kappa_{i-1}^2 \tilde{v}_{i-1}(z_{i-1}), \\ \dot{z}_{i,2}(t) &= -\kappa_i \tilde{v}_i(z_i) + \frac{1}{\kappa_i} \omega_i(t), \quad \forall i \in \{2, \dots, n-1\}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} \tilde{v}_i(z_i) &:= \phi_i(z_i(t); \kappa_i = 1), \\ &= \underline{k}_i \left[|z_{i,1}(t)|^{\frac{\alpha_i}{r_i}} + \bar{k}_i |z_{i,2}(t)|^{\frac{\alpha_i}{\bar{r}_i}} \right]^{\frac{r_i v_i}{\alpha_i}}, \end{aligned} \quad (24)$$

where $\bar{k}_i > 0$ can be chosen arbitrarily and $\underline{k}_i > 0$ is largely chosen to ensure the stability of the decoupled closed-loop system when $\omega = 0$.

For the whole system (23), consider the following Lyapunov Function candidate:

$$V(z) = \sum_{i=1}^{n-1} V_i(z_i). \quad (25)$$

where V_i is given in (14). Its derivative along trajectories of (23), when $\omega = 0$ is:

$$\begin{aligned} \dot{V}(z) &= -\sum_{i=1}^{n-1} \kappa_i W_i(z_i) + \sum_{i=1}^{n-2} \frac{\partial V_i(z_i)}{\partial z_{i,1}} z_{i+1,1} \\ &\quad + \sum_{i=2}^{n-1} \kappa_{i-1}^2 \frac{\partial V_i(z_i)}{\partial z_{i,1}} \tilde{v}_{i-1}(z_{i-1}), \end{aligned} \quad (26)$$

Notice that \dot{V} can be rearranged as follows:

$$\begin{aligned} \dot{V}(z) &:= F_1(z_1, \dots, z_{n-1}), \\ &= -\kappa_1 W_1(z_1) + \frac{\partial V_1(z_1)}{\partial z_{1,1}} z_{2,1} \\ &\quad + \sum_{i=2}^{n-2} \left\{ -\kappa_i W_i(z_i) - \frac{\partial V_i(z_i)}{\partial z_{i,1}} z_{i+1,1} \right. \\ &\quad \left. - \kappa_{i-1}^2 \frac{\partial V_i(z_i)}{\partial z_{i,1}} \tilde{v}_{i-1}(z_{i-1}) \right\} \\ &\quad - \kappa_{n-1} W_{n-1}(z_{n-1}) \\ &\quad + \kappa_{n-2}^2 \frac{\partial V_{n-1}(z_{n-1})}{\partial z_{n-1,1}} \tilde{v}_{n-2}(z_{n-2}). \end{aligned} \quad (27)$$

Seeing that κ_{n-1} only appears as a factor of $W_{n-1}(z_{n-1})$ and that the term $-\kappa_{n-1} W_{n-1}(z_{n-1})$ is negative and vanishes when $z_{n-1} = 0$, we evaluate $F_1(z_1, \dots, z_{n-1})$ at $z_{n-1} = 0$ and we obtain:

$$\begin{aligned} F_2(z_1, \dots, z_{n-2}) &:= F_1(z_1, \dots, z_{n-1}) \Big|_{\{z_{n-1}=0\}}, \\ &= -\kappa_1 W_1(z_1) + \frac{\partial V_1(z_1)}{\partial z_{1,1}} z_{2,1} \\ &\quad + \sum_{i=2}^{n-3} \left\{ -\kappa_i W_i(z_i) - \frac{\partial V_i(z_i)}{\partial z_{i,1}} z_{i+1,1} \right. \\ &\quad \left. - \kappa_{i-1}^2 \frac{\partial V_i(z_i)}{\partial z_{i,1}} \tilde{v}_{i-1}(z_{i-1}) \right\} \\ &\quad - \kappa_{n-2} W_{n-2}(z_{n-2}) \\ &\quad + \kappa_{n-3}^2 \frac{\partial V_{n-2}(z_{n-2})}{\partial z_{n-2,1}} \tilde{v}_{n-3}(z_{n-3}). \end{aligned} \quad (28)$$

Notice that $F_2(z_1, \dots, z_{n-2})$ has a similar structure to $F_1(z_1, \dots, z_{n-1})$ but involves only the variables z_1, z_2, \dots , and z_{n-2} .

Now, seeing that κ_{n-2} only appears as a factor of the term $W_{n-2}(z_{n-2})$ and that the term $-\kappa_{n-2} W_{n-2}(z_{n-2})$ is negative and vanishes when $z_{n-2} = 0$, we evaluate $F_2(z_1, \dots, z_{n-2})$ at $z_{n-2} = 0$ to obtain $F_3(z_1, \dots, z_{n-3})$ which shares a similar structure to $F_2(z_1, \dots, z_{n-2})$ but only involves the variables z_1, z_2, \dots , and z_{n-3} . This process needs to be repeated $n-2$ times. In particular, if it is repeated $n-3$ times, we get

$$\begin{aligned} F_{n-2}(z_1, z_2) &:= F_{n-3}(z_1, z_2, z_3) \Big|_{\{z_3=0\}}, \\ &= -\kappa_1 W_1(z_1) + \frac{\partial V_1(z_1)}{\partial z_{1,1}} z_{2,1} \\ &\quad - \kappa_2 W_2(z_2) + \kappa_1^2 \frac{\partial V_2(z_2)}{\partial z_{2,1}} \tilde{v}_1(z_1). \end{aligned} \quad (29)$$

Here κ_2 only appears as a factor of $W_2(z_2)$ and the term $-\kappa_2 W_2(z_2)$ is negative and vanishes when $z_2 = 0$. By evaluating $F_{n-2}(z_1, z_2)$ at $z_2 = 0$, we get

$$\begin{aligned} F_{n-1}(z_1) &:= F_{n-2}(z_1, z_2) \Big|_{\{z_2=0\}}, \\ &= -\kappa_1 W_1(z_1) < 0, \end{aligned} \quad (30)$$

which is negative for any $\kappa_1 > 0$. Then, by increasing $\bar{\kappa}_2$ given in $\kappa_2 := \kappa_0 \bar{\kappa}_2 > 0$ sufficiently large we can render $F_{n-2}(z_1, z_2) < 0$ (from Lemma 2). Similarly, by increasing $\bar{\kappa}_3$ given in $\kappa_3 := \kappa_0 \bar{\kappa}_3 > 0$ sufficiently large we can render $F_{n-3}(z_1, z_2, z_3) < 0$. Repeating this process, we can render

$$\dot{V}(z) := -\bar{W}(z) < 0, \quad (31)$$

by selecting successively $\bar{\kappa}_i > 0, \forall i \in \{1, \dots, n-1\}$, sufficiently large. Moreover, by using Lemma 3, we can prove the existence of $c > 0$ such that:

$$\dot{V}(z) \leq -cV(z)^{\frac{m+\delta}{m}}, \quad (32)$$

from which we deduce that the origin of (23) (equivalently of (11)) is globally **FTS** for any $\kappa_0 > 0$ and the time of convergence T satisfies:

$$T(z_0) \leq \frac{-m}{c\delta} V(z_0)^{\frac{-\delta}{m}}, \quad \forall z_0 \in \mathbb{R}^n. \quad (33)$$

3.3 Stability analysis of coupled disturbed subsystems:

Considering the effect of ω , it is easy to see that for the whole system (23), the time derivative of V given in (25) along trajectories of (23) is:

$$\dot{V}(z) = -\bar{W}(z) + \frac{\partial V_1(z_1)}{\partial z_{1,1}} \omega_0 + \sum_{i=1}^{n-1} \frac{1}{\kappa_i} \frac{\partial V_i(z_i)}{\partial z_{i,2}} \omega_i. \quad (34)$$

Due to homogeneity properties of $\bar{W}(z)$, $\frac{\partial V_1(z_1)}{\partial z_{1,1}}$ and $\frac{\partial V_i(z_i)}{\partial z_{i,2}}$, $\forall i \in \{1, \dots, n-1\}$ and using Lemma 3 and inequality (2), there exist class- \mathcal{K}_∞ functions σ_1 , σ_2 such that:

$$\dot{V}(z) \leq -\sigma_1(\|z\|) + \sigma_2(\|z\|)^{-1} \|\omega\|, \quad (35)$$

from which it is clear that closed-loop system (23) is **ISS**, since for any $\|\omega\| \leq \sigma_3(\|z\|) := \sigma_2(\|z\|)\sigma_1(\|z\|)$, we have

$$\dot{V}(z) \leq 0. \quad (36)$$

This concludes the proof. \square

Remark 3. Note that the existence of the control gains \bar{k}_i , \underline{k}_i and κ_i , $\forall i \in \{1, \dots, n-1\}$, shown in Theorem 1's proof, can also be guaranteed using Lemma 3 instead of Lemma 2. This alternative approach provides insight into the construction of these gains and can be utilized in numerical simulations. For example, to ensure (21), \underline{k}_i can be constructed from Lemma 3 to satisfy the following inequality:

$$\underline{k}_i \geq \min_{x \in \mathcal{T}_i} \left\{ \left(\gamma_i + \frac{m - \bar{r}_i}{r_i} \right) [x_1]^{\frac{m-r_i}{r_i}} x_2 + \frac{m - \bar{r}_i}{r_i} \hat{k}_i^{\frac{\bar{r}_i}{m-\bar{r}_i}} |x_1|^{\frac{m-r_i-\bar{r}_i}{r_i}} |x_2|^2 \right\}, \quad (37)$$

for any $i \in \{1, \dots, n-1\}$, with $\hat{k}_i = \bar{k}_i^{\frac{m-\bar{r}_i}{\alpha_i}}$ and

$$\mathcal{T}_i = \left\{ [s_1, s_2] : \hat{k}_i^{\frac{\bar{r}_i}{m-\bar{r}_i}} \left([s_1]^{\frac{m-\bar{r}_i}{r_i}} + \hat{k}_i [s_2]^{\frac{m-\bar{r}_i}{\bar{r}_i}} \right) \times \left[[s_1]^{\frac{\alpha_i}{r_i}} + \bar{k}_i [s_2]^{\frac{\alpha_i}{\bar{r}_i}} \right]^{\frac{r_{v_i}}{\alpha_i}} = 0 \right\}. \quad (38)$$

where we recall that for all $i \in \{1, \dots, n-1\}$, \bar{k}_i can be chosen arbitrarily. Similarly for all $i \in \{1, \dots, n-1\}$, the high-gain coefficients κ_i can be constructed recursively using Lemma 3 by following the steps in Subsection 3.3 in backward order. More precisely, we start by taking κ_1 arbitrarily from (30), then construct κ_2 in light of inequality (29), and repeat the process until we construct κ_{n-1} in light of (27).

4. NUMERICAL SIMULATIONS

In this section, we present some numerical simulations to demonstrate the effectiveness of the proposed control design. Let us consider system (1)-(5) with $n = 3$, i.e.:

$$\begin{aligned} \dot{x}_0(t) &= x_1(t) + \omega_0(t), \\ \dot{x}_1(t) &= x_2(t) + \omega_1(t), \\ \dot{\eta}_1(t) &= x_2(t) + v_1(t), \\ \dot{x}_2(t) &= -v_2(t) + \omega_2(t), \end{aligned} \quad (39)$$

with inputs $v_1(t)$ and $v_2(t)$ given as in (6). We start first by taking the extra inputs $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ all equal to zero, and choosing the homogeneity parameters and the control gains to ensure the **FTS** of (39) in lights of Remark 3. After that, we redo the simulations with $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ as follows:

$$\begin{aligned} \omega_0(t) &= 1e-3 [x_0(t)]^{\frac{r_1+\delta}{r_1}}, \\ \omega_1(t) &= 1e-3 [x_0(t)]^{\frac{\bar{r}_1+\delta}{r_1}} + 1.5e-3 [x_1(t)]^{\frac{\bar{r}_1+\delta}{\bar{r}_1}}, \end{aligned} \quad (40)$$

$$\begin{aligned} \omega_2(t) &= 1e-3 [x_0(t)]^{\frac{\bar{r}_2+\delta}{r_1}} + 1.5e-3 [x_1(t)]^{\frac{\bar{r}_2+\delta}{\bar{r}_1}} \\ &\quad + 2e-3 [x_2(t)]^{\frac{\bar{r}_2+\delta}{r_{v_1}}} + \varepsilon \sin(20t). \end{aligned} \quad (41)$$

which are homogeneous of degree δ (when $\varepsilon = 0$). In this case, we keep the same homogeneity parameters and same control gains to show the robustness of the closed-loop system with respect to ω . We take the homogeneity parameters in (7) as follows: $\delta = -0.3$, $r_1 = 1.6$, $\bar{r}_1 = r_2 = 1.3$, $r_{v_1} = \bar{r}_2 = 1$, $r_{v_2} = 0.7$. For the control and Lyapunov function free parameters, we take $m = 11.9$, $\alpha_1 = 3.6 \geq r_1$, $\alpha_2 = 1.6 \geq r_2$, $\gamma_1 = 7$, $\gamma_2 = 4$, $\bar{k}_1 = 15$, $\bar{k}_2 = 17$. The rest of the parameters are chosen to ensure the **FTS** property of the closed-loop by following the steps of the proof and in light of Remark 3: $\bar{\kappa}_1 = 25$, $\bar{\kappa}_2 = 35$, $\underline{k}_1 = 15$, and $\underline{k}_2 = 16.4$. The simulations are achieved using an implicit Euler scheme with the initial condition $x(0) = [3, 5, 2]^\top$ and for two different high-gains $\kappa_0 = 2$ and $\kappa_0 = 8$.

Figure 1 shows on the left the evolution of the states $x_0(t)$, $x_1(t)$, and $x_2(t)$ with $\kappa_0 = 2$ and $\omega = 0$ and on the right the evolution of the norm $\|x(t)\|$ of the closed-loop system (1)-(5) for both $\kappa_0 = 2$ and $\kappa_0 = 8$ and with $\omega = 0$. Figure 2 on the other hand shows the evolution of the norm $\|x(t)\|$ of the closed-loop system (1)-(5) subject the extra inputs ω given in (40)-(41) on the left for $\varepsilon = 0$ and on the right for $\varepsilon = 1.5$.

5. CONCLUSION

This paper presented a new control design for chain of integrators of any length subject to external disturbances. The proposed design ensures global finite time stabilization in the undisturbed case as well as input-to-state stabilization in the disturbed case. The proposed control design generalizes the newly established low-power homogeneous high-gain estimation approach given in [Moreno, 2021] into the context of stabilization of linear systems via low-power high-gain methodologies Astolfi et al. [2017]. Future work will extend this result to a more general class of nonlinear systems. Extensions will also aim to combine the proposed low-power high-gain homogeneous approach with the bi-limit homogeneity-based tools introduced in [Andrieu et al., 2008] in attempt to achieve fixed-time stabilization with a time of convergence featuring the high-gain parameter as a tuning parameter.

APPENDIX

Let us recall some technical results and well-known properties of continuous homogeneous functions.

Lemma 1. (Young's inequality) For any real numbers $x, y \in \mathbb{R}$, $p \in (1, +\infty)$, we have:

$$|x||y| \leq \frac{|x|^p}{p} + \frac{(p-1)|y|^p}{p}. \quad (42)$$

Lemma 2. [Cruz-Zavala and Moreno, 2019, Lemma 4] Let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be two continuous homogeneous functions, with vector of weights $r = [r_1, \dots, r_n]$ and degree m , satisfying the following holds:

$$\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subseteq \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\}. \quad (43)$$

Then, there exists $\lambda^* \in \mathbb{R}$ such that, for all $\lambda \geq \lambda^*$ and for all $x \in \mathbb{R}^n \setminus \{0\}$, we have:

$$\eta(x) - \lambda\gamma(x) < 0. \quad (44)$$

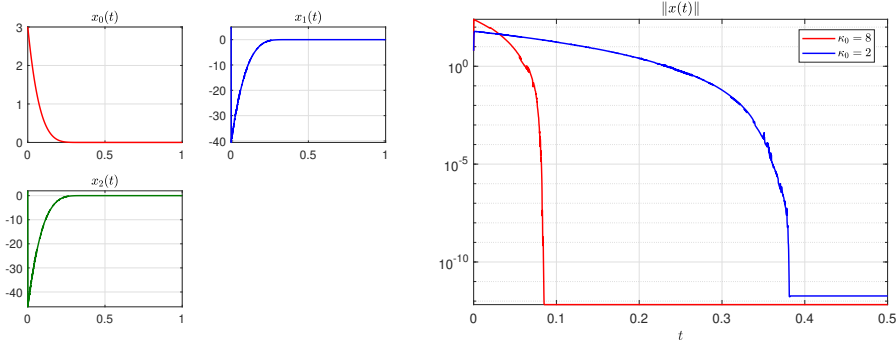


Fig. 1. On the left: the evolution of the states $x_0(t)$, $x_1(t)$, and $x_2(t)$ for the initial condition $[3, 5, 2]^\top$ and for $\kappa_0 = 2$. On the right: The logarithmic scale evolution of the norm $\|x(t)\|$ of the closed-loop system (1)-(5) in blues line for $\kappa_0 = 2$ and in red line for $\kappa_0 = 8$, with the initial condition $[3, 5, 2]^\top$.

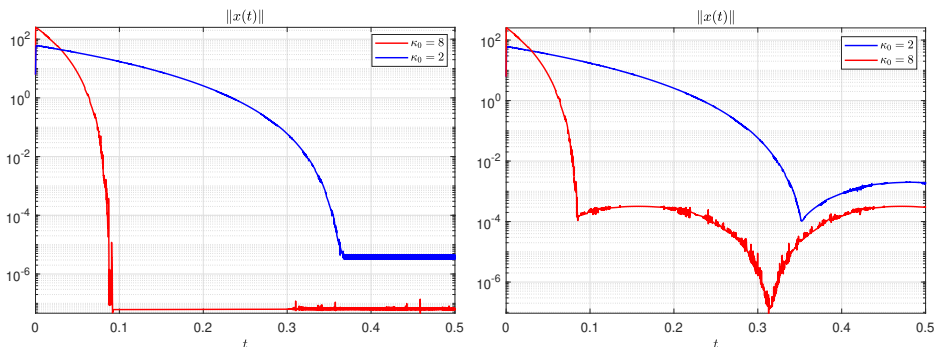


Fig. 2. The logarithmic scale evolution of the norm of the closed-loop system (1)-(5) in blues line for $\kappa_0 = 2$ and in red line for $\kappa_0 = 8$, with the initial condition $[3, 5, 2]^\top$, ω given in (40)-(41) and $\varepsilon = 0$ (on the left) and $\varepsilon = 1.5$ (on the right).

Lemma 3. [Cruz-Zavala and Moreno, 2019, Lemma 5] Let $V_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $V_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be two continuous functions homogeneous of degrees $m_1 > 0$ and $m_2 > 0$, respectively, with vector of weights $r = [r_1, \dots, r_n]$. If V_1 is positive-definite, then for every $x \in \mathbb{R}^n$, we have:

$$c_1 V_1^{\frac{m_2}{m_1}}(x) \leq V_2(x) \leq c_2 V_1^{\frac{m_2}{m_1}}(x), \quad (45)$$

where $c_1 = \min_{\{z:V_1(z)=1\}} V_2(z)$ and $c_2 = \max_{\{z:V_1(z)=1\}} V_2(z)$.

REFERENCES

- V. Andrieu, L. Praly, and A. Astolfi. Homogeneous approximation, recursive observer design, and output feedback. *SIAM Journal on Control and Optimization*, 47(4):1814–1850, 2008.
- V. Andrieu, D. Astolfi, and P. Bernard. Observer design via interconnections of second-order mixed sliding-mode/linear differentiators. *International Journal of Robust and Nonlinear Control*, 31(9):3631–3657, 2021.
- D. Astolfi and L. Marconi. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11):3059–3064, 2015.
- D. Astolfi, A. Isidori, and L. Marconi. Output regulation via low-power construction. *Feedback Stabilization of Controlled Dynamical Systems: In Honor of Laurent Praly*, pages 143–165, 2017.
- J.P. Barbot and T. Floquet. Iterative higher order sliding mode observer for nonlinear systems with unknown inputs. *Dynamics of Continuous, Discrete and Impulsive Systems*, 17(6):1019–1033, 2010.
- E. Cruz-Zavala and J.A. Moreno. Levant’s arbitrary-order exact differentiator: a Lyapunov approach. *IEEE Transactions on Automatic Control*, 64(7):3034–3039, 2019.
- X. Huang, W. Lin, and B. Yang. Global finite-time stabilization of a class of uncertain nonlinear systems. *Automatica*, 41(5):881–888, 2005.
- A. Isidori. *Nonlinear control systems*. Springer, 1995.
- A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International journal of Control*, 76(9-10):924–941, 2003.
- A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5):823–830, 2005.
- Z. Lin. An overview of the development of low gain feedback and low-and-high gain feedback. *Journal of Systems Science and Complexity*, 22(4):697–721, 2009.
- J. A Moreno. Levant’s differentiator by interconnection of low order blocks. *IFAC-PapersOnLine*, 54(14):319–324, 2021.
- J. A Moreno and M. Osorio. Strict Lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4):1035–1040, 2012.
- L. Praly and H.K. Khalil. High-gain observers in nonlinear feedback control. *Int. J. Robust. Nonlinear Control*, 24(6):993–1015, 2014.
- A.R. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33(5):1443–1488, 1995.