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# Fourier integral operator and asymptotics for waves

Olivier Lafitte

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Olivier Lafitte. Fourier integral operator and asymptotics for waves. DEA. Opérateurs pseudodifférentiels, France. 1996. hal-04766556

**HAL Id: hal-04766556**

**<https://hal.science/hal-04766556v1>**

Submitted on 5 Nov 2024

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# Fourier integral operators and asymptotics for waves.

Olivier Lafitte



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## CHAPTER 0

# Introduction

This text comes from a course taught at the University of Paris 13 from February 1996 to June 1996, then from December 1996 to February 1997. Its objective is to bring some clarifications on the asymptotics à used classically in wave propagation.

We give a rigorous and precise mathematical framework which allows to prove the validity of high frequency asymptotics for wave propagation of scalar or or electromagnetic waves (whether in vacuum or in dielectric media), for the reflection of a wave by a boundary (supplemented with a boundary condition), and to calculate the wave around a caustic (or a diffractive point in the last chapter).

This rigorous mathematical framework is the microlocal analysis, and this book is an introduction to some of the techniques used in this branch of analysis. In this introduction, we present the motivations of high frequency asymptotic studies (Section 1) and we explain in a few words some of the micro-local analysis tools used.

More generally, plenty of equations in mathematical physics use geometrical objects; the gradient, the the rotational, the divergence, the laplacian. The equation that shows best the relation between geometry and qualitative properties of the solutions is the wave equation. This will be particularly clear in Chapter 11, written in collaboration with Claude Bardos, devoted to the eigenvalues of the Laplacian associated with a metric characterized by  $A$ . The Fourier integral operators are the natural tools to take into account, in a geometrically intrinsic way, the notions of propagation in the wave equation. The energy in a tube of rays associated with the wave equation (introduced in chapter Chapitre 3 ) is the physical transcription of the notion of half density, which will be introduced in Chapter 11. The concept of half-density has been used as a basis for the study of Fourier operators by Hörmander.

From the equation of waves, we deduce a certain amount of information about other equations : the heat equation, the system of equations of elasticity, the Maxwell's system of equations. Thus the applications of the Fourier integral operators go beyond the study of the wave equation.

### 1. Motivation for asymptotic studies

The calculation of the response of an object to a radar transmission is a problem in the design of an aircraft or any other object which purpose is to make it as undetectable as possible. The response is given by the calculation of the solution of Maxwell's equations and a relation to the boundary depending of the nature of the object, the initial data being the value of the incident electromagnetic field (the radar wave wave, of wavelength  $\lambda$ ).

To avoid the dispersion of the radar beam in the atmosphere, the wavelength must be quite small, of the order of a meter or a centimeter. Since the frequency of the wave (propagating at the speed of light  $c$ ) is  $\omega = 2\pi c/\lambda$ , this frequency is therefore between 1 and 100 GHz. This physics justifies the use of high frequency methods.

Let us suppose moreover that we can associate to the object, or to a part of it, a characteristic length  $L$ . For an airplane, for example,  $L$  is of the order of the ten meters. The usual numerical methods for calculating the solutions of Maxwell's equations consider meshes whose characteristic size is  $\lambda/8$ , that is to say of the order of the millimeter or the ten centimeters. The number of degrees of freedom used is then  $N = (8L/\lambda)^3$ , which is at least of the order of  $10^5$ . This number of degrees of freedom then corresponds to  $6N$  unknowns (coordinates of the electric and magnetic fields). The matrix to be inverted in order to solve the discretized



system is then  $6N \times 6N$ . These large numbers often do not allow for a comprehensive treatment of the problem when the frequency is too high. It is then necessary to find alternative methods to the global numerical calculation; it is asymptotic methods.

From Huyghens to Maxwell, the idea was developed that light and therefore the electromagnetic phenomena propagate as waves on rays. For rays, we can talk about position and velocity or (thanks to the Legendre transform) of position and impulse. The calculation (described below) shows the connection between the Fourier variable and the impulse. The inequality of Heisenberg shows that it is impossible to localize both in position and in velocity, and we are led to localize asymptotically and at high frequency. This leads on the one hand to a seemingly endless refinement of formal asymptotic calculations, on which important qualitative progress has been made by V. Babich [5] and J. B. Keller [39] in the 1950s, probably under the impulse of the problems of radar stealth or detection. Almost all the asymptotic results have been obtained since then. An account of formal asymptotic results in a number of physical situations is presented by D. Bouche and F. Molinet in the volume 16 of the collection "Mathématiques et Applications" [15].

The rigorous justification of these calculations has been the essential preoccupation of at least one generation of mathematicians, and under the influence of Hörmander, the formalism of microlocal analysis has imposed itself.

It is for the moment impossible to justify mathematically all the results of [15]. On the other hand, the proof of the validity of some of these asymptotic calculations is possible, and we present it in this book.

## 2. Mathematical techniques covered

We focus here on the study of linear partial differential equations and systems of linear partial differential equations. We work at **high frequency** for three main reasons :

- 1) the physics of the radar detection problem is a physics at high frequencies (which can be expressed by  $L/\lambda > 10$ ),
- 2) by introducing an asymptotic parameter, we can derive analytical calculations (see for example the chapter 1) hence improving the precision of the solutions,
- 3) finally, one can justify these formal asymptotics in certain cases (see the chapter 2 for an example of proof, see also the work of P. Lax [60]).

Microlocal analysis (introduced by Hörmander [47] in the late 60's) turns out to be the most universal method of treating these high frequency problems, generalizing and justifying for example the calculations made on Gaussian rays [85] and boundary layer calculations [15]. Indeed, microlocal analysis is associated to a pseudo-differential calculus which is an explicit symbolic calculus and to a good asymptotic notion (generated for example by the order of the symbols). This branch of Analysis studies the regularity of distributions, not only locally, but also by distinguishing the regularity with to any direction of derivation. To make a physical analogy, everything happens as if we were to consider the regularity of the distributions in the space of the position-impulse space.

When  $u \in \mathcal{S}(\mathbb{R}^d)$ , the Schwartz class of rapidly decaying functions, its Fourier transform (indifferently denoted by  $\hat{u}(\xi)$  or  $\mathcal{F}(u)(\xi)$ ) is also in the class of Schwartz class:

$$\hat{u}(\xi) = \int e^{-ix \cdot \xi} u(x) dx.$$

We have the inversion formula of Fourier (written formally in the second part of the following equality because the integral is not converging in the Lebesgues sense) :

$$(2.1) \quad u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{u}(\xi) e^{ix \cdot \xi} d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} u(y) e^{i(x-y) \cdot \xi} d\xi dy$$

From the formula  $\mathcal{F}(\partial_x u)(\xi) = i\xi \hat{u}(\xi)$ , we get

$$(2.2) \quad \partial_{x_1} u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} i\xi_1 u(y) e^{i(x-y) \cdot \xi} dy.$$

In the equality (2.2), we will call the function  $i\xi_1$  the **symbol** of the operator  $\partial_{x_1}$ . We study in the chapter 5 the asymptotic calculations on the symbols, which are functions of class  $C^\infty$  for  $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ . A any differential operator is associated with a symbol, and vice versa, to any symbol is associated an operator called pseudodifferential operator (which we will construct here). The calculation of the composition of two pseudo-differential operators defines a calculus on the symbols, hence an algebra, studied in the chapter 7. The elements of this algebra play a particular role; they are the elliptic symbols. A symbol  $a(x, \xi)$  defines intrinsically a pseudo-differential operator; a change of variable in  $x$  induces a change of change of variable in  $\xi$ . We will study these changes of variable in the chapter 7. We show that the notion of geometry preserved in  $\mathbb{R}^d$  by the change of variable in  $x$  in  $\mathbb{R}^d$  is the **symplectic** character of  $\mathbb{R}^d \times \mathbb{R}^d$ , which is then identified with the cotangent bundle  $T^*\mathbb{R}^d$ . Thus, a symbol  $a(x, \xi)$  is a function  $C^\infty$  on  $T^*\Omega$ . The rays that we talked about above are then the traces on  $\Omega$  of geometrically intrinsic in  $T^*\Omega$ , called bicharacteristics.

Computations in micro-local analysis, on the other hand, make constant use of of a result, called the stationary phase theorem. This result is the generalization (at least formally) of a method known since Legendre, the saddle point method. The phase of the saddle point admits a maximum. It is said to be **stationary** at the point where it admits this maximum, hence the name of 'stationary phase'. The difference between the stationary phase method and the saddle point method comes from the type of integral studied; for  $k \in [1, +\infty[$ , the integral of the saddle point method is of the form  $\int e^{-k\phi(x)} dx$ , which is an absolutely convergent integral when  $\phi$  has a minimum on the integration interval whereas the integral used in the in the stationary phase method is  $\int e^{ik\phi(x)} dx$ , which is defined in  $\mathcal{S}'(\mathbb{R}^d)$  (see Chapter 4.)

The stationary phase theorem, studied in chapter 4 is at the center of microlocal analysis, of pseudodifferential and has applications in other fields. We present one in the chapter 5, kindly provided by Eric Pilon, resulting from theoretical physics calculations.

However, the formula (2.2) does not always allow to represent correctly a solution of a partial differential equation. This is the case when the problem has an caustic or a ray which diffracts, or when one wants to have access to properties involving the global structure of rays. It turns out that in these cases it is possible and essential to generalize the formula (2.2) by replacing  $(x - y) \cdot \xi$  by  $\phi(x, y, \xi)$ , homogeneous of degree 1 by  $\xi$ . The operator is then called Fourier integral operator. The global study of Fourier integral operators is a major part of this book (Chapter 6); indeed the representation of solutions of partial differential equations by oscillating integrals is the best adapted to asymptotics which involve phases of the form  $e^{ik\phi(x)}$ ,  $k$  large parameter. parameter. To Fourier integral operators and phases of solutions of partial differential equations are associated some intrinsic geometrical objects, called Lagrangian manifolds. The structure of these manifolds is the subject of the chapter on caustics. Even if a phase (associated to an oscillating integral solution of a linear partial differential equation) is singular, the associated Lagrangian manifold ( $\subset T^*\Omega$ ) is smooth, and its projection on  $\Omega$  is singular. We will thus study some types of singularities. Thanks to the Fourier integral operators, we state and prove the theorems of propagation of singularities in  $\Omega$  for hyperbolic operators (obtained elsewhere, for example in [82]) as well as the theorems of reflection of hyperbolic singularities on  $\partial\Omega$ , for example for electromagnetic waves. Moreover, one can construct a uniform outgoing parametrix for the problem  $Pu = 0$  in  $\Omega$ ,  $\gamma Lu = g$  on the boundary of  $\Omega$  when  $\Omega$  has a boundary,  $P$  and  $L$  denoting two differential operators,  $\gamma$  being the trace on the boundary  $\partial\Omega$  when possible. Another application of Fourier integral operators is global and is presented in the chapter 11, written by Claude Bardos (whose contribution to the other chapters is far from negligible). It is based on a practical problem, which is to estimate the eigenvalues of an elliptic operator, typically the Laplacian in a bounded open. The functions we want to access are for example the number of eigenvalues smaller than a fixed  $R$ , the asymptotic behavior of the eigenvalues in ascending order. We also want to study inverse problems through this; find for example information about the open set  $\Omega$  in which we work through these eigenvalues. We highlight the importance of the global aspect of Fourier integral operators which are the only ones able to take into account closed geodesics. Indeed,

a pseudo-differential operator is pseudo local (see Hörmander [48]), so it will not be able to take into account in particular the application of first return Poincaré map.

### 3. Details of the chapters in this book

In the chapter 1, we introduce the notion of asymptotic expansion; we that any asymptotic series is the Taylor expansion at a point of a function of class  $C^\infty$ , a result known as of Borel's lemma. This result will be used to construct an approximate solution of the wave equation.

In chapter 2, we present the method and results of P. D. Lax [60] to compute the asymptotic expansion associated to the solution of a strictly hyperbolic Cauchy problem. This is a direct introduction to asymptotic calculations.

In chapter 3, we use the equivalence between the wave equation and the Helmholtz equation  $(\Delta + k^2)u = 0$ . We construct, when  $k$  tends to  $+\infty$ , an asymptotic solution of the Helmholtz equation of the form  $a(x, k)e^{ik\phi(x)}$ , where  $a(x, k)$  has an asymptotic expansion in the sense of Chapter 1. More precisely, we compute the amplitude  $a(x, k)$  on a set of rays, once this amplitude is given on a surface where  $\phi$  is constant. The Borel lemma then constructs an asymptotic solution equivalent to the desired exact solution.

Chapter 4 deals with an important notion, which is -in my opinion- at the basis of micro-local analysis : the stationary phase method, closely related to asymptotic expansions. Its application is possible whenever one can use the Morse lemma to represent the phase as a quadratic phase. An application to particle physics of the stationary phase theorem is the subject of chapter ??.

Similarly, chapter 6 presents divergent integrals, called oscillating integrals, to which we will associate a value thanks to the stationary phase theorem. The operators associated with these oscillating integrals are the Fourier integral operators. The oscillating integrals of chapter 6 use functions  $a(x, \xi) \in C^\infty(X \times \mathbb{R}^d)$ , which have some nice behavior in the asymptotic parameter  $|\theta|$ . These properties allow to introduce systematic computational rules. The study of the functions  $a(x, \theta)$  is the subject of the chapter ??. A special case of integrals in chapter 6 allows us to generalize the differential operators into a class of operators, called pseudo-differential operators (studied for example by Kohn and Nirenberg [55]). The study of the properties and regularity of these operators is presented in chapter 8. In particular, we will define a law on the symbols which corresponds to the composition law of the operators.

Chapter 9 studies the intrinsic character of notions related to operators and introduces the symplectic geometrical framework associated to pseudo-differential operators and the Fourier integral operators. Using this geometrical framework, we give the theorem allowing to compose Fourier integral operators.

Chapter ?? links Fourier integral operators and asymptotics calculations of Lax from chapter 2, in the sense that the phase  $\phi$  introduced by Lax, solution of the so-called eikonal equation, can be taken as an oscillating phase in a Fourier integral operator, thus generalizing the asymptotic oscillating solutions. We study more generally the Lagrangian solutions, generalization of the notion of phase solution of the eikonal equation. We deduce the behavior of solutions of hyperbolic problems in the neighborhood of caustics.

In the chapter 11, Claude Bardos details a global application of the Fourier integral operators, which allows, from the eigenvalues of the Dirichlet Laplacian on an open set  $\Omega$ , to know some properties of this open set. This problem is popularized by the sentence "Can we hear the shape of a drum?"<sup>1</sup>.

Chapter 12 proves the theorem of propagation of singularities for waves, which is a generalization of the laws of geometrical optics, using both using both the Fourier integral operators to construct the construct a solution and to transform the propagation problem into a simpler problem by means of a canonical transformation. These two points of view are equivalent.

The reflection of an electromagnetic wave by a dielectric object, first application of asymptotic expansions, is studied in chapter ?? and the reflection theorem of hyperbolic singularities

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<sup>1</sup>Can we hear the shape of a drum?

is proved in the case of an open set containing absorbing dielectric material. We use here a generalization of Lax's asymptotic calculations to hyperbolic pseudodifferential operators, as done by Hörmander [46] and Levitan [64], [65], or more recently by Sjöstrand and Zworski [90].

Finally, chapter 14 studies the diffractive case, where the incident ray arrives tangentially at the boundary of the  $\Omega$  object, in two cases : the Friedlander's model problem [42], original example, and the the case of diffraction by a convex plane, as in Gilles Lebeau's results on the Gevrey 3 regularity [66] and the thesis of the author [57]. In this chapter, written in collaboration with Daniel Bouche, we compare the calculations of the microlocal analysis and the calculations used in the boundary layer method.

#### 4. Acknowledgements

The author would like to thank all those who made it possible for this course: the students of the University of Paris Nord, who had the patience to follow and understand the sometimes complicated calculations, and the analysis group of the University of Paris Nord (Michel Balabane, Laurence Halpern, Patrice Le Calvez) who gave him the opportunity to teach in the DEA of Applicable Mathematics.

The author is also greatly grateful to Johannes Sjöstrand for the course followed at the University of Orsay in 1987-1988. He relied on Max Bezar's notes to write some of the chapters of this of this textbook. He would also like to thank Gilles Lebeau, his thesis advisor, both for the course Gilles taught in 1988-1989 and for the light he cast on reflection of singularities and the calculation in the shadow zone of an object.

The author was also inspired by a text of Jeffrey Rauch, and he hopes that Jeffrey will forgive him to have borrowed several proofs for hyperbolic matrix problems.

The help and advice of Claude Bardos<sup>2</sup> and Jean-Michel Ghidaglia, as well as the rereading of Daniel Bouche and Gérard Meurant have been invaluable in the development of this course. The chapters concerning pseudodifferential operators have been proofreaded with great care by Jean-Marc Delort (University of Paris-Nord), and I thank him a lot. The first chapters (1 to 3) have been corrected and improved by the remarks of B. Bidegaray, from the University Paul Sabatier, and her remarks were very useful.

A special thanks to Souleymane N'Diaye, whose DEA memoir from the University of Paris-Nord (1996) has deepened the case of the Friedlander model problem. Another thanks to Jeanne-Marie Caron, DEA student of the University of Orsay (1998-1999), whose help for the improvement of chapter 1.53 has been essential.

The influence of Richard Melrose and his hosting in 1993-1994 at Massachusetts Institute of Technology has also been fundamental in the genesis of this course.

Finally, all my thoughts go to Christine, Marine and Geoffrey, who have, all, helped me to bring this book to the end.

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<sup>2</sup>and in memoriam of Joseph Bardos, 1906-1997



## Formal asymptotics

In this chapter, we present an asymptotic formal expansion for a number of systems of equations of Physics. We study the wave equation, the Helmholtz equation in a dispersive medium and more generally the More generally, hyperbolic problems. Some of these asymptotics are presented in exercises.

These formal asymptotics will be justified mathematically in the next chapters of this book.

### 1. Introductory examples

We first give two examples which allow to define an asymptotic expansion. The two functions we present are solution of the wave equation, and we can calculate their asymptotic expansion. They have in common that they are oscillating in time and space.

**1.1. Plane waves and the wave equation.** The simplest example traditionally considered is the plane wave ( $\vec{k} \in \mathbb{R}^d, \omega \in \mathbb{R}_+$ )

$$u_p(\vec{x}, t) = e^{i\omega t - i\vec{k} \cdot \vec{x}}, \vec{x} \in \mathbb{R}^d, t \in \mathbb{R}.$$

LEMMA 1.1. (1) *The function  $u_p$  is a non  $L^2(\mathbb{R}^{d+1})$  (but in  $L^2_{loc}$ ) of the wave equation  $(\Delta - c^{-2}\partial_t^2)u = 0$  if and only if*

$$(1.3) \quad \omega^2 - |\vec{k}|^2 c^2 = 0,$$

(2) *The solution of the wave equation in dimension 1*

$$\left(\frac{\partial^2 u}{\partial x^2} - c^{-2}\frac{\partial^2 u}{\partial t^2}\right)(x, t) = \left(\frac{\partial}{\partial x} - c^{-1}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial x} - c^{-1}\frac{\partial u}{\partial t}\right).$$

with Cauchy data  $u_0(x), u_1(x)$  is

$$(1.4) \quad \begin{aligned} u(x, t) &= \frac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \frac{1}{2c}\int_0^{x+ct} u_1(y)dy - \frac{1}{2c}\int_0^{x-ct} u_1(y)dy \\ &= \frac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy. \end{aligned}$$

The relation of the first item is written  $|\vec{k}| = \omega/c$ , and is called the relation of dispersion in the vacuum.

When the dimension of the ambient space is 1, the two possible values of  $k$  are  $\pm k/c$ , which correspond to two plane waves, propagating respectively in the direction of  $x > 0$  and  $x < 0$ . and  $x < 0$ .

PROOF. Let  $u$  be a solution of the wave equation, we associate to it  $v = \partial_x u + c^{-1}\partial_t u$ , which is a solution of  $\partial_x v - c^{-1}\partial_t v = 0$ . We deduce that  $w(z, t) = v(z - ct, t)$  verifies  $\partial_t w(z, t) = 0$ . So  $w(z, t) = w(z, 0) = v(z, 0)$ . The equality  $v = \partial_x u + c^{-1}\partial_t u$  becomes

$$\partial_x u(x, t) + c^{-1}\partial_t u(x, t) = w(x + ct, 0).$$

Let  $h(z) = \frac{1}{2}\int_0^z w(y, 0)dy$ . We verify that  $(\partial_x + c^{-1}\partial_t)h(x + ct) = w(x + ct, 0)$ . It comes then that

$$(\partial_x + c^{-1}\partial_t)(u(x, t) - h(x + ct)) = 0$$

which immediately leads to

$$u(x, t) = h(x + ct) + g(x - ct).$$

The Cauchy conditions at  $t = 0$  are

$$u_0(x, 0) = u(x, 0), \partial_t u_0(x, 0) = u_1(x, t).$$

So we have the two equalities

$$g(x) + h(x) = u_0(x), g'(x) - h'(x) = c^{-1}u_1(x),$$

which gives (5.2) which gives (1.4).  $\square$

Remarks : 1) if the support at  $t = 0$  of the wave, equal to  $\text{Supp } u_0 \cup \text{Supp } u_1$ , is the interval  $[a, b]$ , the support at  $t$  of the wave is included in  $[a - ct, b + ct]$ . This set of points is called the light cone.

2) if there is information at  $t = 0$  at a point  $x_0$ , that is  $u_0$  or  $u_1$  is nonzero, then at  $t_0$  there is information at  $x_0 + ct_0$  and at  $x_0 - ct_0$ . This reflects the propagation of the information at speed  $c$ .

3) We see that two progressive waves appear when  $u_0(x) = \pm(ik)^{-1}u_1(x) = e^{-ikx}$ . We will omit the velocity  $c$ . It will however intervene in a hidden way in solutions of the Maxwell's equations through the relation  $\varepsilon_0\mu_0c^2 = 1$ , and instead of considering the pulsation  $\omega$  we will consider the equations with the wave number  $k = \omega/c$ .

**1.2. Some insight as introduction to asymptotic expansions.** In this paragraph, we use the special functions solution of the wave equation (in dimension 2 or in dimension 3), to see that the notion of asymptotics appears naturally.

For the wave equation in dimension 2, we introduce the Hankel functions.

PROPOSITION 1.1. *The outgoing solutions of the wave equation in  $\mathbb{R}^2 \times \mathbb{R}_t$  are equal to*

$$\sum_{n=0}^{+\infty} a_n e^{in\theta - i\omega t} H_n^1(kr),$$

with  $\sum |a_n|^2 < +\infty$ , where  $H_n^1(x)$  is the Hankel function of the first kind, solution of  $r^2 f'' + r f' + (r^2 - \nu^2) f = 0$  when  $\nu = n$ , uniquely defined by  $k^{\frac{1}{2}} H_n^1(kr) \left(\frac{2}{\pi r}\right)^{-\frac{1}{2}} e^{-ikr} \simeq e^{-\frac{1}{2}n\pi - \frac{1}{4}\pi}$  when  $k$  goes to  $+\infty$ .

PROOF. A solution of the wave equation with speed 1 in  $\mathbb{R}^2 \times \mathbb{R}$  is written  $u(r, \theta, t) = e^{-i\omega t + i\nu\theta} f(r)$ . In writing the Laplacian operator in polar coordinates

$$(1.5) \quad \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta^2},$$

we see that the function  $f$  is a solution of

$$r^2 f'' + r f' + (k^2 r^2 - \nu^2) f = 0$$

Consider the It satisfies

$$x^2 (H_n^1)'(x) + x (H_n^1)''(x) + (x^2 - n^2) H_n^1(x) = 0.$$

The function  $v_n(r, \theta, t) = e^{-i\omega t + in\theta} H_n^1(kr)$  is solution of the wave equation in dimension 2. If we place ourselves at a point  $r \neq 0$ , we write (see Watson [98]) for  $n$  fixed and  $k$  going to  $+\infty$ :

$$(1.6) \quad k^{\frac{1}{2}} H_n^1(kr) = \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} e^{ikr - \frac{1}{2}n\pi - \frac{1}{4}\pi} \sum_{m=0}^{\infty} c_m(r) k^{-m} + O(k^n r^n e^{-2kr}).$$

This expression is called an asymptotic expansion in  $k^{-1}$  for large wave number problem thanks to  $k = \omega c$ . The speed of the wave being fixed, it is equivalent to speak of a **high frequency** problem.

To understand the meaning of the notion of asymptotics for this example, we define  $H_n^M(kr)$  as the function obtained by truncating the series (1.6) at order  $M$  in  $k$ . We have formally, for  $u_M = k^{\frac{1}{2}} H_n^M(kr) e^{-i\omega t + in\theta}$

$$(\Delta - c^{-2} \partial_{t^2}^2) u_M = O(k^{-M+1}),$$

The meaning of the term  $O$  must be specified.

The function  $e^{in\theta} H_n^1(kr)$  is a solution of the Helmholtz equation to which we associate a sequence of functions  $u_M$  which are not solutions of the Helmholtz equation but which are a better candidate when  $M$  tends to  $+\infty$ , in the sense that  $(\Delta - c^{-2} \partial_{t^2}^2) u_M$  will be moreover in smaller and smaller in  $k$ . It turns out that these functions  $u_M$  approach the solution  $e^{in\theta} H_n^1(kr)$  of the Helmholtz equation, but the question is whether such results are general.  $\square$

In this chapter and in the next chapter, we define asymptotic solutions which are not solutions in the usual sense.

## 2. Definitions.

We define an asymptotic expansion as follows :

DEFINITION 1.1. *Let a function  $b \in C^\infty(\Omega \times ]0, 1[)$  (if we choose a small asymptotic parameter  $\varepsilon$ ), or  $b \in C^\infty(\Omega \times ]1, +\infty[)$  (if we choose a large asymptotic parameter  $k$ ).*

*Let  $b_j(y)$  be a sequence of functions of  $C^\infty(\Omega)$ . The relation*

$$b(y, \varepsilon) \simeq \sum b_j(y) \varepsilon^j$$

*means that, for any integer  $m \geq 0$ , for any index  $\alpha \in \mathbb{N}^n$  and for any compact  $K$  included in  $\Omega$ , there exists a constant  $C(K, m, \alpha)$  (whose behavior is not specified here) such that*

$$\forall y \in K, |\partial_{y^\alpha}^\alpha (b(y, \varepsilon) - \sum_{j=0}^{j=m} b_j(y) \varepsilon^j)| \leq C(K, m, \alpha) \varepsilon^m.$$

The definition is easily transcribed for a large asymptotic parameter, as well as the definition for  $k \geq k_0$  or  $\varepsilon \leq \varepsilon_0$ . We also define the notion of asymptotic equivalence of two functions  $b$  and  $c$  of  $C^\infty(]0, 1[ \times \Omega)$ :

DEFINITION 1.2. *We say that  $b(y, \varepsilon)$  and  $c(y, \varepsilon)$  are asymptotically equivalent if*

$$\forall K \subset \Omega, \forall m \geq 0, \forall \alpha, \exists C(K, m, \alpha) > 0, \forall y \in K, |\partial_{y^\alpha}^\alpha (b(y, \varepsilon) - c(y, \varepsilon))| \leq C(K, m, \alpha) \varepsilon^m.$$

This indicates that, **uniformly on any  $\Omega$  compact, the difference  $b - c$  is rapidly decaying in  $\varepsilon$  (or rapidly decaying in  $k$ ).**

Let us define the oscillating asymptotics. We introduce, this time, a large parameter  $k$  in order to emphasize that the notion of asymptotics is linked to a parameter, but that the results written with a ‘small’ parameter are inseparable from those written with a ‘large’ parameter. Let us consider for example

$$(2.7) \quad v(x, k) = a(x, k) e^{ikx},$$

where  $a$  verifies the assumptions of the definition 1.1. Its derivatives in  $x$  sorts out powers of  $k$ , so that

$$\partial_{x^\alpha}^\alpha v(x, k) \simeq e^{ikx} k^{|\alpha|} b_\alpha(x, k),$$

where  $b_\alpha$  is still of the form of Definition 1.1. To take into account in the asymptotic expansions the functions of the form (??), it is then useful to introduce the notion of equivalence between two oscillating functions, expressed in the



DEFINITION 1.3. Let  $a$  and  $b$  be two functions satisfying the assumptions of the definition 1.1. Let  $\phi$  and  $\psi$  be two functions of class  $C^\infty$ .

We say that  $a(x, k)e^{ik\phi(x)} \simeq b(x, k)e^{ik\psi(x)}$  when

$$\begin{cases} \phi(x) = \psi(x) + \psi_0 \\ a(x, k)e^{ik\psi_0} \simeq b(x, k) \end{cases}$$

A result, known as Borel's lemma, indicates that any formal asymptotic expansion is associated with a function  $C^\infty$ . It is the subject of the next section.

### 3. Borel's Lemma

PROPOSITION 1.2. Let  $b_j$  be a sequence of complex numbers. There exists a function  $b(\varepsilon)$ , of class  $C^\infty$ , such that, for all  $M \geq 0$ ,  $b$  has a Taylor expansion in 0 of order  $M$

$$b(\varepsilon) = \sum_{j=0}^{j=M} b_j \varepsilon^j + \varepsilon^M o(1).$$

Let  $b_j(y)$  be a sequence of functions  $C^\infty(\Omega)$ . There exists a function  $b(y, \varepsilon) \in C^\infty(\Omega \times [0, 1])$  such that

$$b(y, \varepsilon) \simeq \sum b_j(y) \varepsilon^j.$$

The *Borel's lemma* proves the existence of a function  $b(y, \varepsilon)$  such that  $b(y, \varepsilon) \simeq \sum_j b_j(y) \varepsilon^j$  for **any** sequence of functions  $C^\infty b_j(y)$ , but does not prove the existence of the sum, even pointwise, of  $b_j(y) \varepsilon^j$ . In fact, we consider a function  $b$  which is equal to an infinite series absolutely convergent as well as all its derivatives, and which approaches well term by term the formal series. There is no uniqueness either, as the example of the function  $f(x) = 1_{x \geq 0} e^{-\frac{1}{x^2}}$ . Indeed,  $\sum 0x^n$  can be associated with  $f$  since the Taylor expansion of  $f$  in 0 is of  $f$  in 0 is 0.

PROOF. This proof is a constructive proof.

We show the first point of the proposition. To do this, we give ourselves a function  $\chi(\varepsilon)$  in  $C_0^\infty(]-1, 1[)$ , identically 1 on  $[-0.5, 0.5]$ . We choose suitably (which is done below) a sequence of positive numbers  $L_j$  greater than 1, and we write

$$b(\varepsilon) = \sum_j \varepsilon^j \chi(L_j \varepsilon) b_j.$$

Its Taylor series in  $\varepsilon$  in 0 is  $\sum_j \varepsilon^j b_j$ .

We choose the  $L_j$ , for  $j \geq 1$ , by the inequality

$$(3.8) \quad \max_{m \leq j-1} \left| \frac{\partial^m}{\partial \varepsilon^m} (\chi(L_j \varepsilon) \varepsilon^j b_j) \right| \leq 2^{-j}.$$

Indeed,

$$\begin{aligned} \left| \frac{\partial^m}{\partial \varepsilon^m} (\chi(L_j \varepsilon) \varepsilon^j b_j) \right| &\leq |b_j| \left| \sum_{p=0}^{p=m} C_m^p L_j^{m-p} \chi^{(m-p)}(L_j \varepsilon) \frac{j^p}{(j-p)!} \varepsilon^{j-p} \right| \\ &\leq |b_j| \left( \sum_{p=0}^{p=m} C_m^p L_j^{m-j} \|\chi^{(m-p)}\| \frac{j^p}{(j-p)!} \right) \\ &\leq \frac{|b_j|}{L_j} \left( \sum_{p=0}^{p=m} C_m^p L_j^{m-j+1} \|\chi^{(m-p)}\| \frac{j^p}{(j-p)!} \right). \end{aligned}$$

Thus, for  $m \leq j-1$  and  $L_j > 1$ , we find

$$\max_{m \leq j-1} \left| \frac{\partial^m}{\partial \varepsilon^m} (\chi(L_j \varepsilon) \varepsilon^j b_j) \right| \leq \frac{|b_j|}{L_j} 2^j j! \|\chi\|_{j-1}$$

where  $\|\chi\|_j$  is the  $H^j$  norm of  $\chi$ . We choose then

$$L_j \geq 2^j j! \|\chi\|_{j-1} |b_j|.$$

We thus obtain (??).

We have then, for  $m$  fixed, uniform convergence of

$$\sum_{j \geq m+1} \frac{\partial^m}{\partial \varepsilon^m} (\chi(L_j \varepsilon) \varepsilon^j b_j).$$

The function  $b$ , sum of  $b_0 \chi(L_0 \varepsilon)$  and of the previous sum for  $m = 0$  exists and is indefinitely differentiable on  $\mathbb{R}$ .

We write

$$\begin{aligned} b(\varepsilon) - \sum_{j=0}^{j=m} b_j \varepsilon^j &= b(\varepsilon) - \sum_{j=0}^m b_j \chi(L_j \varepsilon) \varepsilon^j + \sum_{j=0}^m b_j [\chi(L_j \varepsilon) - 1] \varepsilon^j \\ &= \varepsilon^{m+1} \sum_{j=0}^{\infty} b_{j+m+1} \chi(L_{j+m+1} \varepsilon) \varepsilon^j + \sum_{j=0}^m b_j [\chi(L_j \varepsilon) - 1] \varepsilon^j \\ &= \varepsilon^{m+1} S_1(\varepsilon) + S_2(\varepsilon). \end{aligned}$$

The function  $S_1$  is bounded by

$$\|\chi\|_{\infty} \sum_{j=0}^{\infty} \frac{|b_{j+m+1}|}{L_{j+m+1}^{j+1}} \leq \|\chi\|_{\infty} \sum_{j=0}^{\infty} \frac{|b_{j+m+1}|}{L_{j+m+1}} \leq \|\chi\|_{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{j+m+1} \|\chi\|_{j+m+1} (j+m+1)!} \leq \frac{1}{2^m}.$$

The second term  $S_2(\varepsilon)$  of the decomposition is zero on  $|\varepsilon| \leq \frac{1}{L_m}$ . Thus, for  $1 \geq |\varepsilon| \geq \frac{1}{L_m}$ ,  $|\frac{S_2(\varepsilon)}{\varepsilon^m}| \leq \max_{|\varepsilon| \leq 1} |S_2(\varepsilon)| L_m^m$ , and, for  $\varepsilon \geq 1$ ,  $|\sum_{j=0}^{j=m} \varepsilon^{j-m} (\chi(L_j \varepsilon) - 1)| \leq (\|\chi\| + 1) \sum_{j=0}^m |b_j|$ . We deduce

$$S_2(\varepsilon) \leq D(m) \varepsilon^m.$$

We have proven the inequality:

$$\sup_{[-1,1]} (b(\varepsilon) - \sum_{j=0}^{j=m} b_j \varepsilon^j) \leq C_m \varepsilon^m.$$

The function  $b$  is of class  $C^{\infty}$  and admits  $\sum b_j \varepsilon^j$  as an asymptotic expansion in  $\varepsilon \rightarrow 0$ .

For the second point of the theorem, we give ourselves a compact  $K$  in  $y$  and a truncation order  $m$  of the asymptotic series.

We introduce, as in the proof of the first point of the theorem  $b_j(y) \chi(L_j \varepsilon) \varepsilon^j$ . We verify that, on  $K$  and for  $|\varepsilon| \leq L_j^{-1}$

$$|\partial_{y^{\alpha}}^{\alpha} \varepsilon^m (b_j(y) \chi(L_j \varepsilon) \varepsilon^j)| \leq \|b_j\|_{|\alpha|} \alpha_{m,q} L_j^{-1}.$$

By replacing  $b_j$  in the demonstration of the first part by the norm  $L^{\infty}$  of  $b_j$  on  $K$ , one obtains a choice of  $L_j$  that ensuring the convergence of the infinite series in  $\varepsilon$ . This convergence allows the construction of a function  $b(y, \varepsilon)$  of  $C^{\infty}([-1, 1], C^{\infty}(K))$ . We have moreover a choice of  $L_j^p$  ensuring the uniform convergence of the series constructed with the  $\chi(L_j^p \varepsilon)$  and of all its derivatives in  $\alpha$  of order at most  $p$ . Finally the two sums at order  $m$  are equal when  $|\varepsilon| \leq (2L_j^p)^{-1}$ . We then evaluate

$$b(y, k) - b_M(y, k) = \varepsilon^M \sum_{j=M+1}^{+\infty} \varepsilon^{j-M} \chi(L_j^p \varepsilon) a_j(y) + \sum_{j=0}^M \chi(L_j^p \varepsilon) - 1) b_j(y),$$

as well as all the derivatives up to order  $|\alpha| = p$ .

It is easy to see that the first term in a similar decomposition of  $\partial_x^{\alpha} (b(y, k) - b_M(y, k))$  is bounded by  $\varepsilon^M 2^{-M}$ , and the second term is zero in  $[-(2L_M^p)^{-1}, (2L_M^p)^{-1}]$ . By an argument similar to the previous one, we can find a constant  $C(K, M, p)$  such that the derivative  $\alpha$  of the second term is increased by  $C(K, M, p)$  (we divide and as 0 is not a pole, we have the result).

We can then apply the previous method, which allows us to compare this function  $b(y, \varepsilon)$  with the sum  $\sum_{j=0}^m b_j(y) \varepsilon^j$ . We then control

$$\partial_{y^{\alpha}}^{\alpha} (b(y, \varepsilon) - \sum_{j=0}^{j=m} b_j(y) \varepsilon^j)$$

by the constant used previously  $\sup_{[-1,1]/[-L_j^{-1}, L_j^{-1}]}(\varepsilon^{-m} S_2(y, \varepsilon))$ . This completes the proof of proposition 1.2.  $\square$

#### 4. The Helmholtz equation

Let us consider again the wave equation  $(\Delta - \partial_{t^2}^2)u = 0$ .

DEFINITION 1.4. *Let  $v(x, k)$  be a local asymptotic solution of the Helmholtz equation, which verifies (4.9). Let  $\Sigma_0$  be a hypersurface. We say that  $u$  is a solution of the equation  $(\Delta + k^2)u(x, k) = 0$  with the initial condition<sup>1</sup>  $v(x, k)|_{\Sigma_0} = u(x, k)|_{\Sigma_0}$  when there exists  $r(x, k)$  such that*

$$\begin{cases} r|_{\Sigma_0} = 0 \\ (\Delta + k^2)(v + r) = 0 \end{cases}$$

or, in an equivalent formal way,  $r = (\Delta + k^2)^{-1}(O(k^{-\infty}))$ .

Note that the problem we study here is a Dirichlet condition problem and not a Cauchy problem.

We assume that the solution of this equation is a tempered in time distribution with values in  $\mathcal{D}'(\mathbb{R}^3)$ . Its partial Fourier transform in time, denoted  $\hat{u}(x, k)$ , then exists and we verify that

$$\begin{aligned} \langle \Delta \hat{u}(x, k), \phi(x) \rangle &= \langle \hat{u}(x, k), \Delta \phi(x) \rangle = \int e^{ikt} \langle u(x, t), \Delta \phi(x) \rangle dt \\ &= \int e^{ikt} \langle \Delta u(x, t), \phi(x) \rangle dt = \int e^{ikt} \langle \partial_{t^2}^2 u(x, t), \phi(x) \rangle dt \\ &= \int -k^2 e^{ikt} \langle u(x, t), \phi(x) \rangle dt = -k^2 \langle \hat{u}(x, k), \phi(x) \rangle \end{aligned}$$

so  $(\Delta + k^2)\hat{u}(x, k) = 0$ . This equation is called the scalar Helmholtz equation.

Conversely, if  $\hat{u}(x, k)$  is a solution distribution of the Helmholtz equation, then

$$u(t, x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ikt} \hat{u}(x, k) dk$$

is a solution of the wave equation. An asymptotic solution of the scalar Helmholtz equation is an oscillating function in  $k$ , of the form

$$a(x, k)e^{ik\phi(x)},$$

such that  $a(x, k) \simeq \sum_{j=0}^{\infty} a_j(x)(ik)^{-j}$ . We consider the case where  $a_0$  is not identically zero, which amounts to studying the non-zero solutions (indeed, if there is no  $j_0$  such that  $a_{j_0}$  is not identically zero, then all terms of the asymptotic expansion of  $a$  are zero, i.e.  $a \simeq 0$ , and if there exists  $j_0$  such that, for  $j < j_0$   $a_j = 0$  and  $a_{j_0}$  not identically zero, we can divide  $u(x, k)$  by  $(ik)^{-j_0}$  and then we are back, since the equation is linear, to a solution whose dominant term is non-zero). The notion of asymptotics is understood when  $k$  tends to infinity, and we then speak of an increasingly oscillating solution. In fact, the only result that this asymptotic construction gives directly is, if  $a_M(x, k) = \sum_{j=0}^{j=M} k^{-j} a_j(x)$

$$(4.9) \quad (\Delta + k^2)(a(x, k)e^{ik\phi(x)}) = O(k^{-\infty})$$

or

$$(\Delta + k^2)(a(x, k)e^{ik\phi(x)}) \simeq 0.$$

We first prove the

PROPOSITION 1.3. *If  $ae^{ik\phi}$  is an asymptotic solution, then*

$$(4.10) \quad \begin{cases} a_0(1 - |\text{grad}\phi|^2) = 0 \\ a_1(1 - |\text{grad}\phi|^2) + 2\text{grad}a_0\text{grad}\phi + (\Delta\phi)a_0 = 0 \\ a_j(1 - |\text{grad}\phi|^2) + 2\text{grad}a_{j-1}\text{grad}\phi + (\Delta\phi)a_{j-1} + \Delta a_{j-2} = 0, j \geq 2. \end{cases}$$

<sup>1</sup>Here, the word initial is an abuse of language which will be justified in the chapter 4 when we will study the analytical conormal waves

Introducing the operators  $L_1$  and  $L_2$  defined on  $C^{infy}(\mathbb{R}^n)$  by  $L_1(u) = 2\text{grad}\phi.\text{grad}u + (\Delta\phi)u$  and  $L_2(u) = \Delta u$ , (the index of each operator refers to its order) and the function  $L_0$  on the vector fields in  $\mathbb{R}^n$  such that  $L_0(V) = 1 - |V|^2$ . System (4.10) writes

$$\begin{cases} L_0(\text{grad}\phi) = 0 \\ L_1(a_0) = 0 \\ L_1(a_j) = -L_2(a_{j-1}). \end{cases}$$

The first equality is called the **eikonal** equation.

The second equality is called the **transport** homogeneous.

The third equality is the same transport equation, but inhomogeneous.

PROOF. For the moment, this is only formal asymptotics. By considering the classical form of the Laplacian operator (i.e.  $\Delta = \text{div}(\text{grad})$ ), we verify, given the relations

$$\begin{aligned} \text{grad}(fg) &= f\text{grad}(g) + g\text{grad}(f) \\ \text{div}(fV) &= \text{grad}f.V + f\text{div}(V) \end{aligned}$$

that

$$\begin{aligned} \Delta(ae^{ik\phi}) &= \text{div}(e^{ik\phi}(\text{grad}a + ik\text{grad}\phi)) \\ &= [ik\text{grad}\phi.(\text{grad}a + ik\text{grad}\phi) + \text{div}(\text{grad}a + ik\text{grad}\phi)]e^{ik\phi} \end{aligned}$$

or

$$(4.11) \quad \Delta(ae^{ik\phi}) = e^{ik\phi}(\Delta a + ik(2\text{grad}a.\text{grad}\phi + a\Delta\phi) - k^2|\text{grad}\phi|^2 a).$$

Equality :

$$(\Delta + k^2)(a(x, k)e^{ik\phi(x)}) = O(k^{-\infty}) \simeq 0$$

implies (4.10).

It is clear that when  $a_0$  is not identically zero, then  $|\text{grad}\phi|$  is of norm 1 on the support of  $a_0$ . We then solve the other equalities on the support of  $a_0$ , and we obtain

$$2\text{grad}a_0.\text{grad}\phi + (\Delta\phi)a_0 = 0,$$

then

$$2\text{grad}a_{j-1}.\text{grad}\phi + (\Delta\phi)a_{j-1} + \Delta a_{j-2} = 0.$$

We will make frequent use of these equations. A treatment of the transport equations in the scalar case will be done in the chapter 3.  $\square$

REMARK 1. When  $\phi$  is a solution of the eikonal equation  $|\nabla\phi|^2 = 1$ , the value of  $\phi$  on the surface  $\Sigma_0$ , and the condition

$\nabla\phi(x)$ ,  $x \in \Sigma_0$  is not tangent to  $\Sigma_0$  at  $x$

allow to determine locally (in a generic case) the  $\phi$  phase.

Indeed, we are in the case where the integral curves of the vector field  $\nabla\phi$  are transverse to  $\Sigma_0$ . We parametrize the surface  $\Sigma_0$  by a unitary system of local coordinates  $u_1, \dots, u_{d-1}$ . The phase  $\phi$  restricted to  $\Sigma_0$  is then given, in the neighborhood of  $x_0$ , by  $\psi(u)$ . The condition of transversality is equivalent to  $|\nabla_u\phi| < 1$ . The equation  $\phi(x) = \phi(x_0)$  can be solved locally according to the data  $\psi$ . Let us note  $\Sigma = x, \phi(x) = \phi(x_0)$ .

Let us then give an interpretation of  $\Sigma_0$ . We suppose that the function  $u(x, k) = a(x, k)e^{ik\phi(x)}$  is holomorphic in  $k$  in the complex half-plane  $\text{Im}k < 0$ . By considering the inverse Fourier transform in time (and applying the Paley-Wiener-Schwartz theorem) we verify that  $v(x, t) = \mathcal{F}^{-1}(u)(x, -\phi(x_0))$  is supported on only one side of  $\Sigma$ . Indeed, we can deform the integration contour of the inverse Fourier transform into  $\mathbb{R} - ia$ ,  $a > 0$ . Since the result is independent of  $a$  (by holomorphy) and the limit when  $a$  tends to  $+\infty$  is zero for any  $(x, t)$  such that  $\phi(x) + t < 0$ , we find that  $v$  is zero at a point of  $\phi(x) + t < 0$ . We will see that this amounts to write Cauchy conditions on  $v$  and on its derivative at  $t = -\phi(x_0)$ .

### 5. Generalization of the asymptotic methods to differential operators with non constant coefficients.

We replace here  $\Delta$  by a differential operator  $P(x, \partial_x)$  with real coefficients  $C^\infty$  of order 2, which we write :

$$(5.12) \quad P(x, \partial_x) = \sum_{j,l} a_{jl}(x) \frac{\partial^2}{\partial x_j \partial x_l} + \sum_j b_j(x) \frac{\partial}{\partial x_j} + c(x).$$

We introduce the differential part of  $P(x, \partial_x)$ , noted  $P^b(x, \partial_x)$  (following Taylor's notation [94]), equals to

$$(5.13) \quad P^b(x, \partial_x) = P(x, \partial_x) - c(x).$$

The propagation problem is called hyperbolic (and the operator  $P - \partial_{t^2}^2$  is then said to be hyperbolic) when  $A$  is a positive definite bilinear form. To write the asymptotic expansion we do not need this ellipticity assumption on  $A$ , but we will need it to prove that the asymptotic expansion yields a solution. We associate to  $A$  its canonical bilinear form, at any point  $x$  :

$$(5.14) \quad A(\xi, \eta) = \sum_{j,l} a_{jl}(x) \xi_j \eta_l.$$

The relation

$$\frac{\partial^2}{\partial x_j \partial x_l} (a(x, k) e^{ik\phi(x)}) = \left[ \frac{\partial^2 a}{\partial x_j \partial x_l} + ik \left( \frac{\partial \phi}{\partial x_l} \frac{\partial a}{\partial x_j} + \frac{\partial \phi}{\partial x_j} \frac{\partial a}{\partial x_l} \right) + ik a \partial_{x_j x_l}^2 \phi - k^2 \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_l} a \right] e^{ik\phi}$$

allows to write

$$P(a e^{ik\phi}) = [Pa + ik[2A(\text{grad}\phi, \text{grad}a) + P^b(\phi)a] - k^2 a A(\text{grad}\phi, \text{grad}\phi)] e^{ik\phi}.$$

The first order transport operator associated to the bilinear form is :

$$(5.15) \quad L_1(a) = 2A(\text{grad}\phi, \text{grad}a) + P^b(\phi)a.$$

We verify that the eikonal and transport equations associated à  $P + k^2$  are

$$\begin{cases} 1 - A(\text{grad}\phi, \text{grad}\phi) = 0 \\ L_1(a_0) = 0 \\ L_1(a_j) = -P(a_{j-1}). \end{cases}$$

We notice that the operator  $L_1$  is written  $\frac{\partial A(x, \xi, \xi)}{\partial \xi} |_{\xi = \text{grad}\phi} \cdot \frac{\partial}{\partial x} + P^b(\phi)$ .

We summarize the results

LEMMA 1.2. *Let  $P$  be the operator defined by (5.12). We associate to it the bilinear form  $A$  (5.14), the differential part  $P^b$  of  $P$  (5.14) and the transport operator  $L_1$  (5.15). We have the equality*

$$e^{-ik\phi(x)} P(a(x, k) e^{ik\phi(x)}) \simeq k^2 (1 - A(\text{grad}\phi, \text{grad}\phi)) a(x, k) + ik L_1(a_0) + \sum_{j=0} (ik)^{-j+1} (L_1(a_j) + P(a_{j-1}))$$

when  $a(x, k) \simeq \sum_{j=0} a_j(x) (ik)^{-j}$ .

In the case where the matrix  $a_{ij}(x)$  is positive symmetric, the operator  $P(x, \partial_x)$  is, except for a vector field and a constant function, a Laplacian operator as introduced in exercise 1.2. We consider the metric  $g_{ij}(x) = a_{ij}(x)$ . The Laplacian associated to this metric is

$$\begin{aligned} \Delta_a f &= (\det a(x))^{-\frac{1}{2}} \sum_j \frac{\partial}{\partial x_j} \left[ \sum_l a_{lj}(x) (\det a(x))^{\frac{1}{2}} \frac{\partial f}{\partial x_l} \right] \\ &= \sum_l (\det a(x))^{-\frac{1}{2}} \sum_j \frac{\partial}{\partial x_j} \left[ a_{lj}(x) (\det a(x))^{\frac{1}{2}} \frac{\partial f}{\partial x_l} \right] + \sum_{l,j} a_{lj}(x) \frac{\partial^2 f}{\partial x_l \partial x_j}. \end{aligned}$$

The operator  $P - \Delta_a$  is an operator of order 1, associated to a vector field  $V$ , such that  $a$ ,  $V$ , and  $c$  are uniquely defined by  $P^b - \Delta_a = V.\text{grad} + c$ .

### 6. The harmonic Maxwell equations

In this section, we study the asymptotic solution of a system of coupled partial differential equations: the system of harmonic Maxwell equations. The use of the term ‘‘harmonic’’ comes from the fact that we considered the Fourier transform in time of the solution of the system of Maxwell’s equations in  $\mathbb{R}^3 \times \mathbb{R}_t$ .

We thus consider the equations ( $\varepsilon, \mu$  independent of the position,  $\omega$  denotes the frequency or Fourier variable in time)

$$\begin{cases} \text{rot}E + i\omega\mu H = 0, \\ \text{rot}H - i\omega\varepsilon E = 0, \\ \text{div}E = 0, \\ \text{div}H = 0. \end{cases}$$

LEMMA 1.3. *In the representation*

$$E_j(x) = e_j(x, \omega)e^{i\omega\phi(x)}, H_j(x) = h_j(x, \omega)e^{i\omega\phi(x)},$$

the eikonal equation associated with this system is

$$|\text{grad}\phi|^2 = \varepsilon\mu,$$

and the equations for the principal term are

$$\begin{cases} \text{grad}\phi \wedge \text{rote}^0 - \mu\text{roth}^0 - \text{dive}^0\text{grad}\phi = 0 \\ \text{grad}\phi \wedge \text{roth}^0 - \text{div}h^0\text{grad}\phi + \varepsilon\text{rote}^0 = 0. \end{cases}$$

We deliberately choose in these expressions not to consider the Helmholtz equation consider the vector Helmholtz equation obtained directly by replacing in the equality  $\text{rot}E + i\omega\mu H = 0$   $E = (i\omega\varepsilon)^{-1}\text{rot}H$  then using the zero divergence condition to have  $\text{rot}\text{rot}E = \Delta E - \text{grad}\text{div}E$ .

We present in the chapter 12 dedicated to the properties of Maxwell’s equations written in intrinsic form the calculations of direct asymptotic expansions obtained from the equations of Lemma 1.3.

LEMMA 1.4. *The system of Maxwell’s equations leads to the following system for the leading order term:*

$$(6.16) \quad \begin{cases} \text{grad}\phi \wedge e^0 + \mu h^0 = 0, \\ \text{grad}\phi \wedge h^0 - \varepsilon e^0 = 0, \\ e^0.\text{grad}\phi = 0, \\ h^0.\text{grad}\phi = 0, \\ \text{rote}^j + (\text{grad}\phi \wedge e^{j+1} + \mu h^{j+1}) = 0, \\ \text{roth}^j + (\text{grad}\phi \wedge h^{j+1} - \varepsilon e^{j+1}) = 0, \\ \text{dive}^j + e^{j+1}.\text{grad}\phi = 0, \\ \text{div}h^j + h^{j+1}.\text{grad}\phi = 0. \end{cases}$$

PROOF. As  $\text{rot}(fe^{i\omega\phi}) = (\text{rot}f + i\omega\text{grad}\phi \wedge f)e^{i\omega\phi}$  and as  $\text{div}(fe^{i\omega\phi}) = (\text{div}f + i\omega f.\text{grad}\phi)e^{i\omega\phi}$ , we obtain

$$(6.17) \quad \begin{cases} \text{rote} + i\omega(\text{grad}\phi \wedge e + \mu h) = 0, \\ \text{roth} + i\omega(\text{grad}\phi \wedge h - \varepsilon e) = 0, \\ \text{div}h + i\omega h.\text{grad}\phi = 0, \\ \text{dive} + i\omega e.\text{grad}\phi = 0. \end{cases}$$

The first equation implies  $i\omega\mu h = -\text{rote} - i\omega\text{grad}\phi \wedge e$ . By multiplying the second equation by  $i\omega\mu$ , we find

$$\omega^2 \mu \varepsilon e + i\omega \text{grad}\phi \wedge (i\omega \mu h) + \text{rot}(i\omega \mu h) = 0.$$

We then deduce

$$\omega^2 \mu \varepsilon e - i\omega \text{grad}\phi \wedge (\text{rote} + i\omega \text{grad}\phi \wedge e) - \text{rot}(\text{rote} + i\omega \text{grad}\phi \wedge e) = 0.$$

Using the equality

$$(6.18) \quad \text{grad}\phi \wedge (\text{grad}\phi \wedge e) = -|\text{grad}\phi|^2 e + (e \cdot \text{grad}\phi) \text{grad}\phi$$

we find

$$\omega^2 [\varepsilon \mu e - |\text{grad}\phi|^2 e + (e \cdot \text{grad}\phi) \text{grad}\phi] - i\omega \text{grad}\phi \wedge \text{rote} - \text{rot}(\text{rote} + i\omega \text{grad}\phi \wedge e) = 0.$$

By replacing  $(e \cdot \text{grad}\phi)$  with the fourth equation of the system (6.17), we obtain

$$\omega^2 [\varepsilon \mu e - |\text{grad}\phi|^2 e] + i\omega [\text{div}\text{grad}\phi - \text{grad}\phi \wedge \text{rote} - \text{rot}(\text{grad}\phi \wedge e)] - \text{rot}\text{rote} = 0.$$

When we assume that

$$e = \sum e^j (i\omega)^{-j}, h = \sum h^j (i\omega)^{-j},$$

with  $e^0 \neq 0$ , we deduce the eikonal equation of the lemma 1.3 by cancelling the highest degree term in  $\omega$  of this equality, after having assumed that  $e$  admits an asymptotic expansion.

Obtaining the eikonal equation by this method, as we can see, is relatively easy. The relations between the terms  $e$  and  $h$ , on the other hand are more difficult to obtain owing to these substitution methods. We use a direct method, replacing by their asymptotic expansion of the quantities  $e$  and  $h$  in the system before substitution. We assume that  $e$  and  $h$  give identical contributions in powers of  $\omega$ . If this were not the case, then we would have either  $\text{grad}\phi \wedge e = 0$ , or  $h = 0$  from the first equation, depending on whether  $e$  dominates  $h$  or the other way around. If  $e$  is larger than  $h$ , then, according to the second equation,  $e = 0$ , which gives a null solution. So  $e$  and  $h$  have a principal term of the same order.

The relation (6.18) allows to decompose  $e$  using the unit vector  $t = \frac{\text{grad}\phi}{|\text{grad}\phi|}$  :

$$e = (e \cdot t)t - t \wedge (t \wedge e).$$

It is therefore logical to consider a system where  $t \wedge e$  and  $e \cdot t$  are known simultaneously to compute the solution. By calculating  $\text{grad}\phi \wedge (\text{grad}\phi \wedge (e, h))$  as given by the above system, we obtain as a system from the first two relations

$$\begin{pmatrix} 0 & \mu \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \text{grad}\phi \wedge e^{j+1} \\ \text{grad}\phi \wedge h^{j+1} \end{pmatrix} - |\text{grad}\phi|^2 \begin{pmatrix} e^{j+1} \\ h^{j+1} \end{pmatrix} = \begin{pmatrix} \text{div}e^j \\ \text{div}h^j \end{pmatrix} \cdot \text{grad}\phi - \begin{pmatrix} \text{grad}\phi \wedge \text{rote}^j \\ \text{grad}\phi \wedge \text{roth}^j \end{pmatrix}.$$

We notice that  $e_0$  and  $h_0$  are orthogonal to  $\text{grad}\phi$  then using (6.18) to  $(\varepsilon \mu - |\text{grad}\phi|^2)h^0 = 0$ . If  $h^0$  were equal to 0, then  $e^0$  would be also equal to 0. We find the eikonal equation :

$$|\text{grad}\phi|^2 = \varepsilon \mu$$

Using the eikonal equation, the first term of the system is rewritten as

$$(6.19) \quad e^0 = \varepsilon^{-1} \text{grad}\phi \wedge h^0 = \sqrt{\frac{\mu}{\varepsilon}} t \wedge h^0.$$

Note that, in this case, the cancellation of the 0 order term leads to the eikonal equation as well as to a relation between the first terms of  $e$  and  $h$ . The transport equation is obtained by cancelling the term term of next order. Indeed, if we consider the equations relating  $\text{rote}^0, \text{roth}^0$  to  $e^1, h^1$ , and making  $\text{grad}\phi \wedge$  on these equations, we find

$$\text{grad}\phi \wedge \text{rote}^0 + \text{grad}\phi \wedge (\text{grad}\phi \wedge e^1) + \mu \text{grad}\phi \wedge h^1 = 0,$$

which gives, by replacing  $\text{grad}\phi \wedge h^1$  by  $\varepsilon e^1 - \text{rot}h^0$ , the equation (obtained by using the relation  $\text{div}e^0 + e^1 \text{grad}\phi = 0$ , the eikonal equation and (6.18))

$$(6.20) \quad \begin{cases} \text{grad}\phi \wedge \text{rote}^0 - \mu \text{rot}h^0 - \text{div}e^0 \text{grad}\phi = 0 \\ \text{grad}\phi \wedge \text{rot}h^0 - \text{div}h^0 \text{grad}\phi + \varepsilon \text{rote}^0 = 0 \end{cases}$$

These two equations are the transport equations on  $e^0, h^0$ . We proved Lemma 1.3.  $\square$

To get back to the classical transport equations (each coordinate is a solution of a scalar Helmholtz equation, and the successive equations on each term of the asymptotic expansion are given by (4.10)), it is sufficient to replace the relation  $\mu h^0 = -\text{grad}\phi \wedge e^0$  in the first equation of (6.20). This gives

$$\text{grad}\phi \wedge \text{rote}^0 + \text{rot}(\text{grad}\phi \wedge e^0) - \text{div}e^0 \text{grad}\phi = 0.$$

We verify that  $h^0$  follows the same transport equation as  $e^0$ .

A simple but tedious calculation leads to find, for the first coordinate  $A_1$  of  $\text{grad}\phi \wedge \text{rote}^0 + \text{rot}(\text{grad}\phi \wedge e^0)$ :

$$\begin{aligned} A_1 &= \frac{\partial\phi}{\partial x_2} \left( \frac{\partial e_1^0}{\partial x_2} - \frac{\partial e_2^0}{\partial x_1} \right) - \frac{\partial\phi}{\partial x_3} \left( \frac{\partial e_3^0}{\partial x_1} - \frac{\partial e_1^0}{\partial x_3} \right) + \frac{\partial}{\partial x_2} \left( \frac{\partial\phi}{\partial x_1} e_2^0 - \frac{\partial\phi}{\partial x_2} e_1^0 \right) - \frac{\partial}{\partial x_3} \left( \frac{\partial\phi}{\partial x_3} e_1^0 - \frac{\partial\phi}{\partial x_1} e_3^0 \right) \\ &= \frac{\partial}{\partial x_1} (\text{grad}\phi \cdot e^0) - \Delta\phi e_1^0 + \text{div}e^0 \frac{\partial\phi}{\partial x_1} - 2\text{grad}\phi \cdot \frac{\partial e^0}{\partial x_1}. \end{aligned}$$

We deduce the transport equations for each term  $e_p^0$ ,  $p = 1, 2, 3$  :

$$[2\nabla\phi \cdot \nabla + \Delta\phi]e_p^0 = 0, e^0 \cdot \nabla\phi = 0.$$

This method, as we have seen, is tedious and not very general because the form of the operators divergence, rotational, gradient are very particular. We will see in the chapter 2 how to write in a general way such calculations, using results of Lax and Rauch.

## 7. Exercises

Exercise 1.0 : Fundamental solution of of the Helmholtz equation in  $\mathbb{R}^3$ . 1) Show that the function, defined for  $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ , equal to

$$G(x, y) = \frac{e^{ik|x-y|}}{4\pi ik|x-y|}$$

is a solution of

$$(\Delta_x + k^2)G(x, y) = \delta_{x=y}.$$

What can we say about  $H(x, y) = \frac{e^{-ik|x-y|}}{4\pi ik|x-y|}$  ?

2) Give two solutions of  $(\Delta + k^2)u(x) = f(x)$ . Interpretation?

Exercise 1.1 : asymptotic expansion in the wave equation. We consider the wave equation

$$(\Delta - \partial_{t^2}^2)u(x, t) = 0,$$

in which, let us note, there is no asymptotic parameter. Find a formal solution of this equation in the form

$$u(x, t) = a(x, t, k)e^{ik\phi(x, t)}.$$

Show that this problem is the same as finding the solution of the eikonal equation  $L_0(\text{grad}\psi) = 0$ . We will study the surfaces isophases of  $\phi$ .



Exercise 1.2 : Helmholtz equation in a space with a  $g$  metric. *In this exercise, we introduce the notion of metric Laplacian, which will be useful to study Maxwell's equations in an open space, locally in the neighborhood of an edge. This metric Laplacian is the right formulation of the wave problems; it will be taken up again in the chapters 10 and 11.*

The space  $X$  considered, included in  $\mathbb{R}^n$ , is provided with the metric  $g(x)$ . This translates, by definition, into the fact that the product scalar product on  $T_x X$ , tangent space to  $X$  at  $x$  is

$$\langle u, v \rangle_g = \sum_{ij} g_{ij}(x) u_i v_j,$$

( $g(x)$ ) being a symmetric matrix defined positive at any point  $x$  and  $(u, v)$  are two tangent vectors to  $X$  at  $x$ .

1) Extend the definition of the Laplacian by generalizing the relation

$$\int_{\mathbb{R}^n} h(x) \Delta f(x) dx = - \int (\text{grad} f \cdot \text{grad} h) dx,$$

valid in  $\mathbb{R}^n$  with the (usual) identity metric for functions  $C^\infty$  with compact support. We will note  $\Delta_g$  the Laplacian metric defined in this way.

2) Write the formal asymptotic expansion of  $\Delta_g + k^2$

Exercise 1.3 : Radon transform and Radon transform and wave equation. 1) The Radon transform is defined by  $(\omega, s) \in S^2 \times \mathbb{R}$

$$\mathcal{R}f(\omega, s) = \int_{x \cdot \omega = s} f(x) d\hat{x},$$

$\hat{x}$  being the coordinate on the hyperplane  $x \cdot \omega = s$ .

Verify that  $\mathcal{R}f(-\omega, -s) = \mathcal{R}f(\omega, s)$ , and that the Fourier transform and the Radon transform are related, for  $\omega$  of norm 1, by the relation  $\mathcal{F}f(t\omega) = \int_{\mathbb{R}} e^{-its} \mathcal{R}f(s, \omega) ds$ .

2) Compute  $\mathcal{R}^{-1}g(x)$  and  $\mathcal{R}(\Delta f)$  for  $n$  odd.

3) Write in terms of Radon transform the solution of the equation in  $\mathbb{R}^3$  and prove that if the initial data have their support in  $|x| < R$ , then the solution is null for  $|x| < |t| + R$ . Can we give an identical result with the support of the source term?

Solution of the exercise 1.0. 1) We verify that

$$\partial_{x_j} G(x, y) = \frac{x_j - y_j}{|x - y|} G(x, y) \left( ik - \frac{1}{|x - y|} \right)$$

It comes then

$$\text{partial}_{x_j^2}^2 G(x, y) = \frac{G(x, y)}{|x - y|} \left( ik - \frac{1}{|x - y|} \right) + G(x, y) \frac{x_j - y_j}{|x - y|} \frac{x_j - y_j}{|x - y|^3} + \frac{(x_j - y_j)^2}{|x - y|^2} \left( ik - \frac{1}{|x - y|} \right)^2 G(x, y).$$

Summing up, we obtain

$$\Delta_x G = G \left[ \frac{3}{|x - y|} \left( ik - \frac{1}{|x - y|} \right) - \left( ik - \frac{1}{|x - y|} \right) \frac{|x - y|^2}{|x - y|^3} + \frac{|x - y|^2}{|x - y|^3} + \left( ik - \frac{1}{|x - y|} \right)^2 \right].$$

All calculations done, we obtain  $\Delta_x G(x, y) = -k^2 G(x, y)$ . From Similarly, taking the complex conjugate in this relation,  $\Delta_x H(x, y) = -k^2 H(x, y)$  for  $x \neq y$ . The distribution  $(\Delta_x + k^2)G(x, y)$  is supported in  $x = y$ . On the other hand, we write, using the form of the Laplacian in spherical coordinates, for a test function independent of  $\theta, \phi$  :

$$\int_{\mathbb{R}^3} (\Delta_x + k^2)G(x, y)\phi(|x|)dx = \int_0^{+\infty} \frac{e^{ikr}}{4\pi ikr} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + k^2 \phi \right] 4\pi r^2 dr.$$

The explicit calculation thus gives, after integrations by parts in  $r$

$$\begin{aligned} \int_0^\infty k^2 r \phi(r) dr - \int_0^\infty r \partial_r \phi \partial_r (e^{ikr}) dr &= \int_0^\infty k^2 r \phi(r) dr - \int_0^\infty r \partial_r \phi \partial_r (e^{ikr}) dr \\ &= k^2 \int_0^\infty r \phi(r) dr - ik \int_0^\infty r \partial_r \phi e^{ikr} dr \\ &= k^2 \int_0^\infty r \phi(r) dr + ik \int_0^\infty \phi \partial_r (r e^{ikr}) dr \\ &= ik \int_0^\infty \phi e^{ikr} dr. \end{aligned}$$

The result follows from this. 2) We immediately verify that, for  $f \in L^{inf ty}(\mathbb{R}^3)$

$$u(x, k) = \int_{\mathbb{R}^3} G(x, y) f(y) dy$$

is then a solution of  $(\Delta + k^2)u = f$ , and  $v(x, k) = \int_{\mathbb{R}^3} H(x, y) f(y) dy$ .

Solution of the exercise 1.1. The relations of section 4 give

$$\Delta_x a + ik(2\nabla_x \phi \cdot \nabla_x a + a\Delta_x \phi) - k^2 |\nabla_x \phi|^2 a - \partial_{t^2} a - ik(2\partial_t \phi \cdot \partial_t a + a\partial_{t^2}^2 \phi) + k^2 (\partial_t \phi)^2 a = 0$$

The eikonal equation is, here,

$$(\partial_t \phi)^2 = |\nabla_x \phi|^2.$$

The transport operator corresponding to  $L_1$  is

$$2(\nabla_x \phi, -\partial_t \phi) \cdot (\nabla_x, \partial_t) + (\Delta - \partial_{t^2}^2)$$

We suppose that there exists a point  $(x_0, t_0)$  such that  $\phi(x_0, t_0) = 0$  and such that  $\nabla_t \phi(x_0, t_0) \neq 0$  (this represents a surface which propagates in time). We further assume that the phase is  $C^\infty$  at in the neighborhood of this point. In the neighborhood of this point, there exists a function  $\psi(x)$  such that  $\psi(x_0) = t_0$  and  $\phi(x, t) = 0 \Leftrightarrow t = \psi(x)$ .

As  $\phi(x, u + \psi(x)) = 0 \Leftrightarrow u = 0$ , the function  $a(x, u) = \int_0^1 \partial_t \phi(x, su + \psi(x)) ds$  verifies

$$\phi(x, u + \psi(x)) = ua(x, u).$$

We deduce  $\phi(x, t) = (t - \psi(x))b(x, t)$ , with  $b(x, t) = a(x, t - \psi(x))$ , and  $b(x, \psi(x)) \neq 0$  in a neighborhood of  $x_0$ .

$$\nabla_x \phi(x, t) = -\nabla_x \psi(x)b(x, t) + (t - \psi(x))\nabla_x b(x, t)$$

$$\partial_t \phi(x, t) = b(x, t) + (t - \psi(x))\partial_t b(x, t).$$

The function  $\phi$  is a solution of the eikonal equation, so

$$\begin{aligned} & b^2(x, t) + 2(t - \psi(x))b(x, t)\partial_t b(x, t) + (t - \psi(x))^2(\partial_t b(x, t))^2 \\ &= (\nabla_x \psi(x))^2 b^2(x, t) + (t - \psi(x))^2 (\nabla_x b)^2 - 2(t - \psi(x))\nabla_x b \cdot \nabla_x \psi. \end{aligned}$$

By writing this equality, true for all  $t$ , for  $t = \psi(x)$  and using  $b(x, \psi(x)) \neq 0$  in a neighborhood of  $x_0$ , we find

$$|\nabla_x \psi| = 1$$

and

$$2b\partial_t b + (t - \psi(x))(\partial_t b)^2 = -2\nabla_x b \psi + (t - \psi(x))(\nabla_x b)^2.$$

We have thus verified that  $\phi(x, t) = b(x, t)(t - \psi(x))$ , with  $|\nabla_x \psi| = 1$ . The isophase surfaces of  $\phi$  can be defined in the same way at any point  $(x_0, t_0)$  such that  $\partial_t \phi(x_0, t_0) \neq 0$ . We have the result :

For all  $(x_0, t_0)$ , there exists  $\psi_{\phi(x_0, t_0)}$  such that  $|\nabla_x \phi_a| = 1$  for all  $a$  and such that, locally in the neighborhood of  $(x_0, t_0)$  :

$$\{(x, t) \in \mathbb{R}^d \times \mathbb{R}, \phi(x, t) = \phi(x_0, t_0)\} = S_{\phi(x_0, t_0)} = \{(x, \psi_{\phi(x_0, t_0)}(x)), x \in \mathbb{R}\}.$$

Solution of the exercise 1.2. 1) The metric Laplacian  $\Delta_g$  is defined by

$$(7.21) \quad \int_{\mathbb{R}^n} (\Delta_g f)(x) h(x) d_g x = - \int_{\mathbb{R}^n} \langle \text{grad} f, \text{grad} h \rangle_g d_g x.$$

Let's see what to take for the volume form  $d_g$ . Assuming that the volume is independent by change of variable, we consider a point  $x_0$  and write the eigenvectors of the matrix  $g(x_0)$  (unitary for the identity metric on  $\mathbb{R}^n$ ), which form an orthonormal basis of  $\mathbb{R}^n$ . Writing then  $g(x_0)(u, u) = \sum \lambda_j U_j^2$ , the  $U_i$  being the coordinates of  $u$  in the proper basis of  $g(x_0)$ , the volume element volume is then  $\sqrt{\prod \lambda_i}$ , which gives, by noting  $|g|$  the determinant of the matrix  $g$

$$d_g V = |g(x_0)|^{\frac{1}{2}} dV.$$

We have thus constructed the volume element  $d_g x = |g|^{\frac{1}{2}} dx$ . This local equality is due to, among other things

- i) the fact that a diagonal matrix is associated to a volume element volume as above and  
ii) the fact that the space of  $n$  alternating linear platforms on  $\mathbb{R}^n$  is of dimension 1, so  $d_g x = \lambda dx$ .  
Using the relation (7.21), we have

$$\int_M (\Delta_g f)(x) h(x) |g(x)|^{\frac{1}{2}} dx = - \int \sum_{j,l} g_{jl}(x) \frac{\partial f}{\partial x_j} \frac{\partial h}{\partial x_l} |g(x)|^{\frac{1}{2}} dx.$$

From this, after integration by parts, we immediately derive

$$\Delta_g f(x) = |g(x)|^{-\frac{1}{2}} \sum_l \partial_{x_l} (|g(x)|^{\frac{1}{2}} \sum_j g_{jl}(x) \partial_{x_l} f).$$

We rewrite this equality using the notations  $\text{div}$  and  $\text{grad}$  for the operators  $\partial_{x_1} + \partial_{x_2} + \partial_{x_3}$  et  $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$  :

$$\Delta_g = |g(x)|^{-\frac{1}{2}} \text{div}(|g(x)|^{\frac{1}{2}} g(\text{grad} f)).$$

This formulation of the Laplacian operator is invariant by change of variable. The divergence and rotational operators associated à metric will be defined later, during the study of the Maxwell's equations.

- 2) By introducing  $a(x, k)e^{ik\phi(x)}$ , we verify that

$$\begin{aligned} e^{-ik\phi(x)} \Delta_g (a e^{ik\phi(x)}) &= |g|^{-\frac{1}{2}} \text{div} [|g|^{\frac{1}{2}} \sum_{j,l} g_{jl}(x) (\text{grad} a + ik \text{grad} \phi a)] \\ &= |g|^{-\frac{1}{2}} \text{div} [|g|^{\frac{1}{2}} g_{jl}(x) \text{grad} a] + ik |g|^{-\frac{1}{2}} \text{div} \text{grad} \phi a \\ &\quad + ik \sum_{j,l} g_{jl}(x) \text{grad} a \text{grad} \phi - k^2 \sum_{j,l} g_{jl}(x) \text{grad} \phi \text{grad} \phi \\ &\quad + ik |g|^{-\frac{1}{2}} \text{div} [|g|^{\frac{1}{2}} \sum_j g_{ij}(x) \text{grad}_j \phi] a. \end{aligned}$$

This example shows that the formal calculation of the section (5) is much faster; the canonical bilinear form associated to  $P$  is the metric form and is written  $\sum_{j,l} g_{jl}(x) \xi_j \xi_l$ . Its derivative is written as

$$\partial_{x_{ij}} (A(x, \xi, \xi)) = 2 \sum_l g_{jl}(x) \xi_l$$

As  $\Delta_g(1) = 0$ ,  $\Delta_g$  has no constant term. The result then follows. We will come back in another part of the course to the problem of obtaining an intrinsic writing for this operator and deduce a faster formulation of the eikonal and transport equations..

Solution of the exercise 1.3. 1) We write

$$\mathcal{F}f(t\omega) = \int_{\mathbb{R}^3} e^{-itx \cdot \omega} f(x) dx.$$

We can thus, for any fixed  $\omega \in S^2$ , define in an independent way independently of  $x \cdot \omega$  a measure on  $x \cdot \omega = s$ . This measure is denoted  $d\hat{x}_\omega$ , and  $dx = d\hat{x}_\omega ds$ . We check that  $\int_{x \cdot \omega = s} d\hat{x}_\omega f(x) = \mathcal{R}f(s, \omega)$ , which gives the equality we are looking for.

- 2) We then write the equality on the Fourier transform transform :

$$f(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \mathcal{F}f(\xi) d\xi.$$

We replace the Fourier transform by its expression in function of the Radon transform, writing  $\xi = \rho\theta$ ,  $\theta$  of norm 1, and then :

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^3} e^{i\rho x \cdot \theta} \int_{\mathbb{R}^3} \mathcal{F}f(\rho\theta) \rho^2 d\rho d\theta \\ &= \frac{1}{(2\pi)^3} \int_0^\infty \int_{S^2} \int_{\mathbb{R}} ds e^{i\rho(x \cdot \theta - s)} \mathcal{R}f(s, \theta) \rho^2 d\rho d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{\mathbb{R}} ds e^{i\rho(x \cdot \theta - s)} \rho^2 d\rho \frac{1}{2(2\pi)^2} \int_{S^2} d\theta \mathcal{R}f(s, \theta) \\ &= \frac{1}{2(2\pi)^2} \int_{S^2} d\theta [(-\frac{\partial^2}{\partial s^2}) \mathcal{R}f](x \cdot \theta, \theta). \end{aligned}$$

The last line of this equality comes from the fact that  $|\rho|^2$  is the symbol for the second derivative in  $s$ ; it will be clear that that in dimension  $n$ , this term is replaced by  $|\rho|^{n-1}$ , and so the equivalent operator will be the pseudodifferential operator  $(-\partial_s^2)^{\frac{n-1}{2}}$ .

The inversion formula in  $\mathbb{R}^n$  is

$$(7.22) \quad f(x) = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} d\theta [(-\frac{\partial^2}{\partial s^2})^{\frac{n-1}{2}} \mathcal{R}f](x \cdot \theta, \theta).$$

We then check that, thanks to the correspondence between the transform of Fourier transform and the Radon transform, that

$$\mathcal{F}(\partial_{x_j} f)(t\omega) = \int_{\mathbf{R}} e^{-its} \mathcal{R}(\partial_{x_j} f)(s, \omega) ds$$

On the other hand, the first term is equal to  $it\omega_j \mathcal{F}f(t\omega)$ , equal therefore to  $it\omega_j \int_{\mathbf{R}} e^{-its} \mathcal{R}f(s, \omega) ds$ . It comes

$$\omega_j \int_{\mathbf{R}} \left(-\frac{\partial}{\partial s}\right)(e^{-its}) \mathcal{R}f(s, \omega) ds = it\omega_j \int_{\mathbf{R}} e^{-its} \mathcal{R}f(s, \omega) ds = \int_{\mathbf{R}} e^{-its} \mathcal{R}(\partial_{x_j} f)(s, \omega) ds,$$

and by integration by parts on the first term

$$\omega_j \int_{\mathbf{R}} e^{-its} \left(\frac{\partial}{\partial s}\right) \mathcal{R}f(s, \omega) ds = \int_{\mathbf{R}} e^{-its} \mathcal{R}(\partial_{x_j} f)(s, \omega) ds.$$

The relation

$$\omega_j \left(\frac{\partial}{\partial s}\right) \mathcal{R}f(s, \omega) = \mathcal{R}(\partial_{x_j} f)(s, \omega)$$

allows, with  $|\omega| = 1$ , to have

$$\mathcal{R}(\Delta f)(s, \omega) = \frac{\partial^2}{\partial s^2} (\mathcal{R}f)(s, \omega).$$

3) It is then elementary to calculate the solution of the Cauchy problem. Indeed, we have

$$(\Delta - \partial_t^2)u(x, t) = 0$$

with  $u(x, 0) = u_0(x)$ ,  $\partial_t u(x, 0) = u_1(x)$ , all assumed  $L^2(\mathbb{R}^3)$ . Considering the Radon transform in  $x$  of  $u$ , denoted  $F(s, \omega, t)$ , we obtain the problem :

$$(\partial_{s^2}^2 - \partial_t^2)F(s, \omega, t) = 0$$

$$F(s, \omega, 0) = \mathcal{R}u_0(s, \omega), \partial_t F(s, \omega, 0) = \mathcal{R}u_1(s, \omega).$$

The solution of this problem is explicit; it is given by the expression (1.4)

We find

$$F(s, \omega, t) = \frac{1}{2}(\mathcal{R}u_0(t+s, \omega) + \mathcal{R}u_0(t-s, \omega)) + \frac{1}{2} \int_{t-s}^{t+s} \text{cal} \mathcal{R}u_1(z, \omega) dz.$$

The solution is then the inverse Radon transform of  $F(s, \omega, t)$ , obtained by the expression (7.22). Then we see that we makes intervene  $F(x.\theta, \omega, t)$  in the expression, thus derivatives of  $\mathcal{R}u_0(t+x.\theta, \omega)$  and of  $\mathcal{R}u_0(t-x.\theta, \omega)$  (and the same with  $\mathcal{R}u_1$ ). We suppose that the initial data are supported for  $|x| < r$ . They are null for  $|x| \geq R$ . In particular, the hyperplane  $x.\omega = s$  for  $s > R$  is outside the zone where  $u_0$  is nonzero, so

$$\text{Supp}(u_0) \subset \{|x| < R\} \Rightarrow \mathcal{R}(s, \omega) = 0, s > R.$$

The variable  $s$  considered is then either  $x.\theta - t$ , or  $x.\theta + t$ , so, as  $\theta$  is of norm 1,  $|x.\theta - t| \geq |x| - t$ , so if  $|x| > t + R$ , the transform of the initial data is zero. Thus, we verify that the solution is null for  $|x| > |t| + R$ .



## Asymptotic methods for hyperbolic systems

In this chapter, we present the results of Lax [60] which prove, under sufficient regularity assumptions, that the hyperbolic matrix problems of order 1 admit an asymptotic solution and that this asymptotic solution is quite close to the actual solution. The Lax method allows to solve in generic cases the eikonal and transport equations obtained in the case of systems. We obtained, in the lemma 1.3, that the equations on the principal term  $e^0, h^0$  of  $E, H$  and on the  $\phi$  phase are:

- the eikonal equation on  $\phi$  :

$$|\text{grad}\phi|^2 = \varepsilon\mu,$$

- a compatibility condition between the first terms of  $e$  and  $h$  :

$$e^0 = \varepsilon^{-1}\text{grad}\phi \wedge h^0$$

(equivalent to  $\mu h^0 = -\text{grad}\phi \wedge e^0$ ),

- the transport equations on the first terms

$$\begin{cases} \text{grad}\phi \wedge \text{rote}^0 - \mu\text{roth}^0 - (\text{dive}^0)\text{grad}\phi = 0 \\ \text{grad}\phi \wedge \text{roth}^0 + \varepsilon\text{rote}^0 - (\text{div}h^0)\text{grad}\phi = 0. \end{cases}$$

The situation obtained in this case is different from the one described in section 4 where no compatibility condition appears, the only equation on the first term is a partial differential equation. In the case of the system of Maxwell's equations, we find a **non-differential** relation (called the compatibility condition) and partial differential equations. This chapter clarifies and generalizes the notion of compatibility relation and the induced transport equation.

### 1. Construction of solutions of symmetric hyperbolic systems

We recall in this first paragraph a classical existence and uniqueness result for the Cauchy problem, due to Friedrichs [28]. We introduce, following Rauch [86] and Lax [60], a hyperbolic operator :

DEFINITION 2.1. *A symmetric hyperbolic matricial operator on  $\mathbb{R}^d \times \mathbb{R}_t$  is an operator of  $(C_0^\infty(\mathbb{R}^d \times \mathbb{R}_t))^m$  in itself which takes the general form*

$$Lu = A_0(x, t)\partial_t u + \sum_{j=1}^{j=d} A_j(x, t)\partial_{x_j} u + B(x, t)u.$$

The matrices  $(A_j)_{0 \leq j \leq d}$  are symmetric and verify

$$\forall \alpha \in \mathbb{N}^d, \forall p \in \mathbb{N}, \exists C_{\alpha, p}^j \in \mathbb{R}, \forall (x, t) \in \mathbb{R}^{d+1}, |\partial_x^\alpha \partial_t^p A_j(x, t)| \leq C_{\alpha, p}^j.$$

Moreover,  $A_0$  is positive definite and there exists  $c > 0$  such that  $A_0 - cId \geq 0$ .

We have the existence and uniqueness result

PROPOSITION 2.1. *Let  $L$  be a hyperbolic symmetric matricial operator. Let  $g$  be in  $H^\sigma(\mathbb{R}^d)$ ,  $f \in H_{loc}^1(\mathbb{R}, H^\sigma(\mathbb{R}^d))$  (in the case where this operator is matricial, consider  $g_1, \dots, g_m$  and  $f_1, \dots, f_m$ ,  $m$  being the number of unknown functions).*

*The problem*

$$\begin{cases} Lu = f, \\ u(x, 0) = g(x), \end{cases}$$

admits a unique solution in  $C(\mathbf{R}, H^\sigma(\mathbf{R}^d))$ , which verifies the estimate

$$\|u(t, \cdot)\|_{H^\sigma(\mathbf{R}^d)} \leq C e^{Ct} \|g\|_{H^\sigma(\mathbf{R}^d)} + \int_0^t C e^{C(t-s)} \|f(s, \cdot)\|_{H^\sigma(\mathbf{R}^d)} ds.$$

PROOF. 1) A priori estimate. Let  $v(x, t) = \partial_x^\alpha u$ . One checks that

$$A_0(x, t) \partial_t v + \sum_j A_j(x, t) \partial_{x_j} v = Lv.$$

Moreover

$$\frac{d}{dt} (A_0 v, v)_{L^2(\mathbf{R}^d)} = (\partial_t A_0 v, v)_{L^2(\mathbf{R}^d)} + (A_0 \partial_t v, v)_{L^2(\mathbf{R}^d)} + (A_0 v, \partial_t v)_{L^2(\mathbf{R}^d)}.$$

By replacing the right-hand side of the previous equality  $A_0 \partial_t v$  by  $Lv - \sum_j A_j \partial_{x_j} v$  and by integrating by parts on  $x_j$ , we find

$$(A_0 \partial_t v, v)_{L^2(\mathbf{R}^d)} = (Lv, v)_{L^2(\mathbf{R}^d)} + \sum_j (v, \partial_{x_j} (A_j v))_{L^2(\mathbf{R}^d)}.$$

The matrices  $A_j$ ,  $0 \leq j \leq d$  being symmetric, one gets

$$\frac{d}{dt} (A_0 v, v)_{L^2(\mathbf{R}^d)} = (\partial_t A_0 v, v)_{L^2(\mathbf{R}^d)} + (Lv, v)_{L^2(\mathbf{R}^d)} + (v, Lv)_{L^2(\mathbf{R}^d)} + \left( \sum_j \partial_{x_j} A_j v, v \right)_{L^2(\mathbf{R}^d)}.$$

From  $(A_0 v, v) = (A_0^{\frac{1}{2}} v, A_0^{\frac{1}{2}} v)$  and from  $(v, Lv) = (A_0^{\frac{1}{2}} v, A_0^{-\frac{1}{2}} Lv)$ , the classical Cauchy-Schwartz inequality and the inequality  $A_0^{-\frac{1}{2}} \leq c^{-\frac{1}{2}} Id$  of Definition 2.1 yield

$$(v, Lv)_{L^2(\mathbf{R}^d)} \leq c^{-\frac{1}{2}} \|Lv\|_{L^2(\mathbf{R}^d)} \|A_0^{\frac{1}{2}} v\|_{L^2(\mathbf{R}^d)} = ((A_0 v, v)_{L^2(\mathbf{R}^d)})^{\frac{1}{2}} \|Lv\|_{L^2(\mathbf{R}^d)} c^{-\frac{1}{2}}.$$

We denote by  $h(t) = (A_0 v, v)_{L^2(\mathbf{R}^d)}^{\frac{1}{2}}$ . Thanks to the assumptions of the definition,  $\partial_t A_0$  is bounded by the constant  $C_{0,1}^0$ . Introduce  $C(A) = \max_{1 \leq j \leq d} C_{1,0}^j$ . We thus obtain

$$2h(t) \frac{d}{dt} (h(t)) \leq C_{0,1}^0 c^{-1} h^2(t) + 2c^{-\frac{1}{2}} h(t) \|Lv\|_{L^2} + C(A) h^2(t),$$

which yields, when  $h(t) \neq 0$

$$h'(t) \leq C_1 h(t) + c^{-\frac{1}{2}} \|Lv\|_{L^2}.$$

Gronwall's lemma leads to

$$h(t) \leq h(0) e^{C_1 t} + c^{-\frac{1}{2}} \int_0^t \|Lv\|_{L^2(\mathbf{R}^d)}(s) e^{C_1(t-s)} ds,$$

expression valid even when  $h$  vanishes. Note that this comes from the the fact that the adjoint of  $L$  is explicit, and that  $L - L^*$  is a bounded operator.

We have thus obtained the  $L^2$  estimate :

$$(1.23) \quad \begin{aligned} \|\partial_x^\alpha u\|_{L^2(\mathbf{R}^d)}(t) &\leq c^{-\frac{1}{2}} h(t) \\ &\leq c^{-\frac{1}{2}} \|A_0^{\frac{1}{2}} \partial_x^\alpha u(0)\| e^{C_1 t} + \int_0^t c^{-1} e^{C_1(t-s)} \|L \partial_x^\alpha u\|(s) ds. \end{aligned}$$

The Poisson bracket  $[L, \partial_x^\alpha]$  is equal to  $\sum_{\beta, 1 \leq |\beta| \leq |\alpha|} C_{\alpha, \beta} \partial_x^\beta$ , hence

$$L \partial_x^\alpha \phi = \partial_x^\alpha L \phi + \sum_{\beta, 1 \leq |\beta| \leq |\alpha|} C_{\alpha, \beta}(x, t) \partial_x^\beta \phi(x, t).$$

The inequality (1.23) is rewritten

$$(1.24) \quad \begin{aligned} \|\partial_x^\alpha u\|_{L^2(\mathbf{R}^d)}(t) &\leq c^{-\frac{1}{2}} \|A_0^{\frac{1}{2}} \partial_x^\alpha u(0)\| e^{C_1 t} + \int_0^t c^{-1} e^{C_1(t-s)} \left\| \sum_{\beta} C_{\alpha, \beta}(x, s) \partial_x^\beta u \right\|(s) ds \\ &\quad + \int_0^t c^{-1} e^{C_1(t-s)} \|\partial_x^\alpha f(\cdot, s)\| ds. \end{aligned}$$

We use again the Cauchy-Schwarz inequality, and we denote by  $C^{\alpha,\beta}(s) = \max_{x \in \mathbf{R}^d} C_{\alpha,\beta}(x, s)$ . Inequality (1.24) implies

$$\begin{aligned} \|\partial_x^\alpha u\|_{L^2(\mathbf{R}^d)}(t) \leq & c^{-\frac{1}{2}} \|A_0^{\frac{1}{2}} \partial_x^\alpha u(0)\| e^{C_1 t} + \sum_\beta c^{-1} \int_0^t C^{\alpha,\beta}(s) e^{C_1(t-s)} \|\partial_x^\beta u\| ds \\ & + \int_0^t c^{-1} e^{C_1(t-s)} \|\partial_x^\alpha f(\cdot, s)\|(s) ds. \end{aligned}$$

Denote finally by  $D_N(s) = \max_{|\alpha|, |\beta| \leq N} C^{\alpha,\beta}(s)$ . One gets, summing up for  $|\alpha| \leq N$ , that

$$(1.25) \quad \|u\|_{H^N(\mathbf{R}^d)}(t) \leq c^{-\frac{1}{2}} \|A_0^{\frac{1}{2}}\|_\infty \|u\|_{H^N}(0) e^{C_1 t} + c^{-1} \int_0^t D_m(s) e^{C_1(t-s)} \|u\|_{H^N}(s) ds \\ + \int_0^t c^{-1} e^{C_1(t-s)} \|\partial_x^\alpha f(\cdot, s)\|(s) ds.$$

The inequality *a priori* on the solution in  $H^N$  is again a consequence of Gronwall's lemma.

2) Existence. Let us show the existence of all terms of the approximation by the Courant-Friedrichs-Lévy (CFL) finite difference method [28]. We introduce the operators  $D_j^h$  such that

$$D_j^h \phi(x) = \frac{1}{2h} (\phi(x + he_j) - \phi(x - he_j)).$$

We introduce the solution  $u^h$  of

$$\begin{aligned} L^h u^h &= \partial_t u^h + \sum_j A_j(x, t) D_j^h u^h = f, \\ u^h(x, 0) &= g(x). \end{aligned}$$

We verify the equality, analogous to the discrete integration by parts with the change of variable

$$(D_j^h u^h, A_j u^h) = -(u^h, D_j^h (A_j u^h)).$$

Considering  $N(t) = (u^h, u^h)_{L^2(\mathbf{R}^d)}$ , one finds

$$\begin{aligned} \frac{d}{dt} N(t) &= 2\text{Re} (L^h u^h, u^h) - \sum_j (A_j D_j^h u^h, u^h) - \sum_j (u^h, A_j D_j^h u^h) \\ &= 2\text{Re} (L^h u^h, u^h) + \sum_j (u^h, [D_j^h (A_j u^h) - A_j D_j^h u^h]). \end{aligned}$$

One also has

$$\begin{aligned} & D_j^h (A_j u^h) - A_j D_j^h u^h \\ &= \frac{A_j(x+he_j, t) - A_j(x, t)}{2h} u^h(x + he_j, t) - \frac{A_j(x-he_j, t) - A_j(x, t)}{2h} u^h(x - he_j, t), \end{aligned}$$

which, using the Taylor formula with integral remainder, is written

$$D_j^h (A_j u^h) - A_j D_j^h u^h = \frac{1}{2} \int_0^1 \partial_j A_j(x + she_j, t) u^h(x + he_j, t) + \frac{1}{2} \int_0^1 \partial_j A_j(x - she_j, t) ds u^h(x - he_j, t).$$

We deduce the majoration  $L^2$ , after the inequality of Cauchy-Schwarz inequality on  $u^h$  and  $u^h(x + he_j)$  :

$$|(u^h, [D_j^h (A_j u^h) - A_j D_j^h u^h])| \leq \|u^h\|^2 \|\partial_j A_j\|_\infty.$$

This together with the inequality

$$|2\text{Re} (L^h u^h, u^h)| \leq 2 \|L^h u^h\|_{L^2} \|u^h\|_{L^2}$$

yields

$$\frac{d}{dt} \|u^h\|(t) \leq \|f\|_{L^2} + \frac{1}{2} \left( \sum_j \|\partial_j A_j\|_\infty \right) \|u^h\|_{L^2}.$$

We get the following  $L^2$  estimate :

$$\|u^h\|_{L^2}(t) \leq \|g\|_{L^2} e^{\frac{1}{2}(\sum_j \|\partial_j A_j\|_\infty)t} + \int_0^t e^{\frac{1}{2}(\sum_j \|\partial_j A_j\|_\infty)(t-s)} \|f\|_{L^2(\mathbf{R}^d)}(s) ds.$$

Note that the constant  $C = \sum_j \|\partial_j A_j\|_\infty$  is independent of  $h$ . An similar proof (left to the reader) then shows the regularity  $L_{loc}^\infty(\mathbf{R}, H^N(\mathbf{R}^d))$ .



For  $s = 1$ , we obtain a solution that is in  $L_{loc}^\infty(\mathbb{R}, H^1(\mathbb{R}^d))$ . We deduce that  $\sum_j A_j(x, t) \partial_{x_j} u \in L_{loc}^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$ , hence

$$\partial_t u \in L_{loc}^\infty(\mathbb{R}, L^2(\mathbb{R}^d)).$$

Let us write  $u(x, t) = u(x, 0) + \int_0^t \partial_t u(x, s) ds$ . On the compact set  $[0, t]$ ,  $\partial_t u(x, s)$  is bounded and one has

$$\sup_{s \in [0, t]} \|\partial_t u(x, s)\|_{L^2(\mathbb{R}^d)} < +\infty.$$

We deduce the local continuity in time of  $u(x, t)$ . The inequality

$$\|u\|_{L^2(\mathbb{R}^d)}(t) \leq \|u(x, 0)\|_{L^2(\mathbb{R}^d)} + t \sup_{s \in [0, t]} \|\partial_t u(x, s)\|_{L^2(\mathbb{R}^d)}$$

implies

$$u \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^d)).$$

It is easy to obtain that  $u$  belongs to  $\mathcal{C}(\mathbb{R}, H^\sigma(\mathbb{R}^d))$  thanks to  $u \in L_{loc}^\infty(\mathbb{R}, H^{s+1}(\mathbb{R}^d))$ .

3) Uniqueness. We prove the uniqueness by duality.

Let  $L = \partial_t + G$ , then  ${}^t L = -\partial_t + {}^t G$ . We introduce for any  $\psi$ , rapidly decaying function (of the Schwartz class) a solution of the problem

$${}^t L v = \psi, v|_{t=T} = 0.$$

This solution exists, as we have seen, and we have, for a solution  $u$  of  $Lu = 0$ ,  $u|_{t=0} = 0$ ,

$$\begin{aligned} \int_0^T (\psi, u) dt &= \int_0^T ({}^t L v, u) dt \\ &= \int_0^T (-\partial_t v + {}^t G v, u) dt \\ &= \int_0^T ((-\partial_t v, u) + (v, G u)) dt \\ &= -\int_0^T ((\partial_t v, u) + (u, \partial_t v)) dt \\ &= (u(\cdot, 0), v(\cdot, 0)) - (u(\cdot, T), v(\cdot, T)) = 0. \end{aligned}$$

This holds for any function  $\psi$  in the Schwartz class. As  $u$  belongs to the space  $C(\mathbb{R}, H^\sigma(\mathbb{R}^d))$ , we deduce that there exists, by density, a sequence  $\psi_n$  of  $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$  converging to  $u$ . One has thus  $u = 0$ . This ends the proof of Proposition 2.1.  $\square$

## 2. Asymptotic procedure for hyperbolic systems

We show in this section that we can extend to hyperbolic systems the notion of asymptotic solutions, in particular with the introduction of the eikonal equation. Let  $L(x, t, \partial_x, \partial_t)$  a hyperbolic operator (Definition 2.1), acting on distributions  $u(x, t) \in (\mathcal{D}')^m(\mathbb{R}^d \times \mathbb{R})$ . We introduce a function  $u(x, t, \varepsilon)$  (called Ansatz, which means in German: Conjecture) in the form

$$(2.26) \quad u(x, t, \varepsilon) = a(x, t, \varepsilon) e^{i\phi(x, t)/\varepsilon},$$

where there exists a sequence  $a_j(x, t)$  such that

$$a(x, t, \varepsilon) \simeq \sum_j a_j(x, t) \varepsilon^j.$$

Indeed, we conjecture that the hyperbolic system admits a solution of of this form. This function is supposed to verify :

$$(2.27) \quad L(x, t, \partial_x, \partial_t) u(x, t, \varepsilon) \simeq 0.$$

Note that one sometimes writes  $u(x, t, \varepsilon) = A(x, t, \varepsilon, \theta)|_{\theta = \varepsilon^{-1}\phi(x, t)} = a(x, t, \varepsilon) e^{i\theta}$  where, under the classical notations of Joly-Metivier-Rauch [51],  $y$  is the ‘‘fast’’ variable and  $(x, t)$  are the ‘‘slow’’ variables.

Introduce the functions  $W_j$ ,  $j \geq -1$ , through :

$$\begin{cases} W_{-1} = iL(x, t, \nabla_x \phi, \partial_t \phi) a_0, \\ W_j = iL(x, t, \nabla_x \phi, \partial_t \phi) a_{j+1} + L(x, t, \partial_x, \partial_t) a_j, \quad j \geq 0. \end{cases}$$

We denote by  $\pi(x, t)$  is the orthogonal projection on  $\text{Ker}(L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t)))$ . We notice that

$$L(x, t, \partial_x, \partial_t)(a(x, t, \varepsilon)e^{i\phi(x, t)/\varepsilon}) \simeq \sum_{j=-1}^{\infty} \varepsilon^j W_j(x, t)e^{i\phi(x, t)/\varepsilon},$$

PROPOSITION 2.2. *The necessary conditions for  $u$  to satisfy the equation (2.27) are the following*

(1) *equations:*

$$\begin{cases} L(x, t, \nabla_x \phi, \partial_t \phi)a_0(x, t) = 0, \\ iL(x, t, \nabla_x \phi, \partial_t \phi)a_{j+1}(x, t) + L(x, t, \partial_x, \partial_t)a_j(x, t) = 0, \quad j \geq 0. \end{cases}$$

(2) *The eikonal equation satisfied by  $\phi$  is*

$$(2.28) \quad \det(L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t))) = 0$$

(3) *On a leaf of the manifold (2.28), necessary conditions on  $a_0$  are*

$$(2.29) \quad \forall(x, t) \in V, a_0(x, t) \in \text{Ker}(L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t))),$$

$$(2.30) \quad \forall(x, t) \in V, L(x, t, \partial_x, \partial_t)a_0(x, t) \in \text{Im}L(\nabla \phi(x, t)).$$

(4) *Introduce  $a^0(x)$  which satisfies  $\pi(x, t_0)a^0(x) = a^0(x)$  for all  $x$  (in the suitable neighborhood). Let  $a_0$  be the unique solution of the symmetric hyperbolic system*

$$(2.31) \quad [\pi(x, t)L(x, t, \partial_x, \partial_t)\pi(x, t) + (I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))]a_0(x, t) = 0, a_0(x, t_0) = a^0(x).$$

*Then  $L(x, t, \nabla_x \phi, \partial_t \phi)a_0(x, t) = 0$  and  $L(x, t, \partial_x, \partial_t)a_0 \in \mathfrak{S}L(x, t, \nabla_x \phi, \partial_t \phi)$ .*

PROOF. At first glance, it seems that the equation  $W_{-1} = 0$  is the same as for the the scalar problem. But it is not so : indeed in the scalar case this equality implied that  $L(x, t, \nabla_x \phi, \partial_t \phi)$  is zero, because it was a number (which gave the eikonal equation). On the other hand in the vector case this equation has a non-trivial solution in  $a_0$  when the determinant is zero, and in this case  $a_0$  belongs to the kernel of the matrix  $L(x, t, \nabla_x, \partial_t)$ . We can call the equation (2.28) the generalized eikonal equation. The equation (2.28) is an equation of degree  $m$  in  $\partial_t \phi$ . This equation has therefore, in general,  $m$  roots. The generalized eikonal equation thus corresponds to  $m$  leaves of the manifold and on each leaf we have a system of transport equations.

Example of Maxwell's equations. Let us treat an example to show the difference between the system of hyperbolic equations and scalar equations. In the case of the system of Maxwell's equations seen in the section 1.3, we obtain

$$\det K = \det \begin{pmatrix} & & & 0 & \partial_{x_3} \phi & -\partial_{x_2} \phi \\ & \varepsilon \partial_t \phi I_3 & & -\partial_{x_3} \phi & 0 & \partial_{x_1} \phi \\ & & & \partial_{x_2} \phi & -\partial_{x_1} \phi & 0 \\ 0 & -\partial_{x_3} \phi & \partial_{x_2} \phi & & & \\ \partial_{x_3} \phi & 0 & -\partial_{x_1} \phi & & \mu \partial_t \phi I_3 & \\ -\partial_{x_2} \phi & \partial_{x_1} \phi & 0 & & & \end{pmatrix} = 0.$$

After elementary calculations (based on the calculation of  $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - BA^{-1}C)$ ), we find

$$\det K = \det((\varepsilon \mu (\partial_t \phi)^2 - (\nabla \phi)^2)I_3 + (\nabla \phi)(\nabla \phi)^t) = \varepsilon \mu (\partial_t \phi)^2 [\varepsilon \mu (\partial_t \phi)^2 - (\nabla \phi)^2].$$

The eikonal equation is equivalent to the three equations

$$\partial_t \phi = c|\nabla \phi|, \partial_t \phi = -c|\nabla \phi|, \partial_t \phi = 0.$$

Each is a leaf of the characteristic manifold.

Let us now study the equations on  $a_0$ .

For a given  $\phi$ , such that, in  $(x_0, t_0)$ ,  $\text{Ker}(L(x_0, t_0, \nabla_x \phi(x_0, t_0), \partial_t \phi(x_0, t_0)))$  is not reduced to  $\{0\}$ , we assume that there exists a neighborhood  $V$  of  $(x_0, t_0)$  such that the codimension of  $\text{Ker}(L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t)))$  is constant. For simplicity the matrix  $L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t))$  is denoted in the sequel by  $L(\partial\phi(x, t))$ .

We notice that the condition on  $a_0$  is not a differential condition on  $a_0$ , but a relation between the different coefficients. This relation is therefore a generalization of the compatibility condition obtained in section 6. Remark that it is not enough to determine  $a_0$ . We had already noticed these two facts for the first term corresponding to the Maxwell harmonic equations. Denote by  $L(\nabla\phi(x, t))$  the matrix  $L(x, t, \nabla_x \phi(x, t), \partial_t \phi(x, t))$ , equal to  $\frac{\partial\phi}{\partial t} A_0(x, t) + \sum_{j=1}^{j=d} \frac{\partial\phi}{\partial x_j} A_j(x, t)$ .

We determine  $a_0$  by adding to the compatibility relations (2.29) the differential equations that we obtain by canceling the next term of the asymptotic expansion  $W_0$ . Indeed, as  $L(\partial\phi(x, t))$  is non injective, the relation  $L(x, t, \partial_x, \partial_t)a_0(x, t) = -iL(\partial\phi(x, t))a_1(x, t)$  implies

The two relations(2.29) et (2.30) form a system of  $m$  equations with  $m$  unknowns, consisting of

$$m - \dim(\text{Ker}(L(\partial\phi(x_0, t_0))))$$

differential equations of order 1 and

$$\dim(\text{Ker}(L(\partial\phi(x_0, t_0))))$$

relations for  $a_0$ . We introduce  $\pi(x, t)$  the orthogonal projection on  $L(\partial\phi(x, t))$ .

LEMMA 2.1. *We have the relations  $\text{Im}\pi = \text{Ker}L(\partial\phi(x, t))$  and  $\text{Ker}\pi = \text{Im}L(\partial\phi(x, t))$ . The restriction of  $L(\partial\phi(x, t))$  to  $\text{Ker}\pi$  is invertible.*

We use the assumptions of the definition 2.1. Indeed, for all  $(x, t)$ , the operator is symmetric, so the linear operator  $L(\partial\phi(x, t))$  is symmetric. Its kernel and its image are therefore in direct orthogonal sum. The restriction of  $L(\partial\phi(x, t))$  to its image is invertible.

Equation (2.29) is equivalent to

$$(2.32) \quad \pi(x, t)a_0(x, t) = a_0(x, t).$$

For knowing  $a_0$ , we add to the compatibility relations (2.32) the transport equation (2.30)

$$L(x, t, \partial_x, \partial_t)a_0 \in \text{Im}L(\partial\phi(x, t)).$$

This transport equation is rewritten as  $\pi(x, t)L(x, t, \partial_x, \partial_t)a_0(x, t) = 0$ , equation equivalent to

$$(2.33) \quad \pi(x, t)L(x, t, \partial_x, \partial_t)(\pi(x, t)a_0(x, t)) = 0.$$

Note that (2.33) is an equation on the vector space  $\text{Im}\pi$ . To find  $a_0$ , we show

LEMMA 2.2. *The system of equations (2.32), (2.33) + an initial data initial in time  $a_0(x, t_0)$  verifying  $\pi(x, t_0)a_0(x, t_0) = a_0(x, t_0)$  is equivalent to the symmetric hyperbolic system*

$$[\pi(x, t)]L(x, t, \partial_x, \partial_t)\pi(x, t) + (I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))a_0(x, t) = 0.$$

From this lemma, we immediately deduce the result of the proposition 2.2. Indeed the symmetric hyperbolic system (2.31) admits a unique solution for any initial data, which proves the uniqueness result of the proposition.

The equality  $\pi(x, t)a_0(x, t) = a_0(x, t)$  comes from the fact that  $(I - \pi(x, t))a_0(x, t)$  is a solution of (2.31) for a zero initial datum, so by uniqueness of the solutions of (2.31) we deduce that  $(I - \pi(x, t))a_0(x, t) = 0$  for all  $(x, t)$ . For more convenience, we will denote by  $G(x, t, \partial_x, \partial_t) = \pi(x, t)L(x, t, \partial_x, \partial_t)\pi(x, t) + (I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))$  which intervenes in the equality (2.31).

Proof of the lemma. From (2.29),  $(I - \pi(x, t))a_0(x, t) = 0$ , so

$$(I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))a_0(x, t) = 0.$$

We thus obtain, by addition of (2.33) to this equality, the equality (2.31).

The coefficient matrix of  $\partial_t$  in the operator  $G$  is  $\tilde{A}_0(x, t) = \pi(x, t)A_0(x, t)\pi(x, t) + (I - \pi(x, t))A_0(x, t)(I - \pi(x, t))$ . Since  $A_0(x, t)$  is positive definite,  $\tilde{A}_0$  is also positive definite. Indeed, let us decompose  $A_0$  on  $\text{Ker}\pi \oplus \text{Im}\pi$  in the form

$$\begin{pmatrix} A_0^{11}(x, t) & A_0^{12}(x, t) \\ {}^t A_0^{12}(x, t) & A_0^{22}(x, t) \end{pmatrix},$$

the matrices  $A_0^{11}(x, t)$  and  $A_0^{22}(x, t)$  being symmetric. The inequality  $A_0 \geq cId$  is rewritten, by decomposing  $u = (u_1, u_2)$ , into the inequality

$${}^t u_1 A_0^{11}(x, t) u_1 + {}^t u_2 A_0^{22}(x, t) u_2 + 2({}^t u_1 A_0^{12}(x, t) u_2) \geq c^t u_1 u_1 + c^t u_2 u_2.$$

Considering successively  $u_2 = 0$  and  $u_1 = 0$ , we find that  $A_0^{11}(x, t) \geq cId$  and  $A_0^{22}(x, t) \geq cId$ , so the matrix

$$\tilde{A}_0(x, t) = \begin{pmatrix} A_0^{11}(x, t) & 0 \\ 0 & A_0^{22}(x, t) \end{pmatrix}$$

verifies  $\tilde{A}_0(x, t) \geq cId^1$ .

In the same way, all the matrices  $\pi(x, t)A_j(x, t)\pi(x, t)$  are positive symmetric if the  $A_j(x, t)$  are. The system (2.31) is thus a hyperbolic system because the operator  $G$  is a strictly hyperbolic in time Cauchy operator .

Conversely. Assume (2.31) is satisfied and show that  $c_0(x, t) = (I - \pi(x, t))a_0(x, t)$  is solution of  $G(x, t, \partial_x, \partial_t)c = 0$ . Since  $(I - \pi(x, t))^2 = (I - \pi(x, t))$  and  $(I - \pi(x, t))\pi(x, t) = 0$ , the equation (2.31) leads to

$$(I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))a_0(x, t) = 0.$$

relation in the equation (2.31), we obtain  $[\pi(x, t)L(x, t, \partial_x, \partial_t)\pi(x, t)]a_0(x, t) = 0$ . In using this time  $\pi^2 = \pi$  and  $(I - \pi)\pi = 0$ , we find successively

$$\begin{aligned} & [\pi(x, t)L(x, t, \partial_x, \partial_t)\pi(x, t) + (I - \pi(x, t))L(x, t, \partial_x, \partial_t)(I - \pi(x, t))](\pi(x, t)a_0(x, t)) \\ & \quad = \\ & \quad \pi(x, t)L(x, t, \partial_x, \partial_t)\pi(x, t)a_0(x, t) = 0. \end{aligned}$$

The vector  $\pi(x, t)a_0(x, t)$  is therefore a solution of the same hyperbolic system as  $a_0$  and moreover,  $\pi(x, t_0)a_0(x, t_0) = a_0(x, t_0)$ . The uniqueness result recalled below for hyperbolic systems gives  $\pi(x, t)a_0(x, t) = a_0(x, t)$ . So (2.31) + initial condition implies both (2.32) and (2.31). This completes the proof of the lemma 2.2, since  $a_0$  is then uniquely determined and concludes the proof of Proposition 2.2.  $\square$

Define the maximum velocity  $c$  so that the differential of  $\phi(x, t)$  does not cancel on

$$\Omega = \{(t, x), 0 < t < T \leq R/c, |x - a| < R - ct\}.$$

Restrict our study to  $\Omega$ . Let  $\phi(x, t)$  be a solution of the eikonal equation (2.28). Assume  $\phi$  is solution of the eikonal equation (2.28), associated with a leaf characterized by its orthogonal projection  $\phi(x, t)$  on  $\mathfrak{S}\pi = \text{Ker}L(\partial\phi(x, t))$ . Introduce the inverse  $Q$  on  $\text{Im}L(\partial\phi(x, t))$  of the the operator  $(I - \pi)L(\partial\phi(x, t))(I - \pi)$ . The following equations

- i) (initial condition)  $b_j(x, t_0) = a_j(x)$ , for  $j \geq 0$ .
- ii) (impedance equation)  $L(\partial\phi(x, t))b_0(x, t) = 0$ ,
- iii) (transport equations)  $iL(\partial\phi(x, t))b_j(x, t) + L(x, t, \partial_x, \partial_t)b_{j-1}(x, t) = 0$ . characterize the asymptotic solutions of (2.27). One has the following result, which constructs an asymptotic solution of the hyperbolic system

<sup>1</sup>The matrices identities used in this paragraph are not equal, we see respectively the identity on  $\text{Ker}\pi$ , the identity on  $\text{Im}\pi$  and the identity on the whole space

THEOREM 2.1. *Assume that the leading order term of the initial value of  $a$  at  $t_0$  satisfies*

$$(2.34) \quad \pi(x, t_0)a^0(x) = a^0(x).$$

- *Let  $a_0$  solution of (2.31), uniquely determined thanks to Proposition 2.2.*
- *Assume that initial (partial) conditions  $a^j(x)$  are such that  $a^j(x) \in \mathfrak{S}\pi(x, t_0)$  and assume that  $a_j(x, t)$  is known for all  $j \leq l-1$ . Let  $\pi(x, t_0)a^j(x) = a^j(x)$  and impose the initial value of*

$$(2.35) \quad (I - \pi(x, t_0))a_j(x) = iQ(x, t_0)[L(x, t, \partial_x, \partial_t)a_{j-1}](x, t_0).$$

- *The function  $a_l$  is uniquely determined by  $a_l(x, t_0) = a^l(x)$  and the equations*

$$(I - \pi)a_l = iQ(I - \pi)L(x, t, \partial_x, \partial_t)a_{l-1},$$

$$G(x, t, \partial_x, \partial_t)b_l = i(I - \pi)Q(I - \pi)L(x, t, \partial_x, \partial_t)a_{l-1} + \pi L(x, t, \partial_x, \partial_t)\pi a_l,$$

- *There exists a unique sequence  $a_j(x, t)$  of functions  $C^\infty(\bar{\Omega})$  satisfying i), ii), iii).*

PROOF. The proof of the theorem is done by induction on  $j$ . It is identical for each  $j$ , so we write it to obtain  $b_1$ . From the equality

$$(2.36) \quad b_1(x, t) = \pi b_1(x, t) + (I - \pi)b_1(x, t),$$

By replacing (2.36) in the transport equation and using  $L(\partial\phi(x, t))\pi(x, t) = 0$ , we get

$$L(\partial\phi(x, t))(I - \pi)b_1 = iL(x, t, \partial_x, \partial_t)b_0.$$

The transport equation is thus inhomogeneous. We deduce

$$(I - \pi)L(\partial\phi(x, t))(I - \pi)b_1(x, t) = i(I - \pi)L(x, t, \partial_x, \partial_t)b_0(x, t).$$

The kernel of the matrix  $(I - \pi)L(\partial\phi(x, t))(I - \pi)$  is  $\text{Ker}L(\partial\phi(x, t))$ . The kernel and the image of  $L(\partial\phi(x, t))$  are supplementary, so the matrix  $(I - \pi)L(\partial\phi(x, t))(I - \pi)$  is invertible in  $\text{Ker}\pi = \text{Im}L(\partial\phi(x, t))$ . We note its partial inverse  $Q$ . We deduce

$$(2.37) \quad (I - \pi)b_1(x, t) = iQ(I - \pi)L(x, t, \partial_x, \partial_t)b_0(x, t)$$

which determines  $(I - \pi)b_1$  as a function of  $b_0$ . It remains to determine  $\pi b_1$ . The system of which  $\pi b_1$  is a solution from the equation on  $b_2$ :

$$iL(\partial\phi(x, t))b_2(x, t) + L(x, t, \partial_x, \partial_t)b_1(x, t) = 0.$$

As  $L(\partial\phi(x, t))b_2(x, t) \in \text{Im}L(\partial\phi(x, t))$  one gets

$$\pi(x, t)L(x, t, \partial_x, \partial_t)b_1(x, t) = 0,$$

Using (2.37) and (2.36) one has

$$(2.38) \quad \pi(x, t)L(x, t, \partial_x, \partial_t)[\pi b_1(x, t) + iQ(I - \pi)L(x, t, \partial_x, \partial_t)b_0] = 0.$$

From (2.37) we deduce

$$(I - \pi)L(x, t, \partial_x, \partial_t)(I - \pi)b_1 = i(I - \pi)L(x, t, \partial_x, \partial_t)Q(I - \pi)L(x, t, \partial_x, \partial_t)b_0$$

Adding (2.38) to these equations, one obtains the following system

$$G(x, t, \partial_x, \partial_t)b_1(x, t) = i(I - 2\pi)L(x, t, \partial_x, \partial_t)Q(I - \pi)L(x, t, \partial_x, \partial_t)b_0.$$

This system is then a hyperbolic system and has a unique solution.

The theorem is then a consequence of the

PROPOSITION 2.3. *Let  $a_0(x)$  satisfying  $\pi(x, t_0)a_0(x) = a_0(x)$ . We determine  $b_0(x, t)$  by the proposition 2.2. Under the condition of compatibility of the initial data*

$$(I - \pi)a_1(x) = iQ(x, t_0)L(x, t, \partial_x, \partial_t)b_0(x, t_0),$$

the system

$$\begin{cases} (I - \pi)b_1 = iQ(I - \pi)L(x, t, \partial_x, \partial_t)b_0, \\ \pi L(x, t, \partial_x, \partial_t)\pi b_1 = -i\pi L(x, t, \partial_x, \partial_t)(Q(L(x, t, \partial_x, \partial_t)b_0)), \end{cases}$$

is equivalent to

$$\begin{cases} G(x, t, \partial_x, \partial_t)b_1 = i(I - 2\pi)L(x, t, \partial_x, \partial_t)(Q(L(x, t, \partial_x, \partial_t)b_0)), \\ b_1(x, t_0) = a_1(x). \end{cases}$$

It has thus a unique solution  $b_1$ .

PROOF. the implication  $\Rightarrow$  has just been proved. The reciprocal is the following. Thus, we assume

$$(2.39) \quad G(x, t, \partial_x, \partial_t)b_1(x, t) = i(I - 2\pi)L(x, t, \partial_x, \partial_t)(I - \pi)QL(x, t, \partial_x, \partial_t)b_0.$$

One notices that  $\pi G(x, t, \partial_x, \partial_t) = \pi L(x, t, \partial_x, \partial_t)\pi$  and that  $(I - \pi)G(x, t, \partial_x, \partial_t) = (I - \pi)L(x, t, \partial_x, \partial_t)(I - \pi)$ . Hence, from equality (2.39), one deduces

$$\begin{cases} \pi L(x, t, \partial_x, \partial_t)\pi b_1(x, t) = -i\pi L(x, t, \partial_x, \partial_t)QL(x, t, \partial_x, \partial_t)b_0, \\ (I - \pi)L(x, t, \partial_x, \partial_t)(I - \pi)b_1 = i(I - \pi)L(x, t, \partial_x, \partial_t)QL(x, t, \partial_x, \partial_t)b_0. \end{cases}$$

From the second equation of this system, one has

$$(I - \pi)L(x, t, \partial_x, \partial_t)[(I - \pi)b_1 - iQL(x, t, \partial_x, \partial_t)b_0] = 0.$$

As  $I - \pi$  is the identity on  $\Im Q$  and  $(I - \pi)^2 = I - \pi$ , one has

$$(I - \pi)L(x, t, \partial_x, \partial_t)(I - \pi)[(I - \pi)b_1 - iQL(x, t, \partial_x, \partial_t)b_0] = 0.$$

As  $\pi Q = 0$ , one has  $\pi L(x, t, \partial_x, \partial_t)\pi[(I - \pi)b_1 - iQL(x, t, \partial_x, \partial_t)b_0] = 0$ . Hence, for all  $t$

$$G(x, t, \partial_x, \partial_t)[(I - \pi)b_1 - iQL(x, t, \partial_x, \partial_t)b_0] = 0.$$

The compatibility condition is the zero Cauchy condition for this hyperbolic system. We deduce for all  $t$  that  $(I - \pi(x, t_0))b_1(x, t) = i[QL(x, t, \partial_x, \partial_t)b_0]$ . We deduce the two equalities required. The proposition is proved.  $\square$

This proves by induction Theorem 2.1.  $\square$

Remark 1. The system satisfied by  $\pi b_1$  is

$$G(x, t, \partial_x, \partial_t)(\pi b_1) = -i\pi LQLb_0$$

and the one satisfied by  $b_1$  is

$$G(x, t, \partial_x, \partial_t)b_1 = i(I - 2\pi)LQLb_0.$$

These two systems are identical only when  $L(QLb_0)$  is in  $\Im\pi = \text{Ker}L(\partial\phi)$ . The only relation implied by the system on  $b_0$  is  $\pi(Lb_0) = 0$ , i.e.  $Lb_0 \in \text{Ker}\pi = \Im L(\partial\phi)$ .

Remark 2. The condition on  $a^j(x)$  reduces to an equality on  $\pi(x, t_0)a^j(x)$ , since  $(I - \pi(x, t_0))a^j(x)$  is determined by the compatibility condition. On the other hand, we do not automatically have  $\pi(x, t_0)a_j(x) = a_j(x)$  contrary to the case  $j = 0$ . We underline the difference of presentation with [86] (Theorem 5.4) even if the result is identical: we prefer to emphasize the condition of compatibility condition induced on the rank  $j$  by the data of  $a_k$  for  $k \leq j - 1$ .

### 3. Application to Maxwell equations

The system of Maxwell equations is hyperbolic symmetric in the sense of the definition 2.1. In Indeed, Maxwell equations can be written as Maxwell equations are written

$$\varepsilon\partial_t E = \text{rot}H, -\mu\partial_t H = \text{rot}E.$$

Let  $u = (E, H)$  and denote the matrices

$$A_0 = \begin{pmatrix} \varepsilon I_3 & 0 \\ 0 & \mu I_3 \end{pmatrix}, A_1 = \begin{pmatrix} & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & -1 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & -1 & & & \\ 0 & 1 & 0 & & & \end{pmatrix}$$

$$A_2 = \begin{pmatrix} & & & 0 & 0 & -1 \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & 0 \\ -1 & 0 & 0 & & & \end{pmatrix}, A_3 = \begin{pmatrix} & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & \end{pmatrix}$$

Maxwell equations write

$$(3.40) \quad A_0 \partial_t u + A_1 \partial_{x_1} u + A_2 \partial_{x_2} u + A_3 \partial_{x_3} u = 0.$$

Matrices  $A_j$  are symmetrical, and the matrix  $A_0$  is bounded below by  $\min(\varepsilon, \mu)I_6$ . The eikonal equation (2.28) is then

$$(3.41) \quad \det A_0 \partial_t \Phi + A_1 \partial_{x_1} \Phi + A_2 \partial_{x_2} \Phi + A_3 \partial_{x_3} \Phi = 0.$$

Let  $\text{rot}^* \Phi$  be the matrix

$$\text{rot}^* \Phi = \begin{pmatrix} 0 & \partial_{x_3} \Phi & -\partial_{x_2} \Phi \\ -\partial_{x_3} \Phi & 0 & \partial_{x_1} \Phi \\ \partial_{x_2} \Phi & -\partial_{x_1} \Phi & 0 \end{pmatrix}$$

Equation (3.41) rewrites

$$\det \begin{pmatrix} \varepsilon \partial_t \Phi I_3 & \text{rot}^* \Phi \\ -\text{rot}^* \Phi & \mu \partial_t \Phi I_3 \end{pmatrix} = 0,$$

equivalent to

$$[\varepsilon \mu (\partial_t \Phi)^2 - |\nabla \Phi|^2]^2 \varepsilon \mu (\partial_t \Phi)^2 = 0.$$

The two leaves of the characteristic manifold are thus  $\partial_t \Phi = 0$  et  $|\nabla \Phi| = (\varepsilon \mu)^{\frac{1}{2}} |\partial_t \Phi|$ . We are interested in the second leaf. Say that  $u_0$  is in the kernel is equivalent to

$$\begin{pmatrix} (\frac{\varepsilon}{\mu})^{\frac{1}{2}} |\nabla \Phi| I_3 & \text{rot}^* \Phi \\ -\text{rot}^* \Phi & (\frac{\mu}{\varepsilon})^{\frac{1}{2}} |\nabla \Phi| I_3 \end{pmatrix} \begin{pmatrix} e^0 \\ h^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One gets the relations (6.19) :

$$e^0 = \left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}} \frac{\nabla \Phi}{|\nabla \Phi|} \wedge h^0$$

$$h^0 = -\left(\frac{\varepsilon}{\mu}\right)^{\frac{1}{2}} \frac{\nabla \Phi}{|\nabla \Phi|} \wedge e^0$$

It is for this reason that the relation (2.29) is sometimes called impedance condition, the number  $Z = (\frac{\mu}{\varepsilon})^{\frac{1}{2}}$  which appears here is the impedance of the medium. This notation is, of course, an abuse of language because the impedance condition is a condition at the boundary, whereas the relation (2.29) is a valid relation for all  $x$  in the domain  $\Omega$ .

We therefore deduce the eikonal relation  $c^2 (\nabla \Phi)^2 = (\partial_t \Phi)^2$  by the very simple calculation coming from the eikonal equation by rewriting  $k = \partial_t \Phi$ . It is thus (fortunately) the same eikonal equation as for the Helmholtz problem.

By writing the relations induced by (2.29) in (3.40), we find that the transport equations linking  $(e_1, h_1, e_0, h_0)$  which are those given by (2.30) are those written in (6.20).

#### 4. Asymptotic existence for an elliptic problem

In this section, we change the type of operator, by considering an elliptic operator instead of a hyperbolic operator. We wish to show an application of asymptotic analysis for elliptic systems of order 1, and in particular to obtain thanks to asymptotic calculations the gain of regularity of  $Pu \in H^s \rightarrow u \in H^{s+1}$ . Let us notice, in the hyperbolic case, that it is enough to solve

$$L(x, t, \partial_x, \partial_t)c(x, t, \varepsilon) = -r(x, t, \varepsilon) \simeq 0$$

with the Cauchy condition  $c|_{t=0} = 0$ .

In the case where  $L$  is an elliptic operator, the above equality is solvable in the neighborhood of any point, and the regularity of  $c$  is known as a function of the regularity of  $r$  (we can compare this result with this result can be compared to the result from Taylor stated later in Proposition 7.8).

We first introduce the elliptic operators of order 1.

**DEFINITION 2.2.** *Let  $P(x, t, \partial_x, \partial_t) = \sum_{j=1}^d A_j(x, t)\partial_{x_j} + A_0(x, t)\partial_t$ . We say that  $P$  is elliptic in the neighborhood of neighborhood of the point  $(x_0, t_0)$  if there exists a function  $\chi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$  (such that  $\chi = 1$  on  $|x - x_0|^2 + (t - t_0)^2 \leq \varepsilon^2$ ) and a constant  $c > 0$  such that the matrix*

$$\sum_{j=1}^d \chi(x, t)\xi_j A_j(x, t) + \tau \chi(x, t)A_0(x, t)$$

is bounded below by  $c(|\xi|^2 + \tau^2)^{\frac{1}{2}} Id$  for all  $(\xi, \tau) \in \mathbb{R}^d \times \mathbb{R} - \{(0, 0)\}$ .

J. Rauch demonstrates in his course [86] the proposition :

**PROPOSITION 2.4.** *We suppose that in a neighborhood of  $(x_0, t_0)$ ,  $P(x, t, \partial_x, \partial_t)$  is an elliptic operator of order 1.*

- (1) *We can solve  $Pc = r$  for  $r \simeq 0$ , and there exists a  $c$  which is asymptotically zero*
- (2) *Regularity  $H^\sigma$  :*

$$Pu \in H^\sigma \Rightarrow u \in H^{\sigma+1},$$

- (3) *Regularity  $C^\infty$*

$$Pu \in C^\infty \Rightarrow u \in C^\infty$$

**PROOF.** Let us first note that there are other proofs of this proposition. We choose this proof in connection with asymptotic expansions. Let  $P(x, t, \partial_x, \partial_t)$  be an elliptic operator in the neighborhood of a point  $(x_0, t_0)$ . Consider  $\chi P$ , and we introduced  $b(x, t) = \chi Pu(x, t)$ .

The proof proceeds in two steps:

- we construct an asymptotic solution and show that this asymptotic solution "starts one step above", i.e. the first term of the asymptotic expansion is zero
- Then, we show that the Fourier transform of a solution can be characterized by the solution can be characterized by the asymptotic parameter  $k = (|\xi|^2 + \tau^2)^{\frac{1}{2}}$   $k \rightarrow +\infty$  and we check the asymptotic solution thus calculated.

##### Asymptotic solution of an elliptic problem

In this part of the proof, the asymptotic parameter is  $\varepsilon \rightarrow 0$ . Let us first notice that if  $v(x, t, \varepsilon) = V(x, t, \varepsilon)e^{\frac{i\phi(x, t)}{\varepsilon}}$ , with  $V(x, t, \varepsilon) \simeq \sum_l V_l(x, t)(\varepsilon)^l$ , we get

$$Pv(x, t, \varepsilon) = [i(\sum_{j=1}^d \frac{A_j(x, t)}{\varepsilon} \frac{\partial \phi}{\partial x_j} + \frac{A_0(x, t)}{\varepsilon} \frac{\partial \phi}{\partial t})V(x, t, \varepsilon) + (\sum_{j=1}^d A_j(x, t) \frac{\partial V}{\partial x_j} + A_0(x, t) \frac{\partial V}{\partial t})]e^{i\frac{\phi(x, t)}{\varepsilon}}.$$

For all  $(\xi_0, \tau_0) \neq (0, 0)$ , and for any function  $b \in C_0^\infty(\mathbb{R}^{d+1} \times ]0, 1])$ , one can construct an asymptotic expansion of  ${}^t P v(x, t, \varepsilon) = \chi(x, t)b(x, t, \varepsilon)e^{i\frac{x \cdot \xi_0 + t\tau_0}{\varepsilon}}$ , the operator  ${}^t P$  being elliptic as well. Hence  $\phi(x, t) = x \cdot \xi_0 + t\tau_0$ , and

$$e^{-\frac{x \cdot \xi_0 + t\tau_0}{\varepsilon}} P v(x, t, \varepsilon) = \frac{i}{\varepsilon} [\sum_{j=1}^d A_j(x, t)\xi_{0,j} + A_0(x, t)\tau_0]V(x, t, \varepsilon) + [\sum_{j=1}^d A_j(x, t) \frac{\partial V}{\partial x_j} + A_0(x, t) \frac{\partial V}{\partial t}].$$



Assuming que  ${}^t P v(x, t, \varepsilon) = b(x, t, \varepsilon) e^{i \frac{x \cdot \xi_0 + t \tau_0}{\varepsilon}}$ ,  $b(x, t, \varepsilon) \simeq \sum_l b_l(x, t) (i\varepsilon)^l$ , one gets the equations

$$\left[ \sum_{j=1}^d A_j(x, t) \xi_{0,j} + A_0(x, t) \tau_0 \right] V_0(x, t) = 0$$

and, for  $l \geq 1$ ,

$$-\left[ \sum_{j=1}^d A_j(x, t) \xi_{0,j} + A_0(x, t) \tau_0 \right] + \left( \sum_{j=1}^d A_j(x, t) \frac{\partial V_{l-1}}{\partial x_j} + A_0(x, t) \frac{\partial V_{l-1}}{\partial t} \right) V_l(x, t) = \chi(x, t) b_{l-1}(x, t).$$

One deduces  $V_0(x, t) = 0$ , et

$$V_1(x, t) = - \left( \sum_j A_j(x, t) \xi_{0,j} + A_0(x, t) \tau_0 \right)^{-1} (b_0(x, t)),$$

$\tau_0 = \xi_{0,0}$  et  $\tau = \xi_0$  then all the successive equations giving  $V_l$  as a function of  $V_{l-1}$ . We introduce the convention and  $x_0 = t$ . Then

$$V_2(x, t) = - \left( \sum_{0 \leq j \leq d} A_j(x, t) \xi_{0,j} \right)^{-1} \left[ b_1(x, t) + \sum_{0 \leq p \leq d} A_p(x, t) \frac{\partial}{\partial x_p} \left( \sum_{0 \leq j \leq d} A_j(x, t) \xi_{0,j} \right)^{-1} (b_0(x, t)) \right].$$

We have constructed an asymptotic solution of class  $C^\infty$  whose first term is zero. We check that all terms, which include  $b_l$  or derivatives of  $b_l$ , have a support contained in the support of  $\chi$  since  $b = \chi P u$ .

### Sobolev regularity of the asymptotic solution

We suppose now  $\|\xi_0\|^2 + \tau_0^2 = 1$ , this in order to distinguish the parameter  $k$  and the direction of the vector  $(\xi, \tau)$ . The construction and Borel's Theorem (Theorem 1.2) provide an asymptotic solution ( $k \rightarrow \infty, k = \varepsilon^{-1}$ )  $v(x, t, k) \in C^\infty(\mathbb{R}^{d+1}, S^d, k)$ , of support included in  $\text{supp} \chi(x, t)$ , of

$${}^t P v(x, t, k) = \chi(x, t) e^{ik(x \cdot \xi_0 + t \tau_0)}.$$

Note that this is equivalent to choosing  $P u = e^{ik(x \cdot \xi + t \tau)}$ .

We give  $M$  such that  $(1 + \eta^2)^s \eta^{-2M}$  tend to 0 at to infinity in  $\mathbb{R}^{d+1}$ . This value of  $M$  gives the order of truncation of the asymptotic series. We note

$$v_M(x, t; \xi, \tau, k) = \sum_{l=1}^M i^l V_l(x, t, k^{-1}) k^{-l}.$$

Then

$$\begin{aligned} \mathcal{F}(\chi u)(k\xi, k\tau) &= \langle u, \chi(x, t) e^{ik(x \cdot \xi + t \tau)} \rangle \\ &= \langle u, {}^t P v_M(x, t; \xi, \tau, k) \rangle + O(k^{-M}) \\ (4.42) \quad &= \langle P u, v_M(x, t; \xi, \tau, k) \rangle + O(k^{-M}) \\ &= \langle f, e^{ik(x \cdot \xi + t \tau)} \sum_{l=1}^M k^{-l} i^l V_l(x, t; \xi, \tau, k) \rangle + O(k^{-M}) \\ &= \sum_{l=1}^{M-1} k^{-l} i^l \mathcal{F}(f V_l) + O(k^{-M}). \end{aligned}$$

The successive equalities on  $V_l$ , identical to the one obtained for  $V_2$ , show that  $V_l f$  can be written as the product of  $f$  by regular derivatives and quotients involving  $\chi$  and its derivatives. Let  $\tilde{\chi}$  be a function  $C^\infty$  with compact support equal to 1 on  $\text{supp} \chi$ . Then  $T = \partial_{x_l} \chi$  is uniformly bounded as well as all its derivatives. We deduce that the Fourier transform of  $\partial_{x_l} \chi f$  is equal to the convolution product of the Fourier transform of  $\tilde{\chi} f$  and  $T$ . We use then the regularity of  $T$ , which implies the Sobolev regularity of  $T$ , to find that the operator  $\hat{g} \rightarrow \hat{T} \star \hat{g}$  is bounded in  $H^s$ . We deduce

$$\int_{\mathbb{R}^{d+1}} |\mathcal{F}(V_l f)(\xi, \tau)|^2 (1 + |\xi|^2 + \tau^2)^s d\xi d\tau \leq C(l, \chi) \|\tilde{\chi} f\|_{H^s(\mathbb{R}^{d+1})}^2.$$

Then the change of variable  $\xi = k\eta$ ,  $\tau = k\sigma$  allows to obtain

$$\int_{S^d \times \mathbf{R}} |\mathcal{F}(V_l f)(k\eta, k\sigma)|^2 k^{2s+d+1} d\eta d\sigma dk \leq C(l, \chi) |\tilde{\chi} f|_{H^s(\mathbf{R}^{d+1})}^2.$$

From the equality (4.42), the sum starting at  $l = 1$ , we find

$$\begin{aligned} & \int_{\mathbf{R}^{d+1}} |\mathcal{F}(\chi u)(\xi, \tau)|^2 (1 + |\xi|^2 + \tau^2)^p d\xi d\tau \\ &= \int_{\mathbf{R}^{d+1}} |\mathcal{F}(\chi u)(k\eta, k\sigma)|^2 (k^{-2} + |\eta|^2 + \sigma^2)^p k^{2p+d+1} d\eta d\sigma \\ &\leq \max_{1 \leq l \leq M-1} (C(l, \chi)) \sum_{l \geq 1} \int_{\mathbf{R}^{d+1}} |\mathcal{F}(fV_l)(k\eta, k\sigma)|^2 (k^{-2} + |\eta|^2 + \sigma^2)^p k^{2p+d+1-2l} d\eta d\sigma. \end{aligned}$$

This sum is convergent for  $p = s + 1$  because  $l \geq 1$ , so we obtain the regularity  $H^{s+1}$  of the solution  $u$ . We have completed the proof of the second paragraph.

The third paragraph of the theorem can be deduced from the second by the inclusions between local Sobolev spaces and  $C^k$ -function spaces. We will come back to this type of method in the paragraph 4.4.  $\square$

## 5. Exercices de chapitre 2

Exercice 2.1. 1) Calculate the asymptotic expansion in  $k$  of the solution  $u(x_1, x_2, k) = \begin{pmatrix} a_1(x_1, x_2, k) \\ a_2(x_1, x_2, k) \end{pmatrix} e^{ik\phi(x_1, x_2)}$  of

$$(5.43) \quad ik u + \begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix} \partial_{x_1} u + \begin{pmatrix} \frac{2x_1+1}{3} & \frac{\sqrt{2}(1-x_1)}{3} \\ \frac{\sqrt{2}(1-x_1)}{3} & \frac{2+x_1}{3} \end{pmatrix} \partial_{x_2} u = 0.$$

2) Consider an initial condition on  $x_1 = 0$  of the form  $u(x_2, k) = a(x_2, k) e^{ik\psi(x_2)}$ . Give the solution of the previous problem with this initial condition.

Exercice 2.2. Assume that  $\phi$  is solution of the eikonal equation (2.28) and that the dimension of the kernel of  $\text{Ker}(L(\partial\phi))$  is 1. Prove that  $a_0$  is solution of a transport equation characterized by a vector field.

Correction de l'exercice 2.1. On applique les résultats précédents. On vérifie que l'on a

$$ik[Id + \begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix} \partial_{x_1} \phi + \begin{pmatrix} \frac{2x_1+1}{3} & \frac{\sqrt{2}(1-x_1)}{3} \\ \frac{\sqrt{2}(1-x_1)}{3} & \frac{2+x_1}{3} \end{pmatrix} \partial_{x_2} \phi] \begin{pmatrix} a_1(x_1, x_2, k) \\ a_2(x_1, x_2, k) \end{pmatrix} + O(1) = 0$$

Il existe une solution non triviale si et seulement si le déterminant de la matrice entre crochets est nul (ce qui correspond à écrire  $\det(L(x, \nabla_x \phi)) = 0$ ). La matrice coefficient de  $\partial_{x_2} \phi$  est symétrique, donc diagonalisable. Ses valeurs propres sont  $x_1$  et 1. Le vecteur propre associé à la valeur propre  $x_1$  est  $(-\sqrt{2}, 1)$ . Le vecteur propre associé à 1 est  $(1, \sqrt{2})$ . On voit ensuite que

$$\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = - \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = 5 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

De ces égalités, on déduit que l'équation eikonale est équivalente à

$$\det(Id + \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \partial_{x_1} \phi + \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \partial_{x_2} \phi) = 0$$

soit

$$(1 - \partial_{x_1} \phi(x_1, x_2) + x_1 \partial_{x_2} \phi(x_1, x_2))(1 + 5\partial_{x_1} \phi(x_1, x_2) + \partial_{x_2} \phi(x_1, x_2)) = 0.$$

Nous introduisons les courbes intégrales respectives des deux champs :

$$\frac{dx_1}{ds} = -1, \quad \frac{dx_2}{ds} = x_1(s)$$

soit  $x_1(s) = x_1^0 - s$ ,  $x_2(s) = x_2^0 - x_1^0 s - \frac{s^2}{2}$  pour le premier, c'est-à-dire  $x_2 = x_2^0 - x_1^0(x_1^0 - x_1) - \frac{1}{2}(x_1 - x_1^0)^2$

$$\frac{dx_1}{ds} = 5, \quad \frac{dx_2}{ds} = 1$$

soit  $x_1(s) = x_1^0 + 5s$ ,  $x_2(s) = x_2^0 + s$  pour le deuxième, c'est-à-dire  $x_2 = x_2^0 + \frac{1}{5}(x_1 - x_1^0)$ .

On contrôle que  $\phi(x_1(s), x_2(s)) = \phi(x_1(0), x_2(0)) - s$ . On connaît donc la phase solution de l'équation eikonale si on la connaît sur une courbe orthogonale aux champs. On considère donc la courbe  $x_1 = 0$ , comme l'énoncé le suggère. Alors on trouve

- dans le premier cas  $\phi(x_1, x_2) = \phi(0, x_2 + \frac{1}{2}x_1^2) + x_1$  (les surfaces isophase sont des paraboles)
- dans le deuxième cas  $\phi(x_1, x_2) = \phi(0, x_2 - \frac{x_1}{5}) - \frac{x_1}{5}$  (les surfaces isophase sont des droites).

Représentons  $\begin{pmatrix} a_1(x_1, x_2, k) \\ a_2(x_1, x_2, k) \end{pmatrix}$  sur la base propre, par

$$\begin{pmatrix} a_1(x_1, x_2, k) \\ a_2(x_1, x_2, k) \end{pmatrix} = \alpha_1(x_1, x_2, k) \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \alpha_2(x_1, x_2, k) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

avec  $\alpha_1(x_1, x_2, k) = \frac{1}{3}(-\sqrt{2}a_1(x_1, x_2, k) + a_2(x_1, x_2, k))$ ,  $\alpha_2(x_1, x_2, k) = \frac{1}{3}(a_1(x_1, x_2, k) + \sqrt{2}a_2(x_1, x_2, k))$ .  
On vérifie alors que

$$\begin{aligned} & ik\alpha_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + ik\alpha_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix} [\partial_{x_1}\alpha_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \partial_{x_1}\alpha_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}] \\ & + \begin{pmatrix} \frac{2x_1+1}{3} & \frac{\sqrt{2}(1-x_1)}{3} \\ \frac{\sqrt{2}(1-x_1)}{3} & \frac{2+x_1}{3} \end{pmatrix} [\partial_{x_2}\alpha_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \partial_{x_2}\alpha_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}] \\ & + ik\alpha_1(-\partial_{x_1}\phi + x_1\partial_{x_2}\phi) \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + ik\alpha_2(5\partial_{x_1}\phi + \partial_{x_2}\phi) \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = 0 \end{aligned}$$

et on aboutit aux deux équations d'ordre 1

$$ik(1 - \partial_{x_1}\phi + x_1\partial_{x_2}\phi)\alpha_1 - \partial_{x_1}\alpha_1 + x_1\partial_{x_2}\alpha_1 = 0,$$

$$ik(1 + 5\partial_{x_1}\phi + \partial_{x_2}\phi)\alpha_2 + 5\partial_{x_1}\alpha_2 + \partial_{x_2}\alpha_2 = 0.$$

On remarque que ces deux équations correspondent à des solutions asymptotiques de problèmes scalaires d'ordre 1. Il n'existe pas de phase  $\phi$  et de couple  $(\alpha_1, \alpha_2)$  non nuls tels que  $\phi, \alpha_1, \alpha_2$  vérifie à la fois les deux équations.

On considère donc comme donnée la valeur de la phase à  $x_1 = 0$ , soit  $\psi(x_2)$ . Cette phase  $\psi$  génère deux phases  $\phi_1(x_1, x_2)$  et  $\phi_2(x_1, x_2)$  telles que

$$\phi_1(x_1, x_2) = \psi(x_2 + \frac{1}{2}x_1^2) + x_1,$$

$$\phi_2(x_1, x_2) = \psi(x_2 - \frac{x_1}{5}) - \frac{x_1}{5}.$$

Si  $\phi = \phi_1$ , la condition nécessaire pour que  $u$  soit solution de (5.43) est que  $a$  soit colinéaire au vecteur  $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ . On en déduit que  $\alpha_2 = 0$ , et  $\alpha_1$  est solution de

$$-\partial_{x_1}\alpha_1 + x_1\partial_{x_2}\alpha_1 = 0$$

et  $\alpha_1$  est donc constant sur les caractéristiques du champ  $(-1, x_1)$ . Il vient donc  $\alpha_1(x_1, x_2, k) = \alpha_1(0, x_2 + \frac{x_1^2}{2}, k)$ .

Lorsque la phase est égale à  $\phi_2$ , nécessairement  $a$  est dans le noyau de la matrice associée à  $\phi_2$ , donc  $\alpha_1 = 0$ . On trouve  $5\partial_{x_1}\alpha_2 + \partial_{x_2}\alpha_2 = 0$ , ce qui donne  $\alpha_2(x_1, x_2, k) = \alpha_2(0, x_2 - \frac{x_1}{5}, k)$ .

La phase  $\phi$  est nécessairement égale à  $\phi_1$  ou à  $\phi_2$  pour que  $u(x_1, x_2, k) = a(x_1, x_2, k)e^{ik\phi(x_1, x_2)}$  soit une solution du système (5.43). Dans ces cas, on a respectivement  $\alpha_2$  ou  $\alpha_1$  nul.

De ces deux résultats, on déduit que si la donnée initiale n'est pas dans l'espace propre associé à une phase particulière, alors la solution générale ne peut pas s'écrire sous la forme  $a(x_1, x_2, k)e^{ik\phi(x_1, x_2)}$ . En revanche, le système différentiel étant linéaire, toute expression de la forme

$$A(x_1, x_2, k)e^{ik\phi_1(x_1, x_2)} + B(x_1, x_2, k)e^{ik\phi_2(x_1, x_2)}$$

est solution lorsque  $A$  est colinéaire à  $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$  et lorsque  $B$  est colinéaire à  $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ , les coefficients vérifiant les lois de  $\alpha_1$  et de  $\alpha_2$ . De plus, il faut que

$$A(0, x_2, k)e^{ik\phi_1(0, x_2)} + B(0, x_2, k)e^{ik\phi_2(0, x_2)} = a(x_2, k)e^{ik\psi(x_2)}.$$

Par des manipulations élémentaires lorsque  $k$  tend vers  $+\infty$  et en prolongeant  $k$  dans le complexe, on déduit que  $\phi_1(0, x_2) - \psi(x_2) = C_1$ ,  $\phi_2(0, x_2) - \psi(x_2) = C_2$ ,  $C_1$  et  $C_2$  étant deux constantes. On peut les prendre nulles, quitte à inclure le terme  $e^{kC_1}$  dans  $A$  ou  $B$ . Alors il suffit de décomposer

$a(x, k) = \begin{pmatrix} A_1(x_2, k) \\ A_2(x_2, k) \end{pmatrix}$  dans la base propre  $\left\{ \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right\}$  pour obtenir une solution à “deux phases” :

$$\begin{aligned} u_1(x_1, x_2, k) &= -\sqrt{2} \left( \frac{-\sqrt{2}A_1 + A_2}{3} \right) \left( x_2 + \frac{x_1^2}{2}, k \right) e^{ik\psi(x_2 + \frac{x_1^2}{2}) + ikx_1} \\ &\quad + \left( \frac{A_1 + \sqrt{2}A_2}{3} \right) \left( x_2 - \frac{x_1}{5}, k \right) e^{ik\psi(x_2 - \frac{x_1}{5}) - ik\frac{x_1}{5}}, \\ u_2(x_1, x_2, k) &= \left( \frac{-\sqrt{2}A_1 + A_2}{3} \right) \left( x_2 + \frac{x_1^2}{2}, k \right) e^{ik\psi(x_2 + \frac{x_1^2}{2}) + ikx_1} \\ &\quad + \sqrt{2} \left( \frac{A_1 + \sqrt{2}A_2}{3} \right) \left( x_2 - \frac{x_1}{5}, k \right) e^{ik\psi(x_2 - \frac{x_1}{5}) - ik\frac{x_1}{5}}. \end{aligned}$$

On voit ainsi deux fronts se propager. On pourra les différencier si on suppose que la singularité de départ est donnée.

Correction de l'exercice 2.2. Si la dimension du noyau est égale à 1 au voisinage d'un point, on en déduit que  $a_0(x, t)$  est proportionnel à un vecteur  $u_0(x, t)$  de ce noyau, qui est connu explicitement puisque le noyau est de dimension 1. On écrit alors

$$a_0(x, t) = \lambda(x, t)u_0(x, t).$$

L'équation de transport générale s'écrit

$$iL(x, t, \nabla_x \phi, \partial_t \phi) a_{j+1}(x, t) + L(x, t, \partial_x, \partial_t) a_j(x, t) = 0.$$

On appelle  $u_0(x, t)$  la polarisation de l'onde.

Notons alors que l'équation de transport est

$$L(x, t, \partial_x, \partial_t) a_0(x, t) \in \text{Im}L(\partial \phi(x, t))$$

ce qui se traduit par  $\pi L(x, t, \partial_x, \partial_t) a_0 = 0$ . Comme  $a_0 = \pi a_0$  (puisque  $a_0 \in \text{Ker}L(\partial \phi(x, t))$ ), on trouve  $\pi L(x, t, \partial_x, \partial_t) \pi a_0 = 0$ , équation déjà obtenue précédemment.

L'égalité  $\pi \frac{\partial}{\partial x_j} = [\pi, \frac{\partial}{\partial x_j}] + \frac{\partial}{\partial x_j} \pi$  conduit à

$$(5.44) \quad \pi L \pi = \sum_{j=1}^d \pi A_j \pi \frac{\partial}{\partial x_j} + \pi A_0 \pi \frac{\partial}{\partial t} + \sum_{j=1}^d \pi A_j [\pi, \frac{\partial}{\partial x_j}] + \pi A_0 [\pi, \frac{\partial}{\partial t}].$$

Comme  $\pi$  est de rang 1, il existe  $d+1$  scalaires  $v_j(x, t)$ , qui sont les valeurs propres de  $\pi A_j$ , avec  $v_0(x, t) \neq 0$  (ceci car  $A_0 > 0$ ) tels que  $\pi A_j \pi(x, t) = v_j(x, t) \pi(x, t)$ . En notant  $\gamma(x, t)$  le scalaire (opérateur différentiel d'ordre 1 - 1 = 0) tel que

$$(5.45) \quad \begin{aligned} \gamma \pi &= \sum_{j=1}^d \pi A_j [\pi, \frac{\partial}{\partial x_j}] + \pi A_0 [\pi, \frac{\partial}{\partial t}], \\ \pi L \pi &= [v_0(x, t) \partial_t + \sum_{j=1}^{j=d} v_j(x, t) \partial_{x_j} + \gamma(x, t)] \pi. \end{aligned}$$

On a donc démontré que  $\pi a_0$  est solution d'une équation régie par le champ de vecteurs sur  $\mathbb{R}^d$   $((v_0(x, t))^{-1} v_j(x, t))_{1 \leq j \leq d}$ .

Cette relation s'obtient aussi immédiatement en remplaçant  $a_0$  par  $\lambda u_0$ . On voit alors que

$$\pi(x, t) (A_j(x, t) \partial_{x_j} (\lambda(x, t) u_0(x, t))) = (\partial_{x_j} \lambda) \pi A_j u_0 + \lambda \pi A_j \partial_{x_j} u_0,$$

qui est colinéaire à  $u_0$ ,  $\pi$  étant de rang 1 et de noyau  $\mathbb{R} u_0$ .

Donc le champ de vecteurs dont  $\lambda$  est solution est parallèle à  $\partial_\xi l \partial_x + \partial_\tau l \partial_t$ , où  $l(x, t, \xi, \tau) = i\tau A_0(x, t) + i \sum_{j=1}^{j=d} A_j(x, t) \xi_j$ . On verra plus loin le rôle de cette fonction de  $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ , dans le chapitre 5.

Conclusion. Le champ de vecteurs  $v_0^{-1}(v_j)$  s'appelle la vitesse de groupe de l'onde de polarisation  $u_0$ . On remarque que l'équation eikonale s'écrit

$$\det(l(x, t, \xi, \tau))|_{\tau=\partial_t \phi, \xi=\partial_x \phi} = 0.$$

Si on résout  $l(x, t, \xi, \tau) = 0$ , on trouve  $\tau = \tau(x, t, \xi)$  et l'équation donnant la polarisation est

$$\tau(x, t, \partial_x \phi) A_0(x, t) u_0(x, t) + \sum_{j=1}^{j=d} \partial_{x_j} \phi(x, t) A_j(x, t) u_0(x, t) = 0.$$



## Wave propagation and bicharacteristics.

This chapter studies the propagation of a solution of the wave equation along the characteristics, as the theory of geometrical optics can teach us. We demonstrate the propagation results of geometrical optics in vacuum as long as the ray does not meet the caustic.

Note that this is a local problem: indeed, this solution of the the wave equation can come from sources located outside the computational domain, and in this case the wave equation is not satisfied globally (because it is not satisfied in the vicinity of the sources).

In order to use the asymptotic theory, we consider the Helmholtz equation. An asymptotic result obtained for the Helmholtz equation is equivalent, after inverse Fourier transform in time, to a result of propagation result for the wave equation. We then consider the system of equations :

$$(0.46) \quad \begin{cases} (\Delta + k^2)u(x, k) = 0, \\ u(x, k)|_{\Sigma} = A(x, k)|_{\Sigma}e^{ik\phi_0}, \\ S(u)(x, k) = O(\frac{1}{|x|}). \end{cases}$$

where  $\Sigma$  is a given smooth surface, locally flat (i.e. tangent plane),  $\phi_0$  is a constant (in other words, the other words, the phase of the wave on  $\Sigma$  is constant). We will restrict ourselves to the two most common cases where  $\Sigma$  is a half space or where  $\Sigma$  is the boundary of an open set  $\Theta$ . The function  $A$  verifies the definition 1.1. The condition

$$(0.47) \quad u(x, k)|_{\Sigma} = A(x, k)|_{\Sigma}e^{ik\phi_0}$$

is an initial condition, with the same abuse of language as in the previous chapter for the representation of  $u(x, t) = \int e^{ik\phi(x)+ikct}a(x, k)dk$  whose wavefront is  $\phi(x) + ct = 0$ . The name "initial condition" should be replaced by the expression "condition at  $t^* = -\phi_0/c$ ". We refer the reader to Section 4 for more details on this equivalence.

The condition  $S(u)(x, k) = O(\frac{1}{|x|})$  is a Sommerfeld condition at infinity, called the outgoing condition. It indicates for example that the wave is defined only on one side of  $\Sigma$  for  $t = t^*$  (with remainder terms, which can be either decreasing faster than any inverse power of  $k$ , or exponentially decreasing in  $k$  depending on the regularity of the solution).

We know that this problem locally admits a unique solution, by the Holmgren's theorem for example, or by uniqueness results of the Dirichlet problem at the boundary. We consider this solution  $u(x, k)$ .

Let  $W(x_0)$  be the Weingarten matrix, matrix of curvature of  $\Sigma_0$  at  $x_0$ , given by  $W(x_0) = \nabla N(x_0)$  where  $N(x_0)$  is the normal unit vector. If  $\Sigma_0$  has at least one negative curvature, denote by  $R_0 < 0$  the largest negative curvature.

### 1. Asymptotic solution before caustic points

All geometrical elements of  $\Sigma_0$  yield the asymptotic solution when  $A$  is given thanks to

**THEOREM 3.1.** *Let  $u$  be the unique solution of (0.46), with (0.47) as boundary condition on  $\Sigma$ . Assume  $A$  has an asymptotic expansion  $a_j^0$ .*

There is an asymptotic representation of the solution  $u$  as  $a(\cdot, k)e^{ik\phi(\cdot)}$  where  $a, \phi$  solve

$$(1.48) \quad \begin{cases} |\nabla\phi|^2 = 1, \\ \nabla\phi \cdot \nabla a_0 + \frac{1}{2}\Delta\phi a_0 = 0, \\ \nabla\phi \cdot \nabla a_p + \frac{1}{2}\Delta\phi a_p = -\frac{1}{2}\Delta a_{p-1}. \end{cases}$$

One has

- (1) The characteristics  $x(x_0, \cdot)$ , solutions of  $\frac{d}{dt}(x(x_0, t)) = \nabla\phi(x(x_0, t))$  with  $x(x_0, 0) = x_0$  are straight lines  $x(x_0, t) = x_0 + t\nabla\phi(x_0)$ , the gradient of the phase is constant on each characteristic, and  $\phi(x(x_0, t)) = \phi_0 + t$ .

- (2) The leading order term  $a_0$  is given by

$$(1.49) \quad a_0(x(t)) = a_0(x(0))\exp\left(-\frac{1}{2}\int_0^t \Delta\phi(x(u))du\right).$$

Remark that  $a_0$  is solution of

$$(1.50) \quad \operatorname{div}(\nabla\phi|a_0|^2) = 0.$$

and we obtain the invariance of  $|a_0|^2$  along sections of tube of rays.

All other terms are given by the relations

$$(1.51) \quad a_p(x(t)) = a_0(x(t))\left[\frac{a_p(x(0))}{a_0(x(0))} - \frac{1}{2}\int_0^t ds \frac{\Delta a_{p-1}(x(s))}{a_0(x(s))}\right].$$

- (3) The amplitude along the ray issued from  $x_0$  is given also thanks to the curvatures of  $\Sigma$  at  $x_0$ . Every term of the asymptotic expansion  $a_j(x_0 + tN(x_0))$  is in  $C^\infty([0, T])$  for a given  $T < -R_0$ , they are all going to  $+\infty$  for the smallest  $t$  such that  $\det(Id + tW(x_0)) = 0$ . The leading order term is given by

$$(1.52) \quad a_0(x_0 + tN(x_0)) = \frac{a_0(x_0)}{(\det(Id + tW(x_0)))^{\frac{1}{2}}}.$$

- (4) The terms  $a_p$ ,  $p \geq 1$  of (1.48) are given by the  $C^\infty$  functions for  $t < T$ :

$$a_p(x(t)) = \frac{a_p(x_0) - \frac{1}{2}\int_0^t \Delta a_{p-1}(x(s))(\det(Id + sW(x_0)))^{\frac{1}{2}} ds}{(\det(Id + tW(x_0)))^{\frac{1}{2}}}.$$

- (5) Assume  $\Omega$  totally characteristic for  $\Sigma_0$  at time  $T$  (i.e. any point of  $\Omega$  is reached by a point of the form  $x(t)$ ,  $x_0 \in \Sigma_0$ ,  $t < T$  and the transformation  $\Sigma_0 \times [0, T[ \rightarrow \Omega$  is a diffeomorphism on its image), there exists  $a(x, k) \in C^\infty([1, +\infty), C^\infty(\Omega))$  such that

$$(k^{-2}\Delta + 1)a(x, k)e^{ik\phi(x)} \simeq 0.$$

This is in particular the case when all curvatures of  $\Sigma_0$  are strictly positive, and in this case  $T = +\infty$ .

PROOF. We construct an expansion induced by the asymptotic expansion of  $A$  on  $\Sigma$ . We write  $u(x, k) = a(x, k)e^{ik\phi(x)}$ . The eikonal and transport equations (1.48) were obtained in the chapter 1.

The introduction of  $\Sigma$  its interpretation and its use will be detailed in the chapter 12. From the relation  $|\nabla\phi|^2 = 1$ , we deduce, by differentiating with respect to each variable, the relation

$$\operatorname{Hess}\phi \nabla\phi = 0,$$

the matrix  $\operatorname{Hess}\phi$  being the hessian matrix, symmetric, given by

$$\operatorname{Hess}\phi = \begin{pmatrix} \frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial x\partial z} \\ \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial y^2} & \frac{\partial^2\phi}{\partial y\partial z} \\ \frac{\partial^2\phi}{\partial x\partial z} & \frac{\partial^2\phi}{\partial y\partial z} & \frac{\partial^2\phi}{\partial z^2} \end{pmatrix}.$$

We introduce the characteristics of the Hamilton-Jacobi equation  $|\nabla\phi|^2 = 1$ , curves  $x(t)$  solutions of the system

$$\frac{d}{dt}x(t) = \nabla\phi(x(t)), x(0) = x_0 \in \Sigma.$$

We verify that  $\frac{d}{dt}(\nabla\phi(x(t))) = \text{Hess}\phi(x(t))\frac{d}{dt}x(t) = \text{Hess}\phi\nabla\phi = 0$ , hence  $\nabla\phi$  is constant on the characteristics, and

$$x(t) = x_0 + t\nabla\phi(x_0).$$

The characteristic curves are straight lines, and  $\nabla\phi$  is constant on these lines. Moreover, as

$$\frac{d}{dt}(\phi(x(t))) = \nabla\phi \cdot \frac{d}{dt}x(t) = 1,$$

one checks that  $\phi(x(t)) = \phi_0 + t \text{ car } x_0 \in \Sigma$ .

Finally, as  $\phi(x) = \phi_0$  sur  $\Sigma$ ,  $s = 0$ , we give ourselves a curve  $\gamma(s) \subset \Sigma$  passing through the point  $x_0$  at  $\nabla\phi \cdot \gamma'(0) = 0$  which indicates, since  $\gamma$  is arbitrary, that  $\nabla\phi(x_0)$  is orthogonal to the plane tangent to  $\Sigma$  at  $x_0$ .

Choosing an orientation, defining a side of the surface  $\Sigma$  thanks to  $\nabla\phi(x_0) = \vec{N}_0$  ( $\vec{N}_0$  will be called normal outgoing normal to  $\Sigma$  at  $x_0$ ) and admitting the continuity of  $\phi$ , we verify that, in the neighborhood of  $x_0$ ,  $\nabla\phi(x)$  is the normal vector normal exiting  $\Sigma$  at  $x$ . The choice of the orientation is very dependent on the the choice of the outgoing condition, we do not go into the details of the results that the reader can find for example in [57].

The vector  $\nabla\phi(x(t))$ , equal to  $\nabla\phi(x(0))$ , is also the unit normal vector exiting at  $\Sigma_t = x_0, \phi(x) = \phi_0 + t$  at point  $x(t)$ .

- The transport equation rewrites

$$\frac{d}{dt}(a_0(x(t))) + \frac{1}{2}\Delta\phi(x(t))a_0(x(t)) = 0,$$

hence one gets (1.49). Similarly, considering the inhomogeneous transport equation of which  $a_p$  is a solution, and using the method of variation of the constant, we find (1.51). Indeed, the equation characterizing the term  $a_p$  is then

$$\frac{d}{dt}(a_p(x(t))) + \frac{1}{2}\Delta\phi(x(t))a_p(x(t)) = -\frac{1}{2}\Delta a_{p-1}(x(t)).$$

The solution of the homogeneous equation is  $a_0(x(t))$ , so we write

$$a_p(x(t)) = C(t)a_0(x(t)),$$

which gives

$$C'(t) = -\frac{1}{2} \frac{\Delta a_{p-1}(x(t))}{a_0(x(t))}.$$

The result (1.51) follows.

We can therefore calculate all the terms of the transport equation in terms of function of  $\Delta\phi$ , or, which is equivalent, as a function of  $a_0$ .

In the system (1.48), we multiply the equation determining  $a_0$  by  $\bar{a}_0$ . We obtain

$$2\nabla\phi\nabla a_0\bar{a}_0 + \Delta\phi|a_0|^2 = 0.$$

Taking the conjugated expression (we suppose that there exists  $\alpha \in \mathbb{C}$  such that  $\phi(x)/\alpha \in \mathbb{R}$  for all  $x$  and we divide the equation by  $\alpha$  to get back to  $\phi$  real):

$$2\nabla\phi\nabla\bar{a}_0a_0 + \Delta\phi|a_0|^2 = 0.$$

Summing the two equalities, we obtain  $\nabla\phi\nabla(|a_0|^2) + \Delta\phi|a_0|^2 = 0$ , so we obtain (1.50). We consider a tube of radius  $T$ , supported on isophase surfaces  $\Sigma_-$  (corresponding to  $t = t_1$ ) and  $\Sigma_+$  (corresponding to  $t = t_+$ ), the normal vector outside  $T$  on  $\Sigma_-$  is  $-\nabla\phi(x(t_-))$ , and the unit normal vector exterior to  $T$  on  $\Sigma_+$  is  $\nabla\phi(x(t_+))$ . It is moreover bounded by definition, so  $\partial T = \Sigma_- \cup \Sigma_+ \cup \tilde{T}$ , with, at any point of  $\Sigma$ ,  $n \cdot \nabla\phi = 0$ .



The Stokes formula on  $T$  gives

$$\int_T \operatorname{div} (|a_0|^2 \nabla \phi) dx = \int_{\partial T} (|a_0|^2 \nabla \phi) \cdot n d\sigma = \int_{\Sigma_+} |a_0|^2 d_{\Sigma_+} \sigma - \int_{\Sigma_-} |a_0|^2 d_{\Sigma_-} \sigma.$$

Equality  $\operatorname{div} (|a_0|^2 \nabla \phi) = 0$  yields

$$\int_{\Sigma_+} |a_0|^2 d_{\Sigma_+} \sigma = \int_{\Sigma_-} |a_0|^2 d_{\Sigma_-} \sigma,$$

which is also expressed as the conservation of energy on the tubes of rays. We now interpret  $\exp(\int_0^t \Delta \phi(x(u)) du)$  as being the measure function associated to

- Explicit calculations of  $\Delta \phi$  and of  $a_0$ :

Relations (1.49) and (1.51) show that  $\Delta \phi$  plays an essential role in the calculation of the coefficients. We rely here on the relation

$$\Delta \phi = \operatorname{Tr}(\operatorname{Hess} \phi),$$

where  $\operatorname{Hess} \phi$  is the Hessian matrix of the  $\phi$  phase. Let  $\tau$  be a vector tangent to  $\Sigma_0$  at  $x_0$ . The matrix of curvature matrix of  $\Sigma_0$ , traditionally called Weingarten matrix and denoted by  $W(x_0)$ , is given by

$$\operatorname{grad} N(x_0) = W(x_0).$$

its eigenvectors on  $\Sigma$  are the directions of curvature, its eigenvalues are the principal curvatures. This matrix is defined on the space tangent to the surface  $\Sigma$  at  $x_0$ .

Using this matrix, we determine the Hessian matrix of the phase  $\phi$ , which will give, by computing the trace of this hessian matrix, the Laplacian of  $\phi$ .

Let  $P$  be a plane passing through  $x_0$  containing  $N(x_0)$ ,  $x_0 \in \Sigma_0$  and be the curve  $\gamma$  drawn on  $\Sigma_0$ , parametrized by the curvilinear abscissa curvilinear  $u$ , equal to  $P \cap \Sigma_0$  with  $\gamma(0) = x_0$ . The vector  $\gamma'(0)$  is a tangent vector to  $\Sigma_0$  at  $\gamma(0)$  (and  $N(x_0), \gamma'(0)$  is a basis of  $P$ ). Then the components of the vector  $\gamma''(0)$  on the basis given by  $(\gamma'(0), N(\gamma(0)) \wedge \gamma'(0))$  are respectively the curvature and torsion of the  $\gamma$  curve. We have the relation  $\gamma''(0) = W(\gamma(0))\gamma'(0)$ .

The relation  $N(\gamma(u)) = \nabla \phi(\gamma(u))$  indicates that

$$W(\gamma(u))\gamma'(u) = \operatorname{grad} N(\gamma(u))\gamma'(u) = \operatorname{Hess} \phi(\gamma(u))\gamma'(u).$$

This implies that  $W(\gamma(0))$  and  $\operatorname{Hess}(\gamma(0))$  coincide on the tangent plane to  $\Sigma_0$  at  $\gamma(0)$ . Moreover, as  $\nabla \phi(\gamma(u))$  is of norm 1, we can derive with respect to  $u$  the equality

$$\|\nabla \phi(\gamma(u))\|^2 = 1,$$

which yields  $\operatorname{Hess} \phi(\gamma(u))(\nabla \phi(\gamma(u)), \gamma'(u)) = 0$ .

The vector  $\nabla \phi(\gamma(u))$  is in its kernel (because  $\operatorname{Hess} \phi \nabla \phi = 0$ ) and the Hessian matrix of  $\nabla \phi$  is a symmetric matrix, so it is diagonalizable in an orthonormal basis. The tangent plane to the manifold at  $x_0$ , which is orthogonal to the line  $\mathbb{R}N(x_0)$  is stable. If we decompose  $\operatorname{Hess} \phi$  on the normal vector  $N(x_0)$  and on the vectors which diagonalize  $W$ . As  $W$  and  $\operatorname{Hess} \phi$  coincide on the plane, we find that  $\operatorname{Tr} \operatorname{Hess} \phi = 0 + \kappa_1 + \kappa_2$ . We have thus proved that  $\Delta \phi(x_0) = \operatorname{Tr} W(x_0)$ . We deduce that  $\Delta \phi(x_0)$  is the sum of the principal curvatures of  $\Sigma_0$  in  $x_0$ . The Hessian of the phase can be identified with the application on the tangent plane given by the Weingarten matrix.

Let us finally study  $\Sigma_t$ . A point of  $\Sigma_t$  can be written as  $x_0 + t\nabla \phi(x_0)$ . We can construct the curve  $\gamma_t(v) \subset \Sigma_t$  from the curve  $\gamma(u) \subset \Sigma_0$  by constructing the points of the form  $\gamma_t(u) = \gamma(u) + tN(\gamma(u))$ . As  $u$  is not a curvilinear abscissa on  $\gamma_t(u)$ , we normalize  $u$  to obtain a curvilinear abscissa. To do this, we write

$$\frac{d}{du}(\gamma_t(u)) = \gamma'(u) + t \operatorname{grad} N(\gamma(u))\gamma'(u).$$

As  $\operatorname{grad} N(\gamma(u)) = W(\gamma(u))$ , one gets

$$\frac{d}{du}(\gamma_t(u)) = (Id + tW(\gamma(u)))\gamma'(u).$$

Moreover, the equality  $\nabla\phi(\gamma_t(u)) = \nabla\phi(\gamma(u))$  yields, after derivation

$$\text{Hess}\phi(\gamma_t(u))\frac{d}{du}(\gamma_t(u)) = \text{Hess}\phi(\gamma(u))\gamma'(u),$$

hence

$$\text{Hess}\phi(\gamma_t(0))(Id + tW(\gamma(0)))\gamma'(0) = \text{Hess}\phi(\gamma(0))\gamma'(0).$$

To obtain the Hessian of  $\phi$  on  $\Sigma_t$ , we assume that the matrix  $Id + tW(\gamma(0))$  is invertible. In this case, we find

$$(1.53) \quad \text{Hess}\phi(\gamma_t(0))\tau = W(\gamma(0))(Id + tW(\gamma(0)))^{-1}\tau$$

for any  $\tau$  tangent vector to  $\Sigma_t$ , otherwise it is verified only on  $\text{Im}(Id + tW(\gamma(0)))$ .

Proving (1.52) is now possible. Recall that  $x(t)$  depends on  $x_0$ , in the sense that  $x(0) = x_0 = \gamma(0)$ . We deduce from the equality (1.53), using again  $\text{Hess}\phi(\gamma_t(0))N(\gamma_t(0)) = 0$ , the equality

$$(1.54) \quad \text{Hess}\phi(x(t)) = \text{Hess}\phi(x_0)(Id + tW(x_0))^{-1}.$$

We deduce from (1.54) the value of  $a_0$ . Indeed,

$$\Delta\phi(x(s)) = \text{Tr}(\text{Hess}\phi(\gamma_s(0))) = \text{Tr}(\text{Hess}\phi(x_0)(Id + sW(x_0))^{-1}).$$

We show that

$$(1.55) \quad \frac{d}{ds_1}\text{Log}(\det[Id + (s+s_1)W(x_0)])|_{s_1=0} = \frac{\det(Id + sW(x_0))\text{Tr}[W(x_0)(Id + sW(x_0))^{-1}]}{\det(Id + sW(x_0))}$$

Indeed, equality

$$\det(Id + (s + s_1)W(x_0)) = \det(Id + sW(x_0))[1 + s_1\text{Tr}[W(x_0)(Id + sW(x_0))^{-1}]] + O(s_1^2),$$

and the limited expansion of the determinant of a matrix allows us to obtain the derivative with respect to  $s_1$ . As

$$\frac{d}{ds}\text{Log}(\det[Id + sW(x_0)]) = \frac{\det(Id + sW(x_0))\text{Tr}[W(x_0)(Id + sW(x_0))^{-1}]}{\det(Id + sW(x_0))},$$

one deduces  $\int_0^t \Delta\phi(x(s))ds = \text{Log}(\det(Id + tW(x_0)))$ . We have the equality (1.52) and obtained the principal term.

To obtain the term  $a_1$ , we recall the equation satisfied by the term  $a_j$ ; it is

$$2\frac{da_j}{ds} + \Delta\phi a_j = -i\Delta a_{j-1}$$

which rewrites

$$\frac{db_j}{ds} = -\frac{i}{2}a_0^{-1}\Delta a_{j-1}$$

where  $b_j = a_j/a_0$ . Hence

$$b_j(x(s)) = b_j(x_0) - \frac{i}{2}\int_0^t a_0(x(s))^{-1}\Delta a_{j-1}(x(s))ds.$$

With the hypothesis  $t \in [0, T[$ ,  $T$  being the first value of  $t$  for for which  $\det(Id + tW(x_0)) = 0$ , on the characteristic from  $x_0$ . Assuming the two principal curvatures of  $\Sigma_0$  equal to  $\kappa_1$  and  $\kappa_2$ , which can be negative,  $\det(Id + tW(x_0)) = (1 + t\kappa_1)(1 + t\kappa_2)$  and it comes  $T = +\infty$  if the two curvatures are positive,  $T = \min(-\kappa_1^{-1}, -\kappa_2^{-1})$  otherwise. When  $T < +\infty$ , **we say that the point  $x(T)$  is a point of the caustic associated with the phase**. The last item of Theorem 3.1 comes from Borel's theorem (Theorem 1.2) that a sequence of functions of  $C^\infty(\Omega)$  allows to construct a function  $a(x, k)$ , in  $C^\infty([1, +\infty], C^\infty(\Omega))$ , which admits as asymptotic expansion in  $k$  the formal series of  $a_p(x)(ik)^{-p}$ . We can then evaluate  $(k^{-2}\Delta + 1)(a(x, k)e^{ik\phi(x)})$  by (4.11). Fixing then  $M$ , order of truncation of the series series and noting  $a_M(x, k)$  the sum

of the first  $M$  terms of the series  $a_j(x)(ik)^{-j}$ , we apply the result of Borel's Lemma (Theorem 1.2). This proves Theorem 3.1.  $\square$

## 2. Explicit expression after a caustic point

We now consider the solution of the system

$$(\Delta + k^2)u(x, k) = 0, u(x, k)|_{\Sigma} = A(x, k)|_{\Sigma} e^{ik\phi_0}$$

and that this solution exists everywhere. For any point  $x_0$  of  $\Sigma$ , consider the associated radius  $x_0 + t\nabla\phi(x_0) = x_0 + tN(x_0)$ . We assume that, on  $[0, T]$ , the matrix  $(I + tW(x_0))$  is singular only in  $t = t_0$ ,  $0 < t_0 < T$ . The radius does not have a singularity (this is consistent with the definition of caustics, which correspond to an accumulation of rays and not to the singularity of a ray).

We show that the amplitude can be computed for any  $t \in ]t_0, T]$  knowing the amplitude in the neighborhood of  $x(T)$ . We prove

LEMMA 3.1. *Let  $t$  be such that  $t_0 < t \leq T$ . The solution  $a_0$  of (1.48) is given by the relation*

$$a_0(x(t)) = \frac{a_0(x(T))}{(\det(Id + (t - T)W(x_0)))^{\frac{1}{2}}} e^{ik(T-t)}.$$

PROOF. We suppose that  $u(x + TN(x), k)$  admits an asymptotic expansion (asymptotic in  $k$ ) in the neighborhood of  $x_0$ . We express the amplitude at any point of the form  $x + tN(x)$ ,  $t_0 < t \leq T$  from the expansion of  $u(x + TN(x), k)$  in the neighborhood of the point  $x + tN(x)$ . From Theorem 3.1, we deduce that  $(\Delta + k^2)u = 0, u(x, k)|_{\Sigma_T} = A^T(x, k)|_{\Sigma_T} e^{ik(\phi_0 + T)}$  admits a solution as a function of  $Id + tW(x)$ . We choose for that the backward direction on the characteristic whose equation is  $\frac{d}{dt}x(t) = -N(x + TN(x))$ . Let  $t_0$  be the smallest positive solution of  $\det(Id + t_0W(x_0)) = 0$ . Thus the amplitude  $a_0(x(t))$  of Theorem 3.1 diverges when  $t \rightarrow t_0, t < t_0$ . Consider the solution of the system of eikonal and transport equations (1.48), parameterized by  $s$  :

$$(2.56) \quad \begin{cases} \nabla\phi(y(s)) = -\nabla\phi(x_0), \\ y(s) = x(T) - s\nabla\phi(x_0), \\ \nabla\phi(y(s)) \cdot \nabla a_0(y(s)) + \frac{1}{2}\Delta\phi(y(s))a_0(y(s)) = 0. \end{cases}$$

Let  $\Sigma_T$  be the set of points  $x$  such that  $\phi(x) = t_0 + T$ . Let  $W_T$  be the Weingarten matrix of  $\Sigma_T$ . For  $s = 0$ , we notice that  $y(0) = x(T) = x_0 + TN(x_0) \in \Sigma_T$  and that  $\nabla\phi(y(0)) = -\nabla\phi(x_0)$ . The point  $y(0)$  belongs to the surface  $\Sigma_T$ . From Theorem 3.1, we deduce  $y(s) = x(T - s)$  and

$$a_0(y(s)) = \frac{a_0(y(0))}{(\det(Id - sW_T(x_0)))^{\frac{1}{2}}}.$$

Let us also notice that when  $\Gamma(u)$  is a curve on  $\Sigma_T$  and that  $\Gamma_s$  is its image in the translation by  $-s\nabla\phi(x_0)$ ,

$$\frac{d}{du}(\Gamma_s(u)) = (Id - sW_T(\Gamma(u)))\Gamma'(u).$$

The 0 order transport equation of (2.56) is transformed into

$$\frac{d}{ds}(\text{Log}(a_0(y(s)))) + \frac{1}{2} \frac{d}{ds} \text{Log}(\det(Id - sW_T(x_0))) = 0.$$

This completes the proof of the lemma.  $\square$

To determine the expression of  $a_0(x(T))$  as a function of function of  $a_0(x_0)$  we need the tool called **stationary phase**. Go to the chapter 4, Section 8 for the solution of this problem.

REMARK 2. *The energy flow*

$$(2.57) \quad \Phi(t) = \int_{S_t} |a_0|^2(x) d\sigma,$$

where  $d\sigma$  is the surface measure on  $S_t = \Sigma_t \cap T_{t_1, t_2}$  is conserved along the rays.

Indeed, for  $(t_1, t_2) \in [0, t_0[$  or  $(t_1, t_2) \in ]t_0, T]^2$ , we define a tube of radius by its section  $V_{t_1}$  (which is a neighborhood of  $x(t_1)$  in  $\Sigma_{t_1}$ , and which can also be be constructed as the intersection of a neighborhood of  $x(t_1)$  in  $\mathbb{R}^3$ ) and by

$$T_{t_1, t_2} = \{x + tN_{t_1}(x), 0 \leq t \leq t_2 - t_1, x \in \Sigma_{t_1} \cap V_{t_1}\}.$$

We note then that (2.57) is independent of  $t$  for  $t \in [0, t_0[$  as for  $t \in ]t_0, T]$  F We prove in section 8 that  $\phi(0)$  and  $\phi(T)$  are propotional (and  $\phi(T) = \phi(0)$ , result used in [57]).

### 3. Generalization to the case of a wave equation with a metric $A(x)$

Consider a positive symmetric matrix  $A(x)$ . We consider the following wave equation :

$$Pu = [\partial_{t^2}^2 - \sum_{j,i} \partial_{x_j} (A_{ji}(x) \partial_{x_i})]u = 0.$$

We look for the solution of the eikonal equation in time and space, as well as and of the transport equation. For this purpose, we consider a function  $u(x, t, k) = a(x, t, k)e^{ik\phi(x, t)}$  where  $a$  has the asymptotic expansion

$$a(x, t, k) \simeq \sum_{j \geq 0} a_j(x, t)(ik)^{-j}.$$

Hence

$$Pu(x, t, k) = k^2 U(x, t, k)e^{ik\phi(x, t)}$$

where  $a(x, t, k) \simeq \sum_{j \geq 0} a_j(x, t)(ik)^{-j}$  implies

$$U(x, t, k) \simeq P_j(a)(x, t)(ik)^{-j}.$$

One checks

$$P_0(a)(x, t) = a_0(x, t)((A(x)\nabla_x\phi, \nabla_x\phi) - (\partial_t\phi)^2)$$

$$P_1(x, t) = a_1(x, t)((A(x)\nabla_x\phi, \nabla_x\phi) - (\partial_t\phi)^2) - 2\partial_t\phi\partial_t a_0 + 2(A(x)\nabla_x\phi, \nabla_x a_0) + a_0(x, t)P\phi.$$

The eikonal equation is thus

$$\partial_t\phi(x, t) = -(A(x)\nabla_x\phi, \nabla_x\phi)^{\frac{1}{2}}.$$

We consider particular functions  $\phi$ , adapted to the wave propagation:  $\phi(x, t) = \psi(x) - t$ . Recall that this choice amounts to evaluate the surface  $\phi(x, t) = 0$  and to solve it in time.

Consider  $f$  a solution of

$$\partial_t f + (A(x)\nabla_x\psi(x)) \cdot \nabla_x f = 0$$

and  $h$  the solution of the transport equation with initial condition

$$\begin{aligned} \partial_t h + (A(x)\nabla_x\psi) \cdot \nabla_x h + \nabla_x \cdot (A(x)\nabla_x\psi)h &= 0, \\ h(x, 0) &= 1 \end{aligned}$$

We introduce the application of  $V \subset \mathbb{R}^n$  in  $W \subset \mathbb{R}^n$  which, at  $x$  associates  $X(x, t)$ , the solution of

$$\begin{aligned} \frac{dX}{dt} &= A(X(x, t))\nabla_x\psi(X(x, t)), \\ X(x, 0) &= x. \end{aligned}$$

We demonstrate

LEMMA 3.2. *We consider the eikonal equation (which solution is  $\psi(x)$ ) and the transport equation deduced from the wave operator in a Riemannian metric  $A$ . The characteristics  $X(x, t)$ , solution of*

$$\frac{dx}{dt} = A(x(t))\nabla_x\psi(x(t)), x(0) = x$$

satisfy

$$a_0(x(t), t)(\det \frac{dX}{dx})^{-\frac{1}{2}} = a_0(x, 0).$$

This reflects the conservation of energy on the ray tubes.

PROOF. First, checks

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} f(x, t)h(x, t)dx &= \int_{\mathbf{R}^n} (\partial_t fh + \partial_t hf)dx = - \int_{\mathbf{R}^n} \{ (A(x)\nabla_x\psi) \cdot \nabla_x(fh) + \nabla_x \cdot (A(x)\nabla_x\psi)(fh) \} dx \\ &= - \int_{\mathbf{R}^n} \partial_x(A(x)\nabla_x\psi fh)dx = 0. \end{aligned}$$

On the other hand, at  $t$  fixed,  $X(x, t)$  is a diffeomorphism of  $\mathbf{R}^n$ . We know that, since  $f$  is invariant on the characteristics, namely  $f(X(x, t)) = f(x)$ , we find that

$$\int_{\mathbf{R}^n} f(x, 0)dx = \int_{\mathbf{R}^n} f(X(x, t), t)dx = \int_{\mathbf{R}^n} f(X, t) \frac{dX}{|\frac{dX}{dx}(X, t)|},$$

This implies, using  $h(x, 0) = 1$

$$\int_{\mathbf{R}^n} f(x, t)h(x, t)dx = \int_{\mathbf{R}^n} f(x, 0)h(x, 0)dx = \int_{\mathbf{R}^n} f(x, 0)dx = \int_{\mathbf{R}^n} \frac{f(X, t)}{|\frac{dX}{dx}|} dX.$$

In particular, this equality is true in the case

$$(h(x, t))^{-1} = \left| \frac{dX}{dx} \right|.$$

We will prove this equality by calculating  $h$  and  $|\frac{dX}{dx}|$  separately.

Let  $\sigma(x) = \operatorname{div}[A(x)\nabla_x\psi(x)]$ . The function  $h(X(x, t), t)$  is a solution of

$$\begin{aligned} \partial_t(h(X(x, t), t)) &= \partial_t h(X(x, t), t) + \nabla_x h(X(x, t), t) \cdot \frac{dx}{dt} \\ &= -(A(X(x, t))\nabla_x\psi(X(x, t))) \cdot \nabla_x h(X(x, t), t) \\ &\quad - \operatorname{div}(A(X(x, t))\nabla_x\psi(X(x, t)))h(X(x, t), t) \\ &\quad + \nabla_x h(X(x, t), t) \cdot (A(X(x, t))\nabla_x\psi(X(x, t))) \\ &= -\sigma(X(x, t))h(X(x, t), t). \end{aligned}$$

Hence

$$h(x(t), t) = h(x, 0)e^{-\int_0^t \sigma(X(x, s))ds}.$$

The transport equation on  $a_0$ , which was obtained above, is

$$-2\partial_t\phi\partial_t a_0 + 2(A(x)\nabla_x\phi, \nabla_x a_0) - a_0(x, t)[\partial_{t^2}^2\phi - \sum_{j,l} \partial_{x_j}(A_{jl}(x)\partial_{x_l}\phi)] = 0.$$

By multiplying by  $\bar{a}_0$  and taking the real part of the the expression obtained, we find

$$-\partial_t\phi\partial_t(|a_0|^2) - \partial_{t^2}^2\phi|a_0|^2 + (A(x)\nabla_x\phi, \nabla_x|a_0|^2) + \sum_{j,l} \partial_{x_j}(A_{jl}(x)\partial_{x_l}\phi)|a_0|^2 = 0.$$

One is left with

$$-\partial_t[|a_0|^2\partial_t\phi] + \sum_{j,l} \partial_{x_l}[A_{jl}(x)\partial_{x_l}\phi|a_0|^2] = 0.$$

When the phase  $\phi(x, t)$  is equal to  $\psi(x) - t$ , one deduces that

$$\partial_t|a_0|^2 + \sum_{j,l} \partial_{x_j}[A_{jl}(x)\partial_{x_l}\psi|a_0|^2] = 0.$$

The equation of characteristics is

$$\frac{dX_i}{dt} = \sum_j A_{ij}(X(x, t)) \frac{\partial \psi}{\partial x_j}(X(x, t)).$$

By deriving it with respect to  $x_p$ , we find

$$\frac{d}{dt} \frac{\partial X_i}{\partial x_p} = \sum_q \frac{\partial}{\partial X_q} \left[ \sum_j A_{ij}(X) \frac{\partial \psi}{\partial x_j}(X) \right]_{X=X(x,t)} \frac{\partial X_q}{\partial x_p}(x, t).$$

Denote by  $B_{iq}(x, t) = \frac{\partial}{\partial X_q} \left[ \sum_j A_{ij}(X) \frac{\partial \psi}{\partial x_j}(X) \right]_{X=X(x,t)}$ . From the identity

$$\frac{d}{dt} \left| \frac{dX}{dx} \right| = \sum_l \begin{vmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \cdots & \frac{\partial X_1}{\partial x_d} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \cdots & \frac{\partial X_2}{\partial x_d} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_q B_{ql} \frac{\partial X_q}{\partial x_1} & \sum_q B_{ql} \frac{\partial X_q}{\partial x_2} & \cdots & \sum_q B_{ql} \frac{\partial X_q}{\partial x_d} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_d}{\partial x_1} & \frac{\partial X_d}{\partial x_2} & \cdots & \frac{\partial X_d}{\partial x_d} \end{vmatrix},$$

we deduce, by developing with respect to the  $q$ -th line and using the fact that a determinant with two identical lines is zero, the equality

$$\frac{d}{dt} \left| \frac{dX}{dx} \right| = \sum_l B_{lq} \delta_{lq} \left| \frac{dX}{dx} \right|.$$

Note finally that

$$\sum_l B_{ll}(x, t) = \sum_{l,j} \frac{\partial A_{lj}}{\partial x_l}(X(x, t)) \frac{\partial \psi}{\partial x_j}(X(x, t)) + A_{lj}(X(x, t)) \frac{\partial^2 \psi}{\partial x_j \partial x_l}(X(x, t)) = \sigma(x).$$

Quantity  $\left| \frac{dX}{dx}(t) \right|$  and  $h^{-1}(t)$  are solution of the same first order differential equation, and as for  $t = 0$ ,  $X(x, 0) = x$ ,  $\left| \frac{dX}{dx}(x, 0) \right| = 1 = h^{-1}(0)$ . One obtains thus  $\left| \frac{dX}{dx}(t) \right| = h^{-1}(t)$ . The equation for  $|a_0(X(x, t), t)|^2$  being the same as the equation on  $h(x, t)$ , one deduces

$$\begin{aligned} |a_0(X(x, t), t)|^2 &= |a_0(x(0), 0)|^2 e^{-\int_0^t \sigma(X(x, s)) ds} \\ \Rightarrow \\ a_0(X(x, t), t) \left( \left| \frac{dX(x, t)}{dx} \right| \right)^{-\frac{1}{2}} &= a_0(x, 0). \end{aligned}$$

□

#### 4. Exercises of chapter 3

Exercice 3.1 : Propagation of a wave. calculate, for all point of the space  $\mathbb{R}^3$  :

1) a wave, propagating at velocity 1, centered at  $O$  at the time  $t = -T$ , which amplitude is known at  $t = 0$  on the sphere of radius  $T$ .

2) a wave, propagating at velocity 1, which wavefront at  $t = 0$  is the sphere  $S_0$  of center  $O$  and of radius  $T$  propagating in the direction of the normal unit vector directed towards the center of  $S_0$ .

Exercice 3.2 : Wave equation with non constant velocity. We consider the wave equation which velocity depends only on the position  $x \in \mathbb{R}^d$  :

$$(\partial_{t^2}^2 - c^{-2}(x)\Delta)u = 0.$$

1) What is the eikonal equation? What are the transport equation? Write these equations in the set of variables  $(x, t)$  and in the set of variables  $x$ .

2) Propose an equation for the characteristics such that  $\phi(x)$  satisfies  $\phi(x(s)) = \phi(x(0)) - s$ . Give the equation of the characteristics and solve the transport equations.

3) Do again the analysis for the equation  $Pu = 0$ ,  $P = \partial^2 - \text{div}(A(x)\nabla)$ .

4) Propose a change of variable in the wave equation that could be useful.

Solution de l'exercice 3.1. Nous imaginons une onde, centrée en 0 à  $t = -T$ , se propageant à la vitesse 1 dans le vide. A l'instant  $t = 0$ , cette onde a son front d'onde situé sur la sphère de rayon  $T$ . La surface  $\Sigma_0$  est donc  $\{x, |x| = T\}$ . Le vecteur  $\nabla\phi$  est donc  $\vec{e}_r$ . Comme  $\Delta\phi = \text{div}(\text{grad}\phi)$ , on vérifie que

$$\Delta\phi = \partial_x\left(\frac{x}{r}\right) + \partial_y\left(\frac{y}{r}\right) + \partial_z\left(\frac{z}{r}\right) = \frac{2}{r},$$

Utilisant le résultat précédent, on trouve que  $x(t) = x_0 + t\vec{e}_r = x_0 + t\frac{x_0}{|x_0|}$ , et donc lorsque  $x_0 \in B_T$ ,  $x(t) \in B_{t+T}$ .

On retrouve l'équation de transport

$$\frac{d}{dt}a_0(x(t)) = -\frac{a_0(x(t))}{t+T}$$

d'où

$$a_0(x(t)) = a_0(x_0)\left(1 + \frac{t}{T}\right)^{-1}$$

puis

$$a_p(x(t)) = \left(1 + \frac{t}{T}\right)^{-1} \left[ a_p(x_0) - \frac{1}{2} \int_0^t \Delta a_{p-1}(x(s)) \left(1 + \frac{s}{T}\right) ds \right]$$

Dans la deuxième application, nous supposons au contraire que le front est  $\Sigma_0$ , muni de la deuxième orientation. On vérifie alors que

$$\Sigma_t = \{x, |x| = T - t\}, x(t) = x_0 - t\frac{x_0}{|x_0|}$$

puis que  $a_0(x(t)) = a_0(x_0)\left(1 - \frac{t}{T}\right)^{-1}$ .

On vérifie donc que  $a_0(x(t))$  devient singulier pour  $t = T$ . Les rayons se focalisent tous au point 0, et tous les termes de l'équation de transport deviennent singuliers. Nous avons donc exhibé deux cas où on pouvait résoudre les équations de transport, et dans un cas  $t$  peut aller jusqu'à  $+\infty$ .

Solution de l'exercice 3.2. 1) Nous introduisons donc un petit paramètre  $\varepsilon$ . Alors on recherche une solution sous la forme

$$\sigma(x, t, \varepsilon) e^{i\psi(x, t)/\varepsilon}.$$

On obtient alors

$$\begin{aligned} & -\varepsilon^{-2} [(\partial_t \psi)^2 - c^{-2}(x)(\nabla_x \psi)^2] \\ & + i\varepsilon^{-1} [2\partial_t \psi \partial_t \sigma - 2c^{-2}(x)\nabla_x \psi \nabla_x \sigma + (\partial_t^2 \psi - c^{-2}(x)\Delta \psi)\sigma] \\ & + (\partial_t^2 - c^{-2}(x)\Delta)\sigma = 0. \end{aligned}$$

L'équation eikonale est alors

$$(\partial_t \psi)^2 = c^{-2}(x)(\nabla_x \psi)^2.$$

On peut écrire, au voisinage d'un point  $(x_0, t_0)$  avec l'hypothèse  $\partial_t \psi(x_0, t_0) \neq 0$ ,

$$\psi(x, t) - \psi(x_0, t_0) = a(x, t)(t - \phi(x)),$$

où  $t_0 = \psi(x_0)$  et  $a(x_0, t_0) \neq 0$ . On vérifie que l'équation satisfaite par  $\phi$  et par  $a$  se met sous la forme

$$\begin{aligned} & a^2(x, t)[1 - c^{-2}(x)(\nabla_x \phi(x))^2] = \\ & -(t - \phi(x)) \left[ \frac{2}{c^2} a \nabla_x a \nabla_x \phi + 2a \partial_t a + (t - \phi(x))(\partial_t a)^2 - \frac{1}{c^2} (t - \phi(x))(\nabla_x a)^2 \right], \end{aligned}$$

ce qui donne, en  $t = \phi(x)$  au voisinage de  $x = x_0$

$$(\nabla_x \phi(x))^2 = c^2(x).$$

et l'équation sur  $a$  correspondant à annuler le terme de droite entre crochets.

Obtenir l'équation de transport n'est pas évident sous cette forme; il faut plutôt revenir à une formulation du type

$$b(x, t, \varepsilon) e^{i\frac{t - \phi(x)}{\varepsilon}}$$

et obtenir l'équation de transport sur  $b$ . En effet, cela revient à imposer  $\psi(x, t) = t - \phi(x)$ , auquel cas  $\nabla_x \psi = -\nabla_x \phi$  et  $\Delta \psi = -\Delta \phi$ ,  $\partial_t^2 \psi = 0$ . Les équations de transport deviennent

$$\partial_t b + c^{-2}(x)\nabla_x \phi \nabla_x b + \frac{1}{2}c^{-2}(x)\Delta \phi b - \frac{i\varepsilon}{2}(\partial_{t^2}^2 - c^{-2}\Delta)b = 0.$$

Cette équation de transport, simple, sera étudiée plus loin. Pour l'instant, concentrons nous sur les définitions des caractéristiques. Celles-ci doivent être orthogonales aux surfaces d'onde, donc on doit avoir

$$\frac{dx}{ds} = \alpha(x(s))\nabla_x \phi(x(s)).$$

La fonction  $\phi(x(s))$  est, suivant l'hypothèse de l'énoncé, linéaire en  $s$  donc

$$\frac{d}{ds}(\phi(x(s))) = 1 = \nabla_x \phi(x(s))\frac{dx}{ds} = \alpha(x(s))(\nabla_x \phi(x(s)))^2,$$

ce qui donne  $\alpha(x) = c^{-2}(x)$ . Si les courbes caractéristiques sont définies par

$$\frac{dx}{ds} = c^{-2}(x(s))\nabla_x \phi(x(s)), \quad , x(0) \text{ donné}$$

alors  $\phi(x(s)) = \phi(x(0)) + s$ .

Notons que, contrairement au cas scalaire (vitesse constante), les rayons ne sont pas des droites. En effet, soit  $\vec{t}(x(s)) = \frac{\nabla_x \phi(x(s))}{c}$ . C'est le vecteur unitaire tangent au rayon. Alors, lorsqu'on le dérive, en utilisant  $Hess\phi \nabla_x \phi = c\nabla_x c$ , égalité qui provient de l'équation eikonale, on trouve

$$\frac{d}{ds}(\vec{t}(x(s))) = \frac{Hess\phi \frac{dx}{ds}}{c} - \frac{\nabla_x c \frac{dx}{ds}}{c^2} \nabla_x \phi = \frac{1}{c^2}(\nabla_x c - (\vec{t} \cdot \nabla_x c)\vec{t}),$$

qui est la projection orthogonale de  $\nabla_x c$  dans le plan orthogonal à  $\vec{t}$ . Ce vecteur n'a aucune raison d'être nul.





## Stationary phase theorem

In this section, we introduce one of the essential tools for a deeper understanding of wave propagation: the stationary phase theorem. We will use this name in most of the cases studied, even when dealing with the saddle point method.

Traditionally, the stationary phase methods are applied to so-called oscillatory integrals, and allow integrals, and allow to find, under a certain number of conditions, an equivalent when  $k$  tends to  $+\infty$ , of  $\int_{\Omega} a(x, k)e^{ik\phi(x)}$ . Historically, this integral was not the first to be studied. The saddle point method, on the other hand, gives an equivalent of integrals of the form  $\int_a^b \sigma(x)e^{k\psi(x)}dx$ , where the  $\psi$  phase has a non-degenerate maximum at a point  $x_0$  of the interval  $[a, b]$ . The stationary phase method differs significantly in principle from the saddle point method. Indeed, in the stationary phase theorem, the critical point considered is a stationary point for the function  $\phi(x)$  (it can be a maximum, a minimum or a saddle point) whereas the saddle point method applies only to the neighborhood of a maximum.

The integral  $\int_{\Omega} a(x, k)e^{ik\phi(x)}dx$ , with  $\phi$  real, is called an oscillatory integral, and a detailed theory is given in the chapter 8. To justify the use of this name, note that when  $\phi$  is real, the function  $e^{ik\phi(x)}$  is oscillatory. For example, if we consider  $\phi(x) = x^2$ , for  $x \in [-1, 1]$ , when  $k$  is large, the intervals where  $\phi$  takes the same values are the intervals  $[(\frac{2\pi n}{k})^{\frac{1}{2}}, (\frac{2\pi(n+1)}{k})^{\frac{1}{2}}]$ .

Thus, the stationary phase method is more general than the Laplace method, since the phase can be of the form  $e^{ik\phi(x)}$ . In all that follows, we will often call the stationary phase theorem all the results that formally amount to the computation of a non-degenerate extremum of a phase.

### 1. Laplace's method

We give the result of Laplace's method which can be found for example in the exercise book of Polya and Czego [84].

Laplace [59] introduced this method for the computation of  $\int e^{-a^2x^2} \cos rxdx$  when the bounds are  $-\infty$  and  $+\infty$  (p 107). For the calculation of an equivalent of  $\int ydx$ , Laplace introduced the representation  $y = Ye^{-t^{\mu+1}}$  where  $Y$  has a Taylor expansion (p 101). He then studied the special case  $\mu = 1$  (p 112), from which he deduced the expansion of  $\int ydx$ .

Polya and Czego present this method in [84] for the computation of an equivalent in  $n$  of  $\int_a^b [f(x)]^n dx$ , where  $f$ , strictly positive, has a maximum inside  $[a, b]$ . If we rewrite  $f(x) = e^{\log f(x)}$ , denoting by  $x \rightarrow \phi(x) = \log f(x)$ , we formulate the result of Polya and Czego as

**THEOREM 4.1.** *Let  $\phi(x)$  be of class  $C^2$  on  $]a, b[$ ,  $a < a_1 < a_2 < b$ , which reaches its maximum at a single point  $x_0$  of the interval  $]a_1, a_2[$ , such that  $\phi''(x_0) < 0$ . Let  $a(x)$  be a continuous function on  $[a, b]$ , such that  $a(x_0) \neq 0$ .*

*An asymptotic equivalent when  $k \rightarrow +\infty$  of  $\int_{a_1}^{a_2} a(x)e^{k\phi(x)}dx$  is  $(\frac{2\pi}{k(-\phi''(x_0))})^{\frac{1}{2}} e^{k\phi(x_0)}a(x_0)$ .*

**PROOF.** There exist  $\varepsilon_0 > 0$  and  $\varepsilon$  such that, for  $x \in I = ]a_1, x_0 - \varepsilon[ \cup ]x_0 + \varepsilon, a_2[$ , then  $\phi(x) \leq \phi(x_0) - \varepsilon_0$ . We write

$$e^{-k\phi(x_0)} \int_{a_1}^{a_2} a(x)e^{k\phi(x)}dx = \int_I a(x)e^{k(\phi(x)-\phi(x_0))}dx + \int_{x_0-\varepsilon}^{x_0+\varepsilon} a(x)e^{k(\phi(x)-\phi(x_0))}dx.$$

One notices that

$$\phi(x) - \phi(x_0) = (x - x_0)^2 \int_0^1 (1-t)\phi''(tx + (1-t)x_0)dt.$$

Assume that  $a(x_0) \neq 0$ . For  $\varepsilon$  small enough, there exist  $\alpha, \beta > 0$  such that  $\beta < |a(x_0)|$  and  $\alpha + \phi''(x_0) < 0$  and for any  $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$

$$(x - x_0)^2[\phi''(x_0) - \alpha] \leq \phi(x) - \phi(x_0) \leq (x - x_0)^2[\phi''(x_0) + \alpha],$$

$$a(x_0) - \beta \leq a(x) \leq a(x_0) + \beta,$$

which allows to have a lower bound and an upper bound of  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} a(x)e^{k(\phi(x)-\phi(x_0))}dx$  by

$$\begin{aligned} (|a(x_0)| - \beta)e^{k\phi(x_0)} \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-k(-\phi''(x_0)+\alpha)(x-x_0)^2} dx \\ \leq \\ \left| \int_{x_0-\varepsilon}^{x_0+\varepsilon} a(x)e^{k(\phi(x)-\phi(x_0))} dx \right| \\ \leq \\ (|a(x_0)| + \beta)e^{k\phi(x_0)} \int_{\mathbf{R}} e^{-k(-\phi''(x_0)-\alpha)(x-x_0)^2} dx. \end{aligned}$$

Recall that

$$\int_{-\infty}^{+\infty} e^{-k[-\phi''(x_0) \mp \alpha](x-x_0)^2} dx = \left( \frac{2\pi}{k|\phi''(x_0) \pm \alpha|} \right)^{\frac{1}{2}}.$$

Moreover, one checks that

$$(1.58) \quad \int_{x_0+\varepsilon}^{+\infty} e^{-kC(x-x_0)^2} dx \leq -\frac{1}{2kC\varepsilon} \int_{x_0+\varepsilon}^{\infty} (-2kC(x-x_0))e^{-k(x-x_0)^2C} dx \\ \leq \frac{1}{2kC\varepsilon} e^{-k\varepsilon^2C},$$

hence the inequalities

$$\begin{aligned} (|a(x_0)| - \beta)e^{k\phi(x_0)} \left( \frac{2\pi}{k(-\phi''(x_0)+\alpha)} \right)^{\frac{1}{2}} - 2 \int_{x_0+\varepsilon}^{+\infty} e^{-k(-\phi''(x_0)+\varepsilon)(x-x_0)^2} dx \\ \leq \\ \left| \int_{x_0-\varepsilon}^{x_0+\varepsilon} a(x)e^{k(\phi(x)-\phi(x_0))} dx \right| \\ \leq \\ (|a(x_0)| + \beta)e^{k\phi(x_0)} \int_{\mathbf{R}} e^{-k(-\phi''(x_0)-\varepsilon)(x-x_0)^2} dx. \end{aligned}$$

We also have the inequality  $|\int_I a(x)e^{k(\phi(x)-\phi(x_0))} dx| \leq Me^{-k\varepsilon_0}$ .

We deduce that  $k^{\frac{1}{2}}e^{-k\phi(x_0)} \int_{a_1}^{a_2} a(x)e^{k\phi(x)} dx - a(x_0) \left( \frac{2\pi}{(-\phi''(x_0)+\alpha)} \right)^{\frac{1}{2}}$  tends to 0 when  $k$  tends to  $+\infty$ . This completes the proof of Theoreme 4.1.  $\square$

We generalize this result to a compact interval  $K$  of  $\mathbb{R}^n$  and a phase  $\phi(x)$  defined on this compact. We study

$$\int_K e^{k\phi(x)} a(x) dx.$$

Consider the set of points where  $\phi'$  vanishes.

In the first case, we assume that this set is empty. Since  $\phi'$  does not cancel on the compact  $K$ , there exists  $\delta > 0$  such that  $|\phi'| \geq \delta$ . Then we have

$$\begin{aligned} \int_K e^{k\phi(x)} a(x) dx &= \frac{1}{k} \int_K \frac{d}{dx} (e^{k\phi(x)} \frac{a(x)}{\phi'(x)}) dx \\ &= -\frac{1}{k} \int_K e^{k\phi(x)} \frac{d}{dx} \left( \frac{a(x)}{\phi'(x)} \right) dx + \frac{1}{k} \int_{\partial K} e^{k\phi(x)} \frac{a(x)}{\phi'(x)} d\sigma(x). \end{aligned}$$

The maximum of the phase  $\phi$ , denoted by  $l$ , is achieved on the boundary  $\partial\Omega$ .

We verify that  $\frac{d}{dx} \left( \frac{a(x)}{\phi'(x)} \right) = \frac{a'(x)}{\phi'(x)} - \frac{a(x)\phi''(x)}{(\phi'(x))^2}$ , thus  $|\frac{d}{dx} \left( \frac{a(x)}{\phi'(x)} \right)| \leq \frac{|a'|_{\infty}}{\delta} + \frac{|a'|_{\infty}|\phi''|_{\infty}}{\delta^2}$ . There is therefore a constant  $C$ , equal to  $\mu(K) \left( \frac{|a'|_{\infty}}{\delta} + |a'|_{\infty} + \frac{|a'|_{\infty}}{\delta^2} \right) + \mu(\partial K) \frac{|a'|_{\infty}}{\delta}$  ( $\mu(K)$  denotes in this paragraph the Lebesgue measure of  $K$ ), such that

$$\left| \int_K e^{k\phi(x)} a(x) dx \right| \leq \frac{C}{k} e^{kl}.$$

We have obtained a majoration whose main term is in  $k^{-1}e^{k \max \phi}$ . It is therefore negligible compared to the term of the Lemma 4.1.

We have the

LEMMA 4.1. *Assume that the points where the derivative of  $\phi$  is zero are isolated points and contained in the interior of  $K$ , denoted by  $x_1, \dots, x_N$ . Assume in addition (and this is a generic condition) that  $\phi''(x_j) \neq 0$  for all  $j$ .*

*Let  $J \subset \{1, \dots, N\}$  the set of indices of the points where  $\phi$  reaches its maximum, equal to  $l$ . Then*

$$e^{-kl} \int_K e^{k\phi(x)} a(x) dx \simeq \left(\frac{2\pi}{k}\right)^{\frac{1}{2}} \sum_{j \in J} \frac{a(x_j)}{(-\phi''(x_j))^{\frac{1}{2}}}.$$

PROOF. There exist  $N + 1$  functions  $\chi_1, \dots, \chi_N$  dans  $C_0^\infty(K)$  such that

$$\chi + \sum_{j=1}^N \chi_j = 1,$$

each function  $\chi_j$  has its support in a compact  $K_j$  containing  $x_j$ , is equal to 1 on a neighborhood  $K'_j$  of  $x_j$ , such that, on  $K'_j$ ,  $\phi''(x)$  does not vanish and the sets  $K'_j$  have an empty intersection. The function  $\chi$ , on the other hand, has its support in the complementary of a bounded open set  $\tilde{K}$  which does not contain any  $x_j$ . Thus the restriction of  $\chi$  on  $K$  has its support in a compact set which does not contain any  $x_j$ .

We thus write

$$e^{-kl} \int_K e^{k\phi(x)} a(x) dx = \sum_{j \in J} e^{-kl} \int_K e^{k\phi(x)} a(x) \chi_j(x) dx + \sum_{j \notin J} e^{-kl} \int_K e^{k\phi(x)} a(x) \chi_j(x) dx + e^{-kl} \int_K e^{k\phi(x)} a(x) \chi(x) dx.$$

Consider  $a_j = \int_K e^{k\phi(x)} a(x) \chi_j(x) dx$ . The compact  $K_j$  contains only the critical point  $x_j$ . If  $\phi(x_j)$  is a maximum of  $\phi$  on the considered interval, then we apply Lemma 4.1. When  $j \notin J$ , the maximum is strictly smaller than  $l$ , so  $e^{k(\phi(x_j) - l)}$  is exponentially decreasing, so is  $O(k^{-1})$ .

Let us study the case  $x_j$  minimum of  $\phi$ . In this case, there exists a point  $x_{j'}$ ,  $j' \neq j$ , which is a local maximum, such that  $\phi(x_{j'}) \geq \max_{x \in K_j} \phi(x) + b$ ,  $b$  being a strictly positive constant. Thus  $\phi(x) \leq \phi(x_{j'}) - b$ , from which we deduce

$$a_j e^{-k\phi(x_{j'})} = \int_{K_j} a(x) \chi_j(x) e^{k(\phi(x) - \phi(x_{j'}))} dx,$$

uniformly bounded by  $e^{-bk} \mu(K_j) \sup_{K_j} |a|$ .

On the other hand there is no critical point on the support of  $\chi$ . The maximum of  $\phi$  on the support of  $\chi$ , denoted by  $l_0$  is strictly smaller than  $l$ . The integral  $\int a(x) \chi(x) e^{k(\phi(x) - l_0)} dx$  is therefore negligible, and we deduce the result of the Lemma 4.1.  $\square$

## 2. Non-stationary phase theorem

We state a first result of asymptotic regularity for an oscillatory integral, associated to an integrand including the term  $e^{ik\phi(x)}$ . Such phases will be studied in detail in Section 3.

PROPOSITION 4.1. *Let  $\phi(x)$  be a function of class  $C^\infty$  on  $\Omega$  open bounded subset of  $\mathbb{R}^n$ ,  $x = (x_1, \dots, x_n)$ . We assume that  $\phi$  does not have any critical point on  $\Omega$ .*

*Let  $a$  be in  $C_0^\infty(\Omega)$ , of support contained in  $K$  compact. We introduce  $u(k) = \int_\Omega a(x) e^{ik\phi(x)} dx$ .*

*For all  $N$ , there exists a constant  $C_N$  such that*

$$|u(k)| \leq C_N k^{-N}.$$

PROOF. As  $\partial_{x_j} [e^{ik\phi(x)}] = ik \partial_{x_j} \phi(x) e^{ik\phi(x)}$ , we deduce

$$e^{ik\phi(x)} = \left[ \frac{1}{ik} \sum_j \frac{\partial_{x_j} \phi}{|\nabla \phi|^2} \frac{\partial}{\partial x_j} \right] (e^{ik\phi(x)}).$$

Let  $L$  denote the operator

$$L(f) = - \sum_j \frac{\partial}{\partial x_j} \left( \frac{f(x)(\partial_{x_j} \phi)}{|\nabla \phi|^2} \right).$$

The function  $|\nabla \phi|$  has no zero on the compact  $K$ , so there exists  $c > 0$  such that  $|\nabla \phi| \geq c > 0$ , so the operator  $L$  has no singularity. From the equality

$$a(x) = \sum_j \frac{a(x)(\partial_{x_j} \phi)}{\sum_p (\partial_{x_p} \phi)^2} \frac{\partial \phi}{\partial x_j}$$

one deduces, after an integration by parts for  $a \in C_0^\infty(\Omega)$ :

$$\int_{\Omega} a(x) e^{ik\phi(x)} dx = \frac{1}{ik} \int_{\Omega} L(a) e^{ik\phi(x)} dx.$$

The coefficients of  $L$  are  $C^\infty$  and depend on the derivatives of  $\phi$  of order 2. By composition, the coefficients of  $L^k$  depend on the derivatives of  $\phi$  of order  $k+1$  at most. We then verify that, if  $|a|_p$  denotes the maximum on  $\Omega$  of the derivatives of  $a$  of order  $p$  at most, then for all  $M$  there exists a constant  $C_M$  such that

$$(2.59) \quad |L^M(a)| \leq C_M (|\phi|_{M+1}, |a|_M).$$

After successive integration by parts, one has

$$\int_{\Omega} a(x) e^{ik\phi(x)} dx = \left(\frac{1}{ik}\right)^M \int_{\Omega} L^M(a) e^{ik\phi(x)} dx$$

which yields the inequality of the Lemma, using (2.59).

It has therefore been shown that, when the phase is not stationary, the oscillatory integral is bounded by any negative power of  $k$ , which ends the proof.  $\square$

### 3. Saddle point method for a complex phase.

#### 3.1. A preliminary calculation.

LEMMA 4.2. *Let  $a \in \mathcal{S}(\mathbb{R})$  and introduce*

$$I(a, \lambda) := \int_{\mathbb{R}} e^{-\frac{\lambda}{2}(x-i)^2} a(x) dx.$$

One has

$$\forall N, \exists C_N > 0, |I(a, \lambda)| \leq C_N \|a\|_{C^N(\mathbb{R})} \lambda^{-N} e^{\lambda/2}.$$

PROOF. Note that, on  $\mathbb{R}$ , the phase has no critical point. On the other side

$$-\frac{\lambda}{2}(x-i)^2 = -\frac{\lambda}{2}x^2 + i\lambda x + \frac{\lambda}{2},$$

from which one deduces the estimate

$$|I(a, \phi)| \leq \|a\|_{\infty} \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}} e^{\lambda/2},$$

from which we could deduce that the integral behaves in  $e^{\frac{\lambda}{2}}$ , and thus that we would not have an asymptotic expansion. This is not the case; in fact we write

$$I(a, \lambda) = e^{\frac{\lambda}{2}} \left[ \int_{\mathbb{R}} e^{-\frac{\lambda}{2}x^2} a(x) \cos(\lambda x) dx + i \int_{\mathbb{R}} e^{-\frac{\lambda}{2}x^2} a(x) \sin(\lambda x) dx \right].$$

Let  $M$  be the operator

$$Mf(x) = \frac{1}{(x-i)} \frac{\partial f}{\partial x}(x).$$

Its adjoint operator  $L$ , given by  $\int Lfg dx = -\int Mgf dx$  for all  $f, g$  compactly supported is  $Lf(x) = \frac{\partial}{\partial x} \left( \frac{f(x)}{x-i} \right)$ . These two operators act on functions of  $C^\infty(\mathbb{R})$ . Identity  $L(e^{-\frac{\lambda}{2}(x-i)^2}) = -\lambda e^{-\frac{\lambda}{2}(x-i)^2}$  implies

$$I(a, \lambda) = -\frac{1}{\lambda} \int_{\mathbf{R}} M(e^{-\frac{\lambda}{2}(x-i)^2}) a(x) dx = \frac{1}{\lambda} \int_{\mathbf{R}} e^{-\frac{\lambda}{2}(x-i)^2} L(a)(x) dx,$$

hence

$$I(a, \lambda) = \lambda^{-N} I(L^N(a), \lambda).$$

These calculations are similar to the proof of the non stationary phase result. The bound that one obtains is thus

$$\forall N, \exists C_N > 0, |I(a, \lambda)| \leq C_N \|a\|_{C^N(\mathbf{R})} \lambda^{-N} e^{\lambda/2}.$$

□

We will see in the rest of this chapter that we are not in the framework of the stationary phase theorem (in which case we could conclude that this phase is non-stationary): one of the crucial hypotheses (the behavior of the phase), which is that  $\operatorname{Re}(-\frac{1}{2}(x-i)^2) \leq 0$ , is not verified.

To continue the analysis, let  $F(\lambda) = I(1, \lambda)$ . This function, which is defined for all  $\lambda > 0$ , verifies, after derivation of the integral, the differential equation:

$$F'(\lambda) = -\frac{1}{2\lambda} F(\lambda)$$

from which one deduces  $\lambda^{\frac{1}{2}} F(\lambda) = F(1)$ . One has thus

$$\lambda^{\frac{1}{2}} F(\lambda) = \int_{\mathbf{R}} e^{-\frac{1}{2}(u-\lambda^{\frac{1}{2}}i)^2} du.$$

When  $\lambda \rightarrow 0$ , the limit of this quantity is  $(2\pi)^{\frac{1}{2}}$ . As the function  $\lambda^{\frac{1}{2}} F(\lambda)$  is constant and continuous on  $\mathbb{R}_+^*$ , it is then equal to its limit  $(2\pi)^{\frac{1}{2}}$ . Hence

$$I(1, \lambda) = \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}}.$$

We will see below that this result is similar to the one given by the Bergmann transform.

If we want to calculate  $I(1, \lambda)$  using the Laplace method (also called the saddle point method), we introduce the contour in the complex plane which is the rectangle whose one side is  $[-R, R]$  and whose opposite side passes through the point  $i$ . The function  $e^{-\frac{\lambda}{2}(x-i)^2}$  is holomorphic in this rectangle, so the integral over the rectangle is zero. This leads to

$$\int_{-R}^R e^{-\frac{\lambda}{2}(x-i)^2} dx = \int_{-R}^R e^{-\frac{\lambda t^2}{2}} + i \int_0^1 e^{-\frac{\lambda}{2}(-R-iu-i)^2} du + i \int_0^1 e^{-\frac{\lambda}{2}(R+iu-i)^2} du.$$

The first term is equivalent to  $(\frac{2\pi}{\lambda})^{\frac{1}{2}}$  when  $R$  goes to  $+\infty$  thanks to Lemma 4.1.

The two other terms are bounded by  $e^{-\frac{\lambda}{2}(R^2-1)}$ , which tend to 0 faster than any inverse power of  $\lambda$  when  $\lambda$  goes to  $+\infty$ , as soon as  $R > 1$ . Now, let us suppose  $a(z)$  holomorphic in the band  $0 \leq \Im z \leq 1$ , with majorations of  $a$  and all its derivatives analogous to those of  $a \in \mathcal{S}$ .

The calculation is identical, and we find

$$I(a, \lambda) \simeq \int_{\mathbf{R}} e^{-\lambda \frac{t^2}{2}} a(t+i) dt.$$

On this integral, we apply the usual stationary phase theorem.

**3.2. The Laplace method for a general complex phase.** We consider an holomorphic function  $\phi$  verifying the following hypotheses:

- (1) (H1) on the adherence  $\bar{B}$  of a connected open set  $B$  of the complex plane,  $\phi$  has only one critical point  $z_0$  and it is non-degenerate, that is  $\phi'(z_0) = 0$ ,  $\phi''(z_0) \neq 0$ .

- (2) (H2) there exists a path  $z(t)$  verifying  $z(0) = z_0$ ,  $\dot{z}(0) \neq 0$  (and  $t \rightarrow \phi(z(t))$  is oriented in the opposite direction of the trigonometric direction) and a  $\varepsilon > 0$  such that

$$\forall t \in [-\varepsilon, \varepsilon], \operatorname{Re} [\phi(z(t)) - \phi(z_0)] \leq 0.$$

Moreover,  $\operatorname{Re} (\phi(z(\pm\varepsilon)) - \phi(z_0)) < 0$ .

- (3) (H3) By defining  $\bar{B} = [a_-, a_+]$ , there are two paths  $\mathcal{C}_\pm$  such that  $\mathcal{C}_\pm$  joins  $a_\pm$  to  $z(\pm\varepsilon)$ , of length  $L_\pm$  depending only on the open  $B$ , and a constant  $\delta > 0$  such that, on  $\mathcal{C}_+ \cup \mathcal{C}_-$ :

$$\operatorname{Re} [\phi(u) - \phi(z_0)] \leq -\delta.$$

We have the proposition

PROPOSITION 4.2. *Let  $\phi$  satisfy (H1-H2-H3). For any  $a$ , holomorphic symbol on the region  $\Omega$  of boundary  $\partial\Omega = \{[a_-, a_+], \mathcal{C}_+, \mathcal{C}_-, z([-\varepsilon, \varepsilon])\}$*

$$e^{-\lambda\phi(z_0)} \int_{a_-}^{a_+} e^{\lambda\phi(x)} a(x) dx$$

admits an asymptotic expansion in  $\lambda$  which is that of

$$\int_{-\varepsilon}^{\varepsilon} e^{\lambda(\phi(z(t)) - \phi(z_0))} a(z(t)) \dot{z}(t) dt.$$

PROOF. As the functions  $a$  and  $\phi$  are holomorphic on  $\Omega$ , the integral on the contour  $\partial\Omega$  is zero (Cauchy theorem). This is written, assuming that the oriented paths  $\gamma_\pm(s)$  representing  $\mathcal{C}_\pm$  are in the trigonometric direction, one has

$$\begin{aligned} e^{-\lambda\phi(z_0)} \int_{a_-}^{a_+} e^{\lambda\phi(x)} a(x) dx &= \int_{-\varepsilon}^{\varepsilon} e^{\lambda(\phi(z(t)) - \phi(z_0))} a(z(t)) \dot{z}(t) dt \\ &+ \int_{\mathcal{C}_+} e^{\lambda(\phi(\gamma_+(s)) - \phi(z_0))} a(\gamma_+(s)) \dot{\gamma}_+(s) ds \\ &+ \int_{\mathcal{C}_-} e^{\lambda(\phi(\gamma_-(s)) - \phi(z_0))} a(\gamma_-(s)) \dot{\gamma}_-(s) ds \end{aligned}$$

On note  $r(\lambda) = \int_{\mathcal{C}_+} e^{\lambda(\phi(\gamma_+(s)) - \phi(z_0))} a(\gamma_+(s)) \dot{\gamma}_+(s) ds + \int_{\mathcal{C}_-} e^{\lambda(\phi(\gamma_-(s)) - \phi(z_0))} a(\gamma_-(s)) \dot{\gamma}_-(s) ds$ . We apply the usual stationary phase theorem to the first term, since  $\psi(t) = -i(\phi(z(t)) - \phi(z_0))$  verifies  $\Im\psi \geq 0$ , with  $\Im\psi > 0$  on the boundary, and  $\psi$  has a critical point at  $t = 0$  of value 0.

We know that  $|r(\lambda)| \leq \|a\|_\infty (L_+ + L_-) e^{-\lambda\delta}$ . We write

$$\int_{-\varepsilon}^{\varepsilon} e^{\lambda(\phi(z(t)) - \phi(z_0))} a(z(t)) \dot{z}(t) dt = \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}} \sum_{p=0}^{N-1} a_p \lambda^{-p} + r_N(\lambda)$$

with  $|r_N(\lambda)| \leq C_N \lambda^{-N}$  for  $\lambda \geq \lambda_0$ , the constant depending on  $a$  and its derivatives in the domain  $\Omega$ . We then introduce  $r'_N(\lambda) = r_N(\lambda) + r(\lambda)$ . There exists a constant  $C'_N$  such that, for  $\lambda \geq \lambda_0$

$$|r'_N(\lambda)| \leq C'_N \lambda^{-N}.$$

We thus proved the proposition.  $\square$

COROLLARY 4.1. *If, in the open set  $\Omega$ , the phase  $\phi$  admits several critical points  $z_1, \dots, z_d$  such that  $\operatorname{Re} \phi(z_1) > \operatorname{Re} \phi(z_2) > \dots > \operatorname{Re} \phi(z_d)$ , then, under the same assumptions 1. and 2. as before for the critical points*

$$e^{-\lambda\phi(z_1)} \int_a^b a(x) e^{\lambda\phi(x)}$$

admits an asymptotic expansion which is only in the neighborhood of  $z_1$ , provided that one can deform the boundary of the domain into a contour such that  $\operatorname{Re} (\phi(z) - \phi(z_1)) \leq -\delta < 0$ , except in a neighborhood of  $z_1$ .

#### 4. Holomorphic saddle point method and control of all remainder terms

There are as many stationary phase theorems as there are situations in which we can reduce to such an expansion. We will see for example in the chapter 5 another statement, which will also be used in the section 8. It will occur even in cases where the phase is stationary with a degenerate critical point, as it will be considered in the chapter 10, the chapter 11 or the chapter 13: their common point being the existence of a more general oscillatory phase which reduces at almost all the points to the considered phase, and which presents a non-degenerate critical point<sup>1</sup>. The statement of this section is close to Laplace's method, which is also called the saddle point method, which explains the title of the present chapter. This is a result given in the book of J. Sjöstrand [89]:

**THEOREM 4.2.** (1) *There exists a constant  $C_N$  depending only on the dimension  $N$ , such that for all  $n, k > 0$ , and any holomorphic function  $u$  defined on a neighborhood of the ball of radius 1 in  $\mathbb{C}^N$*

$$k^{\frac{N}{2}} \int_{|x| \leq 1, x \in \mathbb{R}^N} e^{-kx^2/2} u(x) dx = (2\pi)^{\frac{N}{2}} \sum_{p=0}^{p=n} u_p k^{-p} + R_n(k)$$

where the  $u_p$  are equal to  $\frac{1}{(p!)} ((\frac{\Delta}{2})^p u)(0)$  and where the remainder  $R_n$  verifies

$$|R_n(k)| \leq C_N (n+1)^{\frac{N}{2}} k^{-n-1} (n+1)! 2^{n+1} \sup_{|z| \leq 1, z \in \mathbb{C}^N} |u(z)|$$

(2) *One has, for  $u \in C_0^\infty(\mathbb{R}^N)$  of support  $B_N$ ,*

$$(4.60) \quad \int_{B_N} e^{-kq(x)} u(x) dx \simeq \left( \frac{(2\pi)^N}{k^N \det Q} \right)^{\frac{1}{2}} \sum \frac{1}{m! k^m} \left( \frac{\Delta_q}{2} \right)^m (u)(0).$$

where the Laplacian  $\Delta_q$  is given by  $\text{div}(Q^{-1}\nabla)$ ,  $q(x) = \frac{1}{2}(Qx, x)$ .

**PROOF.** The proof of this theorem of the complex stationary phase is done in three steps:

- (1) Bound of the remainder using the maximum principle.
- (2) Compute each term from the Taylor series
- (3) Complementary study for a smooth compactly supported function and an arbitrary quadratic form  $\frac{1}{2}(Qx, x)$ ,  $Q$  constant matrix.

We begin with the proof of the second point.

We verify, on  $z \in \mathbb{C}$ ,  $|z| \leq 1$ , the inequality (coming from  $|u^{(p)}(0)| \leq p! \sup_{|z| \leq 1, z \in \mathbb{C}} |u(z)|$ )

$$|u(z) - \sum_{p=0}^{p=2n-1} \frac{u^{(p)}(0)}{p!} z^p| \leq (2n+1) \sup_{|z| \leq 1, z \in \mathbb{C}} |u(z)|.$$

By the holomorphic maximum principle on  $u$ , we deduce

$$|u(z) - \sum_{p=0}^{p=2n-1} \frac{u^{(p)}(0)}{p!} z^p| \leq (2n+1) \sup_{|z| \leq 1, z \in \mathbb{C}} |u(z)| |z|^{2n}.$$

We can generalize this inequality on  $\mathbb{C}^N$  by considering all complex lines of  $\mathbb{C}^N$  passing through the origin and the functions  $f(\lambda) = u(\lambda z)$ , where  $z \in \mathbb{C}^N$  et  $\lambda \in \mathbb{C}$ . One gets (using  $\sum_{\alpha+\delta_j=\beta} \frac{p!}{\alpha!} = \frac{p!}{\beta!} \sum \beta_j = \frac{(p+1)!}{\beta!}$  in the induction)

$$\partial^p f(\lambda) = \sum_{|\alpha|=p} \frac{p!}{\alpha!} z^\alpha \partial^\alpha u(0),$$

which yields

<sup>1</sup>More precisely, if the  $\phi(x)$  phase admits a degenerate critical point at  $x_0$ , we can find a  $\psi(x, \theta)$  which is equivalent to  $\phi$  and which has no degenerate critical point degenerate



$$(4.61) \quad \left| \sum_{\alpha, |\alpha|=p} \frac{\partial^\alpha u}{\alpha!} z^\alpha \right| \leq \sup_{|z| \leq 1, z \in C} |u(z)|$$

and, using the result for  $N = 1$ ,

$$(4.62) \quad \left| u(z) - \sum_{p=0}^{2n-1} \sum_{|\alpha|=p} \frac{u^{(\alpha)}(0)}{\alpha!} z^\alpha \right| \leq (2n+1)^N [\sup_{|z| \leq 1, z \in \mathbb{C}^N} |u(z)|] |z|^{2n}.$$

The estimate of the remainder term in 4.2 comes from the previous estimate.

Calculation of the holomorphic integral when the symbol is a polynomial. One checks, for a homogeneous function  $p_{2l}$  of degree of homogeneity  $2l$  that

$$(4.63) \quad I(p_{2l}) = \int_{\mathbf{R}^N} e^{-x^2/2} p_{2l}(x) dx = (l!)^{-1} (2\pi)^{\frac{N}{2}} \left(\frac{\Delta}{2}\right)^l (p_{2l})(0).$$

En effet,  $\int e^{-\frac{x^2}{2}} \frac{\Delta}{2} p_{2l} dx = \frac{1}{2} \int x \cdot \nabla_x p_{2l} e^{-\frac{x^2}{2}} dx$  after integration by parts. One notices that  $x \cdot \nabla_x p_{2l} = 2l p_{2l}(x)$ , from which one deduces  $\int e^{-\frac{x^2}{2}} \left(\frac{\Delta}{2} p_{2l}\right)(x) dx = l \int e^{-\frac{x^2}{2}} p_{2l}(x) dx$ . On the other side,  $\left(\frac{\Delta}{2}\right)^m (p_{2l})(0) = \delta_{ml} C_l (2l)!$ , where  $C_l$  is the sum of the coefficients of  $x_j^{2l}$  dans  $p_{2l}$ . We then apply a reasoning by recurrence on the order of homogeneity. We notice moreover that for a homogeneous polynomial of odd order, the integral is zero.

We deduce

$$\int_{\mathbf{R}^N} e^{-k \frac{x^2}{2}} \left( \sum_{|\alpha| \leq 2n+1} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right) dx = \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \sum_{l=0}^n \frac{1}{l! k^l} \left[ \left(\frac{\Delta}{2}\right)^l \left( \sum_{|\alpha|=2l} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right) \right] (0).$$

Remark that

$$\left(\frac{\Delta}{2}\right)^l \left( \sum_{|\alpha|=2l} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right) (0) = \left(\left(\frac{\Delta}{2}\right)^l u\right)(0),$$

hence

$$\int_{\mathbf{R}^N} e^{-k \frac{x^2}{2}} \left( \sum_{|\alpha| \leq 2n+1} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right) dx = \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \sum_{l=0}^n \frac{1}{l! k^l} \left(\frac{\Delta}{2}\right)^l u(0).$$

Let us introduce

$$R_n(k) = k^{\frac{N}{2}} \int_{|x| \leq 1} e^{-k \frac{x^2}{2}} u(x) dx - (2\pi)^{\frac{N}{2}} \sum_{l=0}^n \frac{1}{l! k^l} \left(\frac{\Delta}{2}\right)^l u(0).$$

One has

$$R_n(k) = k^{\frac{N}{2}} \int_{|x| \leq 1} e^{-k \frac{x^2}{2}} [u(x) - \sum_{|\alpha| \leq 2n+1} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha] dx - k^{\frac{N}{2}} \int_{|x| \geq 1} e^{-k \frac{x^2}{2}} \left[ \sum_{|\alpha| \leq 2n+1} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right] dx.$$

Denote by  $S_n(k) = k^{\frac{N}{2}} \int_{|x| \geq 1} e^{-k \frac{x^2}{2}} \left[ \sum_{|\alpha| \leq 2n+1} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right] dx$ . After the change of variable  $x = r\omega$ ,  $\omega \in S^{N-1}$ , one finds

$$S_n(k) = k^{\frac{N}{2}} \sum_{p=0}^{2n+1} \int_1^\infty e^{-k \frac{r^2}{2}} r^{N-1+p} dr \sum_{|\alpha|=p} \int_{S^{N-1}} \frac{u^{(\alpha)}(0)}{\alpha!} \omega^\alpha d\omega.$$

Using the estimate (4.61), one finds

$$|S_n(k)| \leq k^{\frac{N}{2}} \sum_{p=0}^{2n+1} \int_1^\infty e^{-k \frac{r^2}{2}} r^{N-1+p} dr \text{Vol}(S^{N-1}) \sup_{|z| \leq 1} |u(z)|.$$

From (4.62), one deduces

(4.64)

$$|R_n(k)| \leq \text{Vol}(S^{N-1}) \sup_{|z| \leq 1} |u(z)| \left[ k^{\frac{N}{2}} \int_0^1 (2n+3)r^{2n+2}r^{N-1}e^{-k\frac{r^2}{2}} dr + \sum_{p=0}^{2n+1} k^{\frac{N}{2}} \int_1^\infty e^{-k\frac{r^2}{2}} r^{N-1+p} dr \right].$$

As  $\sum_0^{2n+1} 1 = 2n+2$  and as, for  $r \geq 1$  and  $p \leq 2n+1$ ,  $r^{N+p-1} \leq r^{2n+N+1}$ , one gets

$$|R_n(k)| \leq \text{Vol}(S^{N-1}) \sup_{|z| \leq 1} |u(z)| (2n+3)k^{\frac{N}{2}} \int_0^\infty r^{2n+N+1} e^{-k\frac{r^2}{2}} dr.$$

We want to calculate this integral and find an equivalent when  $n$  is infinite. Let us introduce

$$F(k) = \int_0^\infty e^{-k\frac{r^2}{2}} dr, G(k) = \int_0^\infty r e^{-k\frac{r^2}{2}} dr.$$

One has

$$F(k) = \left(\frac{2\pi}{k}\right)^{\frac{1}{2}}, G(k) = \frac{1}{k}.$$

Moreover

$$\int_0^\infty r^{2p} e^{-k\frac{r^2}{2}} dr = (-2)^p F^{(p)}(k), \int_0^\infty r^{2p+1} e^{-k\frac{r^2}{2}} dr = (-2)^p G^{(p)}(k).$$

One deduces from these equalities

$$\int_0^\infty r^{2p} e^{-k\frac{r^2}{2}} dr = (2\pi)^{\frac{1}{2}} k^{-p-\frac{1}{2}} \frac{(2p)!}{p!} \frac{1}{2^p}, \int_0^\infty r^{2p+1} e^{-k\frac{r^2}{2}} dr = k^{-p-1} p!.$$

We are therefore looking for an equivalent, when  $n$  is large, of  $2^{\frac{N}{2}+n} (n + \frac{N}{2})!$  when  $N$  is even, and of  $(2\pi)^{\frac{1}{2}} \frac{(2n+N+1)!}{(n+\frac{N+1}{2})!} 2^{-n-\frac{N+1}{2}}$  when  $N$  is odd. To fix the ideas, let us divide by  $n!$  and use Stirling's formula. For  $N$  even, we find

$$\begin{aligned} \frac{2^{\frac{N}{2}+n} (n + \frac{N}{2})!}{n!} &\simeq 2^{\frac{N}{2}+n} \left(n + \frac{N}{2}\right)^{n+\frac{N}{2}} e^{-n-\frac{N}{2}} (2\pi(n + \frac{N}{2}))^{\frac{1}{2}} n^{-n} e^n (2\pi n)^{-\frac{1}{2}} \\ &= 2^{\frac{N}{2}+n} e^{-\frac{N}{2}} \left(1 + \frac{N}{2n}\right)^{\frac{1}{2}} (n+1)^{\frac{N}{2}} \left(1 + \frac{\frac{N}{2}-1}{n+1}\right)^{\frac{N}{2}} \left(1 + \frac{N}{2n}\right)^n, \end{aligned}$$

which yields the equivalent  $G^{(n+\frac{N}{2})}(k) \simeq C'_N 2^n (n+1)^{\frac{N}{2}} n! k^{-n-\frac{N}{2}-1}$ . Similarly, for  $N$  odd, we find as equivalent

$$\begin{aligned} (2\pi)^{\frac{1}{2}} \frac{(2n+N+1)!}{n!(n+\frac{N+1}{2})!} 2^{-n-\frac{N+1}{2}} &\simeq \frac{(2n+N+1)^{2n+N+1+\frac{1}{2}} e^{-2n-N-1}}{n^{n+\frac{1}{2}} e^{-n} (n+\frac{N+1}{2})^{n+\frac{N+1}{2}+\frac{1}{2}}} e^{n+\frac{N+1}{2}} 2^{-n-\frac{N+1}{2}} \\ &\simeq \sqrt{2} (n+1)^{\frac{N}{2}} \left(1 + \frac{N+1}{2n}\right)^{n+\frac{1}{2}} \left(\frac{n+\frac{N+1}{2}}{n+1}\right)^{\frac{N}{2}} e^{-\frac{N+1}{2}} 2^{n+\frac{N+1}{2}} \end{aligned}$$

one has, similarly

$$F^{(n+\frac{N+1}{2})}(k) \simeq C''_N 2^n (n+1)^{\frac{N}{2}} n! k^{-n-\frac{N+1}{2}-\frac{1}{2}}.$$

The limit of the expression  $(2n+3)k^{\frac{N}{2}} \int_0^\infty r^{2n+N+1} e^{-k\frac{r^2}{2}} dr / 2^{n+1} (n+1)! (n+1)^{\frac{N}{2}} k^{n+\frac{N}{2}+1}$  is  $C'_N$  when  $n$  goes to  $+\infty$ . It is thus bounded for all  $n$  by  $C_N$ . One deduces

$$|R_n(k)| \leq C_N \text{Vol}(S^{N-1}) \sup_{|z| \leq 1} |u(z)| (n+1)! (n+1)^{\frac{N}{2}} 2^{n+1} k^{-n-1}.$$

The proof of the third item is now easy through a change of variable in the integral on a compact set. We use the orthonormal change of variable given by  $q(x) = \frac{1}{2} \sum a_i v_i^2$  characterized by  $Pv = x$  where  $P$  is a unit matrix. Note that  $\det Q = \prod a_i$ . Since the change of variable is orthonormal, we have

$$\int_{B_N} e^{-kq(x)} u(x) dx = \int_{B_N} e^{-\frac{k}{2} (\sum_i a_i v_i^2)} u(Pv) dv.$$

Let  $U(v) = u(Pv)$ . It is immediate that

$$\int_{B_N} e^{-kq(x)} u(x) dx = \int_{\tilde{B}_N} e^{-\frac{k}{2} \sum_i t_i^2} U\left(\frac{t_i}{\sqrt{a_i}}\right) \frac{dt}{(\prod a_i)^{\frac{1}{2}}}$$

where the ball  $\tilde{B}_N$  is an ellipsoid induced by the  $\sqrt{a_i}$ . As  $q(x) = \frac{1}{2}(Qx \cdot x) = \frac{1}{2}({}^t P Q P v, v) = \frac{1}{2} \sum a_i v_i^2$ , one finds  $Q = P \text{Diag}(a_i) {}^t P$  and then

$$\int_{\tilde{B}_N} e^{-\frac{k}{2} \sum_i t_i^2} U\left(\frac{t_i}{\sqrt{a_i}}\right) \frac{dt}{(\det Q)^{\frac{1}{2}}} \simeq \frac{(2\pi)^{\frac{N}{2}}}{k^{\frac{N}{2}} (\det Q)^{\frac{1}{2}}} \sum_{p \leq n} k^{-p} \left(\frac{L}{2}\right)^p U(0)$$

where

$$L = \sum_{i=1}^{i=N} \frac{\partial^2}{\partial t_i^2} = \sum_{i=1}^{i=N} \frac{1}{a_i} \frac{\partial^2}{\partial v_i^2}.$$

Note that  $\frac{\partial^2 U}{\partial v_i^2} = \sum_{l,m} P_{il} P_{im} \frac{\partial^2 u}{\partial x_l \partial x_m}(Pv)$ , hence

$$LU = \sum_{l,m,i} \frac{1}{a_i} P_{il} P_{im} \frac{\partial^2 u}{\partial x_l \partial x_m}(Pv)$$

which defines the Laplace operator  $Q$  under the form

$$(4.65) \quad \Delta_q = \text{div}(Q^{-1} \text{grad}).$$

Theorem 4.2 is proved. □

Note that this is a very powerful theorem, for the first two items thanks to holomorphy: we have found an majorant whose behavior in  $n$  is  $2^{n+1}(n+1)!(n+1)^{\frac{N}{2}} k^{-n-1}$ . The result also applies in the third item when  $u$  is analytical.

Remark: for  $N = 1$ , one finds  $\int_0^\infty r^{2n} e^{-kr \frac{r^2}{2}} dr = (2\pi)^{\frac{1}{2}} \frac{(2n+2)!}{(n+1)! 2^{n+1}} = (2\pi)^{\frac{1}{2}} (n+1)! 2^{n+1} \frac{C_{2n+2}^{n+1}}{2^{2n+2}}$ , hence the constant  $C_1$  is  $(\frac{\pi}{2})^{\frac{1}{2}}$ . For  $N = 2$ , the integral calculation gives  $2^{n+1}(n+1)!$ , hence  $C_2 = 2$ .

LEMMA 4.3. *We assume that  $\phi$  admits a non-degenerate critical point  $x_0$  and that the Hessian matrix of  $\phi$  is positive definite. We obtain the Laplacian associated to  $\phi$  in  $x_0$  thanks to the equality*

$$(4.66) \quad \int e^{-k\phi(x_0)} e^{-k\phi(x)} a(x) dx \simeq \frac{(2\pi)^{\frac{N}{2}}}{k^{\frac{N}{2}} (\det \text{Hess} \phi(x_0))^{\frac{1}{2}}} \sum_{m \geq 0} \frac{1}{m! k^m} \left(\frac{\Delta \phi}{2}\right)^m u(x_0).$$

We apply Morse's lemma to the neighborhood of a non-degenerate critical point by noting  $p$  the number of positive eigenvalues (counting their multiplicity) of  $\text{Hess} \phi(x_0)$ . There exists a system of coordinates  $\tilde{x}$  such that, locally

$$(4.67) \quad \phi(x) = \phi(x_0) + \frac{1}{2} \left( \sum_{i \leq p} (\tilde{x}_i)^2 - \sum_{j > p} (\tilde{x}_j)^2 \right),$$

The application  $x \rightarrow \tilde{x}$  is a local diffeomorphism of  $\mathbb{R}^N$ . In the case studied below  $p = N$ , and we compute

$$\int_{B_N} e^{-k\phi(x)} u(x) dx = e^{-k\phi(x_0)} \int_{\tilde{B}_N} e^{-\frac{k}{2} \sum_{j=1}^N (\tilde{x}_j)^2} u(x^{-1}(\tilde{x})) |Jac\left(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}}\right)| d\tilde{x}.$$

To do this, we use the result of the proposition 4.60 since this phase is quadratic (according to Morse's lemma and the assumption in  $x_0$ ). The asymptotic expansion of

$$e^{k\phi(x_0)} \int_{B_N} e^{-k\phi(x)} u(x) dx$$

is thus

$$(4.68) \quad \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \sum_{p=0}^n \frac{1}{p!k^p} \left(\frac{\Delta_{\tilde{x}}}{2}\right)^p [u(x^{-1}(\tilde{x})) |Jac(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}})|](x_0).$$

Identifying the terms in  $k^{-p}$  of the equality (4.68) and noting that  $\det \phi''(x_0) = |\frac{\partial \tilde{x}}{\partial x}|^2(x_0)$ , one finds

$$\left(\frac{\Delta_{\tilde{x}}}{2}\right)^p [u(x^{-1}(\tilde{x})) |Jac(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}})|](x_0) = |Jac(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}})(x_0)| \left(\frac{\Delta_{\phi}}{2}\right)^p u(x_0).$$

We deduce a formal definition of the laplacian  $\Delta_{\phi}$  associated with to the  $\phi$  phase in the neighborhood of  $x_0$ :

$$\Delta_{\phi} u = |Jac(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}})|^{-1} \Delta_{\tilde{x}} (|Jac(\frac{\partial x^{-1}(\tilde{x})}{\partial \tilde{x}})| u)$$

In dimension 1, we will compute explicitly the Laplacian in the section 6.

### 5. Stationary phase theorem

The above results are given when the integral to be computed contains a term of the form  $e^{-k\phi(x)}$ . These are generalizations of Laplace's method or the saddle point method.

The asymptotic expansions obtained in the previous sections can be written formally for integrals of the form  $\int a(x) e^{ik\phi(x)} dx$ . We show in this section that the expressions found **are** the asymptotic expansions of the integral  $\int a(x) e^{ik\phi(x)} dx$ . The two main results (from Hörmander's treatise, Theorem 7.7.5 of [47]) are stated in the following proposition.

**PROPOSITION 4.3.** *Let  $\phi$  be a phase on  $\mathbb{R}^N$  admitting at  $x_0$  a non degenerate critical point, that is  $\nabla_x \phi(x_0) = 0$  and  $\text{Hess}\phi(x_0)$  is invertible. We suppose for (i) that  $\text{Re } \phi \geq 0$  in a neighborhood of  $x_0$  and for (ii) that  $\Im \phi \geq 0$  on  $B(x_0, r)$ .*

*Then the following two results are true, for  $\chi$  function  $C^\infty$  with support in a ball of center  $x_0$  (the determinations of the roots of  $\det \text{Hess}\phi(x_0)$  are chosen to be positive real part)*

$$(i) \quad \int_{\mathbb{R}^N} \chi(x) e^{-k\phi(x)} u(x) dx \simeq \left(\frac{(2\pi)^N}{k^N \det \text{Hess}\phi(x_0)}\right)^{\frac{1}{2}} \sum_{p \geq 0} k^{-p} (p!)^{-1} \left(\frac{1}{2} \Delta_{\phi}\right)^p (u)(x_0) e^{-k\phi(x_0)},$$

$$(ii) \quad \int_{\mathbb{R}^N} \chi(x) e^{ik\phi(x)} u(x) dx \simeq \left(\frac{(2\pi)^N}{k^N \det \text{Hess}\phi(x_0)}\right)^{\frac{1}{2}} e^{\frac{iN\pi}{4}} \sum_{p \geq 0} \left(\frac{i}{k}\right)^p (p!)^{-1} \left(\frac{1}{2} \Delta_{\phi}\right)^p (u)(x_0) e^{ik\phi(x_0)}.$$

This proposition is only true for a phase such that  $\Im \phi(x) > 0$  on the boundary of the open set (see [89], Theorem 2.8). On the other hand, if  $u$  satisfies the hypotheses of Lemma 7.7.3 of Hörmander[48], we recover the theorem.

**PROOF.** Since  $\text{Hess}\phi_0$  is nondegenerate, we apply (4.67). If, moreover, the Hessian matrix is symmetric positive definite at  $x_0$ , there exists a change of variable such that, in the neighborhood of  $x_0$ , there is a change of variable

$$(5.69) \quad \phi(x) = \phi(x_0) + \frac{1}{2}(\tilde{z}, \tilde{z}).$$

From the result obtained in the case where the matrix  $\text{Hess}\phi(x_0)$  is positive definite, we deduce the result in the general case where  $\phi$  is given by (4.67) by successively separating the integrals on the variables  $\tilde{x}_j$  for  $j > p$  and  $\tilde{x}_j$  for  $j < p$ . One will then notice that the result on  $\int a(\tilde{x}) e^{-i\frac{k}{2}\tilde{x}^2} d\tilde{x}$  is obtained by taking the conjugate expression of the asymptotic expansion of  $\int \bar{a}(\tilde{x}) e^{i\frac{k}{2}\tilde{x}^2} d\tilde{x}$ .

**LEMMA 4.4.** *In the case (5.69), there exists a matrix  $Q(x)$  such that  $\phi(x) = \phi(x_0) + \frac{1}{2}(Q(x)z, z)$ ,  $z = x - x_0$ .*

We apply the formula of Taylor formula with integral remainder:

$$\phi(x) = \phi(x_0) + \sum_{i,j} \int_0^1 (1-t) \partial_{x_i x_j}^2 \phi(tx + (1-t)x_0)(x - x_0, x - x_0) dt.$$

On note alors

$$Q_{jk}(x) = 2 \int_0^1 (1-t) \partial_{x_j x_k}^2 \phi(tx + (1-t)x_0) dt,$$

which is proven by noticing that for  $g(t) = \nabla \phi(tx + (1-t)x_0)(x - x_0)$ ,

$$\int_0^1 (1-t) \sum_{i,j} \partial_{x_i x_j}^2 \phi(tx + (1-t)x_0)(x - x_0, x - x_0) dt = \int_0^1 (1-t) g'(t) dt$$

and after integration by parts, using  $\int_0^1 g(t) dt = \phi(x) - \phi(x_0)$ , one gets  $\int_0^1 (1-t) g'(t) dt = \phi(x) - \phi(x_0) + g(0)$ .

We verify that  $Q_{j,k}(x_0) = 2 \partial_{x_j x_k}^2 \phi(x_0) \int_0^1 (1-t) dt$  from which the matrix  $(Q_{j,k}(x_0))$  is symmetric positive definite. Since  $\phi$  is of class  $C^3$ , there exists  $\varepsilon_0 > 0$  such that, for  $|x - x_0| \leq \varepsilon_0$  the matrix  $(Q_{jk}(x))$  is positive definite. The root of the matrix  $Q(x_0)$  is the matrix characterized by the eigenvalue  $\sqrt{\lambda_i}$  on the subspace  $E_i$ . We write then

$$Q(x) = (Q(x_0))^{\frac{1}{2}} [Id + (Q(x_0))^{-\frac{1}{2}} (Q(x) - Q(x_0)) (Q(x_0))^{-\frac{1}{2}}] (Q(x_0))^{\frac{1}{2}}.$$

The square root of the matrix  $Id + R(x)$ , where  $\|R(x)\| \leq \varepsilon$  for a usual norm, can be computed by the expansion in integer series associated to  $(1+x)^{\frac{1}{2}}$ . This expansion is absolutely convergent because  $R^p$  commutes with  $R^q$  for all  $p, q$ . Hence one can construct  $B(x)$  symmetric such that  ${}^t B(x) B(x) = Id + R(x)$ . Let  $A(x)$  be such that  ${}^t A(x) A(x) = Q(x)$ . One can choose  $A(x) = B(x) (Q(x_0))^{\frac{1}{2}}$ . One can then define the change of variable  $\tilde{z} = A(x)x$  which gives

$$\phi(x) = \phi(x_0) + \frac{1}{2} (A(x)x \cdot A(x)x) = \phi(x_0) + \frac{1}{2} (\tilde{z} \cdot \tilde{z}).$$

By definition, the Laplacian associated with the metric is

$$\tilde{\Delta} = \sum_j \partial_{\tilde{z}_j}^2.$$

The first item is proven.

The following proof of the second paragraph is directly inspired by the proof of lemma 7.7.3 of Hörmander. It is based on the Fourier transform of the distribution  $e^{i\frac{x^2}{2}} \in \mathcal{S}'(\mathbb{R})$ .

LEMMA 4.5. *The Fourier transform of  $e^{i\frac{x^2}{2}} \in \mathcal{S}'(\mathbb{R}^N)$  is the tempered distribution  $(2\pi)^{\frac{N}{2}} e^{i\frac{\xi^2}{2}}$ .*

PROOF. Consider  $u(x) = e^{i(\sum_j x_j^2)/2}$ . Then  $D_j u(x) = ix_j u$ .

Suppose  $v$  distribution verifying the equalities  $D_j v(x) = i\xi_j v$ . By considering the distribution  $\frac{1}{u} v$ , which is well defined because  $u^{-1}$  is a  $C^\infty$  function, we find that  $D_j(\frac{1}{u} v) = u^{-2}(D_j v u - D_j u v) = 0$ . So the distribution  $\frac{1}{u} v$  is a constant.

The Fourier transform of the equality  $D_j u = ix_j u$  yields

$$\xi_j \hat{u}(\xi) = -i D_j(\hat{u}).$$

We apply the previous result, which gives  $\hat{u}(\xi) = \hat{u}(0) e^{i\sum_j (\xi_j^2)/2}$ . The coefficient  $\hat{u}(0)$  is obtained by introducing a simple modification of  $u(x)$  as  $u_\varepsilon(x) = e^{i(\sum_j (1+i\varepsilon x_j^2)/2)}$ . The function  $u_\varepsilon$  is in  $\mathcal{S}(\mathbb{R}^d)$  and in the sense of the topology of  $\mathcal{S}'$ ,  $u_\varepsilon$  converges to  $u$ . Moreover,

$$\int_{\mathbb{R}^N} u_\varepsilon(x) e^{-ix \cdot \xi} dx = \prod_{j=1}^{j=N} \left( \int_{\mathbb{R}} e^{-ix_j \xi_j + i\frac{x_j^2}{2} - \varepsilon \frac{x_j^2}{2}} dx_j \right).$$

As  $\int_{\mathbb{R}} e^{-ix_j \xi_j + i\frac{x_j^2}{2} - \varepsilon \frac{x_j^2}{2}} dx_j = \frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}} dy e^{-\frac{y^2}{2} + \frac{i}{\varepsilon} \frac{y^2}{2} - iy \frac{\xi_j}{\sqrt{\varepsilon}}}$ , it is enough to study the integral

$$F(\lambda) = \int_{\mathbf{R}} dy e^{-\frac{y^2}{2} + i\lambda\frac{y^2}{2} - iy\zeta}.$$

After deformation in the complex plane of the integral on  $\mathbf{R}$  into the integral on  $\mathbf{R} + i\zeta$ , one finds

$$F(0) = e^{-\frac{\zeta^2}{2}} (2\pi)^{\frac{1}{2}}.$$

The function  $F$  is differentiable, and its derivative is

$$\begin{aligned} F'(\lambda) &= \int_{\mathbf{R}} dy \frac{i}{2(1-i\lambda)^2} [(1-i\lambda)y + i\zeta]^2 - 2i\zeta((1-i\lambda)y + i\zeta) - \zeta^2] e^{-\frac{y^2}{2} + i\lambda\frac{y^2}{2} - iy\zeta} \\ &= -i \frac{\zeta^2}{2(1-i\lambda)^2} F(\lambda) + \frac{i}{2(1-i\lambda)} F(\lambda) \end{aligned}$$

after integration by parts.

We have then

$$\frac{d}{d\lambda} (e^{\frac{\zeta^2}{2(1-i\lambda)}} F(\lambda)) = \frac{i}{2(1-i\lambda)} e^{\frac{\zeta^2}{2(1-i\lambda)}} F(\lambda).$$

Denote by  $G(\lambda) = e^{\frac{\zeta^2}{2(1-i\lambda)}} F(\lambda)$ , we verify that, if the unique root of  $1 - i\lambda$  with positive real part<sup>2</sup> is denoted by  $(1 - i\lambda)^{\frac{1}{2}}$ ,

$$\frac{d}{d\lambda} [(1 - i\lambda)^{\frac{1}{2}} G(\lambda)] = 0.$$

Thus, since  $G(0) = (2\pi)^{\frac{1}{2}}$  and  $G$  is continuous in  $\lambda = 0$ , the integral being normally convergent, it comes

$$F(\lambda) = e^{-\frac{\zeta^2}{2(1-i\lambda)}} \frac{(2\pi)^{\frac{1}{2}}}{(1 - i\lambda)^{\frac{1}{2}}}.$$

Replacing  $\zeta$  by  $\xi_j/\sqrt{\varepsilon}$ ,  $\lambda$  by  $\varepsilon^{-1}$ , one gets

$$\int_{\mathbf{R}} e^{-ix_j\xi_j + i\frac{x_j^2}{2} - \varepsilon\frac{x_j^2}{2}} dx_j = \frac{1}{\sqrt{\varepsilon}} \frac{(2\pi)^{\frac{1}{2}}}{(1 - \frac{i}{\varepsilon})^{\frac{1}{2}}} e^{\frac{\xi_j^2}{2i - 2\varepsilon}}.$$

As  $\beta(\frac{1}{\varepsilon}) = -\frac{1}{2} \frac{1}{\varepsilon} \frac{1}{\alpha(\varepsilon)}$  and as  $\varepsilon^{\frac{1}{2}} \alpha(\varepsilon^{-1})$  goes to  $\frac{1}{\sqrt{2}}$ ,  $\varepsilon^{\frac{1}{2}} \beta(\varepsilon^{-1})$  goes to  $-\frac{1}{\sqrt{2}}$ . Hence the limit in the space  $\mathcal{S}'$  of  $\hat{u}_\varepsilon(\xi)$  is  $(2\pi)^{\frac{N}{2}} e^{i\frac{N\pi}{4}} e^{\frac{\xi^2}{2i}}$ .

The Fourier transform of  $x \rightarrow e^{ikx^2/2}$  is, using an homotethy

$$\xi \rightarrow \left(\frac{2\pi}{k}\right)^{N/2} e^{i\frac{N\pi}{4}} e^{-i\xi^2/2k}.$$

One considers then  $h \in \mathcal{S}(\mathbf{R}^N)$ . One has

$$\langle e^{ik\frac{x^2}{2}}, h \rangle = \int e^{ik\frac{x^2}{2}} h(x) dx = \frac{1}{(2\pi)^N} \int d\xi \hat{h}(\xi) \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} e^{i\frac{N\pi}{4}} e^{-i\frac{\xi^2}{2k}}.$$

We use the expansion in series of the exponential, which leads to the asymptotic expansion of this integral.  $\square$

This ends the proof of Proposition 4.3.  $\square$

**REMARK 3.** *The theorem of the oscillatory stationary phase can also be proved by considering a deformation of the integration contour in the complex plane.*

<sup>2</sup>Notice that  $(1 - i\lambda)^{\frac{1}{2}} = \alpha(\lambda) + i\beta(\lambda) = \left(\frac{1}{2}(1 + (1 + \lambda^2)^{\frac{1}{2}})\right)^{\frac{1}{2}} - \frac{i\lambda}{2} \left(\frac{1}{2}(1 + (1 + \lambda^2)^{\frac{1}{2}})\right)^{-\frac{1}{2}}$ .

We verify that we have to consider  $ix^2 = -(e^{-\frac{i}{4}}x)^2 = -y^2$ . Classically, we consider the integral on the circle of radius  $R$ , which we divide into  $\theta \leq \pi/4 - a$  and  $\pi/4 - a \leq \theta \leq \pi/4$ . In the first case, the phase grows as  $R \sin a$  and in the second case, we use a majoration of the the integral by  $Ra$  (by finite integration domain). We then use the fact that the integrand is compactly supported or fast decaying to suppress this integral on the great circle. Thus we have equality of the integral on  $\mathbb{R}$  and the integral on  $\mathbb{R}e^{\frac{i\pi}{4}}$ . From an historical and documentary point of view, this method is used in quantum physics and is called the Wick rotation.

## 6. Morse lemma for stationary non degenerate points and construction of the associated Laplacian operator

We give in this section a construction of the Laplacian associated to a stationary phase. We rigorously find in the particular case of dimension 1 the result of the proposition 4.60.

PROPOSITION 4.4. *We consider a  $\phi$ -phase with a non-degenerate critical point at  $x_0$ , of signature  $(d, 0)$ , i.e.  $\nabla_x \phi(x_0) = 0$  and  $\text{Hess} \phi(x_0)$  is a quadratic form of signature  $(d, 0)$ . In the neighborhood of  $x_0$ , there exists a diffeomorphism  $h$  from  $\mathbb{R}^d$  into  $\mathbb{R}^d$ , such that*

$$\phi(x) = \frac{1}{2}(h(x))^2 + \phi(x_0).$$

Its Jacobian is  $J(x) = \det(h'(x))$ , where  $h'$  is the gradient of the application  $h$ , hence a matrix of  $\mathcal{M}(d \times d)$ .

The Laplacian associated to this phase in the neighborhood of the point  $x_0$  is given by the following relation

$$\Delta_\phi a(x) = \text{div}[J(x)(h'^t h')^{-1} \nabla \frac{a}{J}].$$

In the case of dimension 1 space, where  $h'(x)$  is a function and  $J(x) = h'(x)$ , we find

$$\Delta_{\frac{1}{2}h^2(x)} a(x) = \partial_x \left[ \frac{1}{h'(x)} \partial_x \left( \frac{1}{h'(x)} a(x) \right) \right].$$

PROOF. We begin with the dimension 1 of space. By translation, we assume  $\phi(0) = \phi'(0) = 0, \phi''(0) > 0$ . The phase  $\phi$  considered rewrites

$$\phi(x) = x^2 \int_0^1 \phi''(tx)(1-t) dt,$$

and one introduces the function  $h(x) = x(2 \int_0^1 \phi''(tx)(1-t) dt)^{\frac{1}{2}}$ , which is a function  $C^\infty$  in a neighborhood  $]-\varepsilon_0, \varepsilon_0[$  of 0, where  $\varepsilon_0$  satisfies  $2 \int_0^1 \phi''(tx)(1-t) dt > \frac{1}{2} \phi''(0)$ ,  $-\varepsilon_0 \leq x \leq \varepsilon_0$ . The function  $h$  is monotonous on  $(-\varepsilon_0, \varepsilon_0)$  hence is a diffeomorphism of  $[-\varepsilon_0, \varepsilon_0]$  on  $[h(-\varepsilon_0), h(\varepsilon_0)]$  which is a neighborhood of 0. Its reciprocal function is denoted by  $g$  and one has

$$\int_{-\varepsilon_0}^{+\varepsilon_0} e^{-k\phi(x)} a(x) dx = \int_{-\varepsilon_0}^{+\varepsilon_0} e^{-\frac{1}{2}k(h(x))^2} a(x) dx = \int_{h(-\varepsilon_0)}^{h(\varepsilon_0)} e^{-\frac{1}{2}ky^2} g'(y) u(g(y)) dy.$$

Using the result of Theorem 4.2, one finds

$$\int_{h(-\varepsilon_0)}^{h(\varepsilon_0)} e^{-\frac{1}{2}ky^2} g'(y) a(g(y)) dy \simeq \left( \frac{2\pi}{k} \right)^{\frac{1}{2}} \sum_{p=0}^{p=n} \frac{k^{-p}}{p!} \left( \frac{1}{2} \frac{\partial^2}{\partial y^2} \right)^p [g' \cdot (a \circ g)](0)$$

Let  $L_1$  be the operator  $g'(h(x)) \frac{d}{dx}$ . The relation  $\frac{df}{dy} = \frac{dx}{dy} \frac{df}{dx}$  implies  $L_1 f(g(y)) = \frac{df}{dy}(y)$  hence  $\frac{d^{2p} f}{dy^{2p}} = L_1^{2p} f(g(y))$  hence

$$\int_{-\varepsilon_0}^{\varepsilon_0} e^{-k\phi(x)} a(x) dx \simeq \left( \frac{2\pi}{k} \right)^{\frac{1}{2}} \sum_{p=0}^n \frac{1}{p! 2^p k^p} L_1^{2p} [g'(h(x)) a(x)]|_{x=0}.$$

Let  $L_2$  be the operator<sup>3</sup> given by  $L_2 F = \frac{d}{dx}[g'(h(x))F(x)]$ . One has  $L_1[g'(h(x))F(x)] = g'(h(x))L_2 F(x)$ , hence  $L_1^k[g'(h(x))F(x)] = g'(h(x))L_2^k F(x)$ , from which one deduces, using  $g'(h(0)) = g'(0) = \frac{1}{h'(0)} = \frac{1}{(\phi''(0))^{\frac{1}{2}}}$

$$\begin{aligned} \int_{-\varepsilon_0}^{\varepsilon_0} e^{-k\phi(x)} a(x) dx &\simeq \left(\frac{2\pi}{k}\right)^{\frac{1}{2}} \sum_{p=0}^n g'(h(0)) \frac{1}{p! 2^p k^p} L_2^{2p} a(0) \\ &\simeq \left(\frac{2\pi}{k\phi''(0)}\right)^{\frac{1}{2}} \sum_{p=0}^n \frac{1}{p! 2^p k^p} (L_2^{2p} a)(0). \end{aligned}$$

Denote finally by

$$\Delta_{\frac{1}{2}h^2} a(x) = \partial_x \left( \frac{1}{h'(x)} \partial_x \left( \frac{1}{h'(x)} a(x) \right) \right),$$

which is a differential operator with variable coefficients, but which is not in no case equal to  $(\text{Hess}\phi(0))^{-1} \partial_x^2 a$ . The formula of the stationary phase is then

$$\int_{-\varepsilon_0}^{\varepsilon_0} e^{-k\phi(x)} a(x) dx \simeq \left(\frac{2\pi}{k\phi''(0)}\right)^{\frac{1}{2}} \sum_{p=0}^n \frac{1}{p! k^p} \left[ \frac{1}{2} (\Delta_{\frac{1}{2}h^2})^p a \right](0).$$

We place ourselves in the case where  $\text{Hess}\phi(x_0)$  is a positive definite matrix. Using the diffeomorphism  $h(x) = \tilde{x}$ , local diffeomorphism of the neighborhood of  $x_0$  on a neighborhood of  $0 \in \mathbb{R}^d$ , of reciprocal application  $g$ , we find, by introducing  $J_{pq}(y) = \frac{\partial g_p}{\partial y_q}(y)$ , et  $j(y) = \det(J_{pq}(y))$  and recalling that  $\phi(x) = \phi(x_0) + \frac{1}{2}(h(x))^2$ ,

$$\frac{\partial}{\partial y_q} [a(g(y))j(y)] = \sum_p J_{pq}(y) \frac{\partial}{\partial x_p} [a(x)j(h(x))]$$

then

$$\Delta_y [a(g(y))j(y)] = \sum_{i,j,p} J_{pj} \frac{\partial}{\partial x_p} \left[ \sum_i J_{ij}(h(x)) \frac{\partial}{\partial x_i} [a(x)j(h(x))] \right].$$

This formula is not very explicit. By integrations by parts of the integral against a test function  $v(y) \in C_0^\infty(\mathbb{R}^d)$ , to which we associate the test function test function  $V(x) = v(h(x))$ , we have the following equalities:

$$\begin{aligned} \int \Delta_y [a(g(y))j(y)] v(y) dy &= - \int \nabla_y [a(g(y))j(y)] \nabla_y v dy \\ &= - \int ({}^t J \nabla_x V \cdot {}^t J \nabla_x (a \cdot (j \circ h))) \det h'(x) dx \\ &= - \int (\nabla_x V \cdot J^t J \nabla_x (a \cdot (j \circ h))) \det h'(x) dx \\ &= \int V(x) \nabla_x \cdot [J^t J \nabla_x (a \cdot (j \circ h))] \det h'(x) dx. \end{aligned}$$

We introduce the differential operator  $D$  given by

$$Da(x) = \nabla_x \cdot [\det h'(x) J^t J \nabla_x [a(x) \cdot j(h(x))]].$$

Then

$$\int \Delta_y [a(g(y))j(y)] v(y) dy = \int V(x) Da(x) dx = \int v(y) Da(g(y)) \frac{dy}{\det h'(g(y))}.$$

One then deduces (thanks to the relation  $j(y) \det h'(g(y)) = 1$ )

$$\Delta_y [a(g(y))j(y)] = \frac{1}{\det h'(x)} \text{div}_x [\det h'(x) J^t J \nabla_x \left( \frac{a}{\det h'(x)} \right)] \Big|_{x=g(y)} = D(a)(g(y))j(y).$$

By induction, one finds

$$\Delta_y^p [a(g(y))j(y)] = D^p(a)(g(y))j(y).$$

One has then

$$\begin{aligned} \int e^{-k\phi(x)} a(x) dx &\simeq \left(\frac{2\pi}{k}\right)^{\frac{d}{2}} e^{-k\phi(x_0)} \sum_{p=0}^{\infty} \frac{1}{k^p 2^p p!} D^p(a)(x_0) j(0) \\ &\simeq \left(\frac{(2\pi)^d}{k^d \det \text{Hess}\phi(x_0)}\right)^{\frac{1}{2}} e^{-k\phi(x_0)} \sum_{p=0}^{\infty} \frac{1}{k^p 2^p p!} D^p(a)(x_0). \end{aligned}$$

The Laplacian associated with the  $\phi$  phase is then naturally

<sup>3</sup>The operator  $-L_2$  is the adjoint of the operator  $L_1$



$$\Delta_\phi a = Da = \nabla_x \cdot [\text{deth}'(x) J^t J \nabla_x \left[ \frac{a(x)}{\text{deth}'(x)} \right]].$$

Proposition 4.4 is proved.  $\square$

In a number of applications, one wants to eliminate only a part of the integration variables, it is then necessary to keep the other integration variables as parameters. The critical point in the variables to be eliminated then depends on these parameters. An easy generalization of the stationary phase theorem is the stationary phase theorem with parameters. This is the subject of the following section

**6.1. Stationary phase theorem with a parameter.** The general situations where one wishes to use the stationary phase theorem the stationary phase theorem usually involve several variables, and it may be useful to apply the theorem to only some of the variables. We have the following Theorem (theorem 7.7.6 of [48]):

**THEOREM 4.3.** *Assume that  $\phi$ , smooth function from  $\mathbb{R}^d \times \mathbb{R}^{n-d}$  to  $\mathbb{R}$ , satisfies  $\nabla_x \phi(0, 0) = 0$ ,  $\text{Hess}_x \phi(0, 0)$  is non-degenerate. Denote by  $\sigma$  the signature of the hessian matrix (difference between the number of positive and negative eigenvalues). Assume that  $a$  is of class  $C^\infty$ , with compact support containing  $x_0$ .*

- (1) *The locus of the critical points of  $\phi$  in  $x$ , given by  $\nabla_x \phi(x, y) = 0$ , takes the form  $x = x(y)$  for  $y$  in a neighborhood of  $y = 0$ , with  $x(0) = 0$ .*
- (2) *We assume  $a$  of class  $C^\infty$ , supported in a neighborhood  $V \times W$  of  $x = y = 0$ . There are differential operators  $L_j^\phi(x, y, \partial_x)$  in  $x$  of order  $2j$ , depending on the parameter  $y$  and the phase  $\phi$ , such that such that*

$$\int a(x, y, k) e^{ik\phi(x, y)} dy$$

$$\simeq e^{i\sigma\pi/4} \left( \frac{2\pi}{k^d \det(\text{Hess}_x \phi)(x(y), y)} \right)^{\frac{1}{2}} e^{ik\phi(x(y), y)} \sum_j L_j^\phi(x, y, \partial_x) a|_{x=x(y)}(y) k^{-j}.$$

- (3) *Under a diffeomorphism  $(x, y) \leftrightarrow (X, Y)$ , the critical points of the phases  $\phi(x, y)$  and  $\phi(x, Y)$  may be distinct, the critical values and the asymptotic expansion are invariant.*

The last paragraph of this theorem is extremely useful when one studies canonical forms of phases with certain properties. We see, for example, that when a phase  $\phi(x, t)$  has two critical points at  $t$  that coincide at  $x = 0$ , then we can represent this phase as

$$\phi_0(x) + \frac{T^3}{3} - a(x)T$$

which allows to express  $\int b(x, T) e^{ik(\phi_0(x) + \frac{T^3}{3} - a(x)T)} dT$  using the Airy function introduced below in the exercise 4.1 (and its derivative). This representation will be used in the section 4. The proof of this theorem is the subject of the exercise 4.2. The theorem 4.3 will also be used in the chapter 11.

## 7. Application to the solution of the wave equation generated by given data on a surface

We consider a smooth surface  $S$ , bounded or not. We compute, for  $x \notin S$

$$(7.70) \quad I_k(x) = \int_S \frac{e^{ik|x-y|}}{|x-y|} c(y) d\sigma_S(y),$$

where  $d\sigma_S(y)$  is the measure on  $S$  induced by the Euclidean measure on the ambient space. We will only be concerned here with local results. The manifold  $S$  can be a free surface in space, on which a solution is known; it can also be the boundary of an obstacle. We assume it to be compact, or we assume  $c$  to be compactly supported.

Case where  $c$  does not depend on  $k$ . As for all  $y \in S$ ,  $(\Delta + k^2) \frac{e^{ik|x-y|}}{|x-y|} = 0$  for  $x \notin S$  (as has been noticed in chapter 1), we have

$$(\Delta + k^2)I_k(x) = 0 \quad \forall x \notin S.$$

We calculate this integral using the phase (Proposition 4.3, paragraph (ii)). For this purpose, let us introduce  $\phi_0(y) = |y - x|$ . We introduce a local parametrization of the surface<sup>4</sup> and the semi-geodesic coordinates  $u \in R^{n-1} \cap V(y_0)$ ,  $y_0 \in S$ ,  $V$  neighborhood in  $\mathbb{R}^n$  of  $y_0$ . A point of  $V(y_0)$  is represented by  $u$  and by  $l$  such that  $y = Y(u, l) = y(u) + l(u)n(y(u))$ . The Jacobian of the transformation is computed later (Section 4, Section 4.2); it involves the geodesic matrix of the boundary. We introduce the matrix  $\tilde{M}(u, l) = (\partial_j y_k, \partial_l y_k)$ .

We introduce  $\Phi(u) = \phi_0(y(u))$ , we check that

$$\frac{\partial \Phi}{\partial u_j} = \frac{y(u) - x}{|y(u) - x|} \cdot \frac{\partial y}{\partial u_j}.$$

The vectors  $(\frac{\partial y}{\partial u_j})$  form a basis of the tangent space  $T_{y(u)}S$ . The phase  $\Phi$  is stationary if and only if  $y(u) - x$  is orthogonal to the vector  $\frac{\partial y}{\partial u_j}(u)$  for all  $j$ , thus orthogonal to the tangent hyperplane  $T_{y(u)}S$ . Let  $y(u_0)$  be a critical point. There exists  $\lambda$  such that

$$(7.71) \quad \frac{y(u) - x}{|y(u) - x|} = \lambda \bar{n}(y(u)).$$

The comparison of the norms gives  $\lambda = \pm 1$  and the orientation gives  $\lambda = -1$ .

Therefore, there exist  $u_0$  and  $l_0$  such that  $x = y(u_0) + l_0 \bar{n}(y(u_0))$ . Then  $u = u_0$  is a solution of (7.71). The corresponding point  $y$  is a point which makes extremum the distance to the boundary. It is not necessarily unique; indeed, if the surface  $S$  is a sphere, and if the point  $x$  is the center of the sphere then all points on the surface are suitable.

Assume that  $S = \partial\Omega$ , where  $\Omega$  is convex. If  $y \in V(y_0) \cap C\Omega$ , then the vector  $\bar{n}(y_0)$  defined above is the exterior normal unit vector to  $\Omega$  at  $y_0$  and there is uniqueness of the solution of (7.71). To show this, let us compute the Jacobian. It is sufficient to show that the matrix  $\frac{\partial^2 \Phi}{\partial u_i \partial u_j}$  is non-degenerate to obtain the local uniqueness of the solution of the system  $\frac{\partial \Phi}{\partial u_i} = 0$ . From  $\frac{\partial \Phi}{\partial u_j} = \frac{y(u) - x}{|y(u) - x|} \cdot \frac{\partial y}{\partial u_j}$ , we deduce

$$(7.72) \quad \frac{\partial^2 \Phi}{\partial u_j \partial u_m} = \frac{y(u) - x}{|y(u) - x|} \cdot \frac{\partial^2 y}{\partial u_j \partial u_m} + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u) - x|} - \frac{(y(u) - x, \partial_{u_j} y)(y(u) - x, \partial_{u_m} y)}{|y(u) - x|^3}.$$

We calculate this value at the critical point  $u_0$ . From the orthogonality of  $\bar{n}$  and the tangent vector to the surface, we deduce

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial u_j \partial u_m}(u_0) &= -\bar{n}(y(u_0)) \cdot \frac{\partial^2 y}{\partial u_j \partial u_m} + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u_0) - x|} - \frac{(\bar{n}(y(u_0)), \partial_{u_j} y)(\bar{n}(y(u_0)), \partial_{u_m} y)}{|y(u_0) - x|} \\ &= -\bar{n}(y(u_0)) \cdot \frac{\partial^2 y}{\partial u_j \partial u_m} + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u_0) - x|}, \end{aligned}$$

using the orthogonality of  $y(u) - x$  to the tangent plane of the boundary at  $y(u)$ . As  $\bar{n}(y(u)) \cdot \partial_{u_j} y(u) = 0$ , on gets the identities

$$\frac{\partial}{\partial u_m} (\bar{n}(y(u)) \cdot \partial_{u_j} y(u) + \bar{n}(y(u)) \cdot \frac{\partial^2 y}{\partial u_j \partial u_m}(u)) = 0.$$

It appears, as in the chapter 1 the Weingarten matrix  $W$  which is the derivative of the normal vector along direction (and whose eigenvalues are the principal curvatures of the surface  $\partial\Omega$ ), such that  $\frac{\partial}{\partial u_m} (\bar{n}(y(u))) = W(y(u)) \frac{\partial y}{\partial u_m}$

$$(7.73) \quad W(y(u)) \cdot \partial_{u_m} y(u) \partial_{u_j} y(u) + \bar{n}(y(u)) \cdot \partial_{u_j \partial u_m}^2 y(u) = 0,$$

<sup>4</sup>as it is detailed later (Chapter 10)

which yields

$$\frac{\partial^2 \Phi}{\partial u_j \partial u_m}(u_0) = W(y(u)) \cdot \partial_{u_m} y(u) \partial_{u_j} y(u) + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u) - x|}.$$

The determinant of  $(\frac{\partial^2 \Phi}{\partial u_j \partial u_m}(u_0))$  is zero when the point  $x$  is located at the center of curvature of  $S$ . We introduce the matrix

$$L(x, y(u)) = W(y(u)) + \frac{1}{|x - y(u)|} Id$$

One obtains

$$(\det(\partial_{u_j u_m}^2 \Phi)(u_0))^{-\frac{1}{2}} d\sigma_S(y)/du = (\det(W(y(u)) + \frac{1}{|x - y(u)|} Id))^{-\frac{1}{2}},$$

and the equivalent of  $I_k$  given by the stationary phase theorem is equal to

$$I_k(x) \simeq \left(\frac{2\pi}{k}\right)^{\frac{n-1}{2}} c(y(u_0)) (\det(W(y(u_0)) + \frac{1}{|x - y(u_0)|} Id))^{-\frac{1}{2}} e^{ik|x - y(u_0)|}.$$

The above calculation was performed when  $c(y)$  is independent of  $k$ .

Case where  $c$  has an oscillatory phase in  $k$ . In a second application we assume that  $c(y, k) = a(y, k)e^{ik\phi(y)}$  where  $a$  is compactly supported in  $y$ . The considered phase is

$$\psi(y) = |x - y| + \phi(y).$$

The gradient of the phase, using the same boundary representation  $y = y(u)$ , is

$$\partial_{u_j} \psi(y(u)) = \left[ \frac{y(u) - x}{|y(u) - x|} + \nabla \phi \right] \cdot \partial_{u_j} y(u).$$

This gradient is null when the vector  $[\frac{y(u) - x}{|y(u) - x|} + \nabla \phi]$  is orthogonal to the vectors  $(\partial_{u_j} y(u))$ . This means that it is collinear à  $\vec{n}(y(u))$ . For a critical point, there exists  $\lambda$  such that

$$\frac{y(u) - x}{|y(u) - x|} + \nabla \phi = \lambda \vec{n}(y(u)).$$

We notice that  $\nabla \phi$  is tangent to  $S$ , thus orthogonal to  $\vec{n}$  at the considered point and moreover  $|\nabla \phi(y(u))| \leq 1$ . We suppose that  $S = \partial\Omega$ ,  $\Omega$  convex and that  $x \in C\Omega$ . It comes  $\lambda^2 + (\nabla \phi(y(u)))^2 = 1$ , so we define a unit normal vector by

$$(7.74) \quad \frac{y(u) - x}{|y(u) - x|} = (1 - \|\nabla \phi(y(u))\|^2)^{\frac{1}{2}} \vec{n}(y(u)) - \nabla \phi(y(u)) = \vec{t}(u).$$

Finding  $u$  is equivalent to solving the equation  $x = y(u) + \mu \vec{t}(y(u))$ .

An interesting special case is the case where  $\nabla \phi(y(u))$  is of norm 1. The point  $x$  is then located on the tangent to  $S$  parallel to the vector  $\nabla \phi(y)$ . So, for a given  $x$ , we have to find the tangent point of any line coming from  $x$  tangent to  $S$ , which will give the admissible points  $y(u)$ .

We use  $|\nabla \phi(y(u))| \leq 1$  to calculate the Jacobian of  $u \rightarrow \phi(y(u))$ . The relation (7.72) allows to obtain

$$\begin{aligned} \partial_{u_j u_m}^2 (\psi(y(u))) &= \frac{y(u) - x}{|y(u) - x|} \cdot \partial_{u_j u_m}^2 y + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u) - x|} + \nabla \phi(y(u)) \cdot \partial_{u_j u_m}^2 y \\ &\quad - \frac{((y(u) - x) \cdot \partial_{u_j} y)((y(u) - x) \cdot \partial_{u_m} y)}{|y(u) - x|^3} + \text{Hess} \phi \partial_{u_j} y \partial_{u_m} y. \end{aligned}$$

eliminating terms through (7.74) and using the orthogonality of  $\vec{n}(y(u))$  with  $\partial_{u_j} y(u)$ , we find

$$\begin{aligned} \partial_{u_j u_m}^2 (\psi(y(u))) &= (1 - (\nabla \phi(y(u)))^2)^{\frac{1}{2}} \vec{n}(y(u)) \cdot \partial_{u_j u_m}^2 y + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u) - x|} \\ &\quad - \frac{(\nabla \phi(y(u)) \cdot \partial_{u_j} y)((\nabla \phi(y(u)) \cdot \partial_{u_m} y)}{|y(u) - x|} + \text{Hess} \phi \partial_{u_j} y \partial_{u_m} y. \end{aligned}$$

Finally, using (7.73), we obtain

$$\begin{aligned} \partial_{u_j u_m}^2 (\psi(y(u))) &= (1 - (\nabla\phi(y(u)))^2)^{\frac{1}{2}} W(y(u)) \partial_{u_j} y \partial_{u_m} y + \frac{\partial_{u_m} y \cdot \partial_{u_j} y}{|y(u) - x|} \\ &\quad - \frac{(\nabla\phi(y(u)) \cdot \partial_{u_j} y)(\nabla\phi(y(u)) \cdot \partial_{u_m} y)}{|y(u) - x|} + \text{Hess}\phi \partial_{u_j} y \partial_{u_m} y. \end{aligned}$$

We introduce the orthogonal projection  $\pi$  parallel to  $\nabla\phi(y(u))$ . We have

$$\begin{aligned} \partial_{u_j u_m}^2 (\psi(y(u))) &= (1 - (\nabla\phi(y(u)))^2)^{\frac{1}{2}} [W(y(u)) + \frac{1}{|y(u) - x|} Id] \partial_{u_j} y \partial_{u_m} y \\ &\quad + \text{Hess}\phi \partial_{u_j} y \partial_{u_m} y + \frac{|\nabla\phi(y(u))|^2}{|y(u) - x|} \pi(\partial_{u_j} y) \cdot \pi(\partial_{u_m} y). \end{aligned}$$

This allows us to obtain, in the case where  $|\nabla\phi| = 1$ , that the Jacobian of the transformation is equal to  $\det[\text{Hess}\phi + \frac{1}{|y(u) - x|} Id]$ .

We state the results of this paragraph:

Let  $c(y, k) = a(y, k) e^{ik\phi(y)}$ , where  $a \simeq \sum_{j=0}^{+\infty} a_j k^{-j}$ , where  $a_j$  and all its derivatives are bounded in  $k$  for  $k \geq 1$ ,  $a$  compactly supported if  $S$  is not compact

**PROPOSITION 4.5.** • *The asymptotic contribution of a source  $c(y, k)$  (assumed to be supported in  $y$  or  $S$  is compact) on an boundary  $S$  to the the integral*

$$I_k(x) = \int_S \frac{e^{ik|x-y|}}{|x-y|} c(y, k) d\sigma_S(y)$$

*depends crucially on the phase  $\phi$ .*

- *The point  $y(u(x))$  where we compute  $a(y, k)$  (which is the point of the boundary contributing to the value of  $I_k(x)$ ) is a solution of*

$$\frac{y(u(x)) - x}{|y(u(x)) - x|} = (1 - (\nabla\phi(y(u(x))))^2)^{\frac{1}{2}} \vec{n}(y(u(x))) - \nabla\phi(y(u(x))).$$

- *The Jacobian of the phase is thus*

$$\det[(1 - |\nabla\phi|^2)W(y(u(x))) + \text{Hess}\phi(y(u(x))) + \frac{1}{|y(u(x)) - x|}].$$

**LEMMA 4.6.** *The two following cases are useful in practical* • *the phase  $\phi$  is zero. The point  $y(u(x))$  is the one which minimizes the distance to  $S$ . The Jacobian is equal to the product  $\Pi(\kappa_i(u(x)) + \frac{1}{d(x, S)})$ , the  $\kappa_i(u(x))$  being the main curvatures of the surface at the point  $y(u(x))$ .*

• *the gradient of the phase  $\phi$  is 1. Let  $y(u)$  be a point on the boundary such that the line  $(xy(u))$  is tangent to  $S$ . The Jacobian of the phase is the product of all  $\beta + \frac{1}{|y(u) - x|}$  where  $\beta$  is an eigenvalue of the hessian matrix of  $\phi$ .*

The previous discussion on the critical points of the phase allows us to obtain an asymptotic expansion of  $I_k(x)$ .

We end this section of applications of the stationary phase method to the to the wave equation by the scattering calculation which can be found in all found in all the scattering matrix courses. It can be found for example in the course of R. B. Melrose, given at Stanford University [78].

For this, we introduce the spectral resolution of the identity, which is the transcription in polar coordinates of the identity

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{iz\xi} \hat{f}(\xi) d\xi$$

qui se réécrit

$$f(x) = (2\pi)^{-n} \int_0^{+\infty} \int_{S^{n-1}} e^{i\lambda x \cdot \omega} \hat{f}(\lambda\omega) \lambda^{n-1} d\lambda d\omega.$$

Either  $E_0(\lambda)$  is the spectral projector, given by

$$E_0(\lambda) f(x, \lambda) = (2\pi)^{-n} \int_{S^{n-1}} e^{i\lambda x \cdot \omega} \lambda^{n-1} \hat{f}(\lambda\omega) d\omega,$$

and thanks to the spectral theorem,  $Id = \int_0^\infty E_0(\lambda) d\lambda$ .

Or  $\Phi_0(x, \omega, \lambda) = e^{i\lambda x \cdot \omega}$  the family of plane waves of pulsation  $\omega$  and of wave vector norm is  $\lambda > 0$ , then

$$E_0(\lambda) = (2\pi)^{-n} \lambda^{n-1} F_0(\lambda) F_0^*(\lambda)$$

where  $F_0$  is an operator from  $\mathcal{C}^\infty(S^{n-1})$  onto  $\mathcal{S}'(\mathbb{R}^n)$  given by

$$(F_0(\lambda))g(x) = \int_{S^{n-1}} \Phi_0(x, \omega, \lambda) g(\omega) d\omega$$

and  $F_0^*(\lambda)$  acts from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{C}^\infty(S^{n-1})$ :

$$(F_0^*(\lambda)h)(\omega) = \int_{\mathbb{R}^n} \Phi_0(x, \omega, -\lambda) h(y) dy$$

By applying the stationary phase theorem, we obtain

$$(7.75) \quad \begin{aligned} F_0(\lambda)g(\theta|x|) \simeq & e^{i\lambda|x|} (\lambda|x|)^{-\frac{1}{2}} e^{-\frac{1}{4}\pi(n-1)i} (2\pi)^{\frac{n-1}{2}} \sum_{j \geq 0} |\lambda x|^{-j} h_j^+(\theta) \\ & + e^{-i\lambda|x|} (\lambda|x|)^{-\frac{1}{2}} e^{\frac{1}{4}\pi(n-1)i} (2\pi)^{\frac{n-1}{2}} \sum_{j \geq 0} |\lambda x|^{-j} h_j^-(\theta). \end{aligned}$$

We write  $x = |x|\theta$ . The phase is then written  $i\lambda|x|\theta \cdot \omega$ . As the integral is invariant by rotation, we choose the coordinates on  $S^{n-1}$  so that  $\theta \cdot \omega = \omega_1$ , with  $\omega_1 = \pm(1 - (\omega')^2)^{\frac{1}{2}}$ . We note  $J(\omega')$  the Jacobian of the transformation  $d_{S^{n-1}}\omega$  into  $d\omega'$ . It comes

$$\begin{aligned} F_0(\lambda)g(\theta|x|) &= \int_{\mathbb{R}^{n-1}} e^{i\lambda|x|\omega_1} g(\omega_1, \omega') J(\omega') d\omega' \\ &= \int_{S^{n-1}} e^{i\lambda|x|(1 - (\omega')^2)^{\frac{1}{2}}} g((1 - (\omega')^2)^{\frac{1}{2}}, \omega') J(\omega') d\omega' \\ &+ \int_{\mathbb{R}^{n-1}} e^{-i\lambda|x|(1 - (\omega')^2)^{\frac{1}{2}}} g(-(1 - (\omega')^2)^{\frac{1}{2}}, \omega') J(\omega') d\omega'. \end{aligned}$$

Thanks to the Taylor expansion  $(1 - (\omega')^2)^{\frac{1}{2}} = 1 - \frac{1}{2}(\omega')^2 + o((\omega')^2)$ , we verify that the stationary phase theorem applies to the stationary phase applies to the point  $\omega' = 0$  and that  $h_0^+(\theta) = g(\theta)$ ,  $h_0^-(\theta) = g(-\theta)$ . The terms  $h_j^\pm$  are results of the action of iterates of the Laplacian on the sphere, which is the operator  $L$  associated to the critical point  $\theta \cdot \omega = \pm 1$  of the phase  $\theta \cdot \omega$ .

From the expression  $\Delta = -\int_0^\infty \lambda^2 E_0(\lambda) d\lambda$ , we deduce that any solution of deduces the fact that any solution of  $(\Delta - \lambda^2)u = 0$  verifies so  $(|\xi|^2 - \lambda^2)\hat{u}(\xi) = 0$ , that is, taking  $\xi = r\theta$ , the equality  $\hat{u}(\xi) = \delta(r - \lambda)g_1(\theta)$ . By inverse Fourier transform, we find

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{irx \cdot \theta} \delta(r - \lambda) g_1(\theta) r^{n-1} dr d\theta$$

donc

$$u = F_0(\lambda)((2\pi)^{-n} \lambda^{n-1} g_1).$$

The above asymptotic expansion extended to  $g_1$  distribution leads to

LEMMA 4.7. *For any function  $h(\theta)$  of class  $\mathcal{C}^\infty(S^{n-1})$ , there exists a solution of  $(\Delta + \lambda^2)u = 0$  admitting for  $|x| \rightarrow \infty$  the expansion*

$$u(|x|\theta) = e^{i\lambda|x|} |x|^{-\frac{1}{2}(n-1)} h(\theta) + e^{-i\lambda|x|} |x|^{-\frac{1}{2}(n-1)} h_1(\theta) + O(|x|^{-\frac{1}{2}(n+1)}).$$

The relation between  $h_1$  et  $h$  is, thanks to the asymptotic expansion,  $h_1(\theta) = i^{n-1} h(-\theta)$ , which is the simplest of scattering matrices.

## 8. Solution of the wave equation across the caustic with the stationary phase method

In this section, we use the results on the complex stationary phase to complete the section 2 of chapter 3. In particular, we compute the value of  $a_0(x(T))$  solution de (1.48) for  $T > t_0$ , where  $t_0$  is the first point for which  $\det(Id + t_0 W(x_0)) = 0$ , if it exists. We introduce  $\psi(t) = \phi_0 + t$ . For  $t < t_0$  we know that  $\psi(t) = \phi(x(t))$  and we notice that  $\psi$  is defined everywhere. is well defined.

We prove the following result:

PROPOSITION 4.6. Let  $\Sigma$  be a surface of class  $C^\infty$  in  $\mathbb{R}^3$ , and let  $a(x, k) \simeq \sum_j a_j(x)k^{-j}$ ,  $a_j$  being of class  $C^\infty$  on  $\mathbb{R}^3$ , and  $a$  being of class  $C^\infty$  on  $\mathbb{R}^3 \times [1, +\infty[$ . Consider a constant  $\phi_0 \in \mathbb{R}$ . Let  $b(x, k)$  be  $\simeq b_j(x)k^{-j}$  solution of

$$\begin{cases} (\Delta + k^2)(b(x, k)e^{ik\phi(x)}) \simeq 0, \\ \phi(x) = \phi_0, x \in \Sigma, \\ b(x, k)|_\Sigma = a(x, k). \end{cases}$$

All the terms  $b(x, k)$  can be calculated  $x(t) = x_0 + tN(x_0)$  out of the points  $t \geq 0$ . such that  $\det(Id + tW(x_0)) = 0$ . In particular

$$b_0(x) = \sum_{x_0 \in \Sigma, x = x_0 + tN(x_0)} \frac{a_0(x_0)}{(|\det(Id + tW(x_0))|)^{\frac{1}{2}}} e^{in\frac{\pi}{2}}$$

where  $-n$  is the Maslov index, equal to the number of points of the caustic crossed between 0 and  $t$  on each line  $x_0 + sN(x_0)$ .

REMARK 4. We have not expressed here the solution in the neighborhood of the point  $t_0$ , point of the caustic; this will be done in the next section.

PROOF. We call the Bargmann transform of the function  $u(x) \in L^1(V)$ ,  $V \subset \mathbb{R}^d$  the holomorphic function

$$Tu(z, k) = \int_V e^{-\frac{k}{2}(z-x)^2} u(x) dx.$$

The Bargmann transform allows us to get rid of the points of the caustic  $\mathcal{C}$  by deforming the parameter on the bicaracteristics in the complex, so that the function obtained is always continuous in  $t$ . In fact, we will deform the parameter up to  $t - i$ . We denote by

$$d(t, k) = Tu(x_0 + (t - i)\nabla\phi(x_0), k).$$

We write

$$\begin{aligned} \psi(x) &= \phi(x) + \frac{i}{2}(x - (x_0 + t\nabla\phi(x_0)) + i\nabla\phi(x_0))^2 \\ &= \phi(x) - \frac{i}{2} + \frac{i}{2}(x - (x_0 + t\nabla\phi(x_0)))^2 - \nabla\phi(x_0) \cdot (x - (x_0 + t\nabla\phi(x_0))) \end{aligned}$$

This phase admits a critical point in  $x$  at the point  $x = x_0 + t\nabla\phi(x_0)$ . Indeed, there exists a matrix  $A(x)$  such that

$$\nabla\phi(x) - \nabla\phi(x_0 + t\nabla\phi(x_0)) = A(x) \cdot [x - (x_0 + t\nabla\phi(x_0))].$$

The matrix  $A(x) + iId$  is invertible since  $A$  is real.

There exists a neighborhood  $V$  of  $t = 0$  such that, for  $t \in V$ , the equation  $abla_x \psi(x) = 0$  has a unique solution equal to  $x = x_0 + t\nabla\phi(x_0)$ . We notice that this result is true for  $x$  in the connected component of  $\mathbb{R}^3 \setminus \mathcal{C}$  containing  $\Sigma$ . Moreover, the Hessian matrix of this phase is  $Id + i\text{Hess}\phi(x)$ . We are in the conditions of application of the complex stationary phase theorem, the critical value of the phase is  $i/2 + \phi(x_0 + t\nabla\phi(x_0)) = i/2 + \phi_0 + t$  and the determinant of the Jacobian is  $i^3 \det(W_t(x_0) - iId)$ .

In the rest of the proof, we consider a point  $x_0 \in \Sigma$  and we study the ray coming from  $x_0$ . The solution is the superposition of the set of solutions, where we consider for each  $x$  the set of points  $x_0 \in \Sigma$  such that there exists  $t$  such that  $x = x_0 + t\nabla\phi(x_0)$ .

One may wonder why there could be several solutions. This will be presented later; let us note for the moment that the phase characterizing a simple caustic (the fold) is given by

$$\Phi(a(x), b(x), T) = \frac{T^3}{3} - a(x)T + b(x)$$

and that a stationary phase calculation leads to consider two critical points when  $a(x) > 0$  (see Exercise 4.1 for the proof of this statement). The result of the Lemma 4.1 gives a contribution for all critical values. This approach is a current research topic: the numerical computation of multivalued solutions of the of the eikonal equation is now operational (see the work of J.D. Benamou, the thesis of I. Sollic, and the search for multivalued viscosity solutions in [11], [91], [12], [1]).

In the rest of the proof, we consider a point  $x_0 \in \Sigma$  and we study the ray study the ray coming from  $x_0$ . Consider the amplitude  $b_0^*(y)$  associated to the part of the wave carried on the ray  $x_0$  ( $y \in \{x_0 + s\nabla\phi(x_0)\}$ ). We then consider the expression  $d^*(t, k)$  coming from the calculation of stationary phase at this point. The stationary phase theorem (Proposition 4.2) leads (in dimension  $d = 3$ ) à

$$d^*(t, k) \simeq e^{\frac{k}{2}} \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} b_0^*(x_0 + t\nabla\phi(x_0)) e^{i\phi_0 + it} (\det(Id - iW_t(x_0)))^{-\frac{1}{2}} (1 + O(k^{-1})).$$

This is true for any  $t$  on the radius, outside  $t_0$ , by modifying the argument the argument (replacing for example  $t$  by  $t - T$  and the point  $x_0$  by the point  $x_0 + T\nabla\phi(x_0)$ ). The expression of  $d^*(t, k)$  for  $t < t_0$  is transformed using the relation

$$W_t(x_0) = (Id + tW(x_0))^{-1}W(x_0)$$

and the relation

$$b_0^*(x_0 + t\nabla\phi(x_0)) = b_0^*(x_0) / (\det(Id + tW(x_0)))^{\frac{1}{2}}.$$

We obtain, for  $t < t_0$  (only the main term is written here)

$$\begin{aligned} d^*(t, k) &= e^{\frac{k}{2}} \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} b_0^*(x_0) e^{i\phi_0 + it} [\det(Id + tW(x_0))(Id - iW_t(x_0))]^{\frac{1}{2}} \\ &= e^{\frac{k}{2}} \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} b_0^*(x_0) e^{i\phi_0 + it} [\det(Id + (t - i)W(x_0))]^{-\frac{1}{2}}. \end{aligned}$$

Similarly, for  $t > t_0$ , we verify that

$$\begin{aligned} d^*(t, k) &= e^{\frac{k}{2}} \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} b_0^*(x_0 + T\nabla\phi(x_0)) e^{i\phi_0 + i(t - T + T)} [\det(Id + (t - T)W(x_0))(Id - iW_t(x_0))]^{\frac{1}{2}} \\ &= e^{\frac{k}{2}} \left(\frac{k}{2\pi}\right)^{\frac{3}{2}} b_0^*(x_0 + T\nabla\phi(x_0)) e^{i\phi_0 + it} [\det(Id + (t - i)W(x_0))]^{-\frac{1}{2}} [\det(Id + TW(x_0))]^{\frac{1}{2}} \end{aligned}$$

The function  $Tu(z, k)$  is holomorphic in  $Imz \neq 0$  by analytical so in particular the function  $Tu(x_0 + (t - i)\nabla\phi(x_0), k)$  is continuous in  $t$ . The determination of the root is chosen so that the function

$$t \rightarrow (\det(Id + tW(x_0)))^{\frac{1}{2}} (\det(Id - iW_t(x_0)))^{\frac{1}{2}}$$

is continuous in  $t$ . We deduce the equality

$$b_0^*(x_0) = b_0^*(x_0 + T\nabla\phi(x_0)) [\det(Id + TW(x_0))]^{\frac{1}{2}}.$$

The choice of the determination of the square root allows us to write, using the Maslov using the Maslov index  $n$ ,

$$(\det(Id + tW(x_0)))^{-\frac{1}{2}} = e^{\frac{in\pi}{2}} (|\det(Id + tW(x_0))|)^{-\frac{1}{2}}.$$

In particular, if we place ourselves in the case  $d = 3$  and  $\kappa_1 < \kappa_2 < 0$ , then

$$(8.76) \quad \begin{cases} 0 < t < -\kappa_1^{-1}, n = 0, \\ -\kappa_1^{-1} < t < -\kappa_2^{-1}, n = -1, \\ t > -\kappa_2^{-1}, n = -2. \end{cases}$$

We thus calculated the amplitude along a ray. To obtain the solution of the Helmholtz equation, we must superimpose the set of solutions, and, in particular, for each  $x$ , consider the set of each  $x$ , consider the set of  $x_0 \in \Sigma$  such that there exists  $t$  such that  $x_0 + t\nabla\phi(x_0) = x$ . At any point  $x$  in space the asymptotic solution is the superposition of the amplitudes corresponding to all the rays from  $\Sigma$  arriving at point  $x$  at time  $t$ . Thus, even if  $b_0^*(x_0) = a_0(x_0)$ , there is no reason why  $a_0(x_0 + t\nabla\phi(x_0)) = b_0^*(x_0 + t\nabla\phi(x_0))$ , but  $a_0(x_0 + t\nabla\phi(x_0))$  can be written  $b_0^*(x_0 + t\nabla\phi(x_0)) + c_0$ , where  $c_0$  will eventually refer to another value of  $a_0$  in  $y_0$  such that  $y_0 + t\nabla_x\phi(y_0) = x = x_0 + t\nabla\phi(x_0)$ .  $\square$

### 9. Exercises of chapter 4

Exercise 4.1: The Airy function. 1) Prove that  $\xi \rightarrow \exp(i\frac{\xi^3}{3})$  is in  $\mathcal{S}'(\mathbb{R})$ . One defines the Airy function  $Ai$  as the inverse Fourier transform in  $\mathcal{S}'(\mathbb{R})$  of  $\exp(i\frac{\xi^3}{3})$ . Prove that  $Ai \in \mathcal{S}'$  is a  $C^\infty$  function such that, for all  $\eta > 0$ ,

$$Ai(x) = \frac{1}{2\pi} \int_{\mathbb{R}+i\eta} \exp(i\xi^3/3 + ix\xi) d\xi.$$

2) Prove that  $Ai$  is solution of the ordinary differential equation

$$Ai''(x) = xAi(x).$$

3) Choosing  $\eta$  accordingly in terms of  $x$  prove that the resulting phase can be written, for all  $x > 0$ :

$$-\xi^2\sqrt{x} + i\frac{\xi^3}{3}.$$

Calculate an asymptotic expansion of  $Ai$  when  $x$  goes to  $+\infty$ .

4) Recover the asymptotic expansion of  $Ai$  by writing the new function  $Ai(k^{\frac{2}{3}}u)$  and applying the stationary phase theorem.

Exercise 4.2 Degenerate stationary phase theorem. Consider a phase  $\phi$  on a manifold  $X$  which has critical points on a submanifold  $W$  of dimension  $d$ , that is  $\nabla_z \phi(z)|_{z \in W} = 0$ .

1) Prove that there is a local chart  $z = (x, y)$ ,  $y \in \mathbb{R}^d$  such that the system of equations  $\partial_x \phi(x, y) = 0$  is equivalent, in the neighborhood of a point  $(x(y_0), y_0) \in W$ , to  $x = x(y)$  (hence locally, the equation of  $W$  is  $x = x(y)$ ).

2) Assume now that, in these new local coordinates  $\text{Hess}_{x^2} \phi$  is non degenerate. Evaluate, for any  $a \in C^\infty(X)$ , the integral

$$\int_X e^{ik\phi(z)} a(z) dz.$$

Solution de l'exercice 4.1. 1) On vérifie que

$$\text{Re}(i\frac{\xi^3}{3} + i\xi x) = \text{Re}(i(\zeta + i\eta)^3/3 + i\zeta x - \eta x) = -\zeta^2\eta + \eta^3/3 - \eta x.$$

L'intégrale converge donc. Il n'y a pas de résidu entre  $\Im \xi = \eta_1$  et  $\Im \xi = \eta_2$  dans le plan complexe, donc elle est indépendante de  $\eta > 0$ .

Elle est alors de classe  $C^\infty$  puisque le comportement en  $\xi$  est en  $e^{-\eta\xi^2}$ .

On vérifie enfin que, au sens des distributions de  $\mathcal{S}'$ ,  $\exp(i(\zeta + i\eta)^3/3)$  converge vers  $\exp(i\zeta^3/3)$  lorsque  $\eta$  tend vers 0 par valeurs positives, donc on peut appliquer la continuité de la transformée de Fourier et donc on a convergence dans  $\mathcal{S}'$ .

2) On vérifie que

$$\int_{\mathbb{R}+i\eta} (\zeta^2 + x) e^{i(\zeta^3/3 + \zeta x)} d\zeta = 0.$$

On remarque que le terme  $\zeta^2$  correspond à  $-Ai''(x)$ , d'où le résultat.

3) La phase stationne en  $\xi = -i\eta \pm ix^{\frac{1}{2}}$  lorsque  $x > 0$ . On choisit  $\eta = x^{\frac{1}{2}}$  pour avoir les deux points critiques 0 et  $2ix^{\frac{1}{2}}$ . On vérifie que la phase s'écrit, pour cette valeur de  $\eta$ :

$$i(\xi^3/3 + ix^{\frac{1}{2}}\xi^2 + \frac{2}{3}ix^{\frac{3}{2}}).$$

On a alors

$$Ai(x) = \frac{1}{2\pi} \exp(-\frac{2}{3}x^{\frac{3}{2}}) \int_{\mathbb{R}} e^{-x^{\frac{1}{2}}\xi + i\xi^3/3} d\xi.$$

En effectuant le développement de Taylor de l'exponentielle, on trouve

$$Ai(x) = \frac{1}{2\pi} \exp(-\frac{2}{3}x^{\frac{3}{2}}) \int_{\xi \in \mathbb{R}} d\xi \left[ \sum_{p=0}^N \left(\frac{i}{3}\right)^p \frac{\xi^{3p}}{(p)!} e^{-x^{\frac{1}{2}}\xi^2} + R_n \right].$$



On a l'égalité  $\int_{\mathbf{R}} \xi^{3q} e^{-x^{\frac{1}{2}} \xi^2} d\xi = x^{-3q/4-1/4} \int_{\mathbf{R}} u^{3q} e^{-u^2} du$ , nulle lorsque  $q$  est impair. Pour  $q = 2p$ , on trouve  $2 \int_0^{+\infty} \frac{dv}{2v^{\frac{1}{2}}} v^{3p} e^{-v} = \Gamma(3p + \frac{1}{2})$ . Chaque terme de la somme est alors égal à

$$x^{-\frac{1}{4}} \left[ \sum_{p=0}^N (-9)^p \frac{\Gamma(3p + \frac{1}{2})}{(2p)!} x^{-3p/2} \right],$$

car le terme  $(q!)^{-1}$  est égal à  $((2p)!)^{-1}$ . Le développement asymptotique de  $Ai(x)$  s'écrit alors

$$Ai(x) = \frac{1}{2\pi} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right) x^{-\frac{1}{4}} \sum_{p=0}^N \left(-\frac{1}{9}\right)^p \frac{\Gamma(3p + \frac{1}{2})}{(2p)!} x^{-\frac{3p}{2}} + \int_{\xi \in \mathbf{R}} d\xi R_N$$

Le terme  $R_n$  est majoré par le premier terme après la troncature de la série définissant  $\Phi$  que l'on n'a pas considéré, soit  $\frac{\xi^{3N}}{N!}$ . Ceci donne alors le majorant du terme général de la série.

4) Nous écrivons

$$Ai(k^{\frac{2}{3}}u) = \frac{1}{2\pi} \int_{\xi \in \mathbf{R} + ik^{\frac{1}{3}}\eta} e^{i(\xi^3/3 + k^{\frac{2}{3}}u\xi)} d\xi.$$

Effectuons le changement de variable  $\xi = k^{\frac{1}{3}}\tau$ , l'intégrale a lieu sur  $\tau \in \mathbf{R} + i\eta$ . Alors

$$Ai(k^{\frac{2}{3}}u) = \frac{k^{\frac{1}{3}}}{2\pi} \int_{\tau \in \mathbf{R} + i\eta} e^{ik(\tau^3/3 + u\tau)} d\tau.$$

La phase  $\phi(s) = u(s + i\eta) + \frac{1}{3}(s + i\eta)^3$  admet deux points critiques complexes qui sont  $s = -i\eta \pm i\sqrt{u}$  pour  $u > 0$ . Le jacobien est alors égal à  $2(s_c + i\eta) = \pm 2i\sqrt{u}$ . La valeur critique est  $\pm iu^{\frac{3}{2}} + \frac{1}{3}(\pm i)^3 u^{\frac{3}{2}} = \pm i\frac{2}{3}u^{\frac{3}{2}}$ . Celle correspondant à  $-i\frac{2}{3}u^{\frac{3}{2}}$  donne une phase qui tend vers  $+\infty$  à  $+\infty$ . Nous la rejetons car  $Ai$  admet une limite lorsque  $x$  tend vers  $+\infty$ . Il reste alors

$$\frac{k^{\frac{1}{3}}}{2\pi} \left(\frac{2\pi}{k2\sqrt{u}}\right)^{\frac{1}{2}} e^{-k\frac{2}{3}u^{\frac{3}{2}}}$$

comme premier terme, c'est-à-dire le  $k^{-\frac{1}{6}} e^{-\frac{2}{3}ku^{\frac{3}{2}}} u^{-\frac{1}{4}} (4\pi)^{-\frac{1}{2}}$ . En remarquant que  $x = k^{\frac{2}{3}}u$ , on retrouve le comportement de la forme  $x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} \frac{\Gamma(\frac{1}{2})}{2}$ . Notons que cette méthode de recherche des points de phase stationnaire complexe sera utilisée dans le chapitre 5.

Une remarque pour terminer cette étude de la fonction d'Airy: on note que  $Ai(x)$  est solution de l'équation  $u'' - xu = 0$ . Cette solution est exponentiellement décroissante pour  $x > 0$ , mais elle n'est pas nulle, alors qu'elle vérifie la condition de Sommerfeld en  $+\infty$ . Ceci nous aide à préciser nos idées sur la notion de support dont nous avons parlé: en effet le support de la fonction d'Airy est  $\mathbf{R}$  tout entier, mais, asymptotiquement, elle contribue uniquement dans la zone  $x \leq 0$ . Il y a donc une différence entre le support et le support asymptotique. Si on forme par exemple l'équation eikonale vérifiée par une phase  $\phi$  correspondant à  $Ai(x)$ , on trouve  $(\phi')^2 + x = 0$ , donc la phase n'est définie que dans la zone  $x \leq 0$  et les rayons associés vivent dans  $x \leq 0$ . En revanche, là encore, la solution a pour support  $\mathbf{R}$  tout entier.

La dernière remarque est la suivante: la fonction d'Airy est l'unique solution dans  $\mathcal{S}'$  de  $u'' = xu$ . En particulier, c'est une solution frontière, dont le graphe sépare le graphe des solutions qui s'annulent de celui des solutions dont la dérivée s'annule. Si on note  $w_0$  et  $w_1$  les solutions respectivement associées aux données de Cauchy  $(u(0) = 1, u'(0) = 0)$  et  $(u(0) = 0, u'(0) = 1)$ , on trouve  $A'(x) = Ai(0)[w_0(x) - \int_0^{\infty} \frac{ds}{(w_0(s))^2} w_1(x)]$ , ce qui fixe la condition de Cauchy sur  $Ai$ . Ceci est l'équivalent du fait qu'une seule condition de Cauchy suffit lorsqu'on a la condition de décroissance à l'infini (Sommerfeld par exemple).

Preuve de l'exercice 4.2. Le résultat (dû à Colin de Verdière), est le résultat global qui permettra, dans le chapitre 11, de tenir compte des géodésiques fermées dans l'influence d'un objet.

1) Nous notons que le système d'équations

$$\nabla_z \phi(z) = 0$$

est un système de rang  $n - d$ , puisque l'espace des  $z$  solutions est de dimension  $d$ . Ceci veut dire qu'il existe  $n - d$  équations indépendantes (s'en convaincre en regardant les équations tangentées). Notons alors les coordonnées en question  $(z_{j_1}, \dots, z_{j_{n-d}})$ . En réordonnant les coordonnées et en considérant que  $(z_k)_{k \notin \{j_1, \dots, j_{n-d}\}}$  est difféomorphe à  $\mathbf{R}^d$ , on peut écrire  $z = (x, y), y \in \mathbf{R}^d$ . Alors les équations

$$\nabla_x \phi(x, y) = 0$$

sont  $n - d$  équations indépendantes. Par le théorème des fonctions implicites, on peut considérer  $y$  comme un paramètre, remarquer que  $\text{Hess}_{x^2}\phi$  est inversible, et écrire la solution comme  $x = x(y)$ . Alors on peut écrire

$$\int_X e^{ik\phi(z)} a(z) dz = \int_{\mathbf{R}^d} dy \int_{\mathbf{R}^{n-d}} dx e^{ik\phi(x,y)} a(x,y) J(x,y).$$

L'application du théorème de la phase stationnaire en  $x$  (Théorème 4.3) conduit à

$$\int_X e^{ik\phi(z)} a(z) dz \simeq \int_{\mathbf{R}^d} \frac{dy (2\pi/k)^{\frac{n-d}{2}}}{|\det \text{Hess}_{x^2}\phi(x(y), y)|^{\frac{1}{2}}} \left( \sum_{n \geq 0} L^n(y, \partial_y, \partial_x) a(x, y) \Big|_{x=x(y)} \right) e^{ik\phi(x(y), y)}.$$

On note alors que  $\partial_{y_j}\phi(x(y), y) = \sum_p \partial_{x_p}\phi(x(y), y) \frac{\partial x_p}{\partial y_j} + \partial_{y_j}\phi(x(y), y)$ . En utilisant le fait que la phase  $\phi(z)$  stationne sur  $W$ , on vérifie que  $\partial_{y_j}\phi(x(y), y) = 0$ , donc

$$\partial_{y_j}\phi(z)|_W = 0.$$

On note alors la valeur constante de  $\phi(z)$  sur  $W$  par  $\phi(W)$ . Il reste alors

$$\int_X e^{ik\phi(z)} a(z) dz \simeq e^{ik\phi(W)} \int_{\mathbf{R}^d} \frac{(2\pi/k)^{\frac{n-d}{2}} dy}{|\det \text{Hess}_{x^2}\phi(x(y), y)|^{\frac{1}{2}}} \left( \sum_{n \geq 0} L^n(y, \partial_y, \partial_x) a(x, y) \Big|_{x=x(y)} \right).$$



## Fréchet space of symbols.

In this section, we present the symbolic calculus, as it has been as it has been introduced by Hörmander [48]. We note indifferently by  $\hat{u}$  or by  $\mathcal{F}(u)$  the Fourier transform of an element  $u \in \mathcal{S}'(\mathbb{R}^d)$ . The symbols were introduced to generalize the ordinary differential operators, based on the relation

$$i^{|\alpha|}\xi^\alpha \hat{u}(\xi) = \mathcal{F}(\partial_x^\alpha u)(\xi)$$

which rewrites, for  $u \in \mathcal{S}(\mathbb{R}^d)$ :

$$\partial_x^\alpha u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} (i\xi)^\alpha \hat{u}(\xi) d\xi.$$

REMARK 5. For  $P = a_0(x, t)\partial_t + \sum_{j=1}^{j=d} a_j(x, t)\partial_{x_j}$ , (operator introduced in the chapter 2 in the case  $m = 1$ ) we obtain

$$Pu(x, t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} [ia_0(x, t)\tau + i \sum_{j=1}^{j=d} \xi_j a_j(x, t)] \hat{u}(\xi, \tau) e^{ix \cdot \xi + it \cdot \tau} d\xi d\tau.$$

PROOF. By writing  $\hat{u}(\xi, \tau) = \int_{\mathbb{R}^{d+1}} u(y, s) dy ds$  and replacing in the previous integral (which is a convergent integral),  $Pu(x, t)$  is written formally

$$(0.77) \quad Pu(x, t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{2d+2}} a(x, t, \xi, \tau) e^{i(x-y)\xi + i(t-s)\tau} u(y, s) dy ds d\xi d\tau,$$

where  $a(x, t, \xi, \tau) = [ia_0(x, t)\tau + i \sum_{j=1}^{j=d} \xi_j a_j(x, t)]$ . Indeed, it is only formally because this integral is not the integral of a function of  $L^1(\mathbb{R}^{2d+2})$ .  $\square$

In exercise 2.2, this function had been introduced and noted  $l(x, t, \xi, \tau)$ , and its null manifold corresponded to the locus of points  $(x, t, \nabla_x \phi, \partial_t \phi)$ ,  $\phi$  being the solution phase of the the eikonal equation.

We perform the change of variable  $\xi = \lambda \Xi$ ,  $\tau = \lambda \sigma$ , and we verify that we boil down to integrals of the type of those which will be studied in 4, where the asymptotic parameter is  $\lambda$ . In the chapter 6, we will consider integrals of the form

$$(0.78) \quad I_{a, \phi}(u) = \int_{\mathbb{R}^d \times \mathbb{R}^N} dy d\theta e^{i\phi(x, y, \theta)} a(x, y, \theta) u(y)$$

which are the generalization of the representation (0.77) of the differential operators, the variable  $\theta$  replacing  $(\xi, \tau)$  and the variable  $y$  replacing  $(y, s)$ .

We want to be able to apply the non-stationary phase theorem, so we want to be able to differentiate as many times as we want in  $y$  and  $\theta$  in the the integrals defined by (0.78). Moreover, we want the behavior of this integral to be identical to that of (0.77), so a derivation with respect to the variable  $\theta$  of the function  $a(x, y, \theta)$ , analogous to  $a(x, t, \xi, \tau)$  in (0.77), must lead to a decrease of 1 in the power of  $\theta$ , while a derivation in  $y$  or in  $x$  should not change anything in the behavior in  $|\theta|$  for  $\theta$  large.

Based on these remarks, we define a set of functions  $C^\infty$ , having decay properties in the variable  $\theta$ , which we call symbols. In the original presentation of L. Hörmander, the oscillating integrals of the type (0.78) were studied first. The symbols form an algebra, and we will see

in the next chapter that this algebra is associated to an algebra of operators, called pseudo-differential operators, which contains the differential operators with variable coefficients. We generalize in the chapter 8 this algebra of operators into what L. Hörmander calls the Fourier integral operators, and which are one of the essential tools that this course wishes to present.

In this chapter we study the topological properties of this algebra of symbols, we defer to chapter 7 the fact of defining the symbolic calculus (product of two symbols compatible with the composition of the operators). Note that the historical motivation for introducing of the symbols is not the one used here, the symbols were created for transcribe easily on functions the rules of calculation on the differential operators. We prefer this presentation because it is the logical continuation of the theorems of the stationary phase.

We will prove the analog of Borel's lemma (Theorem 1.2) for symbols of decreasing degree of homogeneity (or order). This part is inspired by the course given by J. Sjöstrand at the University of Orsay in 1987-1988 [43].

### 1. Definition of the space of symbols

Let  $X$  be an open set of  $\mathbb{R}^d$  and consider the set  $X \subset \mathbb{R}^d$ . We define the space of symbols of order  $m$  on  $X \times \mathbb{R}^d \times \mathbb{R}^d$  and of weight  $\rho, \delta$  through

DEFINITION 5.1.

(1.79)

$$S_{\rho, \delta}^m(X \times \mathbb{R}^d) = \{a(x, \theta) \in C^\infty(X \times \mathbb{R}^d) \forall K \subset\subset X, \forall \alpha \in \mathbb{N}^d, \forall \beta \in \mathbb{N}^d, \exists C(m, K, \alpha, \beta, a) \\ \text{Let } K \text{ be a fixed compact } \sup_{(x, \theta) \in K \times \mathbb{R}^d, |\theta| \geq 1} |\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \leq C(K, \alpha, \beta, a)(1 + |\theta|)^{m - \rho|\beta| + \delta|\alpha|}\}$$

For  $a \in S_{\rho, \delta}^m(X \times \mathbb{R}^d)$ , the smallest constant  $C(m, K, \alpha, \beta, a)$  satisfying the equality 1.79 is denoted by  $N_{\alpha, \beta, K}^m(a)$ .

Such spaces are non empty: indeed, considering  $a(x, \xi) = \xi_1^m$ , we verify that

$$|\alpha| \geq 1, |\beta'| \geq 1 \Rightarrow \partial_\xi^\beta \partial_x^\alpha a = 0.$$

As  $\partial_{\xi_1}^{\beta_1} a = \frac{m!}{\beta_1!} \xi_1^{m - \beta_1}$ , we find that, in cases where this derivative is non-zero ( $\beta_1 \leq m$ )

$$(1 + |\xi|)^{-m + \rho|\beta| - \delta|\alpha|} |\partial_{\xi_1}^{\beta_1} \partial_x^\alpha a| \leq \frac{m!}{\beta_1!} |\xi_1|^{m - \beta_1} (1 + |\xi_1|)^{-m + \rho\beta_1},$$

which is, for any  $\rho$  and for any  $\delta$ , bounded by  $\frac{m!}{\beta_1!}$ . When  $\alpha \neq 0$  or  $\beta' \neq 0$ , the inequality is trivially verified.

We also define the space  $S^{-\infty}$  of symbols which decay faster than any inverse power of  $\theta$ :

$$(1.80) \quad S^{-\infty}(X \times \mathbb{R}^d) = \{a, \sup_K |\partial_x^\alpha \partial_\theta^\beta a| \leq C(a, \alpha, \beta, K, M)(1 + |\theta|)^{-M}\}.$$

Note that the definition of symbols given in this chapter, and which we will use from now on, is more general than the definition in the preamble, since we allow  $\rho \leq 1$  (i.e. a loss of powers of  $\theta$  less than 1 at each derivation) as well as  $\delta \geq 0$  (i.e. a possible gain of powers of  $\theta$  when we differentiate with respect to  $x$ ). Examples of such symbols are used in this course; in particular, one may encounter  $(\rho, \delta) = (\frac{1}{3}, 0)$  and  $(\rho, \delta) = (\frac{1}{3}, \frac{2}{3})$ .

Let us give an example of a symbol of  $S_{1,0}^0$ , which will be the truncation symbol. Let us consider a function  $\chi \in C^\infty(\mathbb{R}^d)$ , such that  $\chi(\theta) = 1$  for  $|\theta| \leq 1/2$  and  $\chi(\theta) = 0$  for  $|\theta| \geq 1$ .

**The symbol  $\chi_j(\theta) = \chi(\frac{\theta}{j})$  is bounded in  $S_{1,0}^1(X \times \mathbb{R}^d)$ .** Indeed, when we compute  $\partial_\theta^\beta \chi_j(\theta)$ , we find

$$\partial_\theta^\beta \chi_j = j^{-|\beta|} (\partial_\theta^\beta \chi) \left(\frac{\theta}{j}\right)$$

The function  $\chi$  is  $C_0^\infty$  so there exists  $c_\beta$  such that  $(\partial_\theta^\beta \chi) \leq c_\beta$ . We verify that, for  $|\theta| \geq 2j$ ,  $|(1 + \theta)^{|\beta|} \partial_\theta^\beta \chi_j| = 0$  and is less than  $(1 + 2j)^{|\beta|} j^{-|\beta|}$  for  $|\theta| \leq 2j$ , thus by  $3^{|\beta|}$ . So we have

$$\partial_\theta^\beta \chi_j \leq 3^{|\beta|} c_\beta (1 + |\theta|)^{-|\beta|}$$

which proves that  $\chi_j$  belongs to  $S_{1,0}^0$ .

## 2. Fundamental properties

The elementary and fundamental properties of the spaces of symbols are summarized in the

- PROPOSITION 5.1. • *If  $a \in S_{\rho,\delta}^m(X \times \mathbb{R}^N)$ , then, for all  $m' \geq m$ , all  $\rho' \leq \rho$  and all  $\delta' \geq \delta$ ,  $a \in S_{\rho',\delta'}^{m'}$ .*
- *The product of an element of  $S_{\rho,\delta}^m$  and  $S_{\rho,\delta}^{m_1}$  is in  $S_{\rho,\delta}^{m+m_1}$ .*

PROOF. The first item is a consequence of the relation  $-\rho|\beta| + \delta|\alpha| \leq -\rho'|\beta| + \delta'|\alpha|$  for  $\rho' \leq \rho, \delta' \geq \delta$ , and of, in addition,  $m' \geq m$ .

For the proof of the second item, by Leibniz' formula, we have

$$\partial_x^\alpha \partial_\theta^\beta (ab)(x, \theta) = \sum_{\alpha', \beta', |\alpha'| \leq |\alpha|, |\beta'| \leq |\beta|} C_{\alpha, \beta}^{\alpha', \beta'} (\partial_x^{\alpha'} \partial_\theta^{\beta'} a) (\partial_x^{\alpha - \alpha'} \partial_\theta^{\beta - \beta'} b).$$

One deduces

$$|\partial_x^\alpha \partial_\theta^\beta (ab)(x, \theta)|_{x \in K, \theta \in \mathbf{R}^d} \leq \sum_{\alpha', \beta', |\alpha'| \leq |\alpha|, |\beta'| \leq |\beta|} C_{\alpha, \beta}^{\alpha', \beta'} N_{\alpha', \beta', K}(a) N_{\alpha - \alpha', \beta - \beta', K}(b).$$

hence

$$N_{\alpha, \beta, K}(ab) \leq \sum_{\alpha' \leq \alpha, \beta' \leq \beta} C_{\alpha, \beta}^{\alpha', \beta'} (\sup_{\alpha' \leq \alpha, \beta' \leq \beta} N_{\alpha', \beta', K}(a)) (\sup_{\alpha' \leq \alpha, \beta' \leq \beta} N_{\alpha', \beta', K}(b)).$$

Together, the first item and the second item of Proposition 5.1 imply

$$(2.81) \quad a \in S_{\rho,\delta}^m, b \in S_{\rho',\delta'}^{m_1} \Rightarrow ab \in S_{\min(\rho,\rho'), \max(\delta,\delta')}^{m+m_1}.$$

□

PROPOSITION 5.2. (1) *The space of symbols with this family of semi-norms is a Fréchet space for the family of semi-norms  $N_{\alpha, \beta, K}^m$  on  $S_{\rho, \delta}^m(X \times \mathbb{R}^d)$ .*

(2) *If  $(a_j)_{j \geq 1}$  is a bounded sequence in  $S_{\rho, \delta}^m$ , which converges pointwise to  $a(x, \theta)$ , the limit  $a$  is in  $S_{\rho, \delta}^m$  and the sequence  $(a_j)$  converges in the sense of symbols in  $S_{\rho, \delta}^{m'}$  for  $m' > m$ .*

(3) *For the family of semi-norms defined in Proposition 5.1, the space  $S^{-\infty}$  is dense, for the topology of  $S_{\rho', \delta'}^{m'}$ , in  $S_{\rho, \delta}^m$  for  $m' > m$ .*

PROOF. We denote by  $C_{\alpha, \beta, K}$  the uniform bound of all norms of  $\partial_\theta^\beta \partial_x^\alpha a_j$  on  $K$ , which is independent of  $j$ .

Thanks to the inequalities allowing to control  $\partial_x f$  in function of  $\partial_{x^2}^2 f$  and  $f$ , we verify that, on any compact  $K$

$$|\partial_x(a_j - a_l)| \leq C_1(K) \|a_j - a_l\|_\infty + C_2(K) (\|a_j - a_l\|_\infty \|\partial_{x^2}^2(a_j - a_l)\|_\infty)^{\frac{1}{2}}.$$

We know that the sequence  $a_j$  is bounded in  $S^m$ , so in particular, on a compact, we know that

$$\|\partial_{x^2}^2(a_j - a_l)\|_\infty \leq 2(1 + |\theta|)^{m+2\delta} C_{2,0,K}.$$

We place ourselves on a compact  $\mathcal{K}$  in  $\theta$  to apply the previous result, so

$$|\partial_x(a_j - a_l)| \leq C_1(K) \|a_j - a_l\|_\infty + C_2(K) C_{2,0,K}^{\frac{1}{2}} (\|a_j - a_l\|_\infty)^{\frac{1}{2}}.$$

There is a uniform bound of the infinite norm of the sequence on the compact  $K \times \mathcal{K}$ , pointwise convergence at any point, so uniform convergence on the compact  $K \times \mathcal{K}$ . We conclude that the sequence  $\partial_x a_j$  is a Cauchy sequence on this compact, uniformly bounded, so it converges and the limit is uniform.

By induction on the length  $|\alpha| + |\beta|$ , we show that the sequence  $\partial_x^\alpha \partial_\theta^\beta a_j$  is a Cauchy sequence, and uniformly convergent on any compact of the form  $K \times \mathcal{K}$ . Unfortunately, we cannot verify the uniform convergence on  $K \times \mathbb{R}^d$ , because it is not a compact. On the other hand, we find that the limit  $a$  is in  $C^\infty(X \times \mathbb{R}^d)$ , and since we have

$$|\partial_x^\alpha \partial_\theta^\beta a_j| \leq C_{\alpha,\beta,K}(1 + |\theta|)^{m-\rho|\beta|+\delta|\alpha|},$$

the pointwise convergence proves that the pointwise limit is in  $S_{\rho,\delta}^m(X \times \mathbb{R}^d)$ . One introduces  $k_j^{\alpha,\beta}(x, \theta) = \partial_x^\alpha \partial_\theta^\beta (a_j - a)(x, \theta)(1 + |\theta|)^{-m'+\rho|\beta|-\delta|\alpha|}$ . One notices that

$$|k_j^{\alpha,\beta}(x, \theta)| \leq (1 + |\theta|)^{m-m'} 2C_{\alpha,\beta,K}.$$

Let us define  $R_\varepsilon^{\alpha,\beta,K} > 0$  such that, for  $|\theta| \geq R_\varepsilon^{\alpha,\beta,K}$ , one has  $(1 + |\theta|)^{m-m'} 2C_{\alpha,\beta,K} \leq \frac{\varepsilon}{2}$ . On the compact  $K \times B(0, R_\varepsilon^{\alpha,\beta,K})$ , the sequence  $a_j$  converges uniformly in  $C^\infty$  to  $a$ . The sequence  $\partial_x^\alpha \partial_\theta^\beta a_j$  therefore converges on this compact to  $\partial_x^\alpha \partial_\theta^\beta a$ , so there exists  $j(\varepsilon, \alpha, \beta, K)$  such that

$$j \geq j(\varepsilon, \alpha, \beta, K) \Rightarrow k_j^{\alpha,\beta}(x, \theta) \leq \frac{\varepsilon}{2}, (x, \theta) \in K \times B(0, R_\varepsilon^{\alpha,\beta,K}).$$

We use a truncation in  $\theta$  to reduce to symbols with compact support. The proof of the last item of the proposition 5.1 is obtained by constructing, for  $a \in S_{\rho,\delta}^m$ , a sequence of  $S^{-\infty}$  converging to  $a$ . As  $\chi_j \in S_{1,0}^0 \cap S^{-\infty}$ , the sequence of symbols  $a_j = a\chi_j$  is bounded in the sense of the topology of  $S_{\rho,\delta}^m$ , is in  $S^{-\infty}$  and converges pointwise to  $a$ . We use the third paragraph of the the proposition 5.2 to obtain that  $a_j$  converges to  $a$  for the topology of  $S_{\rho,\delta}^{m'}$ ,  $m' > m$ . This shows the density of  $S^{-\infty}$  in  $S_{\rho,\delta}^m$  for the topology of  $S_{\rho,\delta}^{m'}$  for  $m' > m$ . Conversely, if a sequence  $a_j$  verifies the hypotheses of the proposition 5.2, we deduce that there is convergence on any compact of  $\partial_x^\alpha \partial_\theta^\beta a_j$  to  $\partial_x^\alpha \partial_\theta^\beta a$ . The proposition 5.2 is proven.  $\square$

**PROPOSITION 5.3.** *Let  $a_j \in S_{\rho,\delta}^{m_j}(X \times \mathbb{R}^N)$ . We suppose that the decreasing sequence  $m_j$  tends to  $-\infty$ . Then there exists  $a \in S_{\rho,\delta}^{m_0}$ , unique modulo  $S^{-\infty}$ , such that*

$$a - \sum_{j < k} a_j \in S_{\rho,\delta}^{m_k}.$$

**PROOF.** We prove uniqueness. Let  $a'$  be another symbol. Then  $a - a'$  is in  $S^{m_k}$  for all  $k$ . It is therefore in  $S^{-N}$  for all  $N$ , which justifies  $a - a' \in S^{-\infty}$ .

The existence of  $a$  comes from a method similar to the one used for the proof of Borel's theorem (Theorem 1.2) (there exists a function  $C^\infty$  whose the Taylor series is given). We construct, for  $L_j \geq 1$  a strictly increasing tending to  $+\infty$  the symbol

$$\tilde{a}_j(x, \theta) = (1 - \chi(\frac{\theta}{L_j}))a_j(x, \theta).$$

If  $\theta$  is given, then for  $|\frac{\theta}{L_j}| \leq \frac{1}{2}$ ,  $\tilde{a}_j = 0$ . So if  $j$  is such that  $L_j \geq 2|\theta|$ , then  $\tilde{a}_j(x, \theta) = 0$ . We can therefore define, for all  $(x, \theta)$ , the sum of  $\tilde{a}_j$ , which is locally finite. The function obtained  $a(x, \theta)$  is a function  $C^\infty(X \times \mathbb{R}^d)$ .

We note in this paragraph  $\chi_j$  the function (which was written above  $\chi_{L_j}$ ):

$$\chi_j : \theta \rightarrow \chi(\frac{\theta}{L_j})$$

is a symbol of  $S_{1,0}^0(\mathbb{R}^d)$ . The symbol  $\tilde{a}_j$  is in  $S_{\rho,\delta}^{m_j}$ , so is in  $S_{\rho,\delta}^{m_j+1}$ . We can then choose the sequence  $L_j$  (as in the proof of Theorem 1.2) such that, for all  $j \leq |\alpha| + |\beta|$ ,

$$(2.82) \quad |\partial_x^\alpha \partial_\theta^\beta \tilde{a}_j(x, \theta)| \leq 2^{-j}(1 + |\theta|)^{1+m_j-\rho|\beta|+\delta|\alpha|}.$$

Indeed, we verify that  $|\tilde{a}_0| \leq |a_0|$  on the support of  $1 - \chi(L_0\theta)$ , which is included in  $\theta \geq \frac{1}{2}L_0$ . Therefore, on the support of  $\tilde{a}_0$ ,

$$(1 + |\theta|)^{-1-m_0} |\tilde{a}_0| \leq \sup_{|\theta| \geq \frac{1}{2}L_0} N_{0,0,K}(a_0)(1 + |\theta|)^{-1} \leq \frac{2N_{0,0,K}(a_0)}{2 + L_0}.$$

The choice of  $L_0$  is  $L_0 \geq 2N_{0,0,K}$ .

We proceed by induction. We assume that the  $L_j$  have been chosen such that, for  $0 \leq j \leq n-1$ , we have

$$|\alpha| + |\beta| \leq l \leq n-1 \Rightarrow N_{\alpha,\beta,K}^{m_l+1}(\tilde{a}_l) \leq 2^{-l}.$$

Let  $j = n$  and study  $\tilde{a}_j$ . We verify that

$$b = \partial_x^\alpha \partial_\theta^\beta (\tilde{a}_j) = \partial_x^\alpha \left[ \sum_{\beta' \leq \beta} C_{\beta,\beta'} \partial_\theta^{\beta'} a_j L_j^{-|\beta|+|\beta'|} (\partial_\theta^{\beta-\beta'} \chi) \left( \frac{\theta}{L_j} \right) \right].$$

As  $N_{\alpha,\beta,K}^{m_j+1}(\tilde{a}_j) = \max |b(x, \theta)| (1 + |\theta|)^{-m_j-1-\delta|\alpha|+\rho|\beta|}$ , one uses

$$\begin{aligned} & b(x, \theta) (1 + |\theta|)^{-m_j-1-\delta|\alpha|+\rho|\beta|} \\ = & (1 + |\theta|)^{-1} \sum_{\beta' \leq \beta} C_{\beta,\beta'} [(1 + |\theta|)^{-m_j-\delta|\alpha|+\rho|\beta'|} |\partial_x^\alpha \partial_\theta^{\beta'} a_j| (1 + |\theta|)^{\rho(|\beta|-|\beta'|)} L_j^{-|\beta|+|\beta'|} (\partial_\theta^{\beta-\beta'} \chi) \left( \frac{\theta}{L_j} \right)]. \end{aligned}$$

We denote by  $D_{j,\beta}$  the constant  $(\sum_{\beta' \leq \beta} |C_{\beta,\beta'}|) \max_{\alpha \leq \beta} |\partial_\theta^\alpha \chi| N_{\alpha,\beta',K}^{m_j}(a_j)$ . From the identity

$$b(x, \theta) (1 + |\theta|)^{-m_j-1-\delta|\alpha|+\rho|\beta|} = 0 \text{ sur } \chi = 1$$

we deduce the inequality

$$|b(x, \theta) (1 + |\theta|)^{-m_j-1-\delta|\alpha|+\rho|\beta|}| \leq \frac{2}{L_j} D_{j,\beta}.$$

We then take  $L_j \geq 2^{j+1} D_{j,\beta}$ , which gives the inequality (2.82). Let us note here that the control of  $\tilde{a}_j$  in  $S_{\rho,\delta}^{m_j+1}$  is a control in  $2^{-j}$ . For example, we notice that the sequence  $1 - \chi_j \in S_{1,0}^0$  tends to 0 in  $S_{1,0}^1$ .

We then consider  $\alpha, \beta, k, K$  given. For  $p \geq N \geq |\alpha| + |\beta|$ , we have

$$|\partial_x^\alpha \partial_\theta^\beta \tilde{a}_p| \leq 2^{-p} (1 + |\theta|)^{1+m_p-\rho|\beta|+\delta|\alpha|}$$

so a fortiori, since  $m_p \leq m_N$  and  $a - \sum_{p=0}^{p=N-1} \tilde{a}_p = \sum_{p=N}^{\infty} \tilde{a}_p$

$$|\partial_x^\alpha \partial_\theta^\beta (a - \sum_{p=0}^{N-1} \tilde{a}_p)| \leq (1 + |\theta|)^{1+m_N-\rho|\beta|+\delta|\alpha|}.$$

We choose  $N$  such that  $m_{k+1} \geq m_N + 1$  and  $N \geq |\alpha| + |\beta|$ . This is possible because the sequence  $m_k$  tends to  $-\infty$ . Then, we write  $c_{k+1} = a - \sum_{j \leq k} a_j$  and we verify

$$c_{k+1}(x, \theta) = \sum_{k+1 \leq j \leq N-1} \tilde{a}_j + (a - \sum_{j \leq N-1} \tilde{a}_j) + \sum_{j \leq k} (\tilde{a}_j - a_j).$$

We use the fact that  $\sum_{j \leq k} (a_j - \tilde{a}_j)$  is in  $S^{-\infty}$ , the inequality

$$|\partial_x^\alpha \partial_\theta^\beta (a - \sum_{p=0}^{N-1} \tilde{a}_p)| \leq (1 + |\theta|)^{1+m_N-\rho|\beta|+\delta|\alpha|}$$

that we have just proven and the inequality (obtained because  $\tilde{a}_j$  is the product of  $a_j \in S_{\rho,\delta}^{m_j}$  and  $(1 - \chi_j) \in S_{1,0}^0$  and  $S_{\rho,\delta}^0$ )

$$|\partial_x^\alpha \partial_\theta^\beta \tilde{a}_j(x, \theta)| \leq D(\alpha, \beta) (1 + |\theta|)^{m_j-\rho|\beta|+\delta|\alpha|}.$$

We deduce the inequality

$$|\partial_x^\alpha \partial_\theta^\beta c_{k+1}(x, \theta)| \leq ND(\alpha, \beta) (1 + |\theta|)^{m_{k+1}-\rho|\beta|+\delta|\alpha|} + (1 + |\theta|)^{m_{k+1}-\rho|\beta|+\delta|\alpha|} + C_1 (1 + |\theta|)^{m_{k+1}-\rho|\beta|+\delta|\alpha|}$$

and this estimate proves that  $c_{k+1}$  belongs to  $S_{\rho,\delta}^{m_{k+1}}$ .

We have thus proved the convergence in the sense of  $S^{m_0}$  norms of  $\sum_{j \leq k} a_j$ . This completes the proof of Proposition 5.3.  $\square$



Remark: there is no uniqueness, since the choice of two functions  $\chi$  or two sequences  $L_j$  for the same function  $\chi$  leads to two different values of  $\sum \tilde{a}_j$ .

PROPOSITION 5.4. *Let  $a_j$  be a sequence in  $S_{\rho,\delta}^{m_j}$ , where  $m_j$  decreases to  $-\infty$ . Assume that there exists a such that, for any compact  $K$ , any multi index, there exist  $M_{\alpha,\beta}$  and  $C_{\alpha,\beta,K}$*

$$|\partial_x^\alpha \partial_\theta^\beta a_j| \leq C_{\alpha,\beta,K}(1 + |\theta|)^{M_{\alpha,\beta}}.$$

Assume that there exists a sequence  $m'_k$  which tends to  $-\infty$  such that

$$|a(x, \theta) - \sum_{j=0}^{k-1} a_j(x, \theta)| \leq C_{K,k}(1 + |\theta|)^{m'_k}.$$

One has  $a = \sum_{j=0}^{j=+\infty} a_j + O(S^{-\infty})$ .

PROOF. By the proposition 5.3,  $a' = \sum_{j=0}^{\infty} a_j$  exists and is in  $S_{\rho,\delta}^{m_0}$ . Then  $b = a - a'$  verifies

$$b = a - \sum_0^{k-1} a_j + \sum_0^{k-1} a_j - a'.$$

The first term is bounded by  $C_{K,k}(1 + |\theta|)^{m'_k}$  thanks to the hypothesis of the proposition and the second term is bounded by  $C_{K,k,0,0}(1 + |\theta|)^{m_k}$  by the result of the proposition 5.3.

For any  $M$  positive integer, there exists  $k$  such that  $m_k$  and  $m'_k$  are all both smaller than  $-M$ . Therefore there exists  $C_{K,M}$  such that

$$|b| \leq C_{K,M}(1 + |\theta|)^{-M}.$$

Recall the following inequality for  $|\alpha| + |\beta| = 1$ :

$$(2.83) \quad |\partial_x^\alpha \partial_\theta^\beta b(x, \theta)| \leq C(K)(\sup|b|)^{\frac{1}{2}}((\sup|\partial_x^{2\alpha} \partial_\theta^{2\beta} b|)^{\frac{1}{2}} + (\sup|b|)^{\frac{1}{2}}).$$

We place ourselves in the case  $|\alpha| + |\beta| = 2$ , to generalize this inequality. We obtain

$$|\partial_x^\alpha \partial_\theta^\beta b| \leq C_{|\alpha|+|\beta|}(K)[\max|b| + \max|b|^{\frac{1}{2}}] \times \max_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|} \partial_{x^{\gamma_1} \theta^{\gamma_2}}^{2(\gamma_2+\gamma_1)} b|^{\frac{1}{2}}].$$

We rely for the study on

LEMMA 5.1.

• For  $f$  of class  $C^4$  on a compact  $K$ ,

$$\|f'''\|_\infty \leq C_1(K)\|f\|_\infty + C_2(K)\|f^{(4)}\|_\infty.$$

•

$$\|\partial_x f\| \leq C_1[\|f\| + \|f\|^{\frac{1}{2}} \|\partial_{x^2}^2 f\|^{\frac{1}{2}}]$$

•

$$\|\partial_{x^2}^2 f\| \leq C_2[\|f\| + \|f\|^{\frac{1}{2}} \|\partial_{x^4}^4 f\|^{\frac{1}{2}}]$$

$$\|\partial_{\theta^2}^2 f\| \leq C_2[\|f\| + \|f\|^{\frac{1}{2}} \|\partial_{\theta^4}^4 f\|^{\frac{1}{2}}].$$

If we restrict  $f$  to  $[-a, a]$ ,  $a > 0$ , then there exist  $\theta_1, \theta_2 \in ]0, 1[$  such that

$$f'''(0) = \frac{f(x) + f(-x) - 2f(0)}{x^2} - \frac{x^2}{24}(f^{(4)}(\theta_1 x) + f^{(4)}(-\theta_2 x)).$$

If  $f^{(4)} = 0$ , we find  $|f'''(0)| \leq \frac{4}{x^2}\|f\|_\infty$ . Otherwise, we have

$$|f'''(0)| \leq \frac{4}{x^2}\|f\|_\infty + \frac{x^2}{12}\|f^{(iv)}\|_\infty$$

and we optimize this majorant for  $|x| \leq a$ . In the case where  $4\sqrt{3}(\frac{\|f\|_\infty}{\|f^{(4)}\|_\infty})^{\frac{1}{2}} > a^2$ , one find the desired inequality

$$|f'''(0)| \leq \frac{8}{a^2}\|f\|_\infty + \frac{2}{\sqrt{3}}\|f\|_\infty^{\frac{1}{2}}\|f^{(4)}\|_\infty^{\frac{1}{2}}.$$

This inequality is not satisfactory when  $a$  is small. It is then necessary to study this case, i.e. when the point where we compute a majorant of  $f'''$  is close to the boundary of the compact

$K$ . For this purpose, we use the function  $\phi : \phi(x) = 5f(x) - 4f(2x) + f(3x)$ , which allows us to obtain  $f'''(0)$  as a function of  $f$  and  $f^{(4)}$ .

For the bound of  $\partial_{x\theta}^2 f$ , we write

$$\begin{aligned} \|\partial_{x\theta}^2 f\| &\leq C_1[\|\partial_x f\| + \|\partial_x f\|^{\frac{1}{2}}\|\partial_{x\theta^2}^3 f\|^{\frac{1}{2}}] \\ \|\partial_{x\theta^2}^3 f\| &\leq C_1[\|\partial_{\theta^2}^2 f\| + \|\partial_{\theta^2}^2 f\|^{\frac{1}{2}}\|\partial_{x^2\theta^2}^4 f\|^{\frac{1}{2}}]. \end{aligned}$$

All the bounds obtained in the case of one variable (the constant  $C_2$  are adapted to the situation of a second derivative in  $x$  and  $\theta$  in a compact through

$$|\partial_{x\theta}^2 f| \leq M(\|f\|, \|\partial_{x^4}^4 f\|, \|\partial_{\theta^4}^4 f\|, \|\partial_{x^2\theta^2}^4 f\|) \leq C_3[\|f\| + \|f\|^{\frac{1}{2}} \max_{|\alpha|+|\beta|=2} \|\partial_{x^{2\alpha}\theta^{2\beta}}^{2\beta} f\|^{\frac{1}{2}}].$$

This argument can be performed for  $|\alpha| + |\beta| = n \in \mathbb{N}$  by induction on  $n$ . The symbol  $b$  is then in  $S^{-\infty}$ . The proposition 5.4 is proven.  $\square$

In the next chapter, we introduce the tool that allows us to construct the algebra of pseudodifferential operators associated to the algebra of the symbols. We would already have all the tools that would allow us to define it formally, but we choose to introduce the law of composition thanks to the composition of the associated operators. The symbols of  $S_{1,0}^m(X \times \mathbb{R}^d)$  are the symbols that look like the most to polynomials in  $\xi$  of degree at most  $m$  with coefficients  $C^\infty$  in  $x \in X$ , which are the symbols associated to the partial differential operators of order at most  $m$  (hence they are called in this book **classical** symbols). The space  $S_{1,0}^m(X \times \mathbb{R}^d)$  is denoted in this book by  $S^m(X \times \mathbb{R}^d)$ .

### 3. The Friedlander Model Problem

In this section, we begin the study of a problem which will serve as a guideline in this manuscript to which will serve as a guideline in this approach to microlocal analysis; it is the Friedlander model problem. Inspired by the 1977 paper of Friedlander [42], it allows to study, in a simple case, of a problem of a tangent ray at an boundary, whose formal treatment is similar to the one at caustics introduced in the chapter 9. The notations introduced here will be followed throughout the study, in particular in the chapters 6,7, 9. The introduction of the operator (3.84) is somewhat arbitrary, we will see in the section 4 why Friedlander introduced this operator. In this chapter, the main purpose of using this example is to show some simple problems of partial differential equations involve in a natural way symbols which are not classical, these symbols belong to  $S_{1/3,2/3}^0$ .

We give ourselves the operator on  $\mathbb{R}_+ \times \mathbb{R}^2$

$$(3.84) \quad Pu(x, y_1, y_2) = \frac{\partial^2 u}{\partial x^2} - (1+x) \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}.$$

Introduce, for  $\theta_1 \in \mathbb{C}$ ,  $\Im \theta_1 < 0$ , and  $\theta_2 \in \mathbb{R}$ , the real numbers  $\lambda$ ,  $Z$ ,  $Z_0$ ,  $\xi$ ,  $\xi_0$  for

$$\theta_1 = |\theta_1| e^{i(\frac{3\pi}{2} + \lambda)}, \quad -\frac{\pi}{2} \leq \lambda \leq \frac{\pi}{2}$$

hence

$$\theta_1^{\frac{2}{3}} = -|\theta_1|^{\frac{2}{3}} e^{\frac{2i}{3}\lambda}, \quad \theta_1^{-\frac{4}{3}} = |\theta_1|^{-\frac{4}{3}} e^{-\frac{4i}{3}\lambda}.$$

One introduces

$$\begin{aligned} \xi &= \theta_1^{-\frac{4}{3}} \theta_2^2 - (1+x) \theta_1^{\frac{2}{3}} \\ \xi_0 &= \theta_1^{-\frac{4}{3}} \theta_2^2 - \theta_1^{\frac{2}{3}} \end{aligned}$$

One denotes by

$$\begin{aligned} |\theta| &= (\theta_1^2 + \theta_2^2)^{\frac{1}{2}} \\ Z &= |\theta_1|^{\frac{2}{3}} - (1+x) \theta_2^2 |\theta_1|^{-\frac{4}{3}} \\ Z_0 &= |\theta_1|^{\frac{2}{3}} - \theta_2^2 |\theta_1|^{-\frac{4}{3}}. \end{aligned}$$

One solves

$$(3.85) \quad \begin{aligned} Pu &= 0, u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2) \\ u(0, y) &= f(y), f \in \mathcal{E}'(\mathbb{R}^2) \\ u(x, y_1, y_2) &= 0, y_1 < 0. \end{aligned}$$

1) Show that the solution of (3.85) is written, for  $f \in \mathcal{S}(\mathbb{R}^2)$

$$u(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{K}_x(\theta_1, \theta_2) \hat{f}(\theta_1, \theta_2) e^{iy_1\theta_1 + iy_2\theta_2} d\theta_1 d\theta_2$$

in the sense of Fourier integrals. One assumes that  $K_x$ , solution of

$$(3.86) \quad \begin{aligned} PK_x &= 0 \\ K_0(y) &= \delta_{y=0} \\ K_x(y_1, y_2) &= 0, y_1 < 0. \end{aligned}$$

has a Fourier transform, equal in  $\Im\theta_1 < 0$  to<sup>1</sup>

$$\hat{K}_x(\theta_1, \theta_2) = \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)},$$

where  $\text{Ai}$  is the classical Airy function, inverse Fourier transform of  $t \rightarrow e^{i\frac{t^3}{3}}$ , which is in  $C^\infty(\mathbb{R})$  and is a solution of  $\text{Ai}''(x) = x \text{Ai}(x)$  for all  $x$ .

2) a) Show that the function  $\Phi$  defined by the equality

$$u^{\frac{1}{4}} \text{Ai}(u) \exp\left(\frac{2}{3}u^{\frac{3}{2}}\right) = u^{\frac{1}{4}} \Phi(u)$$

admits, as well as its inverse, an asymptotic expansion for  $u \in \mathbb{R}_+$  large, in inverse powers of  $u^{\frac{3}{2}}$ , expansion uniformly valid in  $\arg u \in [-\pi + \varepsilon, \pi - \varepsilon]$ .

b) Let  $G$  be a given function of class  $C^\infty$ . Prove by induction the existence of functions  $C^\infty Q_j^{k,\alpha}(x, \theta)$ , homogeneous in  $\theta$  of degree of homogeneity  $\frac{2}{3}(j+k) - |\alpha|$ , such that

$$\partial_{x^k} \partial_{\theta^\alpha} [G(Z)] = \sum_{j=0}^{j=|\alpha|} G^{(k+j)}(Z) Q_j^{k,\alpha}(x, \theta)$$

3) Consider the function  $C^\infty$ , denoted by  $\sigma_0$ , null for  $t \leq 1$ , identically equal to 1 for  $t \geq 2$ . We introduce  $\sigma_2(t) = \sigma_0(\delta_2 t)$ ,  $0 < \delta_2 < 1/2$ . We introduce

$$a_2(x, \theta_1, \theta_2) = \sigma_0(|\theta|) \sigma_2(Z_0) \frac{\Phi(\xi)}{\Phi(\xi_0)}.$$

a) Show that  $\sigma_2(Z_0) \in S_{1/3,0}^0(\mathbb{R}^2)$ .

b) Using the inclusion  $S_{1/3,0}^0 \subset S_{1/3,2/3}^0$ , show that  $a_2 \in S_{1/3,2/3}^0$ .

4) a) Show that

$$(1 - \sigma_0(|\theta|)) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} \in S^{-\infty}.$$

Let  $\sigma_1$  be the even function, null on  $[1 - \delta_1, +\infty[$ , equal to 1 on  $[0, 1 - 2\delta_1]$ , e.g.  $\sigma_1(u) = 1 - \sigma_0(\delta_1|u| + 1 - 3\delta_1)$ . Let  $h$  be the function equal to  $(s^2 + 1)^{-\frac{1}{2}}(s^2 - 1)^{\frac{3}{2}}$  on  $[1, (1+x)^{\frac{1}{2}}]$ , equal to  $(s^2 + 1)^{-\frac{1}{2}}[(s^2 - 1)^{\frac{3}{2}} - (s^2 - 1 - x)^{\frac{3}{2}}]$  on  $[(1+x)^{\frac{1}{2}}, +\infty[$ . It is reduced to a function  $\gamma(x)$ , which we can express, for  $s \geq (1 - \delta_1)^{-1}$ .

b) Show that

$$|\sigma_0(|\theta|) \sigma_1\left(\frac{\theta_1}{|\theta_2|}\right) \exp\left(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})\right)| \leq \exp\left(-\frac{2}{3}\gamma(x)|\theta|\right).$$

<sup>1</sup>The last inequality of (3.85) shows that  $y_1$  plays the role of time, and that we study the reflection of a wave on the boundary  $x = 0$ , which is related by the notion of solution supported in  $y_1 \geq 0$  to the extension  $\text{Im}\theta_1 < 0$  by the theorem of Paley-Wiener-Schwarz theorem.

c) Prove that, for all  $n$ , there exists  $C_n$  such that

$$|\partial_{x^n}(\sigma_0(|\theta|)\sigma_1(\frac{\theta_1}{|\theta_2|})\frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)})| \leq C_n \exp(-\frac{1}{3}\gamma(x)|\theta|).$$

d) Conclude that  $a_1(x, \theta) = \sigma_0(|\theta|)\sigma_1(\frac{\theta_1}{|\theta_2|})\frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)}$  belongs to a space of symbols.

5) Let us denote by

$$\begin{aligned} a_3(x, \theta) &= [\hat{K}_x(\theta) - (1 - \sigma_0(|\theta|))\frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} - \sigma_0(|\theta|)\sigma_1(\frac{\theta_1}{|\theta_2|})\frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} \\ &\quad - a_2(x, \theta)\exp(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}))]\exp(\frac{2i}{3}Z^{\frac{3}{2}}\text{sign}\theta_1)1_{x>\delta} \end{aligned}$$

a) Show that the support  $S$  of  $a_3$  is given by the inequalities

$$|\theta| \geq 1, |\theta_2| \leq (1 - 2\delta_1)^{-1}|\theta_1|, Z_0 \leq 2\delta_2.$$

b) Determine the smallest cone containing  $S$ .

c) Show that  $Z$  has a strictly positive minorant on  $\{(x, \theta), x > \delta, |\theta_1| \geq (1 - 2\delta_1)|\theta_2|\}$ .

d) Show that  $a_3 \in S_{1/3, 2/3}^0$ .

Solution. 1) We assume that  $K_x(y)$ , fundamental solution of  $P$  with support in  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ , which is, thanks to the partial hypoellipticity of the operator  $P$ , in  $C^\infty(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^2))$ , admits a Fourier transform  $\hat{K}_x(\theta)$ . This Fourier transform is thus a solution of

$$\partial_{x^2}^2 \hat{K}_x(\theta) + [(1+x)\theta_1^2 - \theta_2^2]\hat{K}_x(\theta) = 0$$

with  $\hat{K}_0(\theta) = 1$ . We write the change of variable  $X = a_1(\theta)x + a_2(\theta)$  to bring us back to the equation characteristic of the Airy function. We then have

$$a_1^2 \partial_{X^2}^2 \hat{K}_x - [\theta_2^2 - a_1^{-1}\theta_1^2(X - a_2)]\hat{K}_x = 0$$

One choose  $a_2 = -\theta_2^2\theta_1^{-2}a_1$  and  $a_1^3 = -\theta_1^2$ . The equation is written

$$(3.87) \quad \partial_{X^2}^2 \hat{K}_x(\theta) - X\hat{K}_x(\theta) = 0.$$

We know that  $K_x = 0$  for  $y_1 < 0$ . Since

$$\hat{K}_x(\theta) = \int_{\mathbb{R}^2} e^{-iy_1\theta_1 - iy_2\theta_2} K_x(y) dy$$

the function  $\hat{K}_x(\theta)$  extends analytically in  $\theta_1$  in  $\Im\theta_1 < 0$  and remains bounded. We can then determine the roots  $\theta_1^{\frac{2}{3}}$  by the representation indicated in the text, and we have

$$X = \theta_2^2\theta_1^{-\frac{4}{3}} - (1+x)\theta_1^{\frac{2}{3}}.$$

From (3.87), we deduce that there exist two functions  $C_1(\theta)$  and  $C_2(\theta)$  such that

$$\hat{K}_x(\theta) = C_1(\theta) \text{Ai}(X) + C_2(\theta) \text{Bi}(X).$$

When  $\text{Re}(\theta_1^{\frac{2}{3}}) < 0$ , the solution  $\text{Bi}(X)$  is exponentially increasing and is not in  $\mathcal{S}'(\mathbb{R}^2)$ , which gives  $C_2(\theta) = 0$ . Using the condition  $\hat{K}_x(0) = 1$ , one obtains  $C_1(\theta) = (\text{Ai}(\theta_2^2\theta_1^{-\frac{4}{3}} - \theta_1^{\frac{2}{3}}))^{-1}$ . It suffice then to write  $\xi = \theta_2^2\theta_1^{-\frac{4}{3}} - (1+x)\theta_1^{\frac{2}{3}}$  and  $\xi_0 = \theta_2^2\theta_1^{-\frac{4}{3}} - \theta_1^{\frac{2}{3}}$ .

2) a) This asymptotic expansion was done in exercise 1 of the chapter 4.3. To obtain the one of  $(u^{\frac{1}{4}}\Phi(u))^{-1}$ , we invert the series of Taylor series in  $u^{-\frac{3}{2}}$ .

b) The method used is the same as in exercise 6 of this chapter. We prove by induction the existence of homogeneous functions  $Q_j^\alpha(x, \theta)$  such that

$$\partial_\theta^\alpha(G(Z)) = \sum_{j=1}^{|\alpha|} Q_j^\alpha(x, \theta)G^{(j)}(Z).$$

For  $\alpha = (1, 0)$  ou  $(0, 1)$ , one finds:

$$Q_1^{(1,0)}(x, \theta) = \frac{2}{3}|\theta_1|^{-\frac{1}{3}}\text{sign}(\theta_1)[1 - 2(1+x)\frac{\theta_2^2}{\theta_1^2}], Q_1^{(0,1)}(x, \theta) = -2|\theta_1|^{\frac{2}{3}}(1+x)\frac{\theta_2}{\theta_1^2},$$

and for  $|\alpha| \geq 1$ , we have the induction relations

$$Q_1^{\alpha+\alpha_1}(x, \theta) = \partial_\theta^{\alpha_1} Q_1^\alpha(x, \theta)$$

$$Q_{|\alpha+\alpha_1|}^{\alpha+\alpha_1}(x, \theta) = Q_1^{\alpha_1}(x, \theta)Q_{|\alpha|}^\alpha(x, \theta)$$

$$Q_j^{\alpha+\alpha_1}(x, \theta) = \partial_\theta^{\alpha_1} Q_j^\alpha(x, \theta) + Q_1^{\alpha_1}(x, \theta)Q_{j-1}^\alpha(x, \theta), 2 \leq j \leq |\alpha|.$$

The symbol  $Q_1^{\alpha_1}$  is homogeneous of degree  $-\frac{1}{3}$ . We perform an induction to obtain the degree of homogeneity of  $Q_j^{\alpha_1}$ . It is assumed that  $d^0 Q_j^\alpha = \frac{2}{3}j - |\alpha|$ , then

$$d^0 Q_1^{\alpha+\alpha_1} = d^0 Q_1^\alpha - 1 = \frac{2}{3} - |\alpha| - 1$$

$$d^0 Q_{|\alpha|+1}^{\alpha+\alpha_1} = d^0 Q_{|\alpha|}^\alpha - \frac{1}{3} = -\frac{1}{3}|\alpha| - \frac{1}{3}$$

$$d^0 Q_j^{\alpha+\alpha_1} = \frac{2}{3}j - |\alpha| - 1$$

since, for the last item, the two terms have the same degree of homogeneity  $\frac{2}{3}j - |\alpha| - 1 = \frac{2}{3}(j-1) - |\alpha| - \frac{1}{3}$ . Then, we check that

$$\partial_x^k \partial_\theta^\alpha G(Z) = \partial_\theta^\alpha [(\partial_x Z)^k G^{(k)}(Z)].$$

Applying Leibniz's formula proves that

$$\partial_x^k \partial_\theta^\alpha G(Z) = \partial_\theta^\alpha [(-1)^k \theta_1^{\frac{2}{3}k} G^{(k)}(Z)] = (-1)^k \sum_{\alpha'} C_{\alpha'}^{\alpha'} \partial_\theta^{\alpha'} (G^{(k)}(Z)) su \partial_\theta^{\alpha-\alpha'} (\theta_1^{\frac{2}{3}k}).$$

3 a) Apply 2 b) to the function  $G^{(k)}$  to find the decomposition, where

$$Q_j^{k,\alpha} = (-1)^k C_{\alpha'}^{\alpha'} Q_j^{\alpha'} \partial_\theta^{\alpha-\alpha'} (\theta_1^{\frac{2}{3}k}), j \geq 1$$

$$Q_0^{k,\alpha} = (-1)^k \partial_\theta^\alpha (\theta_1^{\frac{2}{3}k}).$$

One easily checks that the order of  $Q_j^{k,\alpha}$  est  $\frac{2k}{3} - (|\alpha| - |\alpha'|) + \frac{2j}{3} - |\alpha'|$ , which is the desired result.

3 a) Similarly, for  $k = 0$  and  $x = 0$ , we have

$$\partial_{\theta^\alpha}^\alpha (\sigma_2(Z_0)) = \sum_{j=0}^{j=|\alpha|} Q_j^{0,\alpha}(0, \theta) \sigma_2^{(j)}(Z_0)$$

The boundedness of  $\sigma_2$  and of all its derivatives (its first derivative is in  $C_0^\infty(\mathbb{R})$  in  $x$ ) implies that the maximum order of  $\theta$  in the derivative is  $-\frac{1}{3}|\alpha|$ . We conclude as in exercise 6 of this chapter since  $\sigma_2 \in S_{0,0}^0$ .

b) We verify that  $\xi = Z e^{i\frac{\pi}{3}\text{sign}(\theta_1)}$ . The function  $\psi$ :

$$u \rightarrow \frac{\sigma_2(u)}{\Phi(u e^{i\frac{\pi}{3}\text{sign}(\theta_1)})} = \frac{\sigma_2}{\Phi^*}(u)$$

is in  $C^\infty$ . We then check that

$$\partial_x^k \partial_\theta^\alpha a_2(x, \theta) = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} C_{\alpha}^{\alpha_1, \alpha_2, \alpha_3} \partial_\theta^{\alpha_1} \sigma_0(|\theta|) \partial_\theta^{\alpha_2} (\psi(Z_0)) \partial_{x^k}^k \partial_{\theta_3}^{\alpha_3} (\Phi^*).$$

This equality leads to a majorant of each term obtained by expanding each derivative into

$$(1 + |\theta|)^{-\frac{1}{3}|\alpha_2| - \frac{1}{3}|\alpha_3| + \frac{2}{3}k},$$

hence the result. We could also have written

$$\sigma_0(|\theta|) \in S_{0,0}^0(\mathbb{R}^2) \subset S_{0,0}^0(\mathbb{R} \times \mathbb{R}^2) \subset S_{1/3,2/3}^0(\mathbb{R} \times \mathbb{R}^2)$$

$$\psi \in S_{1/3,0}^0(\mathbb{R}^2) \subset S_{1/3,0}^0(\mathbb{R} \times \mathbb{R}^2) \subset S_{1/3,2/3}^0(\mathbb{R} \times \mathbb{R}^2)$$

and, thanks to b), we know that

$$\Phi^* \in S_{1/3,2/3}^0(\mathbb{R} \times \mathbb{R}^2)$$

hence  $a_2 \in S_{1/3,2/3}^0(\mathbb{R} \times \mathbb{R}^2)$ .

4) a) This symbol is compactly supported, so we use exercise 1 of this chapter. We have  $a_0(x, \theta) = (1 - \sigma_0(|\theta|)) \frac{Ai(\xi)}{Ai(\xi_0)} \in \mathcal{S}^{-\infty}(\mathbb{R} \times \mathbb{R}^2)$ .

b) The relation

$$(s^2 + 1)^{-\frac{1}{2}}((s^2 - 1)^{\frac{3}{2}} - (s^2 - 1 - x)^{\frac{3}{2}}) = x \cdot \frac{2s^2 - 2 - x + (s^2 - 1)^{\frac{1}{2}}(s^2 - 1 - x)^{\frac{1}{2}}}{(s^2 + 1)^{\frac{1}{2}}((s^2 - 1)^{\frac{1}{2}} + (s^2 - 1 - x)^{\frac{1}{2}})}$$

allows to see that, for  $x > 0$  and  $s \geq (1 + x)^{\frac{1}{2}}$ , this expression is strictly positive, and bounded below by  $x\alpha(x)$ ,  $\alpha(x) > 0$  when  $x > 0$ . Since, moreover, the function  $(s^2 + 1)^{-\frac{1}{2}}(s^2 - 1)^{\frac{3}{2}}$  is strictly increasing on  $[1, (1 + x)^{\frac{1}{2}}]$ , it is bounded below by its value in  $(1 - \delta_1)^{-1}$  for  $s \geq (1 - \delta_1)^{-1}$ . Noting then  $\gamma(x) = \min(x\alpha(x), h((1 + x)^{\frac{1}{2}}), h((1 - \delta_1)^{-1}))$ ,  $\gamma(x) > 0$  for  $x > 0$ .

We verify that

$$|\theta_1| \leq \frac{|\theta_2|}{\sqrt{1+x}} \Rightarrow \operatorname{Re} \left( -\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}) \right) = -\frac{2}{3}[(-Z)^{\frac{3}{2}} - (-Z_0)^{\frac{3}{2}}]$$

$$\frac{|\theta_2|}{\sqrt{1+x}} \leq |\theta_1| \Rightarrow \operatorname{Re} \left( -\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}) \right) = \frac{2}{3}(-Z_0)^{\frac{3}{2}}$$

$$|\theta_2| \leq |\theta_1| \Rightarrow \operatorname{Re} \left( -\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}) \right) = 0.$$

We see that  $|\theta_1| = |\theta|(1 + \frac{\theta_2^2}{\theta_1^2})^{-\frac{1}{2}}$ . One checks  $\sigma_1(\frac{\theta_1}{|\theta_2|}) \neq 0 \Rightarrow |\frac{\theta_1}{\theta_2}| \leq 1 - \delta_1$ . This implies that

$$|\frac{\theta_2}{\theta_1}| \geq (1 - \delta_1)^{-1}.$$

One checks that

$$\frac{2}{3} \operatorname{Re} (\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}) = \frac{2}{3} |\theta| h(\frac{\theta_2}{\theta_1}), |\theta_2| \geq |\theta_1|.$$

One deduces, from the inequalities on  $h$ , that

$$|\exp(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}))| \leq e^{-\frac{2}{3}\gamma(x)|\theta|}.$$

c) We first check that, uniformly in  $|\arg \xi| \leq \pi - \varepsilon$ ,

$$C_1 \leq (1 + |\xi|^{\frac{1}{4}}) |Ai(\xi) \exp(\frac{2}{3}\xi^{\frac{3}{2}})| \leq C_2$$

Since  $\xi$  and  $\xi_0$  are of argument  $\pm \frac{\pi}{3}$ , the inequality is satisfied and we have

$$\left| \frac{\Phi(\xi)}{\Phi(\xi_0)} \right| \leq \frac{C_2}{C_1} \frac{1 + |Z_0|^{\frac{1}{4}}}{1 + |Z|^{\frac{1}{4}}}.$$

We write the right hand side as

$$f(s) = \frac{1 + |\theta_1|^{\frac{1}{6}} |1 - s^2|^{\frac{1}{4}}}{1 + |\theta_1|^{\frac{1}{6}} |1 + x - s^2|^{\frac{1}{4}}}.$$

We check that, in this equality,  $s = \frac{\theta_2}{\theta_1}$  and is, in absolute value, greater than  $(1 - \delta_1)^{-1} > 1$ .

When  $s^2 \geq 1+x$ ,  $s \geq 0$ , the function is decreasing, so increased by its value in  $s^2 = 1+x$ . When  $s^2 \leq 1+x$ , we see that that it is also increased by this value. We have

$$\left| \frac{\Phi(\xi)}{\Phi(\xi_0)} \right| \leq \frac{C_2}{C_1} (1 + x^{\frac{1}{4}} |\theta_1|^{\frac{1}{6}}).$$

Moreover, the derivative with respect to  $x$  of order  $k$  of the function  $\text{Ai}(\xi)$  involves the derivative of  $\text{Ai}$  of order  $k$ , which is written  $P_k(\xi) \text{Ai}(\xi) + Q_k(\xi) \text{Ai}'(\xi)$ . We verify that we have estimates and a similar asymptotic expansion for  $\text{Ai}'$ . Moreover, the the degree of  $P_{2m}$  and  $Q_{2m+1}$  is  $m$ , that of  $P_{2m+1}$  and  $Q_{2m+2}$  is  $m+1$ .

As we have

$$\partial_{x^k}^k (\text{Ai}(\xi)) = (-|\theta_1|^{\frac{2}{3}})k(P_k(\xi) \text{Ai}(\xi) + Q_k(\xi) \text{Ai}'(\xi)),$$

there exists a constant  $C(k)$  and an exponent  $n(k)$  such that

$$\left| \exp\left(\frac{2}{3}\xi^{\frac{3}{2}}\right) \partial_{x^k}^k (\text{Ai}(\xi)) \right| \leq C(k) |\theta|^{n(k)}$$

We deduce, by considering  $D_k$  the majorant for  $|\theta| \geq 1$  of  $|\theta|^{n(k)} e^{-\frac{1}{6}\gamma(x)|\theta|}$ , that

$$(3.88) \quad \left| \partial_{x^k}^k (\sigma_0(|\theta|) \sigma_1\left(\frac{\theta_1}{|\theta_2}\right) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)}) \right| \leq C(k) D_k \exp\left(-\frac{1}{2}\gamma(x)|\theta|\right).$$

d) When we study all the derivatives in  $\theta$  of this symbol, we do not change the behavior. The exponential decay of  $a_1(x, \theta)$  implies the faster decay than any polynomial, hence  $a_1 \in S^{-\infty}(\mathbb{R} \times \mathbb{R}^2)$ .

5) a) Let us introduce

$$\psi_3(\theta) = 1 - (1 - \sigma_0(|\theta|)) - \sigma_0(|\theta|) \sigma_1\left(\frac{\theta_1}{|\theta_2}\right) - \sigma_0(|\theta|) \sigma_2(Z_0).$$

Hence  $\psi_3(\theta) = \sigma_0(|\theta|) [1 - \sigma_1\left(\frac{\theta_1}{|\theta_2}\right) - \sigma_2(Z_0)]$ . For  $\frac{\theta_1^2}{\theta_2^2} \geq (1 - \delta_1)^2$ , one sees that  $\sigma_1$  is zero. As, for  $Z_0 \geq \delta_2$ , one finds  $\frac{\theta_1^2}{\theta_2^2} \geq 1 + \delta_2 \frac{|\theta_1|^{\frac{4}{3}}}{\theta_2^2} \geq 1$ . We deduce that

$$(1 - \sigma_1\left(\frac{\theta_1}{|\theta_2}\right)) \sigma_2(Z_0) = \sigma_2(Z_0)$$

hence

$$\phi_3(\theta) = \sigma_0(|\theta|) (1 - \sigma_1\left(\frac{\theta_1}{|\theta_2}\right)) (1 - \sigma_2(Z_0)).$$

Support of  $\phi_3$  is of the required form, using the supports of  $\sigma_0$ ,  $1 - \sigma_1$  et de  $1 - \sigma_2$ .

b) Relation  $Z_0 \leq 2\delta_2$  writes, denoting by  $\theta_1 = u_1|\theta|$ ,  $\theta_2 = u_2|\theta|$ :

$$u_1^2 + u_2^2 = 1, |u_1|^{\frac{2}{3}} - u_2^2 |u_1|^{-\frac{4}{3}} \leq 2\delta_2 |\theta|^{-\frac{2}{3}} \leq 2\delta_2$$

We conclude that the boundary of the intersection of the unit ball with  $Z_0 \leq 2\delta_2$  is given by

$$u_1^2 + u_2^2 = 1, u_1^2 - u_2^2 = 2\delta_2 u_1^{\frac{4}{3}}.$$

The solution of this system being denoted by  $(\gamma, (1 - \gamma^2)^{\frac{1}{2}})$ , the above inequalities imply that  $|u_1| \leq \gamma$ , hence  $|\theta_1| \leq \frac{\gamma}{(1 - \gamma^2)^{\frac{1}{2}}} |\theta_2|$ . Denote thus by  $\kappa(\delta_2) = \frac{\gamma}{(1 - \gamma^2)^{\frac{1}{2}}} - 1$ . This constant tends to 0 when  $\delta_2$  tends to 0, since the equation of degree 3 in  $u_1^{\frac{2}{3}}$  has only one real root, which is equal to 1 when  $\delta_2 = 0$ .

We see that

$$S \subset C = \{\theta, (1 - 2\delta_1)|\theta_2| \leq |\theta_1| \leq \frac{\gamma}{(1 - \gamma^2)^{\frac{1}{2}}} |\theta_2|\}$$

with equality in at least one point of each generatrix of the cone.

c) We write

$$Z|\theta_2|^{-\frac{2}{3}} = (1+x) \left(\frac{|\theta_1|}{|\theta_2|}\right)^{\frac{2}{3}} - \left(\frac{|\theta_1|}{|\theta_2|}\right)^{-\frac{4}{3}}.$$

Then, as the right hand side of this equality is an increasing function of the variables  $\frac{|\theta_1|}{|\theta_2|}$ , we find

$$Z|\theta_2|^{-\frac{2}{3}} \geq (1+x)(1-2\delta_1)^{\frac{2}{3}} - (1-2\delta_1)^{-\frac{4}{3}}.$$

Let  $\delta > 0$ . Hence, if we have  $(1+\delta)(1-2\delta_1)^2 > 1$ , that is  $\delta_1 < \frac{1}{2}(1 - \frac{1}{\sqrt{1+\delta}})$ , hence for  $x > \delta$ ,  $Z|\theta_2|^{-\frac{2}{3}} \geq \Delta > 0$ , where  $\Delta = (1-2\delta_1)^{-\frac{4}{3}}[(1+\delta)(1-2\delta_1)^2 - 1]$ .

On the support of  $\phi_3$ ,  $|\theta_2| \geq (1-\gamma^2)^{\frac{1}{2}}|\theta|$ . Hence  $\{(x, \theta), x > \delta, |\theta_1| \geq (1-2\delta_1)|\theta_2|\}$ , one finds  $Z \geq D|\theta|^{\frac{2}{3}}$ . We are always outside  $|\theta| \leq 1$ , so  $Z$  has a strictly positive minorant.

#### 4. Exercises of chapter 6

Exercise 1: Compactly supported symbols. *Let  $a$  be  $C_0^\infty(X \times \mathbb{R}^d)$ . Show that  $a \in S^{-\infty}(X \times \mathbb{R}^d)$ . Note that this implies that, for  $\chi$  function  $C_0^\infty(X \times \mathbb{R}^d)$ , and for  $a \in S_{\rho, \delta}^m(X \times \mathbb{R}^d)$ ,  $a - a(1-\chi) \in S^{-\infty}$ .*

Exercise 2: the generic symbol of order  $m$ . *Let  $a$  be a positively homogeneous function of order  $m$  in the region  $|\theta| \geq 1$ . Show that  $a$  is in  $S_{1,0}^m$ .*

Exercise 3: A symbol of  $S_{1/2, 1/2}^0$ . *Show that if  $f$  is a positive function on  $X \times \mathbb{R}^N$ , homogeneous of degree 1 in  $\xi$ , then*

$$e^{-f} \in S_{\frac{1}{2}, \frac{1}{2}}^0.$$

Exercise 4: A symbol of  $S_{0,1}^0$ . *Prove that the function, on  $\mathbb{R}^N \times \mathbb{R}^N$ , equal to  $e^{ix}$ , is in  $S_{0,1}^0$ .*

Exercise 5: A symbol of  $S_{2/3, 1/3}^0$ . *Let  $m(x, \tau) = e^{-\tau^{\frac{1}{3}}x}$ , where  $\tau^{\frac{1}{3}}$  is the reciprocal function of  $y^3$  on  $\mathbb{R}$ . Show that it is a symbol of  $S_{1/3, 2/3}^0$ . We will show that, for  $d \geq 1$ , there exists a polynomial such that*

$$\partial_\tau^d m = \left(-\frac{1}{3}\tau^{-\frac{2}{3}}\right)^d x P_d(x, \tau^{-\frac{1}{3}}) m.$$

Exercise 6: A symbol of  $S_{\frac{1}{3}, 0}^0$  and of  $S_{1,0}^{\frac{2m}{3}}$ . *Let  $r \in C^\infty(\mathbb{R})$ . We construct, on  $\Gamma = \{|\eta| \leq c_0|\xi|\}$ , the function  $a(\xi) = r(|\xi|^{-\frac{1}{3}})$ .*

*Show that  $r \in S_{0,0}^0(\mathbb{R}) \Rightarrow a \in S_{\frac{1}{3}, 0}^0(\Gamma)$  and that  $r \in S_{1,0}^m(\mathbb{R}) \Rightarrow a \in S_{1,0}^{\frac{2m}{3}}(\Gamma)$ .*

PROOF. For exercises 2 and 3:

We rely on the inequality

$$|\theta|^{-\frac{1}{2}} |\nabla_x f(x, \theta)| + |\theta|^{\frac{1}{2}} |\nabla_\theta f(x, \theta)| \leq C_K |f(x, \theta)|^{\frac{1}{2}}.$$

This inequality is a consequence of

$$|\nabla_x f(x, \theta)| = |\theta| \left| \nabla_x f\left(x, \frac{\theta}{|\theta|}\right) \right|$$

and

$$|\nabla_\theta f(x, \theta)| = \left| \nabla_x f\left(x, \frac{\theta}{|\theta|}\right) \right|.$$

For a function  $F$  of class  $C^2$ , assuming  $F > 0$  and being in a compact included in  $\Omega$ , we see that, for  $|y|$  small enough small

$$F(\alpha + y) = F(\alpha) + yF'(\alpha) + C(y)|y|^2 \geq 0.$$

One checks that, if  $y = -rF'(\alpha)$ , with  $C(y)r \leq \frac{1}{2}$ , one gets

$$(F'(\alpha))^2 \leq \frac{F(\alpha)}{r - C(y)r^2},$$

which gives the inequality, valid in a compact such that  $F(\alpha) > 0$ :



$$|F'(\alpha)| \leq C(F(\alpha))^{\frac{1}{2}}.$$

We then have the majoration, valid in  $x$  in a compact  $K$ , since

$$|\nabla_x f(x, \frac{\theta}{|\theta|})| \leq C_{S,K} |f(x, \frac{\theta}{|\theta|})|^{\frac{1}{2}},$$

the constant  $C_{S,K}$  corresponding to the unit sphere in  $|\frac{\theta}{|\theta|}| = 1$ .

By the homogeneity of  $f$ , we then see that

$$|f(x, \frac{\theta}{|\theta|})|^{\frac{1}{2}} = |\theta|^{-\frac{1}{2}} |f(x, \theta)|.$$

In the same way, for  $f'_\theta$ , we take directly the bound with the derivative in  $\theta$  for  $\frac{\theta}{|\theta|}$ , which belongs to a compact where  $f$  is nonzero.

Thus

$$|\nabla_\theta f(x, \frac{\theta}{|\theta|})| \leq C_{K,S}^1 |\nabla f(x, \frac{\theta}{|\theta|})|,$$

which gives immediately, with the homogeneities sought

$$|\nabla_\theta f(x, \theta)| \leq C_{K,S}^1 |\theta|^{-\frac{1}{2}} |f(x, \theta)|^{\frac{1}{2}}.$$

We thus proved the majoration by taking the constant  $C_K$  equal to the sup of  $C_{S,K}$  and  $C_{K,S}^1$ .

So, for  $\theta$  of modulus greater than 1

$$|\nabla_x f(x, \theta)|^k |\nabla_\theta f(x, \theta)| e^{-f(x, \theta)} \leq C^{k+l} (1 + |\theta|)^{k/2-l/2} |f(x, \theta)|^{k/2+l/2} e^{-f(x, \theta)}.$$

As  $x^n e^{-x} \leq n^n e^{-n}$ , for all  $x \geq 0$ , we obtain:

$$|\nabla_x f(x, \theta)|^k |\nabla_\theta f(x, \theta)| e^{-f(x, \theta)} \leq C^{k+l} (1 + |\theta|)^{k/2-l/2} (\frac{k+l}{2})^{k/2+l/2} e^{-\frac{k+l}{2}}.$$

One then shows by induction

$$\partial_x^\alpha \partial_\theta^\beta (e^{-f}) = \sum_{\alpha', \beta'} a_{\alpha', \beta'} (\partial_x f)^{\alpha'} (\partial_\theta f)^{\beta'} e^{-f}$$

where the symbol  $a_{\alpha', \beta'}$  belongs to  $S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}}$ . Assume that, for  $(\alpha, \beta)$  given, the symbol

$\partial_x^\alpha \partial_\theta^\beta (e^{-f})$  belongs to  $S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}}$ . Then

$$\begin{aligned} e^f \partial_{x_j} [a_{\alpha', \beta'} (\partial_x f)^{\alpha'} (\partial_\theta f)^{\beta'} e^{-f}] &= \partial_{x_j} [a_{\alpha', \beta'}] (\partial_x f)^{\alpha'} (\partial_\theta f)^{\beta'} \\ &\quad - a_{\alpha', \beta'} (\partial_x f)^{\alpha'+\delta_j} (\partial_\theta f)^{\beta'} \\ &\quad + \sum_{k, \alpha_k \geq 1} a_{\alpha', \beta'} \partial_{x_j x_k}^2 f (\partial_x f)^{\alpha-\delta_k} (\partial_\theta f)^{\beta'} \\ &\quad + \sum_{l, \beta_l \geq 1} \partial_{\theta_l x_j}^2 (\partial_x f)^\alpha (\partial_\theta f)^{\beta-\delta_l} \end{aligned}$$

Note that  $\partial_{x_j x_k}^2 f \in S_{1,0}^1$  et que  $\partial_{x_j \theta_l}^2 f \in S_{1,0}^0$ , because  $f$  is homogeneous

We have

$$b_{\alpha', \beta'}^j = \partial_{x_j} (a_{\alpha', \beta'}) \in S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}} \subset S_{1,0}^{\frac{|\alpha+\delta_j|-|\alpha'|-|\beta|+|\beta'|}{2}},$$

$$b_{\alpha'+\delta_j, \beta'}^j = -a_{\alpha', \beta'} \in S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}} = S_{1,0}^{\frac{|\alpha+\delta_j|-|\alpha'+\delta_j|-|\beta|+|\beta'|}{2}}$$

$$b_{\alpha'-\delta_k, \beta'}^j = \alpha'_k a_{\alpha', \beta'} \partial_{x_j x_k}^2 f \in S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}+1} = S_{1,0}^{\frac{|\alpha+\delta_j|-|\alpha'-\delta_k|-|\beta|+|\beta'|}{2}}$$

et

$$b_{\alpha', \beta'-\delta_l}^j = \beta'_l a_{\alpha', \beta'} \partial_{x_j \theta_l}^2 f \in S_{1,0}^{\frac{|\alpha|-|\alpha'|-|\beta|+|\beta'|}{2}+1} = S_{1,0}^{\frac{|\alpha+\delta_j|-|\alpha'|-|\beta|+|\beta'-\delta_l|}{2}},$$

which allows to write

$$\partial_{x_j} (\partial_x^\alpha \partial_\theta^\beta e^{-f})$$

$$\begin{aligned}
&= \\
&\sum_{\alpha', \beta', k, l} [b_{\alpha', \beta'}^j (\partial_x f)^{\alpha'} (\partial_\theta f)^{\beta'} + b_{\alpha' + \delta_j, \beta'}^j (\partial_x f)^{\alpha' + \delta_j} (\partial_\theta f)^{\beta'} \\
&+ b_{\alpha' - \delta_k, \beta'}^j (\partial_x f)^{\alpha' - \delta_k} (\partial_\theta f)^{\beta'} + b_{\alpha', \beta' - \delta_l}^j (\partial_x f)^{\alpha'} (\partial_\theta f)^{\beta' - \delta_l}] e^{-f}.
\end{aligned}$$

For this decomposition, we find the same behavior of  $b^j$  as functions of  $|\alpha|, |\alpha'|, |\beta|, |\beta'|$ . Differentiating in  $\theta$ , we use that  $\partial_{\theta_j}^2 f$  is a symbol of order  $-1$ , and we obtain the same equalities for the symbols  $c_{\alpha', \beta'}^k, c_{\alpha', \beta' + \delta_k}^k, c_{\alpha' - \delta_l, \beta'}^k, c_{\alpha', \beta' - \delta_j}^k$  (left to the reader). The induction is complete.

We check that, for  $\beta_i = (\delta_{ij})_j$  or  $\alpha_i = (\delta_{ij})_i$ , the result is true. It is therefore true by induction.

We then combine the information on the symbol  $a_{\alpha', \beta'}$  with the obtained, to check that the derivative  $\partial_x^\alpha \partial_\theta^\beta (e^{-f})$  is bounded by terms of the form

$$(1 + |\theta|)^{\frac{|\alpha| - |\alpha'| - |\beta| + |\beta'|}{2} + \frac{|\alpha'| - |\beta'|}{2}},$$

that is a bound of the form  $(1 + |\theta|)^{\frac{|\alpha| - |\beta|}{2}}$ .

We proved that  $e^{-f}$  belongs to  $S_{\frac{1}{2}, \frac{1}{2}}^0$ .

For the exercise 4, one notices that

$$\partial_x^\alpha \partial_\xi^\beta (e^{ix \cdot \xi}) = i^{|\alpha|} \partial_\xi^\beta (\xi^\alpha e^{ix \cdot \xi}) = \left[ \sum_{\beta', |\beta'| \leq |\beta|} i^{|\alpha| + |\beta'|} a(\beta, \beta') \partial_\xi^{\beta - \beta'} (\xi^\alpha) (x^{\beta'}) \right] e^{ix \cdot \xi}$$

which immediately gives the majoration of this expression, on a compact  $K$  in  $x$ , by

$$C_K(\alpha, \beta)(1 + |\xi|)^{|\alpha|}.$$

Exercise 4 is solved.

Finally, for exercise 5, we verify that

$$\partial_\tau m(x, \tau) = -\frac{1}{3} \tau^{-\frac{2}{3}} x m(x, \tau)$$

Induction shows

$$\begin{aligned}
&\partial_\tau^{d+1} m(x, \tau) = \\
&\left(-\frac{1}{3} \tau^{-\frac{2}{3}}\right)^{d+1} x^2 P_d(x, \tau^{-\frac{1}{3}}) m - \frac{1}{3} \tau^{-\frac{4}{3}} \left(-\frac{1}{3} \tau^{-\frac{2}{3}}\right)^d \partial_\lambda P_d(x, \tau^{-\frac{1}{3}}) m + 2\tau^{-\frac{1}{3}} \left(-\frac{1}{3} \tau^{-\frac{2}{3}}\right)^{d+1} P_d(x, \tau^{-\frac{1}{3}}) \tau^{-\frac{1}{3}}
\end{aligned}$$

Denote by

$$P_{d+1}(x, \lambda) = x P_d(x, \lambda) + \lambda^2 \partial_\lambda P_d(x, \lambda) + 2\lambda P_d(x, \lambda).$$

This equality by induction defines a sequence of polynomials, so the intermediate result is proved.

Then, we have

$$\partial_x^\alpha \partial_\tau^d m = \partial_\tau^d ((-\tau^{\frac{1}{3}})^\alpha m) = \sum_{j=1}^{j=d} C_d^j \left(-\frac{1}{3} \tau^{-\frac{2}{3}}\right)^j x P_j(x, \tau^{-\frac{1}{3}}) m \partial_\tau^{d-j} (-\tau^{\frac{1}{3}})^\alpha + \partial_\tau^d ((-\tau^{\frac{1}{3}})^\alpha) m.$$

Exercise 6. We use a direct method in the case where  $rS_{1,0}^m(\mathbb{R})$ . We verify that

$$|r(|\xi|^{-\frac{1}{3}} \eta)| \leq C_{0,0}(1 + ||\xi|^{-\frac{1}{3}} \eta)|^m$$

which yields

$$|r(|\xi|^{-\frac{1}{3}} \eta)| \leq C_{0,0}(|\xi|^{\frac{1}{3}} + |\eta|)^m |\xi|^{-\frac{m}{3}} \leq C_{0,0}(1 + |\xi| + |\eta|)^{\frac{2m}{3}}.$$

This leads to the exponent  $\frac{2m}{3}$ . Moreover:

$$\partial_\xi^\alpha \partial_\eta^\beta (r(|\xi|^{-\frac{1}{3}} \eta)) = \partial_\xi^\alpha [|\xi|^{-\frac{|\beta|}{3}} (\partial_s^\beta r)(|\xi|^{-\frac{1}{3}} \eta)].$$

We verify that, for  $|\xi| \geq 1$  and  $(\xi, \eta) \in \Gamma$ ,

$$(4.89) \quad |\xi|^{-\frac{1}{3}} \leq (1 + c_0)^{-\frac{1}{3}} 2^{\frac{1}{3}} (1 + |\xi| + |\eta|)^{-\frac{1}{3}}.$$

It is easy to demonstrate, thanks to the equality

$$|\xi| = \left( \sum_{j=1}^{\dim X} \xi_j^2 \right)^{\frac{1}{2}}$$

that the symbol  $|\xi|^{-|\beta|}$  is a symbol of  $S_{1,0}^{-\frac{|\beta|}{3}}(\Gamma)$ .

The symbol  $r$  belongs to  $S_{1,0}^m(X)$ , hence the inequality

$$|\partial_s^\beta r(s)| \leq C_{0,\beta}(1 + |s|)^{m-|\beta|}.$$

Along with inequality (4.89) pour  $\{\xi \geq 1\} \cap \Gamma$ , one sees that

$$|\partial_s^\beta r(|\xi|^{-\frac{1}{3}}\eta)| \leq C_{0,\beta} \sup(1, \left(\frac{2}{1+c_0}\right)^{\frac{1}{3}(|\beta|-m)})(1 + |\xi| + |\eta|)^{\frac{2}{3}(m-|\beta|)}.$$

One is left, for  $(\xi, \eta) \in \{\xi \geq 1\} \cap \Gamma$ , with

$$| [|\xi|^{-|\beta|} (\partial_s^\beta r)(|\xi|^{-\frac{1}{3}}\eta)] | \leq C_{0,\beta} \sup(1, \left(\frac{2}{1+c_0}\right)^{\frac{1}{3}(|\beta|-m)})(1 + |\xi| + |\eta|)^{\frac{2m}{3} - \frac{1}{3}|\beta|}.$$

Notice then that

$$\partial_{\xi_j} l(|\xi|^{-\frac{1}{3}}) = -\frac{1}{3} |\xi|^{-\frac{1}{3}} [\partial_t l(|\xi|^{-\frac{1}{3}})] \frac{\xi_j}{|\xi|},$$

which allow, by induction to write all derivatives in  $\xi$  of

$$|\xi|^{-\frac{|\beta|}{3}} (\partial_s^\beta r)(|\xi|^{-\frac{1}{3}}\eta)$$

as a sum of derivatives of order  $\alpha'$  of  $\partial_s^\beta r$ , the coefficients being in the unit sphere, weighted by terms in  $|\xi|^{-\frac{|\alpha'|}{3}}$ . So we finally have the result, since a majorant of  $|\xi|^{-\frac{|\alpha'|}{3}}$  est  $\sup(1, \left(\frac{2}{1+c_0}\right)^{\frac{1}{3}|\alpha'|})(1 + |\xi| + |\eta|)^{-|\alpha'|}$ . The proof by induction is finished.

The second method is similar to the method used in the exercise 3. We prove by induction the existence of a sequence of functions  $S_j^\alpha(\xi, \eta)$ ,  $\alpha \in \mathbb{N}^2 - \{(0, 0)\}$ ,  $1 \leq j \leq |\alpha|$  such that

$$\partial_{\xi, \eta}^\alpha a(\xi, \eta) = \sum_{j=1}^{|\alpha|} S_j^\alpha(\xi, \eta) r^{(j)}(|\xi|^{-\frac{1}{3}}\eta)$$

avec, de plus,  $S_j^\alpha$  homogène de degré  $\frac{2}{3}j - |\alpha|$ .

Hence

$$\begin{aligned} \partial_\xi(g(|\xi|^{-\frac{1}{3}}\eta)) &= -\frac{1}{3} |\xi|^{-\frac{1}{3}} \frac{\xi}{|\xi|} \frac{\eta}{|\xi|} g'(|\xi|^{-\frac{1}{3}}\eta). \\ \partial_\eta(g(|\xi|^{-\frac{1}{3}}\eta)) &= |\xi|^{-\frac{1}{3}} g'(|\xi|^{-\frac{1}{3}}\eta). \end{aligned}$$

We conclude that

$$\begin{aligned} S_1^{(1,0)}(\xi, \eta) &= -\frac{1}{3} |\xi|^{-\frac{1}{3}} \frac{\xi}{|\xi|} \frac{\eta}{|\xi|} \\ S_1^{(0,1)}(\xi, \eta) &= |\xi|^{-\frac{1}{3}}, \end{aligned}$$

which are homogeneous polynomials of degree  $\frac{2}{3} - 1$ .

Denote by  $\alpha_1 \in \{(1, 0), (0, 1)\}$ . One checks that

$$S_1^{\alpha+\alpha_1} = \partial^{\alpha_1} S_1^\alpha$$

$$S_{|\alpha+\alpha_1|}^{\alpha+\alpha_1} = S_{|\alpha|}^\alpha S_1^{\alpha_1}$$

$$2 \leq j \leq |\alpha|, S_j^{\alpha+\alpha_1} = \partial^{\alpha_1} S_j^\alpha + S_{j-1}^\alpha S_1^{\alpha_1}.$$

Assuming that  $S_j^\alpha$  is a homogeneous polynomial of degree  $\frac{2}{3}j - |\alpha|$ , we verify that  $S_1^{\alpha+\alpha_1}$  is homogeneous of degree  $\frac{2}{3} - 1 - 1$ ,  $S_{|\alpha|+1}^{\alpha+\alpha_1}$  is homogeneous of degree  $\frac{2}{3}|\alpha| - |\alpha| - \frac{1}{3}$ , and that

$S_j^{\alpha+\alpha_1}$  is homogeneous of degree  $\frac{2}{3}j - |\alpha| - 1$  for  $2 \leq j \leq |\alpha|$ . This allows us to conclude on the homogeneity when the order of derivation is  $|\alpha| + 1$ .

In the case where all derivatives of  $r$  are bounded, i.e.  $r \in S_{0,0}^0$ , then we verify that  $\partial^\alpha a$  is the sum, with bounded weights, of homogeneous symbols of order  $\frac{2}{3}j - |\alpha|$  for  $1 \leq j \leq |\alpha|$ . The dominant order is obtained when  $j = |\alpha|$ , which thus gives the bound

$$|\partial^\alpha a| \leq C(r^{(|\alpha|)}, S_j^\alpha, c_0)(1 + |\xi| + |\eta|)^{-\frac{|\alpha|}{3}}$$

which demonstrates the first item. Finally, when  $r \in S_{1,0}^m(\mathbb{R})$ , the inequality on  $\Gamma$  can be verified:

$$|r^{(j)}(|\xi|^{-\frac{1}{3}}\eta)| \leq C_{\alpha,0}|\xi|^{-\frac{m-j}{3}}(1 + |\xi| + |\eta|)^{m-j},$$

which yields that all the terms of the sum are  $2m/3 - 2j/3 + 2j/3 - |\alpha| = 2m/3 - |\alpha|$ . Second part is proved.  $\square$



## Oscillatory integrals

We introduced before in Chapter 5, the symbols as a generalization of the differential operators, described by (0.77).

In this chapter, we define, for any symbol  $a$ , integrals of the form (0.78), using the methods of the chapter 4 on the stationary phase. The presentation of this chapter is essentially based on the founding article of L. Hörmander [47], which introduces the microlocal analysis through the intermediary of oscillatory integrals, which are not a priori defined but for which we describe a procedure to transform these integrals into convergent integrals. We present this study here for  $a \in S_{1,0}^m(\mathbb{R}^d)$ , but this is easily generalized to  $a \in S_{\rho,\delta}^m(\mathbb{R}^d)$ , for  $\delta < 1$  and  $\rho > 0$  in order to ensure the convergence after multiple integrations by parts.

### 1. Definition of Fourier integral operators

We place ourselves on  $X$  openset of  $\mathbb{R}^d$ . We give the main result which allows us to define a Fourier integral operator :

LEMMA 6.1. *Let  $\phi(x, \theta)$  be a phase, homogeneous of degree 1 in  $\theta$ , for  $(x, \theta) \in X \times \mathbb{R}^N$ , with no critical point on  $X \times \mathbb{R}^N - \theta = 0$ . Then*

$$(1.90) \quad I_\phi(av) = \int_{X \times \mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)v(x)dx d\theta$$

is defined for any symbol  $a \in S^m(X \times \mathbb{R}^N)$ ,  $m < -N$ .

(ii) For any  $m$ , the application  $a \rightarrow I_\phi(av)$  extends into a application continuous on  $S^m(X \times \mathbb{R}^N)$ .

(iii) When  $a \in S^m(X \times \mathbb{R}^N)$ , the application  $v \rightarrow I_\phi(av)$  is a distribution  $A(a, \phi)$  of order  $\leq k$  for  $m - k < -N$ .

Let us start with a comment. It is clear that when  $a$  is a symbol of order  $m \geq 1$ , e.g.  $a(x, \theta) = (1 + |\theta|^2)^{\frac{1}{2}}$ , the integral over  $X \times \mathbb{R}^N$  is not normally convergent, since  $(1 + |\theta|^2)^{\frac{1}{2}}$  is not integrable. Thus the expression in ii) is not an expression of a convergent integral in  $L^1$ . In fact, this convergence is the same type of convergence as that of the integral of the function  $\text{sinc}x$  (cardinal sine). The purpose of this and the following chapters is to give a meaning to these integrals which are defined only by their phase.

The lemma ?? can be found in Hörmander [47]. We use the fact that a linear application of  $C^\infty(X \times \mathbb{R}^N)$  in a Frechet space continues for the topology induced by  $S^m(X \times \mathbb{R}^N)$  admits a unique extension (Corollary 1.1.12 of [47]).

Preuve. Let  $a \in S^m(X \times \mathbb{R}^N)$ ,  $m < -N$ . As, for a real phase  $\phi$

$$|e^{i\phi(x,\theta)} a(x,\theta)v(x)| \leq 1_{\text{Supp}v} \|v\|_\infty N_{0,0}^{\text{Supp}v}(a)(1 + |\theta|)^m,$$

the integral (1.90) is well defined because  $m < -N$  and  $\text{Supp}v$  is compact. To define, in all cases on  $m$ , an integral equivalent to  $I_\phi(av)$ , we need to give a way to compute  $I_\phi(av)$  for  $m \geq -N$ . It is a question of extending the definition of this integral. To do this, we use the same method as in the the proof of the non-stationary phase theorem, stated in chapter 4 through lemma 4.1. We slightly modify the argument by introducing a function allowing us to eliminate the neighborhood of 0, where the functions with a negative power of  $\theta$  are not defined. We introduce the homogeneous function in  $\theta$  of degree  $-2$ :

$$\psi(x, \theta) = [|\theta|^2(\nabla_\theta \phi)^2 + (\nabla_x \phi)^2]^{-1}.$$

Let  $\chi(\theta) \in C_0^\infty(\mathbb{R}^N)$ , zero in a neighborhood of  $\theta = 0$ . The operator

$$M = -i(1 - \chi(\theta))\psi(x, \theta) \left[ \sum_j |\theta|^2 \frac{\partial \phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j} + \sum_p \frac{\partial \phi}{\partial x_p} \frac{\partial}{\partial x_p} \right] + \chi(\theta)$$

and its adjoint operator  $L$  which writes :

$$L = i(1 - \chi(\theta))\psi(x, \theta) \left[ \sum_j |\theta|^2 \frac{\partial \phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j} + \sum_p \frac{\partial \phi}{\partial x_p} \frac{\partial}{\partial x_p} \right] + \chi(\theta) - d(x, \theta)$$

( $d(x, \theta)$  constructed using all the elements of  $M$ ), all terms of  $L^t$  are of the form  $\sum a_j \partial_{\theta_j} + \sum b_p \partial_{x_p} + c$ , where all  $a_j$  are 0th order symbols,  $b_p$  and  $c$  are of order -1, and  $Me^{i\phi} = e^{i\phi}$ . We have thus the relation for  $a \in S^m(X \times \mathbb{R}^N)$ ,  $m < -N$ :

$$I_\phi(av) = I_\phi(L^k(av))$$

for all  $k$ . One checks that  $a \rightarrow L^k(av)$  is continuous from  $S^m(X \times \mathbb{R}^N)$  to  $S^{m-k}(X \times \mathbb{R}^N)$ . As the integral  $\int_{|\theta| \geq 1, \theta \in \mathbb{R}^N} (1 + |\theta|)^{-N-\varepsilon} d\theta$  converges when  $\varepsilon > 0$ , the integral  $\int e^{i\phi(x, \theta)} b(x, \theta) d\theta$  is defined for  $b$  of order at most  $-N$ .

Let  $a \in S^m(X \times \mathbb{R}^N)$  (and this time we have no assumption on  $m$ ). For all  $k$  such that  $m - k < -N$ , that is, for  $k > m + N$ ,  $L^k(av) \in S^{m-k}(X \times \mathbb{R}^N)$ , so  $I_\phi(L^k(av))$  exists. Moreover, for all  $k, k' > m + N$ ,

$$I_\phi(L^{k'}(av)) = I_\phi(L^k(av)).$$

This number is therefore independent of  $k > m + N$ . **By definition**, we say that

$$I_\phi(av) = I_\phi(L^k(av)), k > m + N.$$

The expression given by  $I_\phi(L^k(av))$  is well defined, as a convergent integral. It gives a value of the integral  $I_\phi(av)$  which is not defined as an integral. This thus defines a continuous functional on  $C_0^\infty(\mathbb{R}^N)$  depending on  $a \in S^m(X \times \mathbb{R}^N)$ , which is a distribution :

**DEFINITION 6.1.** *Let  $a \in S^m(X \times \mathbb{R}^N)$ , The application from  $C_0^\infty(X)$  to  $\mathbb{R}$  denoted by  $v \rightarrow I_\phi(av)$  is defined by*

$$I_\phi(L^k(av))$$

for all  $k$  such that  $m - k < -N$ .

This is the extension of the definition of (1.90) for  $a \in S^m(X \times \mathbb{R}^N)$ ,  $m < -N$ . The applications thus defined are called the Fourier integral operators.

## 2. Wavefront set of Fourier integral operators

In the representation

$$I_\phi(av) = \iint e^{i\phi(x, \theta)} a(x, \theta) v(x) dx d\theta,$$

we can consider  $x$  as the variable of integration. We define

$$X_\phi = \{x, \forall \theta \in \mathbb{R}^N - \{0\}, \nabla_\theta \phi(x, \theta) \neq 0\}.$$

For any  $x_0 \in X_\phi$ , the stationary phase theorem 4.3 allows to define

$$\int e^{i\phi(x_0, \theta)} a(x_0, \theta) d\theta.$$

It is indeed an oscillatory integral. For this purpose we introduce the differential operator

$$M_{x_0} = i(1 - \chi(\theta)) |\nabla_\theta \phi(x_0, \theta)|^{-2} \sum_{j=1}^{j=N} \frac{\partial \phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j} + \chi(\theta),$$

where the function  $\chi$  has been introduced earlier. The adjoint of  $M_{x_0}$  for the scalar product in  $L^2(\mathbb{R}^N)$  is denoted by  $L_{x_0}$ ; it is a differential operator of order 1. The formula of integration by parts shows that

$$\int e^{i\phi(x,\theta)} L_x^k(a(x,\theta)) d\theta$$

is absolutely convergent for  $m - k < -N$ . It thus defines a distribution, formally denoted by

$$A(x) = \oint a(x,\theta) e^{i\phi(x,\theta)} d\theta$$

for all  $x$  in  $X_\phi$ .

**Note that, each time we can define an integral through this process, we may often denote by  $\oint$  the integral after multiple integration by parts.**

We verify (exercise 6.1) that

$$e^{-i\phi(x,\theta)} \partial_{x^j}^j [L_x^k(a(x,\theta)) e^{i\phi(x,\theta)}]$$

is a symbol of  $S^{m-k+j}(X \times \mathbb{R}^N)$ . When  $k$  is chosen such that  $m - k + p < -N$ , the function  $A(x)$  is of class  $C^p$  on any compact included in  $X_\phi$ . Since the reasoning holds for any  $p$ , and since for  $k > m + N$  the definition of  $\int e^{i\phi} a d\theta$  does not change, we find that  $A(x)$  is  $C^\infty$  on  $X_\phi$ . Recall that the singular support of a distribution  $A$ , denoted by  $SS(A)$ , is the complementary of the largest open set  $O(A)$  where  $A$  is a  $C^\infty$  function. We have the

PROPOSITION 6.1. *Let  $A$  be the distribution defined by*

$$\forall u \in C_0^\infty(X), \langle A, u \rangle = I_\phi(au).$$

1) *The singular support of  $A$  is contained in the complementary of  $X_\phi$ , that is*

$$\{x \in X, \exists \theta \in \mathbb{R}^N - \{0\}, \nabla_\theta \phi(x, \theta) = 0\}.$$

2) *If  $a$  vanish in a conical neighborhood of*

$$C = \{(x, \theta) \in X \times (\mathbb{R}^N - \{0\}), \nabla_\theta \phi(x, \theta) = 0\}$$

*then  $A$  is a function of class  $C^\infty$ .*

PROOF. The distribution  $A$  is  $C^\infty$  on  $X_\phi$ , so  $X_\phi$  is complementary to  $O(A)$ . Thus  $SS(A)$ , complementary to  $O(A)$ , is contained in  $X_\phi$ .

Moreover, for  $a$  identically zero in a conic neighborhood of  $C$ , for all  $x$  in the projection of the support of  $a$ ,  $(x, \theta) \notin C$  for all  $\theta \in \mathbb{R}^N - \{0\}$ . Therefore, if  $x$  is in  $\pi(\text{supp}(a))$ ,  $x \in X_\phi$ , there exists a neighborhood  $V$  of  $x$  in  $X$  for which  $(y, \theta) \notin C$  for all  $y$  in  $V$  and for all  $\theta \neq 0$ . Therefore  $A$  is of class  $C^\infty$  on  $V(x)$ .

Let  $x$  be outside  $\pi(\text{supp}(a))$ . Then, if  $V_1$  is a neighborhood of  $x$  such that  $V_1 \times (\mathbb{R}^N - \{0\})$  is included in the complementary of the support of  $a$ , and that, by definition for  $k$  such that  $m - k < -N$ ,

$$\int \int e^{i\phi(x,\theta)} a(x,\theta) u(x) dx d\theta = \int \int e^{i\phi(x,\theta)} L^k(a(x,\theta) u(x)) dx d\theta,$$

the integral on the right is identically zero since  $a$  is identically zero on  $V_1 \times \mathbb{R}^N - \{0\}$ . The distribution  $A$  is then zero on the complementary of  $\pi(\text{supp}(a))$ .

In summary,  $A$  is a function  $C^\infty$  on  $X$ . It is also called a Fourier integral operator (even if, rigorously, it is not an operator but a function).  $\square$

We define the  $C^\infty$  wavefront set<sup>1</sup> of a distribution  $A$  on  $X \subset \mathbb{R}^N$ , which is denoted by  $WF(A)$ , subset of  $X \times \mathbb{R}^N$ :

<sup>1</sup>we use the term  $C^\infty$  because there exists also an analytic wave front set.



DEFINITION 6.2. We say that  $(x_0, \xi_0) \in X \times \mathbb{R}^N$  is not in the wavefront set of  $A$  if there exists a function  $\chi$ ,  $C_0^\infty$  compactly supported, identically equal to one in a neighborhood of  $x_0$ , and a conical neighborhood  $\Gamma$  of  $\xi_0$  in  $\mathbb{R}^N$  such that the Fourier transform of the distribution  $\chi A$  is rapidly decaying inside  $\Gamma$ .

In other words

$$\forall p, \exists C_p, \forall \xi \in \Gamma, | \langle A(x), e^{-ix \cdot \xi} \chi(x) \rangle | \leq C_p (1 + |\xi|)^{-p}.$$

One has the

PROPOSITION 6.2. Assume that  $a$  is zero near  $\theta = 0$ . Then the wavefront set of  $A(a, \phi)$  is included in

$$\{(x, \nabla_x \phi(x, \theta)), (x, \theta) \in \text{supp}(a), \nabla_\theta \phi(x, \theta) = 0\}.$$

PROOF. To prove Proposition 6.2, we prove first that a point of the complementary of

$$\{(x, \nabla_x \phi(x, \theta)), (x, \theta) \in \text{supp}(a), \nabla_\theta \phi(x, \theta) = 0\}$$

is not in the wavefront set of  $A(a, \phi)$ . So we consider first the case where  $x$  verifies  $\forall \theta, \nabla_\theta \phi(x, \theta) \neq 0$ . Proposition 6.1 implies that the function  $A(x)$  is of class  $C^\infty$  in the neighborhood of this point. We place ourselves in the case where there exists a solution to  $\nabla_\theta \phi(x, \theta) = 0$ . We suppose that  $\chi$  is a compactly supported function of class  $C^\infty$  localized in the neighborhood of an  $x$  of this form. We introduce  $\Gamma = \{\theta, \exists x \in \text{supp} \chi, \nabla_\theta \phi(x, \theta) = 0\}$  which is thus a conical subset of  $\mathbb{R}^N$  thanks to the homogeneity of  $\phi$  in the variable  $\theta$ . Consider  $K_1 = \{\nabla_\theta \phi(x, \theta), \theta \in \Gamma\}$  and  $K_2$  disjoint from  $K_1$ .

We find

$$\mathcal{F}(\chi A)(\xi) = \int \int e^{i\phi(x, \theta) - ix \cdot \xi} a(x, \theta) \chi(x) dx d\theta.$$

We consider here  $\xi$  as a parameter. Then the gradient of the phase in  $x$  is

$$\nabla_x \phi(x, \theta) - \xi.$$

As  $\theta \in \Gamma$  and  $\xi \in K_2$ ,  $K_1$  and  $K_2$  being disjoint and  $\nabla_x \phi(x, \theta)$  being homogeneous in  $\theta$ , there exists  $C(K_1, K_2) > 0$  such that

$$|\nabla_x \phi(x, \theta) - \xi| \geq C(K_1, K_2)(|\theta| + |\xi|).$$

We modify the theorem of the stationary phase by considering only the variables  $x$ .

There exists  $L$  differential operator in  $x$  such that  ${}^t L(e^{i(\phi(x, \theta) - \xi \cdot x)}) = e^{i(\phi(x, \theta) - \xi \cdot x)}$ . We rewrite then the integral  $\mathcal{F}(A)$  as  $\mathcal{F}(L^k(A))$  (condensed notation for the Fourier integral operator of the same phase and symbol  $L^k(\chi A)$ ). We check that the integrand is bounded by

$$C(a)C(K_1, K_2)(|\xi| + |\theta|)^{-k}(1 + |\theta|)^m.$$

When  $k_0$  is fixed such that  $m - k_0 < -N - 1$ , we verify that

$$\mathcal{F}(\chi A)(\xi) \leq c_1(k_0, K_1, K_2, a)|\xi|^{k_0 - k},$$

and the Fourier transform of  $\chi A$  is then rapidly decaying in  $K_2$ . This proves that a point which is outside the closed cone (set of points  $(x, \nabla_x \phi(x, \theta))$ ) is not in the wavefront set of  $A$ . The points of the wavefront set are either of the form  $(x, \xi)$  where  $x$  is not in the singular support of  $A$  (remark at the beginning of this proof), or of the form  $(x, \nabla_x \phi(x, \theta))$  and

$$\nabla_\theta \phi(x, \theta) = 0$$

This ends our proof of the Proposition 6.2.  $\square$

By giving a particular form to the phase  $\phi$ , defined henceforth for  $X = \mathbb{R}^d \times \mathbb{R}^d$  and  $N = d$ , we define a Fourier integral operator of symbol  $a \in S^m(X \times \mathbb{R}^d)$  and of phase  $\phi(x, y, \theta) \in S^1(X \times X \times \mathbb{R}^d)$  by

DEFINITION 6.3. Let  $\phi(x, y, \theta)$  be a phase such that  $\nabla_{(x, \theta)}\phi \neq 0$ . (for  $\theta \neq 0$ ) and  $\phi$  is homogeneous of degree 1 in  $\theta$ . We call the Fourier integral operator of symbol  $a(x, y, \theta)$  and phase  $\phi(x, y, \theta)$ , the operator from  $\mathcal{S}$  to  $\mathcal{S}'$  defined by its action on the functions  $u \in C_0^\infty(X)$  by

$$\langle Au, \psi \rangle = I_\phi(au \otimes \psi) = \oint_{X \times X \times \mathbb{R}^d} e^{i\phi(x, y, \theta)} a(x, y, \theta) \psi(x) u(y) dy d\theta dx.$$

A special case of the previous definition is obtained when  $\phi(x, y, \theta) = s(x, \theta) - y \cdot \theta$  and  $a(x, y, \theta) = \tilde{a}(x, \theta)$ ,  $X \subset \mathbb{R}^d$ :

DEFINITION 6.4. ( $X \subset \mathbb{R}^d$ )

Let  $s(x, \theta)$  be a phase such that  $\nabla_x s(x, \theta) \neq 0$ , and  $a(x, \theta) \in S^m(X \times \mathbb{R}^d)$ . We introduce the phase  $\phi(x, y, \theta) = s(x, \theta) - y \cdot \theta$ .

We call the Fourier integral operator of symbol  $a(x, \theta)$  and of phase  $s(x, \theta)$  the operator from  $\mathcal{S}$  to  $\mathcal{S}'$ :

$$u \rightarrow Au$$

defined by its action on  $C_0^\infty$  functions through

$$\langle Au, \psi \rangle = \int \int e^{is(x, \theta)} a(x, \theta) \psi(x) \hat{u}(\theta) dx d\theta.$$

One checks that

$$\langle Au, \psi \rangle = I_\phi(au \otimes \psi),$$

is an oscillatory integral on  $X \times X \times \mathbb{R}^d$ . One can also write  $\langle Au, \psi \rangle = I_s((a\hat{u})\psi)$  using definition 6.1.

We have the following result on the operator of the definition 6.4 :

THEOREM 6.1. Let  $A$  be a Fourier integral operator associated with the phase  $\phi(x, y, \theta) = s(x, \theta) - y \cdot \theta$ . Then  $(x, \xi) \in WF(Au)$ ,  $\xi \neq 0 \Rightarrow \exists \theta \neq 0$  such that  $\xi = \nabla_x s(x, \theta)$  and  $(\nabla_\theta s(x, \theta), \theta) \in WF(u)$ .

More generally, for  $A$  Fourier integral operator following the definition 6.3,

$$WF(Au) \subset \{(x, \nabla_x \phi(x, y, \theta)), \nabla_\theta \phi(x, y, \theta) = 0\}.$$

This theorem is the wavefront set version of the operator of Proposition 6.2 (Proposition 2.5.7 of [47]).

Let us prove this Theorem. We give ourselves a point  $(x_0, \xi_0)$  of  $\mathbb{R}^d \times \mathbb{R}^d - \{x_i = 0\}$  and we define the closed set  $G$  of  $\theta \neq 0$  such that  $\xi_0 = \nabla_x s(x_0, \theta)$ . Suppose that for any  $\theta$  of this kind,  $(\nabla_\theta s(x_0, \theta), \theta)$  is not in  $WF(u)$ . There then exist two conical neighborhoods of  $G$ ,  $\Gamma \subset \Gamma'$ , and a conical neighborhood  $W$  of  $\xi_0$  such that, for  $x - x_0$  sufficiently small,  $\theta \notin \Gamma$ ,  $\xi \in W$ ,  $|\nabla_x s(x, \theta) - \xi| \geq C(|\theta| + |\xi|)$ , and  $(\nabla_\theta \phi(x, \theta), \theta) \notin WF(u)$  for  $x - x_0$  small enough and  $\theta \in \Gamma'$ ,  $a(x, \theta) \neq 0$  on  $\Gamma'$ .

We represent  $J(\xi) = \mathcal{F}(\chi_1 Au)(\xi) = \int \int e^{i(s(x, \theta) - x \cdot \xi - y \cdot \theta)} a(x, \theta) u(y) \chi_1(x) dx dy d\theta$  and we prove the fast decay in  $\xi$  of  $J(\xi)$ .

The only interesting case is in the neighborhood of the points  $\mathcal{F}(u)(\theta)$  for  $u$  localized in the neighborhood of  $y$  and  $\nabla_\theta s(x, \theta) - y \neq 0$ . Then we apply a version of the stationary phase theorem in integrating by parts in  $\theta$ . The theorem is proved.

We will see in the chapter 8 that the wavefront set of the operator  $A$  defines the class of Fourier integral operators of the same type than  $A$ , associated to the same phase. The phase will be related to what we call the canonical relation  $\mathcal{C}$  of  $A$ , and we will denote by  $A \in I(X \times X, \mathcal{C}')$ . In particular, we will show that any Fourier integral operator of associated to a phase  $\phi(x, y, \theta)$  can be put, in a coordinate system, in the form of a Fourier integral operator associated to the phase  $s(x, \theta) - y \cdot \theta$ .



## Pseudo-différential operators

### 1. Definition and basic properties

The concepts in this chapter are classical. Many authors have presented the theory of pseudo-differential operators, among who one can mention J. Sjöstrand and A. Grigis [43], S. Alinhac and P. Gérard [3], L. Boutet de Monvel [17], L. Hörmander [48], M. Taylor [94], J.J. Kohn and L. Nirenberg [55], J. Rauch [86]. We assume that  $X \subset \mathbb{R}^d$  is the space on which we work. We aim at constructing a **symbolic calculus** on the space of symbols  $S^m(X \times X \times \mathbb{R}^d)$ , thus generalizing the composition of differential operators and allowing the inversion of a certain of differential operators.

The pseudo-differential operators are constructed, as does Hörmander, as a special case of Fourier integral operators. This presentation does not reflect the history, in which the operators were first introduced by the intermediary of their symbol and of the symbolic calculus. We deduce here the symbolic calculus from the stationary phase theorem and not from the generalization of the calculation of the composition of two differential operators, but we check that they lead to the same result.

We therefore introduce a particular phase in the Fourier integral operators defined previously. The phase is then defined for  $(x, y, \xi) \in X \times X \times \mathbb{R}^d$  by :

$$\phi(x, y, \xi) = (x - y) \cdot \xi = \sum_j (x_j - y_j) \xi_j.$$

Then we have the definition :

**DEFINITION 7.1.** *To any symbol  $a \in S^m(X \times X \times \mathbb{R}^d)$  is associated a particular Fourier integral operator, called pseudo-differential operator associated to  $a$ , denoted by  $A$  or by  $Op(a)$ , defined by*

$$Op(a)u(x) = \frac{1}{(2\pi)^d} \oint_{X \times \mathbb{R}^d} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) dy d\xi,$$

for  $u \in C_0^\infty(X)$ . The set of these operators is denoted by  $L^m(X)$ . The order  $m$  of the symbol corresponds to the order  $m$  of the operator.

Let  $A$  be a pseudodifferential operator, as in the definition 7.1. Then the distribution  $Au$  is defined by its action on a test function  $\psi$  by

$$\langle Au, \psi \rangle = I_\phi(a\psi \otimes u)$$

where  $a(x, y, \xi) \in S^m(X \times X \times \mathbb{R}^d)$  and  $\psi \otimes u \in C_0^\infty(X \times X)$ , defined through  $(\psi \otimes u)(x, y) = \psi(x)u(y)$ . The properties of the phase  $(x - y) \cdot \xi$  give immediately :

**LEMMA 7.1.** (1) *Let  $P$  be a differential operator with variable coefficients*

$$P = \sum_{\alpha, |\alpha| \leq m} a_\alpha(x) D_x^\alpha = \sum_{\alpha, |\alpha| \leq m} i^{-|\alpha|} a_\alpha(x) \partial_x^\alpha.$$

*The pseudo-differential operator of symbol*

$$p(x, \xi) = \sum_{\alpha, |\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

*is equal to  $P$ .*

(2) The spaces  $X_\phi$  and  $C$  introduced in the Proposition 6.1 are respectively

$$X_\phi = \{(x, y), x \neq y \in X \times X\} \quad C = \{(x, x, \xi), x \in X, \xi \in \mathbb{R}^d - \{0\}\}.$$

(3) As  $\nabla_{(x, \xi)}((x - y) \cdot \xi) \neq 0$  on  $X \times (\mathbb{R}^d - \{0\})$  for all  $y \in X$ , the distribution

$$A(y) = \int e^{i(x-y)\xi} a(x, y, \xi) dx d\xi$$

is well defined as an oscillatory integral.

(4) Similarly,  $\nabla_{(y, \xi)}((x - y) \cdot \xi) \neq 0$  implies that the application

$$u \rightarrow I_\phi(au)$$

is continuous from  $C_0^\infty(X)$  to  $C^\infty(X)$ .

We prove the first item, remark that the symbol 1 is associated to the identity operator, thanks to the Fourier inversion formula

$$u(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} \hat{u}(\xi) d\xi = \frac{1}{(2\pi)^d} \oint_{\mathbb{R}^{2d}} e^{i(x-y) \cdot \xi} u(y) dy d\xi.$$

Applying then  $P$  to this equality, we can exchange, for  $\hat{u} \in \mathcal{S}$ , the integration in  $\xi$  and the derivation in  $x$ , hence

$$Pu(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} \sum_{\alpha, |\alpha| \leq m} i^{-|\alpha|} (i\xi)^\alpha a_\alpha(x) \hat{u}(\xi) d\xi,$$

which gives the result. A density argument allows us to conclude. The proof of the other items is a simple application of the proposition 6.1.

We notice that, for  $u \in C_0^\infty(X)$ , we can write

$$Au(x) = \int K_A(x, y) u(y) dy$$

with the formal equality defining the distribution  $K_A(x, y) \in \mathcal{D}'(X \times X)$

$$K_A(x, y) = \int e^{i(x-y) \cdot \xi} a(x, y, \xi) d\xi.$$

The distribution  $K_A$  is called the kernel distribution of  $A$ . It has been defined previously.

Extend now the definition of the Fourier integral operator for more general functions than  $u \in C_0^\infty$ . To do this, we consider  $l \in C_0^\infty(\mathbb{R}^d)$  and check that we can define without problem  $A^t l(y) = \int K_A(x, y) l(x) dx$ . It is a function of class  $C^\infty$  by the lemma 7.1. We can then define, for  $u \in \mathcal{E}'$ ,  $\langle u, A^t l \rangle$ . By definition we say, for  $u \in \mathcal{E}'$ , that  $Au$  is the distribution given by

$$\langle Au, l \rangle = \langle u, A^t l \rangle.$$

LEMMA 7.2. 1) The singular support of  $K_A$  is contained in the diagonal of  $X \times X$ .

2) The singular support of  $Au$  is included in the singular support of  $u$ .

PROOF. The application of the proposition 6.1 allows to verify the first item. For the second item, let  $x_0 \notin SS(u)$ , we can find  $\phi$  identically equal to 1 in a neighborhood of  $x_0$  and  $\psi$  identically equal to 1 near  $SS(u)$ , of disjoint supports.

Since  $(1 - \psi)u$  is a  $C^\infty$  function, the function  $A((1 - \psi)u)$  is  $C^\infty$ , so  $Au - A\psi u \in C^\infty(X)$ .

We also check that  $\phi A\psi$  has for distribution kernel  $\phi(x)K_A(x, y)\psi(y)$ , and since the supports of  $\phi$  and  $\psi$  are disjoint, there is no point of the singular support of  $K_A$  in the support of  $\phi \otimes \psi$ . The kernel  $\phi(x)K_A(x, y)\psi(y)$  is therefore  $C^\infty$ , so

$$\phi(x)A(\psi u)(x) \in C^\infty(X)$$

hence

$$\phi(x)Au(x) \in C^\infty(X).$$

We immediately deduce that  $x_0 \notin SS(Au)$ . This completes the proof of the lemma 7.2.  $\square$

DEFINITION 7.2. We say that an operator  $A: C_0^\infty(X) \rightarrow \mathcal{D}'(X)$  linear continuous, which admits a kernel distribution  $K_A(x, y) \in \mathcal{D}'(X)$  is regularizing when  $K_A(x, y) \in C_0^\infty(X \times X)$ .

LEMMA 7.3. We have the equivalence:

$A$  is regularizing,

$A$  extends into an operator from  $\mathcal{E}'(X)$  to  $C^\infty(X)$ .

We deduce that the regularizing operators are, roughly speaking, the pseudo-differential operators whose symbol is in  $S^{-\infty}$ :

PROPOSITION 7.1. If  $A$  is a regularizing operator, then there exists a symbol  $a(x, y, \theta)$  of  $S^{-\infty}(X \times X \times \mathbb{R}^d)$  such that

$$Au(x) = \frac{1}{(2\pi)^n} \int \int e^{i(x-y)\cdot\theta} a(x, y, \theta) u(y) dy d\theta.$$

PROOF. One knows that, for  $u \in C_0^\infty$ ,

$$Au(x) = \int K_A(x, y) u(y) dy.$$

(which we also write  $\langle Au, \phi \rangle = \langle K_A, u \otimes \phi \rangle$ ). Thus, if  $u$  is regularizing, there is no problem to define in integral form the function  $Au \in C^\infty$ .

We introduce a function  $\chi(\theta)$ , of class  $C_0^\infty(\mathbb{R}^n)$ , of integral equal to  $(2\pi)^n$ . We then construct

$$a(x, y, \theta) = K_A(x, y) e^{-i(x-y)\cdot\theta} \chi(\theta).$$

This symbol is in  $S^{-\infty}(X \times X \times \mathbb{R}^d)$ , because the function  $\chi$  is compactly supported in  $\theta$  and it is clear that

$$Au(x) = \frac{1}{(2\pi)^n} \int \int e^{i(x-y)\cdot\theta} a(x, y, \theta) u(y) dy d\theta.$$

□

## 2. Composition of pseudodifferential operators

We introduce, in a first step, the notion of properly supported operator, so that we can compose two pseudo-differential operators. Indeed, if  $u \in C_0^\infty(X)$ , then  $Bu \in \mathcal{D}'(X)$ . One can then apply any differential operator to  $Bu$ , and this defines directly the composition of a differential operator and a pseudo-differential operator, as well as the composition of a pseudodifferential operator and of a differential operator. However, for  $A$  general pseudo-differential operator, one cannot without any precautions apply  $A$  to  $Bu$ , because  $Bu$  is not a compactly supported function. It is therefore necessary to be able to extend pseudodifferential operators to  $C^\infty$  functions, which will not be possible for any pseudo-differential operator. The sufficient notion for this extension (linked with the notion of support of the distribution  $Bu$ ) is given below.

**2.1. Properly supported operators.** For each  $y \in X$ , we define  $C(y) = \{x \in X, (x, y) \in \text{Supp}(K_A)\}$  and, for  $x \in X$ ,  $C^{-1}(x) = \{y \in X, (x, y) \in \text{Supp}(K_A)\}$ . We define  $C(K)$  and  $C^{-1}(K)$  in a similar way when  $K$  is a set.

DEFINITION 7.3. The pseudo-differential operator  $A$  is properly supported if and only if, for any compact  $K$  of  $X$ ,  $C(K)$  and  $C^{-1}(K)$  are compact.

From this definition, we deduce the following nice properties :

LEMMA 7.4. Let  $A$  be properly supported

(1) We have the inclusion

$$\text{Supp}(Au) \subset C(\text{supp}(u)).$$

(2)  $C^{-1}(x_0) \cap \text{supp}(u) = \emptyset \Rightarrow Au = 0$  in a neighborhood of  $x_0$ .

(3)  $A$  is continuous from  $C_0^\infty(X)$  in itself. It can be extended, as was done above for the extension of any pseudo-differential operator into a continuous operator from  $C^\infty(X)$  to  $C^\infty(X)$ , from  $\mathcal{E}'$  to  $\mathcal{E}'$ , or from  $\mathcal{D}'$  to  $\mathcal{D}'$ .

In other words, the action of  $A$  does not extend the support too much, which allows to look more like a differential operator. This is relative: indeed the size of the support of  $Au$  can be "much larger" (for example for a metric on  $u$ ) than that of the support of  $u$ , but the support of  $Au$  remains compact. The second paragraph of the lemma states that if the support of  $u$  is far from  $x_0$ , then  $Au$  is zero near  $x_0$ . Any pseudo-differential operator is, more or less, properly supported. This is expressed in the :

**PROPOSITION 7.2.** *For all  $A \in L^m(X)$ , there exists  $A' \in L^m(X)$ , properly supported, and  $A'' \in L^{-\infty}(X)$  such that  $A = A' + A''$ .*

**PROOF.** The idea is to write  $A = A\psi + A(1-\psi)$ , where  $\psi$  will have localization properties. We construct  $1-\psi$  such that it vanishes in a neighborhood of the diagonal. Moreover, it is sufficient that the support of  $\psi$  defines a relation, in the sense that  $C(K)$  and  $C^{-1}(K)$  are compact.

The construction we adopt here is the one presented by J. Sjöstrand in [43] Consider a partition of unity of  $X$  locally finite (finite on any compact), that is

$$1 = \sum_{j=0}^{\infty} \chi_j(x) \Rightarrow 1 = \sum_{j,k} \chi_j(x) \chi_k(y).$$

If  $C$  is constructed from this partition of the locally finite unit in  $X \times X$ , then this ensures that  $C(K)$  is compact, since there exists a finite number of non-zero  $\chi_j$  on  $K$ . As we want  $1-\psi$  to be zero on the diagonal, we impose for  $1-\psi$  that the remaining terms verify  $\text{supp} \chi_j \cap \text{supp} \chi_k = \emptyset$  for  $j \neq k$ . It is enough to use

$$\psi(x, y) = \sum_{j,k, \text{supp}(\chi_j) \cap \text{supp}(\chi_k) \neq \emptyset} \chi_j(x) \chi_k(y).$$

The operator  $\psi A$  is properly supported since  $C(K)$  and  $C^{-1}(K)$  are compact (locally finite sum). It remains to show to complete the construction that  $(1-\psi)A \in L^{-\infty}$ . This is a consequence of the first paragraph of the lemma 7.2, because  $SS(K_{(1-\psi)A}) = \emptyset$ .  $\square$

**Remark.** Note that operators whose symbol is not in  $S^{-\infty}$  can however be regularizing operators: indeed for example, the symbol  $(1-\chi(x, y))\xi^m$  is in  $S^m(X \times X \times \mathbb{R}^n)$  (and not in  $S^{m-1}$ , so not in  $S^{-\infty}$ ). On the other hand, the associated pseudodifferential operator is written

$$Pu(x) = \int dy (1-\chi(x, y))u(y) \frac{1}{(2\pi)^n} \int \xi^m e^{i(x-y)\cdot\xi} d\xi,$$

which yields

$$Pu(x) = \langle i^m \left(\frac{\partial}{\partial y}\right)^m \delta_{x-y}, (1-\chi(x, y))u(y) \rangle.$$

Since  $\chi$  has been constructed to be identically equal to 1 in a neighborhood of the diagonal,  $P$  is the null operator, so is regularizing. Therefore, when  $A$  is regularizing, there exists a symbol  $a \in S^{-\infty}$  such that  $A$  is the symbol operator  $a$ , but this does not mean that any mean that any symbol  $a$  representing  $A$  is in  $S^{-\infty}$ . This remark also proves that it is not necessary for the symbol to be zero for the pseudo-differential operator to be zero.

**2.2. Reduction of pseudo-differential operators.** The pseudo-differential operators can be represented by symbols depending only on  $x$  and  $\xi$ , as shown in :

**PROPOSITION 7.3.** *Let  $P$  be the properly supported pseudodifferential operator of symbol  $p(x, y, \xi)$  depending on the variables  $(x, y, \xi)$  of order  $m$  on  $X \times X \times \mathbb{R}^d$ , where the local dimension of  $X$  is  $d$ , for example  $X \subset \mathbb{R}^d$ .*

1) *There exists a symbol  $q(x, \xi)$ , of order  $m$  on  $X \times \mathbb{R}^d$  such that  $P - Op(q)$  is an operator in  $L^{-\infty}(X)$ .*

2) *This symbol  $q(x, \xi)$  is given by the relation*

$$(2.91) \quad q(x, \xi) = e^{-ix\cdot\xi} P(e^{i(\cdot)\cdot\xi})$$

since  $P$  is properly supported

3) The symbol  $q$  has the following asymptotic expansion

$$q(x, \xi) \simeq \sum_{|\alpha| \in \mathbb{N}^d} \frac{1}{i^{|\alpha|}} \frac{1}{\alpha!} (\partial_\xi^\alpha \partial_y^\alpha p(x, y, \xi))|_{y=x}.$$

This Proposition allows us to define the symbol of a properly supported pseudo-differential operator:

DEFINITION 7.4. Let  $P$  be the properly supported pseudodifferential operator of symbol  $p(x, y, \xi)$ . We call the symbol of  $P$ , and denote by  $\sigma(P)$ , the symbol  $q(x, \xi) \in S^m(X \times \mathbb{R}^d)$ .

We have

PROPOSITION 7.4. The symbol  $\sigma(P)$  is uniquely defined in the quotient space  $S^m(X \times \mathbb{R}^d)/S^{-\infty}$ .

PROOF. The proof of this result is for example in [43]. We propose in exercise 1 a formal proof relying on the stationary phase theorem. This formal calculation is a direct application of the methods of the previous chapter. The proof of [43] reads as follows :

Restricting to a properly supported operator on the support of  $p$  in the variables  $x$  and  $y$ , the operator  $P$  extends to functions of  $C^\infty$ , so the integral defining the action of  $P$  on  $e^{i(\cdot)\cdot\xi}$  is well defined. Finally

$$e^{-ix\cdot\xi} P(e^{i(\cdot)\cdot\xi}) = \frac{1}{(2\pi)^d} \oint p(x, y, \xi) e^{i(x-y)\cdot(\eta-\xi)} dy d\eta.$$

The only critical point in  $(y, \eta)$  of this phase is  $(x, \xi)$ . We truncate therefore in the neighborhood of  $\eta - \xi$  of the order of  $\xi$ . For this, J. Sjöstrand introduces the function  $L$  on  $\mathbb{R}_+$ , identically equal to 0 for  $x \geq 0.5$ , identically equal to 1 for  $0 \leq x \leq \frac{1}{3}$ . We verify as in the proof of the proposition 6.2 that, thanks to the non-stationary phase theorem (with the operator  $\mathcal{L} = \frac{1}{|\eta-\xi|} \sum_{j=1}^{j=d} (\xi_j - \eta_j) \frac{\partial}{\partial y_j}$ )

$$\frac{1}{(2\pi)^d} \oint \left(1 - L\left(\frac{|\eta - \xi|}{|\xi|}\right)\right) p(x, y, \eta) e^{i(x-y)\cdot(\eta-\xi)} dy d\eta.$$

is a symbol of  $S^{-\infty}(X \times \mathbb{R}^d)$ . This is presented as an exercise of Chapter 5.  $\square$

We then return to the study of

$$I(x, \xi) = \frac{1}{(2\pi)^d} \oint p(x, x + s, \xi + \sigma) L\left(\frac{|\sigma|}{|\xi|}\right) e^{-is\sigma} ds d\sigma.$$

On this integral, the stationary phase method applies, and we have the asymptotic expansion in  $\lambda$  of the integral obtained when  $\xi = \lambda\omega$ ,  $|\omega| = 1$  and  $\sigma = \lambda\tau$ . One checks that this defines an asymptotic expansion through

$$I(x, \xi) = \sum_{|\alpha| \leq N-1} \lambda^{-|\alpha|} i^{-|\alpha|} \frac{1}{\alpha!} \partial_s^\alpha \partial_\sigma^\alpha (p(x, x + s, \lambda(\omega + \tau)) L(|\tau|))|_{s=\tau=0} + S_N(\lambda).$$

We thus bound  $\lambda^N S_N(\lambda)$  by all the derivatives of  $p(x, x + s, \lambda(\omega + \tau)) L(|\tau|)$  up to and including the order  $2N + 1$ .

The result is also true for all derivatives of  $I$ . This completes the calculation since the symbol  $e^{-ix\xi} P(e^{i(\cdot)\xi})$  is therefore equivalent, modulo  $S^{-\infty}$ , to  $I(x, \xi)$ .

We propose a second proof of Proposition 7.3, based on the properties of symbols, extracted from Hörmander [47]. In addition, it will provide a nice expression of the symbol:

LEMMA 7.5. Let  $p$  be a symbol of  $S^m(X \times X \times \mathbb{R}^n)$ , properly supported. The asymptotic expansion of its Fourier transform is given by

$$\int_{\mathbb{R}^{2n}} p(x, y, \eta) e^{i(x-y)\cdot\eta} dy d\eta \simeq \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial_{\eta^\alpha}^2 p|_{y=x}.$$



PROOF. Based on Fourier transform properties, we write

$$u(y) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{u}(\eta) e^{iy \cdot \eta} d\eta.$$

Hence

$$Pu(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{u}(\eta) \int_{\mathbf{R}^{2n}} e^{iy \cdot \eta} e^{i(x-y) \cdot \theta} p(x, y, \theta) dy d\theta.$$

As the symbol  $p$  is properly supported, we can calculate its action on  $x \rightarrow e^{ix \cdot \xi}$ . We can also, without loss of generality, assume that the support of  $p$  in  $(x, y)$  is compact and included in  $\|x - y\| \leq 2$ . Thus, the symbol of  $P$  is

$$\sigma(P)(x, \eta) = e^{-ix \cdot \eta} \oint_{\mathbf{R}^{2n}} e^{iy \cdot \eta} e^{i(x-y) \cdot \theta} p(x, y, \theta) dy d\theta.$$

If one introduces  $u = x - y, \xi = \theta - \eta$ , then

$$\sigma(P)(x, \eta) = \oint_{\mathbf{R}^{2n}} p(x, x + u, \xi + \eta) e^{-iu \cdot \xi} dy d\theta.$$

We recognize the Fourier transform of the function  $q(x, u, \zeta) = p(x, x + u, \zeta)$ . This Fourier transform is  $\hat{q}(x, \theta, \zeta)$ . The symbol  $p \in S^m$ , thus

$$\partial_x^\alpha [(\theta)^\beta \partial_\zeta^\gamma \hat{q}(x, \theta, \zeta)] = \int \partial_\zeta^\gamma \partial_x^\alpha [p(x, x + u, \zeta)] \theta^\beta e^{-iu \cdot \theta} du.$$

The term  $\theta^\beta e^{-iu \cdot \theta}$  is transformed, as  $p$  is localized for  $\|x - y\| \leq 2$  using integrations by parts in  $u$ . Thus, as  $p \in S^m$ , the support of  $p$  in variable  $(x, y)$  being compact, we verify that there exists a constant  $C_{\alpha, \beta, \gamma}$  such that

$$\left| \int \partial_\zeta^\gamma \partial_x^\alpha [p(x, x + u, \zeta)] \theta^\beta e^{-iu \cdot \theta} du \right| \leq C_{\alpha, \beta, \gamma} (1 + |\zeta|)^{m - |\gamma|}.$$

This bound is true for all  $\beta$ , so there exists, for all  $p, \gamma, \alpha$  a constant  $C_{p, \gamma, \alpha}$  such that

$$|\partial_x^\alpha \partial_\zeta^\gamma \hat{q}(x, \theta, \zeta)| \leq C_{p, \alpha, \gamma} (1 + |\zeta|)^{m - |\gamma|} (1 + |\theta|)^{-p}.$$

Then we write a Taylor expansion of  $\hat{q}(x, \theta, \zeta)$  in the neighborhood of  $\zeta = \eta$ . It comes

$$|\hat{q}(x, \theta, \tilde{\theta} + \eta) - \sum_{|\alpha| < N} \partial_\eta^\alpha \hat{q}(x, \theta, \eta) \frac{\tilde{\theta}^\alpha}{\alpha!}| \leq C_M |\tilde{\theta}|^N \int_0^1 dt (1 + |\eta + t\tilde{\theta}|)^{m-N} (1 + |\theta|)^{-M}.$$

We choose in this inequality  $\tilde{\theta} = \theta$ , separating the region  $|\theta| < \frac{|\eta|}{2}$  and its complementary, choosing  $M = N$  large enough in the region  $|\theta| < \frac{|\eta|}{2}$ , and  $M$  large enough  $|\theta| \geq \frac{|\eta|}{2}$ . Thus, integrating with respect to  $\theta$  this inequality, we find

$$|\sigma(P)(x, \eta) - (2\pi)^n \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\eta^\alpha \int d\eta \hat{q}(x, \theta, \eta) \theta^\alpha d\theta| \leq C(1 + |\eta|)^{m+n-N}.$$

The power of  $\theta$  transforms into a derivative in  $y$  using the Fourier inversion formula:

$$i^{-|\alpha|} \partial_y^\alpha q(x, y, \eta)|_{y=x} = (2\pi)^n \int_{\mathbf{R}^n} e^{iy \cdot \theta} \theta^\alpha \hat{q}(x, \theta, \eta) d\theta.$$

We have constructed a sequence  $a_j$  (each term being obtained by collecting in the sum the terms such that  $|\alpha| = j$ ) of symbols of  $S^{m-j}$  such that

$$\begin{aligned} |a - \sum_{j < N} a_j| &\leq C(1 + |\eta|)^{\mu-j} \\ |\partial_x^\alpha \partial_y^\beta a| &\leq C(1 + |\eta|)^\mu. \end{aligned}$$

We can thus apply Proposition 5.4, which implies Lemma 7.5.  $\square$

Remark that the formal result is a consequence of the stationary phase theorem, but in this case, where the phase is the one associated with Fourier transform, it is enough to use classical results on Fourier analysis.

**2.3. Composition of symbols.** We define a symbolic composition in the algebra of symbols, where the symbol of the composition of  $A$  and  $B$  is not defined by a usual product, but by an adapted symbolic calculus, denoted by the symbol  $\sharp$  :

$$\sigma(A \circ B) = \sigma(A) \sharp \sigma(B)$$

DEFINITION 7.5. *The composition of two pseudodifferential operators of respective symbols  $\sigma(A)$  and  $\sigma(B)$  is given by the symbol, in the quotient space  $S^{m+p}(X \times \mathbb{R}^d)/S^{-\infty}(X \times \mathbb{R}^d)$ ,*

$$(2.92) \quad \sigma(A) \sharp \sigma(B) = \sum_{\alpha} \frac{1}{i^{|\alpha|}} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} \sigma(A)(x, \xi) \partial_x^{\alpha} \sigma(B)(x, \xi).$$

We have the following result

PROPOSITION 7.5. *Let  $A \in L^m(X)$ ,  $B \in L^p(X)$ . It is assumed that at least one of the two is properly supported. Then*

$$A \circ B \in L^{m+p}(X)$$

*and the symbol of  $A \circ B$  is  $\sigma(A) \sharp \sigma(B)$ .*

PROOF. We assume that, for  $u \in C_0^{\infty}(X)$ ,  $B$  is properly supported, that is

$$Bu(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} b(x, \xi) \hat{u}(\xi) d\xi \in C_0^{\infty}(X).$$

Since  $Bu \in C_0^{\infty}$ , then  $A(Bu)$  exists for all  $A$ . If we now assume that  $A$  is properly supported, it extends continuously on  $C^{\infty}(X)$ , in particular it can act on  $Bu$  for all  $B$  and for all  $u \in C_0^{\infty}(X)$ .

We write, as in the proof of the proposition 7.2, if  $A$  is prope

$$A = A' + A''$$

where  $A'$  is properly supported and  $A'' \in L^{-\infty}(X)$ . Then  $A \circ B \simeq A' \circ B$  since  $B$  is properly supported. Similarly, if  $B$  is proper, we write  $B = B' + B''$ ,  $B''$  is regularizing so  $A \circ B''$  is regularizing.

To be more precise, we have to evaluate  $\int K_{A'}(x, z) K_B(z, y) dz$  when  $A'$  is properly supported and  $B$  is regularizing, and to show that this integral is  $C^{\infty}(X \times X)$ . If  $x$  is in a compact set, then  $K_{A'}(x, z)$  is nonzero for  $z$  in a compact  $C_0$ , the integral is  $C^{\infty}$  in the variable  $y$ . Since  $A$  is a pseudodifferential operator which has a symbol in  $S^m$ ,  $A'$ , constructed above, is also a operator with a symbol in  $S^m$ , and since  $z$  is in the compact  $C_0$ , the integral  $\int K_{A'}(x, z) K_B(z, y) dz$  is computed through the action of the pseudodifferential operator  $A'$  on the function of  $C_0^{\infty}$  in  $z$   $K_B(z, y)$ . The result is in  $C_0^{\infty}$ , so we have shown that  $\int K_{A'}(x, z) K_B(z, y) dz$  is in  $C^{\infty}$ .

Thus

$$(A' \circ B)(u)(x) = \frac{1}{(2\pi)^d} \oint_{\mathbb{R}^{3d}} e^{i(x-y) \cdot \xi} e^{iy \cdot \eta} \sigma(B)(y, \eta) \hat{u}(\eta) d\eta \sigma(A)(x, \xi) dy d\xi.$$

As the operator is constructed as an action on  $\hat{u}(\eta)$ , it is natural to keep the variable  $\eta$  and to eliminate the variables  $(y, \xi)$ . The critical point of the phase in this integral is the point  $(y, \xi) = (x, \eta)$  and that the critical value is  $x \cdot \eta$ . One then gets

$$(A' \circ B)(u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \eta} \hat{u}(\eta) d\eta \int_{\mathbb{R}^{2d}} e^{i(x-y) \cdot (\xi-\eta)} \sigma(B)(y, \eta) \sigma(A)(x, \xi) dy d\xi.$$

We introduce  $u$  and  $\theta$  such that  $y = x + u$ ,  $\theta = \xi - \eta$ . One obtains

$$\sigma(A' \circ B)(x, \eta) = \int_{\mathbb{R}^{2d}} e^{-iu \cdot \theta} \sigma(B)(x + u, \eta) \sigma(A)(x, \eta + \theta) du d\theta.$$

The symbol  $p(x, u, \eta, \theta) = \sigma(B)(x + u, \eta) \sigma(A)(x, \eta + \theta)$  belongs to  $S^m(X^2 \times \mathbb{R}^{2d})$ . We are thus in the framework of application of the lemma 7.5. We have thus

$$\sigma(A' \circ B) \simeq \sigma(A \circ B) \simeq \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial_u^{\alpha} \partial_{\theta}^{\alpha} p(x, 0, \eta, \eta) \simeq \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial_{\xi}^{\alpha} \sigma(A)(x, \eta) \partial_y^{\alpha} \sigma(B)(x, \eta).$$

□

### 3. Wavefront set of pseudodifferential operators

In this section, we define and describe the properties of the wave front set of pseudodifferential operators.

**DEFINITION 7.6.** *Let  $A$  be a pseudo-differential operator of  $L^m(X)$ , whose principal symbol is  $\sigma_A$ . The wavefront set of  $A$ , denoted by  $WF(A)$ , is the smallest cone  $\Gamma \subset T^*X - 0$  such that  $\sigma_A|_{C\Gamma} \in S^{-\infty}(C\Gamma)$*

We deduce the definition of the wave front set of an integral kernel  $K$ :

**DEFINITION 7.7.** *Let  $K$  be an integral kernel of  $\mathcal{D}'(Y, X)$ . It is associated with a operator of  $C_0^\infty(X)$  in  $\mathcal{D}'(Y)$ . We introduce*

$$\begin{aligned} WF'(K) &= \{(x, \xi; y, -\eta) \in T^*(X \times Y) - 0, (x, \xi; y, \eta) \in WF(K)\} \\ WF'_X(K) &= \{(x, \xi) \in T^*X - 0, (x, \xi; y, 0) \in WF'(K)\} \\ WF'_Y(K) &= \{(y, \eta) \in T^*Y - 0, (x, 0; y, \eta) \in WF'(K)\} \end{aligned}$$

We have

**REMARK 6.** *If  $u \in \mathcal{E}'(Y)$ ,  $WF(u) \cap WF'_Y(K) = \emptyset$ , then  $Ku$  can be defined, and  $WF(Ku) \subset WF'(K)(WF(u)) \cup WF'_X(K)$ .*

This remark will be used in the proof of the theorem of propagation of singularities.

### 4. Elliptic pseudodifferential operators

**4.1. Definition.** Let us define an elliptic operator of order  $m$  :

**DEFINITION 7.8.** *We say that  $Op(p) \in L^m(X)$  is an elliptic operator in the neighborhood of  $x_0$  if there exists a compact  $K$  containing  $x_0$  and a constant  $c > 0$  such that the symbol  $p(x, \xi) \in S^m(X \times \mathbb{R}^N)$  verifies, for  $|\xi| \geq R \gg 1$*

$$(4.93) \quad \forall x \in K, \forall \xi \in \mathbb{R}^N - \{0\}, |p(x, \xi)| \geq c|\xi|^m.$$

*We say that  $Op(p)$  is a microlocal elliptic operator at  $(x_0, \xi_0) \in X \times \mathbb{R}^N$  if there exists a conical neighborhood  $V$  of  $(x_0, \xi_0)$  such that, on  $V$ , we have (4.93).*

**4.2. Inversion of an elliptic operator.** We prove the fundamental proposition of pseudo-differential calculus

**PROPOSITION 7.6.** (1) *Let  $P$  be a properly supported elliptic pseudodifferential operator, of symbol  $p(x, \xi) \in S^m(X \times \mathbb{R}^d)$ . It admits a right-inverse and a left-inverse, which are equal modulo  $L^{-\infty}$ .*

(2) *Let  $P$  be an elliptic operator in the neighborhood of  $(x_0, \xi_0)$ . There exists a conic neighborhood  $V_1$  of  $(x_0, \xi_0)$ , an operator  $B$  whose essential support does not meet  $V_1$  and an operator  $Q$  such that*

$$Q \circ P = Id + B$$

*Note that  $Q$  is called a **parametrix** of the operator  $P$ .*

**PROOF.** First check that  $Q_1 \circ P = Id + R_1$  and  $P \circ Q_2 = Id + R_2$  implies  $Q_1 \circ P \circ Q_2 = Q_2 + R_1 \circ Q_2 = Q_1 + Q_1 \circ R_2$ . If  $Q_1$  and  $Q_2$  are properly supported,  $Q_1 \circ R_2 \in L^{-\infty}$  and  $R_1 \circ Q_2 \in L^{-\infty}$ , then  $Q_1$  is equal to  $Q_2$  modulo  $L^{-\infty}$ .

By analogy with the inverse of a series, we start by proving the result for  $p$  classical symbol. We assume that  $Q$  is a pseudo-differential operator associated to a classical symbol  $q$ . When  $P$  is not associated with a classical symbol, we use  $\frac{1}{p} \sharp p = 1 + \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial_{\xi}^{\alpha} (\frac{1}{p}) \partial_x^{\alpha} p$ . Let  $q(x, \xi) \simeq \sum q_j(x, \xi)$ ,  $q_j(x, \xi) \in S^{-m-j}(X \times \mathbb{R}^N)$ . The equality  $Q \circ P = Id + R_1$  is equivalent to the equalities

$$\begin{cases} q_0 p = 1, \\ \sum_{|\alpha|+l=n} \frac{1}{\alpha! i^{|\alpha|}} \partial_{\xi}^{\alpha} q_l \partial_x^{\alpha} p = 0, n \geq 1. \end{cases}$$

We obtain  $q_0(x, \xi) = (p(x, \xi))^{-1}$ , which is in  $S^{-m}(X \times \mathbb{R}^N)$ . We then get

$$q_1(x, \xi) = -\frac{1}{p} \sum_{|\alpha|=1} \frac{1}{i} \partial_\xi^\alpha ((p(x, \xi))^{-1}) \partial_x^\alpha p(x, \xi) = \frac{1}{i} \frac{1}{(p(x, \xi))^3} \sum_j \partial_{\xi_j} p(x, \xi) \partial_{x_j} p(x, \xi)$$

hence  $q_1 \in S^{-m-1}(X \times \mathbb{R}^N)$ . We continue the induction process on  $l$ . As the symbol  $\partial_x^\alpha p$  is a symbol in  $S^m(X \times \mathbb{R}^N)$ , the symbol  $\partial_\xi^\alpha(q_l)$  is, for  $l < n$ , a symbol of  $S^{-m-l-|\alpha|}(X \times \mathbb{R}^N) = S^{-m-n}(X \times \mathbb{R}^N)$ . We then find that  $q_n$  is a symbol of  $S^{-m-n+m-m}(X \times \mathbb{R}^N)$  and this concludes the proof by induction. The sum of the symbols  $q_j$  is a symbol by the asymptotic completeness proposition (Proposition 5.4).

The general case can be deduced from this. Indeed,  $Op(\frac{1}{p}) \circ P - Id$ , is an operator of order  $-1$  (or in  $S^{-\rho+\delta}$  when  $p$  is in  $S_{\rho, \delta}^m$ ) that we denote  $R$ . We are thus left to inverting  $Id + R$ . As we have

$$(1 - r) \sharp (1 + r) = 1 - r^2 - \sum_{|\alpha| \geq 1} \partial_\xi^\alpha r \partial_x^\alpha r = 1 - r_2$$

where  $r_2$  is of order  $-2$ , we deduce that the Neumann series for the  $\sharp$  law yields the inverse of  $Op(1 + r)$ .

Let  $r_k = 1 - r + r \sharp r - r \sharp r \sharp r + \dots + (-1)^k r \sharp r \sharp \dots \sharp r$ . One has

$$r_k \sharp (1 + r) = 1 + (-1)^k (r_{k+1} - r_k)$$

(the operator  $R_{k+1} - R_k$  is the composition of  $k + 1$  occurrences of  $R$ , it is thus a symbol in  $L^{-k}$ , its symbol  $r_{k+1} - r_k$  is in  $S^{-k}$ ). We then use the result of asymptotic completeness. We prove now item 2 of Proposition 7.6 in the case where  $p$  is, modulo  $S^{-\infty}$ , homogeneous of degree  $m$ . We place ourselves on the conic neighborhood  $V$  of the hypothesis of the definition 7.8. The symbol  $p$  is obviously nonzero on the boundary of  $V \cap X \times S^{N-1}$  of the elements of  $V$  whose norm of  $\xi$  is 1. Let us denote by  $\pi$  the canonical projection on  $X$ . We can then extend  $p$  into a function, whose derivatives are all bounded belonging to  $C^\infty(\pi(V)) \times S^{N-1}$ . We extend the symbol thus obtained to  $\pi(V) \times (\mathbb{R}^N - \{0\})$  by homogeneity of degree  $m$ , using the constants allowing to bound  $p$  on the boundary. We denote by  $p_1$  this extended symbol. It is the symbol of an elliptic operator  $Op(p_1)$ .

By paragraph 1,  $Op(p_1)$  admits a properly supported inverse  $q$  such that

$$Op(q) \circ Op(p_1) = Id + r$$

which yields

$$Op(q) \circ Op(p) = Id + r + Op(q) \circ Op(p - p_1).$$

One chooses  $V_1$  such that  $V_1 \subset V$ . Then, as the essential support of  $p - p_1$  is contained in the complementary of  $V$ , and as  $Op(q)$  is properly supported, the operator  $r + Op(q) \circ Op(p - p_1)$  has its essential support distinct from  $V$ . This completes the proof of the proposition 7.6.  $\square$

**4.3. Elliptic regularity.** In this section we seek to quantify the relationship between the wavefront set of  $u$  and the wavefront set of  $Pu$  for an elliptic pseudodifferential operator  $P$ . We first recall a characterization of the wavefront :

LEMMA 7.6. *For  $u \in \mathcal{D}'(X)$ , one has*

$$WF(u) = \cap \{ \text{Car}(p), p \in S^0(X \times \mathbb{R}^N), Op(p)u \in C^\infty(X) \},$$

$\text{Car}(p)$  denoting the set of points  $(x, \xi) \in X \times \mathbb{R}^N$  such that  $p(x, \xi) = 0$ . We call this set the characteristic manifold of  $p$ .

Note that this result can also be written

$$WF(u) = \cap \{ \text{Car}(\sigma(P)), P \in L^0(X), Pu \in C^\infty(X) \}.$$

PROOF. Let  $(x_0, \xi_0) \in WF(u)$ . There exists  $\chi$ , equal to 1 in a neighborhood of  $x_0$  and  $\Gamma$  a conical neighborhood of  $\xi_0$  such that  $\mathcal{F}(\chi u)$  is rapidly decreasing. We denote by

$$Pu(x) = \frac{1}{(2\pi)^N} \int \psi(\xi) \mathcal{F}(\chi u)(\xi) e^{ix\xi} d\xi$$

where  $\psi$  is homogeneous of degree 0 outside a compact,  $\psi \equiv 1$  on  $\Gamma' \cap \{|\xi| > R\}$ ,  $\Gamma'$  conical neighborhood of  $\xi_0$ . Then  $Pu \in C^\infty$  and we verify that  $p(x, y, \xi) = \psi(\xi)\chi(y)$  is an element of  $S^0(X \times X \times \mathbb{R}^N)$ . We deduce  $\sigma(p)(x, \xi) \simeq \psi(\xi)\chi(x) + r(x, \xi)$ ,  $r \in S^{-1}(X \times \mathbb{R}^N)$ , and the symbol principal of  $p$  is therefore nonzero in the neighborhood of  $(x_0, \xi_0)$ . We have proved

$$\cap\{\text{Car}(p), \text{Op}(p) \in L^0(X), \text{Op}(p)u \in C^\infty(X)\} \subset WF(u).$$

Conversely, let  $(x_0, \xi_0)$  be such that there exists  $p \in S^0(X \times \mathbb{R}^N)$  such that  $\text{Op}(p)u \in C^\infty$  and  $p(x_0, \xi_0) \neq 0$ . Then  $p$  is elliptic in a neighborhood of  $(x_0, \xi_0)$  and we note  $q$  and  $B$  as in the proposition 7.6. Then

$$\text{Op}(q) \circ \text{Op}(p)u = u + ru \in C^\infty.$$

One has  $u = -ru + \text{Op}(q)[\text{Op}(p)u] + \text{Op}(q) \circ \text{Op}(p - p_1)u$ . Modulo  $C^\infty(X)$ ,  $u = \text{Op}(q) \circ \text{Op}(p - p_1)u$ . We use the inclusion (demonstrated below by using different arguments)  $WF(Av) \subset WF(v) \cap \text{Supp}(A)$ . Then we have  $WF(\text{Op}(q) \circ \text{Op}(p - p_1)u) \subset \text{Supp}(q) \cap \text{Supp}(p - p_1)$ . Since  $(x_0, \xi_0) \notin \text{Supp}(p - p_1)$ , we deduce  $(x_0, \xi_0) \notin WF(u)$ . The lemma is proved.  $\square$

We generalize into the

**PROPOSITION 7.7.** *Let  $P$  be an operator of order 0, of symbol  $p$ . Let  $u$  be a distribution such that there exists  $U$ , a conic set containing  $\text{Supp}(P)$ , with  $U \cap WF(u) = \emptyset$ . Alors  $Pu \in C^\infty$ .*

**PROOF.** Let  $P$  be an operator of order 0 of symbol  $p$  and suppose  $WF(u)$  disjoint from a conic set  $U$  which contains the essential support of  $P$   $\text{Supp}(P)$ .

Consider  $\chi \in S^0$  such that  $\text{supp}\chi \cap WF(u) = \emptyset$ ,  $\chi \equiv 1$  near  $\text{Supp}P$ . The equality

$$Pu = P\chi u + P(1 - \chi)u$$

where  $P\chi u$  is in  $C^\infty$  and where  $P \circ (1 - \chi)$  is of order  $-\infty$  by symbolic calculation leads to  $Pu \in C^\infty$ .  $\square$

**Alternative proof** One can, according to Taylor, construct  $\chi$  using a microlocal partition of the unit.

- Since the wavefront is the complementary of the intersection of characteristic sets of  $P$  such that  $Pu$  is  $C^\infty$ , for any  $(x, \xi)$  in  $\text{Supp}(p)$  there exists  $Q$  such that  $Qu \in C^\infty$  and  $(x, \xi)$  is non characteristic for  $Q$ . We use the homogeneity of the symbol, and we restrict to a compact  $K$  in  $x$ , and to a compact  $\mathcal{K}$  on the unit sphere. Then, we construct, for any point  $(x, \xi)$  of  $\text{Supp}(P)$ , the operator  $Q$  as before. We check thus that, for any point  $(x, \xi) \in K \times \mathcal{K}$ , there exists a neighborhood  $V_{(x, \xi)}$  such that the operator  $\tilde{Q}$  that we can choose in any point of this neighborhood is  $Q$ . We then consider the open cover  $\cup_{(x, \xi) \in K \times \mathcal{K}} V_{(x, \xi)}$  of the compact  $K \times \mathcal{K}$ . We extract a finite subcollection, which yields a finite list of operators  $Q_j$ . By considering  $Q = \sum Q_j^* Q_j$ ,  $Qu \in C^\infty$  and  $\text{Car}(Q) \cap \text{Supp}(P) = \emptyset$ .
- We then consider  $Q_1$  an elliptic operator coinciding with  $Q$  on a neighborhood  $W$  of  $\text{Supp}(P)$ . It can be constructed by using a localizing function, because it is sufficient that  $Q_1$  is nonzero on the complementary to the neighborhood  $W$  and equal to  $Q$ , elliptic on the neighborhood  $W$ . A parametrix of  $Q_1$  exists (because  $Q_1$  is elliptic). We note  $A = PQ_1^{-1}$ . We check that there exists  $R_3 \in L^{-\infty}$  such that

$$AQ = PQ_1^{-1}(Q_1 + (Q - Q_1)) = P + R_3 + PQ_1^{-1}(Q - Q_1)$$

Since the essential support of a product of operators is contained in the intersection of the essential supports, and  $Q - Q_1 = 0$  on the support of  $P$ , we see that  $\text{Supp}(Q - Q_1) \cap \text{Supp}(P) = \emptyset$ . This operator is therefore in  $L^{-\infty}$ . Adding  $R_3$  to it, we find an operator  $R_4$  of  $L^{-\infty}$ , which gives

$$AQ = P + R_4,$$

and so  $Pu = AQu - R_4u$ , which gives  $Pu \in C^\infty$ .

We are now ready to prove the

- PROPOSITION 7.8.** (1)  $WF(Pu) \subset WF(u) \cap \text{Supp}(P)$   
 (2) If  $P$  est elliptic,  $WF(Pu) = WF(u)$

PROOF. The second equality of the theorem comes from the first one and from the existence of a **parametrix** of an elliptic operator. Indeed, let  $E$  be such a parametrix. Then  $WF(EPu) = WF((Id + R)u) = WF(u)$  so

$$WF(EPu) \subset WF(Pu) \Rightarrow WF(u) \subset WF(Pu) \subset WF(u).$$

Let us prove, for a general operator, both inclusions.

To prove  $WF(Pu) \subset \text{Supp}(P)$ , we consider a point  $(x_0, \xi_0) \notin \text{Supp}(P)$ . There exists a conic neighborhood  $V$  of  $(x_0, \xi_0)$  and an operator  $q$ , identically equal to 1 on  $V$ , such that  $\text{Supp}(Q) \cap \text{Supp}(P) = \emptyset$ . Then  $QP \in L^{-\infty}$  (because  $\text{Supp}(QP) \subset \text{Supp}(P) \cap \text{Supp}(Q)$ ) and  $QPu \in C^\infty$ . Using the first definition of the wavefront of  $v$ , we see that the wavefront of  $Pu$  is in the characteristic set of  $Q$ , which does not contains  $(x_0, \xi_0)$ . We have thus proved the first inclusion.

To prove  $WF(Pu) \subset WF(u)$ , we give ourselves a conic neighborhood  $\Gamma$  of  $WF(u)$ . We write  $P = P_1 + P_2$  such that  $\text{Supp}(P_1) \subset \Gamma$  and  $\text{Supp}(P_2) \cap WF(u) = \emptyset$ . Then  $P_2u \in C^\infty$  and  $WF(Pu) = WF(P_1u)$ . We have just seen  $WF(P_1u) \subset \text{Supp}(P_1)$  so  $WF(P_1u) \subset \Gamma$ . This is true for any conical neighborhood of  $WF(u)$  so  $WF(Pu) \subset WF(u)$ . This completes the proof of the proposition 7.8.  $\square$

#### 4.4. Local resolubility of an elliptic operator. We have

PROPOSITION 7.9. *Let  $A$  be an elliptic differential operator of order  $m$  with coefficients  $C^\infty$  on the open  $X \subset \mathbb{R}^n$  and let  $x_0 \in X$ . There exists an open neighborhood  $V \subset X$  of  $x_0$  such that, for all  $v \in \mathcal{D}'(V)$  and any  $W \subset V$ , there exists  $u \in \mathcal{D}'(V)$  such that  $Au = v$  in  $W$*

PROOF. As in the case of the hyperbolic problem, we use the method of a priori estimates. Indeed, we verify that, as  $A$  is of order  $m$ , for any  $u$  distribution with compact support  $K \subset X$ ,  $u \in H^{s+m}(K)$ , we have

$$\|u\|_{H^{s+m}(K)} \leq C_{K,s} (\|A^*u\|_{H^s(\tilde{K})} + \|u\|_{H^s(\tilde{K})}).$$

This is shown by using a properly supported parametrix  $B \in L^{-m}$  of  $A^*$  and the fact that, for  $R \in L^{-\infty}$ ,  $R$  properly supported is continuous from  $H^s(\tilde{K})$  into  $H^{s+m}(\tilde{K})$ . The equality  $u = B \circ A^*u + Ru$  then gives the result.

Now, for any  $\varepsilon > 0$ , there exists an  $r(\varepsilon, m)$  such that, if the diameter of the support of  $u$  is smaller than  $r(\varepsilon, m)$  then the norm  $L^2$  of  $u$  is bounded by  $\varepsilon \|u\|_{H^m(\mathbb{R}^n)}$ .

Consider  $u \in H^m(\mathbb{R}^n) \cap \mathcal{E}'(V_1)$ ,  $V_1$  neighborhood of  $x_0$  of diameter less than  $r(\varepsilon, m)$ . Then, using the inequality for  $s = 0$  and  $K \subset V_1$

$$\|u\|_{H^m(K)} \leq C_{K,0} \|A^*u\|_{L^2(\tilde{K})} + C_{K,0} \varepsilon \|u\|_{H^m(K)}.$$

Consider  $\varepsilon_0 > 0$  given. We consider  $K \subset V_1$  and choose  $\varepsilon = \min(\frac{1}{2C_{K,0}}, \varepsilon_0)$  to obtain

$$(4.94) \quad \|u\|_{H^m(K)} \leq 2C_{K,0} \|A^*u\|_{L^2(\tilde{K})}.$$

We now consider a parametrix  $C$  of  $A$ , properly supported in  $V$ . Thus, for all  $v \in \mathcal{D}'(V)$ ,  $ACv = v + Tv$ ,  $Tv \in C^\infty(X)$ . Let  $u = Cv$ , so  $Au = v + Tv$ . If we find  $w$  such that  $Aw = Tv$ , then  $u - w$  is a solution of  $A(u - w) = v$  and is a solution of  $AU = v$ .

This brings us back to the problem of solving  $Aw = Tv$  in  $C^\infty$ . The associated variational problem is written

$$\forall \phi \in C_0^\infty(K), (Tv, \phi) = (Aw, \phi) = (w, A^*\phi).$$

By definition of the norm on  $H^{-m}$  (dual of the norm in  $H^m$ ),

$$|(Tv, \phi)| \leq D_K \|Tv\|_{H^{-m}(W)} \|\phi\|_{H^m(W)}$$

Using (4.94) stated for  $\phi$ , we obtain

$$|(Tv, \phi)| \leq 2D_K C_{K,0} \|Tv\|_{H^{-m}(W)} \|A^*\phi\|_{L^2(W)}.$$

The application  $\phi \rightarrow (Tv, \phi)$  is then a linear form continuous with respect to  $A^*\phi \in L^2(\tilde{K})$  for  $\phi \in H^m(K)$ . We then associate a continuous extension for  $A^*\phi \in L^2(\tilde{K})$  by the Hahn-Banach theorem. There then exists  $w \in L^2(\tilde{K})$  such that  $(Tv, \phi) = (w, A^*\phi)$ , for  $\phi \in H^m(K)$ . This

distribution is denoted by  $(A)(Tv)$ . The solution of  $Au = v$  is then  $u = Cv - (A)(Tv)$ . This is a local equality since the inversion has been done for  $\phi \in H^m(K)$ . The proposition 7.9 is proved.  $\square$

### 5. Change of variable in pseudo-differential operators.

We now consider a symbol  $a(x, \xi)$ . We show that the notion of pseudo-differential operator  $Op(a(x, \xi))$  (sometimes denoted by  $a(x, D)$ ) is invariant by change of variable in  $x$ . We assume  $a(x, \xi) \in S^m(X \times \mathbb{R}^d)$ .

Let  $\chi^{-1}$  be a diffeomorphism of  $K$  on  $K'$ ,  $K$  being compact and included in  $X$ . Let  $b(y, D)$  be the operator defined by the equality

$$b(y, D)u(\chi(y)) = (a(x, D)u)(\chi(y)).$$

We use the following Proposition, which makes the link between the asymptotic expansion and the pseudo-differential operators (it can be found in particular in the book of S.Alinhac and P. Gérard [3]) :

PROPOSITION 7.10. *Let  $\psi$  be a function  $C^\infty(\mathbb{R}^d)$ , such that  $d\psi \neq 0$ .*

*Let  $u \in C_0^\infty(\mathbb{R}^d)$  and  $a \in S^m(\mathbb{R}^d)$ .*

*Then  $e^{-ik\psi(x)}a(x, D)(u(x)e^{ik\psi(x)}) = I(x, k)$  has the following asymptotic expansion in  $k$ , locally uniformly in  $x$*

$$I(x, k) \simeq \sum_{\alpha} \frac{1}{\alpha!} \partial_{y^\alpha} [e^{ik(\psi(y) - \psi(x) - d\psi(x)(y-x))} u(y)]_{y=x} D_{\xi^\alpha} a(x, kd\psi(x)).$$

PROOF. One writes

$$I(x, k) = \int \int e^{i(x-y)\eta + ik(\psi(y) - \psi(x))} a(x, \eta) u(y) dy d\eta.$$

The critical point  $(y_c, \eta_c)$  in the variables  $(y, \eta)$  of the phase of this oscillatory integral verifies the equalities  $x - y_c = 0$ ,  $kd\psi(y_c) - \eta_c = 0$ , or  $y_c = x$ ,  $\eta_c = kd\psi(x)$ . We perform the change of variable  $(y, \eta) \rightarrow (z, \xi)$  given by  $y = x + z$ ,  $\eta = \xi + kd\psi(x)$ . The oscillatory integral is rewritten as

$$I(x, k) = \int e^{-iz\xi + ik(\psi(y) - \psi(x) - d\psi(x)(y-x))} a(x, \xi + kd\psi(x)) u(y) dz d\xi.$$

Then we denote by  $t(y, k) = e^{ik(\psi(y) - \psi(x) - d\psi(x)(y-x))}$ . The integral can be written as

$$I(x, k) = \int \int e^{-iz\xi} t(x+z, k) a(x, \xi + kd\psi(x)) u(y) dz d\xi.$$

The phase  $z \cdot \xi$  is quadratic because  $z \cdot \xi = \frac{1}{4}((\xi + z)^2 - (\xi - z)^2)$ . The associated Laplacian is  $\partial_{z_j \xi_j}^2$  for the critical point  $z = 0$ ,  $\xi = 0$ .

It is easy to generalize the stationary phase theorem to the case where the symbol is  $at$ , even if it is not a classical  $k$ -asymptotic symbol. This method is correct because the terms we add in the phase are of order greater or equal to 2, so they will not produce terms with positive powers of  $k$ . We can see it below by noticing that

$$\partial_{y^\alpha}^\alpha [t(y, k)u(y)] = \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \sum_{\beta \leq \alpha} C_\beta \partial_{y^\beta}^\beta u \cdot (\partial_{y^{\alpha-\beta}}^{\alpha-\beta} t(y, k)) \partial_{\eta^\alpha}^\alpha a.$$

In this expansion, the nonzero terms in  $z = 0$  coming from  $(\partial_{y^{\alpha-\beta}}^{\alpha-\beta} t(y, k))$  are of order  $\frac{|\alpha-\beta|}{2}$  at most since the phase is quadratic. Thus, combined with the term  $\partial_{\eta^\alpha}^\alpha a$ , we find that the order in  $k$  is  $k^{m-|\alpha| + \frac{|\alpha-\beta|}{2}}$  or  $k^{m - \frac{|\alpha+\beta|}{2}}$ : it is an asymptotic sum. We deduce that the symbol  $I(x, k)$  rewrites :

$$\sum_{\alpha} \frac{1}{\alpha! i^{|\alpha|}} \partial_{y^\alpha}^{|\alpha|} [t(y, k)u(y)]_{y=x} \partial_{\eta^\alpha}^{|\alpha|} a(x, \eta)|_{\eta=kd\psi},$$

which is the expected result, since  $D_\xi = \frac{1}{i} \frac{\partial}{\partial \xi}$ . The reader interested in more precise estimates of the terms of this symbol will refer to [3], p 57, Section 8.3, where the estimates are presented in detail.  $\square$

**Another proof**

Let us take up the calculation in  $y_c = x$ ,  $\eta_c = kd\psi(y_c)$ . The matrix jacobian matrix of the phase at this critical point is then

$$\begin{pmatrix} \text{Hess}\psi & -Id \\ -Id & 0 \end{pmatrix}.$$

We check that it is non-degenerate because its inverse is

$$\begin{pmatrix} 0 & -Id \\ -Id & -\text{Hess}\psi \end{pmatrix}$$

The operator involved in the expansion of the stationary phase (the half metric Laplacian), is then

$$-\sum_j \partial_{y_j \eta_j}^2 - \frac{1}{2} \frac{\partial^2 \psi}{\partial y_j \partial y_k} \partial_{\xi_j \xi_k}^2,$$

The asymptotic formula is deduced from this, by application of the section 4.3.

This allows us to obtain the

**PROPOSITION 7.11.** *Let  $\chi$  be a  $C^\infty$  diffeomorphism of  $\Omega$  on  $\Omega'$ , open sets of  $\mathbb{R}^d$  and let  $a$  be a symbol of  $S^m(\times \mathbb{R}^d)$ . We assume that the integral kernel of the operator associated to  $a$  has compact support.*

*We define the function on  $\Omega' \times \mathbb{R}^d$ ,  $b(y, \eta)$ , by the equality*

$$(5.95) \quad b(\chi(x), \eta) = e^{-i\chi(x)\eta} a(x, D)[e^{i\chi(x)\eta}].$$

- 1) *It is a symbol of  $S^m(\Omega' \times \mathbb{R}^d)$ .*
- 2) *The kernel associated to  $b$  has compact support in  $\Omega' \times \mathbb{R}^d$ .*
- 3) *If  $u \in \mathcal{S}'(\Omega')$ , then*

$$(5.96) \quad a(x, D)(u \circ \chi) = (b(y, D)u) \circ \chi.$$

We have imposed that the kernel of the operator associated to  $a$  is compactly supported to be able to define  $a(x, D)(u \circ \chi)$ .

**PROOF.** We first check that

$$a(x, D)e^{ix \cdot \xi} = e^{ix \cdot \xi} a(x, \xi).$$

Indeed, we verify that, for  $\hat{u}$  with compact support,

$$a(x, D)[u(\varepsilon x)e^{ix \cdot \xi}](x, \varepsilon, \xi) = (2\pi)^{-d} \iint e^{i(x-y) \cdot \eta} a(x, \xi) e^{iy \cdot \xi} u(\varepsilon y) dy d\eta.$$

(note the difference between the variable  $\eta$  of integration and the parameter  $\xi$  of the symbol).

As

$$\begin{aligned} \iint e^{i(x-y) \cdot \eta} a(x, \xi) e^{iy \cdot \xi} u(\varepsilon y) dy d\eta &= (2\pi)^{-n} \int d\eta e^{ix \cdot \eta} \int dy u(\varepsilon y) e^{iy \cdot (\xi - \eta)} \\ &= \int d\eta e^{ix \cdot \eta} \varepsilon^{-n} \hat{u}\left(\frac{-\xi + \eta}{\varepsilon}\right), \end{aligned}$$

this last integral being compactly supported for  $\varepsilon > 0$ , we find, by denoting by  $\tau = \frac{-\xi + \eta}{\varepsilon}$ ,  $\eta = \xi + \varepsilon\tau$ , so

$$a(x, D)[u(\varepsilon x)e^{ix \cdot \xi}](x, \varepsilon, \xi) = (2\pi)^{-n} \int e^{ix(\xi + \varepsilon\tau)} \hat{u}(\tau) d\tau.$$

When  $\varepsilon$  tends to 0, the right-hand term tends in  $\mathcal{S}'$  to  $e^{ix \cdot \xi} (2\pi)^{-n} \int \hat{u}(\tau) d\tau = u(0)e^{ix \cdot \xi}$ . The source term  $u(\varepsilon x)e^{ix \cdot \xi}$  tends in the sense of  $\mathcal{S}'$  to  $u(0)e^{ix \cdot \xi}$ . We then use the fact that a pseudo-differential operator extends into a continuous operator from  $\mathcal{S}'$  to  $\mathcal{D}'$ , continuous, so



we have the result by continuity. We introduce the symbol  $b(\chi(x), \eta) = e^{-i\chi(x)\eta} a(x, D) e^{i\chi(x)\eta}$  associated to a pseudo-differential operator). We have

$$a(x, D)[e^{i\chi(x)\cdot\xi}] = e^{i\chi(x)\cdot\xi} b(\chi(x), \xi).$$

Let  $f_\xi$  be the function  $x \rightarrow e^{ix\cdot\xi}$ , one has

$$a(x, D)[f_\xi \circ \chi] = [e^{iy\cdot\xi} b(y, \xi)](y = \chi(x)).$$

So we verify that (5.95) implies

$$a(x, D)[f_\xi \circ \chi] = [b(\cdot, D)f_\xi] \circ \chi.$$

Since the space generated by  $f_\xi$  is dense in  $\mathcal{S}'$ , this equality implies

$$a(x, D)[u \circ \chi] = [b(y, D)u] \circ \chi,$$

which we had to prove. We have shown the existence of  $b$ , which is a symbol. We evaluate it now with the proposition 7.10. We regularize  $e^{i\chi(x)\eta}$  by a function compactly supported (which we include in  $a$ ). We then write

$$b(\chi(x), \eta) = e^{-i\chi(x)\cdot\eta} \int e^{i(x-y)\cdot\xi + i\chi(y)\cdot\eta} a(x, \xi) d\xi dy.$$

We can consider this integral as an asymptotic integral in  $|y|$  for  $\eta \neq 0$ , using the proposition 7.10. The first method allows to verify that the phase studied is  $\psi(y) = \chi(y)\cdot\eta = \sum_j \chi_j(y)\eta_j$ . One checks that

$$\frac{\partial \psi}{\partial y_k} = \sum_j \partial_{y_k} \chi_j \cdot \eta_j = \sum_j \chi'_{jk} \eta_j = \sum_j {}^t \chi'_{jk} \eta_j.$$

We derive from the proposition 7.10 the calculation of  $b(\chi(x), \eta)$ . Indeed, the value of the phase is  $\chi(x)\eta$ . It is compensated with the  $e^{-\chi(x)\cdot\eta}$  which is in coefficient of the symbol, and we find

$$b(\chi(x), \eta) = \sum_\alpha \frac{1}{\alpha!} \partial_y^\alpha [e^{i\chi(x)\cdot\eta - i\chi(y)\cdot\eta - ({}^t \chi'(x)\eta)\cdot(y-x)}] D_\xi^\alpha a(x, {}^t \chi'(x)\eta).$$

Finally, if we calculate directly the critical point in  $(y, \xi)$  of the phase  $\psi(x, y, \xi, \eta) = (x-y)\cdot\xi + \chi(y)\cdot\eta - \chi(x)\cdot\eta$ , we find the equalities  $-\xi_j^c + \sum_k \partial_{y_j} \chi_k(y^c) \eta_k = 0$ ,  $x_j - y_j^c = 0$ , from which we deduce the change  $y = x + z$ ,  $\xi_j = \sum_k \partial_{y_j} \chi_k(x) \eta_k + \beta_j$ . We thus obtain the relation (5.96) by noting that  $\xi^c = {}^t \chi'(x)\eta$ . □

## 6. Exercices du chapitre 7

**Exercice 7.1 :** Reduction of operators. *Recover formally, using the stationary phase theorem on the phase  $(x-y)\cdot\xi$  in  $X \times X \times \mathbb{R}^{\dim T_{x_0} X}$ , the action of symbols  $p(x, y, \xi)$  and  $q(x, \xi)$  on a  $C^\infty$  compactly supported function  $u$  as in Lemma 7.3. Is it more than a formal calculation?*

**Exercice 7.2 :** Calculate the symbol of the adjoint operator of an Fourier integral operator  $A$ .

**Exercice 7.3 :** Change of variable using the Kuranishi trick. *We assume that we have the hypotheses of Proposition 7.11. Prove the result of this proposition using the change of variable  $\Sigma(y, t, \xi)$  from  $X \times X \times \mathbb{R}^d$  to  $\mathbb{R}^d$  which allows, for all diffeomorphism  $\chi$  of  $X$ , to write in the neighborhood of  $y = t$*

$$(\chi(y) - \chi(t))\cdot\xi = (y - t)\cdot\Sigma(y, t, \xi).$$

Exercise 7.4 : Inversion of a Fourier integral operator. Let  $A$  be a Fourier integral operator, given by its phase  $(x - y) \cdot \xi + s(x, \xi)$  and its classical symbol  $a(x, \xi) \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$ , elliptic. Let  $B$  the operator defined by the phase  $(x - y) \cdot \xi - s(y, \xi)$  and by the symbol  $b(y, \xi)$ .

1) Prove that there exists a classical symbol  $b \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$  such that  $A \circ B = Id + R$ ,  $R \in L^{-\infty}$ .

2) Dédue that, for all  $P \in L^m(\mathbb{R}^d)$  classical pseudodifferential operator, of classical symbol  $p \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ , the operator  $Q = A \circ P \circ B$  is a classical pseudodifferential operator  $\mathbb{R}^d$ , and that the principal symbols of  $P$  and  $Q$  satisfy :

$$q_m(y, \Sigma) = p_m(x + \nabla_{\theta} s(x, \theta(x, x, \Sigma)), \theta(x, x, \Sigma))$$

where  $\theta(x, y, \Sigma)$  is solution in the neighborhood of  $x = y$  of  $\Sigma \cdot (x - y) = \theta \cdot (x - y) + s(x, \theta) - s(y, \theta)$ .

Solution de l'exercice 7.1. La phase dans l'intégrale

$$Pu(x) = \int \int e^{i(x-y)\xi} p(x, y, \xi) u(y) dy d\xi$$

pourrait sembler être une phase linéaire. En réalité, il s'agit d'une phase quadratique de signature  $(d, d)$ . Ceci est vérifié grâce à l'égalité :

$$(6.97) \quad (x - y) \cdot \xi = \frac{1}{4} \sum_j (x_j - y_j + \xi_j)^2 - \frac{1}{4} \sum_j (x_j - y_j - \xi_j)^2.$$

Nous introduisons le difféomorphisme de  $\mathbb{R}^{2d}$  dans lui-même défini par

$$(y, \xi) \rightarrow (z_1, z_2) = (\xi - y + x, \xi + y + x).$$

On voit que  $dy d\xi = \frac{1}{2^d} dz_1 dz_2$  par un simple calcul de jacobien. On a donc

$$\begin{aligned} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} p(x, y, \xi) u(y) dy d\xi &= \int_{\mathbb{R}^{2d}} e^{i\frac{1}{4}z_1^2 - i\frac{1}{4}z_2^2} p(x, x - \frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}) u(x - \frac{z_1 + z_2}{2}) \frac{dz_1 dz_2}{2^d}. \\ \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} q(x, \xi) u(y) dy \wedge d\xi &= \int_{\mathbb{R}^{2d}} e^{i\frac{1}{4}z_1^2 - i\frac{1}{4}z_2^2} q(x, \frac{z_1 - z_2}{2}) u(x - \frac{z_1 + z_2}{2}) \frac{dz_1 dz_2}{2^d}. \end{aligned}$$

L'application du théorème de la phase stationnaire à l'une et à l'autre des intégrales est possible. On applique donc successivement le théorème de la phase stationnaire, sous la forme

$$\int_{\mathbb{R}^{2d}} e^{i\frac{1}{4}z_1^2 - i\frac{1}{4}z_2^2} q(x, \frac{z_1 - z_2}{2}) u(x - \frac{z_1 + z_2}{2}) \frac{dz_1 dz_2}{2^d} =$$

$$\frac{1}{2^d} \int_{\mathbb{R}^d} dz_1 e^{i\frac{(z_1/\sqrt{2})^2}{2}} \int_{\mathbb{R}^d} dz_2 e^{-i\frac{(z_2/\sqrt{2})^2}{2}} q(x, \frac{1}{2}(z_1 - z_2)) u(x - \frac{1}{2}(z_1 + z_2)) dz_1 dz_2.$$

Nous vérifions que  $\Delta_{z/\sqrt{2}} = 2\Delta_z$ . Appliquons alors le deuxième alinéa de la proposition 4.3. L'opérateur permettant de connaître le développement du symbole dans les coordonnées  $z_2$  est

$$\sum_l \frac{1}{l!} (i)^{-l} (-1)^l \left( \frac{\Delta_{z_2/\sqrt{2}}}{2} \right)^l$$

et celui dans les coordonnées  $z_1$  est

$$\sum_l \frac{1}{l!} (i)^{-l} \left( \frac{\Delta_{z_1\sqrt{2}}}{2} \right)^l.$$

On obtient ainsi

$$Pu(x) \simeq \sum \left( \frac{1}{i} \right)^l (l!)^{-1} \partial_{y\xi}^{2l} (p(x, y, \xi) u(y))|_{y=x, \xi=0}$$

$$Op(q)u(x) \simeq \sum \left( \frac{1}{i} \right)^l (l!)^{-1} \partial_{y\xi}^{2l} (q(x, \xi) u(y))|_{y=x}.$$

Calcul des deux premiers termes. Ils sont obtenus grâce à

$$(1 + i^{-1}\Delta_{z_1})(1 - i^{-i}\Delta_{z_2})$$

Il reste ainsi

$$u(x - \frac{1}{2}(z_1 + z_2))p(x, \frac{1}{2}(z_1 - z_2))|_{z_1=z_2=0} + \frac{1}{2}i^{-1}(\Delta_{z_1} - \Delta_{z_2})[u(x - \frac{1}{2}(z_1 + z_2))q(x, \frac{1}{2}(z_1 - z_2))].$$

En étudiant coordonnée par coordonnée, on trouve facilement

$$q(x, 0)u(x) + \sum_j \partial_{x_j} u(x) i^{-1} \partial_{\xi_j} q(x, 0).$$

Ceci correspond au développement asymptotique associé à l'opérateur  $q$ . Si, par exemple,  $q$  est un opérateur différentiel d'ordre 1, alors  $q$  est un polynôme homogène de degré 1, égal à  $\sum_j a_j(x)\xi_j$ , qui vaut 0 pour  $\xi = 0$ . On voit alors que le terme que l'on vient de calculer est

$$\sum_j \partial_{x_j} u(x) i^{-1} a_j(x).$$

Ceci achève l'analyse. Pour le cas général, on utilise la formule de Leibniz, et on obtient

$$\sum_{\alpha_1 + \alpha_2 = \alpha} (\partial_{\xi^{\alpha_1} y^{\alpha_1}}^{|\alpha| + |\alpha_1|} p)(x, x, 0) \partial_{x^{\alpha_2}}^{|\alpha_2|} u(x)$$

On en déduit donc la formule de dérivation en sommant sur les  $\alpha_1$  :

$$P(u) \simeq \sum_{\beta} i^{-|\beta|} (\beta!)^{-1} \partial_{y^{\beta}}^{|\beta|} u(x) \sum_{\alpha, \alpha_1/\beta = \alpha - \alpha_1} i^{-|\alpha_1|} (\alpha_1!)^{-1} \partial_{\xi^{\beta}}^{|\beta|} (\partial_{\xi^{\alpha_1} y^{\alpha_1}}^{2|\alpha_1|} p)(x, x, 0).$$

Nous considérons alors le symbole

$$q_1(x, \xi) = \sum_{\alpha_1} i^{-|\alpha_1|} (\alpha_1!)^{-1} \frac{\partial^{2\alpha_1} p}{\partial \xi^{\alpha_1} y^{\alpha_1}}(x, x, \xi).$$

Comme  $p$  est un symbole de  $S^m$ , le symbole associé à  $\partial_{y^{\alpha_1} \xi^{\alpha_1}}^{2\alpha_1} p(x, x, \xi)$  est un symbole de  $S^{m-|\alpha_1|}$ . Nous appliquons le théorème de complétude asymptotique (Proposition 5.3), pour voir que cette somme asymptotique définit un symbole de  $S^m$ .

On vérifie que  $Pu \simeq Op(q_1)u$ . Il suffit alors de prendre  $q - q_1 \in S^{-\infty}$ , en particulier  $q = q_1$  pour obtenir l'égalité du lemme de réduction.

Solution de l'exercice 7.2. On considère, sur  $L^2(X)$ , le produit scalaire canonique  $(u, v) = \int u \bar{v} dx$ . L'adjoint d'un opérateur (pseudo-différentiel)  $A$  continu de  $C_0^\infty(X)$  dans  $\mathcal{D}'(X)$  est l'opérateur défini par  $(Au, v) = (u, A^*v)$  pour  $u, v$  dans  $C_0^\infty(X)$ . Lorsque  $A$  est un opérateur pseudo-différentiel, il admet un noyau  $K_A(x, y)$ , et  $A^*$  a pour noyau la conjuguée de  $K_A(y, x)$ . On suppose que  $A$  est un opérateur pseudo-différentiel dont un symbole est  $a(x, y, \theta)$ . Alors la représentation par une intégrale oscillante du noyau distribution de  $A$  est donnée par

$$K_A(x, y) = \frac{1}{(2\pi)^n} \int a(x, y, \theta) e^{i(x-y)\theta} d\theta,$$

ce qui donne

$$K_{A^*}(x, y) = \frac{1}{(2\pi)^n} \int \bar{a}(y, x, \theta) e^{i(x-y)\theta} d\theta$$

(dans cette dernière ligne, il y a à la fois l'échange de  $x$  et de  $y$  et la conjugaison pour obtenir la même phase); Ainsi, l'opérateur  $A^*$  est un opérateur pseudo-différentiel, dont un symbole est  $\bar{a}(y, x, \theta)$ .

Pour obtenir le symbole principal associé, on commence par prendre pour  $a(x, y, \theta)$  le symbole  $\sigma_A(x, \theta)$ , symbole principal de  $A$ . On constate alors que  $\bar{a}(y, x, \theta) = \bar{\sigma}_A(y, \theta)$ , et on applique l'alinéa 3) de la proposition 7.3 pour obtenir le symbole

$$\sigma_*(x, \theta) \simeq \sum \frac{1}{i^{|\alpha|}} \frac{1}{\alpha!} \left(\frac{\partial}{\partial y}\right)^\alpha \left(\frac{\partial}{\partial \theta}\right)^\alpha \bar{\sigma}_A(x, \theta).$$

Solution de l'exercice 7.3.

Calcul de  $b$  par l'utilisation de l'astuce de Kuranishi. Nous présentons ici une méthode s'appuyant sur le comportement de phases de la forme  $\phi(x, y, \xi)$  telles que  $\phi(x, x, \xi) = 0$ . Elles se factorisent sous la forme  $(x - y)g(x, y, \xi)$ . L'exemple que nous abordons ici pour l'étude du changement de variable est  $\phi(x, y, \xi) = (\chi(x) - \chi(y))\xi$ . Nous vérifions en effet que

$$(a(x, D)u)(x) = \frac{1}{(2\pi)^d} \int \int e^{i(x-z)\xi} a(x, \xi) u(z) dy d\xi,$$

qui se réécrit

$$(a(x, D)u)(\chi(y)) = \frac{1}{(2\pi)^d} \int \int e^{i(\chi(y)-z)\xi} a(\chi(y), \xi) u(z) dz d\xi.$$

Cette égalité se transforme en

$$(6.98) \quad (a(x, D)u)(\chi(y)) = \frac{1}{(2\pi)^d} \int \int e^{i(\chi(y)-\chi(t))\xi} a(\chi(y), \xi) u(\chi(t)) (d\chi(t)) dt d\xi.$$

La phase oscillante considérée est donc

$$\phi(y, \xi, t) = (\chi(y) - \chi(t)) \cdot \xi.$$

Du théorème des fonctions implicites on déduit le lemme suivant.

**LEMMA 7.7.** *Si une phase  $\phi(y, t, \xi)$  s'annule pour  $y = t$ , et si  $\nabla_y \phi(y_0, y_0, \xi_0) \neq 0$  ou  $\nabla_t \phi(y_0, y_0, \xi_0) \neq 0$ , il existe une fonction  $\Sigma \in C^\infty(K' \times K' \times \mathbb{R}^d)$  telle que, pour  $y - t$  dans un voisinage de 0 et  $\xi$  dans un voisinage de  $\xi_0$  on ait*

$$\phi(y, \xi, t) = (y - t) \cdot \Sigma(y, t, \xi).$$

On vérifie que  $\Sigma(y, y, \xi) = \nabla_t \phi(y, y, \xi)$  dans ce voisinage. L'application de  $X \times \mathbb{R}^d \times X$  dans lui-même, qui à  $(y, \xi, t)$  fait correspondre  $(y, \Sigma(y, \xi, t), t)$ , est un difféomorphisme local au voisinage de  $(y_0, \xi_0, y_0)$ .

La formule d'intégration prouve que

$$\chi_j(y) - \chi_j(t) = \sum_k (y_k - t_k) \int_0^1 \partial_{y_k} \chi_j(st + (1-s)y) ds$$

ce qui donne

$$\Sigma_k(y, t, \xi) = \sum_j \xi_j \int_0^1 \partial_{y_k} \chi_j(st + (1-s)y) ds = \sum_k (y_k - t_k) \int_0^1 \xi_j \partial_{y_k} \chi_j(st + (1-s)y) ds.$$

La relation  $d\chi_j = \sum_k \partial_{y_k} \chi_j dy_k = \sum_k \chi'_{jk} dy_k$  donne alors  $\chi'_{jk} = \partial_{y_k} \chi_j$ , et comme

$$\Sigma_k(y, y, \xi) = \sum_j \xi_j \partial_{y_k} \chi_j = \sum_j \xi_j \chi'_{jk} = \sum_j ({}^t \chi')_{kj} \xi_j$$

on trouve la relation

$$\Sigma(y, y, \xi) = ({}^t \chi'(y)) \cdot \xi.$$

Cette notation est cohérente car  $\chi'$  est une matrice, et on l'applique à un élément  $\xi$  de  $\mathbb{R}^d$ . L'application  $\chi$  est un difféomorphisme, donc  $\chi'$  est une matrice inversible. Localement, au voisinage de la diagonale on peut retrouver  $\xi$  en fonction de  $\Sigma$  et de  $y$ . On peut alors considérer le système d'équations  $\Sigma(y, t, \xi) = \Sigma$ , et il existe une solution  $\xi = \Theta(y, t, \Sigma)$ . On calcule alors l'intégrale (6.98) par

$$(a(x, D)u)(\chi(y)) = \frac{1}{(2\pi)^d} \int \int e^{i(y-t)\Sigma} a(\chi(y), \Theta(y, t, \Sigma)) u(\chi(t)) \det(\chi'(t)) dt d\Sigma d\Theta(x, t, \Sigma).$$

Si  $d(y, t, \Sigma)$  est le symbole  $a(\chi(y), \Theta(y, t, \Sigma)) \det(\chi'(t)) d\Sigma \Theta(x, t, \Sigma)$ , une application du lemme 7.3 de réduction permet alors d'obtenir

$$(a(x, D)u)(\chi(y)) = \frac{1}{(2\pi)^d} \int \int e^{i(y-t)\Sigma} b(y, \Sigma) d\Sigma dt,$$

Le terme dominant de  $b_0$  est égal à

$$b_0(y, \Sigma) = a_0(\chi(y), ({}^t \chi')^{-1}(\Sigma)).$$

Solution de l'exercice 7.4. Nous ferons ici un usage constant du lemme 7.7 démontré dans l'exercice 7.2. En effet, une phase de la forme  $(x - y).\xi + \psi(y) - \psi(x)$  vérifie l'hypothèse dès que  $\xi_0 \neq \nabla\psi(y_0)$ .

1) On peut vérifier que, au sens des intégrales oscillantes

$$A \circ B(u)(x) = \frac{1}{(2\pi)^{2d}} \int \int \int \int e^{i(x-z).\xi + is(x,\xi) + i(z-y).\theta - is(y,\theta)} a(x,\xi) b(y,\theta) u(y) dy d\theta dz d\xi.$$

Nous appliquons la méthode du col (Théorème 4.3) à la phase totale de cet opérateur dans les variables  $z$  et  $\xi$ , qui est

$$\phi_1^{x,y,\theta}(z,\xi) = (x-z).\xi + (z-y).\theta + s(x,\xi) - s(y,\theta).$$

Le point critique est

$$\begin{cases} \xi_c = \theta \\ z_c = x + \partial_\xi s(x, \xi_c) \end{cases}$$

et la matrice jacobienne est

$$\begin{pmatrix} 0 & -Id \\ -Id & \text{Hess}_\xi s(x, \xi) \end{pmatrix}$$

qui est inversible (air désormais connu). La valeur critique de la phase est

$$(x - y)\theta + s(x, \theta) - s(y, \theta)$$

Le théorème de la phase stationnaire avec paramètre affirme l'existence d'un symbole  $L(a, b)(x, y, \theta)$  tel que

$$(A \circ B)(u)(x) = \int e^{i((x-y)\theta + s(x,\theta) - s(y,\theta))} L(a, b)(x, y, \theta) u(y) dy d\theta.$$

De plus, on vérifie, puisque les symboles et la phase ne dépendent pas de  $t$  dans l'égalité qui suit, que

$$L(a, b)(x, y, \theta) = \sum_\alpha \frac{1}{\alpha!} \partial_t^\alpha D_\beta^\alpha [a(x, \theta + \beta) b(y, \theta) e^{i[(x-y).\theta + s(x,\theta) - s(y,\theta)] + i[s(x,\theta + \beta) - s(x,\theta) - \beta \partial_\theta s(x, \theta)]}]|_{t=0, \beta=0}.$$

Il existe  $r \in S^{-1}(X \times Y \times \mathbb{R}^d)$  tel que

$$L(a, b)(x, y, \theta) = [a(x, \theta) b(y, \theta) + r(x, y, \theta)] e^{i(x-y).\theta + i(s(x,\theta) - s(y,\theta))}.$$

Nous appliquons l'**astuce de Kuranishi** (Lemme 7.7). On vérifie que, dans un voisinage de  $x$ , l'égalité  $\theta + V(x, y, \theta) = \Sigma$  est inversible, et on peut trouver  $\theta = \theta(x, y, \Sigma)$ . On voit alors que

$$(A \circ B)(u)(x) = \int e^{i(x-y)\Sigma} L(a, b)(x, y, \theta(x, y, \Sigma)) \left| \frac{\partial \theta}{\partial \Sigma} (x, y, \Sigma) \right| dy \wedge d\Sigma.$$

Le lemme de réduction 7.3 implique l'existence de  $c(x, \Sigma)$  tel que  $A \circ B = Op(c)$ ,  $c$  étant un symbole dont l'ordre est la somme des ordres de  $a$  et de  $b$ . On trouve

$$c(x, \theta) = \sum_\alpha \frac{1}{\alpha!} D_\Sigma^\alpha \partial_y^\alpha a(x, \theta(x, y, \Sigma)) b(y, \theta(x, y, \Sigma))|_{y=x}.$$

Egaler le symbole principal à 1 et tous les autres à 0 se fait de proche en proche; ici on identifie  $b_0$  en fonction de  $(a_0)^{-1}$ , qui est bien défini puisque  $a_0$  est le symbole principal d'un opérateur elliptique. On procède de proche en proche pour identifier  $b_j$ . De cette façon, on construit un symbole  $b$  tel que le symbole de  $A \circ B$  soit  $\simeq 1 \pmod{S^{-\infty}}$ . Ceci achève la preuve du résultat de l'exercice 7.3. Ceci est la même démarche que celle employée dans la démonstration de la proposition 7.6.

2) On applique la même méthode de calcul pour évaluer le symbole principal de l'opérateur  $A \circ P \circ B$ , où on connaît le symbole principal de  $P$ . Le symbole principal obtenu après l'application du théorème de la phase stationnaire est

$$a(x, \theta) b(y, \theta) p(x + \nabla_\theta s(x, \theta), \theta)$$

et la phase que nous considérons est

$$(x - y)\theta + s(x, \theta) - s(y, \theta)$$

L'astuce de Kuranishi conduit au changement de variable

$$\Sigma = \theta + \nabla_x s(x, \theta).$$

Le symbole principal obtenu pour l'opérateur  $A \circ P \circ B$  est donc

$$q(x, \Sigma) = p(x + \nabla_{\theta} s(x, \theta), \theta), \Sigma = \theta + \nabla_x s(x, \theta).$$

Nous reviendrons sur ce résultat dans la partie consacrée à la géométrie.



## Operators and symplectic geometry

In this chapter, we study some objects related to operators: the characteristic set, the bicharacteristic flow. We show that the natural framework in which this is possible is called symplectic geometry (essentially thanks to the proposition 7.11 which shows that a change of variable in  $y$  induces a change of variable in  $\xi$ ). We deduce results on the integral operators, in particular we introduce the canonical relation associated to a Fourier integral operator. This part is very important for the general analysis of a partial differential equation, since any property of the symbol of the operator will be true for any similar operator (obtained by conjugation by an invertible operator or by diffeomorphism). In particular, the bicharacteristic flow is the generalization of the rays of geometric optics studied in chapter 3.

The conjugation of a pseudodifferential operator by a Fourier integral operator leads to a new pseudodifferential operator, and the above-mentioned geometric objects (which are defined on a variable space of dimension  $2d$ ) of the old operator and the new operator are deduced from each other by a transformation adapted to symplectic geometry which is the canonical relation.

### 1. Solutions of a pseudo-differential equation

Consider a pseudodifferential operator  $P$  on  $\mathbb{R}^d$  of order  $m$ , properly supported, of symbol  $p(x, \xi) = \sum_{j=-N}^m p_j(x, \xi) \in S^m(\mathbb{R}^{2d})$ , each  $p_j$  being homogeneous of degree  $j$ . The following proposition allows to compute the action of  $P$  on a function  $x \rightarrow u(x, k)$  which has an asymptotic expansion.

**PROPOSITION 8.1.** *Let  $u(x, k) = a(x, k)e^{ik\phi(x)}$ , such that there exist functions  $(a_j)_{0 \leq j}$  of class  $C^\infty$  in  $x$  such that*

$$a(x, k) \simeq \sum_{j \geq 0} a_j(x)(ik)^{-j}.$$

*There exist functions  $b_j(x)$ , of class  $C^\infty$  in  $x$ , such that*

$$e^{-ik\phi(x)}(ik)^{-m}P(u)(x, k) \simeq \sum_{j \geq 0} b_j(x)(ik)^{-j}.$$

*More precisely, the  $b_j$  are the terms in the expansion of*

$$(1.99) \quad \sum_{l \geq 0} \frac{1}{l!(ik)^l} \sum_{|\alpha|=l} \partial_{u^\alpha} [e^{ikuA(x, x+u)u} a(x+u, k)]|_{u=0} \partial_{\xi^\alpha} p(x, \nabla_x \phi(x), k).$$

*One checks that  $b_0(x) = a_0(x)p(x, \nabla \phi(x))$  and that  $b_1(x) = a_1(x)p(x, \nabla \phi(x)) + (\frac{1}{2} \sum_{i,j} \frac{\partial^2 p}{\partial \xi_i \partial \xi_j}(x, \nabla \phi(x)) a_0(x) + \sum_i \frac{\partial p}{\partial \xi_i}(x, \nabla \phi(x)) \partial_j a_0(x))$ .*

**PROOF.** Let us first consider a particular case.

**A linear phase and a polynomial symbol.** We choose  $\phi(y) = y \cdot \eta_0$ ,  $\eta_0 \in \mathbb{R}^d \times \{0\}$ ,  $p(y, \eta) = \eta^\alpha$ ,  $a(y, k) = 1$ . We find

$$\int_{\mathbb{R}^{2d}} e^{ik((x-y)\eta + y \cdot \eta_0)} \eta^\alpha dy d\eta = e^{ikx \cdot \eta_0} \int_{\mathbb{R}^{2d}} e^{ik(x-y)(\eta - \eta_0)} \eta^\alpha dy d\eta.$$

Let  $u = y - x$  and  $\xi = \eta - \eta_0$ .



$$\int_{\mathbf{R}^{2d}} e^{ik((x-y)\eta+y\cdot\eta_0)} \eta^\alpha dy d\eta = e^{ikx\cdot\eta_0} \int_{\mathbf{R}^{2d}} e^{-iku\xi} (\eta_0 + \xi)^\alpha dud\xi.$$

We notice then that, when  $\beta \neq 0$ ,  $\int_{\mathbf{R}^{2d}} \xi^\beta e^{-iku\xi} dud\xi = 0$ . Remark that  $\int_{\mathbf{R}^{2d}} e^{-iku\xi} dud\xi = \int_{\mathbf{R}^d} \hat{1}(k\xi) d\xi$  (using a formal notation for the integral of a Fourier transform in  $\mathcal{S}'$ ). Hence again  $\int_{\mathbf{R}^d} \hat{1}(k\xi) d\xi = k^{-d} \int_{\mathbf{R}^d} \hat{1}(\xi) d\xi = (\frac{2\pi}{k})^d$ . As the Fourier transform of 1 at  $k\xi_j$  is the distribution  $\delta_0$ , we obtain  $\langle \hat{1}(k\xi_j), \xi_j^{\beta_j} \rangle = 0$ .

Thus  $\int_{\mathbf{R}^{2d}} e^{-iku\xi} (\eta_0 + \xi)^\alpha dud\xi = \eta_0^\alpha \int_{\mathbf{R}^{2d}} e^{-iku\xi} dud\xi = (\frac{2\pi}{k})^d \eta_0^\alpha$ , because we keep only the 0 order term in  $\xi$ .

We deduce

$$\int_{\mathbf{R}^{2d}} e^{ik((x-y)\eta+y\cdot\eta_0)} \eta^\alpha dy d\eta = (\frac{2\pi}{k})^d e^{ikx\cdot\eta_0} \eta_0^\alpha.$$

We use for that the Fourier integral operator representation of the pseudo-differential operator  $P$ . We evaluate

$$\int_{\mathbf{R}^{2d}} e^{i(x-y)\xi + ik\phi(y)} p(x, \xi) a(y, k) dy d\xi.$$

The change of variable  $\xi = k\eta$  yields

$$\int_{\mathbf{R}^{2d}} e^{i(x-y)\xi + ik\phi(y)} p(x, \xi) a(y, k) dy d\xi = k^{m+d} \int_{\mathbf{R}^{2d}} e^{ik((x-y)\cdot\eta + \phi(y))} p(x, \eta, k) a(y, k) dy d\eta,$$

where  $p(x, \eta, k) = \sum_{-N \leq l \leq m} p_l(x, \eta) k^{-l}$ . We evaluate this integral by the stationary phase method. The gradient of the phase in  $(y, \eta)$  is

$$(-\xi + \nabla_y \phi(y), x - y).$$

We generalize this result. To apply the method of the stationary phase, we come back to the critical points on a compact set. More precisely, it is enough to reduce to a compact set in  $y$  and to a conic neighborhood in  $\eta$ . This is done in the exercise 8.0.

The Jacobian of the phase is

$$\begin{pmatrix} -\text{Hess}\phi(y) & -\text{Id} \\ -\text{Id} & 0 \end{pmatrix}.$$

It is invertible thanks to the formula

$$\begin{pmatrix} -\text{Hess}\phi(y) & -\text{Id} \\ -\text{Id} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\text{Id} \\ -\text{Id} & \text{Hess}\phi(y) \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}.$$

The determinant of the Jacobian is equal to  $(-1)^d$ . We can apply the stationary phase theorem in the neighborhood of the critical point  $(x, \nabla_x \phi(x))$  in  $\mathbf{R}_y^d \times \mathbf{R}_\xi^d$ . The value of the critical phase is  $\phi(x)$ , and the integral admits an expansion in  $(y, \eta)$  whose first term is  $p_m(x, \nabla_x \phi(x)) (\frac{2\pi}{k})^{\frac{2d}{2}} a_0(x)$ . We propose here a method based on the lemma 7.5 to compute the functions  $b_j$  resulting from the stationary phase expansion. We use the classical change of variable in the neighborhood of the critical point  $y = x$ ,  $\xi = \nabla_y \phi(x)$ . Indeed, it will be easier to express the stationary phase expansion operator in the vicinity of the point  $(0, 0)$ . We write

$$y = x + u, \xi = \nabla_x \phi(x) + \theta.$$

Then there exists (Taylor expansion with integral remainder) a matrix  $A(x, y)$  such that

$$\phi(y) - \phi(x) - (y - x) \cdot \nabla_x \phi(x) = (y - x) A(x, y) (y - x).$$

The integral calculated is

$$k^{m+d} \int_{\mathbf{R}^{2d}} dud\theta e^{ik[(-u) \cdot (\nabla_x \phi(x) + \theta) + \phi(x) + u \cdot \nabla_x \phi(x) + u A(x, x+u) u]} a(x + u, k) p(x, \nabla_x \phi(x) + \theta, k),$$

or

$$k^{m+d} e^{ik\phi(x)} \int_{\mathbf{R}^{2d}} dud\theta e^{-iku\theta} e^{ikuA(x, x+u)u} a(x + u, k) p(x, \nabla_x \phi(x) + \theta, k).$$

We have already evaluated in the lemma 7.5 this oscillating integral, where the symbol is  $a(x, u, \theta) = p(x, \nabla_x \phi(x) + \theta, k) a(x + u, k) e^{ikuA(x, x+u)u}$ . The expression (1.99) follows and we

have explicitly all the terms of the asymptotic expansion. By evaluating the derivatives in  $u$  of  $e^{ikuA(x,x+u)u}$ , we show that the term generated by the derivative  $\partial_{x^\alpha}^\alpha$  on  $e^{ikuA(x,x+u)u}$  is of order  $k^{m+d-\frac{\alpha}{2}}$ . The sum of these terms then defines an asymptotic series (as in the proof of the proposition 7.11). This ends the proof of Proposition 8.1.  $\square$

An easy consequence of this Proposition is

LEMMA 8.1. *If the function  $u$  is a non-trivial asymptotic solution of  $P$ , i.e.  $P(u) \simeq 0$ , then  $\{(x, \nabla_x \phi(x))\} \subset \text{Car}(p) \cap (\text{supp} a_0 \times \mathbb{R}^d)$ .*

There is a generalization of this lemma for Fourier integral operators, as in [47]. This generalization is used in the chapter 9 which studies the caustics. We state it in the Proposition:

PROPOSITION 8.2. *Let  $P$  be a pseudo-differential operator of order  $m$  and let  $A$  be a Fourier integral operator (see definition 6.4) of phase  $s(x, \theta)$  and symbol  $a(x, \theta) \in S^p(\mathbb{R}^d \times \mathbb{R}^d)$ . Then*

$$P(A(u))$$

*defines an Fourier integral operator of the same phase and of symbol  $b(x, \theta) \in S^{m+p}$ . Each term  $b_j$  is computed using the terms of  $P$  and of  $a$  of sum of orders  $\leq j$ .*

This proposition is a consequence of exercise 7.3. It is stated in the article of Duistermaat and Hörmander[32] in the theorem 4.2.2.2. We will see later that the canonical relation  $\mathcal{C}$  of  $PA$  is the canonical relation of  $A$ , if it exists.

PROOF. We detail this proof to show once again a formal application (the word formal is due to the fact that we do not study here the behavior in  $k$  of each term of the formal expansion obtained, but this expansion can be justified) of the stationary phase theorem. For this purpose, let us write the oscillatory integral

$$(1.100) \quad Pv(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{v}(\xi) d\xi = \oint e^{i(x-z)\xi} p(x, \xi) v(z) dz d\xi.$$

We then write the composition of operators

$$(1.101) \quad P(Au)(x) = \int \int e^{i(x-z)\xi + s(z, \theta) - y \cdot \theta} p(x, \xi) a(z, \theta) u(y) dz d\theta dy d\xi.$$

The associated phase is  $\phi(x, y, z, \xi, \theta) = (x - z)\xi - y\theta + s(z, \theta)$ . To reduce to a Fourier integral operator, whose phase depends only on  $(x, \theta, y)$ , we apply the stationary phase method in the variables  $(z, \xi)$ . The critical point  $(z_c, \xi_c)$  is solution of

$$\begin{cases} -\xi_c + \partial_z s(z_c, \theta) = 0 \\ z_c = x \end{cases}$$

and the Hessian of the phase is

$$\begin{pmatrix} \text{Hess}_{s_z}(z, \theta) & -Id \\ -Id & 0 \end{pmatrix}$$

This phase is non-degenerate, of critical value  $s(x, \theta) - y\theta$ . Thus, using the stationary phase theorem (Proposition 4.3) and introducing the Laplacian associated to the phase, we have

$$\Delta_s a(z, \xi) = \text{Tr} \left( \begin{pmatrix} 0 & Id \\ Id & \text{Hess}_{z_s}(z, \theta) \end{pmatrix} \begin{pmatrix} \partial_{z_2}^2 & \partial_{z_2 \xi}^2 \\ \partial_{z_2 \xi}^2 & \partial_{\xi^2}^2 \end{pmatrix} \right) a = 2\partial_{z_2 \xi}^2 a + \text{Hess}_{s_s}(z, \theta) \partial_{\xi^2}^2 a.$$

The operator used in the stationary phase expansion is thus

$$(i)^{-1} [\partial_{z_2 \xi}^2 + \frac{1}{2} \text{Hess}_{s_s}(z, \theta) \partial_{\xi^2}^2].$$

The integral (1.101) rewrites

$$P(Au)(x) = \int e^{is(x, \theta) - y\theta} B(x, y, \theta) u(y) dy d\theta$$

where the symbol  $B$  is equal to

$$B(x, y, \theta) = \sum_l (l!)^{-1} (i)^{-l} [\partial_{z\xi}^2 + \frac{1}{2} \text{Hess}_s(z, \theta) \partial_{\xi^2}^2]^l (p(x, \xi) a(z, \theta))|_{z_c=x, \xi_c=\nabla_z s(z_c, \theta)}.$$

This symbol is of the form  $b(x, \theta)$ . It does not depend on  $y$  (which is not always the case in the application of the stationary phase theorem).  $\square$

REMARK 7. From the relations  $\partial_{\xi_j \xi_k}^2 (p(x, \xi) a(z, \theta)) = \partial_{\xi_j \xi_k}^2 p(x, \xi) a(z, \theta)$  et  $\partial_{z_j \xi_j}^2 (p(x, \xi) a(z, \theta)) = \partial_{\xi_j} p(x, \xi) \partial_{z_j} a(z, \theta)$ , one deduces the term of order  $-1$ , which is

$$b_1(x, \theta) = \left[ \sum_j \partial_{\xi_j} p(x, \xi) \partial_{z_j} a(z, \theta) + \frac{1}{2} \left( \sum_{j,k} \partial_{\xi_j \xi_k}^2 p(x, \xi) \partial_{z_j z_k}^2 s(z, \theta) a(z, \theta) \right) \right].$$

It is an operator of order 1 on a whose coefficients are those of the transport operator  $\nabla_\xi p \cdot \nabla_z$ . In this operator appears  $\text{Hess} p \cdot \text{Hess} s$  which corresponds to the term  $\Delta \phi a_0$  in the transport equation for the Helmholtz equation.

PROOF. We continue the calculation of (1.101). In the  $\phi$  phase, the change of variable

$$(z, \xi) = (x + u, \nabla_z s(x, \theta) + \eta)$$

leads to the phase  $-u \cdot \eta + [s(x + u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta$ . Let

$$(1.102) \quad \tilde{p}(x, \theta; u, \eta) = p(x, \nabla_z s(x, \theta) + \eta) a(x + u, \theta) e^{i([s(x+u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta)}.$$

It is a symbol of  $S_{1,0}^m((u, \eta) \in \mathbb{R}^{2d})$ , the points  $x$  and  $\theta$  being fixed, since the derivatives in  $u$  do not involve additional powers of  $\eta$ . The asymptotic expansion of the integral  $\int e^{-iu \cdot \eta} \tilde{p}(x, \theta; u, \eta) du d\eta$  exists through the lemma 7.5. It is necessary to complete this proof, we have to calculate the symbol obtained in the variable  $\theta$ . Indeed, a derivation in  $u$  of the symbol  $\tilde{p}$  leads to additional powers of  $\theta$ . The result is true thanks to a method analogous to the one used in the proof of the proposition 7.10.

We have

$$(1.103) \quad A_1 = \partial_{\eta u}^2 \tilde{p}(x, \theta; u, \eta) = \sum_j \frac{\partial p}{\partial \xi_j} (x, \nabla_z s(x, \theta) + \eta) \left[ \frac{\partial a}{\partial x_j} (x + u, \theta) + i a(x + u, \theta) \left( \frac{\partial s}{\partial x_j} (x + u, \theta) - \frac{\partial s}{\partial x_j} (x, \theta) \right) \right] \times e^{i([s(x+u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta)}.$$

The symbol involved in (1.103) is of the form

$$r(x, u, \theta, \eta) = \sum_j \frac{\partial p}{\partial \xi_j} (x, \nabla_z s(x, \theta) + \eta) a(x, u, \theta) e^{i([s(x+u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta)}.$$

From the equality

$$r(x, 0, \theta, 0) = \sum_j \frac{\partial p}{\partial \xi_j} (x, \nabla_z s(x, \theta)) \frac{\partial a}{\partial x_j} (x, \theta) e^{-iy \cdot \theta},$$

we deduce that in the term  $A_1$  there is no term induced by the phase  $s$ . We verify that  $A_1$  is of order  $m - 1$  because the derivative of a homogeneous symbol of  $S_{1,0}^m$  is of order  $m - 1$ .

The phase  $s$  appears in the following terms of the stationary phase expansion. Indeed, let us introduce

$$A_2 = \sum_l \frac{\partial^2}{\partial \eta_l \partial u_l} \left[ \sum_j \frac{\partial p}{\partial \xi_j} (x, \nabla_z s(x, \theta) + \eta) [a(x, u, \theta) e^{i([s(x+u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta)}] = \sum_{j,l} \frac{\partial^2 p}{\partial \xi_j \partial \xi_l} (x, \nabla_z s(x, \theta) + \eta) \left[ \frac{\partial a_{1,j}}{\partial u_l} (x, u, \theta) + a_{1,j}(x, u, \theta) \left( \frac{\partial s}{\partial x_l} (x + u, \theta) - \frac{\partial s}{\partial x_l} (x, \theta) \right) \right] e^{i([s(x+u, \theta) - u \nabla_z s(x, \theta)] - y \cdot \theta)}.$$

One finds

$$\frac{\partial a}{\partial u_l} (x, u, \theta) = \frac{\partial^2 a}{\partial x_j \partial x_l} (x + u, \theta) + a(x + u, \theta) \frac{\partial^2 s}{\partial x_j \partial x_l} (x + u, \theta).$$

In this term, appears the second derivative of  $s$  in  $u$ , which is of order 1 because  $s$  is homogeneous of order 1 in  $\theta$ . Thus, this term appears in the term of order  $m - 1$  of the

symbol  $a$  because it is a term of order  $m - 2 + 1 = m - 1$  (the  $m - 2$  comes from the second derivative of the symbol  $p$ ). The term  $A_2$  is thus a term of homogeneity  $m - 1$ .

We recall the expressions

$$A_1(x, \theta) = \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial_x^\alpha a(x, \theta) \partial_\theta^\alpha p(x, \theta) = \sum_j \partial_{x_j} a \partial_{\theta_j} p,$$

$$A_2(x, \theta) = \sum_{|\alpha|=2} [\frac{1}{\alpha!} \partial_x^\alpha a(x, \theta) + a(x, \theta) \partial_z^\alpha s(x, \theta)] \partial_\theta^\alpha p(x, \theta) = \frac{1}{2} \sum_{i,j} (\frac{\partial^2 a}{\partial x_j \partial x_k} a + a \frac{\partial^2 s}{\partial x_j \partial x_k} \frac{\partial^2 p}{\partial \theta_j \partial \theta_k}).$$

We show in what follows that the term  $A_j$  of the stationary phase expansion is of order of homogeneity  $\leq m - [\frac{j+1}{2}]$  in the variable  $\theta$ . The asymptotic sum exists by the asymptotic completeness theorem (Proposition 5.3). This proves the proposition 8.2. **Treatment of the term  $A_j$ .** In the general case, we have  $A_j = \sum_{|\alpha|=j} D_\alpha$ , avec

$$\begin{aligned} D_\alpha(x, \theta) &= \frac{1}{\alpha!} \partial_u^\alpha \partial_\eta^\alpha [p(x, \nabla_z s(x, \theta) + \eta) a(x + u, \theta) e^{i(s(x+u, \theta) - u \nabla_z s(x, \theta))}] \\ &= \sum_{\alpha' + \alpha'' = \alpha} \frac{1}{\alpha'! \alpha''!} \partial_\eta^\alpha p(x, \nabla_z s(x, \theta)) \partial_x^{\alpha'} a(x, k) \partial_u^{\alpha''} [e^{i(s(x+u, \theta) - u \nabla_z s(x, \theta))}]. \end{aligned}$$

One uses the equality

$$\partial_{u_j} [e^{i(s(x+u, \theta) - u \nabla_z s(x, \theta))}] = i \left( \frac{\partial s}{\partial x_j}(x + u, \theta) - \frac{\partial s}{\partial x_j}(x, \theta) \right) [e^{i(s(x+u, \theta) - u \nabla_z s(x, \theta))}]$$

from which one deduces, using  $t(u, x, \theta) = e^{i(s(x+u, \theta) - u \nabla_z s(x, \theta))}$ , the identity

$$\partial_{u_j} t = i u_j w(u, x, \theta) t(u, x, \theta).$$

Hence, for  $|\alpha'| \geq 2$ , one finds, if  $\alpha'$  contains only derivatives of order 1,  $\partial_{u^{\alpha'}} t(0, x, \theta) = 0$ , and if  $\alpha'$  contains a derivative of order 2, one finds

$$\partial_{u^{\alpha'}} t(0, x, \theta) = \sum_{\text{derivatives of order 2}} b_j \frac{\partial^2 s}{\partial u_j^2}.$$

The successive calculation of all the terms is a very technical application of lemma 7.5.  $\square$

## 2. Change of variable and geometrical objects

We define a first geometrical object: the bicharacteristics. They are defined in the space of positions and impulsions associated to the space of symbols. Remark that it is a classical presentation in the wave propagation theory, it corresponds in the geometrical and mathematical context to the cotangent bundle. They are the generalization of the rays of geometric optics introduced in chapter 3, in the sense that the bicharacteristics associated to the operator  $\xi^2 - k^2$  are the curves whose first coordinates are the rays and where the second coordinates are the vectors  $\vec{p}$  (of modulus  $k$ ) giving the direction of the rays. The impulse is then  $\vec{p}$  by analogy with the Hamiltonian  $|\vec{p}|^2 - k^2$ .

**2.1. The bicharacteristics.** Let us start with an example from the result of Section 3, Chapitre 1.

**PROPOSITION 8.3.** *Consider a solution of the Helmholtz equation  $(\Delta + k^2)u(x, k) = 0$  associated with the 'initial' condition  $u(x, k) = A(x, k)e^{ik\phi_0}$ ,  $x \in \Sigma_0$ .*

• *For each  $(x(y), u)$  in  $\Sigma_0 \times (-a, a)$ ,  $a_0(x(y) + uN(x(y))) = A_0(x(y))(\det(Id + uW(x(y))))^{-\frac{1}{2}}$ ,  $\phi((x(y) + uN(x(y)))) = \phi_0 + u$  (proven in Section 3 of Chapter ??).*

• *Let  $v_0$  be the inverse Fourier transform in  $k$  of  $a_0(x)e^{ik\phi(x)}$ . Its wave front set is*

$$(2.104) \quad WF(v_0) \cap B(x_0, \varepsilon) = \{(x(y) + uN(x(y)), \phi_0 + u, -\tau N(x(y)), \tau)\} \cap B(x_0, \varepsilon).$$

**PROOF.** In section3 Chapitre 1, we have computed the formal asymptotic solution of  $(\Delta + k^2)u(x, k) = 0$  under the condition (which we named initial condition)  $u(x, k) = A(x, k)e^{ik\phi_0}$ ,  $x \in \Sigma_0$ . It was thus a constant phase on the manifold  $\Sigma_0$  of codimension 1. We suppose  $A(x, k) \simeq \sum_j A_j(x)(ik)^{-j}$ . To fix our ideas, let us consider a point  $x_0 \in \Sigma_0$ . Then, for all  $x \in \Sigma_0 \cap B(x_0, \varepsilon)$ , small enough, we can define a local coordinate system  $(y_1, y_2)$  on  $\Sigma_0$  such that  $x = x(y)$ . For  $u$  small enough (so that the matrix  $\det(Id + uW(x))$  is invertible for  $x \in \Sigma_0 \cap B(x_0, \varepsilon)$ ,  $W$  designating, as we recall, the matrix of curvatures or Weingarten

matrix of the  $\phi$  phase solution of  $|\nabla\phi| = 1$ ,  $\phi|_{\Sigma_0} = \phi_0$  on the isophase  $\Sigma_0$ ), we can define a diffeomorphism of a neighborhood of  $x_0$  in  $\mathbf{R}^3$  by the relation

$$(2.105) \quad x(y, u) = x(y) + uN(x(y)).$$

One thus knows that

$$a_0(x(y, u)) = A_0(x(y))(\det(Id + uW(x(y))))^{-\frac{1}{2}}$$

and

$$\phi(x(y, u)) = \phi_0 + u.$$

This is the statement of the first item of the Proposition.

Let  $u_0(x(y, u), k) = a_0(x(y, u))e^{ik(\phi_0+u)}$ . This is the term of the formal asymptotic solution. As (2.105) defines a diffeomorphism, the function  $\phi$  such that  $\phi_0 + u = \phi(x(y, u))$  is a well defined function. Let

$$(2.106) \quad u_0(x, k) = a_0(x)e^{ik\phi(x)}.$$

The inverse Fourier transform in  $k$  of (2.106) is the distribution (formal notation)

$$v_0(x, t) = a_0(x) \int_{\mathbf{R}} e^{ik\phi(x)+ikt} dk = a_0(x)\delta(t - \phi(x)).$$

Recall that the wave front set of a distribution is the complementary of the set of points  $(x_0, t_0, \xi_0, \tau_0)$  such that there exists  $x_0$  located in the neighborhood of  $(x_0, t_0)$  such that  $\mathcal{F}(\chi v_0)$  is fast decaying in a neighborhood of  $(\xi_0, \tau_0)$ . We suppose that  $a_0(x_0) \neq 0$ . There exists a neighborhood  $B(x_0, \varepsilon)$  such that  $|a_0(x)| \geq \frac{1}{2}|a_0(x_0)|$  on this neighborhood (and therefore is non-zero). We choose  $\chi$  so that it has its support contained in this neighborhood. When  $t_0 \neq \phi(x_0)$ , we can choose  $\chi$  so that  $\chi v_0$  is identically zero, so the points  $(x_0, t_0, \xi_0, \tau_0)$ ,  $t_0 \neq \phi(x_0)$  are not in the wave front set of  $v_0$ . The Fourier transform of  $\chi v_0$  is (with a clear abuse of notation)

$$I(\xi, \tau) = \int_{\mathbf{R}^3} a_0(x)e^{-i\xi \cdot x} \int_{\mathbf{R}} \delta(t - \phi(x))e^{-it\tau} dt dx = \int_{\mathbf{R}^3} a_0(x)e^{-i\xi \cdot x - i\tau\phi(x)} dx.$$

The diffeomorphism (2.105) yields

$$\int_{\mathbf{R}^3} dud\sigma_{\Sigma_0}(y)\det(Id + uW(x(y)))a_0(x(y, u))e^{ix(y) \cdot \xi + iuN(x(y)) \cdot \xi + i\tau\phi_0 + i\tau u}.$$

Its phase has a stationary point  $\xi$  orthogonal to  $\Sigma_0$  and with  $N(x(y)) \cdot \xi + \tau = 0$ . It yields  $\xi = \lambda N(x(y))$  et  $\tau = -\lambda$ . The points where the phase is not stationary correspond to the points  $(x(y, u), \phi_0 + u, \xi, \tau)$  which are not in the wave front set of  $v_0$ . The wave front set of  $v_0$  thus is given by (2.104). This ends the proof  $\square$

We give the definition of bicharacteristic curves:

DEFINITION 8.1. *The bicharacteristic curves of the differential operator  $P$  (or of the homogeneous symbol  $p$ , principal symbol of  $P$ ) are the integral curves of the Hamiltonian vector field  $H_p \in T(T^*\mathbf{R}^d)$  associated to the symbol  $p$  on  $T^*\mathbf{R}^d$ :*

$$(2.107) \quad H_p = \sum_{j=1}^{j=d} \frac{\partial p}{\partial \xi_j}(x, \xi) \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}(x, \xi) \frac{\partial}{\partial \xi_j}$$

denoted also by

$$H_p = \left( \frac{\partial p}{\partial \xi}, -\frac{\partial p}{\partial x} \right).$$

They satisfy

$$(2.108) \quad \begin{cases} \frac{d}{ds}((x(s), \xi(s))) = H_p((x(s), \xi(s))) \\ (x(0), \xi(0)) = (x_0, \xi_0). \end{cases}$$

This definition will be clearer from a geometrical point of view in the section 3 devoted to the symplectic geometry. In the simple example of the wave operator, its symbol is  $p(t, x, \tau, \xi) = \tau^2 - \xi^2$ . The associated Hamiltonian is

$$H_p = (2\tau, -2\xi, 0, 0).$$

The integral curves of  $H_p$  verify  $\tau = \tau_0, \xi = \xi_0$ . One has, of course  $\tau^2 - \xi^2 = \tau_0^2 - \xi_0^2 = 0$ . Then  $t(s) = t_0 + 2\tau_0 s, x(s) = x_0 - 2s\xi_0$ . One deduces easily that

- the principal symbol is invariant on the bicharacteristics,
- the solution phase of the eikonal equation can be computed on the bicharacteristics of the d'Alembert operator. Indeed, the phase of the solution of the Helmholtz equation obtained after Fourier transformation in time of the wave equation is  $\phi(s') = \phi_0 + s'$  and  $x(s') = x_0 + s'\nabla\phi(x_0)$ , with  $\|\nabla\phi(x_0)\| = 1$ . We take  $s' = -2s\frac{\xi_0}{\tau_0}$  to check that the projection on the physical space of the bicharacteristics of the d'Alembert operator is coincides with the characteristics and that the phase associated with the Dalemberian, equal to  $\psi(x, t) = \phi_0 + s - t$ , is computed along the bicharacteristics.

**2.2. Change of variable and transformation of the wave front set, the bicharacteristics and the eikonal phase.** In this section we use the change of variable proposition 7.11 which gives the relation between the change of variable in  $X \subset \mathbb{R}^d$  and the change of variable induced in  $X \times \mathbb{R}^d \subset \mathbb{R}^{2d}$ . We have the

PROPOSITION 8.4. *Let  $\chi$  be a diffeomorphism of  $\mathbb{R}^d$  in  $\mathbb{R}^d$ , such that  $\chi(x_0) = y_0$ . We define a diffeomorphism  $h_\chi$  of  $\mathbb{R}^d$  in  $\mathbb{R}^d \times \mathbb{R}^d$  by*

$$h_\chi(x, \xi) = (\chi(x), {}^t\chi'(x)^{-1}\xi).$$

We have

- 1) *Invariance of the wave front set: For all  $u$  distribution in  $\mathcal{S}'(\mathbb{R}^d)$*

$$WF(u \circ \chi^{-1}) = h_\chi(WF(u))$$

- 2) *Invariance of bicharacteristics and of the phase solution of the eikonal equation: For all  $P$  be a classical pseudodifferential operator of principal symbol  $p_m(x, \xi)$  homogeneous of degree  $m$ , define the classical pseudodifferential operator  $Q$  (thanks to proposition 7.11) by*

$$Q(v) \circ \chi^{-1} = P(v \circ \chi^{-1}),$$

*of principal symbol  $q_m(y, \eta)$ , where  $y = \chi(x), \eta = {}^t\chi'(x)^{-1}\xi$ .*

a) *If  $(x(s), \xi(s))_s$  is an integral curve of the Hamiltonian field  $H_{p_m}$ , then  $h_\chi(x(s), \xi(s))$  is the integral curve of the Hamiltonian field  $H_{q_m}$  passing through  $h_\chi(x(0), \xi(0))$ .*

b) *If  $\phi$  is a solution of the eikonal equation associated with  $p_m$ , then  $\phi \circ \chi^{-1}$  is a solution of the eikonal equation for  $q_m$ .*

The proof is the subject of exercise 1 of this section. Let us note that this result can be written locally, with some precautions.

In what follows, we generalize the approach already used in the section expressing asymptotic solutions of the the Helmholtz equation (Section 1.4), or when we expressed in chapter 2 the solution of a hyperbolic matrix problem of order 1 and its eikonal equation (2.28). We use here the result proved in Section 1, in particular paragraph 2) b) of the proposition 8.4. We generalize the notion of eikonal equation. Indeed, we have shown that, for the Helmholtz equation as for a hyperbolic matrix problem with variable coefficients, there exists a scalar partial differential equation, for which a function  $\phi$  **phase function** is solution, equation that is called **eikonal equation**. We have seen in the lemma 8.1 that we could also introduce an eikonal equation for a usual pseudodifferential operator thanks to the operators integral Fourier operators. Finally, the proposition 8.4 shows that the set of points  $(x, \nabla_x\phi)$  where  $\phi$  is a solution of the eikonal equation associated to the pseudo-differential operator of principal symbol  $p_m$  is transformed in the same way as the bicharacteristic curves when we consider the operator  $Q$  such that  $Q(u\chi^{-1}) = (Pu) \circ \chi^{-1}$ . These intrinsic considerations lead us to study

not the phase itself, but the sets  $\{(x, \nabla_x \phi)\}$  and to generalize them. Such sets are **Lagrangian** manifolds, which are **Lagrangian** solutions of the characteristic equation  $p_m(x, \xi) = 0$ .

We introduce the geometrical framework in which these sets are well defined, it is the **symplectic geometry**. In particular, we will lead to a rigorous and intrinsic definition of the object introduced in the Definition 8.1 by the relation (2.107). We rely here on J. Sjöstrand's course [43]. The interested reader may refer to Stenberg's treatise [92].

### 3. Symplectic geometry

We choose in this work to introduce the cotangent bundle as the space of jets on a manifold. It is also natural to define the tangent bundle first.

**3.1. Geometric definition of the cotangent bundle.** Let  $X$  be a  $C^\infty$  manifold of dimension  $d$ .

**DEFINITION 8.2.** *The cotangent bundle of  $X$  in  $x_0$ , denoted  $T_{x_0}^*(X)$ , is the set of equivalence classes of the equivalence relation  $f \sim g \Leftrightarrow f - g = o(|x - x_0|)$  in the set of functions vanishing at  $x_0$ , and the element associated to  $f \in C^1(X, \mathbb{R})$  is denoted by  $df$ .*

$$T_{x_0}^*(X) = \{f \in C^1(X, \mathbb{R}), f(x_0) = 0\} / \sim$$

The element  $df$  is then the first order germ of  $f$ . This definition is abstract; it has the merit of being intrinsic from the geometrical point of view.

To make it more explicit, place ourselves in a local coordinate system. For each point  $x_0$  of  $X$ , there exists a neighborhood  $W$  of this point and a (well, non-unique) local coordinate system  $(x_1, \dots, x_d)$ . The natural basis of  $T_{x_0}^*(X)$  associated to this local coordinate system is then  $(dx_1, \dots, dx_d)$ . Indeed, let  $f \in C^1(X, \mathbb{R})$ . Then there exists a diffeomorphism  $\chi$  of  $X$  in  $\mathbb{R}^d$  such that a point  $u$  of  $X \cap W$  is written  $\chi(u) = (x_1, \dots, x_d)$ . Then, by Taylor's formula, the function of  $\mathbb{R}^d$  in  $\mathbb{R}$  equal to  $f \circ \chi^{-1}$  verifies

$$(3.109) \quad (f \circ \chi^{-1})(x) = f(x_0) + \sum_{j=1}^{j=d} \frac{\partial}{\partial x_j} (f \circ \chi^{-1})(\chi(x_0))(x_j - (\chi(x_0))_j) + o(|x - \chi(x_0)|).$$

Noting that  $x_j - (\chi(x_0))_j$  is associated with  $dx_j$ , one has the decomposition of  $f$  in  $T_{x_0}^*(X)$ . The cotangent bundle is

$$T^*X = \cup_{x_0 \in X} T_{x_0}^*(X).$$

The canonical projection  $\pi$ , from  $T^*X$  to  $X$  is defined by

$$l \in T^*X \Rightarrow \exists x = \pi(l), l \in T_x^*(X).$$

When  $X$  is provided with a local coordinate system, then any point of a section  $T_x^*X$  of the cotangent bundle can be characterized by its coordinates in the basis  $dx_1, dx_2, \dots, dx_d$ . Thus, a point of  $T^*X$  is determined by  $(x, \xi)$  where  $x = \pi(\rho)$ ,  $\pi$  being the natural projection of  $T^*X$  on  $X$ . In  $T_{\pi(\rho)}^*(X)$ , we have

$$\rho = \sum_{j=1}^{j=d} \xi_j dx_j.$$

This proves that there exists an application  $T^*(X) \cap \pi^{-1}(W)$  into  $\chi(W) \times \mathbb{R}^d$  defined through  $\rho \rightarrow (x, \xi)$ ,  $\rho = \sum \xi_j dx_j$ . This application is  $C^\infty$ . Let  $x^1$  and  $x^2$  two distinct points of  $X$ . As  $\pi^{-1}(x^1)$  and  $\pi^{-1}(x^2)$  have the same dimension and are disjoint in  $T^*X$ , one deduces that  $T^*(X)$  is a vectorial bundle  $C^\infty$ .

**3.2. Change of local coordinates.** We use the relation (3.109) to identify the representation of  $\rho$  in the  $dx_j$  basis. We see that

$$(3.110) \quad \xi_j = \frac{\partial}{\partial x_j} (f \circ \chi^{-1})(\chi(x_0)).$$

We assume that, in  $W$ ,  $X$  is represented by a second local coordinate system  $(y_1, \dots, y_d)$ . Then we introduce the diffeomorphism  $\psi$  which to a point of  $X \cap W$  makes correspond  $(y_1, \dots, y_d) \in \psi(X \cap W)$ . To the point  $x_0$  is associated the element  $\eta$  whose coordinates in the base  $dy_1, \dots, dy_d$  are

$$(3.111) \quad \eta_j = \frac{\partial}{\partial y_j} (f \circ \psi^{-1})(\psi(x_0)).$$

Denote by  $\phi = \chi \circ \psi^{-1}$ . It is the application allowing to pass from coordinates  $y$  to the coordinates  $x$ . As  $\psi^{-1} = \chi^{-1} \circ \phi$ , we deduce from (3.111) and (3.110) the equalities

$$\begin{aligned} \eta_j &= \frac{\partial}{\partial y_j} ((f \circ \chi^{-1}) \circ \phi)(\psi(x_0)) \\ &= \sum_k \frac{\partial}{\partial x_k} (f \circ \chi^{-1}) \frac{\partial \phi_k}{\partial y_j} \\ &= \sum_k \frac{\partial \phi_k}{\partial y_j} \xi_k \\ &= ({}^t \frac{\partial x}{\partial y}) \xi_j \end{aligned}$$

where  $\frac{\partial x}{\partial y}$  is the transfer matrix  $(\frac{\partial \phi_j}{\partial y_k})$ , Jacobian matrix of the transition application  $y \rightarrow x$ . We thus summarize the relation characterizing the change of variable in  $T_{x_0}^*(X)$ :

**LEMMA 8.2.** *Let  $x_0$  be a point of  $X$ . In the neighborhood of  $x_0$ , we can define a system of local coordinates  $(x)$ , which allows to define a canonical basis of  $T_{x_0}^*(X)$ . This construction can be done for any point in a neighborhood  $W$  of  $x_0$ . The manifold  $T^*(X)$  is thus a manifold of class  $C^\infty$ . When  $(x)$  and  $(y)$  are two systems of local coordinates,  $(x, \xi)$  and  $(y, \eta)$  describe the same point of  $T^*(X)$  when  $x$  and  $y$  describe the same point of  $X$  and*

$$\xi = ({}^t \frac{\partial x}{\partial y})^{-1}(\eta).$$

We notice that we find here, in projection on  $\mathbb{R}^d$ , the diffeomorphism  $h_{chi}$  defined in paragraph 1 of the proposition 8.4. In particular, this proves that the wave front set  $WF$  of a distribution on a manifold  $X$  is identified with a conic closure of the cotangent bundle  $T^*X$ .

**3.3. Tangent bundle.** We define the tangent space to  $X$  at  $x_0$  as the dual of  $T_{x_0}^*(X)$  in the canonical duality of  $\mathbb{R}^d \times \mathbb{R}^d$ . Note that the dual base associated with  $(dx_1, \dots, dx_d)$  by  $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ . For now, these are ratings. We also define  $TX = \cup_{x_0} T_{x_0}(X)$ . Then, by duality,  $TX$  is also a vector bundle and  $(x, t)$  and  $(y, s)$  describe the same point of  $TX$  when  $x$  and  $y$  describe the same point of  $X$  and when  $t = (\frac{\partial x}{\partial y})s$ .

Let us assume a section of the bundle  $TX$ , i.e. a local application of  $X$  in  $TX$  such that its composite with the projection  $\pi$  is the identity. Such elements are called **fields of vectors**. They can be written, in a system of local coordinates

$$V(\chi^{-1}x) = \sum_{j=1}^{j=d} a_j(x) \frac{\partial}{\partial x_j}$$

Since, in this system

$$df = \sum_{j=1}^{j=d} \frac{\partial}{\partial x_j} (f \circ \chi^{-1}) dx_j$$

the duality relation gives

$$\langle V, df \rangle = \sum_{j=1}^{j=d} a_j(x) \frac{\partial}{\partial x_j} (f \circ \chi^{-1}).$$



We then identify a vector field with a differential operator of order 1, and we say, by duality, since  $f$  is a function,  $df$  is a 1-differential form.

### 3.4. The canonical forms and the Hamiltonian field.

DEFINITION 8.3. *The canonical 1-form on  $T^*X$  is  $\omega_\rho$ , uplifting from the canonical projection of  $T^*X$  to  $X$ , which is in cartesian coordinates  $\sum_j \xi_j dx_j$ . The canonical 2-form on  $T^*X$  is  $\sigma = d\omega$ , where  $\omega$  is the application  $\rho \rightarrow \omega_\rho$ . It is, in cartesian coordinates,  $\sum_j d\xi_j \wedge dx_j$ .*

These definitions are technical, but they allow to reduce the demonstrations of the proposition 8.4 considerably. We detail the construction of these two objects.

We consider a local coordinate system  $(x)$  on  $X$ . The bundle  $T^*X$  is identified with  $\pi(W) \times \mathbb{R}^d$  thanks to the local coordinates and the natural projection.

For  $\rho \in T^*X$ , we can then define the space  $T_\rho^*(T^*X)$ , since  $T^*X$  is a manifold  $C^\infty$ . An element of  $T_\rho^*(T^*X)$  is an equivalence class for  $\sim$  in  $T^*X$  in the neighborhood of  $\rho$ . We define, by the commutative scheme

$$\begin{array}{ccc} T^*X & \leftarrow & T_\rho^*(T^*X) \\ \pi \downarrow & & \uparrow \pi^* \\ X & \leftarrow & T_{\pi(\rho)}^*X \end{array}$$

the uplifting  $\pi^*$ . We apply this dual uplifting  $\pi^*$  to the element  $\rho T_{\pi(\rho)}^*(X)$ , and we define the **1-canonical form** by

$$\omega_\rho = \pi^*(\rho).$$

This is an element of  $T_\rho^*(T^*X)$ . In other words,  $\rho \rightarrow \omega_\rho$  is a differential form on  $T^*X$ . The coordinates of  $T^*X$  are  $(x, \xi)$ , the associated canonical basis is  $(dx_1, \dots, dx_d, d\xi_1, \dots, d\xi_d)$ , and we have

$$\omega_\rho = \sum_{j=1}^{j=d} \xi_j dx_j + \sum_{j=0}^{j=d} 0 d\xi_j.$$

Recall then that we can define a duality between  $(T^*X)^2$  and  $(TX)^2$  by

$$(3.112) \quad \langle \rho_1 \wedge \rho_2, t_1 \wedge t_2 \rangle = \langle \rho_1, t_1 \rangle \langle \rho_2, t_2 \rangle - \langle \rho_1, t_2 \rangle \langle \rho_2, t_1 \rangle .$$

For  $v \in T^*(T^*X)$  one writes

$$v = \sum_{j=1}^{j=d} v_{j,0}(x, \xi) dx_j + \sum_{j=1}^{j=d} v_{j,1}(x, \xi) d\xi_j$$

and one defines

$$dv = \sum_{j=1}^{j=d} dv_{j,0}(x, \xi) \wedge dx_j + \sum_{j=1}^{j=d} dv_{j,1}(x, \xi) \wedge d\xi_j,$$

the element  $dv_{j,0}$  or  $dv_{j,1}$  being in  $T^*X$ , and  $\wedge$  being the notation used for the the element of  $(T^*X)^2$  defined by the duality (3.112).

We thus introduce the **2-canonical form**  $\sigma = d\omega$ . Its definition is intrinsic, does not depend on the coordinate system.

We verify that, for  $\omega$  the differential form on  $T^*X$  defined above, the functions  $v_{j,0}(x, \xi)$  and  $v_{j,1}(x, \xi)$  are known, respectively equal to  $\xi_j$  and 0. So we have

$$dv_{j,0}(x, \xi) = \sum_{k=1}^{k=d} 0 dx_k + \sum_{k=1}^{k=d} \delta_{kj} d\xi_k,$$

which implies

$$(3.113) \quad \sigma = d\omega = \sum_{j=1}^{j=d} d\xi_j \wedge dx_j.$$

By the definitions used previously,  $\sigma$  is a bilinear form on  $(T_\rho T^*X)^2$ , given by

$$\sigma_\rho(t_1, t_2) = \langle \sigma_\rho, t_1 \wedge t_2 \rangle .$$

When, in a coordinate system,  $t_1 = (s_1, \tau_1)$  and  $t_2 = (s_2, \tau_2)$ , we verify that

$$\begin{aligned} \langle \sigma, t_1 \wedge t_2 \rangle &= \sum_{j=1}^{j=d} \langle d\xi_j \wedge dx_j, t_1 \wedge t_2 \rangle \\ &= \sum_{j,k,l} \langle d\xi_j \wedge dx_j, (s_{1,k} dx_k + \tau_{1,k} d\xi_k) \wedge (s_{2,l} dx_l + \tau_{2,l} d\xi_l) \rangle \end{aligned}$$

We notice that  $d\xi_j \wedge dx_j = -dx_j \wedge d\xi_j$  and that  $\langle d\xi_j \wedge dx_j, dx_p \wedge dx_q \rangle = \langle d\xi_j \wedge dx_j, d\xi_p \wedge d\xi_q \rangle = 0$ , then that  $\langle d\xi_j \wedge dx_j, dx_p \wedge d\xi_q \rangle = -\delta_{jq} \delta_{ip}$ . From equality

$$\langle t_1 \wedge t_2 \rangle = \sum_{j,l} s_{1j} s_{2l} dx_j \wedge dx_l + \sum_{j,l} (s_{1j} \tau_{2l} - s_{2j} \tau_{1l}) dx_j \wedge d\xi_l + \sum_{j,l} \tau_{1j} \tau_{2l} d\xi_j \wedge d\xi_l,$$

one deduces the relation

$$\langle \sigma, t_1 \wedge t_2 \rangle = \langle \tau_2, s_1 \rangle - \langle \tau_1, s_2 \rangle .$$

There exists a relation  $H$  between  $T_\rho^*(T^*X)$  and its dual  $T_\rho(T^*X)$  given by

$$\sigma(s, Hu) = \langle s, u \rangle ,$$

for  $u \in T_\rho^*(T^*X)$ . With canonical coordinates, with  $u = u_x dx + u_\xi d\xi$ ,  $Hu = u_\xi \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \xi}$ . One introduces the Hamiltonian vector field of  $f(x, \xi)$  of class  $C^1$  on an openset  $T^*X$  through the relation  $H_f = H(df) \in T(T^*X)$ . With

$$\sigma(s, H(df)) = \langle s, df \rangle .$$

It comes, in canonical coordinates (and we thus find the definition 8.1):

$$H_f = \sum_{j=1}^{j=d} \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} .$$

The intrinsic character of these notions allows us to give a direct proof of Proposition 8.4. Indeed, the integral curves of the Hamiltonian field are curves on  $T^*X$ . The Hamiltonian field is intrinsic, so it is transported by the diffeomorphism  $h_\chi$  (associated to the  $\chi$  diffeomorphism and defined in section 1 of the proposition 8.4), which is the way to identify the points of the cotangent bundle. Its integral curves are therefore transported in the same way. Finally, the symbol of an operator is also defined on the cotangent bundle of in a natural way. This is a consequence of the proposition 7.11. Indeed, a change of variable in the symbol  $a(x, \xi)$  associated to the operator  $a(x, D)$  transforms this operator according to the relation  $a(x, D)(u \circ \chi) = (b(y, D)(u)) \circ \chi$  and its symbol according to the relation  $b(\chi(x), \eta) = a(x, {}^t \chi'(x) \cdot \eta)$ , which is equivalent to  $b(h_\chi(x, \xi)) = a(x, \xi)$ . In other words, the invariance of the symbol by  $h_\chi$  and the fact that the point  $(x, \xi)$  and the point  $h_\chi(x, \xi)$  represent the same element of  $T^*\mathbb{R}^d$  imply that the symbol is well defined on  $T^*\mathbb{R}^d$ .

### 3.5. Some remarks.

LEMMA 8.3. • *One has the duality*

$$(3.114) \quad [H_f, H_g] = H_{\{f, g\}} .$$

*Let  $(p_j)_{j \in J}$ ,  $(q_k)_{k \in K}$  be a family of functions on  $M$ , symplectic manifold (for example  $T^*X$ ), such that*

$$\{p_j, p_k\} = 0, \{p_j, q_k\} = \delta_{jk}, \{q_j, q_k\} = 0.$$

*and  $(dp_j, dq_k)$  linearly independent.*

*Then we can complete this family into a family of symplectic coordinates.*

PROOF. For the first item, we know that the terms of order 2 disappear in the commutator, so we keep in what follows only the terms of order 1 or zero. As

$$H_f H_g = \left( \sum_j \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) \left( \sum_k \frac{\partial g}{\partial \xi_k} \frac{\partial}{\partial x_k} - \frac{\partial g}{\partial x_k} \frac{\partial}{\partial \xi_k} \right),$$

the coefficient of  $\frac{\partial}{\partial x_k}$  in the commutator is

$$\sum_j \left( \frac{\partial f}{\partial \xi_j} \frac{\partial^2 g}{\partial x_j \partial \xi_k} - \frac{\partial f}{\partial x_j} \frac{\partial^2 g}{\partial \xi_j \partial \xi_k} \right) - \left( \frac{\partial g}{\partial \xi_j} \frac{\partial^2 f}{\partial x_j \partial \xi_k} + \frac{\partial g}{\partial x_j} \frac{\partial^2 f}{\partial \xi_j \partial \xi_k} \right).$$

On the other hand, we verify that

$$\{f, g\} = H_f g = \sum_j \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j}.$$

One checks that

$$\frac{\partial}{\partial \xi_k} \{f, g\} = \sum_j \frac{\partial g}{\partial x_j} \frac{\partial^2 f}{\partial \xi_j \partial \xi_k} + \frac{\partial f}{\partial \xi_j} \frac{\partial^2 g}{\partial \xi_k \partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial^2 f}{\partial x_j \partial \xi_k} - \frac{\partial f}{\partial x_j} \frac{\partial^2 g}{\partial \xi_j \partial \xi_k}.$$

By comparing, we get the result. We then check that, for  $p \in C^\infty(T^*X)$ ,  $H_p p = 0$ , which implies that

$$H_p \text{ is tangent to the hypersurface } \{p = 0\}.$$

Indeed, for  $(V, W) \in T_{(x_0, \xi_0)}(\{p = 0\})$ ,  $p(x_0, \xi_0) = 0$ , there exists a curve  $\{(x(t), \xi(t))\} \subset \{p = 0\}$  such that  $x(0) = x_0$ ,  $\xi(0) = \xi_0$ ,  $\dot{x}(0) = V$ ,  $\dot{\xi}(0) = W$ . The equality  $p(x(t), \xi(t)) = 0$  leads then, by derivation with respect to  $t$  and by calculating in  $t = 0$ , to prove that  $(V, W)$  in the hyperplane of equation  $\partial_x p(x_0, \xi_0)V + \partial_\xi p(x_0, \xi_0)W = 0$ . Thus  $T_{(x_0, \xi_0)}(\{p = 0\}) = (\partial_x p(x_0, \xi_0), \partial_\xi p(x_0, \xi_0))^\perp$ . The vector  $H_p$  is then in this tangent space.

The second item is an essential lemma from the geometric point of view to get back to simpler symplectic coordinate systems. It can be found for example in the course of Carathéodory's course [21].

It relies on the relation (3.112). We can, without restricting generality, suppose that  $p_j(\rho_0) = 0 \forall j \in J$ ,  $q_k(\rho_0) = 0 \forall k \in K$ . We will also note  $J$  and  $K$  the respective cardinals of  $J$  and  $K$ . The fields  $H_{p_j}$ ,  $H_{q_k}$  commute with each other, because

$$\begin{aligned} [H_{p_j}, H_{p_{j'}}] &= H_{\{p_j, p_{j'}\}} = H_0 = 0 \\ [H_{q_k}, H_{q_l}] &= 0 \\ [H_{p_j}, H_{q_k}] &= H_{\delta_{jk}} = 0. \end{aligned}$$

Consider a submanifold  $G$  of  $T^*X$ , of dimension  $2n - J - K$ , transverse in  $\rho_0$  to the vector space  $H = \text{Vect}(H_{p_j}, H_{q_k})$ . The idea of this proof is to construct the transport of  $G$  by all fields  $H_{p_j}$ ,  $H_{q_k}$ . From a parameterization of  $G$ , we add a transverse variable propagating the direction of  $H_{p_1}$ . We construct

$$H_{p_1}(G) = \{\rho(x_1, z) \in T^*X, z \in G, \rho(0, z) = z, \frac{d}{dx_1}(\rho(x_1, z)) = H_{p_1}\rho(x_1, z)\}.$$

It is a manifold of dimension  $\dim G + 1$ . We continue the construction by considering  $H_{p_2}(H_{p_1}G) \dots$ . We have thus obtained a local coordinate system  $(x, y, z)$  on  $T^*X$ . The fields  $dp_j, dq_k$  are linearly independent, so the dimension of  $T_{\rho_0}((H_{q_k})_{k \in K} \circ (H_{p_j})_{j \in J}G)$  is equal to  $2n - J - K + J + K = 2n$ .

The fields commute, so we can talk about coordinates (in other words  $H_{p_{j_2}}(H_{p_{j_1}}(z)(x_1))(x_2) = H_{p_{j_1}}(H_{p_{j_2}}(z)(x_2))(x_1)$ , i.e. the order of application of the operators  $H_p, H_q$  is not important). We also show that  $\frac{\partial}{\partial x_1} = H_{p_1}$ , and note that  $G = \{x = y = 0\}$ .

We then need to complete the family  $p_j, q_k$ , which is the objective of our lemma. We already have pairs of symplectic coordinates given by  $(p_j, q_j)$ ,  $j \in J \cap K$  if  $K$  is not empty.

We will complete the family in a first step for all the elements of  $J \cup K - J \cap K$ . Consider, for example,  $j_0 \in J, j_0 \notin K$ . We are looking for the solution of

$$\begin{cases} H_{p_j} q_{j_0} = \delta_{j_0 j} \\ H_{q_k} q_{j_0} = 0 \end{cases}$$

that is

$$\begin{cases} \partial_{x_j} q_{j_0} = \delta_{j_0 j} \\ \partial_{y_k} q_{j_0} = 0 \end{cases}$$

We also impose  $q_{j_0}(0, 0, z) = q(z)$ , any, such that  $dq \neq 0$ . We can then construct  $q_{j_0}(x, y, z)$ . The family  $(dp_j, dq_k, dq_{j_0})$  is a independent family. We therefore consider this new family as a starting point.

We proceed by successive iterations, noting that  $J$  and  $K$  play the same role. We have thus obtained a system of coordinates

$$(p_j, q_j), j \in \{0, \dots, J\}$$

verifying  $(dp_j, dq_k)$  free and  $\{p_j, q_k\} = \delta_{jk}, \{p_j, p_k\} = 0, \{q_j, q_k\} = 0$ . By the same method as before, we have constructed a local coordinate system on local coordinate system on  $T^*X$ , of the form  $(x, y, z)$ , where  $x$  and  $y$  are the  $2\dim(J \cup K)$  independent coordinates including the indices of  $J \cup K$ . We want to complete the system of coordinates  $(x, y, z)$ . We assume  $d < n$  and order the coordinates in the form the form  $(x_1, \dots, x_d, y_1 \dots, y_d)$ . Let  $F = \{x = y = 0\}$ . We look for a function  $q_{d+1}$  such that that  $H_{p_j} q_{d+1} = H_{q_j} q_{d+1} = 0$  for  $1 \leq j \leq d$ , and  $dq_{d+1}|_F \neq 0$ . This is possible because the dimension of  $T_{\rho_0} F$  is at least 2. We then choose  $p_{d+1}$  as the solution of

$$H_{q_{d+1}} p_{d+1} = 1, H_{q_j} p_{d+1} = 0, H_{p_j} p_{d+1} = 0, p_{d+1} = p(z), z \in F \cap \{q_{d+1} = 0\}$$

the function  $p$  being given on  $F \cap \{q_{d+1} = 0\}$ , which is non empty and of dimension greater than or equal to 1. This process is iterated until the dimension is zero. The resulting system is  $(p_1, \dots, p_n, q_1, \dots, q_n)$ .

Let  $f$  and  $g$  be two functions  $C^\infty$  of  $T^*X$ , written in the form  $f(p, q)$  and  $g(p, q)$ , we have  $p = p(x, \xi), q = q(x, \xi)$ . We write  $f(p, q) = F(x, \xi)$  and  $g(p, q) = G(x, \xi)$ .

We then verify

$$\begin{aligned} \sum_j \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial \xi_j} - \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial x_j} &= \sum_{j,j'} \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial p_{j'}} \{p_j, p_{j'}\} + \sum_{k,k'} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial q_{k'}} \{q_k, q_{k'}\} \\ &+ \sum_{j,k} \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_k} \{p_j, q_k\} + \sum_{j,k} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_j} \{q_k, p_j\}. \end{aligned}$$

Using the relations on the brackets of the new coordinates, we find

$$\{f, g\} = \{F, G\} = \sum_j \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j}.$$

The relation  $\{f, g\} = H_f g$  leads to  $H_f = \sum_j \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j}$ . Identification  $\sigma(H_f, H_g) = \{f, g\}$  thus yields

$$\sigma\left(\sum_j -\partial_{q_j} f dp_j + \partial_{p_j} f dq_j, \sum_j -\partial_{q_j} g dp_j + \partial_{p_j} g dq_j\right) = \sum_j \partial_{p_j} f \partial_{q_j} g - \partial_{q_j} f \partial_{p_j} g.$$

One thus has

$$\sigma\left(\sum_j a_j dp_j + b_j dq_j, \sum_j c_j dp_j + d_j dq_j\right) = \sum_j -b_j c_j + a_j d_j,$$

hence  $\sigma = \sum_1^n dp_j \wedge dq_j$ , and the system of coordinates  $(p_j, q_j)$  is symplectic. Lemma 8.3 is thus proven.  $\square$

It is necessary to introduce a geometrical notion associated to a Fourier integral operator, its **canonical relation**. The definition of canonical relations is closely related to particular submanifolds of  $T^*X$ , associated to the symplectic structure symplectic structure, the

Lagrangian manifolds. We introduce them here for the definition of the classes of Fourier integral operators. We will use them again in the chapter 11.

**3.6. Definition of canonical relations and canonical transformations.** Let  $X$  be a manifold of dimension  $n$ .

DEFINITION 8.4. *We call Lagrangian manifold any isotropic submanifold  $\Lambda$  of  $T^*X$  (i.e. satisfying  $\sigma|_{\Lambda} = 0$ ) of maximal dimension  $n$  ( $\dim_{(x_0, \xi_0)} \Lambda = n$ ).*

This definition comes from the work of Maslov [73], we find it in L. Hörmander [47] and in J.J. Duistermaat [30]. The author has also seen (in notes of Chazarain, Scientific Meetings of Cargèse) the name of maximal Lagrangian.

A natural example is the manifold  $\{(x, d\phi(x)), x \in X\} = \Lambda_\phi$ . This manifold is called the Lagrangian manifold associated to the phase  $\phi$ , at least of class  $C^2$ . The maximality hypothesis is trivially satisfied, since  $x$  is a parameterization of the manifold. We check, on  $\Lambda_\phi$  that  $\xi_j = \frac{\partial \phi}{\partial x_j}(x)$ , hence  $d\xi_j = \sum_i \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i$ , which implies

$$\sum d\xi_j \wedge dx_j = \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i \wedge dx_j,$$

thanks to  $\partial_{x_i x_j}^2 \phi = \partial_{x_j x_i}^2 \phi$  because  $\phi$  of class  $C^2$  and  $dx_i \wedge dx_j = -dx_j \wedge dx_i$  hence the result.

DEFINITION 8.5. *A canonical relation  $\mathcal{C}$  of  $T^*(X \times Y)$  is a subspace of  $T^*(X \times Y) \simeq T^*X \times T^*Y$  which is a Lagrangian manifold for the canonical form  $\sigma_X - \sigma_Y = dx \wedge d\xi - dy \wedge d\eta$ . In other words*

$$\mathcal{C} = \{(x, \xi; y, \eta) \in T^*X \times T^*Y\}$$

*is a canonical relation if  $\Lambda_{\mathcal{C}} = \{(x, \xi; y, -\eta), (x, \xi; y, \eta) \in \mathcal{C}\}$  is a Lagrangian manifold for  $\sigma_{X \times Y} = dx \wedge d\xi + dy \wedge d\eta$ .*

Let  $\mathcal{H}$  be a canonical transformation from  $X$  to  $Y$ . It is an application of class  $C^\infty$  from  $T^*Y$  on  $T^*X$  which preserves the symplectic structure (i.e.  $\mathcal{H}^*(dy \wedge d\eta) = dx \wedge d\xi$ ).

We check that the graph  $\mathcal{C}_{\mathcal{H}}$  of the canonical transformation  $\mathcal{H}$ , subset of  $T^*X \times T^*Y$  of  $(\mathcal{H}(\rho), \rho)$ ,  $\rho \in T^*Y$ , is a canonical relation.

Assume that there exists a point  $(x_0, \eta_0)$  such that the canonical projection  $l$  from  $T^*(X \times Y) \subset \mathbb{R}^{2n}$  which associates to  $(x, \xi; y, \eta)$  the point  $(x, \eta)$  is a diffeomorphism in the neighborhood of  $l^{-1}(x_0, \eta_0)$ . Then there exists, by the representation theorem of Hörmander (Theorem 9.1, stated and proved later on, whose proof is based on the 9.2 Lemma of completeness of a canonical basis) a phase  $\phi(x, \eta)$  such that, locally,  $\mathcal{C}_{\mathcal{H}}$  is of the form  $(x, \frac{\partial \phi}{\partial x}(x, \eta), \frac{\partial \phi}{\partial \eta}(x, \eta), \eta)$ . In other words, the canonical transformation  $\mathcal{H}$  is locally of the form

$$\left(\frac{\partial \phi}{\partial \eta}(x, \eta), \eta\right) \rightarrow \left(x, \frac{\partial \phi}{\partial x}(x, \eta)\right).$$

Let  $\mathcal{C}' = \{(x, \xi; y, \eta), (x, \xi; y, -\eta) \in \mathcal{C}\}$ . One checks that, thanks to

$$dx \wedge d\xi + dy \wedge d\eta|_{\mathcal{C}'} = dx \wedge d(\partial_x \phi) - d(\partial_\eta \phi) \wedge d\eta$$

and thanks to the equalities  $d(\partial_{x_i} \phi) = \text{Hess}_{x_i x} \phi \cdot dx + \sum_j \frac{\partial^2 \phi}{\partial x_i \partial \eta_j} d\eta_j$  and  $d(\partial_{\eta_j} \phi) = \sum_{j_i} \frac{\partial^2 \phi}{\partial x_i \partial \eta_j} dx_i + \text{Hess}_{\eta_j \eta} \phi \cdot \eta$ , using that the matrices  $\text{Hess}_{x^2} \phi$  and  $\text{Hess}_{\eta^2} \phi$  are symmetric, one has

$$dx \wedge d\xi + dy \wedge d\eta|_{\mathcal{C}'} = \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial \eta_j} (dx_i \wedge d\eta_j - dx_i \wedge d\eta_j) = 0.$$

This shows that  $\mathcal{C}$  is a canonical relation.

Any canonical relation of  $X \times Y$  has, modulo a symplectic change of variable in  $Y$ , a generating function  $\phi(x, \eta)$ . We find the proof in [43] (Theorem 10.1). This result is a superb result of Egorov, and we recall at the end of this section the original two-page proof.

Let  $A$  be the Fourier integral operator of  $C_0^\infty(Y)$  in  $S'(X)$  given by

$$(3.115) \quad A(x) = \int e^{i(\phi(x,\eta)-y\cdot\eta)} a(x, y, \eta) u(y) dy d\eta.$$

Recalling that this is a formal notation for the action of a distribution of  $S'$ , one should write, for  $\psi \in S'$ ,

$$\langle Au, \psi \rangle = \int e^{i(\phi(x,\eta)-y\cdot\eta)} a(x, y, \eta) u(y) \psi(x) dy d\eta dx.$$

We deduce from the proposition 6.2 that the wave front set of operator  $A$  is contained in

$$WF(Au) \subset T^*X = \{(x, \partial_x(\phi(x, \eta) - y \cdot \eta)), \partial_\eta(\phi(x, \eta) - y \cdot \eta) = 0 / (y, \eta) \in WF(u)\}$$

which writes also

$$WF(Au) \subset \{(x, \frac{\partial\phi}{\partial x}(x, \eta)), (\frac{\partial\phi}{\partial\eta}(x, \eta), \eta) \in WF(u)\}.$$

Let us study explicitly the canonical relation associated to an integral operator. The function  $\phi$  is the generating function of the canonical relation  $\mathcal{C} = \{x, \partial_x\phi; \partial_\eta\phi, \eta\}, x, X, \eta \in \mathbb{R}^d\}$ . To this canonical relation is associated the canonical transformation  $\mathcal{H}$ . It is easy to see that  $WF(Au) \subset \mathcal{H}(WF(u))$ .

REMARK 8. *If  $A$  is a pseudo-differential operator on  $\mathbb{R}^d$ ,*

$$Au(x) = \frac{1}{(2\pi)^d} \int_{T^*\mathbb{R}^d} e^{i(x-y)\cdot\eta} a(x, y, \eta) u(y) dy d\eta.$$

*The phase  $\phi_0(x, \eta)$  is thus  $\phi_0(x, \eta) = x \cdot \eta$ . The application  $\mathcal{H}$  is then the identity of  $T^*\mathbb{R}^d$  and the canonical relation  $\mathcal{C}$  is*

$$\mathcal{C} = \{(x, \eta; x, \eta), (x, \eta) \in T^*\mathbb{R}^d\}$$

*associated with the Lagrangian manifold  $\Lambda_{\phi_0} = \{(x, x, \eta, -\eta)\}$ .*

Indeed, as the phase of the Fourier-inverse Fourier transform is  $\psi(x, y, \xi) = (x - y) \cdot \xi$ , the associated Lagrangian manifold is  $\Lambda_\psi = \{(x, y, \partial_x\psi, \partial_y\psi), \partial_\xi\psi = 0\} = \{(x, y, \xi, -\xi), x - y = 0\} = \{(x, x, \xi, -\xi)\}$ .

REMARK 9. *We now consider the application  $\mathcal{H}_s$  of  $\mathbb{R}^d \times \mathbb{R}^d$  mapping  $(x, \theta)$  to  $(y, \Sigma)$  such that  $x = y + \nabla_\theta s(y, \theta)$ ,  $\theta + \nabla_y s(y, \theta) = \Sigma$ . The Fourier integral operator (according to the definition 6.4) of phase  $\phi(y, \theta) = y \cdot \theta + s(y, \theta)$ , and of symbol  $m(y, z, \theta)$  is written*

$$T_s^m u(y) = \int e^{i(\phi(y, \theta) - z \cdot \theta)} m(y, z, \theta) u(z) dz d\theta = \int e^{i((y-z)\theta + s(y, \theta))} m(y, z, \theta) u(z) dz d\theta.$$

We write the identity  $d\Sigma_j = d\theta_j + \sum_k \frac{\partial^2 s}{\partial y_j \partial \theta_k} d\theta_k + \sum_k \frac{\partial^2 s}{\partial y_j \partial y_k} dy_k$ , hence, as  $\frac{\partial^2 s}{\partial y_j \partial y_k} dy_k \wedge dy_j = 0$  for all  $j, k$ , one deduces

$$dy \wedge d\Sigma = \sum_j dy_j \wedge d\theta_j + \sum_{j,k} \frac{\partial^2 s}{\partial y_j \partial \theta_k} dy_j \wedge d\theta_k.$$

Similarly, from  $dx_j = dy_j + \sum_k \frac{\partial^2 s}{\partial \theta_j \partial y_k} dy_k + \sum_k \frac{\partial^2 s}{\partial \theta_j \partial \theta_k} d\theta_k$  and the identities  $d\theta_j \wedge \frac{\partial^2 s}{\partial \theta_j \partial \theta_k} d\theta_k = 0$  we deduce

$$dx \wedge d\theta = \sum_j dy_j \wedge d\theta_j + \sum_{j,k} \frac{\partial^2 s}{\partial y_j \partial \theta_k} dy_j \wedge d\theta_k.$$

Thus  $dy \wedge d\Sigma = dx \wedge d\theta$ . The transformation  $\mathcal{H}_s$  is a diffeomorphism of the space  $T^*(\mathbb{R}^{2d})$  leaving the symplectic structure invariant. It is a canonical transformation on  $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$  associated to the canonical relation  $\mathcal{C}_s = \{(y, \theta + \nabla_y s(y, \theta); y + \nabla_\theta s(y, \theta), \theta)\}$ .

REMARK 10. (*Order of Fourier integral operators*). Consider the Fourier integral operator  $A$ , whose phase is  $\phi(x, y, \theta)$ ,  $x \in \mathbb{R}^{n_X}$ ,  $y \in \mathbb{R}^{n_Y}$ ,  $\theta \in \mathbb{R}^N$ , homogeneous in  $\theta$  of degree 1, and whose symbol is  $a(x, y, \theta)$ , not yet specified. We apply this Fourier integral operator, defined by the oscillating integral

$$Au(x) = \int e^{i\phi(x, y, \theta)} a(x, y, \theta) u(y) dy d\theta$$

to a oscillatory function  $u(y) = \sigma(y, k) e^{ik\psi(y)}$ . We then denote by  $\tilde{\psi}(x, y)$  the critical value of  $\psi(y) + \phi(x, y, \theta)$  at a critical point  $\eta_c$  solution of

$$\partial_\eta \phi(x, y, \eta_c) = 0.$$

The principal term of  $Au(x, k)$  is written

$$\int_{\mathbb{R}^{n_X}} a(x, y, \eta_c) \sigma(y, k) e^{ik\tilde{\psi}(x, y)} k^{\frac{N}{2}} (\det J(\phi + \psi))^{-\frac{1}{2}} dy.$$

This result is independent of the number of variables used in the phase when, with fixed space dimensions  $n_X$  and  $n_Y$ ,  $a$  is in  $S^{m+l(n_X, n_Y)-\frac{N}{2}}$ .

We find the ordinary pseudodifferential operators when  $n_X = n_Y = N$ , in which case it is sufficient to put  $l(N, N) = -N/2$ .

Then, when we formally compose two Fourier integral operators, respectively  $A_1$  whose symbol  $a_1$  belongs to  $S^{m_1+l(n_X, n_Y)-\frac{N_1}{2}}$  and  $A_2$  of symbol  $a_2$  belonging to  $S^{m_2+l(n_Y, n_Z)-\frac{N_2}{2}}$ ,  $A_1 \circ A_2$  is given by

$$A_1 A_2 u(x) = \int \int e^{i\phi_1(x, y, \theta) + i\phi_2(y, z, \eta)} a_1(x, y, \theta) a_2(y, z, \eta) u(z) dy dz d\eta d\theta.$$

We want to obtain a representation of  $A_1 \circ A_2$  of symbol  $b(x, z, \omega) \in S^{m_1+m_2+l(n_X, n_Z)-\frac{n_\omega}{2}}$ . We formally denote by  $\omega = (y, \theta, \eta) \in \mathbb{R}^{N_1+N_2+n_Y}$  and we find the representation

$$A_1 A_2 u(x) = \int \int \int e^{i\psi(x, z, \omega)} b(x, z, \omega) u(z) dz d\omega$$

To have the same order, we need  $b \in S^{m_1+m_2+l(n_X, n_Z)-\frac{N_1+N_2+n_Y}{2}}$ , which gives the equality

$$l(n_X, n_Y) + l(n_Y, n_Z) = l(n_X, n_Z) - \frac{n_Y}{2}.$$

Considering then the adjoint  $A^* = I_{\tilde{a}, \psi}$  of the operator  $A = I_{a, \phi}$  of phase  $\psi(y, x, \theta) = -\phi(x, y, \theta)$  and of symbol  $\tilde{a}(y, x, \theta) = \bar{a}(x, y, \theta)$ , we find that  $l(n_X, n_Y) = l(n_Y, n_X)$ . From the two equalities, we can then deduce that

$$l(n_X, n_Y) = -\frac{n_X + n_Y}{4}.$$

We then define the space of Fourier integral operators of having the same canonical relation and the same weight  $m$ .

DEFINITION 8.6. Let  $\phi(x, y, \theta)$  be a phase defined on  $\mathbb{R}^{n_X} \times \mathbb{R}^{n_Y} \times \mathbb{R}^N$ . It defines a canonical relation  $\mathcal{C}$  on  $T^*(X \times Y)$ . The space  $I^m(X \times Y, \mathcal{C})$  of the Fourier integral operators of symbol  $a \in S^{m+\frac{n_X+n_Y-2N}{4}}(X \times Y \times \mathbb{R}^N)$  of canonical relation  $\mathcal{C}$  associated to the Lagrangian manifold for  $\sigma_X + \sigma_Y$  of  $C' \text{-} \text{ind} T^*(X \times Y)$  is the set of oscillating integrals in  $\mathcal{S}'$  of the form

$$Au(x) = \int_{\mathbb{R}^{n_Y} \times \mathbb{R}^N} a(x, y, \theta) e^{i\phi(x, y, \theta)} u(y) dy d\theta.$$

This definition does not depend on the number of  $\theta$  variables used to characterize the canonical relationship.

**3.7. Inversion of Fourier integral operators.** We summarize the result of inversion of Fourier integral operators and conjugation of pseudo-differential operators by Fourier integral operators in the

**THEOREM 8.1.** *Let  $\mathcal{C}$  be a homogeneous canonical relation of a conic neighborhood of  $(y_0, \eta_0) \in T^*X$ ,  $X \subset \mathbb{R}^d$  onto a conical neighborhood of  $(x_0, \xi_0) \in T^*Y$ ,  $Y \subset \mathbb{R}^d$  associated to a generating function  $\phi(x, \eta)$ . Its inverse canonical relation, denoted  $\mathcal{C}^{-1}$  is the Lagrangian manifold in  $T^*(X \times Y)$  for  $dy \wedge d\eta - dx \wedge d\theta$  of points  $(y, \eta, x, \theta), (x, \theta, y, \theta) \in \mathcal{C}$ .*

*We consider the operator  $A \in I^0(X \times Y, \mathcal{C}')$ , of symbol  $a \in S^0(X \times Y \times \mathbb{R}^d)$  elliptic*

$$Au(x) = \int_{T^*X} e^{i(\phi(y, \eta) - z\eta)} a(y, z, \eta) u(z) dz d\eta.$$

- (1) *There exists  $b \in S^0(Y \times X \times \mathbb{R}^d)$  such that the Fourier operator operator  $B \in I^0(Y \times X, (\mathcal{C}^{-1})')$  given by*

$$Bv(x) = \int \int e^{i(x \cdot \xi - \phi(y, \xi))} b(x, y, \xi) v(y) dy d\xi$$

*verifies  $AB = Id + R_1 \in L^0(Y)$ ,  $BA = Id + R_2 \in L^0(X)$ ,  $R_1$  and  $R_2$  are in  $L^{-\infty}$ .*

- (2) *Let  $P$  be a pseudodifferential operator on  $X$  of order  $m$  of the principal symbol  $p_m$ . The operator  $Q = A \circ P \circ B$  is a pseudodifferential operator on  $Y$  of order  $m$ , whose principal symbol  $q_m$  is the image of the principal symbol  $p_m$  of  $P$  by the canonical transformation. Moreover, we have  $(x_0, \xi_0, y_0, \eta_0) \notin WF(PA - AQ)$ .*

Note that a pseudo-differential operator on  $X$  is associated to the canonical identity relation on  $T^*(X \times X)$  and to the phase  $x \cdot \eta$ . As we have seen in Remark 3, the space dimension is equal to the dimension in  $\eta$ , so  $I^0$  corresponds to the symbols of  $S^0$ .

On the other hand, when the canonical relation is generated by the phase  $\phi(x, \eta)$ , we have the relation  $p_m(x, \nabla_x \phi) = q_m(\nabla_\eta \phi, \eta)$ , generalization of the relation between the symbols after change of symplectic variable of the exercise 7.3.

This theorem (not proven here) is a direct consequence of the above definitions and of exercise 7.3. This is theorem 10.1 of [43], proved by Egorov [34] and which serves as basis for canonical transformations of pseudodifferential operators in order to

to simple operators.

Egorov considers first a phase function  $S(x, \xi)$  satisfying  $\det(\frac{\partial^2 S}{\partial x_i \partial \xi_j}(x, \xi')) \neq 0$ . He introduces the homogeneous canonical transformation  $(x, \xi) \rightarrow (x', \xi')$  with  $x'_j = \partial_{\xi_j} S(x, \xi')$ ,  $\xi_j = \partial_{x_j} S(x, \xi')$ . Then for any pseudodifferential operator  $P$  and any function  $h$ , there exists  $Q$  such that

$$Ph\Phi u = \Phi hQu + Tu$$

where  $\Phi$  is the Fourier integral operator

$$\Phi v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{iS(x, \xi)} d\xi$$

and where  $T \in L^{-\infty}(\mathbb{R}^n)$ . The canonical relation of  $P\Phi$  is equal to the one of  $\Phi$ , which corresponds to remark 1 above.

**3.8. Composition of Fourier integral operators.** In this last paragraph, we state the composition theorem of the Fourier integral operators through their canonical relations. We refer the interested reader to chapter 11 where the detailed proof of this theorem is presented.

Let  $X, Y, Z$  be three manifolds of dimension  $n_X, n_Y$  and  $n_Z$  and a Fourier integral operator  $A_1$  of  $X$  in  $Y$ , of canonical relation  $C_1 \subset T^*(X \times Y)$ , a Fourier integral operator  $A_2$  of  $Y$  in  $Z$ , of canonical relation  $C_2$ . We assume

$$A_1 \in I^{m_1}(X \times Y, C_1'), A_2 \in I^{m_2}(X \times Y, C_2').$$

It is assumed that  $C_1 \times C_2$  and  $T^*X \times T^*Y \times T^*Z$  have a transverse intersection, and that the natural projection  $C_1 \circ C_2$  of this intersection to  $T^*(X \times Z)$  is clean. We verify that

$$(\rho_1, \rho_2) \in T^*X \times T^*Z \in C_1 \circ C_2 \Leftrightarrow \exists \rho \in T^*Y, (\rho_1, \rho) \in C_1, (\rho, \rho_2) \in C_2.$$



To say that  $A_1$  has canonical relation  $C_1$  is equivalent to saying that, microlocally in the neighborhood of a point of  $C_1$ , we can represent  $C_1$  by a phase  $\phi_1(x, \theta)$  of the form  $C_1 = \{(x, \theta)\}$ . by a phase  $\phi_1(x, \theta)$  in the form  $C_1 = \{(x, \partial_x \phi(x, \theta); (\partial_x \phi(x, \theta), \theta))\}$ . Similarly, to say that  $A_2$  is of canonical relation  $C_2$  is equivalent to saying that, microlocally in the neighborhood of a point of  $C_2$ , we can represent  $C_2$  by a phase  $\phi_2(y, w)$  in the form  $C_2 = \{(y, \partial_y \phi_2(y, w); (\partial_w \phi_2(y, w), w))\}$ .

**THEOREM 8.2.** *Under the previous assumptions, the operator  $A_1 \circ A_2$  is a Fourier integral Fourier integral operator, with canonical relation  $C_1 \circ C_2$ , of order the sum of the orders of the symbols of  $A_1$  and  $A_2$ .*

*Proof.* Equalities

$$\begin{aligned} A_1 v(y) &= \int_{T^*X} e^{i\phi_1(y, \theta) - x \cdot \theta} a_1(y, \theta) v(x) dx d\theta \\ A_2 u(z) &= \int_{T^*Y} e^{i\phi_2(z, w) - y \cdot w} a_2(z, w) u(y) dw dy \end{aligned}$$

one deduces

$$A_2 A_1 u(z) = \int \int_{T^*X \times T^*Y} e^{i(\phi_1(y, \theta) - x \cdot \theta + \phi_2(z, w) - y \cdot w)} a_1(y, \theta) a_2(z, w) u(x) dx dw dy d\theta.$$

We calculate the value of  $A_2 A_1 u$  at point  $z$ , according to the value of  $u$  at point  $x$ . Thus, in the space variables, we choose to eliminate the variable  $y$ . Since, by analogy with Fourier analysis, we have to eliminate jointly a dual variable, we have the choice between  $\theta$  and  $w$ . The choice is indifferent. We formally apply the theorem of the stationary phase of parameter  $(x, z, \theta)$  in the variables  $(y, w)$ . Then the critical point  $(y_c, w_c)$  is a solution of

$$\partial_y \phi_1(y_c, \theta) = w_c, \partial_w \phi_2(z, w_c) = y_c.$$

Let us identify the points of  $C_1$  and  $C_2$ . The point of  $C_1$  is  $(y, \partial_y \phi_1(y, \theta); \partial_y \phi_1(y, \theta), \theta)$ . For  $y = y_c$ , we obtain  $(y_c, w_c; \partial_\theta \phi_1(y_c, \theta), \theta)$ . Similarly, the current point of  $C_2$  associated with the phase  $\phi_2$  is  $(z, \partial_z \phi_2(z, w); \partial_w \phi_2(z, w), w)$ . For  $w = w_c$ , we obtain the point  $(z, \partial_z \phi_2(z, w_c); y_c, w_c)$ . The critical value of the phase is  $\phi_1(y_c, \theta) + \phi_2(z, w_c) - y_c \cdot w_c - x \cdot \theta$ , and  $y_c$  and  $w_c$  depend only on  $z$  and  $\theta$ . We denote then by

$$\tilde{\phi}(z, \theta) = \phi_1(y_c, \theta) + \phi_2(z, w_c) - y_c \cdot w_c.$$

It follows

$$\partial_\theta \tilde{\phi}(z, \theta) = \partial_\theta \phi_1(y_c, \theta) + \partial_\theta y_c [\partial_y \phi_1(y_c, \theta) - w_c] + \partial_\theta w_c [\partial_w \phi_2(z, w_c) - y_c] = \partial_\theta \phi_1(y_c, \theta).$$

Similarly  $\partial_z \tilde{\phi}(z, \theta) = \partial_z \phi_2(z, w_c)$ . The two points of  $C_1$  and  $C_2$  are respectively

$$(y_c, w_c; \partial_\theta \tilde{\phi}(z, \theta), \theta) \quad \text{et} \quad (z, \partial_z \tilde{\phi}(z, \theta); y_c, w_c).$$

We thus recognize the point of  $T^*(X \times Z)$  equal to  $((\partial_z \tilde{\phi}(z, \theta), \theta); (z, \partial_z \tilde{\phi}(z, \theta)))$ . It is then element of the canonical relation  $C_1 \circ C_2$ .

The transversality condition is equivalent to the non-degeneracy of the critical point. The calculation of the stationary phase leads to a symbol, whose order is the sum of the orders, and which is the product of the symbols. We have thus sketched the proof of the theorem 8.2, which is the theorem 4.2.2 of [48], taken up in the theorem 11.12 de [43].

#### 4. Exercises of chapter 8

**Exercice 8.0.** Show that only a neighborhood compact in  $y$ , conical in  $\eta$  of the critical points of the phase  $(x, y, \eta) \rightarrow (x - y) \cdot \eta + \phi(y)$  contributes to the integral of Proposition 1. Show also that this result is true for the calculation of  $P(A(u))$  in Proposition 8.2.

**Exercice 8.1:** Prove Proposition 8.4 with direct methods.

Exercice 8.2: follow-up of the problem on the Friedlander model operator of Section 3. 6) Consider a point  $\rho_0 = (0, 0, 0, \xi^0, \eta_1^0, \eta_2^0) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2) \cap \text{Car}(p)$ . Give the bicharacteristic of  $P$  defined through the system (3.84) passing through the point  $\rho_0$ . For this, one will define

$$q(x, \eta_1, \eta_2) = (1+x)\eta_1^2 - \eta_2^2$$

and one shall express these curves using

$$S(x, \eta_1, \eta_2) = \int_0^x (q(u, \eta_1, \eta_2))^{\frac{1}{2}} du.$$

7) Consider the function, defined on  $\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$ , through

$$\phi(x, y_1, y_2, \theta_1, \theta_2) = y_1\theta_1 + y_2\theta_2 - S(x, \theta_1, \theta_2)\text{sign}(\theta_1).$$

Prove that the bicharacteristics, starting at the origin  $y_1 > 0$  span the space

$$\Sigma = \left\{ (x, y, \xi, \eta) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2), x > 0, |\theta_1| \geq |\theta_2|, \begin{array}{l} \xi = \partial_x \phi(x, y, \theta) \\ \eta = \nabla_y \phi(x, y, \theta) \\ \nabla_\theta \phi(x, y, \theta) = 0 \end{array} \right\}$$

8) Prove that the singular support of the Fourier integral operator  $K^{(2)}$  of symbol  $a_2(x, \theta)$  and of phase  $l(x, y, Y, \theta) = (y - Y)\theta - \frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})$  is contained in the union of bicharacteristics starting at the origin in  $y_1 > 0$ .

Preuve de l'exercice 8.0. On fixe un compact  $K$  dans la variable  $x$ , et on considère  $\psi \in C_0^\infty(K)$ . On introduit aussi  $\tilde{\phi}(y)$  égale à 1 sur le compact  $B(K, 1) = \{z, d(z, K) \leq 1\}$ , de support inclus dans  $B(K, 2)$ . On calcule la distribution

$$A(x, k) = \int_{\mathbb{R}^{2d}} e^{ik((x-y)\eta + \phi(y))} a(y, k) p(x, \eta, k) dy d\eta$$

en considérant son action sur la fonction test  $\psi$ . On considère, de plus, une fonction test  $\chi$  sur  $\mathbb{R}$  qui localise au voisinage de 0. On tronque l'intégrale en  $\eta$  en supprimant un voisinage de  $\eta = 0$ . La distribution obtenue est notée  $A_t$ .

Soit

$$I = \langle A_t, \psi \rangle = \int_{\mathbb{R}^{3d}} e^{ik((x-y)\eta + \phi(y))} \tilde{\phi}(y) a(y, k) p(x, \eta, k) \psi(x) (1 - \chi(|\eta|)) dx dy d\eta \\ + \int_{\mathbb{R}^{3d}} e^{ik((x-y)\eta + \phi(y))} (1 - \tilde{\phi}(y)) a(y, k) p(x, \eta, k) \psi(x) (1 - \chi(|\eta|)) dx dy d\eta$$

Le premier terme de cette somme est noté  $I_1$  et le deuxième terme est noté  $I_2$ .

étudions d'abord le deuxième terme. Comme  $A_t$  est une intégrale ne  $\eta$  ne contenant pas un voisinage de 0, on peut écrire le deuxième terme après intégrations par parties en  $x$ . Il n'y aura pas de termes de bord car la fonction  $\psi$  est à support compact. Ainsi on introduit l'opérateur  $L$  égal à

$$L = \sum_j \frac{\eta_j}{|\eta|^2} \frac{\partial}{\partial x_j}$$

qui vérifie  $L(e^{ik\eta(x-y)}) = ik e^{ik\eta(x-y)}$ . Ainsi, son opérateur transposé étant noté  ${}^tL$ , et c'est aussi un opérateur différentiel, on trouve

$$I_2 = (ik)^{-M} \int_{\mathbb{R}^{3d}} e^{ik((x-y)\eta + \phi(y))} (1 - \tilde{\phi}(y)) a(y, k) ({}^tL)^M [p(x, \eta, k) \psi(x)] (1 - \chi(|\eta|)) dx dy d\eta.$$

Cette intégrale est absolument convergente en  $\eta$  dès que  $M > m + d + 1$ , après avoir supposé  $a$  intégrable. On sépare l'intégrale en  $\eta$  en deux termes, notés  $I_2^R$  et  $I_3$ , avec

$$I_3 = (ik)^{-M} \int_{|\eta| \geq R} e^{ik((x-y)\eta + \phi(y))} (1 - \tilde{\phi}(y)) a(y, k) ({}^tL)^M [p(x, \eta, k) \psi(x)] (1 - \chi(|\eta|)) dx dy d\eta.$$

L'inégalité  $M > m + d + 1$  suffit pour affirmer qu'il existe une constante  $C$  indépendante de  $R$  telle que  $I_3 \leq \frac{C}{R}$ .

On introduit l'opérateur  $D$  dont l'opérateur transposé en  $\eta$  est

$${}^tD = \sum_j \frac{x_j - y_j}{|x - y|^2} \frac{\partial}{\partial \eta_j}.$$

Cet opérateur est régulier sur le support de  $(1 - \tilde{\phi}(y))\psi(x)$ , puisque  $x \in K$  et  $y \in C(B(K, 1))$ , ainsi  $|x - y| \geq 1$ . Des intégrations par parties successives dans  $I_2^R$  conduisent à des termes de bord en  $O(\frac{1}{R})$ , et à un terme intégral dans lequel on peut réaliser autant d'intégrations par parties que l'on souhaite. Finalement

“Pour tout  $N > 0$ , il existe deux constantes  $C_N$  et  $D_N$  telles que

$$I_2 \leq \frac{C_N}{k^N} + \frac{D_N}{R}$$

et donc le terme  $I_2$  est négligeable dans le calcul de phase stationnaire”.

Enfin, on supprime un voisinage en  $\eta$  du complémentaire de  $\{\nabla_y \phi(y), y \in B(K, 2)\}$ . On a le droit de le faire car ces points ne contribuent pas à la phase. On s'est ainsi ramené à un voisinage compact en  $y$  et conique en  $\eta$  des points critiques. La méthode est identique lorsque la phase est  $(x - y) \cdot \theta + s(y, \eta)$ . Cette démonstration est une conséquence du résultat abstrait suivant:

LEMMA 8.4. (*Corollary 1.1.12 de Hörmander [47]*) Soit  $L$  une application linéaire des fonctions de  $C^\infty(X \times \mathbb{R}^n)$ , s'annulant pour  $|\theta|$  grand, sur un espace de Fréchet  $F$ . On suppose que  $L$  est continue pour la topologie de  $S^m(X \times \mathbb{R}^n)$ . Alors  $L$  admet une unique extension continue sur  $S^m(X \times \mathbb{R}^n)$

On peut rendre le support compact en  $\theta$  grâce au résultat de convergence de la proposition 5.1, où on a prouvé que si  $a \in S_{\rho, \delta}^m$ , alors  $a(x, \theta)\chi(\frac{\theta}{j})$  converge vers  $a$  dans la topologie de  $S_{\rho, \delta}^{m'}$  pour  $m' > m$ . On applique ensuite le lemme pour définir l'extension une fois le calcul asymptotique fait avec le symbole tronqué.

Preuve de l'exercice 8.1. Pour le premier alinéa, on sait que  $(x_0, \xi_0) \notin WF(u)$  lorsqu'il existe un opérateur pseudo-différentiel  $Op(a)$  d'ordre 0, tel que  $a_0(x_0, \xi_0) \neq 0$ , vérifiant  $Op(a)u \in C^\infty$ .

On vérifie alors que  $y \rightarrow Op(a)u \circ \chi^{-1}(y)$  est une fonction  $C^\infty$ , et comme

$$(Op(a)u) \circ \chi^{-1} = Op(a')(u \circ \chi^{-1})$$

avec  $a'(\chi(x), \eta)$  calculé par la proposition 7.11, on vérifie que le symbole principal de  $a'$  est

$$a'_0(\chi(x), \eta) = a_0(x, \chi'(x)\eta)$$

ce qui entraîne que  $a'$  est un symbole d'ordre 0, et que  $a'_0(\chi(x_0), (\chi'(x_0))^{-1}\xi_0) = a_0(x_0, \xi_0) \neq 0$ .

Il existe un opérateur pseudo-différentiel d'ordre 0,  $Op(a')$ , dont le symbole est non nul en  $h_\chi(x_0, \xi_0)$ , tel que  $Op(a')(u \circ \chi^{-1}) \in C^\infty$ . Donc  $h_\chi(x_0, \xi_0) \notin WF(u \circ \chi^{-1})$ .

Réciproquement, comme  $h_{\chi^{-1}} = (h_\chi)^{-1}$ ,

$$h_\chi(x_0, \xi_0) \notin WF(u \circ \chi^{-1}) \Rightarrow (x_0, \xi_0) = h_{\chi^{-1}}(h_\chi(x_0, \xi_0) \notin WF((u \circ \chi^{-1}) \circ \chi) = WF(u).$$

On a prouvé l'égalité

$$h_\chi(WF(u)) = WF(u \circ \chi^{-1}).$$

L'égalité  $\sum_j \chi'_{kj}(x) dx_j = d(\chi_k(x))$  permet d'obtenir  $\chi'_{kj}(x) = \partial_{x_j} \chi_k(x)$ .

Supposons  $(x(s), \xi(s))$  solution du système

$$\begin{cases} \frac{dx}{ds} = \partial_\xi p_m(x(s), \xi(s)) \\ \frac{d\xi}{ds} = -\partial_x p_m(x(s), \xi(s)). \end{cases}$$

On introduit  $y(s) = \chi(x(s))$  et  $\eta(s) = ({}^t \chi')^{-1}(x(s))\xi(s)$ .

On a donc

$$\xi(s) = {}^t \chi'(x(s))\eta(s),$$

donc, utilisant  $\chi'_{kj}(x) = \partial_{x_j} \chi_k$ ,

$$\frac{d\xi_j}{ds} = \frac{d}{ds} \left( \sum_k \partial_{x_j} \chi_k(x(s)) \eta_k(s) \right) = \sum_{k,l} \partial_{x_j x_l}^2 \chi_k(x(s)) \eta_k(s) \frac{dx_l}{ds} + \sum_k \partial_{x_j} \chi_k(x(s)) \frac{d\eta_k}{ds},$$

et donc

$$-\partial_{x_j} p_m(x(s), \xi(s)) = \sum_{k,l} \partial_{x_j x_l}^2 \chi_k(x(s)) \eta_k(s) \partial_{\xi_l} p_m(x(s), \xi(s)) + \sum_k \partial_{x_j} \chi_k(x(s)) \frac{d\eta_k}{ds}.$$

Ceci se réécrit

$$-\sum_k \partial_{x_j} \chi_k(x(s)) \frac{d\eta_k}{ds} = \partial_{x_j} p_m(x(s), \xi(s)) + \sum_{k,l} \eta_k(s) \partial_{x_j x_l}^2 \chi_k(x(s)) \partial_{\xi_l} p_m(x(s), \xi(s)).$$

La matrice inverse de  ${}^t\chi'$ , notée  $A(x)$ , est caractérisée  $A_{jk}(x)$  telle que

$$\sum_j A_{pj}(x)\chi'_{jk}(x) = \delta_{pk}.$$

On obtient

$$-\sum_j \sum_k A_{pj}\chi'_{kj} \frac{d\eta_k}{ds} = \sum_j A_{pj}\partial_{x_j} p_m + \sum_{j,k,l} A_{pj}\partial_{x_j x_l}^2 \eta_k(s)\chi_k \partial_{\xi_l} p_m.$$

On a donc

$$-\frac{d\eta_p}{ds} = \sum_j A_{pj}(x(s))\partial_{x_j} p(x(s), \xi(s)) + \sum_{j,k,l} ({}^t\chi')_{pj}^{-1}(\chi^{-1}(y(s)))\partial_{x_j x_l}^2 \chi_k \eta_k(s)\partial_{\xi_l} p_m.$$

Comme on a la relation

$$q_m(y, \eta) = p_m(\chi^{-1}(y), {}^t\chi'(\chi^{-1}(y))\eta),$$

on vérifie que

$$\partial_{y_p} q_m(y, \eta) = \sum_j \partial_{y_p}(\chi^{-1})_j(y)\partial_{x_j} p_m + \sum_{j,k,l} \partial_{x_j x_l}^2 \chi_k(\chi^{-1}(y))\eta_k \partial_{\xi_l} p_m \partial_{x_j}(\chi^{-1}(y)).$$

En comparant, on obtient le résultat

$$(4.116) \quad \frac{d\eta_p(s)}{ds} = -\partial_{y_p} q_m(y(s), \eta(s)).$$

Le résultat sur  $\frac{dy_p}{ds}$  s'obtient en notant que

$$(4.117) \quad \frac{dy_p}{ds} = \sum_j \partial_{y_j}(\chi^{-1})_j \frac{dx_j}{ds} = \sum_j \partial_{y_j}(\chi^{-1})_j \partial_{\xi_j} p_m(x(s), \xi(s)) = \partial_{\eta_p} q_m(y(s), \eta(s)).$$

On a donc montré que  $(y(s), \eta(s))$  est la bicaractéristique de  $q_m$  issue du point  $(y_0, \eta_0)$ .

Le résultat pour la phase solution de l'équation eikonale est plus simple encore; il provient de l'égalité du gradient

$$\nabla_y(\phi \circ \chi^{-1}) = (\chi')^{-1}(\chi^{-1}(y))(\nabla_x \phi)(\chi^{-1}(y)).$$

Choissant  $y = \chi(x)$  dans cette égalité, on obtient

$$(4.118) \quad q_m(y, \nabla_y(\phi \circ \chi^{-1})) = q_m(\chi(x), (\chi')^{-1}(x)\nabla_x \phi(x)) = p_m(x, \nabla_x \phi(x)) = 0.$$

Ces égalités impliquent que le front d'onde, les bicaractéristiques, et la variété des  $\{(x, \nabla_x \phi)\}$  pour  $\phi$  solution de l'équation eikonale sont transportés par  $h_\chi$  dans le changement de variable sur  $\mathbb{R}^d \times \mathbb{R}^d$  induit par le changement de variable sur  $\mathbb{R}^d$  donné par  $\chi$ . C'est un changement de variable symplectique. Montrons qu'il laisse invariant la forme symplectique  $dy \wedge d\eta$ . De l'égalité

$$dy \wedge d\eta = d(\chi(x)) \wedge d({}^t(\chi')^{-1}(x)\eta) = \left(\sum_{j,k} \chi'_{kj} dx_j\right) \wedge \left(\sum_{j,k} \partial_{x_j}({}^t(\chi')^{-1})_k \eta_k dx_j + \sum_{j,k} ({}^t(\chi')^{-1})_{jk} d\eta_k\right)$$

utilisant la commutation des dérivées en  $x$  dans les dérivées secondes de  $\chi$ , il ne reste pas de terme en  $dx_j \wedge dx_k$ . Quant aux termes en  $dx_j \wedge d\eta_k$ , leur coefficient est  $\sum_l \chi'_{lj}({}^t(\chi')^{-1})_{lk} = \delta_{jk}$ . On a vérifié

$$(4.119) \quad dy \wedge d\eta = dx \wedge d\xi.$$

On dit que  $h_\chi$  est un changement de variable symplectique sur  $\mathbb{R}^d \times \mathbb{R}^d$ , et on identifie dans ce cas  $\mathbb{R}^d \times \mathbb{R}^d$  à  $T^*(\mathbb{R}^d)$ , espace muni de la forme symplectique  $d(\xi dx)$  qui est invariant en géométrie par les transformations  $h_\chi$ .

Plus généralement, un changement de variable symplectique est associé à un difféomorphisme  $(x, \xi) \rightarrow (h_1(x, \xi), h_2(x, \xi))$  tel que

$$\left(\sum_j \partial_{x_j} h_1 dx_j + \sum_j \partial_{\xi_j} h_1 d\xi_j\right) \wedge \left(\sum_j \partial_{x_j} h_2 dx_j + \sum_j \partial_{\xi_j} h_2 d\xi_j\right) = \sum_k dx_k \wedge d\xi_k.$$

Ceci donne les conditions nécessaires et suffisantes:

$$\begin{aligned}\partial_{x_i} h_1 \partial_{x_k} h_2 - \partial_{x_k} h_1 \partial_{x_i} h_2 &= 0 \\ \partial_{\xi_i} h_1 \partial_{\xi_k} h_2 - \partial_{\xi_k} h_1 \partial_{\xi_i} h_2 &= 0 \\ \partial_{x_i} h_1 \partial_{\xi_k} h_2 - \partial_{\xi_k} h_1 \partial_{x_i} h_2 &= \delta_{ij}\end{aligned}$$

qui traduisent que les crochets de Poisson des  $h$  soient nuls.

Solution de l'exercice 8.2. 6) Le symbole principal de l'opérateur de Friedlander (3.84) est

$$p(x, \xi, y_1, y_2, \eta_1, \eta_2) = -\xi^2 + (1+x)\eta_1^2 - \eta_2^2 = q(x, \eta_1, \eta_2) - \xi^2.$$

Les bicaractéristiques sont définies par le système

$$\begin{cases} \dot{x} = -2\xi \\ \dot{\xi} = -\eta_1^2 \\ \dot{y}_1 = 2(1+x)\eta_1 \\ \dot{y}_2 = -2\eta_2 \\ \dot{\eta}_1 = 0 \\ \dot{\eta}_2 = 0. \end{cases}$$

Le symbole est nul sur la bicaractéristique, donc

$$(\xi(s))^2 = q(x(s), \eta(s))$$

En particulier, pour  $s = 0$ , on trouve  $\eta_1^2 \geq \eta_2^2$  puisque  $\xi^0$  est défini. De plus,  $\eta_1$  et  $\eta_2$  sont constants sur les bicaractéristiques. Il existe donc  $\varepsilon = \pm 1$  tel que

$$\xi(s) = \varepsilon q(x(s), \eta).$$

Comme l'opérateur est défini pour  $x \geq 0$ , on a, grâce à l'égalité  $\dot{x} = -2\xi$ , directement  $s \leq 0$  et  $\varepsilon = +1$  ou  $s \geq 0$  et  $\varepsilon = -1$ .

On a de plus  $q(x(s), \eta) \geq q(0, \eta) \geq 0$ . Pour  $x(s) > 0$ , ce qui se produit lorsque  $\xi(0) < 0$ ,  $q > 0$ . Il est impossible, puisque  $(\xi^0, \eta_1^0, \eta_2^0) \neq (0, 0, 0)$  que  $q$  soit nul partout, Donc  $q > 0$  hors du point origine même si il est nul au point origine. On vérifie que  $\xi$  est donc non nul, donc  $\dot{x}$  est non nul, et  $x$  peut être choisi comme nouvelle variable. On se place dans un premier temps pour  $\eta_1^2 > \eta_2^2$ , et ensuite, par continuité dans les expressions, on peut étendre les résultats à  $\eta_1^2 = \eta_2^2$ . On obtient

$$\begin{aligned}\frac{dy_1}{dx} &= -\varepsilon(1+x)\eta_1(q(x, \eta))^{-\frac{1}{2}} \\ \frac{dy_2}{dx} &= \varepsilon\eta_2(q(x, \eta))^{-\frac{1}{2}}.\end{aligned}$$

Ce système est exactement  $\frac{dy_j}{dx} = -\frac{1}{2}\varepsilon(q(x, \eta))^{-\frac{1}{2}}\partial_{\eta_j} q(x, \eta)$ , qui se réécrit  $\frac{dy_j}{dx} = -\varepsilon\partial_{\eta_j}(q(x, \eta))^{\frac{1}{2}}$ . Utilisant la fonction  $S$ , on trouve  $\frac{dy_j}{dx} = \frac{d}{dx}\varepsilon\partial_{\eta_j} S(x, \eta)$ . On en déduit l'égalité

$$y_j(x) = -\varepsilon \frac{\partial S}{\partial \eta_j}(x, \eta).$$

La fonction  $S$  est explicite. Il s'agit de calculer

$$\int_0^x ((1+u)\eta_1^2 - \eta_2^2)^{\frac{1}{2}} du = \left[ \frac{2}{3\eta_1^2} ((1+u)\eta_2^2 - \eta_1^2)^{\frac{3}{2}} \right]_0^x = \frac{2}{3\eta_1^2} [((1+x)\eta_2^2 - \eta_1^2)^{\frac{3}{2}} - (\eta_2^2 - \eta_1^2)^{\frac{3}{2}}].$$

Comme de plus le signe de  $\frac{dy_1}{dx}$  est celui de  $-\varepsilon\eta_1$  et que l'on n'est concerné que par les bicaractéristiques dans  $x > 0$  qui vérifient  $y_1 > 0$ , on sait que  $\varepsilon$  est le signe de  $-\eta_1$ . Ceci permet de justifier l'introduction de la fonction  $\phi(x, y, \theta)$ .

7) On vérifie que la relation  $\nabla_{\theta}\phi(x, y, \theta) = 0$  implique

$$y_j = \text{sign}(\theta_1)\nabla_{\theta_j} S(x, \theta)$$

On vérifie de plus que l'expression  $\eta = \nabla_y\phi$  implique que  $\eta = \theta$ , et la relation  $\xi = \nabla_x\phi$  implique  $\xi = -\text{sign}(\theta_1)(q(x, \theta))^{\frac{1}{2}}$ . Toutes ces relations conduisent à

$$\theta = \eta, y(x) = \text{sign}(\eta_1)\partial_{\eta} S(x, \eta), \xi = -\text{sign}(\eta_1)(q(x, \eta))^{\frac{1}{2}}.$$

Le point de  $\Sigma$  considéré est alors le point sur la bicaractéristique issue de  $(0, 0, 0, -\text{sign}(\eta_1)(\eta_1^2 - \eta_2^2)^{\frac{1}{2}}, \eta_1, \eta_2)$  d'abscisse  $x$ . Réciproquement, un point d'une bicaractéristique issue de l'origine est dans  $\Sigma$ .

8) Cette démonstration est une adaptation facile de la démonstration de la proposition 6.2. On se donne une fonction  $\chi$  à support compact dans  $\mathbb{R}_+ \times \mathbb{R}^2$ . On écrit

$$(K^{(2)}, \chi) = \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^2} \int a_2(x, y, \theta) \sigma_\varepsilon(\theta) \exp\left(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})\right) \chi(x, y) e^{iy \cdot \theta} dx dy d\theta.$$

On définit le cône contenu dans  $\Sigma$ :

$$\Gamma_1 = \{(\xi, \eta_1, \eta_2) \in \mathbb{R}^3, \xi = \partial_x \phi, \eta = \nabla_y \phi = \theta, (x, y) \in \text{supp} \chi, \eta_1^2 \geq \eta_2^2 > 0\}.$$

On vérifie tout d'abord que

$$\frac{2}{3}(Z^{\frac{3}{2}} - Z_0^{\frac{3}{2}}) = S(x, \theta_1, \theta_2)$$

(ce sont les quantités  $Z$  et  $Z_0$  introduits dans le 1) de la section 3. On rappelle aussi que sur le support de  $a_2$ ,

$$-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}) = -\frac{2}{3}i \text{sign}(\theta_1)(Z^{\frac{3}{2}} - Z_0^{\frac{3}{2}}).$$

Il reste donc, au sens des distributions et à une troncature près

$$\begin{aligned} (K^{(2)}, \chi) &= \frac{1}{(2\pi)^2} \int a_2(x, y, \theta) \chi(x, y) e^{iy \cdot \theta - i \text{sign}(\theta_1) S(x, \theta)} dx dy d\theta \\ &= \frac{1}{(2\pi)^2} \int a_2(x, y, \theta) \chi(x, y) e^{i\phi(x, y, \theta)} dx dy d\theta \end{aligned}$$

Nous calculons  $\xi - \partial_x \phi$ . Nous trouvons ainsi

$$\xi - \frac{2}{3\theta_1^2} \frac{3\theta_1^2}{2} ((1+x)\theta_1^2 - \theta_2^2)^{\frac{1}{2}} = \xi - ((1+x)\theta_1^2 - \theta_2^2)^{\frac{1}{2}}.$$

De même,  $\partial_y \phi = \theta$ . On suppose  $(\xi, \eta) \in \Gamma_2$  tel que  $\Gamma_2 \cap \Gamma_1 = \emptyset$ . Il existe  $C > 0$  tel que

$$\text{mod}(x, y, \xi, \eta) = (\xi - \partial_x \phi)^2 + |\eta - \nabla_y \phi|^2 \geq C(|\theta| + |\xi| + |\eta|)^2.$$

L'opérateur  $M$  dont l'adjoint est

$${}^t M = (\text{mod}(x, y, \xi, \eta))^{-1} [(\xi - \partial_x \phi) \frac{\partial}{\partial x} + (\eta - \nabla_y \phi) \cdot \frac{\partial}{\partial y}]$$

est alors un opérateur adéquat pour le théorème de la phase non stationnaire, et on a  $e^{i(\phi(x, y, \theta) - x\xi - y \cdot \eta)} = {}^t M e^{i(\phi(x, y, \theta) - x\xi - y \cdot \eta)}$ .

On vérifie ainsi que l'intégrand dans  $(K^{(2)}, \chi)$  peut être remplacé, pour tout  $p$  par  $M^k(a_2 \chi)$ . On utilise le résultat de régularité sur le symbole  $a_2 \in S_{\frac{1}{3}, \frac{2}{3}}^0$  pour conclure que cette intégrale est décroissante aussi rapidement que toute puissance de  $(|\xi| + |\eta|)^{-1}$ .

Il vient donc que

$$\Gamma_2 \cap \Gamma_1 = \emptyset \Rightarrow \text{supp} \chi \cap WF(K^{(2)}) = \emptyset.$$

Nous avons achevé la preuve de cet exercice.



## Lagrangian solutions of the characteristic equation

### 1. Definition of Lagrangian solutions

The Lagrangian manifold  $\Lambda_\phi$  introduced in section 3.6 has an important but restrictive property: the canonical projection from  $T^*X$  to  $X$  is invertible. Relaxing this assumption allows to account for caustics. Indeed, at a point close to the caustic, the manifold will be represented through several phases, which is equivalent to the existence of several rays passing through this point. The corresponding bicharacteristics, on the other hand, do not intersect.

We now consider an operator  $P$  on  $C^\infty(X)$ , with principal symbol  $p(x, \xi)$ . If  $P$  is a differential operator, a phase is a solution of the eikonal equation associated to  $P$  when  $p(x, d\phi(x)) = 0$ . In other words,  $\Lambda_\phi \subset \text{Car}(P)$ .

A generalization of this result has been proved in the previous chapter (lemma 8.1) for pseudo-differential operators.

In this chapter, we study the Lagrangian manifolds associated to a differential or pseudo-differential operator. They are called **Lagrangian solutions of the characteristic equation**  $p = 0$  or **of the operator**  $P$ . The characteristic manifold (or characteristic equation) is defined by  $p(x, \xi) = 0$ ,  $p$  being the principal symbol of the pseudo-differential operator  $P$ . These Lagrangian manifolds are the generalization of the  $\Lambda_\phi$  manifolds where  $\phi$  is a solution of the eikonal equation  $p(x, \nabla_x \phi) = 0$ .

DEFINITION 9.1. *A Lagrangian solution  $\Lambda$  of  $p = 0$  is a manifold*

- maximal ( $\dim T_{(x_0, \xi_0)} \Lambda = \dim T_{x_0}(X)$ ),
- ( $\sigma|_\Lambda = 0$ ),
- solution ( $p|_\Lambda = 0$ ). We will sometimes omit  $p = 0$  to write only  $p$ .

We have the following results

- PROPOSITION 9.1. (1) *If a bicharacteristic curve intersects the characteristic manifold, it is contained in the characteristic manifold,*  
 (2) *If a bicharacteristic curve of  $p$  intersects  $\Lambda$ , it is contained in  $\Lambda$ .*

We noticed before that the principal symbol is invariant on the integral curves of the Hamiltonian. We deduce the first item of the proposition 9.1.

A weaker version of the second item of the proposition 9.1 is:

PROPOSITION 9.2. *Let  $\phi$  be a solution of the eikonal equation. If a bicharacteristic curve intersects  $\Lambda_\phi$ , it is contained in  $\Lambda_\phi$ .*

PROOF. We begin by proving 9.2.

We consider a homogeneous polynomial symbol of order 2,  $p(x, \xi) = \sum_{i,j} a_{i,j}(x) \xi_i \xi_j$ . We assume that the Cauchy data  $(x(0), \xi(0)) = (x_0, \xi_0)$  of Hamilton's equations (2.108), , verify  $\nabla \phi(x_0) = \xi_0$  where  $\phi$  is solution of  $p(x, \nabla_x \phi(x)) = 0$ .

The equation satisfied by  $\phi$  is:

$$(1.120) \quad p(x, \nabla \phi(x)) = 0$$

Derivating along each variable, one obtains the identity

$$(1.121) \quad (\partial_{x_i} p + \sum_j \partial_{x_i x_j}^2 \phi \partial_{\xi_j} p)|_{\Lambda_\phi} = 0.$$

Let  $\omega_i(s) = \xi_i(s) - \partial_{x_i} \phi(x(s))$ ,  $i \leq i \leq n$ ,  $n = \dim(T_x X)$ . The system of ordinary differential equations that  $\omega$  solves is

$$-\frac{d\omega_i}{ds} = \partial_{x_i} p(x(s), \xi(s)) - \sum_j \partial_{x_j x_i}^2 \phi(x(s)) \partial_{\xi_j} p(x(s), \xi(s)).$$



One notices that  $\omega(0) = 0$ . Expressing the relation (1.120) at the point  $x(s)$ , and subtracting the equality obtained to the equation giving  $\frac{d\omega_i}{ds}$ , we obtain:

$$-\frac{d\omega_i}{ds} = -\sum_j \partial_{x_j x_i}^2 \phi(x(s)) [\partial_{\xi_j} p(x(s), \xi(s)) - \partial_{\xi_j} p(x(s), \nabla \phi(x(s)))] \\ + \partial_{x_i} p(x(s), \xi(s)) - \partial_{x_i} p(x(s), \nabla \phi(x(s))).$$

Using  $\xi_i \xi_j - \eta_i \eta_j = \eta_j (\xi_i - \eta_i) + (\xi_i - \eta_i) (\xi_j - \eta_j) + \eta_i (\xi_j - \eta_j)$ , one finds:

$$-\frac{d\omega_i}{ds} = \sum_{j,k} \partial_{x_i} a_{j,k}(x(s)) \omega_j(s) \omega_k(s) + \sum_{j,k} \partial_{x_i} a_{j,k}(x(s)) (\xi_j(s) \omega_i(s) + \xi_i(s) \omega_j(s)) \\ - 2 \sum_{j,k} \partial_{x_j x_i}^2 \phi(x(s)) a_{j,k}(x(s)) \omega_k(s).$$

which is of the form  $\frac{d\omega}{ds} = A(s)\omega(s) - B(s)(\omega(s), \omega(s))$ . As  $\omega(0) = 0$ , the Cauchy-Lipschitz theorem gives that there exists  $s_0 > 0$ , depending on the solutions on  $[0, T]$  and not on the initial point, such that  $\omega(s) = 0$  on  $[0, s_0)$  because of uniqueness of the solution  $\omega = 0$  which is a trivial solution. We note to complete the proof that we can reproduce the previous argument with initial data  $(x(s_0), \xi(s_0))$ , hence the equality  $\xi(s) = \nabla \phi(x(s))$  for  $0 \leq s \leq 2s_0$ . We can continue the argument until  $[0, T]$ , since the extension of the solution is done on an interval independent on the initial datum.

In the case where the symbol is no longer polynomial, by using the Taylor formula with integral remainder, we find that  $\partial_{x_i} p(x(s), \xi(s)) - \partial_{x_i} p(x(s), \nabla \phi(x(s))) = G_i(x(s), \xi(s), \nabla \phi(x(s))) \cdot \omega(s)$  and  $\partial_{\xi_j} p(x(s), \xi(s)) - \partial_{\xi_j} p(x(s), \nabla \phi(x(s))) = R_j(x(s), \xi(s), \nabla \phi(x(s))) \cdot \omega(s)$  and the system is written  $\dot{\omega} = H(s) \cdot \omega(s)$ ,  $H$  being known using the explicit functions  $x(s)$ ,  $\xi(s)$  and  $\phi(x)$ , and we apply again Cauchy-Lipschitz theorem.

The proof of the second item of proposition 9.1 comes from the maximality of the isotropic manifold at  $(x_0, \xi_0)$ . Thus there exists a coordinate system  $y = (y_1, \dots, y_n)$  such that  $\Lambda$  is written, in the neighborhood of  $(x_0, \xi_0)$ ,  $\{(x(y), \xi(y)), y \in \mathbb{R}^n \cap V\}$ . We use the two relations

$$\begin{cases} p(x(y), \xi(y)) = 0 \\ \omega|_{\Lambda} = 0 \end{cases}$$

Equality  $\omega|_{\Lambda} = 0$  writes

$$(1.122) \quad \sum_i \frac{\partial \xi_i}{\partial y_j} \frac{\partial x_i}{\partial y_k} - \sum_i \frac{\partial \xi_i}{\partial y_k} \frac{\partial x_i}{\partial y_j} = 0$$

(this is the equivalent, when  $x(y) = y$ , of  $\nabla \wedge \xi(x) = 0$ , hence  $\xi(x) = \nabla \psi(x)$ .) The manifold is Lagrangian maximal. What changes with the previous proof is that the relation  $x(y)$  is not necessarily invertible (the canonical projection in this case is not surjective). However, since the manifold is maximal, there exists a subset of  $(x(y), \xi(y))$ , denoted  $(x'(y), \xi''(y))$ , where  $x'$  has  $p$  coordinates and  $\xi''$  has  $q$  coordinates with  $p + q = n$ . We have  $\det(\text{Jac}(x'(y), \xi''(y))) \neq 0$  at  $y = 0$ . We then define  $\mathcal{J}^{-1}$  the application such that  $y = \mathcal{J}^{-1}(x'(y), \xi''(y))$ .

Remark that there is no uniqueness of the choice of coordinates  $x', \xi''$ , we thus fix a choice of variables  $x$  and  $\xi$  which is bijective. Let  $J_1$  the set of indices corresponding to the coordinates of  $x'(y)$  and  $J_2$  the set of indices corresponding to the coordinates of  $\xi''(y)$ . If  $J_1 \cap J_2 = \emptyset$ , we go directly to the study of the manifold, knowing that  $J_1 \cup J_2 = \{1, \dots, n\}$ .

If  $J_1 \cap J_2 = J \neq \emptyset$ , then the coordinates forming a system of independent coordinates are then

$$[(x_j)_{j \in J_1 \setminus J}, (x_j)_{j \in J}, (\xi_j)_{j \in J}, (\xi_j)_{j \in J_2 \setminus J}].$$

We note (disjoint unions)

$$\{1, \dots, n\} = J_1 \cup K_1, \{1, \dots, n\} = (J_1 \setminus J) \cup J \cup (J_2 \setminus J) \cup K_2^+, K_2 = (J_1 \setminus J) \cup K_2^+.$$

Let  $k \in K_1$ . By the local inversion theorem based on the fact that the family  $(x', \xi'')$  is maximal, there exists a function  $X_k$  such that

$$x_k(y) = X_k[(x_j)_{j \in J_1 \setminus J}, (x_j)_{j \in J}, (\xi_j)_{j \in J}, (\xi_j)_{j \in J_2 \setminus J}].$$

Similarly if  $l \in K_2$ , there exists a function  $\Xi_l$  such that

$$\xi_l(y) = \Xi_l[(x_j)_{j \in J_1 \setminus J}, (x_j)_{j \in J}, (\xi_j)_{j \in J}, (\xi_j)_{j \in J_2 \setminus J}].$$

Let  $p \in J$ . Then, for  $k \in K_1$ ,  $\{x_p, x_k\} = \frac{\partial x_k}{\partial \xi_p} = 0$ , and therefore the functions  $X_k$  are independent of the variables  $\xi_p$ ,  $p \in J$ . As, moreover,  $\{\xi_p, x_k\} = 0$  for  $p \in J$  and  $k \in K_1$  thanks to  $J \cap K_1 = \emptyset$ ,  $X_k$  does not depend on the variables  $\xi_p$ ,  $p \in J$ . Similarly, the variables  $\xi_l$  for  $l \in K_2$  of  $(x_p, \xi_p)$ ,  $p \in J$  are independent. We write

$$x_k(y) = Y_k[(x_j)_{j \in J_1 \setminus J}, (\xi_j)_{j \in J_2 \setminus J}].$$

$$\xi_l(y) = \Sigma_l[(x_j)_{j \in J_1 \setminus J}, (\xi_j)_{j \in J_2 \setminus J}].$$

The Lagrangian manifold is therefore written (locally)

$$\Lambda = \left\{ \begin{array}{l} [(x_j)_{J_1 \setminus J}, (x_j)_J, (Y_k[(x_j)_{J_1 \setminus J}, (\xi_j)_{J_2 \setminus J}])_{k \in K_1}, (\Sigma_l[(x_j)_{J_1 \setminus J}, (\xi_j)_{J_2 \setminus J}])_{l \in J_1 \setminus J}] \\ , (\xi_p)_J, (\xi_l)_{J_2 \setminus J}, (\Sigma_l[(x_j)_{J_1 \setminus J}, (\xi_j)_{J_2 \setminus J}])_{l \in K_2^+} \end{array} \right\}.$$

Thus we notice that locally  $\Lambda = \tilde{\Lambda} \times \mathbb{R}^{2 \dim J}$ , where  $\tilde{\Lambda}$  is a Lagrangian manifold in  $\mathbb{R}^{n - \dim J}$  (depending only on the coordinates  $(x_j)_{j \in J_1 \setminus J}, (\xi_j)_{j \in J_2 \setminus J}$ ). We are back to the elementary case since  $(J_1 \setminus J) \cap (J_2 \setminus J) = \emptyset$ . The independent coordinates are of the form  $(x', \xi'')$  (no common index). This is the case we will consider from now on.

Let  $(X(s), \Xi(s))$  be the bicharacteristic curve such that  $(X(0), \Xi(0)) = (x_0, \xi_0) = (x(0), \xi(0))$ . We thus deduce  $y(s)$  from  $(X'(s), \Xi''(s))$  through

$$(X'(s), \Xi''(s)) = (x(y(s)), \xi(y(s)))$$

in a neighborhood of  $y = 0$  and therefore in a neighborhood of  $s = 0$  since in this neighborhood, we have a diffeomorphism  $(y(s) = \mathcal{J}^{-1}(X'(s), \Xi''(s)))$ .

Let  $(V(s), W(s))$  be the vector  $(x(y(s)), \xi(y(s)) - (X(s), \Xi(s)))$ . By construction, half of its components are zero.

We verify the equality:

$$\begin{aligned} & \sum_l \frac{\partial \xi_l}{\partial y_m}(y(s)) \frac{dV_l}{ds} - \sum_{l'} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{dW_{l'}}{ds} \\ (1.123) \quad & = \\ & \sum_{l,j} \frac{\partial \xi_l}{\partial y_m} \frac{\partial x_l}{\partial y_j} \frac{dy_j}{ds} - \sum_l \frac{\partial \xi_l}{\partial y_m}(y(s)) \frac{\partial p}{\partial \xi_l}(X(s), \Xi(s)) \\ & - \sum_{l',j} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{\partial \xi_{l'}}{\partial y_j}(y(s)) \frac{dy_j}{ds} - \sum_{l'} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{\partial p}{\partial x_{l'}}(X(s), \Xi(s)) \end{aligned}$$

Using the equality (1.122), valid on  $\Lambda$ , we exchange the derivatives in  $m$  and  $j$  in the first term of the right-hand side. We deduce the equality  $\sum_{l,j} \frac{\partial \xi_l}{\partial y_m} \frac{\partial x_l}{\partial y_j} \frac{dy_j}{ds} = \sum_{l',j} \partial x_{l'}(y(s)) \frac{\partial \xi_{l'}}{\partial y_j}(y(s)) \frac{dy_j}{ds}$ . For the other terms, we derive with respect to all  $y_j$ ,  $1 \leq j \leq n$ , the equality  $p|_{\Lambda} = 0$ . Then

$$\left[ \sum_j \frac{\partial p}{\partial x_j} \frac{\partial x_j}{\partial y_m} + \sum_{n'} \frac{\partial p}{\partial \xi_{n'}} p \cdot \frac{\partial \xi_{n'}}{\partial y_m} \right] |_{(x(y(s)), \xi(y(s)))} = 0.$$

Along with the equality (1.123) one has

$$\begin{aligned} & \sum_l \frac{\partial \xi_l}{\partial y_m}(y(s)) \frac{dV_l}{ds} - \sum_{l'} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{dW_{l'}}{ds} \\ & = \\ & \sum_l \frac{\partial \xi_l}{\partial y_m}(y(s)) \frac{\partial p}{\partial \xi_l}(x(y(s)), \xi(y(s))) - \sum_l \frac{\partial \xi_l}{\partial y_m}(y(s)) \frac{\partial p}{\partial \xi_l}(X(s), \Xi(s)) \\ & + \sum_{l'} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{\partial p}{\partial x_{l'}}(x(y(s)), \xi(y(s))) - \sum_{l'} \frac{\partial x_{l'}}{\partial y_m}(y(s)) \frac{\partial p}{\partial x_{l'}}(X(s), \Xi(s)). \end{aligned}$$

So we have  $n$  equations ( $1 \leq m \leq n$ ), and  $2n$  unknowns of which  $n$  are zero by hypothesis. The assumption of maximality leads to the fact that the system

$$\frac{\partial \xi''}{\partial y_m}(y(s)) \cdot \frac{dV''}{ds} - \frac{\partial x'}{\partial y_m}(y(s)) \frac{dW'}{ds} = T_m(s)$$

is invertible.

We can then write the equations verified by  $(V''(s), W'(s))$  in the form

$$\begin{aligned} \frac{d}{ds}(V''(s), W'(s)) & = C(s, V''(s), W'(s)) \cdot (V''(s), W'(s)) \\ V''(0) & = 0, W'(0) = 0 \end{aligned}$$

which is again a Cauchy problem. Its unique solution is  $(V''(s), W'(s)) = 0$ . The second item of Proposition 9.1 is proven.  $\square$

We denote for later purposes by:

$$\exp(sH_p(x_0, \xi_0)) = (x(s), \nabla \phi(x(s)))$$

the solution of (2.108), notation used because  $\exp(sH_p(x_0, \xi_0))$  is the solution of the Cauchy equation<sup>1</sup>.

## 2. Representation of Lagrangian solutions through phases

**2.1. Lagrangian maximal solution associated with a phase.** We specify here the notion that we had mentioned in the chapter 3 when calculating the asymptotic expansion of the wave solution with a given condition on a hypersurface  $\Sigma_0$ . In fact, we had considered on the set of points  $(y, \nabla\phi(y))$ ,  $y \in \Sigma_0$  (on which we assumed that  $\phi$  was constant).

We consider a pseudodifferential operator  $p$  on  $X$ , a manifold of dimension  $n$ . We give a hypersurface  $S$  in  $X$  and a phase  $\phi_0$  (regular function) defined on  $S$ . Let  $l$  be the natural injection of  $T^*S$  into  $T^*X$ . Let  $\pi$  also be the natural projection of  $\{(x, \xi), x \in S, \xi \in T_x^*X\}$  to  $T^*S$ , defined as  $\pi(x, \xi) = (x, \zeta)$  where the linear form  $\zeta$  on  $T_x^*S$  is the restriction of  $\xi$  to  $T_x^*S$ . It is assumed that  $\pi^{-1}\{(x, d\phi_0(x))\} \cap \text{Carp} \neq \emptyset$ .

It is further assumed that the Hamiltonian field  $H_p$  is transverse to  $l(\Lambda_{\phi_0}^S)$  at  $\rho_0 \in l(\Lambda_{\phi_0}^S) \cap \text{Carp}$ . Then the maximal Lagrangian solution associated to  $S$  and  $\phi_0$  in the neighborhood of  $\rho_0$  is denoted by  $\Lambda(\phi_0)$ , and is the union of the bicharacteristic curves coming from a point of  $\pi^{-1}(\Lambda_{\phi_0}^S)$ .

Indeed,  $\Lambda(\phi_0)$  is maximal because  $H_p$  is transverse to  $l(\Lambda_{\phi_0}^S)$  at  $\rho_0$ , which is of dimension  $n-1$ , and the maximality comes from  $T_{\rho_0}(\Lambda(\phi_0)) = T_{\rho_0}l(\Lambda_{\phi_0}^S) + H_p\mathbb{R}$ . It is solved by the proposition 9.1. It is Lagrangian because the bicharacteristic curves are, always by the proposition 9.1.

The bicharacteristic curves are then projected on the characteristic curves  $x(\sigma)$  solutions of  $\frac{dx}{d\sigma} = \xi(x(\sigma))$  as long as  $\Lambda(\phi_0)$  is transverse at  $x(\sigma)$  to the fiber of the canonical projection of  $T^*X$  on  $X$ . This construction by means of rays fails when the property of transversality is no longer verified. We then say that the point belongs to the caustic subset of the Lagrangian manifold.

**2.2. Generalization to phases with parameters.** This extension should be generalized to phases, solution of the eikonal equation, involving an additional parameter. Now the points of the caustics are the points of the Lagrangian manifold considered where this projection is no longer proper, i.e. where we can no longer consider a phase  $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$  (without parameter) such that  $\Lambda = \Lambda_\psi$ .

Give ourselves a phase  $\phi(x, \theta)$  with  $N$  parameters ( $\theta \in \mathbb{R}^N$ ), non-degenerate ( $\nabla_{x, \theta}\phi(x, \theta) \neq 0$ ) and one associates to it

$$\Lambda_\phi = \{(x, \nabla_{x, \theta}\phi(x, \theta)), \nabla_\theta\phi(x, \theta) = 0\}.$$

LEMMA 9.1. *The manifold  $\Lambda_\phi$  is embedded in a Lagrangian manifold.*

Proof. We verify that the application  $i_\phi$ , which goes from the set of critical points of  $\phi$  in the variable  $\alpha$  denoted by  $C_\phi = (x, \theta), \nabla_\theta\phi(x, \theta) = 0$  to  $T^*X$

$$i_\phi(x, \theta) = (x, \nabla_x\phi(x, \theta)) \in T^*X$$

is a Lagrangian immersion, and its image is  $\Lambda_\phi$ .

We verify that  $d\phi = \sum_j \partial_{x_j}\phi(x, \theta)dx_j + \sum_{p=1}^N \partial_{\theta_p}\phi(x, \theta)d\theta_p \in \Lambda^1(X \times \mathbb{R}^N)$  which gives, thanks to the fact the point  $(x, \theta)$  is a critical point for  $\phi$

$$d\phi|_{C_\phi} = \sum_j \partial_{x_j}\phi(x, \theta)dx_j.$$

We deduce  $i_\phi^*(\sum_j \xi_j dx_j) = d\phi|_{C_\phi}$ , which gives, by commutation of  $d$  and  $i_\phi$

$$i_\phi^*(d(\sum_j \xi_j dx_j)) = d^2\phi = 0,$$

hence  $i_\phi^*(\sum_j d\xi_j \wedge dx_j) = 0$ . The manifold is locally isotropic. Since  $\phi$  is non-degenerate,  $C_\phi$  is a submanifold of dimension  $n$  and we deduce that  $\Lambda_\phi$ , image of a submanifold of dimension  $n$  by an immersion, is maximal. It is thus locally a Lagrangian manifold.

**2.3. The Hörmander representation theorem.** One has

THEOREM 9.1. *Let  $\Lambda$  be a Lagrangian submanifold of  $T^*X$  and let  $\rho_0$  be a point of this Lagrangian.*

*There exists a symplectic coordinate system on  $T^*X$ , denoted by  $(x, \xi)$  and a non-degenerate phase  $\phi(x, \xi)$  such that,  $\mathcal{O}$  being a neighborhood of  $\rho_0$  in  $T^*X$ ,*

$$\Lambda \cap \mathcal{O} = \{(x, \nabla_x\phi(x, \xi)), \nabla_\xi\phi(x, \xi) = 0\}.$$

This theorem is stated in [48]. One deduces

<sup>1</sup>This notation is easy to understand; when  $H_p$  is a scalar function  $H_p(\rho(s)) = a(\rho(s))$ , the Cauchy problem above has as solution  $a(\rho(s)) = a(\rho_0)e^s$ .  $\frac{d}{ds}(\rho(s)) = H_p(\rho(s))$ ,  $\rho(0) = \rho_0$ .

COROLLARY 9.1. *There exists a diffeomorphism  $\chi$  of  $\mathbb{R}^n$  in  $X$ ,  $n$  being the dimension of  $T_{\rho_0}X$ , such that*

$$h_\chi(\Lambda_\psi \cap h_\chi^{-1}(\mathcal{O})) = \Lambda \cap \mathcal{O}$$

where  $\psi(y, \eta) = y \cdot \eta - H(\eta)$ . This diffeomorphism is constructed with Darboux's lemma and the representation of the Lagrangian in the neighborhood of  $\rho_{h_0}$  by

$$(2.124) \quad \Lambda \cap \mathcal{O} = \{(x(y), \xi(y)), y \in \mathbb{R}^n \cap \chi^{-1}(\mathcal{O})\}.$$

PROOF. The proof of the theorem is based on the following technical lemma, which will be proved below:

LEMMA 9.2. *There exists a coordinate system of  $X$  in the neighborhood of  $x_0$  such that, in the associated canonical symplectic coordinate system  $(x, \xi)$ , the Lagrangian is defined by  $x = X(\xi)$ .*

Assuming that this lemma is proved, we know that  $\sigma = \sum_j \xi_j dx_j$  is zero on  $\Lambda$ . This gives

$$\sum_{j=1}^n \xi_j \frac{\partial X_j}{\partial \xi_p}(\xi) = 0.$$

Relation  $d(\sum_{j=1}^{j=n} \xi_j x_j) = \sum_{j=1}^n \xi_j dx_j + \sum_{j=1}^n x_j d\xi_j$  yields, on  $\Lambda$

$$d\left(\sum_{j=1}^{j=n} \xi_j X_j\right) = \sum_{j=1}^{j=n} X_j(\xi) d\xi_j.$$

Let  $H(\xi) = \sum_{j=1}^{j=n} \xi_j X_j(\xi)$ . One checks that  $\partial_{\xi_p} H(\xi) = \sum_{j=1}^{j=n} \xi_j \frac{\partial X_j}{\partial \xi_p} + X_p(\xi) = X_p(\xi)$  and  $H$  is homogeneous of degree 1.

If one introduces  $\phi(x, \xi) = x \cdot \xi - H(\xi)$ , one finds  $\nabla_x \phi(x, \xi) = \xi$  et  $\nabla_{\xi_p} \phi(x, \xi) = x_p - X_p(\xi)$ . Hence the Lagrangian manifold is locally represented through  $\phi$ , that is

$$\Lambda \cap \mathcal{O} = \{(x, \xi), x = X(\xi)\} = \{(x, \nabla_x \phi), \nabla_{\xi} \phi(x, \xi) = 0\} \cap \mathcal{O} = \Lambda_\phi \cap \mathcal{O}.$$

We have proved the representation theorem.

Let us now prove the lemma 9.2. For this, we use the representation (2.124) of  $\Lambda$ . The family  $dx_1(y), \dots, dx_n(y), d\xi_1(y), \dots, d\xi_n(y)$  defines a space of dimension  $n$  in the neighborhood of the point  $\rho_0$  considered, since  $\Lambda$  is maximal (hence of dimension  $n$ ). Let us denote by  $J$  and  $K$  respectively the set of independent coordinates of  $dx_1, \dots, dx_n$  and  $d\xi_1, \dots, d\xi_n$ . We denote, from now on, by  $(y, \eta)$  the symplectic coordinate system on  $\mathbb{R}^n$  associated to  $(y)$ , and we introduce the natural injection  $j$  of  $C(\mathbb{R}^n)$  into  $C(T^*\mathbb{R}^n)$  by  $j(f)(y, \eta) = f(y)$ . The coordinate system  $(x, \xi)$  is symplectic, so we have the relations

$$\{j(x_j)(y, \eta), j(x_{j'})(y, \eta)\} = 0, \{j(\xi_k)(y, \eta), j(\xi_{k'})(y, \eta)\} = 0, \{j(x_j)(y, \eta), j(\xi_k)(y, \eta)\} = \delta_{jk}.$$

Let  $p_j(y, \eta) = j(x_j)(y, \eta) = x_j(y)$  and  $q_k(y, \eta) = j(\xi_k)(y, \eta) = \xi_k(y)$ . These functions on  $T^*\mathbb{R}^n$  satisfy the hypotheses of the lemma 8.3.

We can then complete this family into a family of symplectic coordinates on  $T^*X$ , denoted by  $p_1(y, \eta), p_n(y, \eta), q_1(y, \eta), \dots, q_n(y, \eta)$ .

Let us write the result of Darboux lemma. We check that the family  $(dp_1, \dots, dp_n, dq_1, \dots, dq_n)$  is linearly independent. As it is an application from  $T^*X$  to  $T^*\mathbb{R}^n$ , writing it in matrix form, one has

$$\begin{pmatrix} A_J & 0 \\ B_K & 0 \\ C_{n-J}^1 & C_{n-J}^2 \\ D_{n-K}^1 & D_{n-K}^2 \end{pmatrix}$$

where we reorded the coordinates in  $T^*X$  into  $j \in J, k \in K, j' \in \{1, \dots, n\} - J, k' \in \{1, \dots, n\} - K$ . We used here that  $(p_j)_{j \in J}$  and  $(q_k)_{k \in K}$  depend only on  $y$ .

The matrices  $A_J$  and  $B_K$  are respectively the matrices  $J_n$  and  $K_n$  of the  $(\partial_{y_l} p_j)_{j \in J, 1 \leq l \leq n}$  and  $(\partial_{y_m} q_k)_{k \in K, 1 \leq m \leq n}$ . Matrices  $C_{n-J}^2$  et  $D_{n-K}^2$  are respectively

$$(\partial_{y_l} q_{k'})_{k' \in \{1, \dots, n\} - J, 1 \leq l \leq n}, (\partial_{y_l} p_{j'})_{j' \in \{1, \dots, n\} - K, 1 \leq l \leq n}.$$

Since the family  $(dp_1, \dots, dp_n, dq_1, \dots, dq_n)$  is linearly independent, we deduce that the matrices  $n \times n$

$$\begin{pmatrix} A_J \\ B_K \end{pmatrix}, \begin{pmatrix} C_{n-J}^2 \\ D_{n-K}^2 \end{pmatrix}$$

are invertible in the neighborhood of  $(y_0, \eta_0)$ , image of  $\rho_0$  by the diffeomorphism induced by the symplectic change of coordinates.

We deduce that the system where  $\eta$  is the parameter

$$\begin{cases} x_{j'}(y) = p_{j'}(y, \eta), j' \in \{1, \dots, n\} - J \\ \xi_{k'}(y) = q_{k'}(y, \eta), k' \in \{1, \dots, n\} - K \end{cases}$$

admits, by the implicit functions theorem, a unique solution  $\eta = Y(y)$ . The result of the lemma (by transforming the notations) comes from the fact that  $(\eta, -y)$  is a symplectic coordinate system on  $T^*\mathbb{R}^n$ . The phase is then known, it suffices to write  $\phi(y, \eta) = -\eta y - G(y)$ , with  $G(y) = \sum_j y_j Y(y)$ .

We have completed the proof of the lemma 9.2.  $\square$

The caustics (points where the stationary phase theorem does not apply in an oscillating integral defining a solution) can be studied thanks to Lagrangian solutions. Lagrangian solutions.

### 3. Caustic points

We give ourselves a phase  $\phi(x, \theta)$  with  $N$  parameters, non-degenerate ( $\text{Hess}_\theta \phi(x, \theta) \neq 0$ ), such that

$$\Lambda_\phi = \{(x, \nabla_x \phi(x, \theta)), \nabla_\theta \phi(x, \theta) = 0\}$$

is a Lagrangian solution of  $P$  (pseudo)differential operator.

We define (using a notion identical to that of a Fourier integral operator) an oscillatory function

$$(3.125) \quad u_{\phi, a}(x, k) = \int_{\mathbb{R}^N} e^{ik\phi(x, \theta)} a(x, \theta, k) d\theta$$

where the amplitude  $a$  has an asymptotic expansion (see the definition 1.1).

We give the definition of two equivalent phases:

DEFINITION 9.2. *Let  $\phi_1(x, \alpha)$  and  $\phi_2(y, \beta)$  be two phases defined on  $X \times \mathbb{R}^N$  ( $N$  is the number of parameters of the phase).*

*They are equivalent respectively in  $(x_0, \alpha_0)$  and  $(y_0, \beta_0)$  if there exists a diffeomorphism*

$$X \times \mathbb{R}^N \rightarrow X \times \mathbb{R}^N$$

$$(x, \alpha) \rightarrow (A(x), \Phi(x, \alpha))$$

*such that  $A(x_0) = y_0$ ,  $\Phi(x_0, \alpha_0) = \beta_0$  and a function  $\psi(x)$ ,  $C^\infty$  in the neighborhood of  $x_0$ , such that*

$$\phi_1(x, \alpha) = \phi_2(A(x), \Phi(x, \alpha)) + \psi(x)$$

Two equivalent Lagrangians (in the neighborhood of a point, the classical terminology is to speak of a germ of Lagrangians) are, in an analogous way, defined by

DEFINITION 9.3. *Two Lagrangians  $\Lambda_1$  (in the neighborhood of  $\lambda_1$ ) and  $\Lambda_2$  (in the neighborhood of  $\lambda_2$ ) are equivalent if and only if there exists a symplectic diffeomorphism  $\chi$  that leaves invariant the fibers of the canonical projection  $\pi$  of  $T^*X$  on  $X$  and such that  $\chi(\Lambda_1) = \Lambda_2$ .*

We have the (local) uniqueness of the representation by the two following propositions:

PROPOSITION 9.3. *Let  $\phi$  and  $\phi_1$  be two phases defined respectively on  $X \times \mathbb{R}^N$  and on  $X \times \mathbb{R}^{n_1}$ . For  $x_0 \in X$ , the oscillatory functions of the form  $u_{\phi, a}$  and  $u_{\phi_1, a_1}$  define the same class when there exists  $\theta_0 \in \mathbb{R}^N$ ,  $\theta_0^1 \in \mathbb{R}^{n_1}$  such that  $i_\phi(x_0, \theta_0) = i_{\phi_1}(x_0, \theta_0^1)$ . The point  $(x_0, \nabla_x \phi(x_0, \theta_0)) = (x_0, \nabla_x \phi_1(x_0, \theta_0^1))$  is denoted by  $\lambda_0$ .*

*Then one has  $\Lambda_\phi = \Lambda_{\phi_1}$  in the neighborhood of  $\lambda_0$ .*

PROPOSITION 9.4. *Two Lagrangian manifolds are equivalent  $\Leftrightarrow$  if there exist two phases  $\phi_1$  and  $\phi_2$ , equivalent in the sense of Definition 9.2, such that  $\Lambda_1 = \Lambda_{\phi_1}$  and  $\Lambda_2 = \Lambda_{\phi_2}$ .*

We can then choose the parameters of the phase  $\phi$  associated to a Lagrangian manifold in the neighborhood of a point through the proposition

PROPOSITION 9.5. *Let  $\phi$  be a non-degenerate phase defining a local map of  $\Lambda$  in the neighborhood of  $\lambda_0 = (x_0, \nabla_\theta \phi(x_0, \theta_0))$ ,  $(x_0, \theta_0) \in C_\phi$  and recall that  $\pi_\Lambda$  is the canonical projection from  $\Lambda$  to  $X$ . If  $N$  is the number of parameters of the phase*

$$N = \dim \text{Ker} T_{\lambda_0}(\pi_\Lambda) + \text{rang}(\text{Hess} \phi_\theta(x_0, \theta_0))$$

where  $(x_0, \theta_0) \in C_\phi$  and  $(x_0, \nabla_\theta \phi(x_0, \theta_0)) = \lambda_0$ .

Moreover, there exists a non-degenerate phase  $\phi_0$  which locally represents  $\Lambda$  with  $N_0 = \dim \text{Ker}(T_{\lambda_0}(\pi_\Lambda))$  parameters.

PROOF. We prove Proposition 9.4. We know that there is a phase  $\phi_1$  representing  $\Lambda_1$  (Theorem 9.1). We assume  $\Lambda_1 \simeq \Lambda_2$ , hence there exists a symplectic diffeomorphism  $g$ , of the form  $g(x, \xi) = (\chi(x), G(x, \xi))$  (since it preserves the fibers of the canonical projection), which transforms  $\Lambda_1$  into  $\Lambda_2$ .

We use the following lemma:

LEMMA 9.3. *A symplectic diffeomorphism preserving the fibers of the canonical projection is of the form*

$$(x, \xi) \rightarrow (\chi(x), (\partial_{x_i} \chi_j(x))^{-1}(\xi - \nabla_x \psi))$$

We recognize the form of  $h_\chi$  introduced in the proposition 8.4, alinéa 1.

PROOF. The diffeomorphism is symplectic, so

$$\sum_j dx_j \wedge d\xi_j = \sum_j d(f_j(x)) \wedge d(h_j(x, \xi)).$$

To simplify our expressions, we denote by  $\mathcal{N}$  the matrix of coefficients  $\mathcal{N}_{ij}(x) = \frac{\partial f_j}{\partial x_i}(x)$ . Then  $df_j(x) = \sum_i \mathcal{N}_{ij}(x) dx_i$ .

One writes similarly

$$dh_j(x, \xi) = \sum_i \partial_{x_i} h_j(x, \xi) dx_i + \sum_k \partial_{\xi_k} h_j(x, \xi) d\xi_k$$

and the equalities that  $f, h$  must verify are

$$\begin{cases} \sum_j \mathcal{N}_{ij}(x) \partial_{x_k} h_j(x, \xi) = \sum_j \mathcal{N}_{kj}(x) \partial_{x_i} h_k(x, \xi), \forall j, k \\ \sum_j \mathcal{N}_{ij}(x) \partial_{\xi_k} h_j(x, \xi) = \delta_{jk}, \forall j, k. \end{cases}$$

Considering the matrix  $M(x, \xi) = (M_{jk})_{j,k}(x, \xi) = (\partial_{\xi_k} h_j)_{j,k}(x, \xi)$ , one deduces

$$(3.126) \quad \mathcal{N}M = Id.$$

One has:

- $M$  is independent of  $\xi$  (the dual part of the diffeomorphism is linear in  $\xi$ )
- There exists a vector field  $X_j(x)$  such that

$$h_j(x, \xi) = \sum_k (\mathcal{N}^{-1})_{jk}(x) (\xi_k - X_k(x))$$

thanks to  $\partial_{\xi_k} h_j(x, \xi) = (\mathcal{N}^{-1})_{jk}(x)$ , from which one deduces

$$\partial_{x_k} h_j(x) = \sum_l \partial_{x_k} ((\mathcal{N}^{-1})_{jl})(\xi_l - X_l(x)) - \sum_l (\mathcal{N}^{-1})_{jl}(x) \partial_{x_k} (X_l)(x).$$

The first equalities of the above system defining the symplectic diffeomorphism are

$$\begin{aligned} & \sum_j \mathcal{N}_{ij}(x) \sum_l \partial_{x_k} (\mathcal{N}^{-1})_{jl} (\xi_l - X_l(x)) + \sum_j \sum_l \mathcal{N}_{ij} (\mathcal{N}^{-1})_{jl} \partial_{x_k} X_l \\ & = \\ & \sum_j \mathcal{N}_{kj}(x) \sum_l \partial_{x_i} (\mathcal{N}^{-1})_{jl} (\xi_l - X_l(x)) + \sum_j \sum_l \mathcal{N}_{kj} (\mathcal{N}^{-1})_{jl} \partial_{x_i} X_l. \end{aligned}$$

Reordering the terms into  $\xi_l - X_l(x)$ , we find

$$\sum_l (\xi_l - X_l(x)) \sum_j \mathcal{N}_{ij}(x) \partial_{x_k} (\mathcal{N}^{-1})_{jl} - \sum_j \mathcal{N}_{kj}(x) \partial_{x_i} (\mathcal{N}^{-1})_{jl} = \frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k}.$$

The left hand term writes

$$A = \partial_{x_k} \left( \sum_j \mathcal{N}_{ij}(x) (\mathcal{N}^{-1})_{jl}(x) \right) - \sum_j \partial_{x_k} \mathcal{N}_{ij}(x) (\mathcal{N}^{-1})_{jl}(x) - \partial_{x_i} \left( \sum_j \mathcal{N}_{kj}(x) (\mathcal{N}^{-1})_{jl}(x) \right) + \sum_j \partial_{x_j} \mathcal{N}_{kj}(x).$$

As  $\mathcal{N}_{ij}(x) = \partial_{x_i} f_j(x)$ , one finds  $\sum_j \mathcal{N}_{ij} (\mathcal{N}^{-1})_{jl} = \delta_{il}$  and using  $\sum_j \partial_{x_i x_k}^2 f_j - \partial_{x_k x_i}^2 f_j = 0$ , one obtains

$$\frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k} = 0$$

which shows that  $X$  is the gradient of a function  $\psi$  of class  $C^\infty$  in a neighborhood of  $x_0$ . The reciprocal implication (the diffeomorphism written in the lemma is a symplectic diffeomorphism preserving the fibers of the canonical projection) can be obtained by writing these equalities from  $\frac{\partial X_k}{\partial x_i} - \frac{\partial X_i}{\partial x_k} = 0$ . The proof of the lemma is complete.  $\square$

This lemma being proved,  $\Lambda_1 = \{(x, \nabla_x \phi_1(x, \alpha)), \nabla_\alpha \phi_1(x, \alpha) = 0\}$  because  $\phi_1$  represents locally  $\Lambda_1$ . We deduce

$$\Lambda_2 = \{(f(x), (\nabla_x f)^{-1}(\nabla_x \phi_1(x, \alpha) - \nabla_x \phi(x))), \nabla_\alpha \phi_1(x, \alpha) = 0\}$$

The phase

$$\phi_2(y, \beta) = \phi_1(f^{-1}(y), \beta) - \psi(f^{-1}(y))$$

is well defined as  $f$  is a diffeomorphism in a neighborhood of  $x_0$ . Hence

$$\begin{cases} \nabla_y \phi_2(y, \beta) = \nabla_\alpha \phi_1(f^{-1}(y), \beta) \\ \nabla_x \phi_2(y, \beta) = (\nabla_x f)^{-1}(\nabla_x \phi_1(f^{-1}(y), \beta) - \nabla_x \psi(f^{-1}(y))) \end{cases}$$

The manifold  $\Lambda_2$  is thus

$$\Lambda_2 = \{(y, \nabla_y \phi_2(y, \beta)) \in T^*X, \nabla_\beta \phi_2(y, \beta) = 0\}.$$

This demonstrates  $\Rightarrow$  in Proposition 9.4.

For  $\Leftarrow$  in Proposition 9.4, we have two equivalent phases  $\phi_2(A(x), \Phi(x, \alpha)) = \phi_1(x, \alpha) - \psi(x)$ , and we verify that

$$\Lambda_{\phi_2} = \{(y, \nabla_y \phi_2), \nabla_\beta \phi_2(y, \beta) = 0\}$$

which imply the relations

$$\begin{cases} \nabla_\beta \phi_2(A(x), \Phi(x, \alpha)) \cdot \nabla_\alpha \Phi(x, \beta) = \nabla_\alpha \phi_1(x, \alpha) \\ \nabla_y \phi_2(A(x), \Phi(x, \alpha)) \cdot \nabla_x A = \nabla_x \phi_1(x, \alpha) - \nabla_x \psi. \end{cases}$$

As  $(x, \alpha) \rightarrow (A(x), \Phi(x, \alpha))$  is a diffeomorphism and as

$$(x, \xi) \rightarrow (A(x), (\nabla_x A)^{-1}(\xi - \nabla_x \psi))$$

is a symplectic diffeomorphism complete the proof of the reciprocal.

PROOF. Let us prove Proposition 9.5. We place ourselves at the point  $(x_0, \alpha_0)$  which characterizes a point of the Lagrangian manifold  $\rho_0 = (x_0, \nabla_x \phi(x_0, \alpha_0))$ .

One assumes  $\text{rang Hess}_\alpha \phi(x_0, \alpha_0) = p < N$ . There is a orthogonal linear change of variables (which diagonalizes  $\text{Hess}_\alpha \phi(x_0, \alpha_0)$ ) such that  $\beta = \chi(\alpha)$  and if  $\psi(x, \beta) = \phi(x, \chi^{-1}(\beta))$

$$\text{Hess}_\beta \psi(x_0, \beta_0) = \begin{pmatrix} 0 & 0 \\ 0 & \text{Hess}_{\beta''} \psi(x_0, \beta_0) \end{pmatrix}$$

where  $\beta'' \in \mathbb{R}^p$ . Locally, the equation  $\nabla_{\beta''} \psi = 0$  has, through the implicit functions theorem (which can be applied here because the Hessian in  $\beta''$  is non-zero), a unique solution  $\beta'' = B(x, \beta')$  with  $\beta''_0 = B(x_0, \beta'_0)$ .

The phase

$$\psi_1(x, \beta') = \psi(x, \beta', B(x, \beta'))$$

has  $N - p$  parameters and whose gradient is zero at  $\beta'_0$  and its Hessian is non zero at this point thus locally represents  $\Lambda_\psi$ , as  $\nabla_\beta \phi(x, \beta) = 0$  is equivalent to  $\nabla_{\beta'} \psi(x, \beta) = 0$  and  $\nabla_{\beta''} \psi(x, \beta) = 0$  and that, on the other hand

$$\nabla_x \psi_1(x, \beta') = \nabla_x \psi(x, \beta', B(x, \beta')) + \nabla_{\beta''} \psi(x, \beta', B(x, \beta')) \nabla_x B(x, \beta').$$

Using  $\nabla_{\beta''} \psi(x, \beta', B(x, \beta')) = 0$  for all  $(x, \beta')$ , one has  $\nabla_x \psi_1 = \nabla_x \psi$  on  $\nabla \beta \psi = 0$ . We have therefore represented  $\Lambda$  by a phase with  $N - p$  parameters.  $\square$

Locally,  $\Lambda = \{(x, \nabla_x \psi_1(x, \theta)), \nabla_\theta \psi_1(x, \theta) = 0\}$  where  $\psi_1$  is a non degenerate phase defined on  $X \times R^{N-p}$  such that  $\text{Hess}_\theta \psi_1(x_0, \theta_0) = 0$ . At  $(x_0, \xi_0) = (x_0, \nabla_x \psi_1(x_0, \theta_0))$ , the equalities defining  $\Lambda$  allow to write  $T_{(x_0, \xi_0)} \Lambda$  as

$$\{(u, \text{Hess}_x \psi_1(x_0, \theta_0)u + \sum_j \partial_{\theta_j} \nabla_x \psi_1(x_0, \theta_0) \Sigma_j)\} \cap \{\nabla_x \partial_{\theta_j} \psi_1(x_0, \theta_0)u = 0\}.$$

The space tangent at  $x_0$  to the canonical projection  $\pi_\Lambda$  from  $\Lambda \subset T^*X$  onto  $X$  is thus characterized by the independent equations  $\forall j, \nabla_x \partial_{\theta_j} \psi_1(x_0, \theta_0)u = 0$ . Its kernel is therefore of dimension the number of corresponding equations, that is  $N - p$ . We deduce

$$\dim \text{Ker} T_{x_0}(\pi_\Lambda) = N - p$$

which completes the two parts of the proof of the proposition 9.5. We have then built the phase  $\psi_1$  with  $N - p$  parameters representing the Lagrangian near the point  $(x_0, \xi_0)$ .  $\square$

When  $\dim \text{Ker} T_{x_0}(\pi_\Lambda) \neq 0$ , we say that the point  $x_0$  is in the caustic of  $\Lambda$ . The classification of caustics is not our goal, we only study in this book the fold caustics, but we recall the elementary properties of caustics (which the reader can find in Duistermaat's article ([30] Section 3).

DEFINITION 9.4. • *The caustic set of the Lagrangian  $\Lambda$  is the set of projections on  $X$  of the critical points of the canonical projection  $\pi_\Lambda$  of  $\Lambda$  into  $X$ . The type of the caustic is characterized by the dimension  $\text{Ker} T_{x_0} \pi_\Lambda$ .*

*We suppose that the dimension of the space  $X$  is  $n \leq 4$ . The seven elementary catastrophes in Thom's sense are*

• *The points whose dimension of the kernel of the canonical projection is 1 are:*

- the fold ( $n = 1$ ),*
- the cusp ( $n = 2$ ),*
- the swallowtail ( $n = 3$ ),*
- the butterfly ( $n = 4$ ).*

• *The points whose dimension of the kernel of the projection is 2 are the umbilical type points ( $n = 3$ ).*

*In particular, a caustic point of type fold if  $\dim \text{Ker} T_{x_0}(\pi_\Lambda) = 1$  and  $\dim \text{Ker} T_{x_0}(\pi_\Lambda) \cap T_{x_0} C_1 = 0$ .*

We easily obtain a representation theorem:

THEOREM 9.2. *Let  $u \in O(\Lambda)$ ,  $\Lambda$  maximal Lagrangian manifold.*

- (1) *If the point  $x_0$  is not in the projection of  $\Lambda$ ,  $u$  is rapidly decaying in  $k$  in a neighborhood of  $x_0$ .*
- (2) *If the point  $x_0$  is in  $\pi(\Lambda)$  but is not in the caustic of the Lagrangian,  $u$  admits a classical asymptotic expansion in  $k^{-j}$  of first term  $k^{-N/2}$ .*
- (3) *If the point  $x_0$  is in the caustic set of  $\Lambda$ ,  $u$  defines a representation of the caustic in the neighborhood of  $x_0$ .*

PROOF. For the first item, there is no  $\theta$  such that  $\nabla_\theta \phi(x_0, \theta) = 0$  (if there was one, then we would construct a point of the Lagrangian which projects on  $x_0$ ). We can apply the non-stationary phase theorem (Theorem 4.1) to the integral defining  $u$ . We then have  $u \simeq 0$ .

For the second item, we consider  $\theta_0$  such that  $\nabla_\theta \phi(x_0, \theta_0) = 0$ . Let us consider a neighborhood of  $x_0$  in which we solve  $\nabla_\theta \phi(x, \theta) = 0$ . As  $x_0$  is not in the caustic, the point  $x_0$  is not a critical point for the Lagrangian projection  $\pi_\Lambda$ . We deduce that  $\dim \text{Ker} T_{x_0} \pi_\Lambda = 0$ , and therefore  $p = \text{rank} \text{Hess}_\alpha \phi(x_0, \alpha_0) = N$ . The Hessian matrix is invertible and we apply the stationary phase theorem (Proposition 4.3).

The general study of caustics is based on the following result. We prove this result in the following section only in the case of a singularity of type fold for the canonical projection. A notion of stability or generality of the phase, expressed in the definition below, is useful:

DEFINITION 9.5. *We say that the phase  $\phi(x, \alpha)$  is stable if, for any function  $g(x, \alpha)$  "small enough", we have the equivalence of  $\phi(x, \alpha)$  and  $\phi(x, \alpha) + g(x, \alpha)$ . It is equivalent to say that any function  $f(x, \alpha)$  of class  $C^\infty$  is written in the form*

$$f(x, \alpha) = a(x) + \sum_l b_l(x) \partial_{x_l} \phi + \sum_k c_k(x, \alpha) \partial_{\alpha_k} \phi$$

*all coefficients being  $C^\infty$ .*



Duistermaat [30] then proves the

PROPOSITION 9.6. *Let  $\phi(x, \alpha)$  be a stable phase and let  $g(\alpha) = \phi(x_0, \alpha)$ .*

*We denote by  $g_1(\alpha), \dots, g_q(\alpha)$  a basis of the quotient space  $\frac{C^\infty(\mathbf{R}^N)}{\{\sum g_k(\alpha) \partial_{\alpha_k} g(\alpha)\}}$ . There is a submersion  $x \rightarrow y(x)$  such that*

$$\phi(x, \alpha) = \sum_{1 \leq l \leq q} y_l(x) g_l(\alpha) + g(\alpha)$$

To each type of catastrophe, Thom [95] associated a generating phase and a number of parameters required in the phase of the Fourier integral operator. Of course, one can always consider (thanks to the representation theorem 9.1) as many parameters as the dimension  $N$  of the space  $X$ , but a lemma of Hörmander ([47], Lemma 3.2.3), which is a generalization of the Morse lemma with parameters, show that any germ  $f(\alpha)$  can be represented by  $f_1(\alpha_1, \dots, \alpha_{k_0}) + Q(\alpha_{k_0+1}, \dots, \alpha_N)$ ,  $f_1$  being a germ and  $Q$  a non-degenerate quadratic form. The function  $f_1$  can be evaluated in the cases of the above mentioned catastrophes. In particular, for the fold,  $f_1(\alpha_1) = \alpha_1^3$ , for the cusp,  $f_1(\alpha_1) = \alpha_1^4$ , for the sparrowtail  $f_1(\alpha_1) = \alpha_1^5$ , for the butterfly,  $f_1(\alpha_1) = \alpha_1^6$ , and for the umbilicals, we have respectively  $f_1(\alpha_1, \alpha_2) = \alpha_1^2 \alpha_2 + \alpha_1^3$  (elliptic elliptic),  $f_1(\alpha_1, \alpha_2) = \alpha_1^2 \alpha_2 - \alpha_1^3$  (hyperbolic umbilicus) (hyperbolic umbilicus) and  $f_1(\alpha_1, \alpha_2) = \alpha_1^2 \alpha_2 + \alpha_1^4$  (parabolic umbilicus). The base of the germs is then

- for the fold  $\alpha_1^3 + a(x)\alpha_1$ ,
- for the cusp,  $\alpha_1^4 + a(x)\alpha_1 + b(x)\alpha_1^2$ ,
- for the sparrowtail  $\alpha_1^5 + a(x)\alpha_1 + b(x)\alpha_1^2 + c(x)\alpha_1^3$ ,
- for the butterfly  $\alpha_1^6 + a(x)\alpha_1 + b(x)\alpha_1^2 + c(x)\alpha_1^3 + d(x)\alpha_1^4$ ,
- for, respectively, the elliptic and hyperbolic umbilic  $\alpha_1^2 \alpha_2 \pm \alpha_2^3 + a(x)\alpha_1 + b(x)\alpha_2 + c(x)\alpha_2^2$ ,
- for the parabolic umbilic  $\alpha_1^2 \alpha_2 + \alpha_2^4 + a(x)\alpha_1 + b(x)\alpha_2 + c(x)\alpha_2^2 + d(x)\alpha_2^3$ . This ends a sketch of the proof of Theorem 9.2 □

#### 4. The fold caustic

In this section we study the intrinsic representation associated with the fold type germ. The first study of such a germ has been made by Ludwig [69]. The intrinsic representation uses the Airy function (already introduced by Airy [2] for the study of the rainbow). This caustic type allows us, in the next section, to compute the uniform solution in the neighborhood of the rainbow.

The fold singularity, as we have seen, is characterized by the two following relations:

if  $C_1 = \{x, \dim \text{Ker} T_x \pi_\Lambda = 1\}$ , then

- the point  $x_0$  is a fold with  $x_0 \in C_1$
- $\text{Ker} T_{x_0} \pi_\Lambda$  is transverse to  $T_{x_0} C_1$ . There exists thus a 1-parameter phase  $\phi(x, \alpha)$  such that

$$\Lambda = \{(x, \nabla_x \phi(x, \alpha)), \phi'(x, \alpha) = 0\}.$$

We will explicitly construct in the case of the fold the change of variable allowing to return to the generic phase of the previous section.

We know (since  $x_0$  is on the caustic) that  $\phi''(x_0, \alpha_0) = 0$ . The caustic is the set of points  $x$  such that there exists  $\alpha$  with  $(x, \nabla_x \phi(x, \alpha)) \in \Lambda$  and

$$\frac{\partial^2 \phi}{\partial \alpha^2}(x, \alpha) = 0.$$

The tangent space to this caustic at  $(x, \alpha)$  is given by

$$\{u, \exists \tau, \sum_j \frac{\partial^3 \phi}{\partial x_j \partial \alpha^2}(x, \alpha) u_j + \phi'''(x, \alpha) \tau = 0\}.$$

The singularity is of type fold if the equations defining  $\text{Ker} T_{x_0} \pi_\Lambda$  and those defining  $T_{x_0} C_1$  have only one trivial solution. This is the case if, at  $(x_0, \alpha_0)$ ,  $\phi'''(x_0, \alpha_0) \neq 0$ .

PROPOSITION 9.7. *The lagrangian  $\Lambda$  defined by the phase  $\phi$  at a parameter in the neighborhood of the point  $x_0$  of the caustic  $C_1$  is of type fold at  $x_0$  if and only if there exists  $\alpha_0$  such that*

$$\phi'_\alpha(x_0, \alpha_0) = 0, \phi''_\alpha(x_0, \alpha_0) = 0, \phi'''_\alpha(x_0, \alpha_0) \neq 0$$

*The phase is then a universal deformation of  $y_1 + \alpha y_2 - \frac{\alpha^3}{3}$ . There are two functions  $f(x)$  and  $g(x)$  and a change of variable of variable  $\alpha \rightarrow t$  such that the phase is written*

$$\phi(x, \alpha) = f(x) + (x - x_0)g(x)t - \frac{t^3}{3}$$

Note that we shall often call  $\phi_0 = f$  and  $\rho$  given by  $\rho(x) = (x - x_0)g(x)$ .

**4.1. Proof of the explicit change of variable for the phase (Malgrange's preparation lemma).** We prove here the fundamental representation theorem for oscillatory functions associated with a fold point.

**THEOREM 9.3.** *An oscillatory integral  $I$  of phase  $\phi$  (and of symbol  $\sigma(x, \alpha, k)$ ) which has a fold singularity at  $(x_0, \alpha_0)$ ,  $\alpha \in R$ , is characterized by the functions*

$$\phi_0(x) = \frac{1}{2}(\phi_+(x) + \phi_-(x)) \quad g(x)(x - x_0) = \left[\left(\frac{3}{4}(\phi_-(x) - \phi_+(x))\right)^{\frac{1}{3}}\right]^2$$

and there are two symbols  $\sigma_0(x, k)$  and  $\sigma_1(x, k)$  such that, locally

$$I = k^{-\frac{1}{3}} e^{ik\phi_0(x)} [\sigma_0(x, k) Ai(k^{\frac{2}{3}} e^{\frac{i\pi}{3}} g(x)(x - x_0)) + k^{-\frac{1}{3}} e^{\frac{i\pi}{3}} \sigma_1(x, k) Ai'(k^{\frac{2}{3}} e^{\frac{i\pi}{3}} g(x)(x - x_0))].$$

**PROOF.** The existence of  $\alpha_0$  has been proved above when assuming the existence of a caustic point. We assume that this caustic is of type fold. Then the equation

$$\phi''_\alpha(x, \alpha) = 0$$

can be solved locally in the neighborhood of  $(x_0, \alpha_0)$  by applying the implicit functions theorem in  $\alpha$ . There exists a function  $\theta(x)$  continuous such that  $\theta(x_0) = \alpha_0$  and  $\phi''_\alpha(x, \theta(x)) = 0$ .

Critical points. One can then write

$$\phi(x, \alpha) = \phi(x, v) + \phi'_\alpha(x, \theta(x))(\alpha - \theta(x)) + \frac{1}{3} B(\alpha, x)(\alpha - \theta(x))^3.$$

As  $B(\alpha_0, x_0) \neq 0$ , there exists  $C(\alpha, x)$  such that  $(C(\alpha, x))^3 = B(\alpha, x)$  where  $C(\alpha, x)$  is of the same sign<sup>2</sup> as  $B(\alpha_0, x_0)$  in the neighborhood of this point. The critical points  $\beta = \alpha - \theta(x)$  of the phase are solution of

$$\phi'_\alpha(x, \theta(x)) + B(\beta + \theta(x), x)\beta^2 + \frac{1}{3}\beta^3 \partial_\alpha B(\beta + \theta(x), x) = 0$$

In the neighborhood of  $\beta = 0$ , this equation has two solutions. The phase  $\phi(x, \alpha)$  thus admits two critical values  $\phi^+(x)$  and  $\phi^-(x)$ . There are then two functions, noted  $\kappa$  and  $\gamma$ , such that

$$\phi^\pm(x) = \phi(x, \theta(x)) + (\phi'_\alpha(x, \theta(x)))^2 \gamma(x) \pm \frac{2}{3} (\phi'_\alpha(x, \theta(x)) \kappa(x))^{\frac{3}{2}}.$$

Indeed, if we write  $\beta = (\phi'(x, \theta(x)))^{\frac{1}{2}} t$  in the area where  $\phi'(x, \theta(x)) > 0$  (knowing that it is null for  $x = x_0$ ), we verify that the equation in  $t$  has two solutions, since  $\phi'(x, \theta(x))$  is small. These two solutions are of the form (in the neighborhood of  $x_0$ ):

$$t = \pm (-B(\theta(x), x))^{-\frac{1}{2}} + O((\phi'(x, \theta(x)))^{\frac{1}{2}})$$

which leads to the fact that the functions  $\kappa(x)$  and  $\gamma(x)$ , defined by the equalities

$$\begin{aligned} \gamma(x) &= \frac{1}{2(\phi'(x, \theta(x)))^2} (\phi^+(x) + \phi^-(x) - 2\phi(x, \theta(x))) \\ \frac{4}{3} (\phi'(x, \theta(x)) \kappa(x))^{\frac{3}{2}} &= \phi^+(x) - \phi^-(x) \end{aligned}$$

are indeed  $C^\infty$  functions. They are determined thanks to the critical values. **First change of variable** We want to construct the change of variable such that

$$(4.127) \quad \phi(x, \alpha) = \frac{1}{2}(\phi^+(x) + \phi^-(x)) + (\phi'(x, \theta(x)) \kappa(x)) z - \frac{z^3}{3}.$$

First, note that this equality is consistent since the critical values of the phase written in (4.127) are those of  $\phi$ .

Consider  $\psi(x, \alpha) = \phi'_\alpha(x, \alpha)$ . We notice that  $\psi''_{\alpha^2}(x, \theta(x)) \neq 0$  and  $\psi'_\alpha(x, \theta(x)) = 0$ . By the Taylor formula with integral remainder, we can write  $\psi(x, \theta(x)) = \psi(x, \theta(x)) + (\alpha - \theta(x))^2 \int_0^1 ds (1 - s) \psi''_{\alpha^2}(\theta(x) + s(\alpha - \theta(x))) ds$ . Assuming  $\psi''_{\alpha^2}(x, \theta(x)) < 0$  we apply Morse's lemma for a non-degenerate stationary phase and we find

$$\psi(x, \alpha) = \phi'(x, \theta(x)) - \rho^{-1}(x) t^2,$$

<sup>2</sup>Note that it is not the classical determination of the cubic root

where  $t = (\alpha - \theta(x))[-\rho(x) \int_0^1 ds(1-s)\psi''_{\alpha^2}(\theta(x) + s(\alpha - \theta(x)))ds]^{\frac{1}{2}}$  in a neighborhood of  $\alpha = \theta(x)$ . Then we denote by  $\omega$  the function

$$\omega(x, t) = \phi(x, \alpha(x, t)) - \frac{1}{2}(\phi^+(x) + \phi^-(x)).$$

The functions  $g_1(t, x)$  and  $g_2(t, x)$  equal respectively to  $\frac{1}{2}(\omega(x, t) + \omega(x, -t))$  and  $\frac{1}{2}\frac{\omega(x, t) - \omega(x, -t)}{t}$  are even. We introduce the first auxiliary variable

$$\beta = \phi'(x, \theta(x))\kappa(x).$$

The critical points in  $t$  of  $\omega(x, t)$  in the variable  $t$  are the solutions of  $t^2 = \beta$ , from which the critical values are equal to  $\pm \frac{2}{3}\beta^{\frac{3}{2}}$  (indeed  $\partial_t \omega(x, t) = \partial_t \alpha(x, t) \partial_\alpha \phi(x, \alpha(x, t))$ ). We find directly (for example by Taylor series of even functions and assuming that the correspondence  $x \leftrightarrow \beta$  is a local diffeomorphism) that  $g_{1,2}(x, t) = A_{1,2}(t^2 - \beta, \beta)$ . Hence

$$\phi(x, \alpha(x, t)) = \frac{1}{2}(\phi^+(x) + \phi^-(x)) + tA_1(t^2 - \beta, \beta) + A_2(t^2 - \beta, \beta)$$

where  $A_1$  and  $A_2$  are two functions to be estimated. From equality  $\phi(x, \alpha(x, \pm\beta^{\frac{1}{2}})) = \frac{1}{2}(\phi^+(x) + \phi^-(x)) \pm \frac{2}{3}\beta^{\frac{3}{2}}$  one deduces

$$\pm \beta^{\frac{1}{2}} A_1(0, \beta) + A_2(0, \beta) = \pm \frac{2}{3}\beta^{\frac{3}{2}} \text{ et } 0 = \partial_\alpha t[A_1(0, \beta) + 2\beta \partial_1 A_1(0, \beta) \pm 2\beta^{\frac{1}{2}} \partial_1 A_2(0, \beta)]$$

which yields  $A_1(0, \beta) = \frac{2}{3}\beta$ ,  $2\beta \partial_1 A_1(0, \beta) + A_1(0, \beta) = 0$ ,  $\partial_1 A_2(0, \beta) = 0$ ,  $A_2(0, \beta) = 0$ . One writes Taylor expansions of  $A_1$  and of  $A_2$  in the first coordinate. One has  $A_2(u, \beta) = u^2 H_2(u, \beta)$  and  $A_1(u, \beta) = \frac{2}{3}\beta - \frac{1}{3}u + u^2 H_1(u, \beta)$ , soit  $A_1(t^2 - \beta, \beta) = \beta - \frac{1}{3}t^2 + H_1(t, \beta)(t^2 - \beta)^2$ . Soit  $H(t, \beta) = tH_1(t^2 - \beta, \beta) + H_2(t^2 - \beta, \beta)$ . One obtains

$$\phi(x, \alpha) = \frac{1}{2}(\phi^+(x) + \phi^-(x)) + (t^2 - \beta)^2 H(t, \beta) + \beta t - \frac{t^3}{3}.$$

One notices that  $z = t + (\beta - t^2)U$ ,  $\beta z - \frac{z^3}{3} = \beta t - \frac{t^3}{3} + (\beta - t^2)^2[U - tU^2 - \frac{\beta - t^2}{3}U^3]$ . If  $U(\beta, t)$  is the unique solution in the neighborhood of  $t = 0$  of  $u - u^2 t - (\beta - t^2)u^3/3 = H(t, \beta)$ , we obtain the required equality:

$$\phi(x, \alpha) = \phi(x, \theta(x)) + \beta z - \frac{z^3}{3}.$$

**4.2. Integral representation of the associated solution.** The function representing the caustic of type fold is the Airy function. From a division lemma of Boutet de Monvel (Preparation Lemma p 26 of [18]), there exists three functions  $\sigma_0(x, k)$ ,  $\sigma_1(x, k)$ ,  $h(x, z, k)$  such that

$$\sigma(x, \alpha, k) \frac{d\alpha}{dz} = \sigma_0(x, k) + z\sigma_1(x, k) + (\partial_z h(x, z, k) + (z^2 + (x - x_0)g(x))ikh(x, z, k))$$

We deduce the relation, which allows to write a general representation of the oscillating solution associated to a caustic of type fold, where we denote by  $\rho$  the function  $x \rightarrow (x - x_0)g(x)$ :

$$\int e^{ik\phi(x, \alpha)} \sigma(x, \alpha, k) d\alpha = 2\pi k^{-\frac{1}{3}} e^{\frac{i\pi}{3}} e^{ik\phi(x, \theta(x))} \left[ \begin{array}{l} \sigma_0(x, k) Ai(k^{\frac{2}{3}} e^{\frac{i\pi}{3}} \rho(x)) \\ + k^{-\frac{1}{3}} e^{\frac{i\pi}{3}} \sigma_1(x, k) Ai'(k^{\frac{2}{3}} e^{\frac{i\pi}{3}} \rho(x)) \end{array} \right] + r$$

As the critical values of  $z \rightarrow f(x) + (x - x_0)g(x)z - \frac{z^3}{3}$  are intrinsic and equal to

$$f(x) \pm \frac{2}{3}(s(x))^{\frac{3}{2}} = f(x) \pm \frac{2}{3}((x - x_0)g(x))^{\frac{3}{2}} = \phi_{\pm}(x)$$

we finish the proof of Theorem 9.3. Indeed, if  $x \neq x_0$ , then the stationary phase theorem used in the representation of the theorem 9.3 allows to find

$$I(a, \phi) \simeq a_1(x, k) e^{ik\phi_+(x)} + a_2(x, k) e^{ik\phi_-(x)}.$$

The points of the Lagrangian that are not on the caustic are usual points of stationary phase.

We can thus introduce the notion of **boundary layer** associated with this type of caustic. Indeed, the integral  $I$  is decreasing when the parameter tends to  $+\infty$ , the  $I$  is controlled when  $|g(x)(x - x_0)| \leq Ck^{-\frac{2}{3}}$  for all  $C$ , the Airy function being then considered in a compact  $|u| \leq C$ . We verify that  $g(x_0)$ , given by a power of  $\phi'''(x_0, \alpha_0)$  is nonzero. This implies that the representation of

the solution shown in the theorem 9.3 is uniform in any open set of the form  $|x - x_0| \leq Dk^{-\frac{2}{3}}$ ,  $D$  (large) positive constant given.  $\square$

In the following paragraph we use the expressions obtained for the solution of the wave equation to find the Airy function by a boundary layer method. We thus present the way in which Ludwig introduced caustics.

### 4.3. Ludwig's Ansatz for a fold caustic.

PROPOSITION 9.8. *Let  $V$  be a solution of  $V''(t) + tV(t) = 0$  and let  $\theta, \rho$  be two functions,  $\beta$  a parameter.*

*If  $u(x, k) = e^{ik\theta(x)}[g_0(x, k^\alpha)V(k^\beta\rho(x)) + g_1(x, k^\alpha)V'(k^\beta\rho(x))]$  is a solution of the Helmholtz equation  $(\Delta + k^2)u(\cdot, k) = 0$  (asymptotically) for  $k$  large, then*

$$\beta = \frac{2}{3}, (\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1, \nabla\theta \cdot \nabla\rho = 0.$$

In addition, if

$$(4.128) \quad \begin{cases} 2\nabla\theta\nabla g_0 + \Delta\theta g_0 + \frac{1}{ik}\Delta g_0 + [2\rho\nabla\rho\nabla\tilde{g}_1 + \tilde{g}_1\rho\Delta\rho + \tilde{g}_1(\nabla\rho)^2] = 0 \\ 2\nabla\theta\nabla\tilde{g}_1 + \Delta\theta\tilde{g}_1 + \frac{1}{ik}\Delta\tilde{g}_1 + [2\rho\nabla\rho\nabla g_0 + g_0\Delta\rho] = 0, \end{cases}$$

then

$$(\Delta + k^2)(e^{ik\theta(x)}[g_0(x, k)V(k^{\frac{2}{3}}\rho(x)) + ik^{\frac{1}{3}}\tilde{g}_1(x, k)V'(k^{\frac{2}{3}}\rho(x))]) \simeq 0.$$

PROOF. We consider solutions of the Helmholtz equation  $(\Delta + k^2)u(x, k) = 0$  of the form

$$u(x, k) = e^{ik\theta(x)}[g_0(x, k^\alpha)V(k^\beta\rho(x)) + \frac{1}{ik^{\frac{1}{3}}}\tilde{g}_1(x, k^\alpha)V'(k^\beta\rho(x))].$$

It is assumed that the function  $V$  satisfies the ODE in Proposition 9.8 (and thus Ludwig assumes a priori that the solution depends on solutions of the the Airy equation). We suppose that  $\theta$  is not a solution of the classical eikonal equation, so  $\nabla\theta$  is not of norm 1 on a neighborhood of a point  $x_0$ .

We find that there are two symbols ( $\alpha$  being identified later)  $\sigma_0(x, k^\alpha, k^\beta)$  and  $\sigma_1(x, k^\alpha, k^\beta)$  such that

$$\Delta u = -k^2 e^{ik\theta(x)}[\sigma_0(x, k^\alpha, k^\beta)V(k^\beta\rho(x)) + \sigma_1(x, k^\alpha, k^\beta)V'(k^\beta\rho(x))].$$

A calculation, left to the reader, shows that

$$\begin{aligned} \sigma_0(x, k^\alpha, k^\beta) &= [(\nabla\theta)^2 g_0 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_0 + \Delta\theta g_0) + \frac{1}{(ik)^2}\Delta g_0] \\ &\quad + k^{3\beta-2}\rho(\nabla\rho)^2 g_0 + 2ik^{2\beta-1}g_1\rho\nabla\rho \cdot \nabla\theta \\ &\quad + k^{2\beta-2}[2\rho\nabla\rho \cdot \nabla g_1 + g_1\rho\Delta\rho + g_1(\nabla\rho)^2], \\ \sigma_1(x, k^\alpha, k^\beta) &= [(\nabla\theta)^2 g_1 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_1 + \Delta\theta g_1) + \frac{1}{(ik)^2}\Delta g_1] \\ &\quad + k^{3\beta-2}\rho(\nabla\rho)^2 g_1 + \frac{2}{i}k^{\beta-1}g_0\nabla\rho \cdot \nabla\theta \\ &\quad - k^{\beta-2}[2\rho\nabla\rho\nabla g_0 + g_0\Delta\rho]. \end{aligned}$$

Writing  $(\Delta + k^2)u = 0$  implies that, asymptotically  $\sigma_0 \simeq g_0, \sigma_1 \simeq g_1$ .

We assume  $g_0$  of order 0 in  $k$  and  $g_1$  of order of magnitude  $k^\gamma$  (we will have  $\gamma = -\frac{1}{3}$ ).

One compares the asymptotic expansions

- For  $3\eta - 2 < 0$ ,
  - if  $2\beta - 1 + \gamma < 0$ , the leading order term term of  $\sigma_0$  leads, either to  $g_0 = 0$ , or to the classical eikonal equation  $(\nabla\theta)^2 = 1$ . When  $g_0 = 0$ , we examine the following term. Since  $3\beta - 2 < 0$ , it comes  $(\nabla\theta)^2 g_1 = g_1$ , hence  $(\nabla\theta)^2 = 1$ ,
  - if  $2\beta - 1 + \gamma > 0$ , writing the leading order term term of  $\sigma_0$  gives  $g_1\rho\nabla\rho\nabla\theta = 0$ . Assuming  $\rho \neq 0$ , we see that  $\sigma_1 \simeq g_1$  gives the eikonal equation,
  - if  $2\beta - 1 + \gamma = 0$ , we find  $(\nabla\theta)^2 g_0 + 2i(k^{-\gamma}g_1)\rho\nabla\rho\nabla\theta \simeq g_0$  and, by  $\sigma_1$ , the term  $g_0\nabla\rho\nabla\theta$  being of lower order, the eikonal equation is obtained.
- For  $3\beta - 2 > 0$ , we use the same method, by comparing  $3\beta - 2, 2\beta - 1 + \gamma$  and  $\beta - 1 - \gamma$ . If the third is superior to the first one we find  $g_1\rho\nabla\rho\nabla\theta = 0$  and  $g_0\nabla\rho\nabla\theta = 0$ , then  $\rho(\nabla\rho)^2 g_1 = 0, \rho(\nabla\rho)^2 g_0 = 0$ , then finally after having eliminated the terms of too high of too high order, the eikonal equation  $(\nabla\theta)^2 = 1$ .

We must therefore have  $\beta = \frac{2}{3}$  to be able to find a different expression from the usual asymptotic expansions. In this case, we find:

$$\begin{aligned}\sigma_0(x, k^\alpha, k^\beta) &= [((\nabla\theta)^2 + \rho(\nabla\rho)^2)g_0 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_0 + \Delta\theta g_0) + \frac{1}{(ik)^2}\Delta g_0] \\ &\quad + 2ik^{\frac{1}{3}}g_1\rho\nabla\rho \cdot \nabla\theta \\ &\quad + k^{-\frac{2}{3}}[2\rho\nabla\rho \cdot \nabla g_1 + g_1\rho\Delta\rho + g_1(\nabla\rho)^2], \\ \sigma_1(x, k^\alpha, k^\beta) &= [((\nabla\theta)^2 + \rho(\nabla\rho)^2)g_1 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_1 + \Delta\theta g_1) + \frac{1}{(ik)^2}\Delta g_1] \\ &\quad + \frac{2}{i}k^{-\frac{2}{3}}\nabla\rho \cdot \nabla\theta g_0 \\ &\quad - k^{-\frac{4}{3}}[2\rho\nabla\rho \cdot \nabla g_0 + g_0\Delta\rho].\end{aligned}$$

When  $\gamma > -\frac{1}{3}$ , we deduce from the first equality  $\sigma_0 \simeq g_0$  that  $\rho\nabla\rho\nabla\theta = 0$  and that  $(\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1$ . The second equality is thus verified for its first term. We then find that  $\nabla\rho\nabla\theta = 0$ .

When  $-\frac{2}{3} > \gamma$ , it is from the second equality that we deduces from the second equality  $\sigma_1 \simeq g_1$  that  $\nabla\rho\nabla\theta = 0$ . In this case, we deduce again the eikonal equation  $(\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1$ .

Finally, if  $-\frac{2}{3} < \gamma \leq -\frac{1}{3}$ , we find from the second equality the eikonal equation  $(\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1$ . Thus, replacing  $c$  in the first equality, we still find  $\rho\nabla\rho\nabla\theta = 0$ . We have excluded a neighborhood  $\rho = 0$  because in this neighborhood we would have the eikonal equation on  $\theta$ .

We thus obtained:

$$\beta = \frac{2}{3}, \quad (\nabla\theta)^2 + \rho(\nabla\rho)^2 = 1, \quad \nabla\theta\nabla\rho = 0.$$

One is left with  $\sigma_0(x, k^\alpha, k^\beta) = g_0 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_0 + \Delta\theta g_0) + \frac{1}{(ik)^2}\Delta g_0 + k^{-\frac{2}{3}}[2\rho\nabla\rho \cdot \nabla g_1 + g_1\rho\Delta\rho + g_1(\nabla\rho)^2]$  and  $\sigma_1(x, k^\alpha, k^\beta) = g_1 + \frac{1}{ik}(2\nabla\theta \cdot \nabla g_1 + \Delta\theta g_1) + \frac{1}{(ik)^2}\Delta g_1 - k^{-\frac{4}{3}}[2\rho\nabla\rho \cdot \nabla g_0 + g_0\Delta\rho]$ . For more simplifications, we suppose that  $\gamma = -\frac{1}{3}$  (as written in Proposition 9.8. The term of order  $k^{-1}$  of  $\sigma_0 \simeq g_0$  gives

$$(2\nabla\theta\nabla g_0 + \Delta\theta g_0) + \frac{1}{ik}\Delta g_0 + i[2\rho\nabla\rho\nabla(k^{\frac{1}{3}}g_1) + (k^{\frac{1}{3}}g_1)\rho\Delta\rho + (k^{\frac{1}{3}}g_1)(\nabla\rho)^2] = 0$$

while the one of  $\sigma_1 \simeq g_1$  leads to

$$(2\nabla\theta \cdot \nabla(k^{\frac{1}{3}}g_1) + \Delta\theta(k^{\frac{1}{3}}g_1)) + \frac{1}{ik}\Delta(k^{\frac{1}{3}}g_1) - i[2\rho\nabla\rho \cdot \nabla g_0 + g_0\Delta\rho].$$

One denotes by  $\tilde{g}_1(x, k^{\frac{1}{3}}) = \frac{1}{ik^{\frac{1}{3}}}g_1(x, k^{\frac{1}{3}})$  and one deduces the system of coupled equations (4.128)

$$\begin{cases} 2\nabla\theta \cdot \nabla g_0 + \Delta\theta g_0 + \frac{1}{ik}\Delta g_0 + [2\rho\nabla\rho \cdot \nabla\tilde{g}_1 + \tilde{g}_1\rho\Delta\rho + \tilde{g}_1(\nabla\rho)^2] = 0 \\ 2\nabla\theta \cdot \nabla\tilde{g}_1 + \Delta\theta\tilde{g}_1 + \frac{1}{ik}\Delta\tilde{g}_1 + [2\rho\nabla\rho\nabla g_0 + g_0\Delta\rho] = 0. \end{cases}$$

On the other hand, if  $\rho > 0$ , it comes  $(\nabla\theta \pm \sqrt{\rho}\nabla\rho)^2 = 1$ , which yields two solutions of the eikonal equation which are identical on  $\rho = 0$ . These two solutions are

$$\phi_\pm(x) = \theta(x) \pm \frac{2}{3}(\rho(x))^{\frac{3}{2}},$$

and if the above system of transport equations corresponds to the classical system of transport equations associated to each of these phases for the symbols  $G_\pm(x, k^{\frac{1}{3}}) = g_0(x, k^{\frac{1}{3}}) \pm \sqrt{\rho}\tilde{g}_1(x, k^{\frac{1}{3}})$ .  $\square$

We now construct  $\theta$  and  $\phi$  in the neighborhood of a caustic. According to Ludwig, every point outside the caustic lives on a ray that has left the caustic as well as on a ray that goes on the caustic, both rays being tangent to the caustic. We parameterize, in dimension 2, the caustic by a abscissa on the caustic  $\sigma$  and by the tangent distance to the caustic  $\tau$ . Then, as  $u$  is a solution of the Helmholtz equation and we have

$$u_P(x, k) = a_-(x, k)e^{ik\sigma - (x) - ik\tau - (x)} + a_+(x, k)e^{ik\sigma + (x) + ik\tau + (x)} = a_-(x, k)e^{ik\psi - (x)} + a_+(x, k)e^{ik\psi + (x)}$$

one checks that

$$(\nabla\psi_\pm)^2 = 1$$

These phases must correspond to  $\theta \pm \frac{2}{3}\rho^{\frac{3}{2}}$ , which yields  $\theta = \frac{1}{2}(\psi_+ + \psi_-)$  and  $\rho = [\frac{3}{4}(\psi_+ - \psi_-)]^{\frac{2}{3}}$ . We have completed Ludwig's analysis of caustics. We now present what happens in relation to the usual calculation. We represent the caustic from the integrals in the form of the theorem 9.3.

**4.4. Calculation in the vicinity of the caustic.** We know that, if  $\Sigma_0$  is given, surface on which  $\phi$ , solution of the eikonal equation, is constant equal to  $\phi_0$ , then we can define the characteristic curves  $x(t) = x + tn(x)$  for  $x \in \Sigma_0$ . The associated Lagrangian manifold of  $\mathbb{R}^3 \times \mathbb{R}_t$  is

$$\Lambda^* = \{(x + t\vec{n}(x), t, \vec{n}(x), -1), x \in \Sigma_0\}.$$

The canonical projection is proper if the application  $(x, t) \rightarrow (x + t\vec{n}(x), t)$  is a bijection. This imposes that its gradient must be non-degenerate, so degenerate, so that  $Id_{T\Sigma_0} + tW$  has a non zero.

We can see that the points of the caustic (as we defined them above) are the points which correspond to the definition of the caustics of this chapter. We assume that the two radii of curvature of the Weingarten matrix are distinct. The Lagrangian manifold  $\Lambda^*$  is a maximal Lagrangian solution in the neighborhood of any point which is not on the caustic. Therefore there exists a maximal Lagrangian solution  $\Lambda$  such that  $\Lambda^*$  is included  $\Lambda$ . The singularity of the projection  $\pi$  is of type fold, since 0 is, at time  $t = -\kappa_1^{-1}$ , a simple eigenvalue. We can therefore apply the theorem of representation 9.3, and there exists a phase  $\phi(x, \theta)$  equal to  $\theta_0(x) + \theta\alpha(x) - \frac{\theta^3}{3} - t$  representing  $\Lambda$ . The manifold  $\Lambda$  is then

$$\Lambda = \{(x, t, \nabla\theta_0(x) + \nabla_x\alpha\theta), -1, \alpha(x) - \theta^2 = 0\}.$$

The points of  $\Lambda$  where the projection is not proper are the points where  $\theta = 0$ , that is  $\alpha(x) = 0$  (the Jacobian in  $\theta$  of the phase is  $2\theta$  which is equal, at the critical point, to  $2\sqrt{\alpha}$  when  $\alpha(x) \geq 0$ ). As  $\Lambda = \Lambda^*$  in the vicinity of the points which do not belong to the caustic, we see that the points of the form  $(x, t, \nabla\theta_0(x) \pm \sqrt{\alpha(x)}\alpha(x), -1)$  are points of the smooth part of  $\Lambda$ , so in  $\Lambda^*$ .

This allows the introduction of  $\theta$  and  $\rho$  from the previous section, which are obviously  $\theta_0$  and  $\alpha$ . The geometric interpretation is then the one of the previous paragraph.

We have thus connected the intuitive definition of caustics (points where the classical asymptotic expansion explodes) and the geometrical definition of this chapter.



## Propagation and transverse reflections of singularities.

We prove in this chapter the generalization of the laws of geometrical optics, also called Snell-Descartes laws. We want to understand the notion of propagation along rays in the case of the two rays drawn below; ray 1 is the incoming ray and ray 2 is the ray that reflects.

The two results that we generalize are the following : light propagates along straight lines, called rays (example ray 1), and when a ray intersects a boundary, it is reflected (example ray 2) and the angle of incidence is equal to the angle of reflection. The first result is the aim of theorem 10.1, while the second is proved in theorem 10.2. We use here the Fourier integral operators to prove these results; there are other proofs. The advantage of the approach used here, besides the fact that it gives another application of the Fourier integral operators and allows to use their geometrical properties, is that it constructs an outgoing parametrization for hyperbolic problems of order 2 with a boundary condition. We can thus generalize Descartes' laws for any incident wave and any regular boundary.

In a first part, we consider the propagation of a wave in the vacuum (or in a material) when there is no caustic point (as it will be explained below). We can then show that the typical problem considered is a strictly hyperbolic Cauchy problem, i.e. the characteristic manifold is composed of distinct real leaves in the Fourier variable associated with the normal coordinate to the boundary. Note that this does not prevent that there is a caustic; the caustic depends on the form of the incident wave and is a non microlocal phenomenon. Moreover there is propagation along the rays even in the presence of a caustic, as we have seen in Chapter 8. We prove the propagation of singularities theorem (traditional name given to the propagation along a ray) in the case of given to the propagation along a ray) in the case of the derivation operator along a coordinate (Section 10.2). We then use Darboux's lemma to reduce any problem to this particular problem, using a transformation in the cotangent manifold (Chapter 3). We prove finally Theorem 10.1. We use the Fourier integral operators introduced in the chapter 8 to compute the phase of a reflected wave and the reflection coefficient for different boundary conditions. The obtention of the coefficients is simplified by the introduction of the two parametrization of the problem, which are respectively the Fourier integral operator associated with the incoming and with the outgoing wave.

We generalize here the result obtained for the equation of waves with constant coefficients in the chapter 3, where we computed the solution of the Helmholtz equation knowing its value on a  $\Sigma_0$  surface. We had shown that the solution was known on particular curves, called characteristic curves of the Helmholtz operator.

We proved in the section 3 that the "good" objects to consider when studying the pseudo-differential operators were not the characteristic curves but the bicharacteristic curves, flows of the Hamilton-Jacobi vector field associated with the symbol of the pseudo-differential operator. We have verified that the projection on  $\mathbb{R}^d$  of these bicharacteristic curves were the characteristic curves of the wave operator.

We therefore want to generalize the propagation results for a pseudo-differential operator. These propagation results are true in the case where the differential operator studied is a Cauchy strictly hyperbolic operator. The operators studied in this chapter are said to be of real principal type :

**DEFINITION 10.1.** *We say that  $p$  is of real principal type if  $p(y, \eta) \in \mathbb{R}$  for  $(y, \eta) \in \mathbb{R}^{2n}$ , and  $dp$  and  $\eta dy$  are two independent linear forms.*

### 1. Hyperbolic operator: definition, characteristic manifolds

In chapter 2 (definition 2.1), we defined a matricial hyperbolic operator as follows :

$$L = A_0 \partial_t + \sum_j A_j \partial_{x_j}$$

where the matrices  $A_j$  are symmetric, and, in addition,  $A_0$  is positive definite.



PROPOSITION 10.1. (1) In the case  $d = 1$ , consider  $L = a_0(x, t)\partial_t + a_1(x, t)\partial_x$ . Its characteristic manifold is  $\{(x, t, \xi, -\frac{a_1(x, t)}{a_0(x, t)}\xi)\}$ .

(2) In the case  $d = 2$  with constant coefficients, consider the system

$$(1.129) \quad \begin{cases} \partial_t u_1 + a_{1,11}\partial_{x_1} u_1 + a_{1,12}\partial_{x_1} u_2 + a_{2,11}\partial_{x_2} u_1 + a_{2,12}\partial_{x_2} u_2 = 0 \\ \partial_t u_2 + a_{1,12}\partial_{x_1} u_1 + a_{1,22}\partial_{x_1} u_2 + a_{2,12}\partial_{x_2} u_1 + a_{2,22}\partial_{x_2} u_2 = 0. \end{cases}$$

Denote by  $A_j = (a_{j,kl})_{kl}$  the matrices in this system.

The characteristic manifold of the system is  $\{(x_1, x_2, \tau, \xi_1, \xi_2), p(\tau, \xi_1, \xi_2) = 0\}$  where  $p(\tau, \xi_1, \xi_2) = \det(\tau Id + A_1\xi_1 + A_2\xi_2)$ .

(3) If the coefficients in (1.129) are non constant, we define the characteristic manifold of the system, using the **principal symbol** of the equation as  $\{(x_1, x_2, \tau, \xi_1, \xi_2), p(\tau, \xi_1, \xi_2) = 0\}$  where  $p(\tau, \xi_1, \xi_2) = \det(\tau Id + A_1(t, x_1, x_2)\xi_1 + A_2(t, x_1, x_2)\xi_2)$ .

PROOF. Proof of Item 1. The equation on  $u$  is equivalent to  $(\partial_t + \frac{a_1(x, t)}{a_0(x, t)})(u) = 0$ , of symbol is  $l(x, t, \xi, \tau) = i\tau + \frac{a_1(x, t)}{a_0(x, t)}i\xi$ . For the item 2, call  $P = \partial_t + a_{1,11}\partial_{x_1} + a_{2,11}\partial_{x_2}$ . The first equation of (1.129) writes  $Pu_1 + a_{1,12}\partial_{x_1} u_2 + a_{2,12}\partial_{x_2} u_2 = 0$ .

Applying  $P$  to the second equation, we obtain  $\partial_t Pu_2 + a_{1,12}\partial_{x_1} Pu_1 + a_{1,22}\partial_{x_1} Pu_2 + a_{2,12}\partial_{x_2} Pu_1 + a_{2,22}\partial_{x_2} Pu_2 = 0$ . Replacing  $Pu_1$  by its expression in  $u_2$ , one obtains

$$\partial_t Pu_2 - a_{1,12}\partial_{x_1}(a_{1,12}\partial_{x_1} u_2 + a_{2,12}\partial_{x_2} u_2) + a_{1,22}\partial_{x_1} Pu_2 - a_{2,12}\partial_{x_2}(a_{1,12}\partial_{x_1} u_2 + a_{2,12}\partial_{x_2} u_2) + a_{2,22}\partial_{x_2} Pu_2 = 0,$$

which rewrites

$$\frac{\partial^2 u_2}{\partial t^2} + (a_{1,11} + a_{1,22})\frac{\partial^2 u_2}{\partial t \partial x_1} + (a_{2,11} + a_{2,22})\frac{\partial^2 u_2}{\partial t \partial x_2} + (a_{1,11}a_{1,22} - (a_{1,12})^2)\frac{\partial^2 u_2}{\partial x_1^2} + (a_{2,11}a_{2,22} - (a_{2,12})^2)\frac{\partial^2 u_2}{\partial x_2^2} + (a_{1,22}a_{2,11} - 2a_{1,12}a_{2,12} + a_{2,22}a_{1,11})\frac{\partial^2 u_2}{\partial x_2 \partial x_1} = 0.$$

The symbol of this second order operator is then

$$-p(\tau, \xi_1, \xi_2) = \tau^2 + (a_{1,11} + a_{1,22})\tau\xi_1 + (a_{2,11} + a_{2,22})\tau\xi_2 + (a_{1,11}a_{1,22} - (a_{1,12})^2)\xi_1^2 + (a_{1,22}a_{2,11} - 2a_{1,12}a_{2,12} + a_{2,22}a_{1,11})\xi_1\xi_2 + (a_{2,11}a_{2,22} - (a_{2,12})^2)\xi_2^2.$$

It writes

$$\begin{aligned} p(\tau, \xi_1, \xi_2) &= (\tau + a_{1,11}\xi_1 + a_{2,11}\xi_2)(\tau + a_{1,22}\xi_1 + a_{2,22}\xi_2) \\ &\quad - ((a_{1,12})^2\xi_1^2 + 2a_{1,12}a_{2,12}\xi_1\xi_2 + (a_{2,12})^2\xi_2^2) \\ &= (\tau + a_{1,11}\xi_1 + a_{2,11}\xi_2)(\tau + a_{1,22}\xi_1 + a_{2,22}\xi_2) - (a_{1,12}\xi_1 + a_{2,12}\xi_2)^2. \end{aligned}$$

One recognizes the determinant of the matrix  $\tau Id + A_1\xi_1 + A_2\xi_2$ , hence the proof of the second item.

This yields the proof of the second item.

For the proof of the third item, we can try to apply the previous method; If  $P$  has variable coefficients, applying  $P$  to the second equation, note that it writes

$$P\partial_t u_2 + P(a_{1,12}\partial_{x_1} u_1) + P(a_{1,22}\partial_{x_1} u_2) + P(a_{2,12}\partial_{x_2} u_1) + P(a_{2,22}\partial_{x_2} u_2) = 0$$

and using that  $\partial_{x_1} Pu_1 = \partial_t \partial_{x_1} u_1 + a_{1,11}\partial_{x_1}^2 u_1 + \partial_{x_1} a_{1,11}\partial_{x_1} u_1 + \dots$  (we highlight only the derivative of the first term) we obtain that

$$\partial_{x_1} Pu_1 - P\partial_{x_1} u_1 = \partial_{x_1} a_{1,11}\partial_{x_1} u_1 + \dots$$

where in the right hand side appear only first order terms in  $u_1, u_2$ . We can thus replace  $Pu_1$  in terms of  $u_2$  and we obtain thus an equation of the form  $Qu_2 + L_1(u_1) = 0$ , where  $Q$  is a differential operator (with variable coefficients) on  $u_2$  of order 2 and  $L_1$  is a differential operator of order 1 on  $u_1$ . It is not clear then that  $L_1(u_1)$  is a lower order operator with respect to  $Qu_2$ . We notice then that, using the Fourier identity  $u_j(t, x_1, x_2) = \frac{1}{(2\pi)^3} \oint e^{i(t\tau + x_1\xi_1 + x_2\xi_2)} \hat{u}_j(\tau, \xi_1, \xi_2) d\tau d\xi_1 d\xi_2$ , that the system on  $u_1, u_2$  rewrites

$$\frac{1}{(2\pi)^3} \oint e^{i(t\tau + x_1\xi_1 + x_2\xi_2)} (i\tau + iA_1(t, x_1, x_2)\xi_1 + iA_2(t, x_1, x_2)\xi_2) \begin{pmatrix} \hat{u}_1(\tau, \xi_1, \xi_2) \\ \hat{u}_2(\tau, \xi_1, \xi_2) \end{pmatrix} d\tau d\xi_1 d\xi_2 = 0$$

and the high frequency approximation of this equality yields that the leading order term of  $\begin{pmatrix} \hat{u}_1(\tau, \xi_1, \xi_2) \\ \hat{u}_2(\tau, \xi_1, \xi_2) \end{pmatrix}$  is in the kernel of the matrix  $\tau Id + A_1(t, x_1, x_2)\xi_1 + A_2(t, x_1, x_2)\xi_2$ . This justifies the definition of the **characteristic manifold** for this case.  $\square$

We study now the type of the roots of the characteristic equation.

In the case of item 1, we check that  $l(x, t, \xi, \tau + \tau_0) = 0$  has only as root in  $\tau_0$  the real  $-\tau - \frac{a_1(x, t)}{a_0(x, t)}\xi$ . In the case of item 2, we will distinguish the quadratic form in  $\tau$  and the one in  $\xi_1, \xi_2$  for example. So we have

$$p(\tau, \xi_1, \xi_2) = \left(\tau + \frac{1}{2}(\text{Tr } A_1 \xi_1 + \text{Tr } A_2 \xi_2)\right)^2 + \det(\xi_1 A_1 + \xi_2 A_2) - \frac{1}{4}(\text{Tr } A_1 \xi_1 + \text{Tr } A_2 \xi_2)^2.$$

The matrix  $\xi_1 A_1 + \xi_2 A_2$  is symmetric and therefore diagonalizable. Its eigenvalues are real and denoted by  $\lambda_1(\xi_1, \xi_2)$  and  $\lambda_2(\xi_1, \xi_2)$ . It comes then

$$p(\tau, \xi_1, \xi_2) = \left(\tau + \frac{1}{2}(\lambda_1(\xi_1, \xi_2) + \lambda_2(\xi_1, \xi_2))\right)^2 - \frac{1}{4}(\lambda_1(\xi_1, \xi_2) - \lambda_2(\xi_1, \xi_2))^2.$$

We then check that the roots of  $p(\tau + s, \xi_1, \xi_2) = 0$  are of the form

$$s = -\tau - \lambda_1(\xi_1, \xi_2), s = -\tau - \lambda_2(\xi_1, \xi_2).$$

We note that they are not automatically simple, because we can have for example  $A_1 = A_2 = Id$ , in which case  $\lambda_1 = \lambda_2 = \xi_1 + \xi_2$ . We then notice that in this case, the problems on  $u_1$  and on  $u_2$  are decoupled.

By introducing a new time variable, which could be interpreted as the propagation along the diagonal part of  $A_1$  and  $A_2$ , that is

$$\tau' = \tau + \frac{1}{2}(\lambda_1(\xi_1, \xi_2) + \lambda_2(\xi_1, \xi_2))$$

the associated operator writes  $p(\tau', \xi_1, \xi_2) = (\tau')^2 - A(\xi_1, \xi_2)$ , where  $A(\xi_1, \xi_2)$  is the symbol of an elliptic differential operator (the coefficients depend indeed on  $a_{i,jk}$  only and it is a polynomial; only the positivity is more easily expressed with  $\lambda_1$  and  $\lambda_2$ ) in the sense that we have  $A(\xi_1, \xi_2) \geq C(\xi_1^2 + \xi_2^2)$ , with  $C > 0$ .

The eigenvalues of the matrix  $\tau Id + \xi_1 A_1 + \xi_2 A_2$  are then  $\tau + \lambda_1(\xi_1, \xi_2)$ ,  $\tau + \lambda_2(\xi_1, \xi_2)$ . The hyperbolicity assumption of the definition 2.1 does not exclude multiple eigenvalues. On the other hand, we note that Lax [60] imposes that  $\xi_1 A_1 + \xi_2 A_2$  has **two distinct real eigenvalues** (p 628).

The definition 2.1 of a hyperbolic matrix operator implies, in the case with constant coefficients, that the operator  $P$  of order  $m$  scalar deduced from the matrix operator  $\tau A_0 + \sum_{j=1}^{j=m} \xi_j A_j$  verifies  $\sigma(P)(x, t, \xi, \tau + s) = 0$  has only real solutions. If, moreover, the matrix problem associated to  $\tau A_0 + \sum_{j=1}^{j=m} \xi_j A_j$  satisfies the additional (restrictive) condition, then  $\sigma(P)(x, t, \xi, \tau + s) = 0$  has only real solutions of multiplicity 1. In all the above cases, we will say that the operators are hyperbolic with respect to the surfaces of time-type. This implies two distinct definitions of hyperbolicity :

**DEFINITION 10.2.** *We say that  $P$ , differential operator of order  $m$  on  $\mathbb{R}^d$ , is a hyperbolic operator with respect to  $N \in T_x \mathbb{R}^d$  when its principal symbol  $\sigma(P)$  satisfies*

$$\sigma(P)(x, \xi + sN) = 0 \text{ has only real roots.}$$

This definition comes from chapter 12.3 of [48]. In particular an operator  $P(D)$  is hyperbolic according to Hörmander when

$$P(\xi + i\tau N) \neq 0 \text{ pour } \xi \in \mathbb{R}^d \text{ et pour } \tau < \tau_0$$

This is equivalent, for a homogeneous polynomial, to  $\sigma(P)(x, \xi + sN) = 0$  has only real roots. (Theorem 12.4.3 and Theorem 12.4.6 of [48], volume II)

We also define a notion of strict hyperbolicity :

**DEFINITION 10.3.** *We say that  $P$ , differential operator of order  $m$  on  $\mathbb{R}^d$ , is a strictly hyperbolic operator with respect to  $N \in T_x \mathbb{R}^d$  when its principalsymbol  $\sigma(P)$  verifies*

$$\sigma(P)(x, \xi + sN) = 0 \text{ has only real roots of multiplicity 1.}$$

which is the definition 12.4.11 of [48], Tome II.

In the case we studied earlier, the coordinates are  $(x, t)$ , the dual coordinates are  $(\xi, \tau)$  and the vector  $N$  is (collinear to)  $(0, 1)$ . The hyperbolic operators of Definition 2.1 are the hyperbolic operators with respect to  $N = (0, 1)$ , a vector conormal to any surface  $t = t_0$ .

Hyperbolic operators are extremely important. Indeed, according to the theorem 12.5.1 of [48] Tome II, a hyperbolic operator admits a single fundamental solution supported in the hyperspace  $x.N \geq 0$ .

## 2. Eikonal equation and strictly hyperbolic Cauchy problem

In what follows, we study an operator  $P$  of real principal type which is strictly hyperbolic with respect to  $(0, \xi_n)$  (we also say with respect to  $x_n = 0$ ). We want to study the asymptotic solutions associated to  $P$ , by the same method as the one used in chapter 2 and chapter 3. This section is inspired by the book of F. Trèves [96].

The operator is strictly hyperbolic, which implies (as the roots are simple) that  $\partial_\xi p(x, \xi) \neq 0$  on a leaf of the characteristic manifold  $p(x, \xi) = 0$ . We can thus write a change of coordinates in  $X$  such that  $\partial_{\xi_n} p(x, \xi) \neq 0$  on the leaf considered. This leaf has the equation  $\xi_n = q(x, \xi')$ , and there exists a symbol  $e(x, \xi)$  (of order  $m - 1$  when  $p$  is of order  $m$ ) such that

$$p(x, \xi) = e(x, \xi)(\xi_n - q(x, \xi')),$$

The symbol  $p$  (and the associated operator) is strictly Cauchy hyperbolic for the leaf  $\xi_n = q(x, \xi')$  when there exists  $c$  such that  $e(x, \xi', q(x, \xi')) \geq c|\xi'|^{m-1}$  for  $\xi'$  large.

We study the bicharacteristics of  $e(x, \xi)(\xi_n - q(x, \xi'))$ . We denote by  $p_1(x, \xi) = \xi_n - q(x, \xi')$ . A generic point on the bicharacteristic is denoted by  $\rho(s) = (x(s), \xi(s))$  and we assume that  $\rho(0)$  belongs to  $\xi_n - q(x, \xi') = 0, \xi' \neq 0$ .

The system in  $T^*\mathbb{R}^n$  of bicharacteristic curves is

$$(2.130) \quad \begin{cases} \frac{d\xi_n}{ds} = e(\rho(s)) \frac{\partial q}{\partial x_n}(x(s), \xi'(s)) - \frac{\partial e}{\partial x_n}(x(s), \xi(s)) p_1(\rho(s)) \\ \frac{dx_n}{ds} = e(\rho(s)) + \frac{\partial e}{\partial \xi_n}(\rho(s)) p_1(\rho(s)) \\ \frac{d\xi'}{ds} = e(\rho(s)) \frac{\partial q}{\partial x'}(x(s), \xi'(s)) - \frac{\partial e}{\partial x'}(\rho(s)) p_1(\rho(s)) \\ \frac{dx'}{ds} = -e(\rho(s)) \frac{\partial q}{\partial \xi'}(x(s), \xi'(s)) + \frac{\partial e}{\partial \xi'}(\rho(s)) p_1(\rho(s)). \end{cases}$$

One checks that

$$e(\rho(s)) p_1(\rho(s)) = 0$$

Moreover, since  $\rho(0) \in \{\xi_n - q(x, \xi') = 0\}$ ,  $e(\rho(0)) \neq 0$ . There then exists  $s_0 > 0$  verifying  $e(\rho(s)) \neq 0$  for  $s \in [0, s_0]$ . This indicates that, for  $s \in [0, s_0]$ ,  $\rho(s)$  is in the manifold  $p_1(\rho(s)) = 0$ .

For  $s \in [0, s_0]$ , the system of bicharacteristic curves (2.130) is equivalent to

$$(2.131) \quad \begin{cases} \frac{d\xi_n}{ds} = e(\rho(s)) \partial_{x_n} q(x(s), \xi'(s)) \\ \frac{dx_n}{ds} = e(\rho(s)) \\ \frac{d\xi'}{ds} = e(\rho(s)) \partial_{x'} q(x(s), \xi'(s)) \\ \frac{dx'}{ds} = -e(\rho(s)) \partial_{\xi'} q(x(s), \xi'(s)). \end{cases}$$

Using the change of variable  $S(s)$  (which is a diffeomorphism of  $[0, s_0]$  over  $[0, S(s_0)]$  and whose inverse diffeomorphism is denoted by  $s(S)$ ) such that

$$S'(s) = e(\rho(s)), S(0) = 0$$

we verify that (2.131) is equivalent to

$$(2.132) \quad \begin{cases} \frac{d\xi_n}{dS} = \partial_{x_n} q(x(s(S)), \xi'(s(S))) \\ \frac{dx_n}{dS} = 1 \\ \frac{d\xi'}{dS} = \partial_{x'} q(x(s), \xi'(s)) \\ \frac{dx'}{dS} = -\partial_{\xi'} q(x(s), \xi'(s)). \end{cases}$$

This is the system giving the bicharacteristic curves of the pseudodifferential operator of symbol  $p_1$ . A bicharacteristic curve of  $p_1$  is a bicharacteristic curve of  $p$  as long as  $\xi'(s) = 0$ . We consider now the pseudo-differential operator  $P$  of symbol :

$$p(x, \xi) = \xi_n - q(x, \xi')$$

where  $q$  is a homogeneous symbol of degree 1. Its bicharacteristic flow is given by (2.132). The eikonal equation associated to this pseudodifferential operator is

$$\partial_{x_n} \phi(x', x_n) = q(x_n, x', \partial_{x'} \phi(x_n, x')).$$

We also impose the initial condition on  $x_n = 0$   $\partial_{x'} \phi(0, x') = \xi'_0(x')$ .

LEMMA 10.1. *Let  $\phi$  be the solution phase of the eikonal equation  $\partial_{x_n}\phi(x) = q_1(x, \partial_{x'}\phi)$ . We give  $\phi(x', a)$  and  $\partial_{x'}\phi(x', a)$  (i.e. we impose  $\phi$  on the hypersurface  $x_n = a$  and its normal derivative on this hypersurface). There exists a function  $\Psi$ , determined in a neighborhood of the point  $(x'_0, a)$  thanks to the integral curves, by*

$$\phi(x', x_n) = \phi(x', a) + \int_a^{x_n} q_1(x', u, \xi'(u, x', \Psi(x', u, \partial_{x'}\phi(x', a))), \partial_{x'}\phi(x', a)) du.$$

Preuve. Consider the bicharacteristic from the point

$$x'(0) = x', x_n(0) = 0, \xi'(0) = \xi'_0, \xi_n(0) = q(x', 0, \xi'_0(x'))$$

The parameter  $s$  on the bicharacteristic is equal to  $x_n$ . We choose this new parameter as a variable. A general point on the bicharacteristic is denoted by

$$(x'(s, x', \xi'_0), s, \xi'(s, x', \xi'_0), q(x'(s, x', \xi'_0), s, \xi'(s, x', \xi'_0))).$$

The result of the proposition 9.1 (inclusion of a bicharacteristic in a Lagrangian solution if a point of this bicharacteristic is in it) allows to write

$$\frac{\partial\phi}{\partial x'}(x'(s, x', \xi'_0), s) = \xi'(s, x', \xi'_0).$$

One then deduces

$$(2.133) \quad \frac{\partial\phi}{\partial x_n}(x'(s, x', \xi'_0), s) = q(x'(s, x', \xi'_0), s, \xi'(s, x', \xi'_0))$$

which is an equation on the behavior of the phase in  $x_n$ . We assume that the neighborhood of the considered point is totally characteristic, i.e. any point  $(y', x_n)$  is can be reached by a unique bicharacteristic in this neighborhood.

There exists then a function  $\Psi(x_n, y', \xi'_0)$  such that the equation in  $x'_0$ :  $y' = x'(x_n, x'_0, \xi'_0)$  admits locally for solution:

$$x'_0 = \Psi(x_n, y', \xi'_0).$$

The bicharacteristic passing through  $(\Psi(x_n, y', \xi'_0), 0, \xi'_0, q(\Psi(x_n, y', \xi'_0), 0, \xi'_0))$  goes through the point  $(y', x_n, \xi'(x_n, \Psi(x_n, y', \xi'_0), \xi'_0), q(y', x_n, \xi'(x_n, \Psi(x_n, y', \xi'_0), \xi'_0)))$ . Equation (2.133) is equivalent to :

$$\frac{\partial\phi}{\partial x_n}(y', x_n) = q(y', x_n, \xi'(x_n, \Psi(x_n, y', \xi'_0), \xi'_0)).$$

We deduce the solution from  $\phi(y', 0)$ . This last term is calculated by noting that

$$\partial_{x'}\phi(y', 0) = \xi'(0, \Psi(0, y', \xi'_0), \xi'_0) = \xi'_0.$$

It is then sufficient to express the relation giving  $\xi'_0$  as a function of  $x'$  to deduce  $\phi(y', 0)$ . We obtain the relation of Lemma 10.1.

### 3. Theorem of propagation of singularities

The aim of this Section is to prove the

THEOREM 10.1. *Let  $P$  be a classical pseudodifferential operator of degree 1, whose principal symbol  $p$  is of real principal type. Then*

$$\gamma_p(\rho_0) \cap WF(Pu) = \emptyset \rightarrow \gamma_p(\rho_0) \cap WF(u) = \emptyset \text{ or } \gamma_p(\rho_0) \subset WF(u).$$

Before proving this general result, let us consider a simple case. Consider the derivation operator in  $\mathbb{R}^n$  with respect to the last coordinate. The coordinates in  $\mathbb{R}^n$  are denoted  $(y', y_n)$  and the coordinates in  $T^*(\mathbb{R}^n)$  are denoted  $(y', y_n, \xi', \xi_n)$ , and we give  $y_n^0 \in \mathbb{R}$ . Let  $\tilde{u}_0(y')$  be a function of class  $C^\infty(\mathbb{R}^{n-1})$ . We note in the traditional way  $D_{y_n}$  the operator  $\frac{1}{i} \frac{\partial}{\partial y_n}$ , so that the symbol of the operator  $D_{y_n}$  is  $\xi_n$ . We study the problem model in  $\mathbb{R}^n$  as follows :

$$\begin{cases} D_{y_n} \tilde{u}(y) = \tilde{v}(y) \\ \tilde{u}(y', y_n^0) = \tilde{u}_0(y') \end{cases}$$

which solution is

$$\tilde{u}(y) = \tilde{u}_0(y') + i \int_{y_n^0}^{y_n} \tilde{v}(y', t) dt$$

We have

LEMMA 10.2. *Let  $\gamma$  a bicharacteristic of  $D_{y_n}$  (of  $\xi_n$ ).*

$$\gamma \cap WF(D_{y_n}\tilde{u}) = \emptyset \Rightarrow \gamma \cap WF(\tilde{u}) = \emptyset \quad \text{or} \quad \gamma \subset WF(\tilde{u}).$$

PROOF. Let  $H$  be the Heaviside function, indicator function of  $\mathbb{R}_+$ . We introduced the distribution  $w(y', y_n) = i\delta(y') \otimes H(y_n)$  and the kernel operator  $\tilde{E}$  such that  $\tilde{E}\phi = w \star \phi$ , more precisely  $\tilde{E}\phi(x) = \langle w, \phi(x - \cdot) \rangle$ . One has the equality

$$\tilde{u} = \tilde{u}_0 + \tilde{E}\tilde{v}.$$

We also check that the operator  $\tilde{E}$  is a parametrix of  $D_{y_n}$ . The wavefront set of  $w$  is

$$WF(w) = \{(0, y_n, \xi', 0), y_n > 0\} \cup \{(0, \xi)\}.$$

This fact can be easily verified by performing the Fourier transform of this distribution, in the vicinity of a point such that  $y' = 0$  (indeed, the points  $y' \neq 0$  do not contribute to the front).

Let us now search for the wavefront of the kernel  $K$  associated to the operator  $w\star$ . As  $w\star u(x) = \int w(x-y)u(y)dy$ , we have  $K(x, y) = w(x-y)$ .

We consider a function  $\chi$  which localizes in  $t$  in the neighborhood of  $t_0$ , and we give  $x_0, y_0$  such that  $x_0 - y_0 = t_0$ . We localize  $y$  in the neighborhood of  $y_0$  through  $\phi_0$ ,  $\chi$  and  $\phi_0$  are compactly supported. We deduce  $WF(K)$  by the equality :

$$\mathcal{F}(\phi_0\chi K)(\xi, \eta) = \int \int \chi(t)w(t)e^{-it\xi} dt \int dy\phi_0(y)e^{-iy(\xi+\eta)}.$$

Since  $\chi$  is compactly supported, the distribution  $\chi w$  is of finite order, so its Fourier transform is at most polynomial.

First, since  $\phi_0$  is compactly supported and  $C^\infty$ , if  $(\xi, \eta)$  belongs to a conic neighborhood of  $(\xi_0, \eta_0)$ , with  $\xi_0^2 + \eta_0^2 \neq 0$ , then the integral in  $y$  is rapidly decaying in  $(|\xi|^2 + |\eta|^2)^{\frac{1}{2}}$ . Thus, with the growth of the Fourier transform of  $w$ , we have rapid decay. The points of the form  $(x_0, y_0, \xi_0, \eta_0)$  with  $\xi_0^2 + \eta_0^2 \neq 0$  are not in the the wavefront of  $K$ .

We are now concerned with  $\xi_0 + \eta_0 = 0$ . Then, if  $(x_0 - y_0, \xi_0)$  is not in the wavefront of  $w$ , the point  $(x_0, y_0, \xi_0, \eta_0)$  is not in the wavefront of  $K$ . We have obtained

$$WF(K) \subset \{(x, \xi, y, -\xi), (x-y, \xi) \in WF(w)\}.$$

We deduce

$$WF'(K) \subset \{(x, \xi, y, \xi), (x-y, \xi) \in WF(w)\}.$$

On the other hand, the bicharacteristics of the operator  $D_{y_n}$ , noted  $\gamma(s) = \exp sH_{x_n}(m_0)$ , are given, for  $m_0 = (y_0, \xi'_0, 0)$ , by  $\gamma(s) = (y'_0, y_n^0 + s, \xi'_0, 0)$ .

We use the remark of the section 3, to obtain the wavefront of  $\tilde{E}\tilde{v}$ . The relation  $WF(w\star u) = WF(Ku) \subset WF'(K)(WF(u)) \cup WF'_X(K)$  and the fact that  $WF'_X(K) = \{(x, \xi), \exists y, (x, \xi, y, 0) \in WF'(K)\} = \emptyset$  gives, using the notations of the equality giving  $\tilde{u}$  :

$$WF(\tilde{E}\tilde{v}) \subset WF(\tilde{v}) \cup \{(Y, \xi), \exists(y, \xi) \in WF(\tilde{v}), (Y-y, \xi) \in WF(w)\}.$$

We prove the singularity propagation theorem in the case of the operator  $\partial_{x_n}$ .

Let  $\gamma$  be a bicharacteristic of  $\xi_n$ . We assume that  $\gamma \cap WF(\tilde{v}) = \emptyset$ . The previous study shows that  $\gamma \cap WF(\tilde{E}\tilde{v}) = \emptyset$ . It remains to study the term  $\tilde{u}(x', 0)$ . The wavefront of  $\tilde{u}_0$  is included in  $T^*(\mathbb{R}^{n-1})$ . To characterize the wavefront in  $\mathbb{R}^n$  of  $\tilde{u}(x', 0)$ , we evaluate the Fourier transform of  $\chi(x_n)\phi_0(x')\tilde{u}(x', 0)$ . We find

$$\mathcal{F}(\chi\phi_0\tilde{u}|_{x_n=0})(\xi', \xi_n) = \hat{\chi}(\xi_n)\mathcal{F}(\tilde{u}_0\phi_0)(\xi').$$

Let  $\xi_n^0 \neq 0$  and  $(\xi_n^0)^2 + ((\xi')^0)^2 = 1$ . We consider  $(\xi_n, \xi')$  in a (small) sphere around  $(\xi_n^0, (\xi')^0)$ . Then there exists  $\epsilon > 0$  such that  $|\xi_n - \lambda\xi_n^0| \leq \epsilon\lambda$  for all  $\lambda$ , and as  $\chi$  is of class  $C^\infty$ , its Fourier transform is rapidly decaying in  $\xi_n$  so in  $\lambda$ . We deduce that  $(x', \xi', x_n, \xi_n)$ ,  $\xi_n \neq 0$  is not in the wavefront of  $\tilde{u}(x', 0)$ . From this we deduce

$$WF(\tilde{u}(x', 0)) \subset \{(x', x_n, \xi', 0)\}.$$

Moreover, if  $(x'_0, (\xi')^0) \in WF(\tilde{u}_0)$ , we easily obtain  $(x'_0, x_n, (\xi')^0, 0) \in WF(\tilde{u}(x', 0))$  for all  $x_n$ . We then assume that a bicharacteristic intersects the wavefront of  $\tilde{u}$  solution of  $D_{x_n}\tilde{u} = \tilde{v}$  at a point denoted by  $\rho'_0$ , but does not meet the wavefront of  $\tilde{v}$ . This point of intersection is then equal to  $((y')^0, y_n^0, (\eta')^0, 0)$ . ( $\eta_n^0 = 0$  because the bicharacteristic is included in the characteristic manifold).

Thus, since  $\tilde{E}\tilde{v}$  is regular, we deduce that  $((y')^0, (\eta')^0)$  is in the wavefront of  $\tilde{u}_0$ , and the front wavefront of  $\tilde{u}(x', 0)$  then contains all points of the form  $((y')^0, y_n^0, (\eta')^0, 0)$ . The bicharacteristic from  $((y')^0, y_n^0, (\eta')^0, 0)$  is  $\{((y')^0, y_n^0 + s, (\eta')^0, 0), s \in \mathbb{R}\}$ , so it is entirely contained in the wavefront of  $\tilde{u}$ . The proof is finished and we have either  $\gamma \cap WF(\tilde{u}) = \emptyset$ , or  $\gamma \subset WF(\tilde{u})$ . Now consider the point  $\gamma(s_0) = (y'_0, y_n(s_0), \xi'_0, 0)$  and we study  $\tilde{U}(y) = -\tilde{u}(y', 2y_n(s_0) - y_n)$ . Alors  $D_{y_n}\tilde{U}(y) = \tilde{v}(y', 2y_n(s_0) - y_n) = \tilde{V}(y)$ . We verify that  $\gamma \cap WF(\tilde{V}) = \emptyset$  and that  $\gamma(s_0) \in WF(\tilde{U})$ . Hence  $\gamma(s), s \geq s_0$  is contained  $WF(\tilde{U})$ , which is equivalent to "  $\gamma(s), s \leq s_0$  contained in  $WF(\tilde{u})$ ".

We just proved the result of Lemma 10.2.  $\square$

For proving theorem 10.1, consider a classical pseudo-differential operator of order 1 ( $\in L_{cl}^1$ )

PROOF. By abuse of language, and by similarity with the strictly hyperbolic Cauchy problem, we call this theorem the hyperbolic theorem of propagation of singularities. Indeed, the operator  $\xi_n - e(x, \xi')$ , where  $e$  is homogeneous in  $\xi'$  of degree 1, verifies  $dp = d\xi_n - \frac{\partial e}{\partial x} dx - \frac{\partial e}{\partial \xi'} d\xi'$ , and the coefficient 1 in front of  $d\xi_n$  implies that  $p$  is of real principal type. This result is Theorem 6.1.4 of the paper by Duistermaat and Hörmander [32]. When  $(x_0, \xi_0)$  is a point of the characteristic manifold of  $p$ , operator of real principal type, the authors construct a Fourier integral operator  $A$  such that the point  $(x_0, \xi_0; X_0, \Sigma_0)$  is not in the operator wavefront of  $PA - AD_{x_n}$ , where the Fourier integral operators  $A$  and  $B$  characterize the canonical transformation such that  $BA = Id + R_1, AB = Id + R_2, R_{1,2} \in S^{-\infty}$  and such that the principal symbol of the operator  $Q := BPA$  is  $\eta_n$ . This principal symbol is the symbol of the operator  $\frac{1}{i} \frac{\partial}{\partial y_n}$  is traditionally denoted by  $D_{y_n}$ . There therefore exists an operator  $R$  of order 0 such that  $Q = D_{y_n} + R$ . From  $R$ , we construct a pseudo-differential elliptic operator of order 0 ( $|C(y, \eta)| \geq c > 0$  for  $(y, \eta)$  in a neighborhood of the point  $\chi(x_0, \xi_0)$ ) such that  $(D_{y_n} + R)C = CD_{y_n}$ . The symbol of this operator is solution of the equations

$$\begin{cases} C_0(y', \eta, y_n^0) = 1 \\ C_p(y', \eta, y_n^0) = 0 \\ \partial_{y_n} C_p(y, \eta) = i(RC)_p(y, \eta), p \geq 0 \end{cases}$$

equivalent to  $C = 1$  on  $y_n = y_n^0$  and  $[D_{y_n}, C] + RC = 0$ .

Since  $C$  is elliptic, it admits an inverse  $C^{-1}$  such that  $CC^{-1} = Id + R_3, C^{-1}C = Id + R_4, R_{3,4} \in S^{-\infty}$ . Then there exists an operator  $R_5$  of  $L^{-\infty}$  such that

$$(3.134) \quad C^{-1}BPAC = D_{y_n} + R_5.$$

We constructed a parametrix  $\tilde{E}$  of  $D_{y_n}$ , such that

$$Id = \tilde{E}D_{y_n} = \tilde{E}C^{-1}BPAC + R_6$$

which implies  $(AC)^{-1} + R_7 = \tilde{E}C^{-1}BP$  or again

$$AC\tilde{E}C^{-1}BP = Id + R_8.$$

A parametrix of  $P$  is then  $AC\tilde{E}C^{-1}B$ , and we can also, modulo the  $S^{-\infty}$  terms, write

$$P = ACD_{y_n}C^{-1}B.$$

Let  $\gamma_p(\rho_0)$  be the bicharacteristic of  $P$  from  $\rho_0$ . It is assumed that the bicharacteristic is not in the wavefront of  $Pu$ .

$$\gamma_p(\rho_0) \cap WF(Pu) = \emptyset \Rightarrow \gamma_p(\rho_0) \cap WF(ACD_{y_n}C^{-1}Bu) = \emptyset.$$

Canonical transformations transforming  $P_1$  into  $P_2$  send bicharacteristics of operator  $P_1$  to bicharacteristics of operator  $P_2$ . This result in the case of a symplectic transformation associated to a change of variable in the  $x$ -space comes from the Proposition 8.4. The generalization to a general transformation is a consequence of Theorem 9.1 and of the invariance of the principal symbol after canonical transformation:  $p_m(y, \nabla_y \phi) = p_m(\nabla_\theta \psi, \theta)$  (relation (9.4.9)). We obtain  $\chi(\gamma_p(\rho_0)) \cap WF(CD_{y_n}C^{-1}(Bu)) = \emptyset$ . Now  $\chi(\gamma_p(\rho_0)) = \gamma_{\xi_n}(\chi(\rho_0))$ , and, as  $C$  is elliptic as well as  $C^{-1}$ ,  $WF(CD_{y_n}C^{-1}(Bu)) = WF(D_{y_n}(Bu))$  (application of the proposition 7.8).

So we have  $\gamma_{\xi_n}(\chi(\rho)) \cap WF(D_{y_n}Bu) = \emptyset$ . By application of the lemma 10.2,  $\gamma_{\xi_n}(\chi(\rho)) \cap WF(Bu) = \emptyset$  or  $\gamma_{\xi_n}(\chi(\rho)) \subset WF(Bu)$ .

The wave front set is preserved under the canonical transformation (in the case of a symplectic transformation, it is a result of the Proposition 8.4). If  $B$  is a quantization of this canonical transformation,  $WF(Bu) = \chi(WF(u))$ . Indeed, from the relation of paragraph 9.4.1  $WF(Au) \subset \{(x, \nabla_x \phi), (\nabla_\eta \phi, \eta) \in WF(u)\}$ . We have then the inclusion  $WF(Au) \subset T_\chi(WF(u))$  and we deduce

$WF(BAu) \subset T_x^{-1}(WF(Au))$ . On the other hand,  $BA = I + R_2$  so  $WF(BAu) = WF(u)$ , and we have the equality we are looking for. We deduce

$$\gamma_p(\rho_0) \subset WF(u) \quad \text{ou} \quad \gamma_p(\rho_0) \cap WF(u) = \emptyset.$$

We have completed the proof for an operator of symbol  $p_1(x, \xi)$  homogeneous of degree 1.

When  $p$ , of order  $m$ , is of real principal type, in the neighborhood of a strictly hyperbolic point  $\rho$ , there exists an elliptic operator  $E$  (in the neighborhood of the point  $\rho$ ) such that  $P = EP_1 + R$ ,  $E$  elliptic,  $P_1$  of order 1 and  $R \in S^{-\infty}$ . The wavefront of  $Pu$  in the neighborhood of  $\rho$  is equal to  $WF(P_1u)$  in this neighborhood (consequence of the proposition 7.8). The bicharacteristics of  $P$  are the same as the bicharacteristics of  $P_1$  (see the section 2), so

$$\gamma_p(\rho_0) \cap WF(Pu) = \emptyset \Rightarrow \gamma_{p_1}(\rho_0) \cap WF(P_1u) = \emptyset.$$

One uses then

$$\gamma_{p_1}(\rho_0) \cap WF(P_1u) = \emptyset \Rightarrow \gamma_{p_1}(\rho_0) \cap WF(u) = \emptyset \quad \text{ou} \quad \gamma_{p_1}(\rho_0) \subset WF(u).$$

Thus we proved the theorem 10.1 for any operator.  $\square$

Duistermaat and Hörmander state this theorem (Theorem 6.1.1 of [32]) in the following way :  
*If  $P \in L_{cl}^m(X)$  is properly supported, of real principal type and if  $u \in \mathcal{D}'(X)$  and  $Pu = f$ , then  $WF(u) - WF(f) \subset p^{-1}(0)$  and this set is invariant by  $H_p$ .*

The goal of the following sections is to generalize this theorem of propagation of singularities by the Hamiltonian flow to the case with boundary, i.e. the case where the considered bicharacteristic meets the boundary of the domain (the problem  $Pu = 0$  is solved in  $M$ , typically  $x_n > 0$ ). We restrict ourselves to the case of the application we are interested in : the waves problem.

#### 4. Boundary problems for the wave equation.

In this section we study problems at the boundary. The typical problem that we will study is the following :

Let  $P$  be a second order operator defined on  $\mathbb{R}^n$  and let  $\Omega$  be an open set of  $\mathbb{R}^n$ . We want to solve, locally near the boundary, the problem  $(P - \partial_t^2)u = 0$  in the complementary of  $\Omega \times \mathbb{R}_t$ , and a boundary condition on the boundary  $\partial\Omega \times \mathbb{R}_t$ , which can be either

- Dirichlet boundary condition (D)  $u|_{\partial\Omega \times \mathbb{R}_t} = 0$ ,
- Neumann boundary condition (N)  $\partial_n u|_{\partial\Omega \times \mathbb{R}_t} = 0$  (this in the case where  $u \in H_{loc}^1((\mathbb{R}^d - \Omega) \times \mathbb{R}_t) \cap \{Pu \in L^2\}$ , since we can then define the normal derivative),
- mixed boundary condition (M)  $\partial_n u|_{\partial\Omega \times \mathbb{R}_t} + z(x, t)\partial_t u|_{\partial\Omega \times \mathbb{R}_t} = 0$ .

We also give the two Cauchy conditions,  $u|_{t=0}$  and  $\partial_t u|_{t=0}$ . **Note that these two Cauchy conditions can be interpreted as  $u|_{t<0} = u_i|_{t<0}$ , where the function  $u_i$  is an incident solution of the wave equation.**

It is assumed that, locally in the neighborhood of  $y_0$ , this boundary can be straightened, that is, there exists a system of coordinates  $(x)$  in which the boundary is  $x_n = 0$ . We will come back to this for the problem of diffraction of waves by a convex open set. We start by showing the jump formula, which allows us to know the solution of a problem in the exterior of an obstacle as a function of the traces on the boundary. It is a generalization of Green's formula and single and double layer potentials. We apply this result, stated for a differential operator of order  $m$ , to the operator Laplacian operator, of order 2. Let us note immediately, without demonstration, that there is no redundancy between the Cauchy condition and the boundary condition.

**4.1. Jump formula and boundary wave front set  $WF_b$ .** We give ourselves a differential operator of order  $m$ , in the form  $P = \sum_{j=0}^{j=m} P_j(x_n, x', D_{x'}) (D_{x_n})^j$ . We define an extendible distribution solution.

**DEFINITION 10.4.** • *Let  $V \subset \mathbb{R}^n$ , and  $S$  be a smooth hypersurface of equation  $s = 0$ . We define, in the vicinity of a point such that  $\nabla_x s \neq 0$ , the two open sets  $V_{\pm} = \{x \in V, \pm s(x) > 0\}$ . We can also define  $\tilde{V}_+ = \{x \in V, s(x) \geq 0\}$ . The set  $\tilde{V}_+$  is the manifold with boundary considered here. We say that  $u$  is a extendible distribution of  $V_+$ , and we denote  $u$  as a  $\mathcal{D}'(\tilde{V}_+)$ , when there exists  $\tilde{u} \in \mathcal{D}'(\mathbb{R}^n)$  such that*

$$\forall \phi \in C_0^\infty(V_+), \langle u, \phi \rangle = \langle \tilde{u}, \phi \rangle.$$

The space of extendible distributions is thus the dual space of  $C_0^\infty(V_+)$ , space of  $C^\infty$  test functions with compact support vanishing at any order on  $\partial V_+$  (see Melrose [75]).

As  $\nabla_x s \neq 0$ , we choose a coordinate  $x_j$  such that  $\frac{\partial s}{\partial x_j}(x_0) \neq 0$ . We reorder the coordinates so that  $j$  becomes  $n$ . Let  $V$  be a neighborhood of  $x_0$  where  $|\frac{\partial s}{\partial x_n}(x)| \geq \frac{1}{2}|\frac{\partial s}{\partial x_n}(x_0)|$ . By the implicit function theorem,  $s = 0$  is equivalent to  $x_n = \psi(x')$ ,  $\psi$  being a function of class  $C^\infty$  in  $V$ , and  $s > 0$  is equivalent to  $\frac{\partial s}{\partial x_n}(x_0)(x_n - \psi(x')) > 0$ . We then choose  $X_n = \frac{\partial s}{\partial x_n}(x_0)(x_n - \psi(x'))$ . The application  $(x', x_n) \rightarrow (x', X_n)$  is a diffeomorphism of  $V$  on its image, and  $V_\pm = \{\pm X_n > 0\}$ .

We have therefore reduced ourselves to  $s(x) = x_n$ .

PROPOSITION 10.2. *Let  $u$  be an extendible distribution of  $\mathcal{D}'(V_+)$  such that  $Pu = 0$  in  $V_+$ . We assume*

$$p(x_0, 0, 0, \dots, 1) \neq 0.$$

*There exists a unique distribution  $\underline{u}$ , extending  $u$  by 0 in  $V_-$ , such that there exists  $m > 0$  with  $x_n^m P\underline{u} = 0$ .*

PROOF. Let  $x_0$  be a point of  $\partial V$  and  $U$  a compact set containing an open set  $W$  containing  $x_0$ . We now restrict all distributions to  $W$ . Thus  $\tilde{u}$ , extending  $u$ , is compactly supported and therefore of finite order. In this analysis we take any extension of  $u$ . Thus there exists  $M > 0$  such that

$$(1 - \Delta)^{-M} \tilde{u}(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} \frac{\hat{u}(\xi)}{(1 + |\xi|^2)^M} d\xi$$

and one is in the case where it is continuous on  $\mathbb{R}^n$ . One then defines

$$\check{u}(x) = (1 - \Delta)^M [1_{x_n \geq 0} (1 - \Delta)^{-M} \tilde{u}(x)].$$

This distribution satisfies  $\check{u}(x) = 0, x_n < 0$ ,  $\check{u}(x) = u(x), x_n > 0$ . Moreover,  $P\check{u} = 0$  for  $x_n \neq 0$ , because  $u$  is solution of  $Pu = 0$ . The distribution  $P\check{u}$  is of finite order, so there are  $a_j, 0 \leq j \leq j_0$ ,  $a_j(x') \in \mathcal{D}'_{j_0}(|x'| < r)$  such that

$$P\check{u} = \sum_{j=0}^{j_0} a_j \otimes \delta_{x_n=0}^j.$$

Let  $\mathcal{D}'_S$  be the space of distributions with support in  $S = \partial V_+ = \{x_n = 0\}$ ,  $\mathcal{D}'_{S,l}$  the subspace of  $l$ -layer distributions with coefficients on  $S$ , of the form  $a_l \otimes \delta_{x_n=0}^l$  (distribution defined, for  $\phi \in C_0^\infty(\mathbb{R}^n)$  through the restriction  $\partial_{x_n}^l \phi(x', 0) := \psi_l(x')$ ,  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ ), by  $\langle a_l \otimes \delta_{x_n=0}^l, \phi \rangle = (-1)^l \langle a_l, \psi_l \rangle$ , duality of distributions in  $\mathcal{D}'(\mathbb{R}^{n-1})$ ). Since  $P$  is a differential operator

$$b \rightarrow P(b)$$

defines a linear form  $T$  on  $\mathcal{D}'_S$ . This application is injective. Moreover we verify that, for any  $a \in \mathcal{D}'_S$ , there exists a distribution  $b \in \mathcal{D}'_S$  such that  $T(b) - a \in \mathcal{D}'_{S,m-1}$ .

We prove this explicitly. Let  $P = \sum p_{j\alpha}(x) \partial_{x'}^\alpha \partial_{x_n}^j$ . We write  $b = \sum_{j=0}^{j_0} b_j(x') \otimes \delta_{x_n=0}^j$ , so

$$\begin{aligned} \langle P(b), \phi \rangle &= \sum_{j+|\alpha| \leq m, l \leq j_0} (-1)^{j+|\alpha|} \langle b_l \otimes \delta_{x_n=0}^l, \partial_{x'}^\alpha \partial_{x_n}^j [p_{j,\alpha}(x) \phi(x)] \rangle \\ &= \sum_{j+|\alpha| \leq m, l \leq j_0} (1^{-|\alpha|+j+l} \langle b_l, \partial_{x'}^\alpha \partial_{x_n}^{j+l} [p_{j,\alpha}(x) \phi(x)]|_{x_n=0} \rangle \end{aligned}$$

After applying the Leibniz formula for the derivation in  $x_n$ , it remains

$$T(b) = \sum_{p=0}^{p=m+j_0} \sum_{j+l \geq p} (-1)^{j+l-p} C_{j+l}^p \partial_{x_n}^{j+l-p} p_{j,\alpha}(x', 0) \partial_{x'}^\alpha b_l \otimes \delta_{x_n=0}^p.$$

We check that, for  $p = m + j_0$ , the coefficient is obtained by taking  $j = m, l = j_0$ , which gives  $p_{m,0}(x', 0) b_{j_0}(x')$ . The coefficient  $p_{m,0}$  is equal to 1, so the term of order  $m + j_0$  of  $T(b)$  is equal to the term of order  $j_0$  of  $b$ . Similarly, the term of order  $m + j_0 - 1$  is  $b_{j_0-1} + \sum_{|\alpha|=1} p_{m-1,\alpha} \partial_{x'}^\alpha b_{j_0} + p_{m-1,0}(x', 0) b_{j_0} - \partial_{x_n} p_{m,0}(x', 0) C_{m+j_0}^1 b_{j_0}$ . Thus the term of order  $m + j_0 - q$  will include the term  $b_{j_0-q}$  with for coefficient 1, and all terms  $b_{j_0-q'}$  for  $q' \leq q$ .

When  $a$  is given, of order  $r \geq m$ , we can construct a sequence of distributions  $b_j, j \leq r - m$ , such that  $T(b) = a$ . Indeed,  $b_{r-m} = a_r$ , and we construct  $b_{q-m}$  with the  $b_{q'-m}$  for  $q' \geq q$ .

We return to  $P(\check{u})$ . There exists  $(g_j(u))_{0 \leq j \leq m}$  such that  $T(\sum_{j=0}^m g_j(u) \otimes \delta_{x_n}^{(j)}) - P(\check{u}) \in \mathcal{D}'_{S,m-1}$ , and  $b$  is unique. We then deduce that  $x_n^m [T(b) - P(\check{u})] = 0$  by comparing the orders of the distributions.

We denote by  $\underline{u} = \check{u} - b$ . This is a solution to the problem of Proposition 10.2. If there are two solutions, we check that  $\underline{u}_1 - \underline{u}_2 \in \mathcal{D}'_S$  and  $T(b) \in \mathcal{D}'_{S,m-1}$ . This implies  $b = 0$  by studying the order of  $T(b)$ . The extension is therefore unique.  $\square$



REMARK 11. *Extending  $u$  by 0 does not uniquely define an extended solution, and even can lead to a distribution which is not well defined.*

PROOF. Indeed, the application  $\phi \rightarrow \langle u, \phi|_{\partial\Omega} \rangle$  is not well defined because  $\phi|_{\partial\Omega}$  is not  $C^\infty$  (assume for example that the support of  $\phi$  contains strictly  $\Omega$ , there might be a jump for  $\phi|_{\partial\Omega}$ ). As for the first part of the remark, let  $u_1$  and  $u_2$  be two distributions such that

$$u = u_1 = u_2 \text{ in } \mathcal{D}'(V_+), 0 = u_1 = u_2 \text{ in } \mathcal{D}'(V_-).$$

Then  $u_1 - u_2$  is supported on  $x_n = 0$ , so on a compact subset of  $\{x_n = 0\}$ , there exist  $M \in \mathbb{N}$  and  $M + 1$  distributions  $a_j$  of  $\mathcal{E}'(\mathbb{R}^{n-1})$  such that  $u_1 - u_2 = \sum_{j=0}^M a_j \otimes \delta_{x_n}^{(j)}$ . This idea is used in the proof of the proposition 10.2. In particular, if  $D = \delta_{x_n=0}$  is the Dirac distribution on  $x_n = 0$ , defined by  $\langle D, \phi \rangle = \int_{\mathbb{R}^{n-1}} \phi(x', 0) dx'$ , we see that for  $\phi = 0$  on a compact subset near  $\{x_n = 0\}$ ,  $\langle D, \phi \rangle = \langle 0, \phi \rangle$  yet  $D \neq 0$ . On the other hand, we define the space  $C_{(0)}^\infty(V_+)$  as the space of restrictions to  $V_+$  of functions of  $C_0^\infty(\mathbb{R}^n)$ , then, for  $\chi_{test}(x', x_n) = \chi(x_n)\chi(x_1)\dots\chi(x_{n-1})$ ,  $\chi$  being a positive function of integral 1, we have  $\langle D, \chi_{test} \rangle = \chi(0) \neq \langle 0, \chi_{test} \rangle$ .  $\square$

When  $P$  is a differential operator of order 2, we have the following result

LEMMA 10.3. *Let  $u$  be an extendible distribution solution of  $Pu = 0$  in  $V_+$ . Let  $\underline{u}$  be its unique extension in the sense of the proposition 10.2. There exist two distributions  $g_0(u)$  and  $g_1(u)$  of  $\mathcal{D}'_S$ , such that*

$$P\underline{u} = g_0(u) \otimes \delta_{x_n=0} + g_1(u) \otimes \delta'_{x_n=0}.$$

*This formula is called the jump formula at order 2.*

For this, we recall that there exists a unique  $\underline{u}$ , canonical extension of  $u$ ,  $\underline{u} \in \mathcal{D}'(V)$ ,  $\underline{u}|_{V_+} = u$ ,  $\underline{u}|_{V_-} = 0$ ,  $x_n^2 P\underline{u} = 0$ . By writing  $P\underline{u} = \sum_{j=0}^{j=m} b_j \otimes \delta_{x_n}^j$ , since  $P\underline{u}$  is supported on  $x_n = 0$  and is compactly supported, we check that, for  $j \geq 2$ ,  $x_n^2 b_j \otimes \delta_{x_n}^j = j(j-1)b_j \otimes \delta_{x_n}^{j-2}$ , so  $b_j = 0$  for  $j \geq 2$ . One has  $b_0 = g_0(u)$ ,  $b_1 = g_1(u)$ . Finally, we define the wavefront up to the boundary, which we note  $WF_b(u)$  for an extendible distribution  $u$ . There are several definitions see Hörmander [48], [75], Melrose-Sjöstrand [77]. For this, we consider the canonical injection  $i$  of  $C_0^\infty(V_+)$  into  $C_{(0)}^\infty(V_+)$ , associated to the dual surjection  $i^*$  of  $\mathcal{D}'(V_+)$  onto  $\mathcal{D}'(\tilde{V}_+)$ . The regular distributions up to the boundary  $\mathcal{D}'_{\partial}(\tilde{V}_+)$  are the distributions  $u$  such that  $u \in C^\infty([0, \varepsilon], \mathcal{D}'(\mathbb{R}^{n-1}))$ . It is equivalent to say that there exists  $A(x', D_{x'}) \in L^0(\mathbb{R}^{n-1})$  with compact support such that  $A(x', D_{x'})u(x', x_n) \in C^\infty([0, \varepsilon] \times \mathbb{R}^{n-1})$ . The conormal bundle at  $\partial V_+$ , denoted by  $N(\partial V_+)$ , is by definition  $\{(x', 0, 0, \xi_n), \xi_n \neq 0\}$ . We denote then by  $B\tilde{V}_+ = T^*\tilde{V}_+/N(\partial V_+)$ , which is a topological space whose null section is well-defined and denoted by  $\{0\}$ . This space naturally projects onto  $T^*(V_+)$  and  $T^*(\partial V_+)$ . (the interior projects onto the interior and a point of  $T^*(\partial V_+)$  is written  $(x', \xi')$ , to which we associate the equivalence class of  $(x', 0, \xi', 0)$  in  $B\tilde{V}_+$ ). The canonical projection of  $T^*\tilde{V}_+$  into  $T^*\tilde{V}_+/N(\partial V_+)$  is denoted by  $b$ . The definition of the wavefront at the boundary given by R. Melrose in [75] is the following

DEFINITION 10.5. *Let  $u$  be an extendible distribution. The wavefront at the boundary  $WF_b(u) \subset B\tilde{V}_+$  is*

$$\begin{aligned} WF_b(u) &= \{\rho \in T^*(\tilde{V}_+), \rho \in WF(i^*u) = WF(u|_{V_+})\} \cup \\ &\quad \{\rho_0 \in T^*(\partial V_+)\{0\}, \text{ for any conic neighborhood of } \rho_0, \\ &\quad \text{there exists } \rho = (x', 0, \xi', \xi_n) \text{ such that } (x', \xi') \in \Gamma \text{ and } \rho \in WF(\underline{u})\} \end{aligned}$$

The second part (part of  $WF_b(u)$  contained in  $T^*(\partial V_+)$ ) is obtained in the following way, as indicated by Melrose and Sjöstrand [77] :

DEFINITION 10.6. *If  $\rho \in T^*\partial V_+ - \{0\}$ , then  $\rho \notin WF_b(u)$  if and only if  $u$  is regular up to the boundary as defined above.*

There is also a definition using the characteristic manifolds of operators, analogous to the definition of the wavefront :

$$(4.135) \quad WF(u) = \cap \text{Car} B, Bu \in C^\infty.$$

This definition is more technical and involves the distributions at the boundary  $\partial X$  of the manifold  $X$ . The reader who wants more details will usefully refer to Hörmander [48], Tome III, Definition 18.2.6 for  $I^m(X, \partial X, T^*X)$ , (183.328) to define  $\dot{A}(X)$ , conormal distributions with respect to Definition 18.3.18 to define  $\Psi_b^0(X)$ , operators locally of symbol  $a(x, \xi', x_n \xi_n)$ ,  $\partial X$  being  $x_n = 0$ , and in these notations  $WF_b(u) = \cap \text{Char} B, B \in \Psi_b^0(X), Bu \in \dot{A}(X)$ .

For a differential operator of order 2 with coefficients of class  $C^\infty$ , we prove (see for example G. Lebeau [68] in the case of analytical coefficients):

PROPOSITION 10.3. *Let  $u$  be a solution of  $Pu = 0$  in  $x_n > 0$ . Let  $U$  be its canonical extension, as obtained in the proposition 10.2. We have the equality  $WF_b(u) = b(WF(U))$ .*

We give two results that allow us to understand a little better the wavefront at the boundary. The first one concerns the wavefront at the boundary of a boundary layer distribution (Lemma 10.4, thanks to Hörmander). The second one allows to link in part the wavefront at the boundary of  $u$  and the wavefront of traces of  $u$  on this boundary (when we can define them, for example if the boundary writes  $\{x_n = 0\}$ , for  $u \in H^{\frac{3}{2}+\varepsilon}(x_n > 0)$ ) when  $u$  is solution in  $\{x_n > 0\}$  of  $Pu = 0$ , see [48]. )

LEMMA 10.4.

$$WF\left(\sum_{j=0}^{m-1} a_j(x') \otimes \delta_{x_n}^j\right) = \cup WF(a_j) \times T^*S.$$

PROOF. Consider a point  $(x'_0, x_n^0, \xi'_0, \xi_n^0)$ . In the neighborhood of a point where  $x_n^0 \neq 0$ , the distribution is zero, so the front is included in  $x_n^0 = 0$ . On the other hand, we consider  $(x'_0, 0, \xi'_0, \xi_n^0)$  and localize for  $x'$  in the neighborhood of  $x'_0$ . We find that the Fourier transform of each term is

$$\chi a_j(\xi') (i\xi_n)^j.$$

If the point  $(x'_0, \xi'_0)$  is not in the wavefront of  $a_j$ , then  $\chi a_j(\xi')$  is rapidly decaying in a conical neighborhood of  $\xi'_0$ . We multiply by a polynomial, so the result is rapidly decaying in  $\lambda = (\xi_n^2 + |\xi'|^2)^{\frac{1}{2}}$

$$\hat{a}_j(\xi') \xi_n^j = \lambda^j \hat{a}_j\left(\lambda \frac{\xi'}{(|\xi'|^2 + \xi_n^2)^{\frac{1}{2}}}\right) \left(\frac{\xi_n}{(|\xi'|^2 + \xi_n^2)^{\frac{1}{2}}}\right)^j.$$

Let  $(\eta'_0, \eta_n^0) = \frac{1}{((\xi_n^0)^2 + |\xi'_0|^2)^{\frac{1}{2}}}(\xi'_0, \xi_n^0)$ . Then we know that, for  $(\xi', \xi_n)$  in a conic neighborhood of  $(\xi'_0, \xi_n^0)$ ,  $\frac{\xi'}{(|\xi'|^2 + \xi_n^2)^{\frac{1}{2}}}$  is in a conic neighborhood of  $\xi'_0$  (since a conic neighborhood of  $\eta'_0$  is a conic neighborhood of  $\xi'_0$ ) and therefore  $\hat{a}_j\left(\lambda \frac{\xi'}{(|\xi'|^2 + \xi_n^2)^{\frac{1}{2}}}\right)$  is rapidly decaying in  $\lambda$ , so that  $\lambda^j \hat{a}_j\left(\lambda \frac{\xi'}{(|\xi'|^2 + \xi_n^2)^{\frac{1}{2}}}\right)$  is rapidly decaying in  $\lambda$ . This shows that  $WF(a_j \otimes \delta_{x_n}^{(j)}) \subset WF(a_j) \times \{0\} \times \mathbb{R}^*$ .

Conversely, let  $(x'_0, 0, \xi'_0, \xi_n^0) \notin WF(a_j \otimes \delta_{x_n}^{(j)})$ . Then  $\hat{a}_j(\xi') \xi_n^j$  is rapidly decaying in the cone given by  $(\xi'_0, \xi_n^0)$ . From this we deduce that, for all  $N$ , there exists  $C_{N+j}$  such that we have the rapid decay inequality for the power  $N + j$ . We use the homogeneity of  $\xi_n^j$  to obtain

$$|\hat{a}_j(\lambda \eta') \eta_n^j| \leq C_{N+j} \lambda^{-N}.$$

When  $\eta_n^0 \neq 0$ , we see that this implies the rapid decay of  $\hat{a}_j$  in the cone generated by  $\eta'_0$ , so  $\xi'_0$ , and so the point  $(x'_0, \xi'_0)$  is not in the wavefront of  $a_j$ .

When  $\eta_n^0 = 0$ , we choose a point  $\eta_n = \frac{\varepsilon}{2}$ , and we have  $(x'_0, \xi'_0)$  outside the wavefront of  $a_j$ .

Thus  $WF(a_j \otimes \delta_{x_n}^{(j)}) = WF(a_j) \times (\{0\} \times \mathbb{R})$ .

We finally study the wavefront of the sum, using the successive orders  $p$  of  $\delta_{x_n}^{(p)}$ . Indeed, we have (for example)

$$x_n \delta_{x_n}^j = -j \delta_{x_n}^{(j-1)}$$

for  $j \geq 1$ .

We use  $WF(fT) \subset WF(T)$  if  $f$  is a function of class  $C^\infty$  and  $T$  is a tempered distribution. We thus deduce that, for  $T = a_M \delta_{x_n}^{(M)}$

$$WF(x_n^{M-1} T) = (-1)^M WF((M-1)! a_M \delta_{x_n}) = WF(a_M) \times (\{0\} \times \mathbb{R}).$$

This shows that  $WF(a_M) \times (\{0\} \times \mathbb{R}) \subset WF(T)$ . To simplify the proof, we restrict ourselves to  $M = 1$ . We verify that

$$\begin{aligned} x_n T &= -a_1 \otimes \delta_{x_n} \\ \left(1 + \frac{\partial}{\partial x_n} \circ x_n\right) T &= a_0 \otimes \delta_{x_n} \end{aligned}$$

which gives, using the fact that  $(1 + \frac{\partial}{\partial x_n} \circ x_n)$  is a differential operator, the inclusion  $WF(a_0) \cup WF(a_1) \times (\{0\} \times \mathbb{R}) \subset WF(T)$ . The inverse inclusion is immediate. We deduce the result of the lemma 10.4.  $\square$

Any partial differential operator of order 2 writes as:

$$P = a_{nn}(x) \frac{\partial^2}{\partial x_n^2} + \sum_{1 \leq j \leq n-1} b_j(x) \frac{\partial^2}{\partial x_j \partial x_n} + \sum_{k,l \leq n-1} c_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l} + b(x) \frac{\partial}{\partial x_n} + C(x, \frac{\partial}{\partial x'}).$$

Introducing

$$A(x_n, x', D_{x'}) = - \sum_{1 \leq j \leq n-1} \frac{1}{2} \frac{b_j(x)}{a_{nn}(x)} D_{x_j} + \frac{i}{2} \frac{b(x)}{a_{nn}(x)},$$

one finds

$$\frac{1}{a_{nn}} P = \frac{\partial^2}{\partial x_n^2} + \frac{2}{i} A(x_n, x', D_{x'}) \frac{\partial}{\partial x_n} + B(x_n, x', D_{x'})$$

where the operator  $B$  is a differential operator of order 1 in the coordinates  $x'$  whose coefficients depend on  $x$ . Then we have the

LEMMA 10.5. *We consider  $u$  an extendible distribution such that  $Pu$  is an extendible distribution.*

*Let  $Pu = 0$  in  $x_n > 0$  and let  $U$  be the extension of  $u$  by 0 in  $x_n < 0$  and  $(Pu)^*$  be the extension of  $Pu$  by 0 in  $x_n < 0$ .*

(1) *When  $u$  is a function of class  $C^\infty$ ,*

$$PU = (Pu)^* + \frac{\partial u}{\partial x_n}(0, x') \delta_{x_n=0} + u(x', 0) \delta'_{x_n=0} + \frac{2}{i} A(0, x', D_{x'}) u(x', 0) \delta_{x_n=0}.$$

(2) *If  $u$  is an extendible solution, there exist two distributions  $g_0(u)$  and  $g_1(u)$  such that  $PU = g_0(u) \otimes \delta_{x_n=0} + g_1(u) \otimes \delta'_{x_n=0}$  where  $U$  is the extension of  $u$ .*

(3) *For  $u \in H^1(x_n > 0)$ ,  $Pu \in L^2(x_n > 0)$ , then  $PU = (\frac{\partial u}{\partial x_n}(x', 0) + \frac{2}{i} A(0, x', D_{x'}) u(x', 0)) \delta_{x_n=0} + u(x', 0) \delta'_{x_n=0}$ . The regularity of  $Pu$  allows to extend  $Pu$  by zero and the distribution given by  $Pu$  on  $x_n > 0$ , 0 on  $x_n < 0$  is in  $L^2$ . The regularity of  $u$  allows to define the trace of  $u$ ,  $\gamma u \in H_{loc}^{\frac{1}{2}}(\mathbb{R}^{n-1})$ , as well as the normal derivative, given in this case by*

$$\langle PU, \phi \rangle + \frac{2}{i} \langle \gamma u, A(0, x', D_{x'}) \phi \rangle + \langle \gamma u, \frac{\partial \phi}{\partial x_n} \rangle = \langle \frac{\partial u}{\partial x_n}, \phi \rangle.$$

The first item is the particular form of the expressions (20.1.4) and (20.1.5) of [48].

PROOF. We prove the jump formula for the operators  $\frac{\partial^2}{\partial x_n^2}$  and for  $A(x_n, x', D_{x'}) \frac{\partial}{\partial x_n}$ . Let  $\phi \in C_0^\infty(\mathbb{R}^n)$ . Then

$$\begin{aligned} \int (\frac{\partial^2 u}{\partial x_n^2})^*(x) \phi(x) dx &= \int dx' \int_0^\infty \frac{\partial^2 u}{\partial x_n^2} \phi(x) dx_n \\ &= \int dx' [-\frac{\partial u}{\partial x_n}(x', 0) \phi(x', 0) - \int_0^\infty \frac{\partial u}{\partial x_n} \frac{\partial \phi}{\partial x_n} dx_n] \\ &= \int dx' [-\frac{\partial u}{\partial x_n}(x', 0) \phi(x', 0) + u(x', 0) \frac{\partial \phi}{\partial x_n}(x', 0) + \int u(x) \frac{\partial^2 \phi}{\partial x_n^2} \phi(x) dx_n] \end{aligned}$$

Similarly, denoting by  $A^\perp$  the adjoint of the operator  $A$  (in variables  $x'$ ),  $x_n$  being a parameter, one has

$$\begin{aligned} \int (A(x_n, x', D_{x'}) \frac{\partial u}{\partial x_n})^* \phi(x) dx &= \int_0^\infty dx_n \int dx' A(x_n, x', D_{x'}) \frac{\partial u}{\partial x_n} \phi(x) \\ &= \int_0^\infty dx_n \int dx' \frac{\partial u}{\partial x_n} A^\perp(x_n, x', D_{x'}) \phi(x) \\ &= \int dx' \int_0^\infty dx_n \frac{\partial u}{\partial x_n} A^\perp(x_n, x', D_{x'}) \phi(x) \\ &= \int dx' [-u(x', 0) A^\perp(0, x', D_{x'}) \phi(0, x') \\ &\quad - \int_0^\infty dx_n u(x) \frac{\partial}{\partial x_n} \circ A(x_n, x', D_{x'}) \phi \\ &= - \int dx' A(0, x', D_{x'}) u(x', 0) \phi(x', 0) \\ &\quad - \int_0^\infty dx_n \int dx' u(x) \frac{\partial}{\partial x_n} \circ A(x_n, x', D_{x'}) \phi \end{aligned}$$

Looking at the equalities and noticing that the adjoint in  $\mathcal{D}(\mathbb{R}^n)$  of  $A(x, D_{x'}) \frac{\partial}{\partial x_n}$  is  $-\frac{\partial}{\partial x_n} \circ A^\perp(x_n, x', D_{x'})$  the equality of the lemma since

$$\int_0^{+\infty} dx_n \int dx' u P^\perp \phi = \int dx U P^\perp \phi \int PU \phi.$$

This completes the proof of the first point of the lemma 10.5. The second point comes from the fact that if  $u$  is a solution,  $Pu = 0$  in  $x_n > 0$  so the unique extension of 0 being 0 in  $L^2$ ,  $(Pu)^* = 0$  as a distribution in  $\mathbb{R}^n$ .  $\square$

We then have additional information about the wavefront at the boundary of  $u$ , extendible distribution solution of  $Pu = 0$  in  $x_n > 0$ :

PROPOSITION 10.4. *The wavefront at the boundary of  $u$ , extendible distribution solution of  $Pu = 0$  in  $x_n > 0$  is*

$$WF_b(u)|_{x_n=0} = WF(u|_{x_n=0}) \times T^*S \cup WF(\partial_{x_n} u|_{x_n=0}) \times T^*S.$$

When  $u$  is regular enough ( $H^1$  for example), the notations used are correct. When  $u$  is not regular enough, we replace  $u|_{x_n=0}$  by  $g_1(u)$  and  $\partial_{x_n} u|_{x_n=0}$  by  $g_0(u)$ , the distributions  $g_0(u)$  and  $g_1(u)$  of  $\mathcal{D}'(\mathbb{R}^{n-1})$  being defined in the lemma 10.3.

PROOF. Since, for any differential operator  $B$ , we have  $WF(Bu) \subset WF(u)$ , we deduce that  $WF(Pu) \subset WF(u)$ , so as  $WF(g_0(u) \otimes \delta_{x_n=0} + g_1(u) \otimes \delta'_{x_n=0}) = WF(g_0(u)) \times T^*S \cup WF(g_1(u)) \times T^*S$ , we have the inclusion

$$WF(g_0(u)) \cup WF(g_1(u)) \subset WF_b(u).$$

□

**4.2. Reduction of the wave operator to a normal form.** We prove the following proposition of the Laplacian representation, in the case where  $\Omega$  is totally geodesic in the neighborhood of  $x_0$ , that is, any point of  $\mathbb{R}^n$  can be put in the form  $M = N + X_n n(N)$ ,  $N \in \partial\Omega$  and  $n(N)$  exterior normal to  $\partial\Omega$  at the point  $N$ .

PROPOSITION 10.5. *There exists a local coordinate system  $(X_n, X)$  in the neighborhood of  $x_0 \in \partial\Omega$  ( $x_0$  is characterized by  $X_n = 0, X = 0$  and  $\partial\Omega$  is locally  $X_n = 0$ ) and there exists a function  $K$  identically equal to 1 on  $\partial\Omega$  such that*

$$K^{-1} \Delta K f = \frac{\partial^2 f}{\partial X_n^2} + Q(X_n, X, \frac{\partial}{\partial X}) f.$$

The operator  $Q$  is differential and its restriction on  $\partial\Omega$  is the Laplacian associated to the boundary metric.

PROOF. We consider  $x_0 \in \partial\Omega$  and we associate to the boundary of  $\Omega$  a local map in the neighborhood of  $x_0$ . This local map is characterized by the coordinates  $(X_1, \dots, X_{n-1}) = X'$  and each point of  $\partial\Omega$  can be written  $M(X_1, \dots, X_{n-1})$ , each coordinate being  $x_j(X_1, \dots, X_{n-1})$  being  $x_j(X_1, \dots, X_{n-1})$ ,  $1 \leq j \leq n$ .

To each point  $M \in \partial\Omega$  we associate the unit outgoing normal vector  $n(M)$ , of coordinates  $n_j(X_1, \dots, X_{n-1})$ ,  $1 \leq j \leq n$ . In the usual Cartesian coordinate system, we then write

$$x_j(X_1, \dots, X_n) = x_j(X_1, \dots, X_{n-1}) + X_n n_j(X_1, \dots, X_{n-1}).$$

This change of variable defines a diffeomorphism of a neighborhood of  $x_0 \in \mathbb{R}^n$  which transforms a neighborhood of  $x_0$  in  $\partial\Omega$  into a neighborhood of  $(X_n = 0, X' = 0)$  in  $X_n = 0$ . We note that  $X \in \mathbb{R}^{n-1}$ . The equality

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j = \sum_j \frac{\partial f}{\partial X_j} dX_j$$

translates into, using  $dx_j = \sum_{k \geq n-1} (\frac{\partial x_j}{\partial X_k} + X_n \frac{\partial n_j}{\partial X_k}) dX_k + n_j(X_1, \dots, X_{n-1}) dX_n$ ,

$$(4.136) \quad \nabla_X f = {}^t B(X_n, X') \nabla_X f.$$

The components of  $B$  are given, for  $1 \leq i \leq n, 1 \leq j \leq n-1$  by

$$\begin{aligned} B_{ij}(X_n, X) &= \frac{\partial x_j}{\partial X_k} + X_n \frac{\partial n_j}{\partial X_k} \\ B_{in}(X_n, X) &= n_i(X) \end{aligned}$$

Let  $C = {}^t B^{-1}$ . One has, from (4.136)

$$\nabla_x f = C(X_n, X) \nabla_X f.$$

The Laplacian operator is

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i^2} &= \sum_j C_{ij} \partial_{X_j} (\sum_p C_{ip} \partial_{X_p} f) \\ \Delta f &= \sum_{j,p} [\sum_i C_{ij} C_{ip}] \frac{\partial^2 f}{\partial X_p \partial X_j} + \sum_p (\sum_{i,j} C_{ij} \frac{\partial C_{ip}}{\partial X_j}) \frac{\partial f}{\partial X_p} \end{aligned}$$

The coefficient of  $\frac{\partial^2 f}{\partial X_j \partial X_p}$  in this sum is then  $({}^tCC)_{pj} = ({}^tBB)_{pj}^{-1}$ . The resulting matrix is symmetric. The matrix  ${}^tBB$  verifies the following equalities :

$$\begin{aligned} ({}^tBB)_{nn} &= \sum_i (n_i(X, X_n))^2 = 1 \\ ({}^tBB)_{ni} &= \sum_p ({}^tB)_{np} B_{pi} = \sum_p B_{pn} B_{pi} = \sum_p n_p(X) \left[ \frac{\partial x_p}{\partial X_i} + X_n \frac{\partial n_p}{\partial X_i} \right] \end{aligned}$$

As  $x(X) \in \partial\Omega$ , the vector  $\partial X_i(x(X))$  is tangent to  $\partial\Omega$  at  $x(X)$ . As  $(n_p)$  is the normal vector at  $x(X)$ , the sum  $\sum_p n_p(X) \frac{\partial x_p}{\partial X_i}$  is zero. The rest of the coefficient is zero because it is  $X_n \frac{1}{2} \frac{\partial n^2}{\partial X_i}$ . The matrix  ${}^tBB$  can be then written as form

$$\begin{pmatrix} G^{-1}(X_n, X) & 0 \\ 0 \dots 0 & 1 \end{pmatrix},$$

where  $G$  is a matrix. As the inverse of  ${}^tBB$  is the matrix

$$\begin{pmatrix} G(X_n, X) & 0 \\ 0 \dots 0 & 1 \end{pmatrix},$$

one has

$$\Delta f = \frac{\partial^2 f}{\partial X_n^2} + \sum_{j,k \leq n-1} g_{jk}(X_n, X) \frac{\partial^2 f}{\partial X_j \partial X_k} + a(X_n, X) \frac{\partial f}{\partial X_n} + \sum_{j \leq n-1} b_j(X_n, X) \frac{\partial f}{\partial X_j}$$

Let  $K$  be the function solution of

$$\begin{cases} K|_{X_n=0} = 1 \\ 2\partial_{X_n} K + a(X_n, X)K = 0. \end{cases}$$

We then deduce that  $f \rightarrow K^{-1} \Delta K f - \frac{\partial^2 f}{\partial X_n^2}$  is a differential operator of order 2  $Q$  in the variables  $X$ , with coefficients depending also on  $X_n$ , such that

$$K^{-1} \Delta K f = \frac{\partial^2 f}{\partial X_n^2} + Q(X_n, X, \frac{\partial}{\partial X}) f.$$

The principal symbol of  $-Q$  is the quadratic form associated to the matrix  $G(X_n, X)$ . We can then write

$$(4.137) \quad Q(X_n, X, \frac{\partial}{\partial X}) = \sum_{j,k} g_{j,k}(X_n, X) \frac{\partial^2}{\partial X_j \partial X_k} + \sum_j V_j(X_n, X) \frac{\partial}{\partial X_j} + S(X_n, X),$$

where the indices  $j, k$  belong to  $\{1, \dots, n-1\}$ . This operator is constructed from the  $b_j(X_n, X)$  and  $K$ . We have  $S(X_n, X) = (\Delta K/K)$  and  $KV_j(X_n, X) = \Delta(X_j K) - X_j \Delta K$ . This completes the proof of the proposition 10.5.  $\square$

We will abandon in the following the notations  $(X_n, X)$  that we used here to make the difference between between the usual Cartesian coordinates of  $\mathbb{R}^n$ , noted  $(x)$ , and the semi-geodesic coordinates introduced here  $(X_n, X)$ . We now consider, in the coordinates  $(x', x_n)$ , the rectified boundary boundary  $x_n = 0$  and the Laplacian with variable coefficients  $\frac{\partial^2}{\partial x_n^2} + Q$ .

**4.3. Elliptic, hyperbolic and glancing.** Let  $P$  be a second-order, differential operator defined on  $\mathbb{R}^n \times \mathbb{R}_t$ . We study it in the half space  $\Omega = \{(x', x_n, t), x_n > 0\}$  in the neighborhood of the hypersurface  $x_n = 0$ . Here, we consider directly coordinates adapted to the boundary.

The cotangent space to the boundary  $T^*(\partial\Omega) = T^*(\mathbb{R}^{n-1} \times \mathbb{R}_t)$  can be divided into three regions (elliptic, hyperbolic, *glancing*), defined as follows : Let  $\pi$  be the canonical projection of  $T^*\mathbb{R}^{n+1}$  onto  $\mathbb{R}^{n+1}$  and let  $\pi$  be the canonical projection of  $\pi^{-1}(\partial\Omega) \cap T^*(\mathbb{R}^{n+1})$  over  $T^*(\partial\Omega)$  which associates to  $(x'_0, 0, t_0, \xi_0, \tau_0) \in T^*\mathbb{R}^{n+1}$  the point  $(x'_0, t_0, \xi'_0, \tau_0)$ . The equation  $p(x'_0, 0, t_0, \xi'_0, \xi_n, \tau_0) = 0$  is an equation of degree 2 in  $\xi_n$ , which has 0, 1, or 2 real roots. We introduce the classification

DEFINITION 10.7. *The elliptic region  $\mathcal{E}$  is the set of points  $\rho_0 \in T^*(\partial\Omega)$  such that*

$$\Pi^{-1}(\rho_0) \cap \text{Car}(p) = \emptyset.$$

*The hyperbolic region  $\mathcal{H}$  is the set of points  $\rho_0 \in T^*(\partial\Omega)$  such that*

$$\text{Card}(\Pi^{-1}(\rho_0) \cap \text{Car}(p)) = 2.$$

*Finally, the glancing region is  $\mathcal{G}$  is  $T^*\partial\Omega - (\mathcal{E} \cup \mathcal{H})$ .*

The word *glancing* comes from the first study of Friedlander [42]. One finds in particular p148 "The front of disturbance in the shadow is orthogonal to diffracted rays which are *glancing* rays emerging from the boundary".

Interpretation. For the wave operator  $\Delta - \partial_{t^2}^2$  and the  $\partial\Omega$  boundary defined by  $x_n = 0$ , the points of  $T^*(\partial\Omega \times \mathbb{R}_t)$  are then of the form  $(x', t, \xi', \tau)$ . The characteristic manifold of the D'Alembertian is  $\xi_n^2 + (\xi')^2 - \tau^2 = 0$ . Then

$$\mathcal{E} = \{(x', t, \xi', \tau), |\xi'| > |\tau|\}$$

$$\mathcal{H} = \{(x', t, \xi', \tau), |\xi'| < |\tau|\}$$

$$\mathcal{G} = \{(x', t, \xi', \tau), |\xi'|^2 = \tau^2\}.$$

An incident plane wave  $e^{ik \cdot x - i\tau t}$  verifies necessarily (since it is a solution of the wave equation)  $|k| = |\tau|$ . Then, necessarily,  $|k'| \leq |\tau|$ , and the wave is tangent to the boundary when  $k \cdot n = 0$ . An incident plane wave is therefore never associated to an elliptic point, on the other hand, hyperbolic points are those where the reflection is transverse (so  $k \cdot n \neq 0$ ), and the *glancing* points are those where the wave arrives tangentially. We explain and clarify these remarks in the next section.

## 5. Reflection of singularities

In this part, we prove the result of reflection of hyperbolic singularities, which is the generalization of the result of propagation of singularities in the vacuum for generalized rays. A generalized ray is the projection of the union of the two half bicharacteristic curves passing through  $\rho$  whose projection is in the boundary, contained in the exterior of  $\Omega$ . It comes from the construction of two Fourier integral operators which are the two parametrix, outgoing and incoming with respect to  $\partial\Omega$ , of the considered second order operator. These Fourier integral operators are denoted  $A_+$  and  $A_-$ . They are solution of  $P$  and can be considered respectively as outgoing and incoming (with respect to with respect to  $x_n > 0$ ).

**5.1. Parametrix for calculate the reflection.** We place ourselves in the case where  $P = \frac{\partial^2}{\partial x_n^2} + R(x_n, x', \frac{\partial}{\partial x'}, \partial_t)$ ,  $R$  being an operator in the variables  $(x', t)$  with parameter  $x_n$ . When we study the wave equation, this is equivalent to considering a system of coordinates  $x' \in \mathbb{R}^{n-1}$  locally on  $\partial\Omega$ , the variable normal to the boundary  $x_n$  and the time  $t$ . The operator  $R$  in this equality is obtained, for the wave equation, from the operator  $Q(x_n, x', \partial_{x'})$  of the proposition 10.5 by considering

$$R(x_n, x', \partial_{x'}, \partial_t) = Q(x_n, x', \partial_{x'}) - \partial_{t^2}^2,$$

since the boundary  $\partial\Omega$  is assumed to be of the form  $\partial C \times \mathbb{R}_t$ , independent of time and  $K$  commutes then with  $\partial_t$ .

We prove the proposition, which describes the solution of a Cauchy problem with data on  $x_n = 0$  :

**PROPOSITION 10.6.** *Let  $f \in \mathcal{S}'(\{x_n = 0\})$  and  $\rho_0 \in WF(f) \cap \mathcal{H}$ . We denote by  $\rho_+$  and  $\rho_-$  the two points of *Carp* which project on  $\rho_0$ .*

*Let  $A_+$  and  $A_-$  be the Fourier integral operators which describe the outgoing and incoming parametrix of  $P$  in the neighborhood of  $\rho_+$  and  $\rho_-$ .*

*The solution of the problem*

$$\begin{cases} Pu = 0, x_n > 0 \\ u|_{x_n=0} = 0 \\ \frac{\partial u}{\partial x_n}|_{x_n=0} = f(x', t) \end{cases}$$

*is described in the vicinity of  $\rho_0$  by*

$$u(x', x_n, t) = (A_+ - A_-)(g)(x', x_n, t)$$

*where  $g(x', t)$  is a solution of the boundary problem, elliptic in the vicinity of  $\rho_0$ :*

$$(\partial_{x_n} \circ A_+ - \partial_{x_n} \circ A_-)|_{x_n=0}(g)(x', t) = f(x', t).$$

The two operators  $T_{\pm} = \partial_{x_n} \circ A_{\pm}|_{x_n=0}$  are pseudo-differential operators on  $T^*(\mathbb{R}^{n-1} \times \mathbb{R}_t)$ .

Construction of  $A_{\pm}$ . According to the notations of the previous section, we write the operator  $P$  in the form

$$P = \frac{\partial^2}{\partial x_n^2} + G(x_n, x', \partial_{x'}^2) - \partial_t^2 + V(x_n, x') \cdot \nabla_{x'} + S(x_n, x').$$

This representation is the wave operator in semi-geodesic coordinates, and is less general than the representation of a hyperbolic operator. The operator  $G(x_n, x', \partial_{x'}^2)$  is written

$$G(x_n, x', \partial_{x'}^2) = \sum_{i,j=1..n-1} g_{ij}(x_n, x') \frac{\partial^2}{\partial x_i \partial x_j}$$

where  $(g_{ij})$  is a symmetric positive definite matrix (uniformly). The principal symbol of  $P$  is equal to

$$(5.138) \quad p(x_n, x', \xi_n, \xi', \tau) = -\xi_n^2 + \tau^2 - \sum_{i,j=1..n-1} g_{ij}(x_n, x') \xi_i \xi_j.$$

If  $\phi(x_n, x', t, \xi', \tau)$  is a homogeneous phase of degree 1 in  $\xi', \tau$  and if  $\sigma(x, t, \xi', \tau)$  is a symbol of  $C^\infty(\mathbb{R}, S^0(\mathbb{R}^{n-1} \times \mathbb{R}_t))$ , Lemma 1.2 leads to the eikonal equation, which expresses the vanishing at first order of homogeneity in  $\tau, \xi$  of  $e^{-i\phi} P(\sigma e^{i\phi})$  :

$$(5.139) \quad p(x_n, x', \nabla_{x_n} \phi, \nabla_{x'} \phi, \nabla_t \phi) = 0.$$

We write the operators of the lemma 1.2 which will intervene in the transport equations :

$$\begin{cases} P^b(x, \phi) = \frac{\partial^2 \phi}{\partial x_n^2} + \sum_{j,k=1..n-1} g_{jk}(x_n, x') \frac{\partial^2 \phi}{\partial x_j \partial x_k} - \frac{\partial^2 \phi}{\partial t^2} + \sum_{j=1..n-1} V_j(x_n, x') \frac{\partial \phi}{\partial x_j} \\ L_1(x, \partial_x, \partial_t) = -2 \frac{\partial \phi}{\partial x_n} \frac{\partial}{\partial x_n} + 2 \frac{\partial \phi}{\partial t} \frac{\partial}{\partial t} - 2 \sum_j (\sum_k g_{jk}(x) \frac{\partial \phi}{\partial x_k}) \frac{\partial}{\partial x_j} + P^b(x, \phi) \end{cases}$$

Eikonal equation (5.139) rewrites

$$(\partial_t \phi)^2 = (\partial_{x_n} \phi)^2 + \sum_{i,j} g_{i,j}(x_n, x') \partial_{x_i} \phi \partial_{x_j} \phi.$$

We want the phase  $\phi$  to be the identity phase on the boundary  $x_n = 0$ . This corresponds to the condition

$$(5.140) \quad \phi(0, x', t, \xi', \tau) = x' \xi' + t\tau.$$

The tangential derivatives of this phase are then known on the boundary through

$$\partial_t \phi = \tau, \partial_{x_j} \phi = \xi_j,$$

which yields

$$(5.141) \quad \tau^2 = (\partial_{x_n} \phi|_{x_n=0})^2 + \sum_{i,j} g_{i,j}(0, x') \xi_i \xi_j.$$

The bicharacteristic system for the operator  $P$  is ( $\dot{f}$  denotes the derivative with respect to the parameter on the bicharacteristics of the function  $s \rightarrow f(s)$ )

$$\begin{cases} \dot{t} = 2\tau \\ \dot{\tau} = 0 \\ \dot{x}_n = -2\xi_n \\ \dot{\xi}_n = -\sum_{i,j} \partial_{x_n} g_{ij}(x_n, x') \xi_i \xi_j \\ \dot{x}_i = 2 \sum_{i,j} g_{i,j}(x_n, x') \xi_j \\ \dot{\xi}_i = -2 \sum_{j,k} \partial_{x_i} g_{j,k}(x_n, x') \xi_j \xi_k. \end{cases}$$

In particular, we deduce that  $\tau(s) = \tau(0)$  and  $t(s) = t(0) + 2\tau(0)s$ .

We apply the proposition 9.1 on the identity between the Lagrangian manifold  $\Lambda \subset \text{Carp}$  and the manifold generated by the bicharacteristics of  $P$ . This theorem states that if

$$(x'(0), x_n(0), t(0), \xi'(0), \xi_n(0), \tau(0)) \in \Lambda_\phi,$$

where the phase  $\phi$  is a solution of the eikonal equation (5.141) associated with the symbol  $p$ , then the bicharacteristic curve

$$\{(x'(s), x_n(s), t(s), \xi'(s), \xi_n(s), \tau(s)), s \in [-a, b]\}$$

is contained in the Lagrangian manifold  $\Lambda_\phi$ . We choose the initial point  $(0, x', t, \partial_{x_n}\phi(0, x', t), \partial_t\phi(0, x', t))$  in the Lagrangian manifold associated to the phase  $\phi$  introduced above by (5.141), (5.140).

This initial point is

$$(0, x', t, \partial_{x_n}\phi, \xi', \tau).$$

We then check that the point on the bicharacteristic curve passing through this initial point is  $(x_n(s), x'(s), t - 2\tau s, \xi'(s), \tau)$ . In particular, we verify that, identically

$$\partial_t\phi(x_n(s), x'(s), t(s), \xi', \tau) = \tau.$$

There is thus a phase  $\psi(x_n, x', \xi', \tau)$ , independent of  $t$  such that

$$\phi(x', x_n, t, \xi', \tau) = \psi(x', x_n, \xi', \tau) + t\tau.$$

This phase  $\psi$  is a solution of the eikonal equation (note that it is not the same as (5.141) is not the same as (5.141)) :

$$\tau^2 = (\partial_{x_n}\psi)^2 + \sum_{i,j} g_{i,j}(x_n, x') \partial_{x_i x_j}^2 \psi.$$

Let us consider a point on the boundary  $\partial\Omega \times \mathbb{R}^{n+1}$  which is hyperbolic:  $(x'_0, t_0, \xi'_0, \tau_0) \in \mathcal{H}$ . This means that

$$\tau_0^2 - \sum_{i,j} g_{ij}(0, x'_0) (\xi_i)_0 (\xi_j)_0 > 0.$$

There exist two values  $\xi_n^\pm$  such that  $(x'_0, 0, t_0, \xi'_0, \xi_n^\pm, \tau_0)$  are in the set  $\text{Carp}$ . Such points are stable, so in a neighborhood of  $(x'_0, \xi'_0, t_0, \tau_0)$  and  $x_n = 0$  we can write

$$-\xi_n^2 + \tau^2 - \sum_{i,j} g_{ij}(x) \xi_i \xi_j = -(\xi_n - \xi_n^+(x, \xi', \tau))(\xi_n - \xi_n^-(x, \xi', \tau)).$$

In the case studied on symbol  $p$ ,  $\xi_n^- = -\xi_n^+$  and  $\xi_n^+(x, \xi', \tau) = (\tau^2 - \sum_{i,j} g_{ij}(x) \xi_i \xi_j)^{\frac{1}{2}}$ . The phase  $\psi$  is solution of the eikonal equation (5.141) which writes

$$(\partial_{x_n}\psi)^2 = (\xi_n^+(x, \partial_{x'}\psi, \tau))^2.$$

In the neighborhood of the hyperbolic point  $(x'_0, 0, t_0, \xi'_0, \tau_0)$  of  $T^*(\partial\Omega \times \mathbb{R})$ , thanks to Cauchy-Lipschitz theorem there exist two phases  $\psi_+$  and  $\psi_-$ , solutions in the neighborhood of this point of

$$(5.142) \quad \begin{cases} \partial_{x_n}\psi_\pm = \pm \xi_n^\pm(x, \partial_{x'}\psi, \tau) \\ \psi_\pm(x', 0, \xi', \tau) = x' \cdot \xi'. \end{cases}$$

These two phases are homogeneous with respect to  $(\xi', \tau)$ . We denote by  $L_1^\pm(x, \xi', \tau, \partial_x, \partial_t)$  the transport operator associated with the phase  $\psi_\pm + t\tau$  and  $c_\pm(x, \xi', \tau)$  the function  $P^b(\psi_\pm + t\tau)$ . We obtain, for any function  $\sigma(x_n, x', t, \xi', \tau)$  which admits an homogeneous asymptotic expansion :

$$\begin{aligned} c_\pm(x, \xi', \tau) &= \partial_{x_n^2}^2 \psi_\pm + \sum_{j,k} g_{jk}(x) \partial_{x_j x_k}^2 \psi_\pm + \sum_j V_j(x) \partial_{x_j} \psi_\pm \\ L_1^\pm \sigma &= -2\tau \frac{\partial \sigma}{\partial t} \pm 2\xi_n^\pm(x, \partial_{x'}\psi_\pm, \tau) \frac{\partial \sigma}{\partial x_n} + 2 \sum_{j,k} g_{jk}(x) \frac{\partial \psi_\pm}{\partial x_j} \frac{\partial \sigma}{\partial x_k} + c_\pm \cdot \sigma. \end{aligned}$$

Assume, on the other side, that

$$\sigma(0, x', t, \xi, \tau) = 1$$



(this condition is the counterpart of the condition on the phase which allows to obtain the Fourier transform on  $x_n = 0$ ). We write  $\sigma(x, t, \xi', \tau)$  as an asymptotic sum of homogeneous symbols  $\sigma_j$  of degree  $-j$  in  $(\xi', \tau)$ . We check

$$\sigma_j(0, x', t, \xi', \tau) = \delta_{j0}$$

and the homogeneous transport equation on  $\sigma_0$  is  $L_1^\pm \sigma_0 = 0$ . As  $L_1^\pm$  commutes with  $\partial_t$  and as  $\partial_t \sigma_0(0, x', t, \xi', \tau) = 0$ , the Cauchy problem on  $\partial_t \sigma$  has a unique solution, which is zero. We deduce that  $\sigma_0(x, t, \xi', \tau)$  does not depend on  $t$ . For each  $j \geq 1$ , we proceed by induction, the source term being independent on  $t$  and the coefficients being independent on  $t$  in  $L_1^\pm \sigma_j = iP(\sigma_{j-1})$ , so  $\partial_t \sigma_j = 0$ . We omit the dependence in  $t$  and we can write the transport operator

$$R_1^\pm = \pm 2\xi_n^\pm(x, \partial_{x'} \psi_\pm, \tau) \frac{\partial}{\partial x_n} + 2 \sum_{j,k} g_{jk}(x) \frac{\partial \psi_\pm}{\partial x_j} \frac{\partial}{\partial x_k} + c_\pm.$$

Since  $\xi_n^\pm$  is nonzero in a neighborhood of the hyperbolic point considered, the transport equations

$$R_1^\pm(\sigma^\pm(x, \xi', \tau)) = iP(\sigma^\pm(x, \xi', \tau))$$

with initial condition  $\sigma^\pm(0, x', \xi', \tau) = 1$  have a unique solution. We have the

PROPOSITION 10.7. *Let  $(x'_0, t_0, \xi'_0, \tau_0) \in T_{x'_0, t_0} \mathbb{R}^{n-1} \times T^* \mathbb{R}_t \cap \mathcal{H}$ .*

- *There are two symbols  $\sigma^+$  and  $\sigma^-$  and two phases  $\psi_+(x, \xi', \tau)$  and  $\psi_-(x, \xi', \tau)$  such that*

$$\begin{cases} \psi_\pm|_{x_n=0} = x' \cdot \xi' \\ \frac{\partial \psi_\pm}{\partial x_n} = \pm \xi_n^\pm(x, \partial_{x'} \psi_\pm, \tau) \\ \sigma^\pm(0, x', \xi', \tau) = 1 \\ R_1^\pm(\sigma^\pm) - iP\sigma^\pm = 0. \end{cases}$$

- *The functions*

$$u_{\xi', \tau}(x, t) = \sigma^\pm(x, \xi', \tau) e^{it\tau + i\psi_\pm(x, \xi', \tau)} = \sigma^\pm(x, \xi', \tau) e^{i\tau(t + \psi_\pm(x, \frac{\xi'}{\tau}, 1))}$$

are two solutions of  $(P - \partial_{t_2}^2)u = 0$  satisfying  $u(x', 0, t) = e^{i\tau(t + \frac{\xi'}{\tau} x')}$ . We define them microlocally in a conical neighborhood of  $\rho_0^\pm = (x'_0, 0, t_0, \xi'_0, \pm \xi_n^\pm(x'_0, \xi'_0, \tau_0), \tau_0)$  in  $T_{(x'_0, 0, t_0)} \mathbb{R}^{n+1}$ .

From these two solutions, we introduce the Fourier integral operators defined on the functions  $f \in \mathcal{S}'(\mathbb{R}^{n-1} \times \mathbb{R}_\tau)$  whose wavefront is located in the vicinity of the point  $\rho_0^\pm$ :

$$(5.143) \quad A_\pm(f)(x, t) = (2\pi)^{-n} \int e^{i\phi_\pm(x, t, \xi', \tau)} \sigma^\pm(x, \xi', \tau) \mathcal{F}(f)(\xi', \tau) d\xi' d\tau$$

which rewrite

$$A_\pm(f)(x, t) = (2\pi)^{-n} \int e^{i\phi_\pm(x, t, \xi', \tau) - i\tau s - iy' \xi'} \sigma^\pm(x, \xi', \tau) f(y', x) d\xi' d\tau dy' ds.$$

We recognize here the notation of the Fourier integral operators of the Section 6, and in particular Definition 6.4. We verify that the operators  $A_\pm$  yield solutions of  $P$ . In  $x_n > 0$ , if we introduce two elements of  $\mathcal{S}'(\mathbb{R}^{n-1} \times \mathbb{R})$ , denoted by  $f_+$  et  $f_-$ :

$$P(A_\pm(f_\pm)) = 0, x_n > 0$$

We also notice that, when  $f \in \mathcal{S}'(\mathbb{R}^{n-1} \times \mathbb{R})$ , then the Fourier integral operators  $A_\pm$  construct very regular distributions in  $x_n$  in a neighborhood of the boundary:

$$A_\pm(f) \in C^\infty([0, \varepsilon], \mathcal{D}'(\mathbb{R}^{n-1} \times \mathbb{R})).$$

This allows us to verify that the distributions  $g_0(u)$  and  $g_1(u)$  of the jump formula (Lemma 10.5) are therefore the trace and the normal derivative of the solution  $u$ , which are the pseudo-differential operators

$$g_1(A_+(f)) = A_+(f)(x', 0, t) = \frac{1}{(2\pi)^n} \int e^{i(x' \xi' + t\tau)} \sigma^+(x', 0, \xi', \tau) \mathcal{F}(f)(\xi', \tau) d\xi' d\tau,$$

$$g_0(A_+(f)) = \partial_{x_n}(A_+(f))(x', 0, t) = \frac{1}{(2\pi)^n} \int e^{i(x'\xi' + t\tau)} [i\partial_{x_n}\phi_+(x', 0, \xi')\sigma^+ + \partial_{x_n}\sigma_+](x', 0, \xi', \tau)\mathcal{F}(f)(\xi', \tau)d\xi'd\tau.$$

We deduce, thanks to the jump formula

$$P(A_+(f_+) + A_-(f_-)) = (A_+(f_+) + A_-(f_-))|_{x_n=0}\delta'_{x_n=0} + \left(\frac{\partial}{\partial x_n}(A_+(f_+) + A_-(f_-))\right)|_{x_n=0}\delta_{x_n=0}.$$

Let us define the operators

$$T_{\pm} = \left[\frac{\partial}{\partial x_n} \circ A_{\pm}\right]|_{x_n=0}.$$

One checks that

$$\partial_{x_n} \circ A_+(f)(x, t) = \frac{1}{(2\pi)^n} \int e^{i\phi_{\pm}(x, t, \xi', \tau)} [i\partial_{x_n}\phi_+(x, \xi')\sigma^+(x, \xi', \tau) + \partial_{x_n}\sigma_+(x, \xi', \tau)]\mathcal{F}(f)(\xi', \tau)d\xi'd\tau.$$

The operator  $A_+$  has been constructed so that its restriction à  $x_n = 0$  is the identity, the phase  $\phi_+$  restricted to  $x_n = 0$  is  $x' \cdot \xi' + t\tau$  and the symbol  $\sigma_+|_{x_n=0}$  being identically 1. Then one has

$$T_+f(x', t) = \frac{1}{(2\pi)^n} \int e^{ix' \cdot \xi' + t\tau} [i\xi_n^+(x, \partial_{x'}\psi_+(x', \xi')) + \partial_{x_n}\sigma_+(x', 0, t, \xi', \tau)]\mathcal{F}(f)(\xi', \tau)d\xi'd\tau.$$

**The operators  $T_+$  and  $T_-$  are classical operators with respective principal symbols  $\pm i\xi_n^{\pm}(x, \partial_{x'}\psi(x', \xi'))$ . We notice that these are operators of order 1, elliptic at hyperbolic points.**

We introduce  $V = A_+(f_+) + A_-(f_-)$ . According to Hörmander,  $U$  is in  $C^{\infty}([0, \varepsilon], \mathcal{D}')$  (with the order for the variables  $x_n, (x', t)$ ). It is a solution of the equation in  $\mathcal{S}'$ :

$$PU = \frac{\partial u}{\partial x_n}(x', 0, t)\delta_{x_n=0}.$$

The equality  $PU = PV$  implies

$$\begin{cases} A_+(f_+)(x', 0, t) + A_-(f_-)(x', 0, t) = 0 \\ T_+(f_+)(x', 0, t) + T_-(f_-)(x', 0, t) = \partial_{x_n}u(x', 0, t). \end{cases}$$

The coefficient of the  $\delta_{x_n=0}$  is characterized by the symbol

$$\xi_n^+(x', 0, \xi', \tau)(\mathcal{F}(f_+) - \mathcal{F}(f_-))(\xi', \tau).$$

This yields the system, equivalent to  $P(U - V) = 0$ :

$$(5.144) \quad \begin{cases} f_- = -f_+ \\ \left[\frac{\partial}{\partial x_n} \circ A_+ - \frac{\partial}{\partial x_n} \circ A_-\right](f_+) = \frac{\partial u}{\partial x_n}(x', 0, t). \end{cases}$$

The point  $(x'_0, \xi'_0)$  is a hyperbolic point. Thus  $r_2(0, x', \xi') \geq c > 0$  for  $|\xi'| = 1$  in a neighborhood of  $(x'_0, \xi'_0/|\xi'_0|)$ , the hyperbolicity property being stable, and the homogeneity of order 1 of  $\sqrt{r_2}$  implies the ellipticity of  $\sqrt{r_2}(0, x', \xi')$  in a neighborhood of the point  $(x'_0, \xi'_0) \in \mathcal{H}$ . It follows that the principal symbol of the operator  $T_+ - T_-$  is elliptic.

The second equation of (5.144) is an elliptic equation in the neighborhood of the hyperbolic points of  $P$ . Assuming that  $\frac{\partial u}{\partial x_n}(x', 0)$  has its wavefront included in a small conical neighborhood of  $(x'_0, \xi'_0)$ , there exists a solution  $f_+(x')$  whose wavefront is included in a small conical neighborhood of  $(x'_0, \xi'_0)$ . We have thus determined  $f_+(x') = (T_+ - T_-)^{-1}\left(\frac{\partial u}{\partial x_n}(x', 0)\right)$  modulo  $C^{\infty}$ .

It remains to show that if  $u$  and  $v$  are two distributions solution in  $x_n < 0$  and that  $PU = PV$ , then  $u$  and  $v$  coincide microlocally in the vicinity of the hyperbolic points. It is equivalent to show that if  $W = v1_{x_n > 0}$  if it is possible,  $PW = 0$  implies  $w = 0$  in the neighborhood of the hyperbolic points.

The wavefront of  $W$  does not intersect  $x_n < 0$  because  $w$  is zero on  $x_n < 0$ . As the point  $\rho_0 = (x'_0, t_0, \xi'_0, \tau_0)$  is in the hyperbolic region, the two bicharacteristics  $\gamma_+$  and  $\gamma_-$  which pass through the point  $(x'_0, 0, t_0, \xi_0^0, \pm\xi_n^{\pm}(x'_0, 0, \xi'_0), \tau_0)$  enter or exit the domain  $\{x_n < 0\}$ .

By construction, they do not meet the wavefront of  $W$ . Thus  $W$  is microlocally zero in a neighborhood of the boundary. Finally  $U = A_+(f_+) - A_-(f_+) + W$ , where  $WF(W) \cap (\gamma_+ \cup \gamma_-) = \emptyset$ .

Proposition 10.6 is therefore proven. Recall that this proposition allows us to know the solution of a Dirichlet problem for which the normal derivative is known on the boundary, microlocally, and such that its wavefront contains only hyperbolic points.

We now intend to solve the Dirichlet problem with Cauchy data  $u_0$  and  $u_1$  at  $t = 0$  :

$$(5.145) \quad \begin{cases} Pu = 0, x_n > 0 \\ u|_{x_n=0} = 0 \\ u(x, 0) = u_0(x) \\ \partial_t u(x, 0) = u_1(x). \end{cases}$$

This problem is locally well posed. Let  $K^{-1}\Delta K = \frac{\partial^2}{\partial x_n^2} + Q(x_n, x, \frac{\partial}{\partial x})$  and  $P = \Delta - \frac{\partial^2}{\partial t^2}$ . One then introduces  $\tilde{Q}(x_n, x, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x}) = \frac{\partial^2}{\partial x_n^2} + Q(x_n, x, \frac{\partial}{\partial x})$ . Using more traditional notation, we change  $X_n$  and  $x_n$  and  $X$  to  $x'$ . The coordinates  $(x', x_n)$  are denoted  $x$ . Let  $p_0 \in WF(u_0) \cup WF(u_1) \subset T^*(\mathbb{R}^n) (\simeq T^*(\mathbb{R}^{n+1}) \cap \{t = 0\})$ . We construct the bicharacteristic of  $p$  passing through  $(p_0, t = 0, \tau = (\sigma(\tilde{Q})(p_0))$ .

Let us start with the solution of the Cauchy problem. To the Cauchy data are associated two Fourier integral operators, noted  $B_{\pm}$ , whose construction is similar to the one of the Fourier integral operators  $A_+$  and  $A_-$ . We introduce

$$B_{\pm}g(x, t) = \frac{1}{(2\pi)^n} \int e^{i\phi_{\pm}(x, \xi, t) - iy \cdot \xi} s_{\pm}(x, \xi, t) g(y) dy d\xi$$

where  $\phi_{\pm}$  and  $s_{\pm}$  are solutions of the eikonal and transport equations generated by the time variable, i.e.

$$\begin{cases} \frac{\partial \phi_{\pm}}{\partial t} = \pm(\sigma(\tilde{Q})(x, \nabla_x \phi_{\pm}(x, \xi, t)))^{\frac{1}{2}} \\ \phi_{\pm}(x, \xi, 0) = x \cdot \xi. \end{cases}$$

The two problems for  $\tilde{\phi}_{\pm}$  are Hamilton-Jacobi problems. To simplify the notations, we introduce  $\tilde{q}(x, \xi) = \sigma(\tilde{Q})(x, \xi)$ .

We write, in the neighborhood of  $p_0$ , the solution  $u$  of (5.145) under the form

$$u = B_+g_+ + B_-g_-.$$

We have to determine  $g_+$  and  $g_-$  as a function of  $u_0$  and  $u_1$  (microlocally of course). The system obtained is then

$$(5.146) \quad \begin{cases} u_0 = g_+ + g_- \\ u_1 = \frac{1}{(2\pi)^n} \int [i(\tilde{q}(x, \xi))^{\frac{1}{2}} + \partial_t s_+(x, \xi, 0)] \hat{g}_+(\xi) d\xi \\ \quad + \frac{1}{(2\pi)^n} \int [-i(\tilde{q}(x, \xi))^{\frac{1}{2}} + \partial_t s_-(x, \xi, 0)] \hat{g}_-(\xi) d\xi \end{cases}$$

As  $\tau$  is constant on the bicharacteristic, equal to  $-(\tilde{q}(p_0))^{\frac{1}{2}}$ , we deduce that on the bicharacteristic, the symbol  $\tilde{q}(x(s), \xi(s))$  is also constant, so that in a tubular neighborhood of this bicharacteristic, this symbol remains bounded below, which leads to the ellipticity of the system (5.146). More precisely, let  $R$  be the inverse of the pseudodifferential operator of symbol  $2i(p(x, \xi))^{\frac{1}{2}} + \partial_t s_+(x, \xi, 0) + \partial_t s_-(x, \xi, 0)$ , and  $\tilde{T}$  the pseudodifferential operator of symbol  $i(p(x, \xi))^{\frac{1}{2}} - \partial_t s_-(x, \xi, 0)$ . On a

$$g_+ = R(u_1 + \tilde{T}u_0), g_- = (I - R\tilde{T})u_0 - Ru_1.$$

One obtains the solution of problem (5.145)

$$(5.147) \quad u = (B_+R - B_-R)u_1 + (B_+R\tilde{T} + B_-(I - R\tilde{T}))u_0.$$

This solution is valid **only in a neighborhood of  $t = 0$** , or, more precisely, as long as the bicharacteristics  $\gamma$  and  $\tilde{\gamma}$  (the second one passing through the point  $(p_0, 0, (\tilde{q}(p_0))^{\frac{1}{2}}$ ) do not do not meet the boundary.

From equality (5.147), we deduce the solution of the Dirichlet problem. Indeed, we know that the operator  $B_+$  propagates on the bicharacteristic included in  $\tau = (\tilde{q}(x, \xi))^{\frac{1}{2}}$  and the operator  $B_-$  propagates on the bicharacteristic included in  $\tau = -(\tilde{q}(x, \xi))^{\frac{1}{2}}$ . These two bicharacteristics are projected on the same ray on  $\mathbb{R}_x^n$ , the main difference coming from the direction of propagation on the ray. To calculate the solution of (5.145), we use the previous notations, considering  $p_0 \in WF(u_0) \cap WF(u_1)$ ,  $u_0$  and  $u_1$  being the data at the boundary of the problem (5.145). The bicharacteristic  $\gamma$ , corresponding to  $B_-$ , meets the boundary at  $\tilde{\rho}_0$ , if it exists. The projection of  $\tilde{\rho}_0$  on  $T^*(\partial\Omega \times \mathbb{R}_t)$  is denoted by  $\rho_0$ . We have  $p_0 = (x_0, \xi_0)$ , we associate to it  $\tilde{p}_0 = (x_0, \xi_0, 0, \tau_0 = [((\xi_0)_n)^2 + \sigma(Q)(x_0, \xi'_0)]^{\frac{1}{2}})$ , then  $\tilde{\rho}_0 = (x(s_0), \xi(s_0), 2\tau_0 s_0, \tau_0)$  with  $x_n(s_0) = 0$ , and  $\rho_0 = (x'(s_0), 2\tau_0 s_0, \xi'(s_0), \tau_0)$ .<sup>1</sup> We assume that the point  $\rho_0$  is an element of  $\mathcal{H}$ . Microlocally in the neighborhood of  $\rho_0$ , we know that the two traces generated, namely  $u|_{x_n=0}$  and  $\frac{\partial u}{\partial x_n}|_{x_n=0}$  are known. For  $t > t_0$ ,  $u$  is not the solution of the Dirichlet problem, because the trace on the boundary exists and is nonzero. On the other hand, as long as the ray has not met the boundary, microlocally  $u$  is the solution of the Dirichlet problem since  $u$  is  $C^\infty$  in the neighborhood of the boundary, the ray has not yet touched the boundary. The solution of the Dirichlet problem is written

$$(5.148) \quad \tilde{u} = A_+(h) - A_-(h)$$

which verifies the Dirichlet condition. On the other hand, we verify that the reentrant microlocal contribution is given by  $A_-(h)$ , and we know that  $h|_{x_n=0} = u|_{x_n=0}$ , and  $T_-(h) = \frac{\partial u}{\partial x_n}|_{x_n=0}$ . Considering, microlocally in the neighborhood of  $\rho_0$  hyperbolic point<sup>2</sup>

Since  $h$  is known, the microlocal contribution of  $u$ , in the neighborhood of a ray reflected by the boundary  $\{x_n = 0\}$  in the neighborhood of  $\rho_0$ , is given by the equality (5.148). We have thus solved the Dirichlet problem with Cauchy data in  $t = 0$ , under the assumption that the rays intersect transversely the boundary (this is exactly the condition  $(x'_0, t_0, \xi'_0, (\tilde{q}(p_0))^{\frac{1}{2}}) \in \mathcal{H}$ ). Note that the complete expression of the solution is rather complicated, and depends on whether the rays meet the boundary or not. A simple and intrinsic translation of these ideas is presented in the following section, and is called the propagation theorem for reflected singularities.

REMARK 12. *One knows that  $P(A_+(f)) \simeq 0$ . On the other hand, we check that  $P = \partial_{x_n}^2 - (-R(x_n, x', \partial_{x'}, \partial t))$ . The principal symbol of  $R$  is denoted by  $r_2(x_n, x', \xi', \tau)$ . Traditionally we introduce the strictly hyperbolic operators of order 1 equal to  $P_\pm = \partial_{x_n} \mp Op(-r_2(x_n, x', \xi', \tau))^{\frac{1}{2}}$ . There are two operators  $S_-$  of order 1 such that*

$$P_+ \circ P_- = P + S_+, P_- \circ P_+ = P + S_-.$$

It follows that  $P_+[P_-(A_-)] = S_+(A_-)$  et  $P_-[P_+(A_+)] = S_-(A_+)$ , which proves that  $A_+$  does not give a solution of  $P_+f = 0$  because  $A_+$  is a solution of  $Pf = 0$ . The approach we have used here is not the one usually used for the Cauchy problem (as in Taylor [94]).

**5.2. Theorem of transverse reflection of singularities.** We want to show the following theorem, which is the theorem of propagation of singularities on reflected rays.

THEOREM 10.2. *Let  $u$  be a solution, when it exists of the problem (5.145). Let  $\gamma$  be a bicharacteristic of  $p$ , passing through a point of  $Car p \cap \{t = 0\}$  projecting on  $p_0$  element of  $WF(u_0) \cup WF(u_1)$ . Let  $\rho_0$  be the projection on  $T^*(\{x_n = 0\})$  of an intersection point of and  $T^*X \cap \{x_n = 0\}$ .*

<sup>1</sup>We note that we sometimes note the coordinates in  $T^*(X \times Y)$  in the two equivalent forms  $T^*X \times T^*Y$  and  $X \times Y \times T_x X \times T_y Y$ .

<sup>2</sup>In fact, we have assumed that the point is hyperbolic, so the ray associated to  $B_-$  comes from  $x_n < 0^n$ , which implies that  $B_-(I - RT)u_0 - B_-Ru_1$  is not a part of the solution generated by the reflection on the boundary. This reason is heuristic; one should rather say that the solution  $u$  of the Cauchy problem in vacuum does not include a term in  $B_-$  if the associated ray has already traveled, in time, a part of the interior of the reflecting object.

If  $\rho_0 \in \mathcal{H}$ , we construct the two bicharacteristics  $\gamma_+$  and  $\gamma_-$  passing through the two points of  $(\Pi^{-1}(\rho_0) \cap \text{Car}(p))$ . The following equivalences are true:

$$\rho_0 \in WF_b(u) \Leftrightarrow \gamma_+ \subset WF(u) \text{ or } \gamma_- \subset WF(u) \Leftrightarrow \gamma_+ \subset WF(u) \text{ and } \gamma_- \subset WF(u).$$

PROOF. Let us consider the two points of  $\text{Car } p$  which project on  $\rho_0$  (one of them is  $\tilde{\rho}_0$ ). These two points are then

$\rho^+ = (x'_0, 0, 2\tau_0 s_0 \xi'_0, [\sigma(R)(x'_0, 0, \xi'_0, \tau_0)]^{\frac{1}{2}})$  and  $\rho^- = (x'_0, 0, \xi'_0, -[\sigma(R)(x'_0, 0, \xi'_0, \tau_0)]^{\frac{1}{2}})$ . We denote by  $\gamma_+$  the bicharacteristic for  $q_+(x, \xi) = \xi_n - [\sigma(R)(x, \xi'_0, \tau_0)]^{\frac{1}{2}}$  from  $\rho_+$  and  $\gamma_-$  the bicharacteristic of  $q_-(x, \xi) = \xi_n + [\sigma(R)(x, \xi'_0, \tau_0)]^{\frac{1}{2}}$  from  $\rho^-$ .

Thanks to the study on the strictly hyperbolic Cauchy problem (Section 2),  $\gamma_+$  and  $\gamma_-$  are the bicharacteristics of  $P$  passing through  $\rho^+$  and  $\rho^-$  (we have to check the sign of the elliptic coefficient to be sure of the sign of the parameter of the bicharacteristic to see if the bicharacteristic escapes the domain).

- Denote by  $A_+$  and  $A_-$  constructing respectively two solutions of  $P$ , associated respectively to  $q_+$  and to  $q_-$  (without being solutions of  $q_+$  or  $q_-$ ). When  $u \in H^1(x_n > 0)$ ,  $\frac{\partial u}{\partial x_n}$  is well defined. Since  $u$  verifies the Dirichlet condition, we deduce the equality  $PU = \frac{\partial u}{\partial x_n}|_{x_n=0} \delta_{x_n=0}$ .

- Assume that  $\gamma_+$  and  $WF(A_+(f_+))$  (it is sufficient, according to the propagation theorem, to assume that  $\gamma_+ \cap WF(A_+(f_+)) \neq \emptyset$ ). Then  $\gamma_+ \cap \{x_n = 0\} \times T_x \mathbb{R}^{n+1} \subset WF(A_+(f_+)) \cap \{x_n = 0\}$ . Let  $\pi$  be the projection of an element of  $T^*(\mathbb{R}^n)$  onto  $T^*(\mathbb{R}^n)$ . We know that  $WF(f_+) \subset \pi(WF(A_+(f_+)) \cap \{x_n = 0\})$  because  $A_+(f_+)|_{x_n=0} = f_+$ . On the other hand, thanks to rapid decay in a cone around  $(x'_0, \xi'_0)$ ,  $\xi'_0 \neq 0$ , if  $(x'_0, \xi'_0) \notin WF(f_+)$ , then a Taylor formula in the neighborhood of  $x_n = 0$  shows that  $\mathcal{F}[\theta(x_n)\psi(x')A_+(f_+)(x', x_n)](\xi', \xi_n)$  is rapidly decaying in any (small) cone constructed around  $(x'_0, 0, \xi'_0, \xi_n^0)$  because  $\xi'_0 \neq 0$  and therefore  $|\xi'|^2 + |\xi_n^0|^2$  is equivalent to  $|\xi'|^2$  in this case. Thus  $\pi(WF(A_+(f_+)) \cap \{x_n = 0\}) = WF(f_+)$ . As  $\gamma_+ \cap \{x_n = 0\} = \emptyset$ , the point  $\rho_0$  is in  $WF(f_+)$ . Any point  $(\rho_0, 0, \xi_n)$  is then in  $WF(PU)$ , and of  $WF(PU) \subset WF(U)$ , we deduce that  $\rho_0 \in WF_b(u) = b(WF(U))$ . We also deduce that  $\rho_0 \in WF(f_-)$  so  $\gamma_- \subset WF(A_-(f_-))$ , and thus  $\gamma_- \subset WF(U)$ .

Conversely, let  $\rho_0 \notin WF(\frac{\partial u}{\partial x_n}|_{x_n=0})$ . Then, for all  $\xi_n$ ,  $(\rho_0, 0, \xi_n) \notin WF(\frac{\partial u}{\partial x_n}|_{x_n=0} \otimes \delta_{x_n=0})$ . In particular, if  $(\rho_0, 0, \xi_n^\pm)$  are the two antecedents by  $\Pi$  of  $\rho_0$ , we know that these two points are not in  $WF(PU)$  nor in  $WF(u|_{x_n=0})$ . By the theorem of propagation of singularities on  $U$ , the two bicharacteristics  $\gamma_+$  and  $\gamma_-$  do not meet  $WF(U)$  because they are not included in  $WF(U)$ , since the intersection is empty on a half-bicharacteristic ( $U|_{x_n>0} = u|_{x_n>0}$ ). Thus, microlocally in the neighborhood of  $\gamma_+$  and of  $\gamma_-$ ,  $U$  is regular.

Consider a point  $\rho_+^1 \notin WF(A_+(f_+))$ . Let  $\rho_1$  be the point of intersection between the bicharacteristic of  $P$  passing through  $\rho_+^1$  and  $\{x_n = 0\}$ . By application of the propagation theorem of singularities for a strictly hyperbolic Cauchy operator, the projection  $\tilde{\rho}_1$  of  $\rho_1$  into  $T^*(\mathbb{R}^{n-1} \times \mathbb{R}_t)$  is not in  $WF(f_+)$ . As

$$\frac{\partial u}{\partial x_n}|_{x_n=0} = \left(-\frac{\partial}{\partial x_n} \circ A_+|_{x_n=0} + \frac{\partial}{\partial x_n} \circ A_-|_{x_n=0}\right)(f_+)$$

one finds  $\tilde{\rho}_1 \notin WF(\frac{\partial u}{\partial x_n}|_{x_n=0})$  and that  $\frac{\partial u}{\partial x_n} = (T_+ - T_-)(f_+)$ , and  $T_+ - T_-$  is an elliptic operator in the neighborhood of the hyperbolic points. This proves Theorem 10.2.  $\square$

Let us finally state a more powerful theorem than the one 10.2, since it allows to take into account the case of *glancing* in part. Let  $u$  be an extendible distribution solution of  $Pu = 0$  in  $x_n > 0$ , of extended  $U$ , and let  $g_0(u)$  and  $g_1(u)$  be the two distributions of  $\mathcal{D}'_S$  obtained by the lemma 10.5 such that

$$PU = g_0(u) \otimes \delta_{x_n=0} + g_1(u) \otimes \delta'_{x_n=0}.$$

**THEOREM 10.3.** (Hörmander [48]) *Let  $\rho$  be in the characteristic manifold, such that  $\exp(sH_p(\rho))$  meets  $x_n < 0$ . We note  $\rho_0$  the point of intersection of  $\exp(sH_p(\rho))$  and  $x_n = 0$ . We have the equivalence*

$$\rho_0 \in WF_b(u) \Leftrightarrow \rho_0 \in WF(g_0(u)) \cup WF(g_1(u)).$$

A reader interested by such results can refer to Melrose and Sjöstrand [77] which is a theorem of propagation of singularities : Let  $\Sigma_b$  be the projection on  $T^*\partial\Omega$  of  $\text{Car}(\partial_{t^2}^2 - \Delta)$ , and let  $\Sigma_b^\infty$  be the set of points such that  $\partial_{x_n}^l r(0, x', t, \xi', \tau) = 0$  for all  $l$  when the wave operator in semi-geodesic coordinates is written  $\partial_{x_n}^2 + R(x_n, x', D_{x'}, D_t)$ . We have the propagation theorem on the generalized flow :

**THEOREM 10.4.** *Let  $u \in \mathcal{D}'(\Omega^c)$ ,  $(\partial_{t^2}^2 - \Delta)u \in C^\infty(\Omega^c)$  et  $u|_{\partial\Omega} \in C^\infty(\partial\Omega)$ . Then  $WF_b(u) \subset \Sigma_b$  et  $WF_b(u)$  is invariant under the Hamiltonian generalized flow defined in [77].*

**5.3. Construction of the solution of the Dirichlet problem.** We are inspired by R.Taylor for this construction. First, we consider a solution of the Cauchy problem associated to  $u_0$  and  $u_1$ , denoted by  $\tilde{u}$ . This solution is defined microlocally in the neighborhood of the bicharacteristics  $\gamma$  passing through a point of  $WF(u_0) \cup WF(u_1)$ . It is assumed to be defined for  $x_n] - \varepsilon, \varepsilon[$  in a neighborhood of the rays passing at points of  $WF(u_0) \cup WF(u_1)$ . We denote it by  $\tilde{u}$ . We consider the point  $\tilde{\rho}_0 = (x'_0, 0, t_0, \xi'_0, \xi_n^0, \tau_0)$  of the intersection of  $\gamma$  and  $x_n = 0$ , by noting that the point  $(x'_0, t_0, \xi'_0, \tau_0)$  is hyperbolic, that  $\xi_n^0 < 0$  and that the other point of the characteristic manifold is  $\check{\rho}_0 = (x'_0, 0, t_0, \xi'_0, -\xi_n^0, \tau_0)$ . We define the reflected bicharacteristic, denoted by  $\tilde{\gamma}$  passing through the point  $\check{\rho}_0$ . We then consider the mixed problem

$$\begin{cases} Pv = 0 \\ v|_{x_n=0} = -\tilde{u}|_{x_n=0} \end{cases}$$

where  $v$  is defined in a conic neighborhood of the bicharacteristics  $\tilde{\gamma}$ . This mixed problem has a solution (modulo  $C^\infty$ ) denoted  $v$ . Then the wavefront of  $v$  is concentrated in a neighborhood of the reflected rays  $\tilde{\gamma}$  generated by the intersection of the incident rays and of  $WF(u_0) \cup WF(u_1)$  at  $t = 0$ . The solution  $v$  is microlocally zero in the neighborhood of  $WF(u_0) \cup WF(u_1)$  at  $t = 0$ .

The distribution  $\tilde{u} + v$  is a solution, modulo  $C^\infty$ , of  $P(\tilde{u} + v) = 0$  in  $x_n > 0$ ,  $\tilde{u} + v|_{t=0} = u_0$ ,  $\partial_t(\tilde{u} + v) = u_1$  (modulo  $C^\infty$ ), checking  $(\tilde{u} + v)|_{x_n=0} = 0$ .

**5.4. Analysis of the mixed problem.** We call here (unlike the classical terminology of the literature of microlocal analysis where the mixed condition links  $\partial_n u$  and  $u$  on the boundary) the mixed condition a condition mixing  $\partial_t u(x', 0, t)$  and  $\partial_{x_n} u(x', 0, t)$ . We want to solve the problem

$$\begin{cases} Pu = 0 \\ z(x')\partial_t u(x', 0, t) + \partial_{x_n} u(x', 0, t) = 0. \end{cases}$$

Using the representation with the operators  $A_+$  and  $A_-$ , we write a general solution as  $A_+(f_+) + A_-(f_-) = u$ . Thus, the equality on the boundary leads to

$$(5.149) \quad T_+(f_+) + T_-(f_-) + z(x')[\partial_t \circ A_+|_{x_n=0}(f_+) + \partial_t \circ A_-|_{x_n=0}(f_-)] = 0.$$

We notice that the symbol of the operator  $\partial_t \circ A_+|_{x_n=0}$ , which is a pseudo-differential operator, is equal to  $i\tau$ , because the phase and symbol do not depend on time. The equality (5.149) becomes

$$Op(i\xi_n^+(x', \partial_{x'}\psi_+(x', \xi')) + \partial_{x_n}\sigma_+ + i\tau z(x'))(f_+) = -[T_-(f_-) + z(x')Op(i\tau)(f_-)]$$

If the mixed condition verifies the Lopatinskii condition, for example if  $\text{Re } z > 0$ , the operator  $Op(i\xi_n^+(x', \partial_{x'}\psi_+(x', \xi')) + \partial_{x_n}\sigma_+ + i\tau z(x'))$  is an elliptic operator and  $f_+$  is known in terms of  $f_-$ , which allows to identify the two traces  $u$  and  $\partial_{x_n} u$  in terms of  $f_-$ . The result of propagation of singularities applies then as before.

In the next section, we present applications of the parametrix of the wave equation. We will focus in particular on the rigorous proof, in terms of propagation of singularities, of the Snell-Descartes laws (transverse reflection of an incident scalar wave).

We first analyze the notion of wavefront, and we introduce functions adapted to this notion functions adapted to this notion : the conormal waves. We use this representation to

construct explicitly the wavefront of the reflected wave, using the operators  $A_+$  and  $A_-$ , for a Dirichlet condition.

## 6. The reflection coefficient

Here we want to calculate the reflection coefficient. To calculate it more easily, we introduce a representation of the incident wave which generalizes the notion of plane wave. This is the conormal wave. It is characterized by a symbol and a phase  $t - \theta_i(x', x_n)$ . This phase is zero on the front studied (generalization of the hyperplane in the case of a plane wave).

**6.1. Onde conormale.** The right tool to study the propagation of a wave associated to a wavefront put in the form  $t = \theta_i(x)$  is a conormal wave. We give its definition and study its properties :

DEFINITION 10.8. *A wave  $u(x, t)$  is a conormal distribution with respect to the surface  $i(x, t) = t - \theta_i(x', x_n)$  when there is a symbol  $\sigma(x, \tau) \in S^{m+\frac{n-1}{4}}(\mathbb{R}^n \times \mathbb{R}, C)$  such that*

$$u(x, t) = \int_0^{+\infty} e^{i\tau(t-\theta_i(x))} \sigma(x, \tau) d\tau.$$

with

$$\sigma(x, \tau) \simeq \tau^\alpha \sum_j \sigma^j(x) \tau^{-j}.$$

This definition is an application of definition 18.2.6 and of theorem 18.2.8 of [48].

Remark : the wave is said to be analytic conormal when the symbol  $\sigma(x, \tau)$  is holomorphic in  $\Im\tau < 0$  and that it verifies

$$\sup_{s \geq 0} \int_0^\infty (1 + |\tau|^2)^{-\beta/2} |\sigma(x, \tau - is)|^2 d\tau < \infty.$$

We can represent them in an imaged way in the following figure, the function  $\theta_i(x)$  can then be calculated and it is equal to  $-5 + ((x + 5)^2 + y^2)^{\frac{1}{2}}$  (we check that its gradient is of norm 1) :

We verify that the conormal waves are  $C^\infty$  outside the front  $\Sigma$ . For this purpose, separate the integral in  $\tau$  into a fixed neighborhood of  $\tau = 0$  and its complement. In its complement, the bound  $|\sigma(\tau)| \leq \tau^{-1-\varepsilon}$  leads to an absolutely convergent integral.

Let  $N$  be an integer, order of truncation of the asymptotic series, satisfying  $\alpha - N < -1$ . Then we can write, on any compact  $K$  in  $x$ , the estimate

$$|\sigma(x, \tau) - \sigma^N(x, \tau)| \leq C_{N,K} \tau^{-N}$$

which leads to the following integral being absolutely convergent

$$\int e^{i\tau(t-\theta(x))} (\sigma(x, \tau) - \sigma^N(x, \tau)) d\tau.$$

If we consider a derivative of order  $m$  of  $\int e^{i\tau(t-\theta(x))} (\sigma(x, \tau) - \sigma^N(x, \tau)) d\tau$ , this derivative is also associated to an absolutely convergent integral. For this, we fix an order of derivation  $m$ . We then verify by integrations by parts the relation, valid outside  $t = \theta(x)$  :

$$u^{N+m}(x, t) = \int e^{i\tau(t-\theta(x))} \sigma^{N+m}(x, \tau) d\tau = \left(\frac{i}{t-\theta(x)}\right)^p \int e^{i\tau(t-\theta(x))} \partial_{\tau^p}^p (\sigma^{N+m}(s, \tau)) d\tau.$$

This relation indicates, by choosing  $p > \alpha + 1$ , that the integral defining  $u^{N+m}$  is absolutely convergent outside  $t = \theta(x)$ . We also notice that

$$\partial_{x_i}^l \partial_{x_\beta}^{|\beta|} u^{N+m}(x, t) = \int e^{i\tau(t-\theta(x))} \tau^{l+|\beta|} S_{\alpha,\beta}(\sigma^{N+m})(x, \tau) d\tau,$$

and the choice  $p > l + \alpha + |\beta| + 1$  allows to conclude that the integral defining the derivative of  $u^{N+m}$  is a function in  $C^0$ . The function  $u$  is sum of two functions element of  $C^m$  out of

$t = \theta(x)$ . Moreover, as  $\sigma(x, \tau)$  is holomorphic in  $\tau$ , we can rewrite the integral by deforming the contour in the complex plane, on  $\{\tau - is, s > 0\}$ . We then check that

$$u(x, t) = e^{s(t-\theta_i(x))} \int_0^\infty e^{i\tau(t-is)} \sigma(x, \tau - is) d\tau.$$

The uniform growth of  $\sigma(x, \tau - is)$  and the fact that  $e^{-s(\theta_i(x)-t)}$  tends to 0 when  $s$  tends to  $+\infty$  for  $\theta_i(x) - t > 0$  allows us to obtain that  $u(x, t)$  is zero for  $t < \theta_i(x)$ . The function  $u(x, t_0)$  is supported in the half space characterized by the boundary  $t_0 = \theta_i(x)$  not containing  $\nabla_x \theta_i(x_0), \theta_i(x_0) = t_0$ .

We give two examples

Dirac sum. We suppose that  $\alpha$  is a positive integer  $N_0$ . We check that, for  $\phi$  compactly supported in  $\mathbb{R}^d \times \mathbb{R}$

$$\begin{aligned} \int \int e^{i\tau(t-\theta(x))} \tau^l d\tau \phi(x, t) dx dt &= i^{-l} \int dx d\tau \int dt \partial_t^l (e^{i\tau(t-\theta(x))}) \phi(x, t) \\ &= i^l \int dx \int dt e^{i\tau(t-\theta(x))} \partial_t^l \phi(x, t) \\ &= 2\pi i^l \int dx \partial_t^l \phi(x, \theta(x)) \end{aligned}$$

The distribution  $\int e^{i\tau(t-\theta(x))} \tau^l$  is thus the distribution  $2\pi i^l \delta_{t=\theta(x)}^{(l)}$  when  $l$  is positive or zero. The case  $l = 0$  gives what is called a single layer potential, the case  $l = 1$  gives a double layer potential.

To fix the ideas and simplify the notations, we suppose that  $\theta_i(0) = 0$  and that the point  $(0, \nabla_{x'} \theta_i(0), -1)$  is a hyperbolic point of  $R^{d-1} \times \mathbb{R}_t$ .

Wave front set. We decompose  $\sigma_i$  into  $\sigma_i^N$  and  $r_i^N$ . The  $m$ -th derivative in  $t$  (or in  $x$ ) of  $u_i^N$  corresponds to a symbol in  $\tau^{a+m}$ . We then perform sufficient integrations by parts (possible on  $t = \theta_i(x)$ ) so that the integral in  $\tau$  is absolutely convergent in  $\tau = +\infty$ . The term  $r_i^N$  is treated in the same way. From this we deduces that  $u_i(x, t)$  is  $C^\infty$  outside  $t = \theta_i(x)$ .

The wavefront of the distribution  $u_i$  is then concentrated on  $t = \theta_i(x)$ . We localize  $u_i$  thanks to  $\chi(x, t)$  in the neighborhood of a point  $(x_0, t_0)$  such that  $t_0 = \theta_i(x_0)$ . The Fourier transform of  $\chi u_i$  is :

$$\hat{u}(\xi, k) = \int e^{i(\tau(t-\theta_i(x))-kt-\xi x)} \sigma_i(x, \tau) \chi(x, t) dx dt d\tau$$

We verify that the derivative in  $(x, t, \tau)$  of the phase in this integral is  $-\xi - \tau \nabla_x \theta_i(x) = 0$ ,  $\tau - k = 0$ ,  $t - \theta_i(x) = 0$ . The phase has a critical point at  $t = \theta_i(x)$ ,  $\xi = -k \nabla_x \theta_i(x)$ . This implies that

$$WF(u_i) \subset \{(x, \theta_i(x), -k \nabla_x \theta_i(x), k), k \in R^*\}.$$

Solution. We consider the operator  $P$  studied previously

$$P = P_0 - \partial_t^2 = \frac{\partial^2}{\partial x_n^2} + R_2(x_n, x', \frac{\partial}{\partial x'}) + R_1(x_n, x', \frac{\partial}{\partial x'}) + R_0(x_n, x') - \frac{\partial^2}{\partial t^2}.$$

We verify the equality

$$P \left( \int e^{i\tau(t-\theta_i(x))} \right) = \int d\tau e^{i\tau t} (P_0 + \tau^2) (e^{-i\tau \theta_i(x)} \sigma_i(x, \tau)).$$

This corresponds to the application of the Fourier transform in time to  $v = Pu$ .

We then use the section 5. We find explicitly ( $R_2$  is a bilinear form on the tangent bundle  $T_{x'} \mathbb{R}^{n-1}$  with parameter  $x_n$  and  $R_1$  is a vector of this same tangent bundle) :

$$(6.150) \quad \begin{aligned} e^{i\tau \theta_i(x)} P_0 (e^{-i\tau \theta_i(x)} \sigma_i(x, \tau)) &= -\tau^2 \left[ \left( \frac{\partial \theta_i}{\partial x_n} \right)^2 + R_2(\nabla_{x'} \theta_i, \nabla_{x'} \theta_i) \right] \sigma_i(x, \tau) - i\tau \left[ 2 \frac{\partial \theta_i}{\partial x_n} \frac{\partial \sigma_i}{\partial x_n} + 2R_2(\nabla_{x'} \theta_i, \nabla_{x'} \sigma_i) \right. \\ &\quad \left. + (P_0 - R_0)(\theta_i) \sigma_i \right] + P_0 \sigma_i. \end{aligned}$$

There exists  $\theta_i$  solution of the eikonal equation with  $\frac{\partial \theta_i}{\partial x_n} > 0$ :

$$\left( \frac{\partial \theta_i}{\partial x_n} \right)^2 + R_2(x_n, x') \left( \frac{\partial \theta_i}{\partial x'}, \frac{\partial \theta_i}{\partial x'} \right) = 1$$



and  $\sigma_i$  solution of the transport equations

$$2 \frac{\partial \theta_i}{\partial x_n} \frac{\partial \sigma_i^l}{\partial x_n} + \sum_j \partial_{x_j} R \partial_{x_j} \sigma_i^l + \partial_\tau R \partial_t \sigma_i^l + (P_0 - R_0)(\theta_i) \sigma_i^l = iP_0(\sigma_i^{l-1}).$$

We have thus constructed a solution of  $Pu = 0$ .

**6.2. Amplitude and phase after reflection of a conormal wave.** Consider a wave  $u_i(x, t)$ , conormal analytic with respect to the surface of equation  $t = \theta_i(x)$ , supported for  $t = -T$  in the part of  $\mathbb{R}^3$  not containing the open set  $\Omega$ . Under these conditions, by finite speed propagation for the wave equation, there exists a point of  $\partial\Omega$ , denoted by  $x_0$ , such that  $(x_0, \theta_i(x_0), \nabla \theta_i(x_0), -1)$  is in the wavefront of  $u_i$  and that  $WF(u_i) \cup \partial\Omega \times \mathbb{R}^3$  is empty for  $t < \theta_i(x_0)$ . Let  $u_i$  be a solution of  $(P_0 - \partial_{t^2}^2)u_i = 0$ :

$$u_i(x, t) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i\tau(t-\theta_i(x))} \sigma_i(x, \tau) d\tau.$$

We prove in this paragraph the proposition :

PROPOSITION 10.8. *There exists a function  $\theta_r$  on  $\mathbb{R}^n$  and a function  $\sigma_r$  on  $\mathbb{R}^n \times C$ , such that the problem on  $u = u_i + u_r$*

$$\begin{cases} (P_0 - \partial_{t^2}^2)u = 0 \\ u|_{t < 0} = u_i|_{t < 0} \\ (D), (N), (M) \end{cases}$$

(where the condition (D), (N), or (M) is written on  $u$ ) has a unique solution such that  $u_r(x, t) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i\tau(t-\theta_r(x))} \sigma_r(x, \tau) d\tau$ . where  $\theta_r$  is the unique solution of

$$\begin{cases} (\frac{\partial \theta_r}{\partial x_n})^2 + R_2(x_n, x') (\frac{\partial \theta_r}{\partial x'}, \frac{\partial \theta_r}{\partial x'}) = 1 \\ \theta_r(x', 0) = \theta_i(x', 0) \\ \frac{\partial \theta_r}{\partial x_n}(x', 0) = -\frac{\partial \theta_i}{\partial x_n}(x', 0). \end{cases}$$

This function  $\theta_r(x)$  is equal to  $\theta_r(x) = \theta_i(y'_c, 0) + y'_c \eta'_c - \psi_+(x, \eta'_c)$ , where  $y'_c, \eta'_c$  are solution of the system

$$\begin{cases} y'_c = \nabla_{\xi'} \psi_+(x, \eta'_c) \\ \eta'_c = -\partial_{y'} \theta_i(y'_c, 0). \end{cases}$$

The leading order term of the symbol  $\sigma_r$  satisfies

$$\sigma_r^0(x', 0) = R_{(D),(N),(M)} \sigma_i^0(x', 0).$$

In this equality

$$\begin{aligned} R_{(D)} &= -1 \\ R_{(N)} &= 1 \\ R_{(M)}(x') &= \frac{\frac{\partial \phi_{\pm}}{\partial x_n}(x', 0) - z(x')}{\frac{\partial \phi_{\pm}}{\partial x_n}(x', 0) - z(x')} = \frac{\xi_n^+(x', \frac{\partial \psi_{\pm}}{\partial x'}(x', 0)) - z(x')}{\xi_n^+(x', \frac{\partial \psi_{\pm}}{\partial x'}(x', 0)) - z(x')}. \end{aligned}$$

We represent in the following set of figures the reflection of a front by an ellipse. These drawings come from the application of the laws of geometric optics.

Figure 8

PROOF. We perform a semi-geodesic coordinate change, which corresponds to this point  $x_0$  and we perform a translation in time, so that the time  $t'$  is  $t' = t - \theta_i(x_0)$ . The point  $x_0$  will then be associated to the point  $(0, 0)$  in semi-geodesic coordinates.

We suppose that the wavefront of  $u_i$  contains the point

$$\rho_0^- = (0, 0, \nabla_{x'} \theta_i(0), \nabla_{x_n} \theta_i(0), -1)$$

and that, to fix the ideas, this point is the first of the boundary reached by the support of  $u_i$ . We look for the solution of the problem

$$(6.151) \quad \begin{cases} Pu = 0, x_n > 0 \\ u|_{x_n=0} = 0 \\ u - u_i \in C^\infty, t < -T \end{cases}$$

Consider then  $A_+$  and  $A_-$  and apply  $A_+$  to  $u_i(\cdot, 0, \cdot)$   $A_-$  to obtain the solution. Write the solution of this problem in the form  $u = u_i + u_r$ . The problem becomes :

$$\begin{cases} Pu_r = 0, x_n > 0 \\ u_r|_{x_n=0} = -u_i|_{x_n=0} \\ u_r \in C^\infty, t < -T \end{cases}$$

The set of points of  $T^*(\mathbb{R}^3 \times \mathbb{R}_t) \cap \partial\Omega \times \mathbb{R}^3 \times \mathbb{R}$  that project onto  $\rho_0 = (0, \nabla_{x'}\theta_i(0, 0), -1)$  is characterized by  $\xi_n^2 = (\nabla_{x_n}\theta_i(0, 0))^2$ . When  $\nabla_{x_n}\theta_i(0, 0)$  is non zero,  $\rho_0$  is a hyperbolic point for  $p$ , which will be our assumption. Note that we have used here the fact that  $\tau(t - \theta_i(x', x_n))$  is a solution of the eikonal equation. Let us also note that  $WF(u_i - A_-(u_i|_{x_n=0})) = \emptyset$  in a neighborhood of all points of  $\gamma_-(\rho_0^-) = \{\exp(sH_p(\rho_0^-)), s < 0\}$ . The bicharacteristic for  $P$  of principal symbol

$$p = \tau^2 - \xi_n^2 - R_2(x_n, x')(\xi', \xi')$$

outgoing from  $\rho_0^-$  is characterized by the system of equations

$$\begin{cases} \dot{t} = 2\tau \\ \dot{\tau} = 0 \\ \dot{x}_n = -2\xi_n \\ \dot{\xi}_n = \partial_{x_n}R_2(\xi', \xi') \\ \dot{x}' = -2R_2(\xi') \\ \dot{\xi}' = \partial_{x'}R_2(\xi', \xi') \end{cases}$$

with initial condition

$$(6.152) \quad t(0) = 0, \tau(0) = -1, x_n(0) = 0, \xi_n(0) = \nabla_{x_n}\theta_i(0, 0), x'(0) = 0, \xi'(0) = -\nabla_{x'}\theta_i(0, 0).$$

In particular  $\tau(s) = -1$ ,  $t(s) = -2s$ . We thus verify that the last relation of the system, which is  $u_r \in C^\infty, t < -T$  implies

$$WF(u_r) \cap \{-2s < -T, \exp(sH_{-p}(\rho_0^-))\} = \emptyset$$

or

$$WF(u_r) \cap \{s \geq 0, \exp(sH_p(\rho_0^-)), s > T/2\} = \emptyset.$$

In an imaged way,  $u_r$  has no wavefront on the reentrant bicharacteristic of  $P$  arriving at the point  $\rho_0^-$ .

This translates into the fact that, microlocally in the neighborhood of this reentrant bicharacteristic,  $u_r$  is zero. We define the bicharacteristic associated to the point

$$\rho_0^+ = (0, \nabla_{x'}\theta_i(0), -1)$$

By the analysis on the operators  $A_+$  and  $A_-$ , when the point  $(0, \nabla_{x'}\theta_i(0), -1)$  is in the wavefront of  $f$ , the wavefront of  $f$ , then the bicharacteristic  $\gamma_+$  is contained in the wavefront of  $A_+(f)$ .

Microlocally in the neighborhood of  $\gamma_+$ , we proved that  $u_r = -A_+(u_i|_{x_n=0})$ .

We check that  $u_r$  can be determined in two ways: a stationary phase theorem and a stationary phase on one side and a calculation of the reflected solution using the eikonal and transport eikonal and transport equations on the other side. The second point of view is immediate.

We prove in particular that the reflected wave is characterized by the phase  $\theta_r$  given by

LEMMA 10.6. *Let  $\theta_i(x', x_n)$  be the function characterizing the front of the wavefront. Let  $\psi_+(x, \eta')$  be the characteristic phase of the outgoing Fourier integral operator  $A_+$ . Let  $(y'_c, \eta'_c)$  be the unique solution of*

$$\begin{cases} -\eta'_c - \nabla_{y'}\theta_i(y'_c, 0) = 0 \\ \nabla_{\xi'}\psi_+(x, \eta'_c) - y'_c = 0 \end{cases}$$

Then

$$\theta_r(x) = \theta_i(y'_c, 0) + y'_c \eta'_c - \psi_+(x, \eta'_c).$$

For the first method, we recall the existence of  $\sigma_+$  and  $\psi_+$  such that

$$A_+(f)(x) = \int \int e^{i\psi_+(x, \xi') - iy' \xi'} \sigma_+(x, \xi') f(y') dy' d\xi'$$

We replace  $f(y')$  by  $-e^{-i\tau\theta_i(y', 0)} \sigma_i(y', 0, \tau)$ . We obtain

$$u_r(x, \tau) = \int \int e^{i\psi_+(x, \xi') - iy' \xi' - \tau\theta_i(y', 0)} \sigma_+(x, \xi') \sigma_i(y', 0, \tau) dy' d\xi'.$$

Write first, for  $\tau > 0$ ,  $\eta' \tau = \xi'$ . The stationary phase theorem in  $(y', \eta')$  gives

$$\begin{cases} -\eta'_c - \nabla_{y'} \theta_i(y'_c, 0) = 0 \\ \nabla_{\xi'} \psi_+(x, \eta'_c) - y'_c = 0 \end{cases}$$

For  $x_n$  small, there exists a smooth function  $g$  such that  $\psi_+(x, \eta') = x' \eta' + x_n \nabla_{\eta'} g(x, \eta')$  so  $\nabla_{\eta'} \psi_+(x, \eta') = x' + x_n g(x, \eta')$ . The critical point satisfies

$$\begin{cases} y'_c = x' + x_n \nabla_{\eta'} g(x, \eta'_c) \\ \eta'_c = -\nabla_{y'} \theta_i(y'_c, 0) \end{cases}$$

For  $x_n = 0$ , we find  $y'_c = x'$  and  $\eta'_c = -\nabla_{y'} \theta_i(x', 0)$ . The critical value of the phase thus obtained is  $x' \cdot \eta' - x' \eta' - \theta_i(x', 0) = -\theta_i(x', 0)$ .

For  $x_n > 0$ , the equation giving  $y'_c$  is

$$y'_c = x' + x_n \nabla_{y'} g(x, -\nabla_{y'} \theta_i(y'_c, x_n), \eta'_c).$$

In a neighborhood of  $x_n = 0$ , as the differential in  $y'_c$  is  $Id + x_n \text{Hess}_{y'} g \text{Hess}_{y'} \theta_i$ , invertible if  $x_n$  is small enough, one can find  $y'_c$  in terms in  $x$ . This system has a unique solution  $(y'_c(x), \eta'_c(x))$ . The critical value is denoted by  $-\theta(x)$  with

$$\theta(x) = \theta_i(y'_c(x), 0) + y'_c(x) \cdot \eta'_c(x) - \psi_+(x, \eta'_c(x)).$$

One deduces

$$\begin{aligned} \nabla_x \theta(x) &= [\nabla_{y'} (\theta_i(y', 0)) + \eta'_c(x)] \cdot \frac{dy'_c}{dx} + [y'_c(x) - \nabla_{\eta'} \psi_+(x, \eta'_c(x))] \frac{d\eta'_c}{dx} - \nabla_x \psi_+(x, \eta'_c(x)) \\ &= -\nabla_x \psi_+(x, \eta'_c(x)). \end{aligned}$$

The phase  $\theta(x)$  is thus a solution of the eikonal equation, and is associated to associated to  $A_+$ . We have therefore constructed the phase  $\theta_r$  that was expected for the wave reflected by  $x_n = 0$ . This proves Lemma 10.6. The reflection coefficient is identified by writing the equalities

$$u_i(x, t) = A_-(u_i|_{x_n=0}), u_r = -A_+(h(x', t)).$$

The Dirichlet boundary condition is written  $u_i + u_r = 0$ . We thus obtain

$$A_-(u_i|_{x_n=0})|_{x_n=0} - A_+(h(x', t))|_{x_n=0} = 0.$$

The operators  $A_-$  and  $A_+$  are equal to identity on  $x_n = 0$ , so  $h(x', t) = u_i(x', t)$ . We immediately deduce

$$R_{(D)}(x', \tau) = -1.$$

The Neumann boundary condition is  $\partial_{x_n}(u_i + u_r)|_{x_n=0}$ . It is therefore expressed as

$$\frac{\partial}{\partial x_n} \circ A_-(u_i|_{x_n=0})|_{x_n=0} - \frac{\partial}{\partial x_n} \circ A_+(h)|_{x_n=0}.$$

We then use  $\partial_{x_n} \phi_+ = -\partial_{x_n} \phi_-$ . As we are in the vicinity of a hyperbolic point, this quantity is non-zero. It comes then, by considering the asymptotic in  $\tau$ , that

$$\frac{\partial \phi_+}{\partial x_n}(x', 0)[\sigma_i^0(x', 0) + \tilde{h}(x')] = 0.$$

As  $\sigma_r$  is obtained by calculating  $A_+(\tilde{h}e^{-i\tau\theta_r(x',0)})$  and as  $A_+$  is the identity on  $x_n = 0$ , we find

$$R_{(N)} = 1.$$

Finally, the mixed boundary condition is written  $\partial_{x_n}(u_i + u_r)|_{x_n=0} + z(x')\partial_t(u_i + u_r)|_{x_n=0}$ . We then replace  $\partial_t$  by  $i\tau$ ,  $\partial_{x_n}A_+|_{x_n=0}$  by  $-i\tau\frac{\partial_+}{\partial x_n}(x',0)$  and  $\partial_{x_n}A_-$  by  $i\tau\frac{\partial_+}{\partial x_n}(x',0)$ . We obtain

$$-i\tau\frac{\partial\phi_+}{\partial x_n}(x',0)\sigma_i^0(x',0) - i\frac{\sigma_i^0(x',0)}{\partial x_n}(x',0)h(x') + i\tau z(x')[\sigma_i(x',0) - h(x')] = 0$$

We thus obtain

$$\sigma_i^0(x',0) = \frac{z(x') + \frac{\partial\phi_+}{\partial x_n}(x',0)}{z(x') - \frac{\partial\phi_+}{\partial x_n}(x',0)}h(x').$$

The reflection coefficient is immediately deduced

$$R_{(M)}(x') = \frac{\partial\phi_+}{\partial x_n}(x',0) - z(x').$$

□



## Les valeurs propres du Laplacien (C. Bardos)

The determination of the eigenvalues of the Laplacian, or more generally of an elliptic operator on a compact manifold with or without boundary is a "pure" mathematical problem of "pure" mathematics which has many applications in fundamental mathematics in fundamental mathematics, number theory or geometry, as well as in physics geometry, as well as in physics (as it is explained in the introduction of the of the article by Balian and Bloch (1970) [7]), nuclear physics and electromagnetism, and finally in the engineering sciences (acoustics of a concert hall for example).

The first results date from 1911 and are due to Hermann Weyl, they they give an equivalent of the asymptotic behavior of these eigenvalues.

Several ideas appeared afterwards; the heat kernel proved to be a useful a convenient tool (Minakshisundaram and Pleijel (1949) [79]) and the connection with Riemannian geometry became apparent (Mac Kean and Singer (1967) [72]).

But as observed by Keller and Rubinow (1960) [54] and Balian and Bloch [7], the eigenvalue distribution presents oscillations which the heat kernel cannot account for. These oscillations are due to the contribution of the closed geodesics. It It is therefore a question of global effects. Thus, the Fourier integral operators have been proved to be the most suitable tool for the rigorous shaping of these observations.

Without claiming to be new, we propose in this chapter to illustrate these ideas and to illustrate these ideas and to show how the sophistication of the tools goes of with the precision of the results. At the same time, we try to be to be "economical" in this presentation and to introduce the tools only when they become At the same time, an attempt is made to be "economical" in this presentation and to introduce the tools only when they become essential for improving the results.

### 1. Introduction

This chapter is devoted to the evaluation of the eigenvalues of an elliptic operator of the second order elliptic operator, either in a bounded domain of  $\mathbb{R}^n$ , with to to fix the ideas the Dirichlet condition:

$$u = 0 \text{ on } \partial\Omega.$$

be on a compact manifold without boundary.

The most elementary information about this behavior can be obtained by a direct be obtained by a direct calculation (maximum principle) and then thanks to a Lévy series (which is interpreted as a pseudo-differential calculation). To access the "optimal" results it is necessary to take into account the global geometry of the problem and thus to use Fourier Integral Operators. Fourier Integral Operators. It is this approach that we propose to illustrate in this chapter. We therefore denote by  $L$  a second order operator which is written in the form form:

$$(1.153) \quad Lu = - \sum_{ij} \partial_i(a^{ij}(x)\partial_j u) + \sum_i b^i \partial_i u + c = -\nabla_x \cdot (A(x)\nabla_x u) + B(x)\nabla_x u + c(x)u$$

where  $A$  denotes a symmetric real positive definite matrix,  $B \cdot \nabla_x$  a vector field and  $c$  a scalar function.

We suppose that there exists on  $\Omega$  a density  $0 < Y(x) \in C^\infty(\bar{\Omega})$  which symmetrizes  $L$ , i.e. for any pair of functions  $(u, v) \in D(L) \times D(L)$  we have:

$$(1.154) \quad \int_{\Omega} Lu(x)v(x)Y(x)dx = \int_{\Omega} u(x)Lv(x)Y(x)dx, \int_{\Omega} Lu(x)u(x)Y(x)dx \geq 0$$

In this case there are Hilbertian bases of  $L^2(\Omega)$  formed by eigenfunctions  $w_k(x)$  of the operator  $L$ :

$$Lw_k(x) = \lambda_k w_k(x) \text{ in } \Omega, w_k(x) = 0 \text{ on } \partial\Omega \text{ if } \Omega \text{ is not a boundaryless manifold}$$

We therefore denote by

$$0, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$$

the sequence of these eigenvalues counted with their multiplicity. This sequence tends towards infinity with  $k$  and we want to obtain information about its behavior asymptotic behavior when  $k$  tends to infinity.

It is therefore a high frequency analysis (which fits well in the the asymptotic theory). Moreover, in many physical applications we will have to consider will be led to consider not too large" values and so the precision of the asymptotic behavior becomes an important issue.

To simplify the presentation and to emphasize even more the role of of geometry, we limit ourselves the case where  $L$  is identified with the Laplacian on  $\Omega$  provided with the Riemannian structure. In fact, because of the high frequency aspect, only the main part of the operator is involved. We can therefore always reduce ourselves to to the case of the Laplacian which contains the geometrical aspects of the of the IFO theory, so we assume that in a local coordinate system this operator is written in the form (invariant by change of of Riemannian coordinates):

$$(1.155) \quad Lu = -\Delta u = -\frac{1}{\sqrt{\det g}} \sum_{ij} \partial_i(a^{ij}(x)\sqrt{\det g}\partial_j u)$$

In (1.155) intervenes the symmetric positive definite matrix:  $A(x) = \{a^{ij}(x)\}$  associated to the main symbol of the operator  $(A(x)\xi, \xi)$ ,  $g(x)$  is the matrix which defines the Riemannian structure, it is related to  $A(x)$  by the formula (deduced from the variational calculus via Legendre transformation)  $g(x) = (A(x))^{-\frac{1}{2}}$ . The element of volume element (cf (1.154) ) is therefore

$$dv(x) = Y(x)dx = \sqrt{\det g(x)}dx$$

formula which identifies the densities of order  $\frac{1}{2}$  and the functions according to the isomorphism:

$$f \mapsto f\sqrt{\det g(x)}dx$$

To evaluate the asymptotic behavior of the eigenvalues it is convenient to to introduce several objects:

- (1) The enumeration function:

$$N(\lambda) = \text{Card}\{\lambda_k \leq \lambda\}$$

- (2) The trace of the heat operator

$$(1.156) \quad \text{Trace } e^{-tL} = \sum_k e^{-t\lambda_k}$$

- (3) The frequency enumeration function:

$$\tilde{N}(\tau) = N(\tau^2)$$

(4) The trace of the wave operator

$$(1.157) \quad \text{Trace } \cos t\sqrt{L} = \sum_k e^{\pm it\sqrt{\lambda_k}}.$$

The term 1 in the second member of (1.157) corresponds to the eigenvalue 0 which is present (and simple) in the case of an open set without boundary, it disappears in the case of an open set with boundary and does not influence the asymptotic behavior of the eigenvalues. The enumeration function is the inverse of the function  $k \mapsto \lambda_k$ ; it is not obvious and this will be proved below that the expressions in the right hand side of (1.156) and (1.157) are well defined but it should be noted that, at least formally the trace of the heat operator is the Laplace transform of the distribution

$$\sum_k \delta(\lambda - \lambda_k) = \frac{d}{d\lambda} N(\lambda)$$

while the trace of the wave operator is the Fourier transform of the distribution

$$\sum_k \delta(\tau \pm \sqrt{\lambda_k})$$

and that we finally have:

$$\tilde{N}(\tau) = \frac{1}{2\pi} \int_0^\tau d\tau \int_{-\infty}^\infty e^{it\tau} \text{Trace}(\cos t\sqrt{L} dt).$$

The easiest object to study is the heat core and we will prove below we have for  $t > 0$  and close to zero an asymptotic expansion of the following form

$$(1.158) \quad \sum_k e^{-t\lambda_k} = (4\pi t)^{-n/2} \sum_{0 \leq k \leq n} a_k t^{\frac{k}{2}} + O(1).$$

The terms of this expansion have an interpretation geometric interpretation. In particular in the case of the "flat" Laplacian on a bounded open set of  $\mathbb{R}^n$ , the first coefficient is none other than the volume of  $\Omega$ , the second is proportional to the surface of the boundary etc... Moreover for a manifold without boundary only are present in (1.158) the even powers of  $k$ .

The formula (1.158) is obtained by very direct calculations (cf MacKean and Singer [72]) which can also be interpreted as the use of pseudo-differential operators (cf. Taylor [T] paragraph 8.3 ).

The Tauberian theorem of Karamata, [88], [53] (Theorem 11.2) then allows to obtain the Weyl estimate

$$N(\lambda) \sim \frac{a_0}{\Gamma(\frac{n}{2} + 1)(4\pi)^{n/2}},$$

If the enumeration function  $N(\lambda)$  had for  $\lambda \rightarrow \infty$  an expansion in powers:

$$(1.159) \quad N(\lambda) = b_0 \lambda^{s_0} + b_1 \lambda^{s_1} \dots + b_p \lambda^{s_p} + \lambda^{s_p} (1 + o(\lambda)), \quad s_0 > s_1 > \dots > 0$$

then it would be the same for its Laplace transform, (it is the object of a very simple theorem called abelian theorem (Theorem 11.1) the proof of which is given below), for  $t \rightarrow 0_+$  and the coefficients  $b_l$  are deduced from the coefficients  $a_l$  of the expansion of the Laplace transform by a simple identification according to the formula:

$$(1.160) \quad \sum_{k \geq 0} e^{-\lambda_k} = \sum_{0 \leq l \leq (p-1)} \frac{b_l \Gamma(s_l - 1)}{t^{s_l}} + O\left(\frac{1}{t^{s_l}}\right).$$

The pathologies of the asymptotics come from the the fact that the function  $N(\lambda)$  is in general not, up to a suitable order, a sum of powers, it contains oscillating terms which are due to geometric effects geometrical effects non local effects. This is why the trace of the wave operator is necessary. wave operator is necessary.



To finish this introduction we prove the abelian theorem and the Tauberian theorem of Karamata's Tauberian theorem and we give two results allowing to use the Fourier transform of the distribution

$$\sum_k \delta(\tau \pm \sqrt{\lambda_k}).$$

It is worth noting that the last of these statements contains information more precise information than those which use only the Laplace transform.

**THEOREM 11.1.** (*Abelian theorem*) Let  $d\mu$  be a Borel measure on  $[0, \infty)$  having the following asymptotic behavior:

$$(1.161) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-\gamma} \mu[0, \lambda) = C, \quad \text{with } \gamma \geq 0.$$

Then we have

$$\lim_{t \rightarrow 0^+} t^\gamma \int_0^\infty e^{-\lambda} d\mu(\lambda) = C\Gamma(\gamma + 1).$$

Demonstration. We introduce the function

$$G(\lambda) = (\lambda + 1)^{-\gamma} \int_0^\lambda d\mu \equiv (\lambda + 1)^{-\gamma} F(\lambda)$$

By hypothesis  $G(\lambda)$  is a uniformly bounded function on  $[0, \infty)$  which tends to  $C$  when  $\lambda$  tends to infinity. An integration by parts in the sense of Stieltjes then gives

$$\begin{aligned} t^\gamma \int_0^\infty e^{-t\lambda} d\mu(\lambda) &= t^{\gamma+1} \int_0^\infty e^{-t\lambda} F(\lambda) d\lambda \\ &= t^{\gamma+1} \int_0^\infty e^{-t\lambda} (\lambda + 1)^\gamma G(\lambda) d\lambda \\ &= \int_0^\infty e^{-y} (y + t)^\gamma G\left(\frac{y}{t}\right) dy. \end{aligned}$$

For  $t1$ ,

$$e^{-y} (y + t)^\gamma G\left(\frac{y}{t}\right)$$

is uniformly bounded in  $L^1(\mathbf{R}_y^+)$  and for  $t$  tending to zero,  $(y + t)^\gamma G\left(\frac{y}{t}\right)$  converges simply to  $Cy^\gamma$ . Thus (1.161) can be deduced from the dominated convergence.

We notice that the theorem 11.1 can be applied to the second member of (1.159) and that it leads to the identification of the terms of this second member according to the power behavior of the Laplace transform. But as the Laplace transform smoothes the functions, the reciprocal theorem reciprocal theorem is much less precise, in the absence of the hypothesis concerning the expansion in powers. It is a little less trivial and it is the famous

**THEOREM 11.2.** (*Tauberian theorem*). For any  $\mu$  measure on  $[0, \infty)$  the relation

$$(1.162) \quad \int_0^\infty e^{-t\lambda} d\mu(\lambda) \sim t^{-s}, \quad s \geq 0, \quad t \rightarrow 0_+$$

implies for  $\lambda \rightarrow \infty$  the relation:

$$(1.163) \quad \int_0^\lambda d\mu(k) \sim \frac{\lambda^s}{\Gamma(s + 1)}.$$

**Demonstration.**

We use the transformation which, to any measure  $\sigma$ , associates the measure  $\sigma_t$  defined by the formula

$$\sigma_t(A) = t^\sigma (t^{-1}A).$$

This transformation leaves invariant the measure  $d\nu = k^{s-1} dk$  and the conclusion (1.163) is reformulated according to the relation:

$$\lim_{t \rightarrow 0} \mu_t[0, 1) = \frac{1}{\Gamma(s)} \nu([0, 1)).$$

It is therefore sufficient to prove the stronger relationship:

$$\lim_{t \rightarrow 0^+} \int f(x) d\mu_t(x) = \frac{1}{\Gamma(s)} \int f(x) d\nu(x) \quad \forall f \in C_0^\infty[0, \infty).$$

The assumption (1.162 (positivity plays an important role) implies that the family of measures  $e^{-t}$  of  $\mu_t$  is uniformly bounded. Thus it suffices to prove (1.160) for  $f$  belonging to a dense subspace of continuous functions tending to zero at infinity. Finally, we use the density of polynomials in  $e^{-x}$  (Stone Weierstrass!) for for which an exact calculation is immediate. Finally, here are the two statements involving not the Laplace transform but a Fourier analysis.

**PROPOSITION 11.1.** *Let  $S(t)$  be a uniformly finite order distribution on  $R_t$ . We assume that zero is a point isolated from its singular support and that after localization (by a function  $\theta$  equal to 1 in the neighborhood of zero and not meeting the singular support support of  $S(t)$  outside zero) we have:*

$$(\hat{\theta}S)(\tau) \simeq \left(\frac{1}{2\pi}\right)^n |\tau|^{(n-1)} \sum_{k \geq 0} p^k |\tau|^{-k}.$$

Then for  $t \rightarrow 0$  we have

$$(1.164) \quad \langle e^{-t\tau^2} \hat{S}(\tau) \rangle \simeq \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}} \sum_{0 \leq k \leq n} a_k t^{\frac{k}{2}} + O(1).$$

with

$$(1.165) \quad a_k = \left(\frac{1}{2}\right)^{\frac{n}{2}} \Gamma(n-k) p_k.$$

**Demonstration.** So we introduce the function  $\theta(t)$  and we decompose  $S(t)$ . and we decompose  $S(t)$  into the sum of two distributions  $S_1(t) = \theta(t)S(t)$  and  $S_2(t) = (1 - \theta(t))S(t)$ , the first one is located in the the first is localized in the neighborhood of zero, its Fourier transform transform is therefore a regular function which, for  $\tau$  tending to infinity, has an asymptotic behavior asymptotic behavior given by

$$(\hat{\theta}S)(\tau) \simeq \left(\frac{1}{2\pi}\right)^{n-1} |\tau|^{n-1} \sum_{k \geq 0} p^k |\tau|^{-k}$$

Its contribution:

$$\left(\frac{1}{2\pi}\right)^n \int_0^\infty e^{-\tau^2 t} |\tau| \sum_{k \geq 0} p^k |\tau|^{-k} d\tau$$

provides by an obvious calculation (of the abelian theorem type) the relation (1.164) provided that it is established that the contribution of  $S_2(t)$  in the first member of (1.164) is negligible; and indeed indeed we have according to Plancherel

$$\int_{-\infty}^\infty e^{-\tau^2 t} S_2(\tau) d\tau = \sqrt{2\pi} \langle \frac{1}{4t} e^{-\frac{s^2}{4t}}, S_2(s) \rangle, .$$

Now, since  $S_2(s)$  is uniformly of finite order (derivative at most  $m$  times of a continuous bounded function) and since its support does not meet 0, we have

$$\langle \frac{1}{4t} e^{-\frac{s^2}{4t}}, S_2(t) \rangle \leq e^{-\frac{c^2}{t}},$$

which completes the proof of the theorem 11.1. The following statement allows to specify the asymptotic behavior of a measure  $d\mu$  whose Fourier transform is known in a neighborhood  $I$  of zero. To filter we introduce a truncation function (in Fourier)  $\phi$  constructed in the following way. in the following way. We choose  $\hat{\psi} \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$  of norm  $L^2$  equal to 1 and we pose

$$\phi = |\psi|^2 \star |\psi|^2, \quad \phi_a(\tau) = \frac{1}{a} \phi\left(\frac{\tau}{a}\right).$$

Of course the support of  $\hat{\psi}$  is contained in the interval  $] -1, 1[$  and the one of  $\hat{\phi}_a$  in the interval  $]-\frac{1}{a}, \frac{1}{a}[$ . Finally  $\phi$  is a function strictly positive on all  $\mathbb{R}$ . We have the theorem (Lemma 17.5.6 volume III. p 50 of Hörmander [48])

**THEOREM 11.3.** *Let  $\mu$  be a function with tempered growth verifying  $\mu(0) = 0$ . Let on the other hand  $\nu$  be a function locally with bounded variation with  $\nu(0) = 0$ .*

*We suppose that these two functions verify the following estimates*

$$(1.166) \quad \begin{aligned} |d\nu(\tau)| &\leq M_0(|\tau| + a_0)^{n-1} d\tau \\ |(d\mu - d\nu) \star \phi_a(\tau)| &\leq M_1(|\tau| + a_1)^\kappa \end{aligned}$$

with  $\kappa \in [0, n-1]$  and  $a \leq a_0, a \leq a_1$ . Then

$$|\mu(\tau) - \nu(\tau)| \leq C(aM_0(|\tau| + a_0)^{n-1} + M_1(|\tau| + a_1)^\kappa)$$

where  $C$  is a constant that depends only on  $\kappa$  and  $n$ .

**Demonstration** Since  $\phi(\tau)$  is strictly positive on  $\mathbb{R}$ , there exists a constant  $c_0 > 0$  such that, on the interval  $] -\frac{1}{2}, \frac{1}{2}[$  we have  $\phi > c_0$  and so it comes:

$$(1.167) \quad c_0 a^{-1} \int_{\tau-\frac{1}{2}a}^{\tau+\frac{1}{2}a} d\mu \leq d\mu \star \phi_a(\tau) \leq C(aM_0(|\tau| + a_0)^{n-1} + M_1(|\tau| + a_1)^\kappa)$$

indeed it is enough to increase appropriately  $|d\mu \star \phi_a(\tau)|$  which which, with (1.166), results from the relations:

$$|d\mu \star \phi_a(\tau)|(d\mu - d\nu) \star \phi_a(\tau)| + |d\nu \star \phi_a(\tau)| \leq C(aM_0(|\tau| + a_0)^{n-1} + M_1(|\tau| + a_1)^\kappa).$$

By dividing the interval  $(0, s)$  into  $|s|+1$  intervals of length less than 1 we deduce from (1.167) the relation

$$(1.168) \quad |\mu(\tau) - \mu(\tau - as)| C(a(|s|+1)M_0(|\tau| + a_0 + a|s|)^{n-1} + M_1(|\tau| + a_1 + a|s|)^\kappa)$$

Multiplying (1.168) by  $\phi(s)$  and integrating we obtain the essential relationship:

$$|\mu(\tau) - \mu \star \phi_a(\tau)| \leq C(aM_0(|\tau| + a_0)^{n-1} + M_1(|\tau| + a_1)^\kappa)$$

and the demonstration ends by using again the second relation of (1.166).

## 2. Trace of the heat kernel.

The operator  $L$  is (with a suitable definition of its domain) the generator of a strongly continuous semigroup in  $L^2(\Omega)$  and by the kernel theorem we have kernels we have:

$$e^{-tL} f(x) = \int_{\Omega} K(t, x, y) f(y) dy.$$

It also follows from the regularity of the solutions of the heat equation heat that, for all  $t > 0$ , this kernel is a bounded function. On the other hand the decomposition spectral decomposition of  $L$  (with boundary conditions if it is a bounded domain with boundary) leads to the formula:

$$(e^{-tL} f)(x) = \int_{\Omega} K(t, x, y) f(y) dv(y) = \sum_k e^{-t\lambda_k} w_k(x) \int_{\Omega} w_k(y) f(y) dv(y)$$

From the classical theorems on the regularity of solutions of elliptic problems, it follows that the series

$$\sum_k e^{-t\lambda_k} |w_k(x)|^2, \quad x \in \bar{\Omega}$$

$$(2.169) \quad \sum_k e^{-t\lambda_k} = \sum_k e^{-t\lambda_k} \int_{\Omega} |w_k(x)|^2 dv(x) = \int_{\Omega} K(t, x, x) dx = \text{Trace } e^{-tL}$$

are (for all  $t > 0$ , are but not for  $t = 0$ ) convergent and in particular the kernel of the heat operator is a with trace.

We start by treating an elementary case.

**2.1. The Weyl asymptotics for the Dirichlet problem with constant coefficients.** We consider the eigenfunctions and eigenvalues of the Laplacian “flat” We consider the eigenfunctions and eigenvalues of the “flat Laplacian” in a bounded open set  $\Omega$  with Dirichlet conditions:

$$-\Delta w_k = \lambda_k w_k, \quad w_k(x)|_{\partial\Omega} = 0.$$

We introduce the heat kernel in the whole space:

$$E_0(t, x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

For  $t \geq 0$ ,  $x$  and  $y$  in  $\Omega$  this distribution is (with respect to the variables  $x, t$ ) solution of the heat equation; it verifies the same initial conditions but of moreover on the boundary it is strictly positive so, according to the principle of the maximum, it majors the kernel of the heat semigroup with Dirichlet condition according to the formula

$$0 < K(t, x, y) < E_0(t, x, y), \quad \forall (t, x, y) \in \mathbb{R}_+ \times \Omega \times \Omega.$$

From this equation we immediately deduce the relation:

$$\int_{\Omega} K(t, x, x) dx \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \text{vol}(\Omega).$$

Then for any fixed  $\epsilon > 0$  we introduce an open set  $\mathcal{O}$  which approximates  $\Omega$  to within  $\epsilon > 0$  by the interior:

$$\Omega \subset \bar{\mathcal{O}} \subset \Omega, \text{vol}(\Omega - \mathcal{O}) \leq \epsilon$$

Let  $\delta(y)$  be the distance from a point  $y$  to  $\partial\Omega$  and  $\delta = \delta\mathcal{O}$  the distance from  $\mathcal{O}$  to  $\Omega$  and we observe that we have (always according to the principle of the maximum):

$$(2.170) \quad 0 < E_0(t, x, y) - K(t, x, y) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta(y)^2}{4t}}.$$

By studying the behavior of the function

$$s \rightarrow \frac{1}{(4t)^{\frac{n}{2}}} e^{-\frac{\delta^2}{4s}}$$

and by choosing  $t > 0$  small enough ( $\mathcal{O}$  and  $\delta$  fixed,  $t \leq \frac{\delta^2}{2n}$ ) we derive the formula

$$0 < \int_{\mathcal{O}} E_0(t, x, x) dx - \int_{\mathcal{O}} K(t, x, x) dx \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta^2}{4t}}$$

This gives

$$\lim_{t \rightarrow 0_+} (4\pi t)^{\frac{n}{2}} \int_{\mathcal{O}} K(t, x, x) dx = \lim_{t \rightarrow 0_+} (4\pi t)^{\frac{n}{2}} \int_{\mathcal{O}} E_0(t, x, x) dx$$

Reusing (2.170) we conclude that we have:

$$\text{Vol}(\Omega) \geq (4\pi t)^{\frac{n}{2}} \lim_{t \rightarrow 0_+} \int_{\mathcal{O}} K(t, x, x) dx \geq \text{Vol}(\Omega) - \epsilon$$

With the relation (2.169) and the Theorem 11.2 (Theorem tauberian) we obtain the Weyl estimate:

**THEOREM 11.4.** (Weyl estimate) *The asymptotic behavior of the eigenvalue enumeration function eigenvalues of the “flat” Laplacian in a bounded open set is “to first order”. given by the formulas:*

$$(2.171) \quad N(\lambda) = \#\{(\lambda_k \leq \lambda)\} \simeq \frac{\text{Vol}(\mathcal{O})}{\Gamma(\frac{n}{2}+1)(4\pi)^{\frac{n}{2}}} \lambda^{\frac{n}{2}}, \quad \lambda_k \sim \frac{4\pi}{(\text{Vol}(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}},$$

**Note 1** The volume of the unit ball in  $\mathbb{R}^n$  is given by the formula (cf. Schwartz Mathematical Methods of Physics page 350)

$$C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

so the Weyl asymptotic also takes the form:

$$(2.172) \quad N(\lambda) \sim \text{Vol}(\Omega) \frac{C_n}{(2\pi)^n} \lambda^{\frac{n}{2}} = \frac{1}{(2\pi)^n} \int_{\Omega} \int_{(|\xi|^2 \leq \lambda)} dx d\xi$$

formula which will be generalized in the following paragraphs.

**2.2. The method of frozen coefficients and Lévy sums.** To systematize the results on the trace of the heat kernel we proceed in two steps.

First, we generalize to a problem with variable coefficients variable on a manifold without boundary,

then we treat the influence of the boundary.

The calculations made in Mac Kean and Singer [72] and that we reproduce here are essentially explicit. They can be well be interpreted in terms of pseudodifferential operators which the reader will find in Taylor [94](volume 2 page 55). We notice that as the heat kernel is regularizing heat is regularizing the analysis is local and therefore the recourse to Fourier integral operators is not contrary to what will be explained in the section 3 necessary.

Following the notations of the introduction we consider the kernel of heat operator for a compact manifold without boundary. In local coordinates it is written as

$$(2.173) \quad e^{tL} f = \int_{\Omega} K(t, x, y) f(y) dv(y)$$

Of course the kernel  $K(t, x, y)$  is symmetric in  $(x, y)$  and positive.

Finally, as this is both a step in the demonstration and a tool for the continuation (thanks to the Theorem 11.3) we study the series:

$$(2.174) \quad \sum_k e^{-\lambda_k t} |w_k(x)|^2 = K(t, x, x), \quad x \in \Omega$$

keeping in mind that it is a convergent series and that we have:

$$(2.175) \quad \sum_k e^{-\lambda_k t} = \int_{\Omega} K(t, x, x) dv(x).$$

For  $x \neq y$ ,  $K(t, x, y)$  is exponentially decreasing in  $\frac{1}{t}$  for  $t$  tending to zero. Thus for the asymptotics of the second member of (2.174) we use local coordinates and the formula (1.155):

$$Lu = -\Delta u = -\frac{1}{\sqrt{\det g}} \sum_{ij} \partial_i (a^{ij}(x) \sqrt{\det g} \partial_j u)$$

We can extend the operator to  $\mathbb{R}^n$  by assuming that outside a ball of suitable size it coincides with the usual Laplacian, then we then "freeze" the coefficients at the point  $y$  which leads to the equation parabolic equation with constant coefficients:

$$\partial_t u = \text{div}_x (A(y) \text{grad}_x u) = L_y u, \quad u(x, 0) = f(x)$$

whose elementary solution is given (make a change of variables) by

$$u(t, x) = (4\pi t)^{-n/2} \int e^{-\frac{(g(y)(x-z), (x-z))}{4t}} \sqrt{\det g(y)} f(z) dz.$$

So we pose

$$(2.176) \quad K_y(t, x, z) = (4\pi t)^{-n/2} e^{-\frac{(g(y)(x-z), (x-z))}{4t}}$$

On the one hand we have:

$$\begin{aligned}
& \int K(t, x, z) \sqrt{\det g(z)} K_y(0, z, y) \sqrt{\det g(y)} dz \\
&= K(t, x, y) \sqrt{\det g(y)} \int K_y(t, x, z) \sqrt{\det g(y)} K(0, z, y) \sqrt{\det g(z)} dz \\
&= K_y(t, x, y) \sqrt{\det g(y)}
\end{aligned}$$

and on the other hand we also have:

$$(2.177) \quad \begin{aligned} & K(t, x, y) \sqrt{\det g(y)} - K_y(t, x, y) \sqrt{\det g(y)} \\ &= \int_0^t ds \int_{\mathbf{R}^n} K(s, x, z) \sqrt{\det g(z)} K_y((t-s), z, y) \sqrt{\det g(y)} dz \end{aligned}$$

By simplifying (2.177) by  $\sqrt{\det g(y)}$  and using the properties of elementary solutions we finally get

$$(2.178) \quad \begin{aligned} K(t, x, y) - K_y(t, x, y) &= \int_0^t ds \int_{\mathbf{R}^n} L(K(s, x, z)) \sqrt{\det g(z)} K_y((t-s), z, y) dz \\ &\quad - \int_0^t ds \int_{\mathbf{R}^n} K(s, x, z) \sqrt{\det g(z)} L_y K_y((t-s), z, y) dz \end{aligned}$$

The Laplace Beltrami operator being self-adjoint for the volume form  $\sqrt{\det g(z)} dz$  we can rewrite the equation 2.178 in the form:

$$(2.179) \quad K(t, x, y) - K_y(t, x, y) = \int_0^t ds \int_{\mathbf{R}^n} K(s, x, z) (L - L_y) (K_y((t-s), z, y)) \sqrt{\det g(z)} dz.$$

We introduce the following notations:

$$f(t, x, y) = (L - L_y) K_y(t, x, y)$$

and

$$(g \# h)(t, x, y) = \int_{\mathbf{R}^n} \int_0^t g(s, x, z) h(t-s, z, y) \sqrt{\det g(z)} dz ds$$

which allow us to rewrite (2.177) in the following form

$$K(t, x, y) = K_y(t, x, y) + K \# f$$

We denote by  $f^{\#k}(t, x, y)$  the sequence of functions defined by the recurrence:  $f^k(t, x, y)$ .

$$f^{\#1} = f, f^{\#(k+1)} = K \# f^{\#k}.$$

In the remainder of this construction we denote by  $c$  and  $d$  different constants independent of  $(t, x, y)$  and we observe that  $f$  verifies the estimate:

$$(2.180) \quad |f(t, x, y)| \leq c \left( \frac{|x-y|^3}{t^2} + \frac{|x-y|}{t} \right) t^{-n/2} \exp(-d \frac{|x-y|^2}{t})$$

and we deduce the markup:

$$|f^{\#k}(t, x, y)| \leq \frac{c^k}{\Gamma(\frac{n}{2} + 1)} t^{\frac{k-n}{2}} \exp(-d \frac{|x-y|^2}{t})$$

The series (called Lévy's Sum)

$$(2.181) \quad S(t, x, y) = \sum_{k \geq 1} f^{\#k}(t, x, y)$$

converges uniformly on any compact of  $\mathbf{R}_t^+ \times \mathbf{R}^{2n}$  and  $S_N(t)$ , its sum at order  $N$  verifies the relation:

$$K = K_y + S_N + K f^{\#(N+1)},$$

i.e. by making  $N$  tend to infinity, we obtain the uniqueness of the obtain the uniqueness of the kernel for the solution of the heat equation. We finally the relation:

$$K(t, x, y) = K_y(t, x, y) \sum_{k \geq 1} K_y \# f \# f \# \dots \# f \#, k \text{ times}$$

We summarize the above calculation in the

PROPOSITION 11.2. *Let  $L$  be the Beltrami Laplace operator associated to a to an bound-aryless manifold  $\Omega$  then the kernel of the elementary solution is in local coordinates given by the series:*

$$(2.182) \quad K(t, x, y) = (4\pi t)^{-\frac{n}{2}} e^{-(g(y)(x-z), (x-z))} (1 + t^{\frac{1}{2}} p_1(t, x, y) + t p_2(t, x, y) + \dots + t^{\frac{k}{2}} p_k(t, x, y) \dots)$$

avec

$$|p_k(t, x, y)| \leq \frac{e^k}{\Gamma[\frac{n}{2}+1]}$$

Explaining the asymptotic expansion of the second member of (2.182) we obtain (cf. Minakshisundaram and Pleijel [79] for a complete demonstration) the formula

$$(2.183) \quad \sum_{k \geq 0} e^{-\lambda_k t} |w_k(x)|^2 = K(t, x, x) = (4\pi t)^{-\frac{n}{2}} a_k(x) t^{\frac{k}{2}}$$

The following steps of the analysis are then on the one hand the calculation of coefficients

$$a_k = \int_{\Omega} a_k(x) dv(x)$$

intervening in the expansion of the trace of the Laplacian and on the other hand the generalization of this formula to an open set without boundary.

An essential remark due to Mac Kean and Singer is that these coefficients depend only on intrinsic geometric objects intrinsic geometric objects and therefore that we have means of calculation which are also intrinsic. More precisely,  $p_{2k}(g)$  is a homogeneous polynomial in the metric  $g$  and in its covariant derivatives derivatives, associated to a  $2k$ -form differential form. As the computation is local it is done in the neighborhood of a point  $x$  chosen to simplify equal to zero. With geodesic coordinates, we see that the matrix  $g(x)$  develops as a polynomial function of the curvature tensor curvature tensor  $R$  and its covariant derivatives. covariant derivatives. This tensor is defined by the first order expansion according to the Taylor formula:

$$(2.184) \quad g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ij}^{kl} x_k x_l + O(|x|^3)$$

We then observe that  $a_k$  is a homogeneous polynomial of degree  $k$  in  $R$  and in its covariant derivatives, it follows, with the symmetries of the Laplacian that the odd order coefficients are zero.

We then introduce the functions (which are intrinsic on  $\Omega$  i.e. i.e. invariant by change of Riemannian coordinates):

$$K = - \sum_{i < j} R_{ij}^{ij}$$

and

$$(2.185) \quad \begin{aligned} A &= (\sum_{i < j} R_{ij}^{ij})^2 = K^2 \\ B &= \sum_{j,k} (\sum_i R_{ij}^{ik})^2 \\ C &= \sum_{ijkl} (R_{ij}^{kl})^2 \end{aligned}$$

We note, for a manifold with or without boundary (in the framework of a manifold without boundary we do not define  $\mathcal{S}$ ) and  $\mathcal{J}$ ):

$$\begin{aligned} \mathcal{V} &= \int_{\Omega} dv(x) \text{ the Riemannian volume of } \Omega \\ \mathcal{S} &= \int_{\partial\Omega} d\sigma \text{ the Riemannian surface of } \partial\Omega \\ \mathcal{K} &= \int_{\partial\partial\Omega} K(x) dv(x) \text{ the integral curvature of } \Omega \\ \mathcal{J} &= \int_{\partial\partial\Omega} J(\sigma) d\sigma \text{ the average curvature integrated of } \partial\Omega \end{aligned}$$

and we have the

THEOREM 11.5. *Let  $\Omega$  be a compact Riemannian manifold without boundary, of dimension  $n$  and  $L$  the corresponding Beltrami Laplace operator then we have the following trace formula:*

$$(2.186) \quad \begin{aligned} \sum_k e^{-\lambda_k t} &= \int_{\Omega} K^d(t, x, x) dv(x) \\ &= \frac{(4\pi t)^{-\frac{n}{2}} \{ \mathcal{V} + \frac{t}{3} \mathcal{K} + t^2 \\ &\quad \frac{1}{180} \int_{\Omega} (10A - B + 2C) dv(x) + O(t^3) \}}{180 \int_{\Omega} (10A - B + 2C) dv(x) + O(t^3)} \end{aligned}$$

COROLLARY 11.1. *If  $\Omega$  is a compact manifold of dimension 2 the formula (2.186) is rewritten:*

$$(2.187) \quad \sum_k e^{-\lambda_k t} = \int_{\Omega} K^d(t, x, x) dV_g(x) = \frac{\text{The surface}}{4\pi t} + \frac{E}{6} + \frac{\pi t}{60} \int_{\Omega} K^2 dV_g(x)$$

where  $E$  denotes the Euler characteristic [26], equal to  $\frac{1}{2\pi} \int_M K$ .

**Comments and Demonstration** The formula (2.187) is deduced from the formula (2.186) with changes of notation and arguments of geometry arguments. First we speak of surface instead of volume, then we use the Gauss-Bonnet formula:

$$E = \frac{1}{2\pi} \int_{\Omega} \mathcal{K} dV_g(x)$$

and the relation (specific to the 2 dimension):

$$10A - B + 2C = 12K^2$$

The zero order term is calculated directly from the first term of the Lévy series (2.182). We will indicate the calculation of the term in  $t$  and for the term in  $t^2$  we refer to [72]. If we change  $g$  into  $C^2$ , the operator  $L$  is changed into the operator  $C^{-2}L$ , so the series

$$\sum_{k \geq 0} e^{-\lambda_k t} |w_k(x)|^2$$

is changed into the series

$$\sum_{k \geq 0} e^{-\lambda_k \frac{t}{C^2}} |w_k(x)|^2.$$

Thus the coefficients  $a_{2k}$  (the only non-zero ones) are multiplied by  $C^{2k}$ . On the other hand any covariant derivative of order  $l$  of  $R(C^2)g$  is a multiple of  $C^{2l+2}$ . Consequently,  $a_{2k}$  is a homogeneous polynomial of degree  $2k$  in  $R$  and its derivatives covariant, if we agree to assign to a covariant derivative of order  $l$  the degree  $2+l$ . In particular  $a_2$  is a form of degree 1 in  $\mathbb{R}$  while  $a_4$  is a form of degree 2 in  $\mathbb{R}$  plus a form of degree 1 in the covariant derivatives of order 2 of  $\mathbb{R}$ . The coefficients of these expressions depend on  $\Omega$  only through the dimension.

Then we use the fact that the Laplacian commutes with transformations. It is therefore the same for the coefficients of the trace formulas and it follows that  $a_2$  forms invariant form of degree 1 is proportional (H. Weyl [99]) to

$$K(x) = - \sum_{i < j} R_{ij}^{ij}(x).$$

The calculation of the coefficients is done by looking at the sphere  $\Omega = S^2$  and product manifolds.

### 2.3. The calculation of the first terms in the Dirichlet problem.

THEOREM 11.6. *Let  $\Omega$  be a compact Riemannian manifold with boundary, of dimension  $n$  and  $L$  the corresponding Beltrami Laplace operator with Dirichlet condition  $u = 0$  on  $\partial\Omega$ . Then we have the following trace formula:*

$$\sum_k e^{-\lambda_k t} = \int_{\Omega} K^d(t, x, x) dv(x) = (4\pi t)^{-\frac{n}{2}} \left\{ \mathcal{V} - \frac{1}{4} \sqrt{4\pi t} \mathcal{S} + \frac{t}{3} \mathcal{K} - \frac{t}{6} \mathcal{J} + o(t^{\frac{3}{2}}) \right\}$$

**Remark 2** As in dimension 2 we have:

$$\int_{\partial\Omega} J d\sigma = 2(1 - h)$$

with  $h$  designating the number of “holes” of the open set (connected but possibly not simply connected)  $\Omega$ . Thus the asymptotic behavior of the eigenvalues  $\lambda_k$  for  $k \rightarrow \infty$  contain all the information (use the Abelian theorem) about the asymptotic behavior for  $t \rightarrow 0$



of  $k \rightarrow e^{-\lambda_k t}$  and thus allows to know the size of the domain, the surface of its boundary and in dimension 2 the number of “holes”.

**Remark 3** In all cases we find for the first term the expression

$$(4\pi t)^{-\frac{n}{2}} \mathcal{V}$$

Which with the Tauberian theorem gives:

$$(2.188) \quad N(\lambda) \sim \frac{\lambda^{n/2}}{(4\pi)^{n/2} \Gamma(n/2+1)} \mathcal{V}$$

or, by introducing in the calculation the volume  $C_{n-1}$  of the unit ball of  $\mathbb{R}^{n-1}$ ,

$$(2.189) \quad N(\lambda) \sim \left(\frac{1}{2\pi}\right)^n \int_{(A(x), \xi, \lambda)} dx d\xi$$

This formula will be found in the section 3. The structure will be explained.

**Scheme of the proof of the Theorem 11.6**

We start by computing  $K^d(t, x, x)$  for  $x \in \Omega'$  ( $\bar{\Omega}' \subset \Omega$ ). The influence of the boundary is for  $t$  tending to zero in  $exp(-c/t)$  and so the methods of theorem 11.5 apply and we have:

$$(4t) \int_{\Omega'} K(t, x, x) \sqrt{\det g(x)} dx = \int_{\Omega'} [1 + \frac{t}{3} \mathcal{K}] + O(t^2)$$

Then we use in the neighborhood of each point of  $\partial\Omega$  a system of local maps local maps which transforms the open set  $\Omega$  into the open  $x_n > 0$  and we introduce a covering of  $\partial\Omega$  by open set  $U_j$  of  $\mathbb{R}^n$  and we propose to estimate:

$$(4\pi t) \int_{\Omega \cap U} K(t, x, x) \sqrt{\det g(x)} .dx$$

On  $U$  we introduce an involution  $I : U \rightarrow U$  which keeps the Riemannian structure and leaves  $\partial\Omega$  invariant. We observe that the heat kernel with Dirichlet conditions on  $\partial\Omega \cap U$  is given by

$$K^d(t, x, y) = \tilde{K}(t, x, y) - \tilde{K}(t, x, Iy)$$

where  $\tilde{K}(t, x, y)$  denotes the restriction to  $U \times U$  of the kernel of the heat. We have in the neighborhood of the boundary:

$$(2.190) \quad K^d(t, x, x) = \tilde{K}(t, x, x) - \tilde{K}(t, x, Ix)$$

The two terms of the second member of (2.190) are computed by Lévy sum (2.182) in power of  $t^{\frac{1}{2}}$ . We again use the Taylor expansion of  $g(x)$  in terms of the Ricci tensor and its covariant derivatives, this allows us to improve the estimate (2.180) and to prove that we have:

$$(2.191) \quad |f(t, x, y)| \leq c \left( \frac{|x||x-y|^3}{t^2} + \frac{|x||x-y|}{t} + 1 \right) t^{-n/2} \exp(-d \frac{|x-y|^2}{t})$$

and we deduce the markup:

$$(2.192) \quad |K \# \sum_{k \geq 2} f^{\#k}(t, x, y)| \leq ct^{2-\frac{n}{2}}$$

Thus to have the asymptotic behavior up to and including order inclusive, it is sufficient to consider the following terms extracted from (2.192) by sums of Lévy sums:

$$(4\pi t)^{n/2} \int_{\Omega \cap U} e^0(t, x, Ix) dv(x)$$

and

$$(4\pi t)^{n/2} \int_{\Omega \cap U} e^0 f(t, x, Ix) dV_g(x)$$

We use of course the change of semi-geodesic variable by noting  $\partial_n$  the derivative along the exterior normal to  $\Omega$ . The calculations are done in a difficult but completely explicit way by replacing  $g$  by by its Taylor expansion in terms of the Ricci tensor  $R$ .

$$(4\pi t)^{n/2} \int_{U \cap \Omega} e^0(t, x, Ix) dv(x) = \frac{1}{4} \sqrt{4\pi t} \int_{U \cap \partial\Omega} \frac{\partial}{\partial n} \{g_{nnn} \det g\} / \det g dx' + o(t) \text{ volume of } U$$

$$(4\pi t)^{n/2} \int_{U \cap \Omega} e^0(t, x, Ix) dV_g(x) = -\frac{t}{6} \int_{U \cap \Omega} \frac{\frac{\partial}{\partial n} \{g^{nn} \det g\}}{\det g} \sqrt{g_{nn}} + o(t \times \text{vol}U)$$

and

$$(4\pi t)^{n/2} \int_{U \cap \Omega} e^0 \# f(t, x, Ix) dV_g(x) + o(t \times \text{vol}U).$$

**Note 4** To conclude this section we can recall again the meaning” of the results obtained; the information on the eigenvalues provides by the Laplace transform of their enumeration function, precise information about the geometry of the domain: volume, surface, number of holes, geodesic curvature, but inversely the asymptotic behavior asymptotic behavior of the trace of the heat operator allows us to know the first term of the of the asymptotic expansion of the function  $N(\lambda)$ ; it would allow to know other terms if we knew that terms if we knew that this function has an expansion in power of but as it will be illustrated below illustrated below this is generally not the case and so the most accurate most accurate results that we can hope for in the be expected in the general case for this enumeration function are limited to the limited to the first or at best to the first two terms as according to the statements of the corollary 11.2, theorem 11.8, corollary 11.3 and Theorem 11.9 which follow.

### 3. Influence of closed geodesics and of Fourier integral operators.

**3.1. Poisson’s formula and pathologies of the circle and the sphere.** The existence of an expansion of the form (1.159) up to the second term is by no means obvious, and even false in simple cases. Its non-existence is related to the accumulation of closed geodesics which as we will see below contribute to its non-rational character (in  $\frac{\lambda^{n-k}}{2}$ ). The first example is constructed in dimension 1 and is none other than the interpretation in this framework of Poisson’s formula.

$$(3.193) \quad \sum_{k \in \mathbf{Z}} e^{ikt} = 2\pi \sum_{k \in \mathbf{Z}} \delta(t - 2\pi k).$$

Indeed with  $\Omega = T = \mathbb{R}/(2\pi\mathbf{Z})$ , the eigenfunctions of the Laplacian are the complex exponentials  $e^{\pm ikx}$  and the corresponding eigenvalues are the numbers  $\lambda_k = k^2$ , (of multiplicity 2 for  $k \neq 0$ ). We find the relation:

$$N(\lambda) \sim 2\lambda^{\frac{1}{2}}$$

in perfect agreement with (1.161). But, as at the passage of each eigenvalue,  $N(\lambda)$  increases by 2, this function cannot admit an expansion of the form:

$$N(\lambda) = 2\lambda^{\frac{1}{2}} + a_1 + o(1).$$

On the other hand for the function  $\tilde{N}(\tau) = N(\tau^2)$  whose derivative is the distribution:

$$S(t) = \sum \delta(\tau - \sqrt{\lambda_k})$$

we have an even more explicit formula (denoting by  $\tilde{N}(x) = N(x^2)$ ):

$$(3.194) \quad N(\tau) = 2\tau + 2(E(\tau) - \tau).$$

Thus we observe that the second member of (3.194) is the sum of a linear function and a function ( $E(\tau)$  being the integer part of  $\tau$ ). The presence of this periodic function periodic function results in singularities in  $\tilde{N}(\tau)$  or in its derivative:  $\tilde{N}(\tau)$  is the or its derivative:

$$\frac{d}{d\tau} \tilde{N}(\tau) = \delta + 2 \sum_{k>0} \delta_k.$$

Extending by parity, to use the Fourier transformation, we have, according to the ”Poisson formula” (3.193):

$$(3.195) \quad \mathcal{F}\left(\sum_{-\infty}^{+\infty} \delta_{\tau-k}\right) = \sum_{-\infty}^{+\infty} e^{ikt} = 2\pi \sum_{-\infty}^{+\infty} \delta_{\tau-2\pi k}$$

It is then important to notice that the second term of (3.195) coincides with the distribution (3.196)

$$\text{Trace } \cos t\sqrt{-\Delta}.$$

And the formula (3.196) says in particular that the singular support of the Fourier transform of the distribution

$$\frac{d}{d\tau}\tilde{N}(\tau)$$

coincides with the lengths of the closed geodesics drawn on the circle (in this case it is not circle (in this case there is only one that we go through several times either in the positive direction or in the negative direction. the retrograde direction). This singular support translates the presence of oscillations in the asymptotic behavior of the function  $\tilde{N}(\tau)$ .

The second example is the sphere, to fix the ideas we consider it in dimension 3, (the observation being however valid in any any dimension) the Laplace Beltrami operator is then

$$Lu = -\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) + \frac{1}{\sin^2\theta}\partial_\phi^2 u\right]$$

whose eigenfunctions are the spherical harmonics:

$$Y_{k,l}(\theta, \phi) = e^{\pm ik\phi} P_l^k(\cos\theta), \quad 0 \leq l \leq k.$$

corresponding to the eigenvalues

$$\lambda_k = k(k+1)$$

so each eigenvalue has multiplicity  $2k+1$ . Because of this multiplicity we have:

$$N(\lambda_k) = \sum_{l=1}^{l=k} (2l+1) = k(k+1) + k = (k+1)^2$$

Here again we find the relation

$$N(\lambda) \sim \lambda$$

which is no other than (2.188) in dimension 2.

On the other hand at the passage of each eigenvalue  $N(\lambda_k)$  jumps from  $(2k+1)$

$$N(\lambda_{k+0}) - k(k+1) = k+1, N(\lambda_{k-0}) - k(k+1) = -k,$$

and it is impossible to have an expansion of the form:

$$N(\lambda) = (\lambda) + a_1\sqrt{(\lambda)} + o(\sqrt{(\lambda)})$$

**Note 5** The circle is indeed on the sphere of  $\mathbb{R}^2$ , the calculation made above extends to the sphere of  $\mathbb{R}^n$  for all  $n$ . Nevertheless we prefer here to separate the two cases because it is the 2 dimension which gives back the original Poisson formula. The  $\mathbb{R}^n$ -sphere is a manifold where all the closed geodesics have the same primitive length. We then say that the Hamiltonian is periodic and in this framework we can exhibit behaviors of the same nature. This is beyond the scope of this presentation and the reader can refer to the article by Duistermaat and Guillemin [33] or to volume III of Hörmander [48].

### 3.2. Closed geodesics and singularities of the trace of the wave operator.

. Generalizing these observations is an important application of operators integral Fourier operators in particular in particular because the contributions of closed geodesics require the use of global parametrization. This is the subject of the present paragraph where, for simplicity, we always consider the spectrum of the Laplace Beltrami operator, written in local coordinates in the following form:

$$Lu = \frac{1}{\sqrt{\det A^{-1}(x)}} \text{div}_x (A(x)\sqrt{\det A^{-1}(x)} \text{grad}_x u)$$

on a compact manifold in general without boundary  $\Omega$ .

Since the main symbol of the operator is  $(A(x)\xi, \xi)$ , the characteristic manifold of the wave operator

$$u \mapsto \partial_t^2 u - Lu$$

is defined by the relation:  $\mathcal{C} = \{\tau^2 = (A(x)\xi, \xi) \subset T^*(\mathbb{R} \times \Omega) \setminus 0\}$ , the Riemannian metric is defined by the matrix  $A^{-1}$  and the bicharacteristics are the applications

$$s \mapsto (t(s), \tau(s); y(s), \eta(s)) \in \mathcal{C}$$

solutions of the differential system

$$\begin{cases} \dot{t} = 2\tau, & \dot{\tau} = 0 \\ \dot{y} = -2A(x)\xi, & \dot{\eta} = \text{grad}_x(A(x)\xi, \xi) \\ (t, \tau; y, \eta)|_{s=0} = (0, \tau_0; x, \xi) \in \mathcal{C} \end{cases}$$

We have of course

$$\tau = \tau_0 = \pm\sqrt{(A(y)\eta, \eta)} = \pm\sqrt{(A(x)\xi, \xi)}$$

and we say that a bicharacteristic is periodic if its projection  $s \mapsto (x(s), y(s))$  is a periodic application. The running times of one or more loops are called the periods of the bicharacteristic. Let  $\mathcal{L}^+$  be the set of positive periods and let

$$\mathcal{L} = \mathcal{L}^+ \cup -\mathcal{L}^+ \cup 0, .$$

With the parameterization by  $s$  and the metric given by  $A^{-1}$ , the length of a closed geodesic of period  $t > 0$  is given by:

$$l = \int_0^{\frac{t}{2\tau_0}} (A^{-1}\dot{x}(s), \dot{x}(s))ds = \int_0^{\frac{t}{2\tau_0}} 4(A(x(s)\xi(s), \xi(s))ds = l = \int_0^{\frac{t}{2\tau_0}} 4\tau_0^2 ds = t.$$

As in the case of the heat kernel it is natural to consider both the sums both the sums

$$(3.197) \quad S(t, x) = 1 + \frac{1}{2} \sum_k e^{\pm i\sqrt{\lambda_k}t} |w_k(x)|^2$$

and

$$(3.198) \quad S(t) = \int_{\Omega} S(t, x) dv(x).$$

PROPOSITION 11.3. *The formulas (3.197) and (3.198) where the eigenvalues are counted with their multiplicity do define uniformly finite order distributions.*

**Demonstration:** We use the relationship,

$$\frac{d^p}{dt^p} \left( \frac{1}{(i\sqrt{\lambda_k})^p} e^{i\sqrt{\lambda_k}t} \right) |w_k(x)|^2 = e^{i\sqrt{\lambda_k}t} |w_k(x)|^2,$$

the fact that the norm in  $L^\infty(\Omega)$  of  $|w_k(x)|$  is increased by  $C\lambda_k^M$  with  $M$  suitable results from Sobolev's theorem and the relation:

$$-Lw_k(x) = \lambda_k w_k(x).$$

The convergence (uniform in  $t$  and  $x$ ), for  $p$  large enough of the series

$$\sum_k \frac{e^{i\sqrt{\lambda_k}t}}{(i\sqrt{\lambda_k})^p}$$

is therefore a consequence of the Weyl estimate (2.171) obtained previously.

To take into account the influence of the lengths of the closed geodesics we will need the Fourier operators operators and a generalization of the phase theorem stationary adapted to the evaluation of integrals whose phase oscillates on submanifold. Thus we introduce the:

PROPOSITION 11.4. *Theorem of the stationary phase on a sub manifold (Colin de Verdière [27]). Let  $Z$  be a Riemannian manifold of dimension  $d$ , let  $a \in C_0^\infty(Z)$  be a real-valued phase  $Z$  and let  $\Phi \in C^\infty(Z)$  be a real-valued phase  $Z$ . and let  $\Phi \in C^\infty(Z)$ . We assume that the critical points of  $\Phi$  located in the support of  $a$  constitute a related submanifold  $W$  of  $Z$  whose we note  $\nu$  the dimension. We further assume that  $W$  is a non-degenerate critical manifold for  $\Phi$ , i.e. that the Hessian  $\Phi''(z)$  induces on the normal space*

$$N = T_z Z / T_z W$$

*a non-degenerate quadratic form  $\Phi''(z)|_N$  whose signature we note  $\sigma$  the signature.*

Then we have an asymptotic behavior of the following form:

$$(3.199) \quad J(\tau) = \int_Z e^{i\tau\Phi(z)} a(z) dz = \left(\frac{2\pi}{\tau}\right)^{\frac{d-\nu}{2}} e^{i\frac{\pi}{4}} \sigma(W) e^{i\tau\Phi(W)} p(\tau)$$

with

$$(3.200) \quad p(\tau) \sim \sum_{k \geq 0} a_k \tau^{-k} \text{ for } \tau \rightarrow \infty$$

Finally in (3.200) the principal coefficient is given by

$$(3.201) \quad a_0 = \int_W a(z) |\det \Phi''(z)|_N|^{-\frac{1}{2}} dv_w(z)$$

with in (3.201)  $dv_w(z)$  denoting the volume element defined by the metric induced by that of  $Z$  on  $W$ .

The proof of the proposition 11.4 is given at the end of this paragraph and we will observe in the course of this demonstration that the value of  $\Phi(z)$  and the signature of the matrix  $\Phi''(z)$  are constant on  $W$ .

**3.3. The nonsingular support of the trace of the wave operator and its singularity in 0.** To introduce the tools and although the extension to the case of an open set with boundary is a fundamental issue, we concentrate in this paragraph on the case of an open set without boundary.

**THEOREM 11.7.** *i) For  $x \in \Omega$  the singular support of  $S(t, x)$  is contained in  $\text{cal}L(x)$  the set of lengths of the closed geodesics passing through  $x$  and that of  $S(t)$  in  $\text{cal}L$ .*

*ii) The point 0 is isolated in  $\mathcal{L}$  and in its neighborhood (with the introduction of function  $\theta$  which localizes around zero, the Fourier transform of Fourier transform of  $S(t, x)$  admits an asymptotic expansion of the following form:*

$$(3.202) \quad (\hat{\theta}S)(\tau, x) \simeq \left(\frac{1}{2\pi}\right)^{n-1} |\tau| \sum_{k \geq 0} p^k(x) |\tau|^{-k}$$

and similarly we have:

$$(\hat{\theta}S)(\tau) \simeq \left(\frac{1}{2\pi}\right)^{n-1} |\tau| \sum_{k \geq 0} p^k(x) |\tau|^{-k}$$

*iii) The coefficients  $p^k(x)$  appearing in the formula (3.199) are (for  $k < n$ ) related to the coefficients  $a_k$  of the asymptotic expansion of Minakshisundaram and Pleijel (formula (2.183)) by the relation*

$$(3.203) \quad a_k(x) = (2\pi i)^{1-\frac{n}{2}} \Gamma\left(\frac{n-k}{2}\right) p_k(x)$$

The relation (3.203) is integrated to give, with the notations of section 2,

$$a_k = (2\pi)^{1-\frac{n}{2}} \Gamma\left(\frac{n-k}{2}\right) p_k$$

This implies in particular that these coefficients are zero for  $k$  odd.

**Demonstration** We use the asymptotics by integral Fourier operators and more precisely the precisely the

**PROPOSITION 11.5.** *(Representation theorem) There exists an integral integral Fourier operator  $\tilde{F} : \Omega \mapsto \times\Omega$  belonging to the class  $I^{-\frac{1}{4}}(\mathbb{R} \times \Omega, \Omega; C)$  which solves the Cauchy problem:*

$$(3.204) \quad \partial_t^2 \tilde{F} - L\tilde{F} = 0, \tilde{F}|_{t=0} = \text{Identity of } \Omega, \tilde{F}|_{t=0} = 0$$

where  $C$  is the canonical relation:  $(t, \tau; y, \eta)$  is on the bicharacteristic that passes through one of the points  $(0, \tau_0 = \pm\sqrt{(A(x)\xi, \xi)}; x, \xi)$ .

The exhibited canonical manifold is then, if we note  $\Phi^t$  the bicaracteristic flow bicaracteristic of the studied operator  $i^{-1}\partial_t + \sqrt{A(x)}\xi.\xi$ ,

$$(3.205) \quad (t, \tau); (x, \xi); (y, \eta), \tau + q(x, \xi) = 0, (x, \xi) = \Phi^t(y, \eta)\}.$$

We denote by  $F(t, x, y) \in \mathcal{D}(\mathbb{R} \times \times O)$  the kernel distribution of  $F$ . Its wavefront is therefore contained in

$$\{(t, \tau; y, \eta; x, \xi) \in \mathcal{C} \times \mathcal{C} \setminus (t, \tau; y, \eta; x, -\xi) \in \mathcal{C}.$$

By introducing the spectral decomposition of the operator  $L$  we have

$$1_x \otimes 1_y + \sum_k \cos(\sqrt{\lambda_k}t)w_k(x) \otimes w_k(y) = F(t, x, y) \text{ modulo } C^\infty.$$

Moreover we can define the restriction of this distribution to the manifold  $\mathbb{R} \times D \subset \mathbb{R} \times \Omega \times \Omega$  where  $D$  denotes the diagonal of  $\Omega \times \Omega$  and we obtain:

$$\sum_k \cos(\sqrt{\lambda_k}t)|w_k(x)|^2 - \int_\Omega F(t, x, x) \in C^i nfty.$$

Therefore the singular support of  $S(t, x)$  verifies the relation:

$$\text{supp. sing } S(t, x) \subset \{t \setminus \exists (\xi, 0, \tau) \in \mathcal{C}, (t, \tau, x; \xi, x, \xi) \in \mathcal{C}\}$$

which proves point i).

To prove that the point 0 is isolated in  $\mathcal{C}$  (thus in  $\mathcal{L}(x)$  for all  $x \in \Omega$ ) we use the Hamiltonian equation and the fact that  $\Omega$  is compact.

The singularity of  $S(t, x)$  in the neighborhood of zero is thus characterized by the asymptotic asymptotic behavior for  $\tau \rightarrow \pm\infty$  of the expression

$$I(\tau, x) = (\hat{\theta}S)(\tau, x) = \int_{\times \Omega} e^{-i\tau t} \theta(t) F(t, x, x) dt.$$

Since  $S$  is real and even it suffices to study the case  $\tau > 0$

In the neighborhood of  $t = 0$  we can solve in a classical way (there is no of caustic) the eikonal equation

$$(3.206) \quad d_t \phi^2 = (A(x) \text{grad}_x \phi, \text{grad}_x \phi)$$

and the equations of transport. The kernel  $F(t, x, y)$  is written in the form:

$$(3.207) \quad F(t, x, y) = (2\pi i)^{-n} \int_{\mathbb{R}_\eta^n} \exp i(\phi(t, x, \eta) - y \cdot \eta) a(t, x, y, \eta) d\eta$$

where  $a(t, x, y, \eta)$  is a symbol of order zero.

We use the homogeneity of degree 1 of the phase  $\phi$  and of degree 0 of the symbol to rewrite, by changing  $\eta$  into  $\tau\eta$ , (3.207) as

$$(3.208) \quad I(\tau, x) = \left(\frac{\tau}{2\pi}\right)^n \int_{\mathbb{R}_\eta^n} \int_{-\infty}^{\infty} \theta(t) a(t, x, x, \eta) dt \eta dt.$$

The phase stops at most for  $t = 0$  where we have:

$$(3.209) \quad \phi(0, x, \eta) = x \cdot \eta, \text{ and } a(t, x, x, \eta) = 1$$

and

$$(3.210) \quad \begin{aligned} \phi'_t(t, x, \eta) &= 1, \\ \phi'_x &= \eta, \\ \phi'_\eta &= x. \end{aligned}$$

Thus the sub-manifold of  $\mathbb{R}_t \times \mathbb{R}_\eta^n$  where the phase stations is characterized by the relations:

$$(3.211) \quad t = 0, (A(x)\xi, \xi) = 1$$

It is thus a submanifold of dimension  $(n - 1)$  in a space of dimension  $(n + 1)$  it is therefore of codimension 2. Moreover in the neighborhood of zero the corresponding phase is written (use the eikonal equation of (3.206) ) in the form:

$$(3.212) \quad \phi(t, x, \eta) = x \cdot \eta + t\sqrt{A(x)\eta, \eta} + O(t^2)$$

We deduce from (3.212) that the sub-manifold defined by (3.211) is indeed non-degenerate in the sense of the phase theorem stationing on a submanifold. The existence of an expansion of the type (3.202) is therefore a consequence of this last theorem. To finish the demonstration of point (ii) it is enough to observe that 0 is an isolated point of the spectrum spectrum of  $S(t, x)$  and apply the proposition 11.1 in conjunction with the with the results of the section 2.

**COROLLARY 11.2.** *Let  $N(\lambda)$  be the enumeration function of the eigenvalues of the Laplacian on a compact manifold  $\Omega$  of dimension  $n$  (with Dirichlet condition condition if there is there is an boundary) then we have:*

$$N(\lambda) = (2\pi)^{-n} C_n \text{Vol}(\Omega) \lambda^{\frac{n}{2}} + O((\lambda)^{n-1})$$

**Demonstration:** From (i) and (ii) of Theorem RefIII.4, the distribution  $S(t, x)$  is regular on the open set  $0 < t < l(x) = \frac{1}{a(x)}$  (with  $l(x)$  denoting the length of the smallest length of the smallest closed geodesic of non-zero length passing through  $x$ , moreover always by the theorem 11.7 (point iii)), for  $t = 0$  it admits an asymptotic expansion asymptotic expansion of the following form:

$$(3.213) \quad \theta(t)S(\hat{t}, x, x)(\tau) = A(x)|\tau|^{(n-1)} + O(|\tau|^{(n-2)}).$$

Thus we can apply the localization theorem (Theorem 11.3) with the following notations:

$$\begin{aligned} d\nu(\tau, x) &= A(x)\tau^{n-1} \\ \text{and} \\ d\mu(\tau, x) &= \frac{2}{2\pi} \int e^{i\tau t} F(t, x, x). \end{aligned}$$

We get:

$$\mu(\tau, x) = \frac{1}{n} A(x)\tau^n + Ca(x)(\tau^{n-1})$$

The relation (3.213) is deduced by integration.

In fact it is possible to improve this theorem significantly by microlocalizing the operator  $F(t, x, x)$  and in particular which leads to the relevant result of Ivrii for the boundary problem.

To do this we introduce two objects.

On the one hand for any pair  $(x, \xi) \in T^*(X) \setminus 0$  we denote by  $l(x, \xi)$  the length of the smallest closed geodesic passing through this point and then we pose (as above)

$$l(x) = \inf_{\xi \neq 0} l(x, \xi)$$

of course if there is no closed geodesic passing through this point  $l(x)$  is taken equal to  $+\infty$ . Thus defined  $l(x, \xi)$  is a semi continuous function below and the infimum in the definition of  $l(x)$  can be taken on the unit sphere.

On the other hand we introduce a covering of  $T^*(X) \setminus 0$  by an arbitrary but finite number arbitrary but finite of open cônes  $\Gamma_j$  and a family  $B_j$  of pseudo homogeneous differential operators of order zero (cf. Hörmander [48] tome IV page 258 for the details) which realize a resolution (modulo  $C^\infty$ ) of the identity and whose wavefront is contained in  $\Gamma_j$ . Let us just note that this result was first established in 1968 by Hörmander [46]. This is the Hadamard parametrix.

With these tools we will prove the

**THEOREM 11.8.** *The enumeration function of the Laplace Beltrami operator on the manifold without boundary  $\Omega$  verifies the following estimate:*

$$(3.214) \quad |N(\lambda) - C_n \text{Vol}(\Omega) \lambda^{\frac{n}{2}}| C \lambda^{\frac{(n-1)}{2}} (A(x)\xi, \xi) < 1 \frac{1}{l(x, \xi)} dx d\xi$$

Before making the demonstration it is advisable to notice that from this theorem we immediately deduce the

COROLLARY 11.3. *We suppose that the set of points  $(x, \xi)$  through which a closed geodesic passes is of measure zero in  $T^*(X) \setminus 0$  (which is much weaker than assuming that the set of points  $x \in \Omega$  through which a closed geodesic passes is of measure zero in  $\Omega$ ) then we have, for the enumeration function of the Laplace Beltrami operator, the formula:*

$$N(\lambda) = C_n \text{Vol}(\Omega) \lambda^{\frac{n}{2}} + o(\lambda^{\frac{(n-1)}{2}})$$

Proof of the Theorem 11.8. We use the overlay  $\Gamma_j$  and the operators  $B_j$  introduced above by posing in particular

$$l_j = \inf_{(x, \xi) \in \Gamma_j} l(x, \xi).$$

With the function  $\tilde{N}(\tau) = N(\tau^2)$  the formula (3.214) is written

$$\begin{aligned} \tilde{N}(\tau) &= C_n \text{Vol}(\Omega) \lambda^{\frac{n}{2}} + C(\tau) \\ |C(\tau)| &= \mathcal{C}(\tau)^{(n-1)} \int_{\text{int}_{(A(x), \xi, \xi) < 1} \frac{1}{l(x, \xi)}} dx d\xi, . \end{aligned}$$

On the other hand we have

$$\tilde{N}(\tau) = \int_0^\tau \int_\Omega d\mu(\tau', x) dx .$$

We introduce the resolution of the identity:

$$I = \sum_1^N B_j B_j^* = I + R, \text{ with } R \text{ indefinitely regularizing}$$

and so modulo  $\tau^{-\infty}$ , we have:

$$\tilde{N}(\tau) = \sum_j \int_0^\tau \int_\Omega d\mu_j(\tau', x) dx$$

with

$$(3.215) \quad d\mu_j(\tau', x) = \frac{1}{\pi} \int e^{it\tau} \text{Trace}\{\cos(t\sqrt{-\Delta} B_j B_j^*)\} dt ;$$

in (3.215) the traces and Fourier transforms are understood in the sense of the distributions. Finally, we denote by  $F_j(t, x, y)$  any approximation (modulo  $C^\infty$ ) of the kernel of the operator:

$$\cos(t\sqrt{-\Delta} B_j B_j^*)$$

and, as we have:

$$\text{Trace}\{\cos(t\sqrt{-\Delta} B_j B_j^*)\} = \text{Trace}\{B_j^* \cos(t\sqrt{-\Delta} B_j)\}$$

we deduce from the singularity propagation theorem that the function:

$$S_j(t, x) : t \mapsto F_j(t, x, x)$$

is regular for  $0 < t < l_j$ . We assume (which is also possible) that the main symbols  $b_j$  of the operators  $B_j$  verify the relation:

$$\sum_j |b_j|^2 = 1 .$$

To apply the localization theorem, we introduce the measures:

$$(3.216) \quad d\nu_j(\tau) = \frac{1}{\pi} \int e^{-it\tau} \theta(t) \text{Trace}\{\cos(t\sqrt{-\Delta} B_j B_j^*)\} dt$$

In (3.216)  $\theta(t)$  denotes a regular function, with support in support in a small enough neighborhood of 0. (no caustics formed for  $t \in \text{support}\theta$ ) and equal to 1 in an (even smaller)



neighborhood of zero. Thus as in the formula (3.207), the principal term of the kernel of a parametrix of the operator  $\cos(t\sqrt{-\Delta}B_jB_j^*)$  is written:

$$F_j(t, x, y) = \left(\frac{1}{2}\right)^{-n} \int_{\mathbf{R}_\eta^n} e^{i\{\phi(t, x, \eta) - y \cdot \eta\}} a(t, x, y, \eta) |b_j(\eta)|^2 d\eta$$

This allows to explain the measures  $\nu_j(\tau, x)$  and  $\nu_j(\tau)$  according to the formulas:

$$\begin{aligned} d\nu_j(\tau) &= \int d\nu_j(\tau, x) dx \\ d\nu_j(\tau, x) &= \left(\frac{1}{2\pi}\right)^{-n} \int_{\mathbf{R}_\eta^n} e^{i\{\phi(t, x, \eta) - x \cdot \eta\}} a(t, x, x, \eta) |b_j(\eta)|^2 d\eta \end{aligned}$$

from which we deduce as above and by the phase theorem stationary on a sub-manifold that  $d\nu_j(\tau, x)$  has an asymptotic expansion of the form

$$(3.217) \quad d\nu_j(\tau, x) = C_j(x)\tau^{(n-1)} + D_j(x)\tau^{(n-2)} + o(\tau^{(n-2)})$$

with

$$|C_j(x)| \leq C \int_{(A(x)\xi, \xi) \leq 1} |b_j(\xi)|^2 dx d\xi.$$

It remains then to apply the localization theorem 11.3 and to sum with respect to  $j$  to obtain the relation:

$$(3.218) \quad |\tilde{N}(\tau) - \tau^n \sum_j \int_\Omega C_j(x) dx - \tau^{(n-1)} \sum_j \int_\Omega D_j(x) dx| \leq C \sum_j \frac{1}{l_j} \int_{(A(x)\xi, \xi) \leq 1} |b_j(\xi)|^2 dx d\xi, .$$

We use the point ii) for the identification of the coefficients, in particular the term in  $\tau^{(n-1)}$  in the first member of (3.218) is zero in the case, in the the case, considered here, of the open set without boundary. Finally for the second member of (3.218) we have the bound:

$$\sum_j \frac{1}{l_j} \int_{(A(x)\xi, \xi) \leq 1} |b_j(\xi)|^2 dx d\xi \leq \int_{(A(x)\xi, \xi) \leq 1} \sum_j \frac{|b_j(\xi)|^2}{l_j(x, \xi)} dx d\xi \leq C \int_{(A(x)\xi, \xi) \leq 1} \frac{1}{l_j(x, \xi)} dx d\xi,$$

which ends the demonstration.

Inspection of the above demonstration reveals the ingredients the following ingredients:

i) The fact that  $S_j(t, x)$  is regular on the interval  $0 < t < l_j$ ,

ii) The calculation (it is enough to know that we have a power expansion and that we have a suitable majorization of the first term) of the singularity in  $t = 0$  of this distribution.

These two steps can be generalized for a problem with problem. In particular for point i) we speak (cf. Hörmander [48] Tome III) of generalized bicaracteristics (including any interaction physical" interaction with the boundary) and thus we obtain the following statement from Ivrii which we quote without demonstration.

**THEOREM 11.9.** *The enumeration function of the Laplace operator on an open set  $\Omega$  with Dirichlet condition on the boundary verifies the following estimate following:*

$$|N(\lambda) - \frac{C_n}{(2\pi)^n} \mathcal{V} \lambda^{\frac{n}{2}} + \frac{1}{4} \frac{C_{n-1}}{(2pi)^{n-1}} \mathcal{S} \lambda^{\frac{(n-1)}{2}}| \leq C \lambda^{\frac{(n-1)}{2}} \int_{(A(x)\xi, \xi) \leq 1} \frac{1}{l(x, \xi)} dx d\xi$$

and in particular if the closed geodesics form a set of measure zero, we have:

$$N(\lambda) = \frac{C_n}{(2\pi)^n} \mathcal{V} \lambda^{\frac{n}{2}} - \frac{1}{4} \frac{C_{n-1}}{(2pi)^{n-1}} \mathcal{S} \lambda^{\frac{(n-1)}{2}} + o(\lambda^{\frac{(n-1)}{2}})$$

**3.4. Contribution of geodesics to the spectrum of the trace.** Thanks to the localization theorem the previous results are, by with respect to the Riemannian structure **global** structure of a negative nature: it has been proved that if we do not have too many closed geodesics one can improve (a little) the Weyl asymptotics. To prove “positive” results, that is i.e. involving closed geodesics it is necessary to have more precise global parametrization. One can no longer (because of the passage of caustics) represent the the passage of the caustics) represent the integral Fourier operator  $F(x, y, t)$  of the formula (3.204) by a single On the other hand, we have the following statement:

In the neighborhood of any point  $l \in \mathbb{R}_t$  the operator  $\theta(t)F(t, x, y)$  can be expressed as into a finite sum of oscillating integrals.

To do this we define an overlay of the canonical relation  $C$  by domains  $C_\alpha$  of  $T_\alpha$  maps associated to phase functions of the form

$$(3.219) \quad \phi_\alpha(t, x, \eta, y) = \phi_\alpha(t, x, \eta, y) - y \cdot \eta$$

More precisely the open set  $C_\alpha$  is diffeomorphic to a conic  $Z_\alpha$  of  $]l - \epsilon, l + \epsilon[ \times \Omega \times \Gamma_\alpha$  with  $\Gamma_\alpha$  cône open from  $\mathbb{R}^n \setminus 0$  by means of the application:

$$(t, x, \eta) \in Z_\alpha \xrightarrow{T_\alpha} (t, \phi'_{\alpha t}, x, \phi'_{\alpha x}; y = \phi'_{\alpha \eta}, \eta);$$

so we represent locally modulo  $C^\infty$ , (for  $t$  in a neighborhood of  $T$ ) the kernel  $K(t, x, y)$  as a finite sum of oscillating integrals:

$$(3.220) \quad F_\alpha(t, x, y) = \sum_{\alpha \in A} (2\pi)^{-n} \int_{\Gamma_\alpha} e^{i\{\phi_\alpha(t, x, \eta) - y \cdot \eta\}} a_\alpha(t, x, \eta) d\eta$$

where  $a_\alpha(t, x, \eta)$  is a zero-order symbol whose support is a conic part with compact base included in  $Z_\alpha$ . With this representation we have the:

**THEOREM 11.10.** *We suppose that  $(l > 0) \in \mathcal{L}$  is a point isolated in  $calL$  and that, assumption  $(H_l)$ , the set  $W_l^+$  of closed bicaracteristics admitting  $l$  for period is a finite meeting of related related non-degenerate submanifold  $W_{l,j}$  of dimension  $\nu_j$ :*

$$(3.221) \quad W_l^+ = \bigcap_{j \in J_l} W_{l,j},$$

*then in the neighborhood of point  $l$  the singularity of  $S(t)$  is given (in the sense of the wavefront, i.e. by localizing and taking the Fourier transformation) by an asymptotic expansion of the following form:*

$$(3.222) \quad (\hat{\theta}S)(\tau) = \sum_{j \in J_l} e^{-i\tau l} \left(\frac{2\pi}{\tau}\right)^{(1-\nu_j)/2} e^{i\frac{\pi}{4}\sigma_j} p_j^k \tau^{-k}$$

with

$$p_j^0 = \sum_{\alpha} (1/2)^n \int_{T_\alpha^{-1}(W_{l,j} \cap C_\alpha)} |a_\alpha^0(l, x, \eta)| |det\Phi''_{\alpha|N}|^{-\frac{1}{2}} dv(x) d\eta$$

**Demonstration**

Let  $l \in \mathcal{L}$  and  $\theta$  be an indefinitely differentiable function locating in the neighborhood of  $l$ . The singularity of  $S$  at point  $l$  is characterized by the asymptotic asymptotic behavior for  $\tau \rightarrow \pm\infty$  of the expression

$$(3.223) \quad I(\tau) = (\hat{\theta}S)(\tau) = \int_{\mathbb{R} \times \Omega} e^{-i\tau t} \theta(t) F(t, x, x) dt dv(x).$$

Since  $S$  is real, it is sufficient to to study the case  $\tau > 0$ , the other being deduced by conjugation, of même it is enough to limit ourselves by parity to  $l \in \mathcal{L} \cup O$ .

According to the classical theory of integral Fourier operators, we explicit  $\theta F$  into a finite sum (the integral of the second member of (3.223) is on a compact in  $(x, t)$ ) of oscillating integrals using (3.220). The first member of (3.223) is therefore, modulo a fast decaying term in  $\tau$ , a finite sum of oscillating integrals of the following type

$$(3.224) \quad \int \theta(t) \int_{\Gamma_\alpha} \exp\{i\{\phi_\alpha(t, x, \eta) - x \cdot \eta - \tau t\}\} a_\alpha(t, x, \eta) dx dt;$$

we use (as in the passage from (3.207) to (3.208) the homogeneity of degree 1 of the phase  $\phi_\alpha$  and of degree 0 of the symbol  $a_\alpha(t, x, \eta)$  to obtain

(3.225)

$$I_\alpha(\tau) = \left(\frac{\tau}{2\pi}\right)^n \int \theta(t) a_\alpha(t, x, 1) = (a_\alpha(t, x, 1)) = (a_\alpha(t, x, 1)) = (a_\alpha(t, x, 1)) = (a_\alpha(t, x, 1)), a_\alpha(t, x, \tau, \eta) dx dt.$$

Henceforth we will omit the index  $\alpha$  in the calculations. The critical points of the phase of the integral (3.225) are given by (3.235), which implies, given the canonical relation  $C$ , that these critical points coincide with the points of the form:

$$(t, 1; x, \xi; x, \xi), \xi, \xi = \phi'_x(t, x, \xi)$$

Moreover the phase  $\phi$  verifies the ‘‘eikonal’’ equation

$$\phi_t^2 = (A(x) \nabla_x \phi, \phi) = 1$$

We denote by  $W_l^+$ , the part of  $C$  constituted by all points

$$(l, 1; x, \xi; x, \xi)$$

which are obtained as images by the various  $T_\alpha$  maps of the critical points of the various phases  $\Phi_\alpha$  and we introduce the fibered conormal sphere sphere fibric  $S^* \Omega$  of  $\Omega$ :

We now consider  $l \in \mathcal{L}$ , with to fix the ideas  $l > 0$  and one assumes that the set  $W_l^+$  is a finite union of related compact compact manifolds according to the formula (3.221) given below:

$$W_l^+ = \bigcap_{j \in J_l} W_{l,j},$$

Let  $\nu_j = \dim W_{l,j}$ .

By a partition of the unit  $C_\alpha, r_\alpha$  subordinated to the applications  $T_\alpha$  defined by the phases  $\phi_\alpha$  (in finite number) we come back to evaluate (cf Chazarain [23] for the details), modulo of fast decaying terms in  $\tau$

$$(\hat{\theta}S)(\tau) = \left(\frac{\tau}{2\pi}\right)^n \sum_j \sum_\alpha I_{j,\alpha}(\tau)$$

where  $I_{j,\alpha}(\tau)$  corresponds to a card  $C_\alpha$  that meets  $W_{l,j}$ . The oscillating integral  $I_{j,\alpha}(\tau)$  is written as:

$$(3.226) \quad I_{j,\alpha} = \sum_\alpha \int_{Z_\alpha} \exp(i\tau[\phi_\alpha(t, x, \eta) - x \cdot \eta - t]) a_\alpha dt dv(x) d\eta$$

In (3.226) the main symbol  $a_\alpha^0$  is the main part of the symbol  $r_\alpha a$  by the map  $T_\alpha$ , the phase stations on the manifold  $T^{-1}(W_{l,j} \cap C_\alpha)$ , on this manifold it is constant, and according to the eikonal equation equal to  $-\tau l$ ; thus we have (stationary phase with parameter)

$$I_{j,\alpha}(\tau) = e^{-i\tau l} \left(\frac{2\pi}{\tau}\right)^{\frac{1-\nu_j}{2}} e^{i\frac{\pi}{4}\sigma_{j,\alpha}} p_{j,\alpha}(\tau)$$

with

$$p_{j,\alpha} \sim \sum_{k>0} p_{j,\alpha}^k \tau^{-k}$$

and

$$p_{j,\alpha}^0 = \left(\frac{1}{2\pi}\right)^n \int_{T_\alpha^{-1}(W_{l,j} \cap C_\alpha \cap \text{supp}(r_\alpha a))} a_\alpha^0(l, x, \eta) |\det(\Phi''_{\alpha|N})|^{-\frac{1}{2}} dv(x) d\eta$$

To finish the demonstration, it remains to prove that we have:

$$a_\alpha^0(l, x, \eta) = e^{i\frac{\pi}{4}\sigma_j} |a_\alpha^0(l, x, \eta)|$$

with  $\sigma_j$  independent of the order map  $\alpha$  and  $|a_\alpha^0(l, x, \eta)| > 0$ . This results from the following two lemmas explained in the article of Chazarain [23] and which we recall here:

LEMMA 11.1 ([23] Lemma 5.2). *There exists an integer  $n_\alpha$  such that we have on  $T_\alpha^{-1}(W_{l,j} \cap C_\alpha)$*

$$a_\alpha(l, x, \eta) = e^{in_\alpha \frac{\pi}{2}} |a_\alpha^0(l, x, \eta)| \neq 0.$$

and

LEMMA 11.2 ([23] Lemma 5.3). *Let two indices  $\alpha, \beta$  be such that that*

$$(supp r_\alpha) \cap (supp r_\beta) \cap (W_{l,j}) \neq \emptyset$$

*then we have:*

$$e^{(\frac{i\pi}{4}\sigma_{j,\alpha} + in_\alpha \frac{\pi}{2})} = e^{(\frac{i\pi}{4}\sigma_{j,\beta} + in_\beta \frac{\pi}{2})},$$

The first lemma can be proved by noticing that the main symbol is on  $C$ . solution of the transport equation

$$(3.227) \quad \{H, a\} = 0,$$

that the initial conditions imply that the restriction  $a|_{\{0,1\}D^*}$  is identical to 1 and that  $a|_{(\{l,1\} \times D^*) \cap C}$  is a constant section of this restriction.

The second lemma is proved by microlocalizing the integral Fourier operator operator  $F(t, x, x)$  on the intersection of the two maps and computing the asymptotic behavior of these two  $m^{th}$  terms by the theorem 11.2.

The most intuitive case of a sub-manifold formed by two closed features closed bi-features corresponds to a single bicaracteristic  $\gamma$  of primitive length  $T$  passing through through the point  $(x, \xi)$ . The application:

$$(x, \xi) \mapsto (x(s), \xi(s)) = \exp sH(x, \xi)$$

is symplectic so its differential  $D(\exp sH)$  is an application de  $\mathbb{R}^{2n}$  in itself whose spectrum is invariant by the transformation  $\lambda \mapsto \lambda^{-1}$ .

As the trajectory is periodic, at the point  $t = s = T$  the vector  $(\dot{x}(0), \dot{\xi}(0))$  is invariant so 1 is an eigenvalue of multiplicity at least equal to 2 of the application  $D(\exp TH)$ . According to the tradition we said that  $\gamma$  is non-degenerate if the multiplicity of this eigenvalue is exactly equal to 2. We denote by  $K$  the corresponding subspace and by  $P$  the application defined on  $\mathbb{R}^{2n}$  modulo  $K$ , it is identified to an application of  $\mathbb{R}^{2(n-1)}$  in itself called a Poincaré application and denoted by  $P_\gamma$  for which 1 is not an eigenvalue. Thus  $\det(I - P)$  is nonzero.

We then have the

THEOREM 11.11. (Duistermaat Guillemin [33])

*We assume that  $mT$  is an isolated point in  $\mathcal{L}$  which corresponds to a non-degenerate  $\gamma$  geodesic then in the neighborhood of  $t = mT$  the singularity of the distribution trace  $\cos t\sqrt{-\Delta}$  is described, with  $\theta$  a function localizing in the neighborhood of  $t = mT$  and  $\tau$  tending towards infinity, by the formula:*

$$\int_k e^{it\tau} \sum_k e^{i\sqrt{\lambda_k}t} \theta(t) dt = i^{-m\sigma} \frac{T}{2\pi} e^{-imT\tau} |I - P^m|^{-\frac{1}{2}} + O(\tau^{-1}).$$

with  $\sigma$  the Morse index of the  $\gamma$  curve.

Since the point  $mT$  is isolated, the manifold  $W_{mT}$  of the theorem 11.9 is reduced to the bicaracteristic  $\gamma$  traversed  $m$  times. Moreover the fact that the Poincaré application is non-degenerate is equivalent to the fact that this manifold is also non-degenerate.

Thus from the formula (3.222) and with the notations of the theorem 11.9 we have:

$$(3.228) \quad I_m(\tau) = e^{i\frac{\pi}{4}\sigma} e^{-i\tau mT} c_m + O(\tau^{-1})$$

And it remains to observe that  $\sigma$  is identified with the Morse index of the curve and that  $c_m$  is given by the formula:

$$(3.229) \quad c_m = \frac{T}{2\pi} |\det(I - P^m)|^{-\frac{1}{2}}$$

which is the subject of section 3.5.

By introducing an even  $\theta(t)$  function which localizes near the points  $mT$  and by using parity and the reality of the distribution Trace  $\cos t\sqrt{-\Delta}$ , we deduce from the formula (3.228) and under the assumptions of the theorem ??, the relation

$$(3.230) \quad \int e^{-i\tau t} \text{Trace} \cos t\sqrt{-\Delta} \theta(t) dt = (-1)^{\sigma' m} \frac{T}{2\pi} |\det(I - P^m)|^{-\frac{1}{2}} \cos(\tau mT) + O(\tau^{-1})$$

From this statement one comes very close to an explicit generalization of the of the formula of Poisson formula, as shown by the

COROLLARY 11.4. *We denote by  $U$  a bounded open interval of  $]0, \infty[$  and we assume that*

$$\bar{U} \cap \mathcal{L} = \cup \{mT_j | mT_j \in \mathcal{L}\}$$

*is a finite union of lengths of isolated non-degenerate geodesics degenerate geodesics, then for any regular even function  $\phi$  with support in  $U \cup (-U)$  we have:*

$$(3.231) \quad \begin{aligned} & \langle \sum_k \cos t\sqrt{\lambda_k}, \phi(t) \rangle \\ & = \langle \text{Trace} \cos t\sqrt{-\Delta}, \phi(t) \rangle \\ & = \langle \sum_{j,m} (-1)^{\sigma'_j m} \frac{T_j}{2\pi} | \det(I - P_j^m) |^{-\frac{1}{2}} \delta(t - mT_j), \phi(t) \rangle + \langle h(t), \phi(t) \rangle \end{aligned}$$

with  $h(t) \in L^\infty(\mathbb{R})$

The demonstration is done simply by taking the inverse Fourier transform of the Fourier transform of the formula (3.230) observing that only a finite number of terms are involved.

**Remark 6** The above statements have been described in the framework of an open set of an open set without boundary, but since they only use the microlocal analysis along an isolated bicaracteristic they fit here they can be adapted here without too much difficulty to the case of closed bifeatures, after a finite number of reflections reflections on the boundary, and not degenerated.

**3.5. Relation with the Poincaré first return application.** The object of this section is therefore the calculation of coefficient  $c_m$  and of exponent  $c_m$  and their geometrical interpretation. Recall that, in the adapted local map, there exists a phase  $\phi(t, x, \eta)$  such that (according to the equality (3.220))

$$F(t, x, y) = \frac{1}{(2\pi)^n} \int_{\Gamma} e^{i(\phi(t, x, \eta) - y \cdot \eta)} a(t, x, \eta) d\eta.$$

We put the wave operator in the form of its Cauchy term strictly hyperbolic  $i^{-1}\partial_t + q(x, i^{-1}\partial_x)$ , of principal symbol  $\tau + q(x, \eta)$ . We are led to calculate the action of  $Q$  on  $F$ . The theorem of the stationary phase in  $(z, xi)$  on

$$\int dz d\zeta e^{i(x-z) \cdot \zeta + i\phi(t, z, \eta) - iy \cdot \eta} Q(x, \zeta) a(t, z, \eta) d\eta$$

leads to the critical point  $(z_c, \zeta_c) = (x, \partial_x \phi(t, z_c, \eta))$ , the Jacobian of the phase being

$$J = \begin{pmatrix} \partial_{x^2}^2 \phi & -I \\ -I & 0 \end{pmatrix}, \quad \text{of inverse} \quad J^{-1} = \begin{pmatrix} 0 & -I \\ -I & -\partial_{x^2}^2 \phi \end{pmatrix}.$$

The first two terms of the asymptotic expansion of  $Op(q)F$  are

$$(\partial_t \phi + q(x, \partial_x \phi(t, x, \eta)))(a_0 + i^{-1}a_1) + i^{-1}\partial_t a_0 + \frac{1}{2i} [q_1 a_0 + 2 \sum_j \frac{\partial^2}{\partial z_j \partial \zeta_j} + \sum_{j,k} \partial_{x_j x_k}^2 \phi \frac{\partial^2}{\zeta_j}] a_0$$

Then the phase  $\phi$  is solution of the eikonal equation:

$$(3.232) \quad \partial_t \phi(t, x, \eta) + q_0(x, \partial_x \phi(t, x, \eta)) = 0.$$

The symbol  $a_0$  is solution of the transport equation:

$$(3.233) \quad \partial_t a_0 + \sum_j \frac{\partial}{\partial \eta_j} Q(x, \partial_x \phi) \frac{\partial a_0}{\partial x_j} + \frac{1}{2} (\sum_{j,k} \partial_{\eta_j \eta_k}^2 Q(x, \partial_x \phi) \partial_{x_j x_k}^2) a_0 = 0.$$

Let us also note that

$$\sum_j \frac{\partial}{\partial x_j} (\partial_{\eta_j} a_0(x, \partial_x \phi)) = \sum_{j,l} \partial_{\eta_j}^2 Q \partial_{x_j x_l}^2 \phi + \sum_j \partial_{x_j \eta_j}^2 Q$$

which makes the coefficient term of  $a_0$  in (??) closely related to the closely related to the subprincipal symbol of  $Op(Q)$  which is  $q_1 - \sum_j \partial_{x_j}^2 Q$ , and which is invariant by change of symplectic coordinates. In the case of the wave operator, the main sub-symbol is null<sup>1</sup>.

The canonical relation associated with the representation of  $F$  is thus characterized by

$$(y, \eta) \leftrightarrow (x, \eta)$$

with  $y = \partial_\eta \phi(t, x, \eta)$  and  $\zeta = \partial_x \phi(t, x, \eta)$ . Then the graph of the transformation canonical transformation  $\Phi^{t_0}$  associated to the operator  $e^{it_0 P}$  is equal to

$$Phi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_x \phi(t_0, x, \eta) \end{matrix} \right) = \left( \begin{matrix} x \\ \partial_x \phi(t_0, x, \eta) \end{matrix} \right) = \left( \begin{matrix} \Phi_x^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \\ \Phi_\xi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \end{matrix} \right).$$

From this relation, we deduce the Jacobian matrix of  $\Phi^{t_0}$ , which is called Poincaré's first return application, thanks to the relations obtained by deriving with respect to  $x$  and with respect to  $\eta$  the two equalities:

$$\begin{cases} \Phi_x^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) = x \\ \Phi_\xi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) = \partial_x \phi(t_0, x, \eta) \end{cases}$$

or

$$\begin{cases} \nabla_y \Phi_x^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \partial_{x\eta}^2 \phi(t_0, x, \eta) = Id \\ \nabla_y \Phi_x^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \partial_{\eta^2}^2 \phi(t_0, x, \eta) + \nabla_\eta \Phi_x^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \eta \end{matrix} \right) = 0 \\ \nabla_y \Phi_\xi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \partial_{x\eta}^2 \phi(t_0, x, \eta) = \partial_{x^2}^2 \phi(t_0, x, \eta) \\ \nabla_y \Phi_\xi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) \partial_{\eta^2}^2 \phi(t_0, x, \eta) + \nabla_\eta \Phi_\xi^{t_0} \left( \begin{matrix} \partial_\eta \phi(t_0, x, \eta) \\ \partial_\eta \phi(t_0, x, \eta) \end{matrix} \right) = \partial_{x\eta}^2 \phi(t_0, x, \eta). \end{cases}$$

Omitting the set of variables  $(t_0, x, \eta)$ , we obtain

$$P = \begin{pmatrix} \nabla_y \Phi_x^{t_0} & \nabla_\eta \Phi_x^{t_0} \\ \nabla_y \Phi_\xi^{t_0} & \nabla_\eta \Phi_\xi^{t_0} \end{pmatrix} = \begin{pmatrix} (\partial_{x\eta}^2 \phi)^{-1} & -(\partial_{x\eta}^2 \phi)^{-1} \partial_{\eta^2}^2 \phi \\ (\partial_{x\eta}^2 \phi)^{-1} \partial_{x^2}^2 \phi & \partial_{x\eta}^2 \phi - (\partial_{x\eta}^2 \phi)^{-1} \partial_{x^2}^2 \phi \partial_{\eta^2}^2 \phi \end{pmatrix}.$$

This equality is legitimate since the Poincaré first return application is well defined when this Jacobian matrix is nonsingular, which implies that  $\partial_{x\eta}^2 \phi$  is invertible.

We deduce that

$$I - P = \begin{pmatrix} I - (\partial_{x\eta}^2 \phi)^{-1} & (\partial_{x\eta}^2 \phi)^{-1} \partial_{\eta^2}^2 \phi \\ -(\partial_{x\eta}^2 \phi)^{-1} \partial_{x^2}^2 \phi & I - \partial_{x\eta}^2 \phi + (\partial_{x\eta}^2 \phi)^{-1} \partial_{x^2}^2 \phi \partial_{\eta^2}^2 \phi \end{pmatrix}.$$

By elementary algebraic manipulations, we find that

$$-(I - P) \begin{pmatrix} \partial_{x\eta}^2 \phi & 0 \\ 0 & \partial_{x\eta}^2 \phi \end{pmatrix} \begin{pmatrix} \partial_{\eta^2}^2 \phi & I \\ I & 0 \end{pmatrix} = - \begin{pmatrix} \partial_{\eta^2}^2 \phi & I - \partial_{x\eta}^2 \phi \\ I - \partial_{x\eta}^2 \phi & \partial_{x^2}^2 \phi \end{pmatrix} \begin{pmatrix} \partial_{x\eta}^2 \phi & 0 \\ 0 & I \end{pmatrix},$$

from which the relation, in any symplectic coordinate system

$$(3.234) \quad \det(I - P) \det(\partial_{x\eta}^2 \phi) = \det(Hess_{x,\eta}(\phi(t_0, x, \eta) - x.\eta)).$$

We found that the critical points of (3.225) were the points  $(t_c, x_c, \eta_c)$  solution of (3.235)

$$(3.235) \quad \begin{aligned} \partial_t \phi(t_c, x_c, \eta_c) &= 1 \partial_x \phi(t_c, x_c, \eta_c) = \eta_c \\ \partial_\eta \phi(t_c, x_c, \eta_c) &= x_c \end{aligned}$$

<sup>1</sup>Note that we find what was done previously previously, when  $Q = \xi^2$ , where we find the term *Delta* $\phi$ . The solution is proposed by Nirenberg and Treves [81] p 491-493 for example, and we see that

$$a_0(t, x(t), \eta) = a_0(0, x(0), \eta) e^{-\frac{1}{2} \int_0^t \sum_j \frac{\partial}{\partial x_j} (\partial_{ct} a_j q_0(x, \partial_x \phi))|_{x=x(s), t=s} ds}.$$

We will see below a direct method

The relation  $\partial_\eta(\phi(t, x, \eta) - x \cdot \eta)|_{(t_c, x_c, \eta_c)} = 0$  implies that  $(t, \partial_t \phi); (x, \partial_x \phi); (x, \xi)$  is in  $\mathcal{C}$ , that is, according to the definition of *calC* given by (3.205), the relation  $(x, \partial_x \phi(t, x, \eta)) = \Phi^t(x, \xi)$ , the other relation  $\tau + q(x, \partial_x \phi) = 0$  being already verified because  $\phi$  is solution of the eikonal equation.

The third relation of (??) allows to have

$$(x_c, \partial_x \phi(t_c, x_c, \eta_c)) = \Phi^{t_c}(x_c, \eta_c).$$

From the first two relations, we deduce that

$$q(x_c, \xi_c) = 1, (x_c, \eta_c) = \text{Phi}^{t_c}(x_c, \eta_c).$$

We thus verify that the points  $(t_c, x_c, \eta_c)$  where the phase stations in (3.225) are the points  $t_c = T$ , period of the characteristic characteristic flow, and  $(x_c, \eta_c)$  fixed point of  $\Phi^{t_c}$  with the normalizing relation  $q(x_c, \eta_c) = 1$ . It is a critical manifold in the sense of sense of integrals with parameter.

Let us consider  $(x, \eta)$  a point of the cotangent fibered, and  $(x(s), \xi(s))$  the solution of the Hamilton solution of the Hamilton equations associated to  $q_0$ :  $(x(s), \xi(s))$  the solution of the Hamilton equations associated to  $q_0$ :  $(x(s), \xi(s))$ ,  $(x(s), \xi(s))$ ,  $(x(s), \xi(s))$ ,  $(x(s), \xi(s))$ ,

$$(3.236) \quad \begin{cases} \frac{dx}{ds} = \partial_\eta q_0(x(s), \xi(s)) \\ \frac{d\xi}{ds} = -\partial_x q_0(x(s), \xi(s)) \\ x(0) = x, \xi(0) = \eta \end{cases}$$

We verify that, from (??)

$$(3.237) \quad \left[ \frac{\partial}{\partial t} + \sum_p \frac{\partial q_0}{\partial \eta_p}(x, \partial_x \phi(t, x, \eta)) \frac{\partial}{\partial x_p} \right] \left[ \frac{\partial \phi}{\partial \eta_j}(t, x, \eta) \right] = 0,$$

$$(3.238) \quad \frac{\partial q_0}{\partial t} + \sum_p \frac{\partial q_0}{\partial \eta_p}(x, \partial_x \phi(t, x, \eta)) \frac{\partial}{\partial x_p} \left[ \frac{\partial \phi}{\partial x_j}(t, x, \eta) \right] = -\frac{\partial q_0}{\partial x_j}(x, \partial_x \phi(t, x, \eta)).$$

The equation (??) is the one allowing to obtain  $\xi(t)$ , which is justified by what follows. Since  $\phi$  is a solution of the eikonal equation, the Lagrangian manifold  $\Lambda_\phi$  is decomposed laminarly by the bicaracteristics. Now  $(x, \frac{\partial}{\partial x}(\phi(0, x(0), \eta))) = (x, \eta)$ , and we verify that

$$\frac{d}{ds} \left( \frac{\partial \phi}{\partial x_j}(s, x(s), \eta) \right) = \left[ \frac{\partial}{\partial t} + \sum_p \frac{\partial q_0}{\partial \eta_p}(x, \partial_x \phi(s, x(s), \eta)) \frac{\partial}{\partial x_p} \right] \left( \frac{\partial \phi}{\partial x_j}(s, x(s), \eta) \right).$$

The initial point corresponds to  $s = 0$  on the two-characteristic (??), so  $\frac{\partial \phi}{\partial x}(t, x(t), \eta) = \xi(t)$ . We deduce from (3.237) that

$$\frac{d}{ds} \left[ \frac{\partial \phi}{\partial \eta_j}(s, x(s), \eta) \right] = 0 \Rightarrow \frac{\partial \phi}{\partial \eta_j}(s, x(s), \eta) = \frac{\partial \phi}{\partial \eta_j}(0, x(0), \eta) = x_j.$$

It comes

$$\frac{\partial}{\partial x_p} \left( \frac{\partial \phi}{\partial \eta_j}(s, x(s), \eta) \right) = \delta_{jp}$$

and by developing the term on the left, it remains

$$\sum_m \frac{\partial^2 \phi}{\partial x_m \partial \eta_j}(z, x(s), \eta) \frac{\partial}{\partial x_p}(x_m(s)) = \delta_{jp}.$$

The two matrices thus exhibited are inverses of each other, which which implies that their determinants are inverses of each other. Thus

$$\frac{a_0(s, x(s), \eta)}{(\det(\partial_{x_m \eta_j}^2 \phi(s, x(s), \eta)))^{\frac{1}{2}}} = a_0(s, x(s), \eta) \left| \frac{\partial}{\partial x_p}(x_m(s)) \right|^{\frac{1}{2}}.$$

The last term is the Jacobian of the change of variable, and we know that  $a_0(s, x(s), \eta) | \frac{\partial}{\partial x_p} (x_m(s))^\frac{1}{2}$  is constant, equal to  $a_0(0, x, \eta) \times 1 = 1$ . We take  $s = t_c$  and  $\eta = \eta_c$ , in which case  $x(t_c) = x_c$ .

To apply the stationary phase theorem with parameter to the integral the integral (3.226), one must represent the phase in a system where the variable on the characteristic manifold is isolated. We thus consider a point  $(x_0, \eta_0)$  of the closed bicaracteristic of length  $T = t_c$ . Then the hypersurface  $H = \{(x - x_0) \cdot \eta_0 = 0\}$  of  $T^*\mathbb{R}^n$  is transverse to the geodesic, and it is parametrized by  $(y, \eta)$ . We introduce

$$(x(s, y, \eta), \eta(s, y, \eta)) = \Phi^s(y, \eta).$$

The considered phase is then

$$\psi(t, s, y, \eta) = \phi(t, x(s, y, \eta), \eta) - x(s, y, \eta) \cdot \eta - t.$$

It stations in  $(t_c, y_c, \eta_c) \simeq (T, x_0, \eta_0)$  for all  $s$  by the relations of (??).

The gradient of this phase is  $(t, s, y_i, \eta_i)$ .

$$\left( \begin{array}{l} \partial_t \phi(t, x(s, y, \eta), \eta) - 1 \\ \sum_j [\partial_{x_j} \phi(t, x(s, y, \eta), \eta) - \eta_j] \frac{\partial x_j}{\partial s} \\ \sum_j [\partial_{x_j} \phi(t, x(s, y, \eta), \eta) - \eta_j] \frac{\partial x_j}{\partial y_i} \\ \sum_j [\partial_{x_j} \phi(t, x(s, y, \eta), \eta) - \eta_j] \frac{\partial x_j}{\partial \eta_i} + \partial_{\eta_i} \phi(t, x(s, y, \eta), \eta) - x_i(s, y, \eta) \end{array} \right)$$

The Jacobian in  $(t, y, \eta)$  is then

$$\left( \begin{array}{ccc} \partial_{t^2}^2 \phi(t, x(s, y, \eta), \eta) & \sum_j \partial_{tx_j}^2 \phi \frac{\partial x_j}{\partial y_i} & \sum_j \partial_{tx_j}^2 \phi \frac{\partial x_j}{\partial \eta_i} + \partial_{t\eta_i}^2 \phi \\ \sum_j \partial_{tx_j}^2 \phi \frac{\partial x_j}{\partial y_i} & \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial y_i} \frac{\partial x_l}{\partial y_k} & \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial y_i} \frac{\partial x_l}{\partial \eta_k} + [\sum_j \partial_{x_j \eta_k}^2 \phi - \delta_{jk}] \frac{\partial x_j}{\partial y_i} \\ \sum_j \partial_{tx_j}^2 \phi \frac{\partial x_j}{\partial \eta_i} + \partial_{t\eta_i}^2 \phi & \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial y_i} \frac{\partial x_l}{\partial \eta_k} & \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial \eta_k} \frac{\partial x_l}{\partial \eta_i} \\ & + [\sum_j \partial_{x_j \eta_k}^2 \phi - \delta_{jk}] \frac{\partial x_j}{\partial y_i} & + [\sum_j \partial_{x_j \eta_k}^2 \phi - \delta_{jk}] \frac{\partial x_j}{\partial \eta_i} \end{array} \right).$$

We use a canonical transformation<sup>2</sup> The first line of the hessian matrix is

$$\left( \sum_j \frac{\partial q_0}{\partial x_j} \cdot \frac{\partial x_j}{\partial s} + \sum_{j,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} \frac{\partial x_i}{\partial s} \frac{\partial x_l}{\partial \eta_i}, - \sum_j \frac{\partial q_0}{\partial x_j} \frac{\partial x_j}{\partial y_i} - \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial y_i} \frac{\partial x_l}{\partial s}, - \sum_j \frac{\partial q_0}{\partial x_j} \frac{\partial x_j}{\partial \eta_i} - \sum_{j,l} \partial_{x_j x_l}^2 \phi \frac{\partial x_j}{\partial \eta_i} \frac{\partial x_l}{\partial s} - \sum_p \partial_{x_p \eta_i}^2 \frac{\partial x_p}{\partial s} \right).$$

The Hessian matrix is equal to (note the two extra terms)

$$t \begin{pmatrix} \partial_s x & 0 \\ \partial_y x & 0 \\ \partial_x & I \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi}{\partial x^2} & \frac{\partial^2 \phi}{\partial x \eta} - I \\ \frac{\partial^2 \phi}{\partial x \eta} - I & \frac{\partial^2 \phi}{\partial \eta^2} \end{pmatrix} \begin{pmatrix} \partial_s x & 0 \\ \partial_y x & 0 \\ \partial_x & I \end{pmatrix} + M_1 \frac{\partial q_0}{\partial x_1} + M_2 \frac{\partial q_0}{\partial x_2}.$$

in which the operator is written  $q_0(x, \eta) = \eta_1$ . Then we obtain, from the eikonal equation

$$\left\{ \begin{array}{l} \partial_{t^2}^2 \phi + \partial_{tx_1}^2 \phi = 0 \\ \partial_{tx_1}^2 \phi + \partial_{x_1^2}^2 \phi = 0 \\ \partial_{tx_1}^2 \phi + \partial_{x_1 x'}^2 \phi = 0 \\ \partial_{t\eta}^2 \phi + \partial_{x_1 \eta}^2 \phi = 0 \end{array} \right.$$

which allows to rewrite, noting that  $\partial_s x_j = \delta_{j1}$  and  $\partial_\eta x_j = 0$ , that the determinant of this hessian matrix is equal to the determinant of the Hessian matrix of  $\phi(t, x, \eta) - x \cdot \eta$  in the variables  $(x, \eta)$ .

Finally, the phase  $\phi$  is constant on the critical manifold, and so we find  $\phi(t_c, x_c, \eta_c) - x_c \cdot \eta_c - t_c = -t_c = -T$ . This completes the proof of the relation

$$I(\mu) \simeq (\det(I - P))^{-\frac{1}{2}} (2\pi)^{-1-n} \left(\frac{2\pi}{\mu}\right)^{2n+1-\frac{1}{2}} \mu^n \int_{Z \cap q(x, \xi)=1} d\omega_{Z \cap q(x, \xi)=1} e^{-i\mu T} \hat{\rho}(T) \frac{a(T, x, \xi)}{(\det(\partial_{x_n}^2))^\frac{1}{2}}.$$

<sup>2</sup>The eikonal equation  $\partial_t \phi(t, x, \eta) + q_0(x, \partial_x \phi(t, x, \eta)) = 0$  implies the relations

$$\partial_{t^2}^2 \phi(t, x(s, y, \eta), \eta) = \sum_j \frac{\partial q_0}{\partial x_j} \cdot \frac{\partial x_j}{\partial s} + \sum_{j,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} \frac{\partial x_i}{\partial s} \frac{\partial x_l}{\partial s} \partial_{tx_j}^2 \phi = - \frac{\partial q_0}{\partial x_j} - \sum_k \frac{\partial^2 \phi}{\partial x_k \partial x_j} \frac{\partial x_k}{\partial s} \partial_{t\eta_j}^2 = - \sum_p \frac{\partial^2 \phi}{\partial x_p \partial \eta_j} \frac{\partial x_p}{\partial s}$$



or

$$I(\mu) \simeq (\det(I - P))^{-\frac{1}{2}} \frac{T}{2\pi} e^{-i\mu T} \text{hat}\rho(T) \frac{a(T, x, x, \mu\xi)}{(\det(\partial_{x\eta}^2 \phi))^{\frac{1}{2}}} = (\det(I - P))^{-\frac{1}{2}} \frac{T}{2\pi} e^{-i\mu T} \text{hat}\rho(T).$$

**3.6. Proof of the stationary phase theorem on a submanifold.** We denote by  $Z$  a manifold of dimension  $n$  and by  $\phi(z)$  a phase stationing on

$$W = \{z \in Z, \nabla_z \phi = 0\}$$

We suppose that  $W \subset Z$  is a connected submanifold of dimension  $d$ . There exists then an application  $\text{cal}W : \mathbb{R}^d \mapsto X$  such that

$$W = \{z \in \mathbb{R}^n, z = \text{cal}W(y), y \in \mathbb{R}^d\}$$

so  $y \mapsto \mathcal{W}(y)$  is a diffeomorphism of  $\mathbb{R}^d$  on  $W$  and there is a set of exists a set of coordinates  $(z_{j_1}, z_{j_2}, \dots, z_{j_1})$  such that the matrix

$$\left( \frac{Z_{j_p}}{dy_l} \right)$$

is invertible. We then reorder the coordinates on  $X$  so that the  $d$  last coordinates are  $(z_{j_1}, z_{j_2}, \dots, z_{j_1})$  and we note  $\Xi$  the inverse diffeomorphism of the application

$$y \mapsto (z_{j_1}(y), z_{j_2}(y), \dots, z_{j_1}(y)) = (\tilde{z}_{n-d+1}(y), \tilde{z}_{n-d+1}(y), \dots, \tilde{z}_{n-d+1}(y)) = \tilde{z}'' = \xi^{-1}(y)$$

We verify that  $W$  is written in the following form:

$$W = \{\text{tldez}'(\text{tldez}''\text{tldez}''\}), \text{ with } \tilde{z}'' = Xi^{-1}(y), \quad \tilde{z}'(\tilde{z}'' = \tilde{z}'(\Xi(y))$$

Thus we construct a representation of  $Z$  under the form  $Z = (\tilde{z}', \tilde{z}''')$  in which we have  $W = \{(\tilde{z}'(\tilde{z}''), \tilde{z}''\}\}$ . Of course for all  $p, 1, \text{thep} \leq n$  we have

$$(3.239) \quad \partial_{\tilde{z}_p} \phi(\tilde{z}'(\tilde{z}''), \tilde{z}'' = 0$$

which implies that the phase is constant on  $W$  and that we have, by deriving (91) with respect to to  $l$ , for  $n - d + 1 \leq l \leq n$

$$(3.240) \quad \sum_{q=1}^{q=n-d} \partial_{\tilde{z}_p \tilde{z}_q}^2 \phi(\tilde{z}'(\tilde{z}''), \tilde{z}'' \frac{\partial \tilde{z}_q}{\partial \tilde{z}_l} + \partial_{\tilde{z}_p \tilde{z}_l}^2 \phi(\tilde{z}'(\tilde{z}''), \tilde{z}'' = 0$$

The hessian of  $\phi$  is thus solution of  $d$  independent equations and is at most of rank  $n - d$ . We say that the submanifold  $W$  is nondegenerate if this hessian is exactly of rank  $n - d$ . Then there exists at least one at least one invertible sub-matrix of size  $(n - d)\text{times}(n - d)$ . The equations (92) also show that for  $l \geq n - d + 1$  we have:

$$\partial_{\text{tildez}_p \tilde{z}_l}^2 \phi(\tilde{z}'(\tilde{z}''), \tilde{z}'' = - \sum_{q=1}^{q=n-d} \partial_{\text{tildez}_p \tilde{z}_q}^2 \phi(\tilde{z}'(\tilde{z}''), \tilde{z}'' \text{''} = \text{''} z'' = \text{''} z'' = \text{''} z'' = \text{''} z'' = \text{''} z'' = \text{''} z'' = \text{''} z''$$

so the invertible matrix  $(n - d) \times (n - d)$  is extracted from the matrix

$$\left( \partial_{\text{tildez}_p \tilde{z}_q}^2 \right)_{1 \leq p \leq n, 1 \leq q \leq n-d}$$

From now on we write  $\text{tildez}'' = y$  and  $\text{tildez}'(\tilde{z}'' = x(y)$ . The critical manifold  $W$  is the set of points of the form  $(x(y), y)$  and the equations of the critical point are

$$(3.241) \quad \begin{aligned} \text{grad}_x \phi(x(y), y) &= 0, \\ \nabla_y \phi(x(y), y) &= 0 \end{aligned}$$

By deriving with respect to  $y$  these two equalities it comes:

$$(3.242) \quad \begin{aligned} \text{Hess}_x \phi(x(y), y) \cdot \frac{\partial x}{\partial y} + \frac{\partial^2 \phi(x(y), y)}{\partial x \partial y} &= 0, \\ \frac{\partial^2 \phi}{\partial y \partial x} \cdot \frac{\partial x}{\partial y} + \text{Hess}_y \phi(x(y), y) &= 0 \end{aligned}$$

This allows us to write that we have:

$$\text{Hess}_{(x,y)}\phi = \begin{pmatrix} l\text{Hess}_x\phi & -\text{Hess}_x\phi \frac{dx}{dy} \\ {}^t\text{Hess}_x\phi \frac{dx}{dy} & +{}^t(\text{Hess}_x\phi \frac{dx}{dy}) \frac{dx}{dy} \end{pmatrix}$$

It follows that if  $\text{Hess}_x$  is of rank strictly less than  $n - d$ , the same is true for the matrix  $\text{Hess}_{(x,y)}\phi$ . So impose that the matrix  $\text{Hess}_{(x,y)}\phi$  be of rank  $n - d$  (which is the non-degeneracy hypothesis) is equivalent to imposing that the matrix  $\text{Hess}_x$  is also of rank  $n - d$  which is the necessary hypothesis necessary to be able to apply the stationary phase theorem with parameter. We observe moreover that if  $W$  is connected  $\phi(x(y), y)$  and the signature of  $\text{Hess}_x$  are both constant on  $W$ , these constant on  $W$ , these two numbers are thus noted  $\phi(W)$  and  $\sigma(W)$ .

According to the stationary phase theorem with parameter for  $a(z)$  an indefinitely differentiable function and by using the changes of variables and variables and notations above:

$$\begin{aligned} (3.243) \quad \int e^{ik\phi(z)} a(z) dz &= \left(\frac{2\pi}{k}\right)^{\frac{n-d}{2}} \int_{\mathbf{R}_y^d} e^{ik\phi(x(y),y)} e^{i\frac{\phi}{4}\sigma(\text{Hess}_x\phi(x(y),y))} (|\text{Hess}_x\phi(x(y),y)|^{-\frac{1}{2}} \\ &\quad \times \{\sum_{j \geq 0} k^{-j} L^j(a)(x(y),y)\}) dy \\ &= \left(\frac{2\pi}{k}\right)^{\frac{n-d}{2}} e^{ik\phi(w)} e^{i\frac{\phi}{4}\sigma(w)} \left\{ \sum_{j \geq 0} \int_{\mathbf{R}_y^d} |\text{Hess}_x\phi(x(y),y)|^{-\frac{1}{2}} L^j(a)(x(y),y) dy \right\}. \end{aligned}$$

Of course in (3.243) the equalities are to be taken in the sense of asymptotic expansions with respect to parameter  $k$  and the  $L^j$  designate differential operators of order  $j$  with respect to the "normal" variables  $y \in \mathbf{R}^{n-d}$ . We have completed the proof of the stationary phase theorem on a submanifold.

### 4. Conclusion

We have tried to show in this section the necessity to use precise and adapted tools to tools to prove fine properties concerning the distribution of the eigenvalues of the distribution of the eigenvalues of the Laplacian.

The reader will observe that there is still a rather important gap between the intuitive results provided by physicists (Balian and Bloch [7] for example) and those rigorously demonstrated. One of the main reasons is that, at the level of the wave equation, we use a double passage to the the limit.

The parametrix (obtained by Fourier integral operators) are in general only valid for a finite time. valid only for a finite time (the stability of closed geodesics allows to bypass geodesics allows to get around this handicap in some cases), while the eigenvalues are eigenvalues are naturally manifested in the "large" time behavior of the solutions. time behavior of the solutions.

More geometrically complex situations (chaotic trajectories) lead to the chaotic trajectories) lead to the quantum chaos or scar theory.

Conversely, one may wonder if the knowledge of the eigenvalues determines the domain and the Riemannian geometry. This is moreover the title of an title of an essential contribution of Kac [52] on the subject (Can we hear the shape of a drum?). Lax and Phillips [61] have thus contributed to to study the answer to this question.

As we saw in section 2, many geometric quantities are already determined by the distribution or the asymptotic behavior of asymptotic behavior of the eigenvalues, nevertheless the answer to the question of Kac's question is negative (even with an isometry).

To know the object, one would need to have not only the eigenvalues but the trace on the boundary of the corresponding eigenfunctions (or what amounts to to the Dirichlet-Neumann operator) Belyshev [10], Sylvester and Ullmann [87]. This is already true in dimension 1 for the Hill operator (Mc Kean [71]).

In more than one dimension of space, an explicit geometric example where the answer to Kac's question is negative is provided by Buser and Bérard [20].

In the case of an unbounded open set (for example the complementary of a compact (obstacle)) the spectrum of the wave operator (with to fix the ideas a ideas a Dirichlet condition on the boundary):

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \mathcal{D}(A) = (H_0^1(\Omega) \times L^2(\Omega))$$

becomes continuous and equal to  $i\mathbb{R}$ . The resolvent  $(\mu I - A)^{-1}$ , defined for  $\text{Re } \mu > 0$ , can - provided that we localize it (i.e. we multiply it on the left and on the right) - be used as a multiplied on the left and on the right by regular functions with support) - be extended to the whole complex plane in a meromorphic function meromorphic function whose poles

$$\mu_k = \alpha_k + i\beta_k, \text{Re } \alpha_k < 0$$

are called radiation frequencies. Under suitable assumptions, these pôles allow to represent the near field (i.e. the solution in the vicinity of the obstacle) in the form

$$(4.244) \quad u(t, x) = \sum_{\text{Re } \mu_k \geq -\sigma, |x| \leq R} e^{\mu_k t} w_k(x) + O(e^{-\sigma t}).$$

Thus the imaginary part of the poles translates the oscillations of the waves and their real part corresponds to the local decay due to the dispersion.

The resonances play for the external problem the role of the eigenvalues for the eigenvalues for the interior problem. Their "applied" meaning is even much more explicit (a plane illuminated by a radar does not fly in a bounded a bounded domain; when listening to music one does not place one's ear inside the instrument but outside it). Without developing this subject it seems good to point out that, progressively of their obtaining, the results concerning the eigenvalues have been transposed to the radiation frequencies.

Weyl's estimation finds its analog first in the estimation of real frequencies (Lax and Phillips [62]) and then in a first estimate of the estimate of the number  $N(R)$  of frequencies of modulus less than  $R$  (Melrose [62]) (Melrose [76]). The calculation of asymptotics with a stationary phase or an asymptotic series has allowed to treat "pathological" examples showing the contributions of captive geodesics (Ralston [85], Bardos-Guillot-Ralston [8] Ikawa [49], [50]).

The introduction of the analytical wavefront and the Gevrey regularity has allowed to systematize the first results of V. B. Filippov [38], V. M. Babic and N.S. Grigoreva [4]. One can thus evaluate the contribution of the geodesics crawling on the obstacle (Bardos-Lebeau-Rauch [9], Sjostrand-Zworski [90]). Finally, the use of inequalities or the  $h$ -pseudo-differential calculus leads to some optimal results on how the radiation frequencies (which are, recall the  $\mu_k = \alpha_k + i\beta_k, \alpha_k < 0$ ) can approach the imaginary axis when  $\beta_k \rightarrow \infty$  (Burq [19]). As it was said in the introduction, the analysis of the spectrum of the Laplacian also appears in arithmetic and group theory. For start, we can extend Poisson's formula to  $n$  (or to simplify to 2) dimensions of space. We denote by  $L$  the network of points

$$L = \{(ma, nb), (m, n) \in \mathbb{Z}^2, a, b \in \mathbb{R}^2\}$$

and the group of translations generated by the vector  $(a, b)$ . From similarly, we denote by  $L'$  the dual network of  $2(\frac{m}{a}, \frac{n}{b})$ . As the translations commute with the Laplacian, by passing to the quotient, this one becomes the Laplace-Beltrami operator on the compact compact manifold  $\Omega = \mathbb{R}^2/L$ . The eigenvalues are the squares of the moduli of the elements of the dual group  $\omega' = 2\pi(\frac{m}{a}, \frac{n}{b}), |\omega'|^2 = 4\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$ , while while the lengths of the closed geodesics are the moduli of the elements of  $L$ :  $\omega = (ma, nb), \omega = (ma, nb), |\omega|^2 = m^2a^2 + n^2b^2$ . One find by Poisson's formula an exact version of (3.80):

$$\sum_{\omega' \in L'} e^{it|\omega'|} = \frac{ab}{4\pi t} \sum_{\omega \in L} \delta(t - |\omega'|).$$

By inverse Fourier and Laplace transforms it gives back the famous formula of Jacobi

$$\sum_{L'} e^{-4\pi^2|\omega|^2 t} = \frac{ab}{4\pi t} \left( \sum_L e^{-\frac{|\omega|^2}{4t}} \right)$$

which corresponds to the asymptotic behavior of the heat kernel trace. Of course, the asymptotic behavior of the counting function  $N(\lambda)$  can be interpreted as the calculation of the number of points with integer coordinates located in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\lambda}{4\pi^2}$$

According to Lax and Phillips [63], Gauss had, in the case of the circle, given the optimal" formula

$$N(\lambda) = M(\lambda^2) + O(\lambda).$$

We immediately notice that the above approach can be generalized to the case where generalized to the case where  $M$  is any Riemannian manifold and where  $\Gamma$  is a subgroup of the transformations of  $M$  which preserve the Riemannian structure. Then the spectrum of the Laplacian will contain intrinsic information about the manifold  $M$  and the subgroup  $\Gamma$ . The case that has been studied the most is probably the one where  $M$  is identified with the Poincaré half-plane  $H = \{(x, y), y > 0\}$  equipped with the Riemannian metric  $ds^2 = \frac{dx^2 + dy^2}{y^2}$ . The group which preserves this metric is identified with  $SL(2, \mathbb{R}) = G$  and the operator of Laplace-Beltrami operator is  $y^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{1}{4}u$ .

For any subgroup  $\Gamma$  of  $G$ , we can therefore consider the Riemannian manifold Riemannian manifold  $H/\Gamma$  and study the spectrum of the Laplace-Beltrami operator in relation to  $\Gamma$ . The case where  $H/\Gamma$  is compact has been developed among others by Mc Kean [74]. But the most most fascinating is the one corresponding to the modular group given by

$$z \rightarrow \frac{az + b}{cz + d}, ad - bc = 1, (a, b, c, d) \in \mathbb{Z}^4.$$

In this situation,  $H/\Gamma$  is no longer compact (although of finite volume). The use of an adaptation of the scattering theory mentioned above is necessary to to get around this difficulty and as in the case of the bounded obstacle the bounded obstacle, we introduce a semi-group  $Z(t)$  corresponding to a expansion of the type (4.244). An explicit calculation of the representation of groups shows that the spectrum of  $B$  corresponds to the poles of the function

$$-\left(\frac{1 - 2\mu}{1 + 2\mu}\right)^2 \frac{\Gamma(-\mu)\zeta(-2\mu)}{\Gamma(\frac{1}{2} - \mu)\zeta(1 - 2\mu)}$$

where  $\zeta$  denotes the Riemann zeta function. Thus an analysis of asymptotic behavior of  $Z(t)$  would give the answer to the Riemann conjecture. Riemann's conjecture. More precisely it would suffice to prove the estimate

$$(4.245) \quad \bar{\lim}_t \frac{1}{t} \log \|(B + i)Z(t)\| \leq -\frac{1}{4}.$$

Unfortunately the authors [62] continuing the calculation show that the proof of (4.245) is in fact based on the Riemann hypothesis, and so the problem remains open. These last remarks end this chapter of presentation of an application of Fourier integral operators.



## Reflection of electromagnetic waves

In this chapter, we calculate the local expression of a relation at the boundary between electric and magnetic fields (called the impedance boundary condition). We introduce an intrinsic formulation of Maxwell's equations, which we will derive in an elementary way from identities on the divergence and the rotational. We then write the Maxwell equations in a system of coordinates adapted to the boundary.

We extend the notion of wavefront set for a distribution  $(E, H)$ . We consider an analytic conormal incident electromagnetic field with respect to a wave surface.

From this form of the electromagnetic field, we will deduce (under certain assumptions on the reflecting object  $\Omega$ ) a relation between the tangential electric and magnetic fields. The external problem obtained by imposing this condition on  $\partial\Omega$  is then a problem of reflection of singularities like the one we have studied for the scalar waves. We will prove the reflection result in the general case.

### 1. Geometry and Maxwell's equations

**1.1. Differential formalism for the system of Maxwell equations.** In this section, we describe the intrinsic representation of the system of Maxwell equations. For this purpose, we introduce the notion of exterior derivative  $d$

DEFINITION 12.1. *Consider a system of (cartesian) coordinates  $(X_1, X_2, X_3)$  on  $\mathbb{R}^3$ , and consider a  $C^\infty$  change of variable  $x := x(X)$ . Denote by*

$$(1.246) \quad M_{ij}(x) = \frac{\partial X_j}{\partial x_i}(x),$$

$$(1.247) \quad N = (\det M)({}^t M)^{-1},$$

*respectively Jacobian matrix of the change of variables and cofactor matrix of  $N$ . The exterior derivative  $d$  which transforms a  $C^1$  function to a 1-differential form, a 1-differential form to a 2-differential form and a 2-differential form to a determinant) is intrinsic.*

It writes:

- i) differential of a function  $df = \sum_j \partial_{X_j} f dX_j = \sum_j \partial_{x_j} f dx_j$  where  $(\partial_{x_j} f)_j = M(\partial_{X_j} f)_j$ ,
- ii) differential of a vector field  $V = \sum V_j dX_j = \sum v_j dx_j$ :

$$dV = (\partial_{X_2} V_3 - \partial_{X_3} V_2) dX_2 \wedge dX_3 + (\partial_{X_3} V_1 - \partial_{X_1} V_3) dX_3 \wedge dX_1 + (\partial_{X_1} V_2 - \partial_{X_2} V_1) dX_1 \wedge dX_2$$

- iii) differential of a 2-differential form  $\omega = \Omega_1 dX_2 \wedge dX_3 + \Omega_2 dX_3 \wedge dX_1 + \Omega_3 dX_1 \wedge dX_2 = \sum_{\{i,j,k\}} \omega_j dx_k \wedge dx_l$ :

$$d\omega = (\partial_{X_1} \Omega_1 + \partial_{X_2} \Omega_2 + \partial_{X_3} \Omega_3) dX_1 \wedge dX_2 \wedge dX_3.$$

Remark that ii) comes from i) and iii) comes from ii) using the rule of differentiation

$$(1.248) \quad d(fT) = df \wedge T$$

when  $dT = 0$  (which is the case when  $T = d\phi$ ).

It writes on the bases adapted to the coordinates  $(x_1, x_2, x_3)$  and  $(X_1, X_2, X_3)$  of  $\Lambda^1, \Lambda^2, \Lambda^3$  (of 1-differential forms, 2-differential forms and determinants respectively) using the correspondence between fields of vectors and differential forms:

$$\begin{cases} df \rightarrow \nabla_x f = M \nabla_X f \\ dV \rightarrow \text{rot}_x v = N \text{rot}_X M^{-1} V \\ d\omega \rightarrow \text{div}_x \omega = \text{div}_X (N^{-1} \Omega). \end{cases}$$

The pull-back  $d^*$  is characterized by  $\int df.gds = -\int f.d^*gd\tau$  for all  $f$  and  $g$  compactly supported.

LEMMA 12.1. *Associating to  $E = (E_1, E_2, E_3)$  in Cartesian coordinates the 1-differential form  $\mathcal{E} = E_1 dX_1 + E_2 dX_2 + E_3 dX_3$  and to  $H = (H_1, H_2, H_3)$  in this same coordinate system the 2-differential  $\mathcal{H} = H_1 dX_2 \wedge dX_3 + H_2 dx_3 \wedge dX_1 + H_3 dX_1 \wedge dX_2$ . Definition 12.1 translates, in the case of the electromagnetic theory, into*

$$\begin{cases} df \rightarrow \nabla_x f = M \nabla_X f \\ de \rightarrow \text{rot}_x e = N \text{rot}_X M^{-1} E \\ dh \rightarrow \text{div}_x h = \text{div}_X (N^{-1} H). \end{cases}$$

Maxwell's equations are written intrinsically in any time independent coordinate system as

$$(1.249) \quad \begin{cases} d\mathcal{E} = -\mu \partial_t \mathcal{H} \\ d^* \mathcal{H} = +\varepsilon \partial_t \mathcal{E} \\ d^* \mathcal{E} = 0 \\ d\mathcal{H} = 0. \end{cases}$$

Introduce for later purposes the following positive definite matrix

$$(1.250) \quad g(x) = (M^t M)^{-1}.$$

Let us recall some explicit formulas of differential calculus. Let be two coordinate systems in  $\mathbb{R}^3$ , the system  $(x_1, x_2, x_3)$  and the system  $(X_1, X_2, X_3)$ . We assume that the coordinate system  $(X_1, X_2, X_3)$  is the Cartesian coordinate system, in which the Maxwell equations have the form known in the literature.

We use the following identity: if  $F_1(X), F_2(X), F_3(X)$  are three functions in  $C^\infty(\mathbb{R}^3, \mathbb{R})$  and if we introduce

$$(1.251) \quad \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = M(x) \begin{pmatrix} F_1(X(x)) \\ F_2(X(x)) \\ F_3(X(x)) \end{pmatrix}$$

one gets

$$(1.252) \quad \begin{pmatrix} \partial_{x_3} f_2(x) - \partial_{x_2} f_3(x) \\ \partial_{x_1} f_3(x) - \partial_{x_3} f_1(x) \\ \partial_{x_1} f_2(x) - \partial_{x_2} f_1(x) \end{pmatrix} = N(x) \begin{pmatrix} \partial_{X_3} F_2(X(x)) - \partial_{X_2} F_3(X(x)) \\ \partial_{X_1} F_3(X(x)) - \partial_{X_3} F_1(X(x)) \\ \partial_{X_1} F_2(X(x)) - \partial_{X_2} F_1(X(x)) \end{pmatrix}.$$

If one defines  $\text{rot}_x \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \partial_{x_3} e_2(x) - \partial_{x_2} e_3(x) \\ \partial_{x_1} e_3(x) - \partial_{x_3} e_1(x) \\ \partial_{x_1} e_2(x) - \partial_{x_2} e_1(x) \end{pmatrix}$ , and  $\begin{pmatrix} e_1(x) \\ e_2(x) \\ e_3(x) \end{pmatrix} = M(x) \begin{pmatrix} E_1(X(x)) \\ E_2(X(x)) \\ E_3(X(x)) \end{pmatrix}$ ,

this equality allows to get the equivalence

$$\text{rot}_x e = i\omega \mu h \Leftrightarrow \text{rot}_X E = i\omega \mu H.$$

From the duality equality

$$\begin{aligned} \int_{\mathbb{R}^3} \text{div}_X (Hu)(X) dX &= - \int_{\mathbb{R}^3} \text{div}_x (hu)(x) dx = - \int_{\mathbb{R}^3} H \cdot \nabla_X u(X) dX \\ &= - \int_{\mathbb{R}^3} H \cdot M^{-1} \nabla_x u(X(x)) (\det M)(x) dx \\ &= - \int_{\mathbb{R}^3} H \cdot (\det M)^{-1t} N \nabla_x u(X(x)) (\det M) dx \\ &= - \int_{\mathbb{R}^3} (NH) \cdot \nabla_x u dx \\ &= \int_{\mathbb{R}^3} \text{div}_x (N(x)H(X(x))) u(x) dx \end{aligned}$$

we deduce the expression of a divergence of a vector field  $h$  associated with  $H$

$$\operatorname{div}_x h = \operatorname{div}_X H.$$

The two equations  $\operatorname{div}_X H = 0$  and  $\operatorname{rot}_X E = i\omega\mu H$  are therefore preserved by the **simultaneous** change of variable and functions. We verify that the transformation  $ME = e$  is a transformation which characterizes the 1--differential forms (by writing the 1-differential form associated to  $e$  as  $e_1 dx_1 + e_2 dx_2 + e_3 dx_3$  and the one associated to  $E$  as  $E_1 dX_1 + E_2 dX_2 + E_3 dX_3$ ). In the same way, the matrix  $N$  being the cofactor matrix of  ${}^t M$ , it characterizes the 2--differential forms (expressed for example as  $h_1 dx_2 \wedge dx_3 + h_2 dx_3 \wedge dx_1 + h_3 dx_1 \wedge dx_2$ ).

This leads to the evaluation of  $\operatorname{rot}_X H$  and  $\operatorname{div}_X E$  as functions of  $E, H$  and of the change of variable. For example, in the equality (1.252), we consider  $F = H$ . Then we find, defining  $f = MH$  as in (1.251), that

$$N\operatorname{rot}_X H = \operatorname{rot}_x MH,$$

As  $h = NH$ , one gets  $N\operatorname{rot}_X H = \operatorname{rot}_x(MN^{-1}h)$ . From the definition of  $N$  and of  $g$ , one has  $MN^{-1} = (\det M)^{-1}M^t M = (\det M)^{-1}g^{-1}$ . Finally,  $\det g = (\det(M^t M))^{-1} = (\det M)^{-2}$ , which yields

$$(1.253) \quad \operatorname{rot}_X H = (\det g)^{\frac{1}{2}t} M[\operatorname{rot}_x((\det g)^{\frac{1}{2}}g^{-1}h)].$$

Finally, after integration by parts

$$\int_{R^3} \operatorname{div}_X Eu(X)dX = - \int_{R^3} E \cdot \nabla_X u dX = - \int (M^{-1}e) \cdot M^{-1} \nabla_x u (\det M) dx,$$

which yields

$$(1.254) \quad \operatorname{div}_X E = (\det g)^{\frac{1}{2}} \operatorname{div}_x((\det g)^{-\frac{1}{2}}g(e)).$$

The two Maxwell equations  $\operatorname{div}_X E = 0$  et  $\operatorname{rot}_X H = -i\omega\varepsilon E$  become

$$(1.255) \quad \begin{aligned} (\det g)^{\frac{1}{2}} \operatorname{div}_x((\det g)^{-\frac{1}{2}}g(e)) &= 0 \\ (\det g)^{\frac{1}{2}t} M[\operatorname{rot}_x((\det g)^{\frac{1}{2}}g^{-1}h)] &= -i\omega\varepsilon E, \end{aligned}$$

which rewrites

$$(1.256) \quad (\det g)^{\frac{1}{2}}g^{-1}[\operatorname{rot}_x((\det g)^{\frac{1}{2}}g^{-1}h)] = -i\omega\varepsilon e.$$

Equalities (1.254) on the 1--differential forms and (1.256) on the 2--differential forms define the action of the operator  $d^*$ . This result is well known (see Bossavit [14], Buldyrev [5]), and it is summarized in Lemma 12.1.

The form of the operators  $d$  and  $d^*$  and the simultaneous change of unknowns (from  $E$  to  $e = ME$ ) allows to write in a simpler way the differential system verified by linear combinations of  $E_j$  (like  $E_n = E_1 n_1 + E_2 n_2 + E_3 n_3$  if  $n$  is the normal vector to a surface). This representation allows to simplify and to rewrite the system associated to the reflection of the electromagnetic singularities.

**1.2. Calderòn operators.** We now consider a  $\Omega$  object (whose boundary we assume to be regular enough for convenience) and regular boundary) and we suppose that the material contained in  $\Omega$  is characterized by its dielectric constants  $\varepsilon$  and  $\mu$ . We write  $\mathbb{R}^3 = \Omega \cup \Omega' \cup \partial\Omega$ . The stationary electromagnetic fields, solution of the Maxwell's equations in  $\Omega$  and in a neighborhood of the boundary  $\partial\Omega$  in  $\Omega'$ , verify the conditions of transmission at the boundary of  $\Omega$ , that is, denoting by  $n(x)$  the normal outside  $\Omega$  at the point  $x \in \partial\Omega$ :

$$\begin{cases} \lim_{y \rightarrow x, y \in \Omega} n(x) \wedge E(y) = \lim_{y \rightarrow x, y \in \Omega'} n(x) \wedge E(y) \\ \lim_{y \rightarrow x, y \in \Omega} n(x) \wedge H(y) = \lim_{y \rightarrow x, y \in \Omega'} n(x) \wedge H(y). \end{cases}$$

Traditionally, when one studies reflection or diffraction by  $\Omega$ , one would like to replace the calculation or the analysis of the solution in  $\Omega$  by a relation linking  $\lim_{y \in \Omega, y \rightarrow x \in \partial\Omega} f$  and  $\lim_{y \in \Omega', y \rightarrow x \in \partial\Omega} f$  for any function  $f$ . The theoretical answer has been given by M. Cessenat ([22]) but also by many authors, thanks to Calderòn operators. When the incident wave is a plane wave, of given frequency, and when the wave vector is normal or close to normal to the boundary, (or when the material contained in  $\Omega$  is of high index), Léontovich introduced in



1948 the notion of impedance coefficient  $Z = \sqrt{\frac{\mu}{\varepsilon}}$  and deduced an approximate local relation between  $E$  and  $H$ , which is  $n \wedge E = Z(n \wedge n \wedge H)$ . We show here that, modulo certain assumptions on the coefficients  $E$  and  $H$ , the Calderòn operator obtained by M. Cessenat, when expressed as a relation between  $n \wedge E$  and  $n \wedge n \wedge H$  is locally a pseudodifferential operator. We give its principal symbol in the high frequency asymptotic regime. This pseudodifferential operator is the generalization of the impedance coefficient obtained by Léontovich in the case of the high index medium. We have the

**THEOREM 12.1.** *Assume that  $\varepsilon\mu\omega^2$  is a real or complex number which is not eigenvalue of the operator  $u \rightarrow Au = -\text{rot} \text{rot} u$ ,  $u \in H_0(\text{rot}, \Omega)$ ,  $\text{rot} \text{rot} u \in (L^2(\Omega))^3$  (this means that the domain of the operator is  $\mathcal{D}(A) = \{u \in H_0(\text{rot}, \Omega), Au \in L^2(\Omega)\}$ ).*

- *Let  $S$  be a function of  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ . The system of equations*

$$(1.257) \quad \begin{cases} \text{rot} E = i\omega\mu H \\ \text{rot} H = -i\omega\varepsilon E \\ n \wedge E|_{\partial\Omega} = S \\ (E, H, \text{rot} E, \text{rot} H) \in (L^2(\Omega))^3 \end{cases}$$

*has a unique solution*

- *If  $\omega \in \mathbb{R}$  and  $\varepsilon\mu$  is complex, the condition on  $\varepsilon\mu\omega^2$  is always verified, which ensures the existence and uniqueness of the solution of the system of equations (1.257).*

In particular, the assumptions of Theorem 12.1 are satisfied when (Fourier transform in time characterized by  $-i\omega$ )

$$(1.258) \quad \Im\varepsilon > 0, \Im\mu > 0, \Im(\varepsilon\mu) > 0.$$

**PROOF.** Under these conditions, the problem in  $\Omega$  is an elliptic problem. The result of Proposition 2.4 allows then to prove the existence of an asymptotic expansion of the solution inside  $\Omega$  when the tangential field  $n \wedge E$  is given on the boundary by its asymptotic expansion. On the other hand, we recall the result given by M. Cessenat of the existence and uniqueness of the solution of the problem of harmonic Maxwell's equations in an open set  $\Omega$ . For this purpose, let us introduce the space  $H_0(\text{rot}, \Omega)$  of distributions  $u$  of  $(L^2(\Omega))^3$  such that  $\text{rot} u \in (L^2(\Omega))^3$  and that  $n \wedge u|_{\partial\Omega} = 0$ , and the boundary space  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega) = \{v \in (H^{-\frac{1}{2}}(\partial\Omega))^3, n.v = 0, \text{div}_{\partial\Omega} v \in H^{-\frac{1}{2}}(\partial\Omega)\}$ . When we study the solution of the system of equations (using  $\text{div}(\text{rot}) = 0$ ,  $\text{div} E = \text{div} H = 0$ ), we come back to the Helmholtz equation (where  $k^2 = \varepsilon\mu\omega^2$ )

$$(\Delta + k^2)E = 0, n \wedge E|_{\partial\Omega} = S, \text{div} E = 0,$$

whose variational formulation is

$$\forall \phi \in C^\infty, a(E, \phi) = \int_{\Omega} (-\nabla E \nabla \bar{\phi} + k^2 E \bar{\phi}) dx + \int_{\partial\Omega} \phi (n \wedge S) d\sigma = 0$$

The sesquilinear form  $a$  is coercive, because  $k^2$  is complex. Indeed, we find, thanks to

$$|a(E, E)|^2 = (\text{Re } a(E, E))^2 + (\Im a(E, E))^2$$

and on respective controls on each term, that

$$|a(E, E)|^2 \geq C(\|E\|^2 + \|\nabla E\|^2).$$

The system has a unique solution for  $S$  given. This ends the proof.  $\square$

The operator giving  $(n \wedge H)|_{\partial\Omega}$  as a function of  $S$  exists and is called the Calderòn operator. We study in what follows an asymptotic expansion of this operator in a coordinate system adapted to the boundary.

At the end of this paragraph, we define, after M. Cessenat, the Calderòn operators for the stationary Maxwell problem in the case of dielectric constants  $\varepsilon > 0$ ,  $\mu > 0$ . We show how this definition is related to the definition of the Dirichlet to Neumann (DTN) operator used for the Helmholtz problem. We do not define the spaces in which these equalities are true;

we refer the reader to chapter 4.2 of [22]. We rely on two existence and uniqueness theorems (respectively Theorem 5 and Theorem 6 of [22] pp 106-107). These theorems are valid when  $\Omega$  is a regular bounded open set of simply connected complement  $\Omega^c$ :

**THEOREM 12.2.** *Let  $\Omega$  be an open set of  $\mathbb{R}^3$ . There exists a unique  $u \in H_{loc}^1(\Delta, C\Omega)$  satisfying*

$$\begin{cases} (\Delta + k^2)u = 0 \\ u|_{\partial\Omega} = u_0 \in H^{\frac{1}{2}}(\partial\Omega) \\ \text{outgoing Sommerfeld condition: } \nabla u \cdot \frac{\vec{r}}{r} - iku = o(\frac{1}{r}). \end{cases}$$

**THEOREM 12.3.** *Let  $\varepsilon, \mu > 0$ . Define  $c = \frac{1}{\sqrt{\varepsilon\mu}}$ . There exists a unique pair  $(E, H)$  such that that (the  $o$  in the Silver-Muller condition is uniform as a function of the direction  $\frac{\vec{r}}{r}$  on the unit sphere)*

$$\begin{cases} \text{rot}H + ikc\varepsilon E = 0, \text{rot}E - ikc\mu H = 0 \text{ dans } C\Omega \\ n \wedge E|_{\partial\Omega} = m \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \\ (E, H) \in (L_{loc}^2(C\Omega))^6, (\text{rot}E, \text{rot}H) \in (L_{loc}^2(C(\Omega)))^6 \\ \text{Silver-Muller conditions : } kc\varepsilon E - kH = o(\frac{1}{r}), kc\mu \frac{\vec{r}}{r} \wedge H + kE = o(\frac{1}{r}). \end{cases}$$

The Dirichlet to Neumann operator for the Helmholtz problem is then the application  $u_0 \rightarrow \partial_n u|_{\partial\Omega}$ , where  $u$  is the solution of the theorem 12.2 problem.

The outer Calderón operator  $C^e$  is the application  $m \rightarrow n \wedge H|_{\partial\Omega}$ , where  $(E, H)$  is the solution of the problem treated in the theorem 12.3.

**1.3. Maxwell equations in semi-geodesic coordinates.** This paragraph reproduces the construction of the coordinates adapted to the boundary already seen in the section 4.2. This part can be read independently from the rest of the book. We consider a point  $x_0 \in \partial\Omega$ . There exists a neighborhood  $V$  of  $x_0$  such that, all points in  $V$  can be represented in a coordinate system  $(x_1, x_2, l)$ ,  $x_1, x_2$  on  $\partial\Omega$  and  $l$  the distance along  $n(x_1, x_2)$ ,  $n$  unit normal outgoing vector from the point  $(x_1, x_2)$  on  $\partial\Omega$ . In other words, any point of  $V$  is characterized by the change of variable

$$X_i(x_1, x_2, l) = Y_i(x_1, x_2) + ln_i(x_1, x_2)$$

where  $(n_1(x_1, x_2), n_2(x_1, x_2), n_3(x_1, x_2))$  is the normal vector to  $\partial\Omega$  at point

$$(Y_1(x_1, x_2), Y_2(x_1, x_2), Y_3(x_1, x_2)).$$

**LEMMA 12.2.** *Let us define four functions  $C_{ij}, i, j = 1..2$  (see below) and two operators  $A_j$  and a vector field vector  $B$ , defined by*

$$\begin{aligned} A_j f &= (\det g)^{\frac{1}{2}} [g_{1j} \partial_{x_1} ((\det g)^{\frac{1}{2}} f) + g_{2j} \partial_{x_2} ((\det g)^{\frac{1}{2}} f)] \\ B(h_1, h_2) &= (\det g)^{\frac{1}{2}} [\partial_{x_1} ((\det g)^{-\frac{1}{2}} (-g_{12}h_1 + g_{11}h_2) + \partial_{x_2} ((\det g)^{-\frac{1}{2}} (-g_{22}h_1 + g_{12}h_2))] \end{aligned}$$

*The system of Maxwell equations writes*

$$(1.259) \quad \begin{cases} (\det g)^{\frac{1}{2}} g^{-1} [\text{rot}_x ((\det g)^{\frac{1}{2}} g^{-1} h)] = -i\omega\varepsilon e \\ \text{rot}_x e = i\omega\mu h. \end{cases}$$

*It rewrites*

$$\begin{cases} -\partial_l h_2 + A_1 h_3 + C_{11} h_1 + C_{12} h_2 = -i\omega\varepsilon e_1 \\ \partial_l h_1 - A_2 h_3 + C_{21} h_1 + C_{22} h_2 = -i\omega\varepsilon e_2 \\ -\partial_l e_2 + \partial_{x_2} e_3 = i\omega\mu h_1 \\ \partial_l e_1 - \partial_{x_1} e_3 = i\omega\mu h_2 \\ B(h_1, h_2) = -i\omega\varepsilon e_3 \\ \partial_{x_1} e_2 - \partial_{x_2} e_1 = i\omega\mu h_3. \end{cases}$$

PROOF. The change of basis matrix  $M$  (where  $x_3$  is denoted by  $l$ ) is given by

$$\begin{aligned} M_{ij}(x_1, x_2, l) &= \partial_{x_i} Y_j(x_1, x_2) + l \partial_{x_i} n_j(x_1, x_2), i, j = 1, 2 \\ M_{3j}(x_1, x_2, l) &= n_j(x_1, x_2) \end{aligned}$$

We note that, since  $n$  is a unit vector and  $Y(x_1, x_2)$  describes  $\partial\Omega \cap V$ , the vector  $n$  is orthogonal to  $\partial_{x_i} n$  as well as to  $\partial_{x_i} Y$ , this second vector being tangent to  $\partial\Omega$ . The third line of  $M$  is therefore orthogonal to the two others. The matrix  $M^t M$  is therefore of the form

$$\begin{pmatrix} g_{11}(x_1, x_2, l) & g_{12}(x_1, x_2, l) & 0 \\ g_{12}(x_1, x_2, l) & g_{22}(x_1, x_2, l) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the relation  $N = (\det g)^{-\frac{1}{2}} g M$ , the last line of  $N$  is  $(\det g)^{-\frac{1}{2}} n_i$  and it is orthogonal to the two other lines of  $N$ .

This allows us to verify that, for  $l = 0$ ,  $e = ME$  and  $h = MH$  verify  $e_3 = E.n$ ,  $h_3 = (\det g)^{-\frac{1}{2}} H.n$ , and  $(e_1, e_2), (h_1, h_2)$  act on the tangent plane of  $\partial\Omega$ . This change of unknowns allows us to have a simple system of equations (since it only depends on the metric at the boundary) where one decompose  $e$  and  $h$  in their tangential and normal components.

We are interested in these components. In particular, there are four functions  $C_{ij}, i, j = 1, 2$  and two operators  $A_j$  and a vector field vector  $B$ , defined by

$$\begin{aligned} A_j f &= (\det g)^{\frac{1}{2}} [g_{1j} \partial_{x_1} ((\det g)^{\frac{1}{2}} f) + g_{2j} \partial_{x_2} ((\det g)^{\frac{1}{2}} f)] \\ B(h_1, h_2) &= (\det g)^{\frac{1}{2}} [\partial_{x_1} ((\det g)^{-\frac{1}{2}} (-g_{12} h_1 + g_{11} h_2) + \partial_{x_2} ((\det g)^{-\frac{1}{2}} (-g_{22} h_1 + g_{12} h_2))] \end{aligned}$$

such that (this defines the functions  $C_{ij}$ )

$$(\det g)^{\frac{1}{2}} g^{-1} [\text{rot}_x ((\det g)^{\frac{1}{2}} g^{-1} h)] = \begin{pmatrix} -\partial_l h_2 \\ \partial_l h_1 \\ B(h_1, h_2) \end{pmatrix} + \begin{pmatrix} A_1 h_3 \\ -A_2 h_3 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_j C_{1j} h_j \\ \sum_j C_{2j} h_j \\ 0 \end{pmatrix}.$$

This ends the proof of Lemma 12.2.  $\square$

We deduce

LEMMA 12.3. *There exist three operators  $P_0, P_1, P_2$  such that the system of Maxwell equations is equivalent to*

$$(1.260) \quad \begin{cases} (P_0 + \frac{1}{i\omega} P_1 + \frac{1}{(i\omega)^2} P_2)(x_1, x_2, l, \partial_{x_1}, \partial_{x_2}) \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} = (i\omega)^{-1} \partial_l \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} \\ -(i\omega)^{-1} B(h_1, h_2) = \varepsilon e_3 \\ (i\omega)^{-1} (\partial_{x_1} e_2 - \partial_{x_2} e_1) = \mu h_3, \end{cases}$$

PROOF. By replacing  $e_3$  and  $h_3$  in the first four equations by their expression as a function of  $h_1, h_2, e_1, e_2$ , we obtain the equivalent system, where  $P_0, P_1, P_2$  are the following differential operators

$$(1.261) \quad \begin{aligned} P_0 &= \begin{pmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \\ 0 & -\varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}, & P_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C_{21} & -C_{22} & 0 & 0 \\ C_{11} & C_{12} & 0 & 0 \end{pmatrix}, \\ P_2 &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \partial_{x_1} \circ \varepsilon^{-1} B \\ \partial_{x_2} \circ \varepsilon^{-1} B \end{pmatrix} \\ \begin{pmatrix} A_1 \circ \mu^{-1} \partial_{x_2} & -A_1 \circ \mu^{-1} \partial_{x_1} \\ A_2 \circ \mu^{-1} \partial_{x_2} & -A_2 \circ \mu^{-1} \partial_{x_1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

We have thus decomposed the Maxwell equations accordingly.  $\square$

LEMMA 12.4. *Maxwell's equations are written as an  $\hbar$ -pseudo-differential problem, and the operators used are develop in  $\hbar = \omega^{-1}$ . The system (1.260), with boundary conditions on  $l = 0$*

$$e_1(x_1, x_2, 0, \omega) = e_1^0(x_1, x_2, \omega), e_2(x_1, x_2, 0, \omega) = e_2^0(x_1, x_2, \omega)$$

and the conditions of decay at infinity  $e_1, e_2$  bounded in  $L^2$  when  $l$  tends to  $-\infty$  is a  $\hbar$ -pseudodifferential system.

We solve the system at the boundary using the techniques introduced by Hörmander and Levitan. However, it is necessary to modify the proof on the boundary conditions to be imposed on the unknowns because the open set is bounded hence we have no way of letting  $l$  go to  $+\infty$ . We have to add some additional conditions on the solution for the proof to be possible. In particular, to an incident wave of wave vector  $\vec{k}_i$  which arrives on the object  $\Omega$ , in classical optics there is a transmitted wave of wave vector  $\vec{k}_{//}$  complex. The continuity of the tangential components of the transmitted wave vector implies the equality of  $\vec{k}_{//}$  and  $-\vec{k}_i \wedge n \wedge n$ . From the relation  $\|\vec{k}_{//}\|^2 + k_n^2 = \varepsilon\mu c^2$ , one deduces

$$k_n = \pm(\varepsilon\mu c^2 - \|\vec{k}_i \wedge n\|^2)^{\frac{1}{2}}.$$

The classical hypothesis  $\text{Re } \varepsilon > 0, \text{Re } \mu > 0$  and the fact that the imaginary parts are small in front of the real parts allows to write  $k_n = \pm(a + ib)$ ,  $a$  and  $b$  being of same sign. We notice that the transmitted wave (the one corresponding to  $-a - ib$ ) is thus characterized by the phase, when  $l$  negative

$$e^{i\omega l(-a-ib)} = e^{-ia\omega l} e^{-b\omega|l|}.$$

The transmitted wave (which propagates in the material) is the damped wave (which justifies the name "dissipative" given to this material). In what follows, we will interpret these remarks by using micro-local analysis.

## 2. Hörmander-Levitan calculus

The aim of this section is to prove

PROPOSITION 12.1. *Under the hypothesis*

$$\Im \varepsilon > 0, \Im \mu > 0, \Im \varepsilon \mu > 0$$

on the dielectric constants, the inner Calderón operator linking the tangential traces of  $E$  and  $H$  on  $\partial\Omega$  admits an asymptotic expansion in  $k$  in the  $k$ -pseudodifferential calculus. The principal symbol of the local pseudo-differential operator which represents  $\mathcal{C}$  is  $Z^0$  given below by (2.267).

Let  $(x_1, x_2, l, \eta_1, \eta_2, \xi_3)$  be an element of  $T^*(\Omega)$ . Using geometry, we associate then  $(\xi_1, \xi_2)$  through  $\xi_i = g_{i1}(x_1, x_2, l)\eta_1 + g_{i2}(x_1, x_2, l)\eta_2$  and the norm of  $|\eta|$  in the metric through

$$(2.262) \quad |\eta|_g^2 = g_{11}\eta_1^2 + 2g_{12}\eta_1\eta_2 + g_{22}\eta_2^2 = \xi \cdot \eta.$$

We verify that, in the  $\omega^{-1}$ -pseudo-differential calculus, the principal symbol of the system on  $e_1, e_2, h_1, h_2$  written above is

$$\xi_3 Id_4 - \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \varepsilon^{-1} M_1 \\ \varepsilon(\varepsilon\mu - |\eta|_g^2) M_1^{-1} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

where  $M_1$  is

$$M_1 = \begin{pmatrix} \eta_1 \xi_2 & \varepsilon\mu - |\eta|_g^2 + \eta_2 \xi_2 \\ \varepsilon\mu - |\eta|_g^2 + \eta_1 \xi_1 & -\eta_2 \xi_1 \end{pmatrix}.$$

Its determinant is thus  $(\varepsilon\mu - |\eta|_g^2 - \xi_3^2)^2$ , and as  $\Im \varepsilon \mu < 0$ , this determinant is bounded below by  $|\Im \varepsilon \mu|^2$ . In the  $\omega^{-1}$ -pseudo-differential calculus, as it is an operator of order 0, it is

elliptic. The Hörmander-Levitan ([64], [65], [46]) calculation for the representation in terms of integral Fourier operators of  $e^{iH}$  can be applied in this case. The operator

$P = P_0 + (i\omega)^{-1}P_1 + (i\omega)^{-2}P_2$  has two eigenvalues  $\lambda_{\pm} = \pm\sqrt{\varepsilon\mu - |\eta|_g^2}$  of multiplicity 2, ( $\pm\Im\lambda_{\pm} \geq |\Im\varepsilon\mu|^{\frac{1}{2}} > 0$ ). Note that we are not exactly in the classical framework developed by Levitan and then Hörmander, since the eigenvalues are not simple, but as the eigenspace is of dimension equal to the multiplicity, the arguments are identical to their case. We denote by  $\psi_{\pm}$  the two phases defined on  $\mathbb{R}_- \times T^*(\partial\Omega)$ , equal to zero on  $l = 0$ , respective solutions of

$$\partial_l \psi_{\pm}(x_1, x_2, l, \eta_1, \eta_2) = \lambda_{\pm}(x_1, x_2, l, \eta_1, \eta_2), \psi_{\pm}(x_1, x_2, 0, \eta_1, \eta_2) = 0.$$

Introducing the Fourier integral operators  $F_{\psi_{\pm}}$  of phase  $\psi_{\pm}$  and of symbol 1, we write a decomposition of the matricial Fourier integral operator  $\mathcal{K}$  solution of (2.263):

$$(2.263) \quad (i\omega)^{-1} \partial_l \mathcal{K} = P \circ \mathcal{K}, \mathcal{K}|_{l=0} = Id_{C^\infty(\partial\Omega)}$$

under the form

$$\mathcal{K} = Op(M_+) \circ Op(F_{\psi_+}) + Op(M_-) \circ Op(F_{\psi_-}).$$

The symbols  $M_{\pm,ii}$  are elliptic in a neighborhood of  $l = 0$ . As we assumed that  $\pm\psi_{\pm}(x_1, x_2, 0, \eta_1, \eta_2)$  are of positive imaginary part, we see that  $\partial_l \psi_{\pm}(x_1, x_2, 0, \eta_1, \eta_2) = \pm\nu$ , where  $\nu$  is the root of positive imaginary part of

$$\varepsilon\mu - (g_{11}(x_1, x_2, 0)\eta_1^2 + 2g_{12}(x_1, x_2, 0)\eta_1\eta_2 + g_{22}(x_1, x_2, 0)\eta_2^2).$$

For  $(x_1, x_2)$  in a neighborhood of  $(x_1^0, x_2^0)$ , there exists two strictly positive constants  $c_1$  and  $c_2$  such that for  $-l \leq \epsilon_0 < 0$  one has

$$\begin{cases} |e^{i\omega\psi_+(x_1, x_2, \eta_1, \eta_2, l)}| \geq e^{-\omega c_1 l} \\ |e^{i\omega\psi_-(x_1, x_2, \eta_1, \eta_2, l)}| \leq e^{\omega l c_2}. \end{cases}$$

It follows that only the Fourier Integral operator  $F_{\psi_-}$  is bounded when  $\omega l$  tends to  $-\infty$ . When  $e_1, e_2, h_1, h_2$  is given on  $l = 0$  one deduces  $e_1, e_2, h_1, h_2$  in a neighborhood of  $l = 0$ . We thus find

$$\begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} (x_1, x_2, l, \omega) = [Op(M_+) \circ Op(F_{\psi_+}) + Op(M_-) \circ Op(F_{\psi_-})] \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (x_1, x_2, l).$$

Thanks to the composition calculus of a pseudodifferential operator and a Fourier integral operator, there are two operators  $Q_+$  and  $Q_-$  (deduced from  $\psi_{\pm}$  and  $M_{\pm}$ ) such that

$$\begin{aligned} \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} (x_1, x_2, l, \omega) &= \iint e^{i\omega\psi_+(x_1, x_2, \eta_1, \eta_2, l) - i\omega y \cdot \eta} Q_+ \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) dy d\eta \\ &+ \iint e^{i\omega\psi_-(x_1, x_2, \eta_1, \eta_2, l) - i\omega y \cdot \eta} Q_- \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) dy d\eta. \end{aligned}$$

We observe that, for all  $\tau$  compactly supported

$$|\tau(x_1, x_2, l) \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} (x_1, x_2, l, \omega) - \tau(x_1, x_2, l) \iint e^{i\omega\psi_-(x_1, x_2, \eta_1, \eta_2, l) - i\omega y \cdot \eta} Q_- \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) dy d\eta| = |\tau(x_1, x_2, l)|$$

If one has, in the neighborhood  $V_0$  of  $y_0$ ,  $Q_+ \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) \neq 0$ , then

$$|\tau(x_1, x_2, l) \iint e^{i\omega\psi_+(x_1, x_2, \eta_1, \eta_2, l) - i\omega y \cdot \eta} Q_+ \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) dy d\eta|$$

is going to  $+\infty$  in  $\omega$  on the set of points  $(x_1, x_2, l)$  such that there exists  $y \in V_0$  and  $\eta = \nabla_y \psi_+(x_1, x_2, \eta_1, \eta_2, l)$ . As the solution is bounded when  $\omega l \rightarrow -\infty$  we deduce, at each point  $y$  of  $V_0$ , the relation

$$Q_+ \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y, \omega) = 0.$$

This pseudo-differential relation between  $e^0$  and  $h^0$  on the boundary is the one we are looking for. We give the principal symbol of the operator that computes  $(e_1^0, e_2^0)$  as a function of  $(h_1^0, h_2^0)$  (and more precisely  $(-e_2^0, e_1^0)$  as a function of  $(h_1^0, h_2^0)$ ). From the inequality  $\Im\nu > 0$  and the relation  $\partial_l \psi_{\pm}|_{l=0} = \pm\nu$ , we deduce  $\mp \Im \psi_{\pm}(x_1, x_2, \eta_1, \eta_2, l) \geq -c_1 l$  for  $l < 0$  small enough thanks to  $\psi_{\pm}|_{\partial\Omega} = 0$  and the continuity of  $|\eta|_g$  as a function of  $x$  (if the boundary is regular enough). As  $\text{Re}(i\omega\psi_-) = -\omega \Im \psi_-$ , the only phase which leads to an exponential decay in  $\omega$  is  $\psi_-$  (more precisely,  $\psi_+$  leads to an exponential growth in  $\omega$  of the associated solution).

As one has

$$(2.264) \quad \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix} (x_1, x_2, l, \omega) = \int dy_1 dy_2 d\eta_1 d\eta_2 \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y_1, y_2, \omega) e^{i\omega(\psi_-(x_1, x_2, l, \eta_1, \eta_2) - y_1 \eta_1 - y_2 \eta_2)},$$

applying  $(i\omega)^{-1} \partial_l - P$  to this solution, we get the system

$$\int dy d\eta [\partial_l \psi_-(x_1, x_2, l, \eta_1, \eta_2) Id_4 - Op(P)] \begin{pmatrix} e_1^0 \\ e_2^0 \\ h_1^0 \\ h_2^0 \end{pmatrix} (y_1, y_2, \omega) e^{i\omega(\psi_-(x_1, x_2, l, \eta_1, \eta_2) - y_1 \eta_1 - y_2 \eta_2)} = 0$$

which gives the relation on the principal symbol of the trace on  $l = 0$  thanks to  $\psi_-(x, \eta, 0) = x_1 \eta_1 + x_2 \eta_2$ , and this relation defines the matrix  $M_1(x, \eta, \omega)$ :

$$(2.265) \quad -\nu \begin{pmatrix} e_1^0(x_1, x_2, 0, \omega) \\ e_2^0(x_1, x_2, 0, \omega) \end{pmatrix} - \varepsilon^{-1} M_1 \begin{pmatrix} h_1^0(x_1, x_2, 0, \omega) \\ h_2^0(x_1, x_2, 0, \omega) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One checks that

$$M_1 \begin{pmatrix} -\eta_2 \\ \eta_1 \end{pmatrix} = \nu^2 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad M_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \varepsilon \mu \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix}.$$

Introduce the matrix  $Z^0$  such that

$$Z^0 \begin{pmatrix} -\eta_2 \\ \eta_1 \end{pmatrix} = \frac{\nu}{\varepsilon} \begin{pmatrix} -\eta_2 \\ \eta_1 \end{pmatrix} \\ Z^0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\mu}{\nu} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Relation (2.265) is equivalent to the pseudodifferential equality where  $Op(Z)$  is a pseudo-differential operator such that

$$(2.266) \quad \begin{pmatrix} -e_2 \\ e_1 \end{pmatrix} |_{l=0} = -Op(Z) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} |_{l=0}.$$

If  $u = \begin{pmatrix} e_1 \\ e_2 \\ h_1 \\ h_2 \end{pmatrix}$  is a solution, bounded in  $\omega$ , of  $[(i\omega)^{-1}\partial_l - P]u = 0$ , then the principal term of the pseudodifferential operator  $Op(Z)$  is  $Z^0$ . We represent the two eigenvectors of  $M_1$  in the figure:

We assume that the conormal analytic wavefront set of the incident wave (which is underlying our study) is contained in  $t - \theta(x, y, l) = 0$ . We know that  $|\nabla\theta| = 1$ , so  $\text{Re}(\varepsilon\mu - |\eta|_g^2) > 0$  (the medium is of index greater than 1). As  $\Im\varepsilon\mu > 0$ , we deduce that  $\text{Re}\nu > 0$  when  $\Im\nu > 0$ .

Remark that, when  $\Im\nu \rightarrow 0$  (weak losses),  $\nu \rightarrow \sqrt{\varepsilon\mu}$ .

The incident wave is assumed to be given by

$$\tilde{E}(x, t) = \int e^{ik(\theta(x)-t)} E(x, k) dx.$$

We denote by  $(x_1, x_2, \eta_1, \eta_2)$  the element of  $T^*(\partial\Omega)$  where we compute the pseudodifferential tangential operators. The impedance matrix  $Z^0$  is the matrix whose eigenvectors are respectively  $\begin{pmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{pmatrix}$  and  $(M^t M)^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 \\ 0 \end{pmatrix}$  in the system of semi-geodesic coordinates and whose eigenvalues are  $\nu/\varepsilon$  et  $\mu/\nu$ .

Let  $\Omega'$  be the complementary set of  $\bar{\Omega}$ . Let  $k = \omega(\varepsilon_0\mu_0)^{\frac{1}{2}}$ .

PROPOSITION 12.2. *The system of Maxwell's equations in  $\Omega'$*

$$\begin{aligned} \text{rot}_X E &= i\omega\mu_0 H, \text{rot}_X H = -i\omega\varepsilon_0 E, \\ &\text{radiation condition at infinity,} \\ &\text{finite local energy} \end{aligned}$$

is equivalent to the system of Helmholtz equations on each component of  $E$ , the radiation condition, the relation  $H = (i\omega\mu_0)^{-1}\text{rot}_X E$  and the relation  $\text{div}_X E|_{\partial\Omega'} = 0$ .

PROOF. We replace  $H$  by its value  $(i\omega\mu_0)^{-1}\text{rot}_X E$  in the equation on  $E$ . We obtain, in  $\mathcal{D}'$

$$\text{rot}_X \text{rot}_X E - \omega^2 \varepsilon_0 \mu_0 E = 0.$$

This equation implies that  $\text{div}_X E = 0$ , and therefore  $(\Delta + (\omega/c)^2)E = 0$  and of course implies  $\text{div}_X E|_{\partial\Omega'} = 0$ .

Now, let us consider the equations (in which R.C designates the radiation conditions at infinity)

$$(\Delta + k^2)E = 0, H = (i\omega\mu_0)^{-1}\text{rot}_X E, \text{div}_X E|_{\partial\Omega'} = 0, R.C.$$

The distribution  $u(X, k) = \text{div}_X E(X, k)$  is a solution of the scalar Helmholtz equation

$$(\Delta + k^2)u(X, k) = 0$$

with the boundary condition  $u(X, k)|_{\partial\Omega'} = 0$  and an outgoing condition at infinity. The boundary  $\partial\Omega$  is non characteristic for the wave operator, so by application of the Holmgren's theorem, the solution of this problem is unique, and  $u(X, k) = 0$  in  $\Omega'$ . This completes the proof of the reciprocal. The same result holds for  $H$ . □

We write the divergence condition on  $e$  in semi-geodesic coordinates. Using the expression (1.254) we obtain

$$(\det g)^{\frac{1}{2}} \partial_l ((\det g)^{-\frac{1}{2}} e_3) + (\det g)^{\frac{1}{2}} (\partial_{x_1} (g_{11} e_1 + g_{12} e_2) + \partial_{x_2} (g_{21} e_1 + g_{22} e_2)) = 0$$

We then use relations  $h_1 = (i\omega\mu_0)^{-1}(\partial_{x_2} e_3 - \partial_l e_2)$ ,  $h_2 = (i\omega\mu_0)^{-1}(-\partial_{x_1} e_3 + \partial_l e_1)$  that we replace in the equality (2.266)

$$\begin{cases} -e_2 = -Op(Z_{11})h_1 - Op(Z_{12})h_2 \\ e_1 = -Op(Z_{21})h_1 - Op(Z_{22})h_2 \end{cases}$$

to obtain, after using the notation  $\mu_0 c = Z_0$  (impedance of the vacuum),

$$(2.267) \quad ikZ_0 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = - \begin{pmatrix} Op(Z_{22}) & -Op(Z_{21}) \\ -Op(Z_{12}) & Op(Z_{11}) \end{pmatrix} \begin{pmatrix} \partial_l e_1 - \partial_{x_1} e_3 \\ \partial_l e_2 - \partial_{x_2} e_3 \end{pmatrix}.$$

Noting that the determinant of the principal symbol  $\begin{pmatrix} Z_{11}^0 & Z_{12}^0 \\ Z_{21}^0 & Z_{22}^0 \end{pmatrix}$  of the matrix operator written in the right hand side of this equality is  $\frac{\mu}{\nu} \times \frac{\nu}{\varepsilon} = \frac{\mu}{\varepsilon}$ , the matricial operator is thus invertible and we can calculate the principal symbol of the inverse which is

$$\frac{\varepsilon}{\mu} \begin{pmatrix} Z_{11}^0 & Z_{12}^0 \\ Z_{21}^0 & Z_{22}^0 \end{pmatrix}.$$

We deduce then two boundary conditions that the solutions of the system of Maxwell's equation must fulfill:

$$\begin{cases} \frac{1}{ik} \partial_l e_1 = \partial_{x_1} e_3 - Z_0 \frac{\varepsilon}{\mu} (Z_{11}^0 e_1 + Z_{12}^0 e_2) + L_0^1(e_1, e_2) \\ \frac{1}{ik} \partial_l e_2 = \partial_{x_2} e_3 - Z_0 \frac{\varepsilon}{\mu} (Z_{21}^0 e_1 + Z_{22}^0 e_2) + L_0^2(e_1, e_2) \end{cases}$$

the operators  $L_0^1(e_1, e_2)$  and  $L_0^2(e_1, e_2)$  being pseudo-differential operators of order 0.

The third boundary condition is given by the divergence relation restricted to the boundary (Proposition 12.2), that is, by applying the relation (1.254) recalled above,  $\frac{1}{ik} \partial_l e_3|_{l=0} + \mathcal{D}(e_1, e_2)|_{l=0} = 0$ , where  $\mathcal{D}$  is a differential operator of principal symbol  $\xi_1 e_1 + \xi_2 e_2$ . The resulting boundary condition that follows is

$$(2.268) \quad \begin{pmatrix} \partial_l e_1 \\ \partial_l e_2 \\ \partial_l e_3 \end{pmatrix} - ikT_E \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0$$

where the operator  $T_E$  is a pseudo-differential matricial tangential operator. We verify that

$$\begin{aligned} T_E^0 \begin{pmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{pmatrix} &= -Z_0^{-1} \frac{\nu}{\mu} \begin{pmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \eta_1 \end{pmatrix} \\ T_E^0 \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix} &= -Z_0^{-1} \frac{\varepsilon}{\nu} \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \eta_2 \end{pmatrix} \\ T_E^0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}. \end{aligned}$$

As  $e(x_1, x_2, l, k) = M(x_1, x_2, l)E(X(x, l))$ , relation between the cartesian and semi-geodesic components of the electric field, expressed in the system of semi-geodesic variables, we deduce

$$\partial_l e(x, 0, k) = (\partial_l M)(x, 0)E(X(x, 0)) + M(x, 0)\partial_l E(X(x, 0)).$$

This implies

$$(\partial_l M)E + M(x, 0)\partial_l E - T_E(ME) = 0$$

equivalent to

$$\partial_l E(X(x, 0)) - [M^{-1}(x, 0)T_E(M(x, 0).)) - M^{-1}(x, 0)\partial_l M(x, 0)]E(x, 0) = 0.$$

Let  $\mathcal{B}$  be the tangential matricial pseudodifferential operator equal to  $M^{-1}(x, 0)T_E \circ M(x, 0)Id - M^{-1}(x, 0)(\partial_l M(x, 0))Id$ . We have

**PROPOSITION 12.3.** *The electromagnetic equations in  $\Omega$  with dielectric coefficients + the equations in  $\Omega'$ , complement of  $\bar{\Omega}$ , is equivalent to*

- the equality  $H = (i\omega\mu_0)^{-1} \text{rot}_X E$  in  $\Omega'$ ,
- the three scalar Helmholtz equations with radiation condition at infinity

$$(\Delta + k^2)E_j = 0,$$

- the boundary condition coupling these three components



$$(2.269) \quad \partial_n E - \mathcal{B}(E|_{\partial\Omega'}) = 0.$$

Remark that one can replace this boundary condition by what is called the impedance boundary condition

$$(2.270) \quad n \wedge E|_{\partial\Omega} + Z[(n \wedge n \wedge H|_{\partial\Omega})] = 0$$

where  $Z$  is deduced from  $\mathcal{B}$  and is called the impedance operator.

To this system of equations, we can apply the theory allowing the reflection of transverse singularities. This is the the object of the next section.

### 3. Reflection of transverse singularities for Maxwell's equations

We prove in this section the following general theorem

**THEOREM 12.4.** *The reflection of transverse singularities for Maxwell equations holds true.*

The following proposition, that follows describes an explicit case. Let us assume that there exists  $j$  such that  $WF(E_j)$  contains the point  $\rho_-$ . Through this point  $\rho_-$  passes a bicharacteristic  $\gamma_-(-s)$  such that  $\pi(\gamma_-)$  meets  $\partial\Omega$  at a point  $x_0$ , the corresponding point on  $\gamma_-$  being  $\rho_0^- \in T^*\mathbb{R}^n \cap \pi^{-1}(\partial\Omega)$ . Assume that the projection on  $T^*(\partial\Omega)$  of  $\rho_0^-$  is an hyperbolic point  $\rho_0$ . Denote by  $\rho_0^\pm$  the point such that  $\rho_0^\pm$  are the two points  $\rho_0^+, \rho_0^-$ . More precisely,  $\rho_0^- = (x_0 = \pi(\rho_0^-), \zeta)$ ,  $\zeta \in T_{x_0}\mathbb{R}^n \simeq T_{x_0}\partial\Omega \times \mathbb{R}$ , so that  $\zeta = (\zeta', \zeta_n)$ , and  $\rho_0 = (x_0, \zeta')$ ,  $\rho_0^+ = (x_0, 0, \zeta', -\eta_n)$ . We denote by  $\gamma_+$  the bicharacteristic which passes through  $\rho_0^+$ .

**PROPOSITION 12.4.** *There exists  $p \in \{1, 2, 3\}$  such that  $\gamma_+ \cap T^*(\Omega') \subset WF(E_p)$ , for all  $q \in \{1, 2, 3\}$  with  $\rho_0 \in WF_b(E_q)$ .*

**PROOF.** It is a matricial generalization of the proof for a condition in the scalar case. We construct the operators  $A_+$  and  $A_-$  from the chapter 10. The solutions of the three Helmholtz equations are written as

$$(3.271) \quad (E_1, E_2, E_3) = (A_-(f_1^-), A_-(f_2^-), A_-(f_3^-)) + (A_+(f_1^+), A_+(f_2^+), A_+(f_3^+)).$$

We write the general jump formula for each component (where  $P$  is the conjugate wave operator in coordinates  $(l, x_1, x_2)$ )

$$PE_j = \partial_l E_j(x, 0, k)\delta_{l=0} + E_j(x, 0, k)\delta'_{l=0}.$$

Applying this equality to (3.271), we find

$$A_+(f_j^+) + A_-(f_j^-)|_{l=0} = E_j, T_+(f_j^+) + T_-(f_j^-) = \partial_l E_j$$

By replacing these equalities in the boundary condition, we find (the pseudo-differential operators  $T_\pm$  were introduced in the chapter in the proposition 10.6 and they are are elliptic at hyperbolic points)

$$\mathcal{B}f^+ + \mathcal{B}f^- - (T_+ Id_3 f^+ + T_- Id_3 f^-) = 0$$

We check that  $\mathcal{B} - T_\pm Id_3$  is an invertible operator. From this we deduce the equality

$$f^+ = (\mathcal{B} - T_+ Id_3)^{-1}(T_- Id_3 - \mathcal{B})(f^-)$$

and the equality

$$E|_{\partial\Omega} = (\mathcal{B} - T_+ Id_3)^{-1}(T_- - T_+) Id_3(f^-).$$

We then conclude that, if  $\rho_- \in WF(E_j)$ , then  $\rho_0 \in WF(f_j^-)$ . The equality on  $E$  allows us to assert that there is at least one coordinate of  $E$  whose wavefront set contains  $\rho_0$ . Thus there is at least one coordinate of  $f^+$ , of index  $p$ . The properties of the operator  $A_+$  allow to state that  $\rho_+$  is in the wavefront set of  $A_+(f_p^+)$ . Since  $\rho_+$  is not in the wavefront set of  $A_-(f_p^-)$ , then  $\rho_+$  is in the wavefront set of  $E_p = A_+(f_p^+) + A_-(f_p^-)$ . This completes the proof of the proposition.  $\square$

Note that we cannot deduce that all components of  $E$  have a wavefront set containing  $\rho_+$ , because it could happen that  $(\mathcal{B} - T_+ Id_3)^{-1}(T_- Id_3 - \mathcal{B})(f^-)$  has one component at least which is zero. In particular, if the wave vector happens to coincide at a given 'moment'  $s$  with one of the axis of coordinates, the relation  $\operatorname{div} E = 0$  (which is verified everywhere) would imply that the component along this axis is zero, and therefore does not contain a point of the wavefront set.

We can overcome this difficulty by noticing that the wavefront set of the 1-differential  $\tilde{E}$  associated to  $E$ , which is  $\tilde{E} = E_1 dX_1 + E_2 dX_2 + E_3 dX_3$  is

$$WF(\tilde{E}) = \cup_p WF(\tilde{E}_p).$$

We have the proposition on the differential forms, identical to the Theorem 10.2:

PROPOSITION 12.5. *Let  $\tilde{E}, \tilde{H}$  be solution of*

$$\begin{cases} \varepsilon \partial_t \tilde{E} = d^* \tilde{H} \\ d\tilde{E} = \mu \partial_t \tilde{H} \\ d\tilde{H} = 0 \\ d^* \tilde{E} = 0 \\ n \wedge \tilde{E} = Z(\tilde{H}^{tan}). \end{cases}$$

- (1) *This system is equivalent to a Helmholtz equation  $(-dd^* - d^*d - \varepsilon\mu\partial_{t^2}^2)\tilde{E} = 0$  and a system of boundary conditions including  $d^*E|_{\partial\Omega} = 0$ . then  $\rho_0 \notin WF_b(\tilde{E})$ ,*
- (2) *For  $\tilde{\rho}_0 \in \mathcal{E}$  for the operator  $P = -dd^* - d^*d - \varepsilon\mu\partial_{t^2}^2$ ,  $\rho_0 \notin WF_b(\tilde{E})$ .*
- (3) *For  $\rho_0 \in \mathcal{H}$ , we construct the two bicharacteristics  $\gamma_+$  and  $\gamma_-$  passing through the two points of  $(\Pi^{-1}(\rho_0) \cap \operatorname{Car}(p))$ . The following equivalence is true*

$$\rho_0 \in WF_b(\tilde{E}) \Leftrightarrow \gamma_+ \subset WF(\tilde{E}) \quad \text{or} \quad \gamma_- \subset WF(\tilde{E})$$

From the proof of Theorem 10.2, one deduces the proposition 12.5, which is the generalization of the proposition 10.6 obtained for waves with mixed boundary conditions

PROPOSITION 12.6. *The matrix of reflection coefficients associated with a hyperbolic point is a pseudodifferential matricial operator  $R_{\varepsilon,\mu}$  such that that*

$$E_r(x_1, x_2, 0, k) = R_{\varepsilon,\mu}(E_i|_{l=0})(x_1, x_2, k).$$

*Its principal symbol is the matrix*

$$(\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) Id_3 - \mathcal{B}^0(x', \xi'))^{-1} (\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) Id_3 + \mathcal{B}^0(x', \xi')).$$

*This matrix is diagonalizable in the image by  $M^{-1}$  of the basis which diagonalizes  $Z^0$ . The two reflection coefficients are respectively  $R_1(x_1, x_2, \eta_1, \eta_2)$  on the vector  $M^{-1}(-\eta_2, \eta_1, 0)$  and  $R_2(x_1, x_2, \eta_1, \eta_2)$  on the vector  $M^{-1}(\xi_1, \xi_2, 0)$ :*

$$R_1 = (\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) + Z_0^{-1} \frac{\mu}{\nu(x, \eta)})^{-1} (\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) - Z_0^{-1} \frac{\mu}{\nu(x, \eta)})$$

$$R_2 = (\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) + Z_0^{-1} \frac{\nu(x, \eta)}{\varepsilon})^{-1} (\xi_n^+(x', 0, \partial_{x'}\psi_+(x', 0, \xi')) - Z_0^{-1} \frac{\nu(x, \eta)}{\varepsilon}).$$

We have only two reflection coefficients for the components which are tangent to the plane because the divergence condition is naturally true for  $E_r$  as well as for  $E_i$  which imply that the vectors  $E_r$  and  $E_i$  are orthogonal to the wave vector thanks to the divergence condition. We thus find the third row of the matrix  $R_{\varepsilon,\mu}$ .



## Diffraction

### 1. The model problem of Friedlander

**1.1. End of the study of the model problem of Friedlander.** Recall that this problem was presented in section 3.

We state the first result of propagation of singularities in the case of a strictly diffractive point. It is the result obtained by Friedlander in the case of the model operator, which we have studied in detail in this work. The last steps before the end of the proof are questions 9), 10), 11) of the exercise 3. To make the result and its demonstration more understandable, we give the whole text of this exercise

Introduce the operator on  $\mathbb{R}_+ \times \mathbb{R}^2$

$$Pu(x, y_1, y_2) = \frac{\partial^2 u}{\partial x^2} - (1+x) \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}.$$

Consider

$$\begin{cases} Pu = 0, u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2) \\ u(0, y) = f(y), f \in \mathcal{E}'(\mathbb{R}^2) \\ u(x, y_1, y_2) = 0, y_1 < 0. \end{cases}^1$$

1) Let  $f \in \mathcal{S}(\mathbb{R}^2)$ . Prove that the solution of (3.85) writes

$$u(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{K}_x(\theta_1, \theta_2) \hat{f}(\theta_1, \theta_2) e^{iy_1\theta_1 + iy_2\theta_2} d\theta_1 d\theta_2$$

(Fourier oscillatory integral as an inverse Fourier transform of a function belonging to  $\mathcal{S}'(\mathbb{R}^2)$ ), where  $K_x$  is solution of

$$\begin{aligned} PK_x &= 0 \\ K_0(y) &= \delta_{y=0} \\ K_x(y_1, y_2) &= 0, y_1 < 0. \end{aligned}$$

In  $\Im\theta_1 < 0$ , one has

$$\hat{K}_x(\theta_1, \theta_2) = \frac{\text{Ai}(\theta_1^{-\frac{4}{3}}\theta_2^2 - (1+x)\theta_1^{\frac{2}{3}})}{\text{Ai}(\theta_1^{-\frac{4}{3}}\theta_2^2 - \theta_1^{\frac{2}{3}})} = \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)}.$$

One identifies  $\theta_1^{-\frac{4}{3}}$  by writing

$$\theta_1 = |\theta_1| e^{i(\frac{3\pi}{2} + \lambda)}, -\frac{\pi}{2} \leq \lambda \leq \frac{\pi}{2}$$

and  $\theta_1^{\frac{2}{3}} = -|\theta_1|^{\frac{2}{3}} e^{\frac{2i}{3}\lambda}$ ,  $\theta_1^{-\frac{4}{3}} = |\theta_1|^{-\frac{4}{3}} e^{-\frac{4i}{3}\lambda}$ .

For the sequel, denote by

$$\begin{aligned} |\theta| &= (\theta_1^2 + \theta_2^2)^{\frac{1}{2}} \\ Z &= |\theta_1|^{\frac{2}{3}} - (1+x)\theta_2^2 |\theta_1|^{-\frac{4}{3}} \\ Z_0 &= |\theta_1|^{\frac{2}{3}} - \theta_2^2 |\theta_1|^{-\frac{4}{3}}. \end{aligned}$$

2) Prove that  $\Phi$ , equal to

$$u^{\frac{1}{4}} \text{Ai}(u) \exp\left(\frac{2}{3}u^{\frac{3}{2}}\right) = \Phi(u)$$

admits, as well as its inverse, an asymptotic expansion for  $u \in \mathbb{R}_+$  large, in inverse powers of  $u^{\frac{3}{2}}$ , expansion valid uniformly in  $\arg u \in [-\pi + \varepsilon, \pi - \varepsilon]$ .

3) Let  $\sigma_0$  be a  $C^\infty$  function, equal to 0 for  $t \leq 1$ , equal to 1 for  $t \geq 2$ . Let  $\sigma_2(t) = \sigma_0(\delta_2 t)$ ,  $0 < \delta_2 < 1/2$ . On introduit

$$a_2(x, \theta_1, \theta_2) = \sigma_0(|\theta|)\sigma_2(Z_0) \frac{\Phi(\xi)}{\Phi(\xi_0)}.$$

a) Prove by induction that there exist functions  $Q_j^{k,\alpha}(x, \theta)$  of class  $C^\infty$ , homogeneous in  $\theta$  of homogeneity degree  $\frac{2}{3}(j+k) - |\alpha|$ , such that

$$\partial_{x^k}^k \partial_{\theta^\alpha}^\alpha [G(Z)] = \sum_{j=0}^{j=|\alpha|} G^{(k+j)}(Z) Q_j^{k,\alpha}(x, \theta)$$

b) Prove  $\sigma_2(Z_0) \in S_{1/3,0}^0(\mathbb{R}^2)$ .

c) Using the inclusion  $S_{1/3,0}^0 \subset S_{1/3,2/3}^0$ , prove that  $a_2 \in S_{1/3,2/3}^0$ .

4) a) Prove that

$$(1 - \sigma_0(|\theta|)) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} \in S^{-\infty}.$$

Let  $\sigma_1$  be an even function, equal to 0 on  $[1 - \delta_1, +\infty[$ , equal to 1 on  $[0, 1 - 2\delta_1]$  (for example  $\sigma_1(u) = 1 - \sigma_0(\delta_1|u| + 1 - 3\delta_1)$ ).

Let  $h(s) = \begin{cases} (s^2 + 1)^{-\frac{1}{2}}(s^2 - 1)^{\frac{3}{2}} & s \in [1, (1+x)^{\frac{1}{2}}] \\ (s^2 + 1)^{-\frac{1}{2}}[(s^2 - 1)^{\frac{3}{2}} - (s^2 - 1 - x)^{\frac{3}{2}}] & s \in [(1+x)^{\frac{1}{2}}, +\infty[ \end{cases}$ . It is bounded below by  $\gamma(x)$ , that one can compute, for  $s \geq (1 - \delta_1)^{-1}$ .

b) Prove

$$|\sigma_0(|\theta|)\sigma_1\left(\frac{\theta_1}{|\theta_2|}\right)\exp\left(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})\right)| \leq \exp\left(-\frac{2}{3}\gamma(x)|\theta|\right).$$

c) Prove, for all  $n$ , that there exists  $C_n$  such that

$$|\partial_{x^n}^n (\sigma_0(|\theta|)\sigma_1\left(\frac{\theta_1}{|\theta_2|}\right) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)})| \leq C_n \exp\left(-\frac{1}{3}\gamma(x)|\theta|\right).$$

d) Deduce values of  $m, \rho, \delta$  such that  $a_1(x, \theta) = \sigma_0(|\theta|)\sigma_1\left(\frac{\theta_1}{|\theta_2|}\right) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)}$ .

5) Denote by

$$a_3(x, \theta) = [\hat{K}_x(\theta) - (1 - \sigma_0(|\theta|)) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} - \sigma_0(|\theta|)\sigma_1\left(\frac{\theta_1}{|\theta_2|}\right) \frac{\text{Ai}(\xi)}{\text{Ai}(\xi_0)} - a_2(x, \theta)\exp\left(-\frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})\right)] \exp\left(\frac{2i}{3}Z^{\frac{3}{2}} \text{sign}\theta_1\right) 1_{x>\delta}.$$

a) Prove that the support  $S$  of  $a_3$  is given by the set of points

$$|\theta| \geq 1, |\theta_2| \leq (1 - 2\delta_1)^{-1}|\theta_1|, Z_0 \leq 2\delta_2.$$

b) Find the smallest cone containing  $S$ .

c) Prove that  $Z$  has a strictly positive lower bound on  $\{(x, \theta), x > \delta, |\theta_1| \geq (1 - 2\delta_1)|\theta_2|\}$ .

d) Prove that  $a_3 \in S_{1/3,2/3}^0$ .

6) Let  $\rho_0 = (0, 0, 0, \xi_0^0, \eta_1^0, \eta_2^0) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2) \cap \text{Car}(p)$ . What is the bicharacteristic of operator  $P$  defined by (3.84) passing through  $\rho_0$ ? It is useful to define

$$q(x, \eta_1, \eta_2) = (1+x)\eta_1^2 - \eta_2^2$$

and to define

$$S(x, \eta_1, \eta_2) = \int_0^x (q(u, \eta_1, \eta_2))^{\frac{1}{2}} du.$$

7) Let the function, defined on  $\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$ , given by

$$\phi(x, y_1, y_2, \theta_1, \theta_2) = y_1\theta_1 + y_2\theta_2 - S(x, \theta_1, \theta_2)\text{sign}(\theta_1).$$

Prove that the two bicharacteristics from the origin in  $y_1 > 0$  form the set

$$\Sigma = \{(x, y, \xi, \eta) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2), x > 0, |\theta_1| \geq |\theta_2|, \begin{array}{l} \xi = \partial_x \phi(x, y, \theta) \\ \eta = \nabla_y \phi(x, y, \theta) \\ \nabla_\theta \phi(x, y, \theta) = 0 \end{array} \}$$

8) Show that the singular support of the Fourier integral operator  $K^{(2)}$  of symbol  $a_2(x, \theta)$  and phase  $l(x, y, Y, \theta) = (y - Y)\theta - \frac{2}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})$  is included in the union of the bicharacteristics coming from the origin in  $y_1 > 0$ .

9) Find for the operator (3.84) the elliptic, hyperbolic, and glancing zones.

10) Let  $X_\delta = \{(x, y), x > \delta\}$ . We consider the restriction of the operator of symbol  $a_3 \exp(-\frac{2}{3}i \text{sign}(\theta_1) Z_0^{\frac{3}{2}})$  à  $X_\delta$ , que l'on note  $K_\delta^{(3)}$ . One considers the phase

$$\tilde{\phi}(x, y, \theta) = y \cdot \theta - \frac{2}{3} \text{sign}(\theta_1) Z^{\frac{3}{2}}$$

Under the condition

$$(1 + \delta)^{-\frac{1}{2}} < 1 - 2\delta_1$$

prove that the wave front of  $K_\delta^{(3)}$  is included in

$$\Lambda_\delta = \{(x, y, \xi, \eta), x > \delta, \theta \in S_3, \xi = \partial_x \tilde{\phi}, \eta = \nabla_y \tilde{\phi}, \nabla_\theta \tilde{\phi} = 0\}.$$

11) Prove the theorem of propagation of singularities for any ray, in other words

**THEOREM 13.1.** • The wave front set of the fundamental solution  $K$  is contained in the union of the outgoing bicharacteristics (in the direction  $y_1 > 0$ ) from the point  $(0, 0, 0)$ .

• The wave front set of the solution of (3.85) is contained in the set of bicharacteristics of  $P$  coming from a point  $(0, z, 0, \eta)$  belonging to the wavefront set of  $f$ .

**PROOF.** 9) We check that

$$p(x, \xi, \theta_1, \theta_2) = -\xi^2 + (1 + x)\theta_1^2 - \theta_2^2.$$

The elliptic manifold  $\mathcal{E}$  is the set of points  $(y, \theta_1, \theta_2)$  such that  $p(0, \xi, \theta_1, \theta_2) = 0$  has no real root, that is  $|\theta_2| > |\theta_1|$ .

The hyperbolic manifold is  $\mathcal{H} = \{|\theta_2| < |\theta_1|\}$ .

The equation of the glancing manifold  $\mathcal{G}$  is  $|\theta_1| = |\theta_2|$ .

Let us write  $P = \frac{\partial^2}{\partial x^2} + R(x, y, \partial_y)$ . On the glancing manifold, the principal symbol  $r$  of  $R$  satisfies  $r = 0$  and  $\partial_x r(0, \theta_1, \theta_2) > 0$ . Indeed,  $\partial_x r(0, \theta_1, \theta_2) = \theta_1^2$  and as  $\theta \neq 0$  and  $\theta_1 = \theta_2$ ,  $\theta_1 \neq 0$ . One says that the points of  $\mathcal{G}$  for the operator  $P$  defined by (3.84) are strictly diffractive points.

10) Let us introduce

$$K_\delta^{(3)}(x, \theta) = a_3(x, \theta) \exp(-\frac{2}{3}\xi^{\frac{3}{2}}) 1_{x > \delta}.$$

One knows already that  $a_3 \in S_{\frac{1}{3}, \frac{2}{3}}^0(\mathbb{R}^2)$  thanks to question 5) of Chapter 3. Moreover, when  $x > \delta$  and

$$(1.272) \quad 0 < \delta_1 < \frac{1}{2}(1 - (1 + \delta)^{-\frac{1}{2}})$$

we know, in a same way, that  $Z \geq D > 0$ , hence  $K_\delta^{(3)}(x, \theta) = a_3(x, \theta) \exp(-\frac{2}{3}i Z^{\frac{3}{2}} \text{sign} \theta_1)$ . The action of the operator  $K_\delta^{(3)}$  is computed through the oscillatory integral

$$\langle K_\delta^{(3)}, \chi \rangle = \int dx dy d\theta 1_{x > \delta} a_3(x, \theta) \chi(x, y) e^{i(y \cdot \theta - \frac{2}{3} Z^{\frac{3}{2}} \text{sign} \theta_1)},$$

because the function

$$(1.273) \quad \tilde{\phi}(x, y, \theta) = y \cdot \theta - \frac{2}{3} Z^{\frac{3}{2}} \text{sign} \theta_1$$

is an admissible phase in the sense of Lemma 6.1. The wavefront set of the distribution is then contained in the intersection  $\Lambda_\delta$  of  $\{x > \delta\}$  and of the Lagrangian manifold  $\Lambda_{\tilde{\phi}}$ , by immediate application of Proposition 6.2. 1) One checks that

$$\begin{aligned} \hat{K}_x(\theta)1_{x>\delta} &= (1 - \sigma_0(|\theta|))1_{x>\delta} \frac{Ai(\xi)}{Ai(\xi_0)} + a_1(x, \theta)1_{x>\delta} \\ &\quad + a_2(x, \theta)1_{x>\delta} \exp\left(-\frac{2i}{3}(\xi^{\frac{3}{2}} - \xi_0^{\frac{3}{2}})\right) \\ &\quad + a_3(x, \theta) \exp\left(-\frac{2i}{3}\xi^{\frac{3}{2}}\right)1_{x>\delta} \end{aligned}$$

Assume  $\delta_1$  and  $\delta_2$  satisfy

$$(1.274) \quad 0 < \delta_2 < \frac{1}{2}, 0 < \delta_1 < \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\delta}}\right),$$

and denote by  $L_{\delta_1, \delta_2, \delta}$  the subspace of  $T^*(\mathbb{R}^3)$  equal to

$$L_{\delta_1, \delta_2, \delta} = \Lambda_\delta \cap \{(x, y, \xi, \theta), (1 - 2\delta_1)|\theta_2| \leq |\theta_1| \leq (1 + \kappa(\delta_2))|\theta_2|\}.$$

Let  $K_\delta$  be the inverse Fourier transform of  $\hat{K}_x(\theta)1_{x>\delta}$ . One has

$$(1.275) \quad WF(K_\delta) \subset (\Sigma \cap \{x > \delta\}) \cup L_{\delta_1, \delta_2, \delta}.$$

One knows that  $WF(K_\delta)$  is a set independent on  $\delta_1$  and  $\delta_2$  satisfying (1.274). Hence

$$WF(K_\delta) \subset \Sigma \cap \{x > \delta\} \cap (\cap_{(\delta_1, \delta_2) \in (1.274)} L_{\delta_1, \delta_2, \delta}).$$

Note that  $\kappa(\delta_2)$  tends to 0 when  $\delta_2$  tends to 0, hence this intersection is included in  $\tilde{\Lambda}_\delta \cap \{|\theta_1| = |\theta_2|\}$ . As  $\partial_{y, \theta, x}(\frac{2}{3}Z_0^{\frac{3}{2}}) = Z_0^{\frac{1}{2}}\partial_{y, \theta, x}Z_0$  et que  $Z_0 = 0$  sur  $|\theta_1| = |\theta_2|$ , one concludes that

$$\tilde{\Lambda}_\delta \cap \{|\theta_1| = |\theta_2|\} \subset \Sigma \cap \{x > \delta\}.$$

Hence  $WF(K_\delta) \subset \Sigma \cap \{x > \delta\}$ , from which one has  $WF(K) \subset \Sigma$ . The first item of Theorem 13.1 is proven.

To prove the second relation, consider the bicharacteristics from a point  $(0, z)$ . We can then define in the same way  $\Sigma_z$ . The result then comes from the equality  $u_x = K_x \star f$ . We define the application from  $\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$  to  $\mathbb{R}_+ \times \mathbb{R}^2$  by  $\mu(x, y, z) = (x, y - z)$ . One has  $\mu^*F = F(x, y - z)$ , and  $u = \mu^*Kf$ . The wave front set of the operator  $\mu^*K$  is the subset of  $T^*(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$  defined through

$$WF'(\mu^*K) = \{(x, y, z, \xi, \eta, \zeta), (x, y, z, \xi, \eta, -\zeta) \in WF(\mu^*K)\},$$

thanks to the relations between the wavefront set of the convolution operator and the wavefront set of its kernel. We know moreover that

$$WF(\mu^*K) = \{(x, y, z, \xi, \eta, \zeta) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2), (x, y - z, \xi, \eta) \in WF(K)\}.$$

By definition of the bicharacteristics passing through a point of  $WF(f) \cap \{|\theta_1| \geq |\theta_2|\}$ , and thanks to the inclusion

$$WF(u) \subset WF'(\mu^*K)WF(f),$$

(applying the relation proven for  $WF'$  to  $WF(f)$ ), one has proven the second item of Theorem 13.1.  $\square$

## 2. The wave equation outside a convex smooth open set of $\mathbb{R}^2$

We present a calculation that uses all the notions introduced in the previous chapters. It is the calculation of the solution diffracted by  $\Omega$ , a strictly convex open set of boundary the closed smooth curve  $\mathcal{C}$  in  $\mathbb{R}^2$ . The interior open set is denoted by  $\Omega$ , it is bounded. The exterior open set is denoted  $\Omega^c$ . This calculation has already been done by Filippov [38], [100]. It is a difficult generalization of the Friedlander model presented in Section 3. This work has been done in collaboration with D. Bouche [16] for the explicit part in dimension

2 and is issued from G. Lebeau for the Dirichlet boundary condition [66] for the theoretical part.

After partial Fourier transformation in time, the problem writes

$$(2.276) \quad \begin{cases} (\Delta + k^2)u = 0 \text{ in } \Omega^c \\ \mathcal{L}u = 0 \text{ on } \partial\Omega \\ u - u_i = 0 \text{ for } \Im k < 0, \end{cases}$$

$\mathcal{L}$  being a differential operator of order 1.

**2.1. Laplace operator in Euler coordinates.** Let us define the coordinates used. Let  $A$  be a point of  $\mathcal{C}$ , fixed. A point of  $\mathcal{C}$  is characterized by its curvilinear abscissa  $s$ , counted from  $A$ , and choosing an orientation for it. The length of  $\mathcal{C}$  being  $L$ , we have  $s \in [0, L]$ , with condition  $u|_{s=L} = u|_{s=0}$ . We replace then the problem with a boundary condition on  $\partial\Omega$  by a problem with a boundary condition on  $\mathbb{R}$  using periodization. The curve  $\mathcal{C}$  is characterized by the function  $R(s)$ , radius of curvature at the point point  $M(s)$ , supposed to be strictly positive and finite, which corresponds to a strictly convex open set.

From a point  $M(s)$  of  $\mathcal{C}$ , we consider in  $\Omega^c$  the distance to  $M(s)$  on the normal unit vector, and denote it by  $n$ . Thus, a point  $M(s, n)$  in a neighborhood of  $\mathcal{C}$  is given by

$$\vec{AM}(s, n) = \vec{AM}(s) + n\vec{n}(s).$$

It is assumed that there exists a neighborhood of  $\mathcal{C}$  which is completely geodesic, which means that all points of this neighborhood are on a unique normal line.

Let  $P$  be the operator  $\Delta + k^2$ , expressed in coordinates  $(s, n)$ . We have

$$(2.277) \quad Pu = \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial}{\partial s} \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial u}{\partial s} + \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial}{\partial n} \left(1 + \frac{n}{R(s)}\right) \frac{\partial u}{\partial n} + k^2 u.$$

In the  $k^{-1}$ -pseudodifferential calculus, its principal symbol is written

$$p(s, n, \sigma, \xi_2) = 1 - \xi_2^2 - \left(1 + \frac{n}{R(s)}\right)^{-2} \sigma^2.$$

The space  $T^*(]0, L])$  is decomposed into elliptic, hyperbolic and glancing as follows:

$$\mathcal{E} = \{(s, \sigma), |\sigma| > 1\}$$

$$\mathcal{H} = \{(s, \sigma), |\sigma| < 1\}$$

$$\mathcal{G} = \{(s, \sigma), |\sigma| = 1\}$$

Note that for  $n = \xi_2 = 0$ , we find propagation on the boundary at constant speed 1 (given by  $\sigma^2 = 1$ ), and we recognize here the equation of the classical light cone in dimension 1. The relation  $\sigma^2 = 1$  entails  $\sigma = \pm 1$ , which implies that the solution contains a factor  $e^{\pm iks}$ . We study the solution diffracted around the point  $\sigma = 1$ . To do so, perform a translation in the variable  $\sigma$  to  $\sigma = 0$  by a conjugation technique. Introduce then the operator  $P_1$  defined by

$$(2.278) \quad P_1(U)(n, s, k) = e^{-iks} P(e^{iks} U(n, s, k)).$$

This operator  $P_1$  is given by:

$$(2.279) \quad P_1(U) = \left(1 + \frac{n}{R(s)}\right)^{-1} \left[ \frac{\partial}{\partial n} \left( \left(1 + \frac{n}{R(s)}\right) \frac{\partial U}{\partial n} \right) + \frac{\partial}{\partial s} \left( \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial U}{\partial s} \right) + ik \left[ \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial U}{\partial s} + \frac{\partial}{\partial s} \left( \left(1 + \frac{n}{R(s)}\right)^{-1} U \right) \right] + k^2 U \left(1 - \left(1 + \frac{n}{R(s)}\right)^{-2}\right). \right.$$

The principal symbol of  $P_1$  is  $p_1(n, s, \xi_2, \sigma) = 1 - \xi_2^2 - \left(1 + \frac{n}{R(s)}\right)^{-2} (\sigma + 1)^2$ , and  $p_1$  vanish in the neighborhood of  $\sigma = 0$  for

$$(2.280) \quad n = \frac{R(s)\sigma}{(1 - \xi_2^2)^{\frac{1}{2}}} + R(s) \left( (1 - \xi_2^2)^{-\frac{1}{2}} - 1 \right).$$

We note that the calculation of the principal symbol can also be done by applying the result of the exercise 7.3, by introducing the function  $S(s, n, k) = ks$ , which satisfies  $\partial_s S = k$ .



Let us introduce the dual change of variable, deduced from (2.280) by the restriction to  $\xi_2 = 0$  of the equation of the characteristic manifold, i.e.  $\tau = R(s)\sigma$ . It corresponds formally to the equality  $\frac{\partial}{\partial s} = R^{-1}(s)\frac{\partial}{\partial \theta}$ . We define the function  $\theta$  by

$$\theta(s) = \int_0^s \frac{du}{R(u)}.$$

This angle  $\theta$  is then the angle between a fixed direction and the tangent line to the boundary (with the precaution of restricting this angle to  $(0, \pi)$ ), called the Euler angle of the curve  $\mathcal{C}$ . The change of variable  $s \rightarrow \theta$  is a diffeomorphism, which allows to write the operator  $P_1$  in the system of variables  $(\theta, n)$ . Denoting by  $s = s(\theta)$  this diffeomorphism, we obtain:

$$(2.281) \quad \begin{aligned} P_1(U) = & k^2 U \left(1 - \left(1 + \frac{n}{R(s)}\right)^{-2}\right) + \left(1 + \frac{n}{R(s)}\right)^{-1} \left[ \frac{\partial}{\partial n} \left( \left(1 + \frac{n}{R(s)}\right) \frac{\partial U}{\partial n} \right) \right] \\ & + (R(s) + n)^{-1} \frac{\partial}{\partial \theta} \left( (n + R(s))^{-1} \frac{\partial U}{\partial \theta} \right) \\ & + ik(n + R(s))^{-1} \left[ \left(1 + \frac{n}{R(s)}\right)^{-1} \frac{\partial U}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \left(1 + \frac{n}{R(s)}\right)^{-1} U \right) \right]. \end{aligned}$$

The principal symbol of this operator is:

$$(2.282) \quad \hat{p}_1(n, \theta, \xi_2, \tau) = 1 - \xi_2^2 - \left(1 + \frac{n}{R(s)}\right)^{-2} \left( \frac{\tau}{R(s(\theta))} + 1 \right)^2.$$

The equation of the characteristic manifold is

$$(2.283) \quad 1 - \xi_2^2 = \left(1 + \frac{n}{R(s(\theta))}\right)^{-2} \left( \frac{\tau}{R(s(\theta))} + 1 \right)^2,$$

and this yields

$$(2.284) \quad n = \tau + \frac{1}{2} \xi_2^2 (R(s(\theta)) + \tau) + O(\xi_2^4).$$

In a first part, we write an asymptotic outgoing solution  $u_k$  (in the sense that it is defined for  $\Im k < 0$  and uniformly bounded in this region) using the phase analysis techniques constructed in Chapter 9, in particular in Lemma 9.3. We compute then the trace  $\mathcal{L}u_k$  on the boundary of this asymptotic outgoing solution.

In the Friedlander model problem, the relation  $u_k|_{n=0} = 1$  previously allowed us to obtain the fundamental solution. Here, this method cannot be used any more; indeed, the trace on the boundary is defined as an integral which is a non local operator. Lafitte [57] and Lebeau [66] have developed techniques allowing to invert this equality on the boundary, and to make the calculation explicit. This method corresponds to the boundary layer method used by Bouche and Molinet [15].

**2.2. Another asymptotically outgoing solution.** The method we present in this paragraph is directly inspired by Lebeau's work [67], [66] where outgoing asymptotic solutions of operators having a strictly diffractive point are studied. It is in particular different from the construction of the solution shown in the chapter 10 which uses the two Fourier integral operators  $A_+$  and  $A_-$ . It is related to the calculations of D. Ludwig presented in the chapter 9 in the section 4.

Let  $(n, \xi_2, \theta, \tau) \in T^*(\Omega_c)$ ,  $\beta \in ]0, \theta(L)[$  in a neighborhood of an boundary point. We give ourselves a symbol  $a(n, \xi_2, \theta, \beta, \tau, k)$  and a phase  $\phi(\theta, \tau, \xi_2)$  and we construct  $J(f)$ , a function defined in a neighborhood of the boundary  $n = 0$  depending on  $f$  defined on the boundary by

$$(2.285) \quad J(f)(n, \theta, k) = \int e^{ik[(\theta-\beta)\cdot\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} a(n, \xi_2, \theta, \beta, \tau, k) f(\beta, k) d\beta d\tau d\xi_2$$

We construct here a phase  $\phi$  and a symbol  $a$  providing a general solution of  $P_1$ . In all this Section, we simplify notations and denote by  $R$  the function  $R(s(\theta))$ .

We introduce the operators  $T_0, T_1, T_2$  of order 0, 1, and 2 equal to:

$$\begin{aligned} T_0(n, \theta, \xi_2, \tau) &= \frac{\xi_2}{n+R} + \partial_\theta \left( \frac{\tau + \partial_\theta \phi}{(n+R)^2} \right) + \frac{R'}{R} \frac{n}{n+R} - \left[ (1 - \xi_2^2) \frac{\partial_{\xi_2}^2 \phi}{R} + 4 \left( 1 - \frac{\partial_{\xi_2} \phi}{R} \right) \xi_2 \right] (n+R)^{-1}, \\ T_1(n, \theta, \xi_2, \tau, \partial_n, \partial_\theta, \partial_{\xi_2}) &= \xi_2 \frac{\partial}{\partial n} + (1 - \xi_2^2) \left( 1 - \frac{\partial_{\xi_2} \phi}{R} \right) \frac{\partial}{\partial \xi_2} + \frac{\tau + \partial_\theta \phi + n + R}{(n+R)^2} \frac{\partial}{\partial \theta}, \\ T_2 &= (n+R)^{-2} \left[ (1 - \xi_2^2) \frac{\partial^2}{\partial \xi_2^2} - 4 \xi_2 \frac{\partial}{\partial \xi_2} - 2 \right] - \left[ \frac{\partial^2}{\partial n^2} + (n+R)^{-1} \frac{\partial}{\partial n} + (n+R)^{-1} \frac{\partial}{\partial \theta} \left( (n+R)^{-1} \frac{\partial}{\partial \theta} \right) \right]. \end{aligned}$$

PROPOSITION 13.1. *Let*

$$J(f)(n, \theta, k) = \int e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} a(n, \xi_2, \theta, \beta, \tau, k) f(\beta, k) d\beta d\tau d\xi_2.$$

For all  $M$  integer,  $P_1(J(f)) = O(k^{-M})$  if and only if

- the function  $\phi$  is solution of

$$\hat{p}_1 \left( -\frac{\partial \phi}{\partial \xi_2}, \theta, \tau + \frac{\partial \phi}{\partial \theta}, \xi_2 \right) = 0$$

(eikonal equation),

- the classical symbol  $a$ , developed in inverse powers of  $ik$ , verifies,

$$2T_1 a_0 + T_0 a_0 = 0,$$

and, for all  $M$ :

$$2T_1 a_j + T_0 a_j = -T_2 a_{j-1}, 1 \leq j \leq M.$$

This representation differs from the one used to study transverse reflection, where the behavior in variables  $(n, \theta)$  was computed by a function  $\phi(\theta, n, \tau)$  such that  $\phi(\theta, 0, \tau) = \theta\tau$ . Here, we consider a different phase in  $n$  because we want to take into account the singularity in  $\sqrt{n}$  which appears in the equation of the characteristic manifold.

PROOF. We consider this problem as an initial value problem associated with a differential equation in  $\xi_2$ , for using a representation of the Lagrangian manifold in coordinates  $(\xi_2, \theta)$ , the initial condition being  $\phi(\theta, \tau, 0) = 0$ . We look for conditions on  $a$  and  $\phi$  such that  $J(f)$  is an asymptotic solution of  $P_1$  for all  $f$  defined on the boundary. More precisely, we compute  $P_1(J(f))$  and reorder its expression according to the powers of  $k^{-1}$ .

The principal term of  $P_1(J(f))$ , which is of order  $k^2$ , is written:

$$k^2 \int \hat{p}_1 \left( n, \theta, \xi_2, \tau + \frac{\partial \phi}{\partial \theta} \right) a(n, \xi_2, \theta, \beta, \tau, k) f(\beta, k) e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} d\beta d\tau d\xi_2$$

If we were to use the traditional approach used for asymptotic expansions, we would look for  $\phi$  solution of

$$\hat{p}_1 \left( n, \theta, \xi_2, \tau + \frac{\partial \phi}{\partial \theta} \right) = 0.$$

But, since the principal symbol depends on  $n$ , one cannot find a  $\phi$  phase independent of  $n$  verifying this partial differential equation. To get rid in part of the dependence in  $n$  of the main symbol principal symbol, we use the two integration by parts equalities written below, valid for any function  $A(n, \xi_2, \theta, \beta, \tau, k)$  (by discarding the boundary terms):

$$\begin{aligned} (2.286) \quad & \int n A e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} d\beta d\tau d\xi_2 \\ &= \\ & - \int \left[ -\frac{\partial \phi}{\partial \xi_2} A + ik^{-1} \frac{\partial A}{\partial \xi_2} \right] e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} d\beta d\tau d\xi_2 \end{aligned}$$

$$\begin{aligned} (2.287) \quad & \int n^2 A e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} d\beta d\tau d\xi_2 \\ &= \\ & \int \left[ \left( \frac{\partial \phi}{\partial \xi_2} \right)^2 A - ik^{-1} \left( \frac{\partial^2 \phi}{\partial \xi_2^2} A + 2 \frac{\partial \phi}{\partial \xi_2} \frac{\partial A}{\partial \xi_2} \right) - k^{-2} \frac{\partial^2 A}{\partial \xi_2^2} \right] e^{ik[(\theta-\beta)\tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]} d\beta d\tau d\xi_2. \end{aligned}$$

For this, we notice that

$$\hat{p}_1 \left( n, \theta, \xi_2, \tau + \frac{\partial \phi}{\partial \theta} \right) = \left( 1 + \frac{n}{R} \right)^{-2} \left[ (1 - \xi_2^2) (1 + 2R^{-1}n + R^{-2}n^2) - (R^{-1}\tau + 1)^2 \right].$$

Considering  $A = A_1$  in the relation (2.286), with  $A_1$  given by:

$$A_1(n, \xi_2, \theta, \beta, \tau, k) = \frac{2}{R} \left(1 + \frac{n}{R}\right)^{-2} (1 - \xi_2^2) a(n, \xi_2, \theta, \beta, \tau, k) f(\beta, k),$$

and considering  $A = A_2$  in (2.287), with  $A_2$  given by

$$A_2(n, \xi_2, \theta, \beta, \tau, k) = R^{-2} \left(1 + \frac{n}{R}\right)^{-2} (1 - \xi_2^2) a(n, \xi_2, \theta, \beta, \tau, k) f(\beta, k),$$

the principal term in  $k$  of  $P_1(J(f))$  becomes:

$$k^2 \int^3 d\beta d\tau d\xi_2 \frac{a}{\left(1 + \frac{n}{R}\right)^2} \left[ (1 - \xi_2^2) \left(1 - \frac{2}{R} \partial_{\xi_2} \phi + \left(\frac{\partial_{\xi_2} \phi}{R}\right)^2\right) - \left(1 + \frac{\tau + \partial_{\theta} \phi}{R}\right)^2 \right] f(\beta, k) e^{ik[(\theta - \beta) \cdot \tau + n\xi_2 + \phi(\theta, \tau, \xi_2)]}.$$

The new eikonal equation (obtained by cancellation of the principal term in  $k^2$  for any function  $f$  and for any symbol  $a$ ) amounts to replacing  $n$  by  $-\frac{\partial \phi}{\partial \xi_2}$  and  $\tau$  by  $\tau + \frac{\partial \phi}{\partial \theta}$  in the characteristic equation (2.283) :

$$(2.288) \quad -\frac{\partial \phi}{\partial \xi_2} = \frac{\tau + \frac{\partial \phi}{\partial \theta}}{(1 - \xi_2^2)^{\frac{1}{2}}} + R((1 - \xi_2^2)^{-\frac{1}{2}} - 1).$$

The Taylor expansion (2.284) leads thus to the relation

$$(2.289) \quad -\frac{\partial \phi}{\partial \xi_2} = \tau + \frac{\partial \phi}{\partial \theta} + \frac{1}{2} (R(s(\theta)) + \tau + \frac{\partial \phi}{\partial \theta}) \xi_2^2 + O(\xi_2^4) ((1 - \xi_2^2)^{-\frac{1}{2}} - 1).$$

Assume  $\phi(\theta, \tau, \xi_2) = \phi_0(\theta, \tau) + \xi_2 \phi_1(\theta, \tau) + \frac{1}{2} \xi_2^2 \phi_2(\theta, \tau) + \frac{1}{3} \phi_3(\theta, \tau) \xi_2^3 + O(\xi_2^4)$ . By replacing this expansion in the relation (2.289), we verify that the phase  $\phi$  admits the following expansion in the neighborhood of  $\xi_2 = 0$ :

$$(2.290) \quad \phi(\theta, \tau, \xi_2) = -\tau \xi_2 - \frac{1}{6} (R(s(\theta)) + \tau) \xi_2^3 + O(\xi_2^4)$$

The symbol  $a$ , on the other hand, verifies the explicit transport equations obtained by cancelling the term of magnitude  $k^1$  term and the term of magnitude  $k^0$  in the form of the operator obtained after the integrations by parts using  $A_1$  and  $A_2$ . We thus identify in the last equality of Proposition 13.1 the vector field  $T_1$ , the function  $T_0$  and the remainder term  $T_2$  of the Laplace operator (which in fact represents the operator  $P_1$  applied to the symbol  $a$ ).

This completes the proof of the proposition 13.1.  $\square$

### 3. Expression of the parametrix through Fourier-Airy integral operators

**3.1. Symbols for the parametrix.** The parametrix written in (2.285) rewrites using the modified Airy functions:

$$(3.291) \quad w_1(\xi) = Ai(e^{\frac{2i\pi}{3}} \xi), \quad w_2(\xi) = Ai'(e^{\frac{2i\pi}{3}} \xi)$$

as stated in the following Proposition:

**PROPOSITION 13.2.** *There are two functions  $\phi_0(n, \theta, \tau)$  and  $\alpha(n, \theta, \tau)$ , holomorphic in the neighborhood of the strictly diffractive point  $(0, 0, 0)$ , as well as two classical symbols  $s_0$  and  $s_1$  such that*

$$J(f)(n, \theta, k) = \int e^{ik((\theta - \beta)\tau + \phi_0(n, \theta, \tau))} f(\beta, k) S(n, \theta, \beta, \tau, k) d\beta d\tau,$$

where the symbol  $S(n, \theta, \beta, \tau, k)$  is given by

$$S(n, \theta, \beta, \tau, k) = s_0(n, k^{\frac{2}{3}} \alpha, \theta, \beta, \tau, k) \omega_1(k^{\frac{2}{3}} \alpha) + k^{-\frac{1}{3}} s_1(n, k^{\frac{2}{3}} \alpha, \theta, \beta, \tau, k) \omega_1'(k^{\frac{2}{3}} \alpha) + R.$$

This result is based on the representation lemma for the phase  $\phi$  solution of the eikonal equation:

LEMMA 13.1. *There exist two functions  $\gamma(\theta, \tau)$  and  $\rho(\theta, \tau)$ , holomorphic in a neighborhood of  $(\theta, \tau) = (0, 0)$ , such that  $\phi_0(0, \theta, \tau) = \tau^2 \gamma(\theta, \tau)$  and  $\alpha(0, \theta, \tau) = -\tau \rho(\theta, \tau)$ , with, in addition  $\rho(\theta, 0) = (\frac{2}{R(s(\theta))})^{\frac{1}{3}}$ .*

PROOF. We note that the phase  $\phi(\theta, \tau, \xi_2) + n\xi_2$  has two critical points (i.e. two values of  $\xi_2$  such that  $\partial_{\xi_2}(\phi(\theta, \tau, \xi_2) + n\xi_2) = 0$  in a neighborhood of  $(\tau, \xi_2) = (0, 0)$ ). These critical points, denoted by  $(\xi_2^c)^+$  and  $(\xi_2^c)^-$ , are solution of  $n - \tau - \frac{1}{2}(R(s(\theta)) + \tau)(\xi_2^c)^2 + O((\xi_2^c)^3) = 0$ . The phase considered here has thus two critical values that we denote respectively by  $\phi_+(n, \theta, \tau)$  and  $\phi_-(n, \theta, \tau)$ . There are two functions  $\phi_0(n, \theta, \tau)$  and  $\alpha(n, \theta, \tau)$  and a holomorphic change of variable  $z \leftrightarrow \xi_2$  (already obtained in the section 4) such that we have

$$(3.292) \quad \phi(\theta, \tau, \xi_2) + n\xi_2 = \phi_0(n, \theta, \tau) - \frac{1}{3}z^3 - \alpha(n, \theta, \tau)z.$$

The two critical values of the phase in  $z$  written in (3.292) are equal to  $\phi_0(n, \theta, \tau) \pm \frac{2}{3}\alpha^{\frac{3}{2}}$ . By invariance of the critical values during a change of variable, we find:

$$\begin{cases} \phi_0(n, \theta, \tau) = \frac{1}{2}(\phi_+ + \phi_-) \\ \alpha(n, \theta, \tau) = (-\frac{3}{4}(\phi_+ - \phi_-))^{\frac{2}{3}} \end{cases}$$

Using the result already used for the study of fold singularities for the Lagrangian manifold (Proposition 9.7), the exact change of variable  $\xi_2 \leftrightarrow z$  in (2.285) yields

$$(3.293) \quad J(f)(n, \theta, k) = \int e^{ik[(\theta-\beta)\cdot\tau + \phi_0(n, \theta, \tau) - \frac{1}{3}z^3 - \alpha(n, \theta, \tau)z]} a(n, \xi_2(z), \theta, \beta, \tau, k) \frac{d\xi_2}{dz} f(\beta, k) d\beta d\tau dz.$$

Using Boutet de Monvel's division lemma [18], we obtain the existence of  $s_0, s_1, h$  such that:

$$(3.294) \quad a(n, \xi_2(z), \theta, \beta, \tau, k) \frac{d\xi_2}{dz} = s_0(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k) + zs_1(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k) + [\partial_z h + ikh(-z^2 - \alpha(n, \theta, \tau))].$$

We notice that the previous integral splits into three terms.

The last term is exactly zero (we integrate a differential).

The first term involves the integral  $\int dz e^{-ik(\frac{1}{3}z^3 + \alpha(n, \theta, \tau)z)}$ , which is an Airy function because it is the inverse Fourier transform of  $e^{i\frac{1}{3}t^3}$ .

Remark that we do not consider the usual Airy function,  $w_1$  is one of the (complex) oscillating solutions of the Airy equation which is used for considering incoming waves once a direction is fixed (see [58] for an explanation).

The second term is the derivative with respect to argument of the previous Airy function. We end up with:

$$(3.295) \quad J(f)(n, \theta, k) = \int e^{ik[(\theta-\beta)\cdot\tau + \phi_0(n, \theta, \tau)]} k^{-\frac{1}{3}} [s_0(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k) w_1(k^{\frac{2}{3}}\alpha) + k^{-\frac{1}{3}} s_1(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k) w_1'(k^{\frac{2}{3}}\alpha)] f(\beta, k) d\beta d\tau + R$$

The definition of critical points yields  $\xi_2^c = \pm((\frac{2}{R(s(\theta))+\tau})(n-\tau))^{\frac{1}{2}} + O(n-\tau)$ , hence  $\phi_{\pm} = O((n-\tau)^2) \pm \frac{2}{3}(\frac{2}{R(s(\theta))+\tau})^{\frac{1}{2}}(n-\tau)^{\frac{3}{2}}$ .

One deduces  $\phi_0(n, \theta, \tau) = O((n-\tau)^2)$  et  $\alpha(n, \theta, \tau) = (\frac{2}{\rho})^{\frac{1}{3}}(n-\tau) + O(\tau) + O((n-\tau)^{\frac{3}{2}})$ . We check that  $\phi_0(0, \theta, \tau)$  is of the form  $O(\tau^2)$  and that  $\alpha(0, \theta, \tau) = O(\tau)$ . This completes the proof of the lemma.  $\square$

The rest  $R$  that we wrote in Proposition 13.2 comes from the fact that the Airy functions are obtained when computing the integral on  $(-\infty, +\infty)$ , while the integral considered here is a local integral in  $\xi_2$ , on a contour connecting  $-\delta_0 - i\delta_0^2\delta_2$  to  $i\delta_0 + \delta_1\delta_0^{\frac{3}{2}}$ . Proposition 3.1 p 1451 of [66] allows to verify that the remainder is controlled when  $0 < \delta_0 < D_1$  and when  $|\delta_1|, \delta_2$  are smaller than  $D_0$ .

We deduce then

COROLLARY 13.1. Let  $\xi_0 = k^{\frac{2}{3}}k^{\frac{2}{3}}(\frac{2}{R(s(\theta))})^{\frac{1}{3}}$  and let  $\nu = k^{\frac{2}{3}}(\frac{2}{R(s(\theta))})^{\frac{1}{3}}n$  (which is therefore the variable  $n$  stretched according to the frequency as in Ludwig [70]). We verify that

$$(3.296) \quad \begin{aligned} k^{\frac{2}{3}}\alpha(n, \tau, \theta) &= k^{\frac{2}{3}}(\frac{2}{R(s(\theta))})^{\frac{1}{3}}n - k^{\frac{2}{3}}(\frac{2}{R(s(\theta))})^{\frac{1}{3}}\tau + \beta(n, \theta, \tau, k) \\ &= \nu - \xi_0 + \beta(n, \theta, \tau, k). \end{aligned}$$

**3.2. Boundary Fourier-Airy operator associated with the Dirichlet boundary condition.** The symbol of the Fourier integral operator (3.295) is expressed through the function  $w_1$ , which is divergent at infinity. We introduce in this paragraph the domain of the plane  $\mathbb{C}^M$  complementary of

$$\{z, |z - Me^{\frac{i\pi}{3}}| < \varepsilon, |\Im(e^{-\frac{i\pi}{3}}z)| \leq \varepsilon(\operatorname{Re}(e^{-\frac{i\pi}{3}}z) - M)\}.$$

We introduce the functions, denoted  $w_1^M$  and  $w_2^M$ , bounded in  $\mathbb{C}^M$ , equal to

$$(3.297) \quad \begin{cases} w_1^M(\theta) = (e^{\frac{2i\pi}{3}}\theta + M)^{\frac{1}{4}}e^{-\frac{2}{3}(e^{\frac{2i\pi}{3}}\theta + M)^{\frac{1}{2}}(e^{\frac{2i\pi}{3}}\theta - M/2)}w_1(\theta) \\ w_2^M(\theta) = (e^{\frac{2i\pi}{3}}\theta + M)^{-\frac{1}{4}}e^{-\frac{2i\pi}{3}}e^{-\frac{2}{3}(e^{\frac{2i\pi}{3}}\theta + M)^{\frac{1}{2}}(e^{\frac{2i\pi}{3}}\theta - M/2)}w_1'(\theta). \end{cases}$$

These functions are the functions  $A_1^M$  and  $A_2^M$  calculated at  $e^{\frac{2i\pi}{3}}\theta$  introduced in [66].

**Canonical manifold associated with the boundary operator and Sjöstrand spaces**

The notations and spaces introduced in this paragraph are written in detail in [89], to which we refer the reader. We introduce the subharmonic functions on  $\mathbb{C}$ :

$$\begin{cases} \psi_0(\theta) = \frac{1}{2}(\Im\theta)^2 \\ \psi_r(\theta) = \psi_0(\theta) - \frac{1}{4}(\sup|\operatorname{Re}\theta| - r, 0)^2 \end{cases}$$

Using the phase  $\phi_0$  we define a special special subharmonic function  $l_0$  as in [66]. By application of the theorem 8.1, we know that the canonical manifold  $\Lambda_{l_0} = \{(\beta, \frac{2}{i}\frac{dl_0}{d\beta})\}$  allowing to compute it is

$$\{(\partial_\tau\phi(\theta, \tau, \xi_2^*), \tau)\},$$

with following conditions on  $\xi_2^*$ :

$$\begin{cases} \partial_{\xi_2}\phi(\theta, \tau, \xi_2^*) = 0 \\ \partial_\theta\phi(\theta, \tau, \xi_2^*) = -\Im\theta. \end{cases}$$

Thanks to the change of variable, we can see that this system is equivalent to

$$\begin{cases} z^2(\theta, \tau, \xi_2^*) = -\alpha(0, \theta, \tau) \\ \tau^2\partial_\theta\gamma(\theta, \tau) = z(\theta, \tau, \xi_2^*)\partial_\theta\alpha(0, \theta, \tau) - \Im\theta. \end{cases}$$

The canonical manifold is written

$$\{(\partial_\tau(\tau^2\gamma) - \partial_\tau\alpha z, \tau)\} = \{(\partial_\tau(\tau^2\gamma) - \partial_\tau\alpha/\partial_\theta\alpha[\tau^2\partial_\theta\gamma(\theta, \tau) + \Im\theta], \tau)\}.$$

G. Lebeau used the Taylor expansion of the phase to give the proof of the existence of the function  $l_0$ . We limit ourselves here to giving an explicit method of construction, but which cannot be applied as is. This subharmonic phase defines the Sjöstrand space on which the Fourier integral operator given by (3.295) extends naturally. We recall then (Lebeau [66] p 1439-1440), that  $H_{l_0}(|\theta| < a)$  is the set of holomorphic functions  $\theta \rightarrow f(\theta, k)$ ,  $\theta \in \mathbb{C}$  on  $|\theta| < a$  such that

$$\forall K \subset \{|\theta| < a\}, \forall \varepsilon, \exists C_{\varepsilon, K} \text{ such that, } \forall x \in K, \forall k \geq 1, |f(\theta, k)| \leq C_{\varepsilon, k}e^{k(l_0(\theta) + \varepsilon)}.$$

Similarly,  $H_{\phi_0}^{+, \kappa}(|\theta| < a)$  is the set of functions  $f(n, \theta, k)$ , holomorphic for  $\theta \in X = \{|\theta| < a\}$  and for  $n$  in a neighborhood  $U$  in  $\mathbb{C}$  of  $[0, \kappa]$ , such that  $f$  is uniformly bounded by  $e^{kC}$  on  $(n, \theta) \in U \times X$ ,  $k \geq 1$  and whose  $p$ -th derivative in  $n$  is bounded on any compact  $K$  in  $X$  and on  $n \in [0, \kappa]$  by  $C_{\varepsilon, p, K}e^{k(\varepsilon + \phi_0(\theta))}$ . The analytical spectrum of  $f$  in  $T^*(]0, \kappa[ \times \Lambda_{\phi_0})$  is then denoted by  $SS_{\delta, \phi_0}^{w, \kappa}$ .

**Symbol of the non local operator** Let us introduce the symbol

$$\sigma^M(\theta, t, k) = (M - k^{\frac{2}{3}}t^2\rho(\theta, t^2))^{\frac{1}{4}}e^{\frac{2}{3}kt^3\rho^{\frac{3}{2}}(\theta, t^2) + \frac{2}{3}(M - k^{\frac{2}{3}}t^2\rho(\theta, t^2))^{\frac{1}{2}}(k^{\frac{2}{3}}t^2\rho(\theta, t^2) + M/2)}$$

and the phase  $F(\theta, t) = t^4\gamma(\theta, t^2) + \frac{2}{3}t^3\rho^{\frac{3}{2}}(\theta, t^2)$ .

The non-local operator that defines the parametrix is

$$I(h)(\theta, k) = \int d\beta \int 2tdte^{ik[(\theta-\beta)t^2+F(\theta,t)]}\sigma^M(\theta, t, k)h(\beta, k).$$

We write the result of Proposition 3.2 of [66]:

LEMMA 13.2. *If  $X$  defines the disk of radius  $D_2r^2$  and if the integral defining  $I$  is on the disk of radius  $r/D_2$ , then  $h \in H_{\psi_r}(B_d) \Rightarrow I(h) \in H_{l_0}(X)$ .*

This lemma allows to define the parametrix that we use which is  $J \circ I$ . We then have the Proposition ([66], propositions 3.1 et 3.2).

PROPOSITION 13.3. *Let  $h \in H_{\psi_r}(B_d)$ . Then  $I(h) \in H_{l_0}(X)$ , and we can calculate  $J(h)$  for  $Dr^2 \leq d_0$ .*

- i) *The distribution  $J \circ I(h)$  belongs to  $H_{\phi_0}^{+,\kappa}(|\theta| < d_2)$  for  $d_2$  and  $\kappa$  small enough.*
- ii)  *$J \circ I(h)$  is an asymptotic solution of  $P$ , that is*

$$SS_{\partial, \phi_0}^{w, \kappa}(P(J \circ I(h))) = \emptyset.$$

PROOF. We obtain a solution depending on a Fourier integral operator of symbol bounded in the variables  $(\nu, \xi_0)$ . This imposes that we restricted to the region where  $\xi_0$  is controlled; more precisely, the mathematical analysis of [66] imposes the majoration  $\Im \xi_0 \leq c + \delta|\text{Re } \xi_0|$ , this majoration will be specified hereafter. On the other hand, we keep the variable  $\nu$  as a notation for the *stretching* of the variable  $n$  from the previous section. The result is an expression in  $\nu$ , which gives the size of the transition zone. The majorations of the Airy functions allow to give a uniform, but oscillating in the neighborhood of  $V = \{(n, \theta, \tau), |k^{\frac{2}{3}}\alpha(n, \theta, \tau)| \leq C\}$ , representation. The calculation,  $\phi_0$  and  $\alpha$  being known, of the solution  $J(f)(n, \theta, k)$  is exact. The operator  $J$  is the classical Fourier integral operator of symbol

$$[s_0(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k)w_1(k^{\frac{2}{3}}\alpha) + k^{-\frac{1}{3}}s_1(n, k^{\frac{2}{3}}\alpha, \theta, \beta, \tau, k)w'_1(k^{\frac{2}{3}}\alpha)]$$

and of phase  $(\theta - \beta).\tau + \phi_0(n, \theta, \tau)$ . □

**3.3. Fourier integral operator for the impedance boundary condition.** Inspired by the section 2, we introduce the impedance operator  $L$ . It is of

$$(3.298) \quad Lu(\theta, k) = \partial_n u(0, \theta, k) + ikZ(\theta)u(0, \theta, k).$$

The application of (3.298) to (2.285) leads to:

$$(3.299) \quad L(J(f))(\theta, k) = \int e^{ik[(\theta-\beta).\tau+\phi(\theta,\tau,\xi_2)]}[\partial_n a + ik\xi_2 a + ikZa]|_{n=0} f(\beta, k) d\beta d\tau d\xi_2$$

This integral is treated in the same way as the integral giving  $J$ , in particular it is expressed with the same changes of variable. We have to calculate the symbol

$$s_Z(\xi_2, \theta, \beta, \tau, k) = \partial_n a(0, \xi_2, \theta, \beta, \tau, k) + ik\xi_2 a(0, \xi_2, \theta, \beta, \tau, k) + ikZa(0, \xi_2, \theta, \beta, \tau, k).$$

One performs the change of variable  $\xi_2(z)$  and we express  $s_Z(\xi_2(z), \theta, \beta, \tau, k) d\xi_2/dz(z)$  using Boutet de Monvel division theorem which writes

$$s \frac{d\xi_2}{dz} = ik(s_0^L + zs_1^L + (\partial_z h^L + ikh^L(-z^2 - \alpha))).$$

The two terms  $s_0^L$  and  $s_1^L$  associated to this new symbol can be evaluated using the Taylor expansion of  $\xi_2(z)$ , and we thus take up the expansion of pages 479-481 of [56]. Lemma 16 of [56] indicates that, for  $n = 0$  (which is the only condition we are interested in when we study the operator at boundary):

$$\xi_2(z) = \tilde{a}(\theta, \tau) + bz + c\frac{z^2}{2} + d\frac{z^3}{6} + O(z^4),$$

with functions  $\tilde{a}, b, c$  satisfying

$$\begin{cases} \phi(\theta, \tau, \tilde{a}) = \tau^2 \gamma(\theta, \tau) \\ b \partial_{\xi_2} \phi(\theta, \tau, \tilde{a}) = -\tau \rho(\theta, \tau) \\ c \partial_{\xi_2}^2 \phi(\theta, \tau, \tilde{a}) = -b^2 \partial_{\xi_2}^2 \phi(\theta, \tau, \tilde{a}). \end{cases}$$

Note then  $\sigma_Z^j(\theta, \tau, k) = \partial_{\xi_2^j}^j (s_Z \frac{d\xi_2}{dz})(\theta, \tau, \tilde{a}, k)$  et  $\xi^0 = k^{\frac{2}{3}} \alpha(0, \theta, \tau) = k^{\frac{2}{3}} \tau \rho(\theta, \tau)$ . One can express in a simple way the symbol  $s_0^L w_1(\xi^0) + k^{-\frac{1}{3}} s_1^L w_1'(\xi^0)$ , and one gets

$$2\pi e^{\frac{i\pi}{3}} k^{-\frac{1}{3}} [(b\sigma_Z^0 - k^{-\frac{2}{3}} \xi^0 (d\sigma_Z^0 + bc\sigma_Z^1 + \frac{1}{2} b^3 \sigma_Z^2)) w_1(\xi^0) + ik^{-\frac{1}{3}} [\sigma_Z^0 c + b^2 \sigma_Z^1 + O(k^{-\frac{2}{3}} \xi_0)] w_1'(\xi^0)].$$

One checks that the principal term of the symbol is

$$2\pi e^{\frac{i\pi}{3}} k^{-\frac{1}{3}} a(0, 0, \theta, \beta, 0, k) [\rho(\theta, 0) \omega_1(\xi^0) + k^{-\frac{1}{3}} e^{-\frac{i\pi}{6}} e^{-\frac{2i\pi}{3}} \rho^2(\theta, 0) [Z(\theta) + \frac{1}{2} \frac{\partial_{\theta} \rho}{\rho}(\theta, 0) \frac{\partial_{\xi_2} a}{a} |_{\beta=\theta, \tau=0}] \omega_1'(\xi^0)].$$

This principal term vanishes for all  $\xi^0$  such that  $\omega_1(\xi^0) = 0$ .

#### 4. 2-microlocal calculation of the diffracted wave

**4.1. Roots of the symbol of the boundary operator.** This section is not "self contained" and is rather a summary of the methods and results of G. Lebeau [66] who set up the algebra of the **unilateral operators** and who constructed the pseudo-differential 2-microlocal calculus allowing the calculation of the inverse of a unilateral operator and the article of the author of this book [56] who explicitly constructed the inverse of the unilateral operator at the boundary obtained here. The complete presentation goes beyond the aim of this book. It gives a flavor of what is needed for the proof.

We introduce the open set  $\Omega(r, \delta, \gamma_1, k)$  included in  $(\mathbb{C} \cap B(0, r))^2$  of points  $(\theta, \tau)$  satisfying  $\Im \tau - \gamma_1 k^{-\frac{2}{3}} < \delta |\operatorname{Re} \tau|$ . Soit

We verify that, for all  $M$ , there exists  $\gamma_1(M, \delta, r)$  such that the two functions  $w_1^M$  and  $w_2^M$  are bounded in  $\Omega(r, \delta, \gamma_1(M, \delta, r), k)$ . Let us introduce the open set  $V_{M, \varepsilon_0}$ , complementary of  $\{\|\Im(e^{\frac{2i\pi}{3}} z)\| < -\varepsilon_0 (\operatorname{Re}(e^{\frac{2i\pi}{3}} z) + M)\} \cup \{|e^{\frac{2i\pi}{3}} z + M| \leq \varepsilon_0\}$ . We have:

LEMMA 13.3. *There exist two constants  $C$  and  $\alpha_0$  such that if  $P$  is the number of zeros of  $w_1$  in  $V_{M, \varepsilon_0}$ , there exist  $P$  analytical functions analytic functions  $\xi_1, \dots, \xi_p$  verifying:*

$$\forall \alpha' \leq \alpha, |w_1^m(\xi^0)| \leq \alpha', \xi^0 \in V_{M, \varepsilon_0} \Rightarrow \exists p \leq P \text{ tel que } |\xi^0 - \xi_p| \leq C \alpha'.$$

The zeroes of  $s_0^L \omega_1(\xi^0) + k^{-\frac{1}{3}} s_1^L \omega_1'(\xi^0)$  are approximated by the zeroes of  $w_1$ .

PROOF. This is the result of lemma 4 of [56] p 433. We enunciate the results of [56].

#### Phase reduction

We verify by a stationary phase calculation that the operator  $I$  introduces a phase term equal to  $-ik\tau^2 \gamma(\theta, \tau) + f(M, \xi^0)$ , and that the operator  $J$  already has a term of the form  $ik\tau^2 \gamma(\beta, \tau) - f(M, k^{\frac{2}{3}} \rho(\beta, \tau))$ , where  $f$  is the function  $\frac{2}{3}(x+M)^{\frac{1}{2}}(x-M/2) + 1/4 \ln(x+M)$ . Let  $W(\theta, \beta, \tau, k)$  be the function defined by

$$(\theta - \beta)\tau W = \tau^2 [\gamma(\beta, \tau) - \gamma(\theta, \tau)] + (ik)^{-1} [f(M, k^{\frac{2}{3}} \tau \rho(\theta, \tau)) - f(M, k^{\frac{2}{3}} \tau \rho(\beta, \tau))].$$

The change of variable  $\eta = \tau(1-W)$  allows to obtain in the definition integral of  $L(J \circ I(h))$  a phase equal to  $ik(\theta - \beta)\eta$  without changing significantly significantly the symbol. This is demonstrated rigorously in any dimension in Lemma 7 p 448 of [56]. Then there exists a symbol  $E(\theta, \beta, \eta, k)$  such that

$$L(J \circ I(f))(\theta, k) = \int e^{i(\theta-\beta)\eta} E(\theta, \beta, \eta, k) f(\beta, k) d\beta d\eta.$$

Symbol  $E$  writes

$$E(\theta, \beta, \eta, k) = E_1(\theta, \beta, \eta, k) \omega_1^M(k^{\frac{2}{3}} \alpha(0, \theta, \tau(\theta, \beta, \eta))) + k^{-\frac{1}{3}} E_2(\theta, \beta, \eta, k) \omega_2^M(k^{\frac{2}{3}} \alpha(0, \theta, \tau(\theta, \beta, \eta))).$$

**Conjugation by the Gevrey 3 exponential operator** The roots of symbol  $E$  are given through the relation

$$(4.300) \quad \tau = k^{-\frac{2}{3}} e^{-\frac{2i\pi}{3}} \xi_p \left( \frac{R(s(\theta))}{2} \right)^{\frac{1}{3}} + O(k^{-1}).$$

Solving  $k^{\frac{2}{3}} \tau \rho(\theta, \tau) = \xi_p$  is equivalent to finding  $g$  telle que  $\tau = k^{-\frac{2}{3}} g(\theta, k)$ . One has then  $g(\theta, k) \rho(\theta, k^{-\frac{2}{3}} g(\theta, k)) = \xi_p$ . Assuming  $g$  bounded for  $k \geq 1$ , one gets  $\rho(\theta, k^{-\frac{2}{3}} g) = \rho(\theta, 0) + O(k^{-\frac{2}{3}})$ . Heuristically, the phase factor obtained then "looks like":

$$ik\tau\theta = ikk^{-\frac{2}{3}} \theta e^{-\frac{2i\pi}{3}} \xi_p \left( \frac{R}{2} \right)^{\frac{1}{3}} = k^{\frac{1}{3}} [e^{-\frac{i\pi}{6}} \theta \xi_p \left( \frac{R}{2} \right)^{\frac{1}{3}}].$$

This gives the idea to conjugate the operator  $L(J \circ I)$  by an operator of the form  $e^{ik^{\frac{1}{3}} H(\theta)}$ , to obtain a power  $k^{\frac{1}{3}}$ . Let  $Op(T)$  be the operator

$$(4.301) \quad Op(T)(f)(\theta, k) = Op(e^{ik^{\frac{1}{3}} H(\theta)}) L[e^{-ik^{\frac{1}{3}} H(\beta)} f(\beta, k)].$$

By application of the lemma 7.7 (Kuranishi's trick), this corresponds to **change of dual variable** given by:

$$(4.302) \quad \Sigma = \eta + k^{-\frac{2}{3}} H'(\theta).$$

**We consider from this point on the function  $H$  associated to the first zero of the function  $\omega_1$ .** Using the first zero of the Airy function  $\omega$  (such that  $Ai(-\omega) = 0$  and  $Ai(x) \neq 0$  for  $x > -\omega$ ), we see that  $\xi_1 = e^{\frac{i\pi}{3}} \omega$ . Let us choose then  $H'(\theta) = -e^{\frac{i\pi}{3}} \xi_1 \left( \frac{R(s(\theta))}{2} \right)^{\frac{1}{3}}$  and  $H(0) = 0$ . The symbol  $E$  vanish for  $\eta = -k^{-\frac{2}{3}} H'(\theta)$ . The principal symbol of the operator  $Op(T)$  cancels<sup>2</sup> for  $\Sigma = 0$ . The change of variable (4.302) corresponds to  $\xi^0 = \xi_1 + e^{\frac{2i\pi}{3}} k^{\frac{2}{3}} \left( \frac{2}{R(s(\theta))} \right)^{\frac{1}{3}} \Sigma$ , which gives:

$$Op(T)(f)(\theta, k) = \int \int e^{ik(\theta-\beta) \cdot \Sigma} S_1(\theta, \beta, \Sigma, k) f(\beta, k) d\beta d\Sigma$$

where

$$S_1(\theta, \beta, \Sigma, k) = 2\pi k^{-\frac{1}{3}} \left( \frac{2}{R(s(\theta))} \right)^{\frac{1}{3}} \left( \frac{kR(s(\theta))}{2} \right)^{-\frac{1}{3}} w_1'(\xi_1) [Z(ik\Sigma) e^{\frac{i\pi}{6}} - 1 + k^{-\frac{1}{3}} r_1].$$

□

**4.2. Algebra of unilateral operators.** Recall that we have introduced the open sets  $\Omega(r, \delta, \gamma_1, k)$  and that the symbols  $w_1^M$  and  $w_2^M$  are bounded on these open sets. Let us notice that the behavior of  $w_1^M$  is the same as the behavior of  $w_1^M(\xi^0)$  in  $\tau$  provided that we chose  $M$  sufficiently large compared to  $\xi_1$ , and that we modify the constant  $\gamma_1$  such that the symbol  $w_1^M(\xi^0)$  is bounded in  $\Omega(\dots, \gamma_1)$  in the variable  $\tau$ , and be bounded in  $\Omega(\dots, \gamma_2)$ ,  $0 < \gamma_2 < \gamma_1$  in the variable  $\Sigma$ .

We define a space of symbols associated to these open sets, which must be provided with an algebraic structure. This space is  $\mathcal{E}_{\gamma_2}^+$ , the set of symbol sequences  $p_n(\Sigma, \theta, k)$  such that

$$\sup_{\Omega(r, \delta, \gamma_2, k)} |(\Sigma - i\gamma_2 k^{-\frac{2}{3}})^n p_n(\Sigma, \theta, k)| \leq AB^n n!$$

with the composition law associated to the formal representation

$$p(\Sigma, \theta, k) = \sum_n p_n(\Sigma, \theta, k) (\Sigma - i\gamma_2 k^{-\frac{2}{3}})^{-n} (ik)^{-n}.$$

In particular

$$(p \circ q)_n(\Sigma, \theta, k) = \sum_{m+m'+j=n} \partial_{\Sigma}^j p_m(\Sigma, \theta, k) \partial_{\theta}^j q_{m'}(\Sigma, \theta, k).$$

We verify that an elliptic element (whose principal symbol is bounded on  $\Omega(r, \delta, \gamma_2, k)$ ) is invertible in this space (Proposition 4.1.3 of [66] p 1465). We define the operator

<sup>2</sup>This approach, which is facilitated by the fact that the boundary is of dimension 1, is the same as Lemma 8 of [56] p 449-450.



$$(4.303) \quad D_{\gamma_2}^{-n} v(\theta) = \int_{-A}^{\theta} e^{-\gamma_2 k^{\frac{1}{3}}(\theta-\beta)} \frac{(\theta-\beta)^{n-1}}{(n-1)!} v(\beta, k) d\beta,$$

and the operator associated to a symbol  $p$  of the space is

$$Op(p)(u) = \sum_n Op^c(p_n)(D_{\gamma_2}^{-n}(u)).$$

the sign  $^c$  designating the action of a classical symbol in variable  $(\theta, \Sigma)$ . We have shown that

$$S_1(\theta, \beta, \Sigma, k) = ik2\pi \left(\frac{kR}{2}\right)^{-\frac{2}{3}} e^{\frac{i\pi}{6}} \omega'_1(\xi_1) [ikZ\Sigma - e^{-\frac{i\pi}{6}} + k^{-\frac{1}{3}} r_1(\theta, \beta, \Sigma, k)].$$

The reduction lemma 7.3 allows to write

$$Op(S_1) = Op(S_2)$$

with

$$S_2(\theta, \Sigma, k) = ik2\pi \left(\frac{kR(s(\theta))}{2}\right)^{-\frac{2}{3}} e^{\frac{i\pi}{6}} \omega'_1(\xi_1) [ikZ\Sigma - e^{-\frac{i\pi}{6}} + k^{-\frac{1}{3}} r_2(\theta, \Sigma, k)].$$

LEMMA 13.4. *Let  $e_0(\theta, k) = k^{-\frac{2}{3}} 2\pi(ik)^2 \left(\frac{kR(s(\theta))}{2}\right)^{-\frac{2}{3}} Z(\theta) e^{\frac{i\pi}{6}} \omega'_1(\xi_1)$ . There exists a symbol  $r_2$  such that*

$$r(\theta, \Sigma, k) = Z^{-1}(\theta) [-e^{-\frac{i\pi}{6}} + k^{-\frac{1}{3}} r_2(\theta, \Sigma, k)]$$

and

$$S_2(\theta, \Sigma, k) = k^{\frac{2}{3}} e_0(\theta, \Sigma, k) [\Sigma_1 + (ik)^{-1} r(\theta, \Sigma, k)].$$

PROOF. Let us introduce  $\tilde{e}_0^{\gamma_2}(\theta, k) = e_0(\theta, \Sigma, k)(\Sigma - i\gamma_2 k^{-\frac{2}{3}})$ . Symbol  $\tilde{e}_0^{\gamma_2}$  is elliptic in  $\mathcal{E}_{\gamma'}^+$ ,  $\gamma' < 0$ . We have the relation

$$(\Sigma - i\gamma_2 k^{-\frac{2}{3}})(\tilde{e}_0^{\gamma_2})^{-1} \circ S_2 = \Sigma + (ik)^{-1} \tilde{r}(\theta, \Sigma, k),$$

the symbol  $\tilde{r}$  satisfying  $\tilde{r}(\theta, 0, k) - r(\theta, 0, k) = O(k^{-\frac{1}{3}})$ . We introduce the symbol  $h(\theta, \sigma, k)$  defined by the system (we find a method used for example for the inversion of an elliptic operator)

$$\begin{cases} \partial_{\theta} h(\theta, \Sigma, k) = (h \circ \tilde{r})(\theta, \Sigma, k) \\ h(\theta_0, \Sigma, k) = 0. \end{cases}$$

One gets

$$(4.304) \quad h_0(\theta, \Sigma, k) = \exp\left(\int_{\theta_0}^{\theta} \tilde{r}(u, \Sigma, k) du\right) = (1 + O(k^{-\frac{1}{3}})) \exp\left(\int_{\theta_0}^{\theta} r(u, \Sigma, k) du\right).$$

□

Let  $R_0^0(\theta, \Sigma, k)$  the symbol equal to  $\int_0^1 \partial_{\Sigma} h_0(\theta, u\Sigma, k) du$ .

LEMMA 13.5. • *The inverse of the operator*

$$\frac{\Sigma}{\Sigma - i\gamma_2 k^{-\frac{2}{3}}} + \frac{1}{ik} \frac{\tilde{r}(\theta, \Sigma, k)}{\Sigma - i\gamma_2 k^{-\frac{2}{3}}}$$

in  $\mathcal{E}_{\gamma'}^+$ ,  $\gamma' < 0$ , writes

$$(\Sigma - i\gamma_2 k^{-\frac{2}{3}}) \circ h^{-1} \circ [R_0^0(\theta, \Sigma, k) + ikD_0^{-1} \circ h_0(\theta, 0, k) + R^0(\theta, \Sigma, k)],$$

where  $R^0$  is a symbol of order  $k^{-\frac{2}{3}}$ .

• *The equality  $Op(S_2)f = g$  is equivalent to*

$$f = (\Sigma - i\gamma_2 k^{-\frac{2}{3}}) \circ h^{-1} \circ [R_0^0 + ikD_0^{-1} \circ h_0 + R^0] \circ Op((\tilde{e}_0^{\gamma_2})^{-1})(k^{-\frac{2}{3}}g).$$

The proof of this lemma is a consequence of proposition 6 pp 443 of [56].

**4.3. Calculation of the trace of the normal derivative of the diffracted wave.**

We use the following consequence of the previous lemma:

COROLLARY 13.2. *Let  $g \in H_{\phi_0}$ . Let  $f$  defined by  $Op(T)f = g$ . Then*

$$f(\theta, k) = \frac{1}{2\pi i k^{\frac{1}{3}} \omega'_1(\xi_1) \times e^{\frac{i\pi}{6}} Z(\theta)} \int_{-A}^{\theta} \left(\frac{R(s(\beta))}{2}\right)^{\frac{2}{3}} \exp\left(e^{-\frac{i\pi}{6}} \int_{\beta}^{\theta} \frac{du}{Z(u)}\right) g_0(\beta, k) d\beta.$$

PROOF. One checks the equality, in  $\mathcal{E}_{\gamma'}^+$ ,  $\gamma' < 0$ :

$$Op((\tilde{e}_0^{\gamma_2})^{-1})(k^{\frac{2}{3}}g) = e_0^{-1}(\theta, 0, k)(-i\gamma_2 k^{-\frac{2}{3}})^{-1}(1 + O(k^{-\frac{1}{3}}))g(\theta, k)$$

that is

$$Op((\tilde{e}_0^{\gamma_2})^{-1})g = [2\pi(ik)^2 Z(\theta) e^{\frac{i\pi}{6}} \omega'_1(\xi_1) (-i\gamma_2 k^{-\frac{2}{3}})]^{-1} \left(\frac{kR(s(\theta))}{2}\right)^{\frac{2}{3}} g_0(\theta, k) (1 + O(k^{-\frac{1}{3}})).$$

Applying (4.303), one finds

$$ikD_0^{-1}h_0(\tilde{e}_0^{\gamma_2})^{-1}g = \frac{(1 + O(k^{-\frac{1}{3}}))}{2\pi(ik)^2 e^{\frac{i\pi}{6}} \omega'_1(\xi_1) (-i\gamma_2 k^{-\frac{2}{3}})} \int_{-A}^{\theta} \left(\frac{kR(s(\beta))}{2}\right)^{\frac{2}{3}} (Z(\beta))^{-1} g_0(\beta, k) h_0(\beta, k) d\beta.$$

The second relation of Lemma 13.5 gives

$$f(\theta, k) = Op(\Sigma - i\gamma_2 k^{-\frac{2}{3}}) \left[ \frac{(1 + O(k^{-\frac{1}{3}}))}{2\pi(ik)^2 e^{\frac{i\pi}{6}} \omega'_1(\xi_1) (-i\gamma_2 k^{-\frac{2}{3}})} \int_{-A}^{\theta} \left(\frac{kR(s(\beta))}{2}\right)^{\frac{2}{3}} \frac{g_0(\beta, k) h_0(\beta, k)}{Z(\beta) h_0(\theta, k)} d\beta \right].$$

The symbol  $\Sigma - i\gamma_2 k^{-\frac{2}{3}}$  is a classical symbol, that we compute at  $\Sigma = 0$ . We replace  $h_0$  by the expression (4.304) and the terms  $-i\gamma_2 k^{-\frac{2}{3}}$  simplify:

$$f(\theta, k) = \frac{(1 + O(k^{-\frac{1}{3}}))}{2\pi(ik)^2 e^{\frac{i\pi}{6}} \omega'_1(\xi_1)} \int_{-A}^{\theta} \left(\frac{kR(s(\beta))}{2}\right)^{\frac{2}{3}} \frac{g_0(\beta, k)}{Z(\beta)} e^{\int_{\theta}^{\beta} r(u, 0, k) du} d\beta.$$

The term  $\gamma_2$ , which indicates in which symbol space the operator  $S_1$  is elliptic, has been simplified in the expression. This is natural since the result should not depend on the space in which the calculation is performed. The definition of  $r$  in the lemma 13.4 gives

$$\exp\left(\int_{\theta}^{\beta} r(u, 0, k) du\right) = \exp\left(e^{-\frac{i\pi}{6}} \int_{\beta}^{\theta} \frac{du}{Z(u)}\right) (1 + O(k^{-\frac{1}{3}})).$$

This completes the proof of the corollary.  $\square$

We obtain the Theorem:

THEOREM 13.2. *The exact solution, in a neighborhood of the shadow zone, defined for  $\theta > \theta_0$  and  $0 \leq n \leq g(\theta)$ ,  $g$  strictly increasing, is given by*

$$u_d(n, \theta, k) = \frac{a_i(0, \theta_0, k) e^{ik\phi(0, \theta_0)} e^{\frac{i\pi}{3}}}{2\pi(Ai'(-\omega))^2} \int e^{ik[(\theta-\beta)\tau + \phi_0(n, \theta, \tau)] + ik^{\frac{1}{3}}(H(\theta_0) - H(\beta))} \\ \times \left(\frac{kR(s(\theta_0))}{2}\right)^{\frac{1}{3}} e^{\int_{\theta_0}^{\beta} \frac{du}{e^{\frac{i\pi}{6}} Z(u)}} S(\theta, \beta, n, \tau, k) d\tau d\beta.$$

where  $S$  is given in the proposition 13.2. Obtaining the solution of  $P$  is obtained by multiplying by the factor  $e^{ik(s-s(\theta_0))}$ . The wave obtained propagates at speed 1 on the boundary.

PROOF. Notice that the relation between  $Op(T)$  and  $L(J \circ I)$  is (4.301). One defines  $F(\theta, k) = [(Op(e^{-ik^{\frac{1}{3}}H(\beta)}))^{-1}f](\theta, k)$ . Then

$$f(\theta, k) = [Op(e^{-ik^{\frac{1}{3}}H(\beta)})F](\theta, k).$$

Equation (3.298) is equivalent to

$$Op(e^{ik^{\frac{1}{3}}H(\theta)})[L(J \circ I)[Op(e^{-ik^{\frac{1}{3}}H(\beta)})F] = -[Op(e^{ik^{\frac{1}{3}}H(\theta)})Lu_i](\theta, k).$$

The function  $u_i$  is a solution of the wave equation in the vacuum. We know that its wavefront set meets the boundary  $\mathcal{C}$  at a strictly diffractive point, characterized by its Euler angle  $\theta_0$ . All the points of the boundary that intersect with a tangent ray are strictly diffractive because the boundary is strictly convex. Then we write

$$u_i(\theta, n, k) = e^{ik\phi(n, \theta)} a_i(n, \theta, k)$$

where the phase  $\phi$ , solution of the eikonal equation, verifies  $\partial_n \phi(0, \theta_0) = 0$ .

Thus ( $Y$  denotes the Heaviside function  $\mathbf{1}_{\mathbb{R}_+}$ )

$$Lu_i(\theta, k) = ik[Z(\theta)a_i(0, \theta, k) + \partial_n \phi(0, \theta)a_i(0, \theta, k) + \frac{1}{ik} \frac{\partial a_i}{\partial n}(0, \theta, k)]e^{ik\phi(0, \theta)}(1 - Y(\theta - \theta_0)).$$

The application of the corollary 13.2 allows to obtain

$$F(\theta, k) = \frac{-k}{2\pi e^{\frac{i\pi}{6}} k^{\frac{1}{3}} \omega'_1(\xi_1)} \int_{-A}^{\theta} \left(\frac{R(s(\beta))}{2}\right)^{\frac{2}{3}} e^{\int_{\beta}^{\theta} \frac{-i\pi}{6} \frac{du}{Z(u)} + ik\phi(0, \beta) - ik^{\frac{1}{3}} H(\beta)} a_i(0, \beta, k) \left[1 + \frac{\partial_n \phi(0, \beta)}{Z(\beta)}\right] (1 - Y(\beta - \theta_0)) d\beta.$$

Noting that, by the following relation allowing to calculate  $\phi$  (lemma 10 p 463 of [56])

$$\phi(0, \beta) = \phi(0, \theta_0) + \frac{1}{6} R(s(\theta_0))(\beta - \theta_0)^3 + O(\beta - \theta_0)^4$$

and that

$$H(\beta) = H(\theta_0) - e^{\frac{i\pi}{3}} \xi_1 \left(\frac{R(s(\theta_0))}{2}\right)^{\frac{1}{3}} (\beta - \theta_0) + O((\beta - \theta_0)^2)$$

one is left with,  $m = k^{\frac{1}{3}}(\beta - \theta_0) \left(\frac{R(s(\theta_0))}{2}\right)^{\frac{1}{3}}$

$$e^{ik\phi(0, \beta) + ik^{\frac{1}{3}} H(\beta)} = e^{ik\phi(0, \theta_0) + ik^{\frac{1}{3}} H(\theta_0)} e^{i\frac{m^3}{3} - e^{\frac{i\pi}{3}} \xi_1 m + k^{-\frac{1}{3}} \psi(m, k^{-\frac{1}{3}})}.$$

We note that the integration in  $\beta$  will allow, for any  $\theta > \theta_0$ , to assign a value to  $F(\theta, k)$ . Indeed, the interval of integration  $[-A, \theta]$  contains  $\theta_0$ . Performing the change of variable  $\beta \rightarrow m$ , we verify that the new bounds of the integral are  $k^{\frac{1}{3}}(\theta - \theta_0) \left(\frac{R(s(\theta_0))}{2}\right)^{\frac{1}{3}}$  and  $k^{\frac{1}{3}}(-A - \theta_0) \left(\frac{R(s(\theta_0))}{2}\right)^{\frac{1}{3}}$ . When  $k$  tends to  $+\infty$ , these respective bounds tend to  $\pm\infty$ , which implies that the difference term between the integral on  $\mathbb{R}$  and the integral studied is uniformly decreasing in  $k$ , of the form  $e^{-ck}$  for  $\theta - \theta_0$  bounded. After using the change of variable and the approximations  $l(\beta, k) = (1 + O(k^{-\frac{1}{3}}))l(\theta_0, k)$  for the terms of the symbol, we find

$$F(\theta, k) - O(e^{-ck}) = -k^{\frac{1}{3}} \frac{(1 + O(k^{-\frac{1}{3}}))}{e^{\frac{i\pi}{6}}} \left(\frac{2}{R(s(\theta_0))}\right)^{\frac{1}{3}} a_i(0, \theta_0) \exp\left(\int_{\theta_0}^{\theta} \frac{du}{e^{\frac{i\pi}{6}} Z(u)}\right) \frac{\omega_1(e^{-\frac{2i\pi}{3}} \xi_1)}{\omega'_1(\xi_1)}.$$

Using  $2\pi Ai(\omega e^{-\frac{i\pi}{3}}) = \frac{e^{\frac{i\pi}{6}}}{Ai'(-\omega)}$  et  $\xi_1 = e^{\frac{i\pi}{3}} \omega$ , one deduces

$$\frac{\omega_1(e^{-\frac{2i\pi}{3}} \xi_1)}{\omega'_1(\xi_1)} = \frac{1}{2\pi i} (Ai'(-\omega))^{-2}.$$

One has then

$$F(\theta, k) = \left(\frac{kR(s(\theta_0))}{2}\right)^{\frac{1}{3}} e^{\frac{i\pi}{3}} a_i(0, \theta_0) \exp\left(e^{-\frac{i\pi}{6}} \int_{\theta_0}^{\theta} \frac{du}{Z(u)}\right) e^{ik\phi(0, \theta_0) + ik^{\frac{1}{3}} H(\theta_0)} \frac{(1 + O(k^{-\frac{1}{3}}))}{2\pi (Ai'(-\omega))^2}.$$

One deduces easily  $f(\theta, k) = e^{-ik^{\frac{1}{3}} H(\theta)} F(\theta, k) (1 + O(k^{-\frac{1}{3}}))$ . Finally, we can calculate the solution at any point by the relation

$$(J \circ I)(f) = u_d(n, \theta, k).$$

We have thus demonstrated Theorem 13.2.  $\square$

We see, in the expression of this solution in the shadow region, that the curve  $n = g(\theta)$  is the expression in coordinates  $(n, \theta)$  of the half line coming from the point  $M(s(\theta_0))$  which direction is  $\vec{t}(s(\theta_0)) = \nabla\phi(0, \theta_0)$ . Indeed, the bicharacteristics of the classical wave operator are straight lines, and the change of variable does not transform these geometrical objects.

Then, we can note that  $\text{Re}(-iH(\theta)) = (\text{Re} ie^{\frac{2i\pi}{3}}\omega)(\frac{R(s(\theta))}{2})^{\frac{1}{3}}$ . Comme  $\text{Re}(ie^{\frac{2i\pi}{3}}) = -\text{Re} e^{\frac{i\pi}{6}} = -\cos\frac{\pi}{6}$ , we can verify that, for  $\theta_0 + 2\delta_0 > \theta > \theta_0 + \delta_0$ , there exists a constant  $C(\delta_0)$ , which can be taken equal to

$$\delta_0 \min_{\theta \in [\theta_0 + \delta_0, \theta_0 + 2\delta_0]} \left(\frac{R(s(\theta))}{2}\right)^{\frac{1}{3}} \omega \cos\frac{\pi}{6}$$

such that, for  $n > g(\theta) - c'\delta_0$ ,

$$|u_d(n, \theta, k)| \leq Ce^{-k^{\frac{1}{3}}C(\delta_0)}.$$

This indicates that, in the region  $n > g(\theta)$ , the calculated wave decays faster than any inverse power of  $k$ . It is therefore  $C^\infty$  in this region. We demonstrate (see [66] or [56]) that there is propagation of analytic singularities.

This generalizes Friedlander’s result (Theorem 13.1) for the singularity propagation  $C^\infty$  for glancing rays at an analytic propagation of singularities on generalized analytic rays which are the union of a bicharacteristic of the vacuum wave operator operator up to the point  $M(s(\theta_0))$ , which are then an integral curve of the wave operator reduced on the boundary (on  $\mathcal{C}$ ) and finally again a bicharacteristic curve of the wave operator in vacuum.

The generalized ray of the operator  $P - \partial_{t^2}^2$ ,  $P = \partial_{n^2}^2 + R(n, s, \partial_s)$  writes

$$\gamma_- \cup \tilde{\gamma} \cup \tilde{\gamma}_+$$

where  $\gamma_-$  is a bicharacteristic of  $P - \partial_{t^2}^2$  in  $T^*(\mathbb{R}^2 \times \mathbb{R}_t)$ ,  $\gamma_- \cap \tilde{\gamma} = \{\rho_0\} \subset \mathcal{G}$ ,  $\tilde{\gamma}$  is the bicharacteristic, in  $T^*(\mathbb{R} \times \mathbb{R}_t)$ , of the operator  $R(0, s, \partial_s) - \partial_{t^2}^2$ , passing through  $\rho_0$ , and  $\tilde{\gamma}_+$  is the bicharacteristic, in  $T^*(\mathbb{R}^2 \times \mathbb{R}_t)$ , of the operator  $P - \partial_{t^2}^2$ , passing through  $\tilde{\gamma} \cap \gamma_-$ . It thus exists an infinite number of generalized bicharacteristics passing through  $\rho_0$ . Let us finally note that we have constantly used the hypothesis  $R \neq 0$ , which is exactly equivalent to the strict convexity of  $\mathcal{C}$ . The points of  $\mathcal{G}$  are then called strictly diffractive points. The singularity propagation theorem is written, in in this case:

**THEOREM 13.3.** *Let  $P$  be a hyperbolic differential operator of order 2. Let  $\Omega$  be a regular open. We suppose that the manifold glancing of  $P$  with respect to  $\Omega^c$ ,  $\mathcal{G}$ , has only strictly diffractive points, that is, if  $\psi$  is an equation of  $\partial\Omega$  such that  $\Omega^c = \{\psi > 0\}$ , then*

$$\frac{\{\{\psi, p\}, p\}}{\{\psi, p\}, \psi} \Big|_{p=\psi=0} > 0$$

*There is propagation of analytic singularities on generalized bicharacteristics. The above calculation proves that the transfer operator is explicit in the case of a differential operator of order 2 admitting in the glancing manifold only strictly diffractive points.*

### 5. Conclusion on the rays

There are three types of generalized bicharacteristics of the operator  $P_0 - \partial_{t^2}^2$  studied in this work:

- "elliptic" bicharacteristics, which coincide with the usual bicharacteristics in  $\mathbb{R}^2 \times \mathbb{R}_t$  of  $P_0 - \partial_{t^2}^2$ . The calculation of the propagated wave is the subject of the chapter 3, the proof of the propagation theorem of singularities (Theorem 10.1) in the chapter 10.
- the hyperbolic bicharacteristics, which are the union of a ray which intersects the boundary in a transverse way and the reflected ray generated by this point of intersection (we consider the other solution of the problem  $\xi_n^2 = r(0, x', \xi')$ ). The calculation of the coefficient of 10.6, and the theorem of reflection of singularities is the theorem 10.2.
- the diffractive bicharacteristics, which are the union of a bicharacteristic bicharacteristic meeting the boundary at a diffractive point, of a segment of length on the boundary in

dimension 2, and then a bicomponent which diffracts which diffracts (i.e. tangent) from the boundary at the end of the segment the end of the segment, which was the subject of this chapter. We have seen that **all generalized bicharacteristics carry information**. This principle is called diffraction (what is happens for the incident wave at one point is reflected on an infinite number of rays for the diffracted wave). The segments of the boundary are called rays.

Let's finish with the figures representing the front that take into account the propagation, transverse reflection, and diffraction:

Figure 12



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# Appendix: Application à la physique des particules

## 0. Introduction

Nous présentons ici, sur une idée d'E. Pilon, un exemple simple d'application du théorème de la phase stationnaire tel que les physiciens des particules peuvent l'utiliser. La section 1 est due à E. Pilon, s'inspirant de [83]. Dans un premier paragraphe, nous définissons la quantité physique que nous étudions; il s'agit en l'occurrence de la distribution de gluons dans un proton. Dans un deuxième paragraphe, nous obtenons la forme intégrale d'un équivalent de la densité de probabilité. La section 3 est consacrée à une généralisation du théorème de la phase stationnaire énoncé plus haut, en insistant sur quelques difficultés dues au fait que le point de phase stationnaire n'est pas sur le contour. Nous donnons alors une interprétation physique du résultat.

## 1. Le contexte physique

Comme le neutron, avec lequel il constitue les noyaux des atomes, le proton n'est pas une particule élémentaire : c'est un état lié de sous-constituants, des *quarks* ( quanta de matière ) et des *gluons* ( quanta du champ de rayonnement associé à l'interaction considérée; ils sont à l'interaction forte au niveau élémentaire ce que les photons sont à l'électromagnétisme ). Ces sous-constituants sont très fortement liés - quarks et gluons sont astreints à se "confiner" en états liés (appelés génériquement *hadrons*) : pour des raisons énergétiques, il leur est interdit de s'échapper librement les uns des autres à des distances supérieures au femtomètre, typiquement le rayon d'un proton.

Néanmoins, lorsqu'un proton subit une collision à haute énergie devant son énergie de masse, et mettant en jeu un transfert d'énergie-impulsion grand devant l'énergie de liaison de ses sous-constituants, ce proton se comporte, de façon la plus probable, non pas comme un tout cohérent réagissant "d'un seul tenant", mais comme un faisceau de quarks et gluons collimés, quantiquement incohérents et quasi-libres ( appelés génériquement *partons* ). Un seul parton participe activement à la collision à grand transfert; les autres n'interviennent qu'ultérieurement lors de la recombinaison des divers quarks et gluons produits en hadrons dans l'état final de la collision.

Parmi les quantités physiquement pertinentes lors de l'étude de ces collisions figurent les "densités partoniques" dans le proton, notées  $G_a(x, Q^2)$ , qui fournissent la densité de probabilité  $x G_a(x, Q^2)$  pour qu'un parton d'espèce  $a$ , ( quark ou antiquark de tel ou tel type, ou gluon ) ayant dans le hadron une "épaisseur transverse"  $\hbar/Q$  ( perpendiculairement à l'axe de la collision;  $Q$  est de l'ordre de grandeur du transfert d'énergie-impulsion de la collision ), porte la fraction  $x$  ( entre 0 et 1 ) d'énergie-impulsion du proton incident dont il fait partie. Le lecteur intéressé par les détails se reportera utilement à [83], [35], [36].

La ChromoDynamique Quantique, théorie de l'interaction forte au niveau élémentaire des quarks et des gluons, permet en principe de calculer ces quantités; toutefois en pratique leur calcul complet est actuellement inextricable.

## 2. Equation régissant $G_a$ .

Lorsque  $Q^2$  est grand devant le carré de l'énergie de masse du proton, et lorsque  $x$  n'est, ni très petit devant 1, ni voisin de 1, la dépendance en  $Q^2$  des densités partoniques  $G_a(x, Q^2)$  est contrôlée par l'équation linéaire de la forme :

$$(2.305) \quad Q^2 \frac{\partial}{\partial Q^2} G_a(x, Q^2) = \bar{\alpha}(Q^2) \sum_b \int_0^1 dy dz \delta(x - yz) P_{ab}(y) G_b(z, Q^2)$$

appelées "équations de Dokshitzer-Gribov-Lipatov-Altarelli-Parisi" [31], du nom de leurs inventeurs ( voir aussi les cours [35], [36], [83] ), où

- le paramètre de couplage  $\bar{\alpha}(Q^2)$  est une fonction connue ( inversement proportionnelle au logarithme de  $Q^2$  );
- les  $P_{ab}(z)$  sont des distributions connues.

Une méthode de résolution standard consiste à effectuer une transformation de Mellin sur  $x$  :

$$(2.306) \quad \bar{G}_a(n, Q^2) = \int_0^1 dx x^{n-1} G_a(x, Q^2).$$

En introduisant la variable dite "d'évolution naturelle"  $\xi$  définie par

$$(2.307) \quad \xi = \int_{Q_0^2}^{Q^2} \frac{dQ^2}{Q^2} \bar{\alpha}(Q^2),$$

on obtient une équation différentielle linéaire du premier ordre pour chaque  $\bar{G}_a(n, Q^2)$ , explicitée en (3.309). Les "conditions initiales", pour un  $Q_0^2$  de l'ordre de grandeur du carré de l'énergie de masse du proton, sont incalculables à l'heure actuelle à partir des premiers principes. Il s'agit plutôt de ce que l'on "mesure" dans ces collisions. Dans cet exemple, nous les paramètrons sous une forme simple. Dans la "représentation des  $x$ ", ce paramétrage est de la forme :

$$(2.308) \quad G_a(x, Q_0^2) \simeq N_a x^{\alpha_a - 1} (1 - x)^{\beta_a} P_a(x)$$

où

- $N$  est une normalisation;
- $\alpha_a > -1$  contrôle le profil à petit  $x$ ;
- $\beta_a > 0$  contrôle le profil à  $x$  voisin de 1;
- $P_a(x)$  est une fonction simple contrôlant le profil aux  $x$  intermédiaires,

auquel correspond, par transformation de Mellin (2.306), la forme en "représentation des  $n$ " des conditions initiales  $\bar{G}_a(n, Q_0^2)$ .

La solution recherchée en "représentation des  $x$ " s'obtient alors par transformée de Mellin inverse. Le traitement usuel de cette transformée de Mellin inverse s'effectue par l'intermédiaire d'un calcul numérique, ou, dans certains cas, grâce à une application du théorème de la phase stationnaire. Rigoureusement parlant, ces équations sont obtenues dans le régime où

$$\bar{\alpha}(Q^2) \ll 1, \quad \bar{\alpha}(Q^2) |\text{Log} x| \ll 1, \quad \bar{\alpha}(Q^2) |\text{Log}(1 - x)| \ll 1$$

et elles sont physiquement incorrectes hors de ce domaine. Il est malgré tout intéressant de l'étudier dans le cas où  $x \ll 1$ , ce qui sera un cas d'application du théorème de la phase stationnaire.

## 3. Formulation du problème lorsque $x \ll 1$

La fonction ( écrite pour le gluon )  $\tilde{H}_g(n, \xi) = \bar{G}_g(n, Q^2(\xi))$  est solution d'une équation de forme :

$$(3.309) \quad \frac{\partial \tilde{H}_g}{\partial \xi}(n, \xi) = \left( \frac{c}{n-1} - b + o(n-1) \right) \tilde{H}_g(n, \xi)$$

où  $b$  et  $c$  sont des constantes positives connues. En supprimant le terme  $o(n-1)$  et en considérant la nouvelle équation ainsi obtenue, la nouvelle fonction inconnue étant notée  $H_g(n, \xi)$ , on trouve la solution en “représentation des  $n$ ” :

$$(3.310) \quad H_g(n, \xi) = e^{\frac{c\xi}{n-1} - b\xi} H_g(n, 0)$$

Utilisant la transformée de Mellin inverse, on trouve son expression en “représentation des  $x$ ” après changement de variable  $\omega = n - 1$  :

$$(3.311) \quad x \mathcal{M}^{-1}(H_g)(x, \xi) = e^{-b\xi} \int_{C_\omega} \frac{d\omega}{2\pi i} x^{-\omega} e^{\frac{c\xi}{\omega}} H_g(1 + \omega, 0)$$

$C_\omega$  est un contour parallèle à l’axe imaginaire pur, et passant à droite de 0, singularité dominante de l’intégrand.

Avec  $\zeta = \text{Log} \frac{1}{x}$ ,  $u = \sqrt{c\xi\zeta}$ ,  $\omega = t\sqrt{\frac{c\xi}{\zeta}}$  et  $f(t) = t + \frac{1}{t}$ , ceci se réécrit :

$$(3.312) \quad x \mathcal{M}^{-1}(H_g)(x, \xi) \simeq e^{-b\xi} \sqrt{\frac{c\xi}{\zeta}} \int_{C_t} \frac{dt}{2\pi i} e^{uf(t)} H_g\left(1 + t\sqrt{\frac{c\xi}{\zeta}}, 0\right)$$

$C_t$  est un contour parallèle à l’axe imaginaire pur, et passant à droite de 0.

Nous souhaitons démontrer dans cette partie l’approximation suivante, lorsque  $u \gg 1$  :

$$(3.313) \quad x \mathcal{M}^{-1}(H_g)(x, \xi) \simeq \sqrt{\frac{c\xi}{\zeta}} \left[4\pi\sqrt{c\xi\zeta}\right]^{-\frac{1}{2}} \bar{G}_g\left(1 + \sqrt{\frac{c\xi}{\zeta}}, 0\right) e^{-b\xi + 2\sqrt{c\xi\zeta}} \left[1 + O\left(\frac{1}{\sqrt{c\xi\zeta}}\right)\right].$$

Remarques.

- (i) L’intégrale  $\int_{C_t} \frac{dt}{2\pi i} e^{uf(t)}$  est la représentation intégrale d’une fonction de Bessel.
- (ii) En “représentation des  $x$ ”, l’équation (3.309) s’écrit

$$\frac{\partial}{\partial \xi} [x H_g(x, \xi)] = \int_x^1 dz \left(\frac{c}{z} - b\right) [z H_g(z, \xi)].$$

En posant  $x H_g(x, \xi) = e^{-b\xi} g(\zeta, \xi)$ , on voit que la fonction  $g$  satisfait une “équation d’onde avec masse imaginaire pure” [35] :

$$\frac{\partial^2}{\partial \zeta \partial \xi} g(\zeta, \xi) = c g(\zeta, \xi)$$

dont une solution pertinente est la fonction de Bessel ci-dessus.

2 - A priori, cette expression n’est pertinente que si  $\sqrt{c\xi/\zeta} \ll 1$ . En effet, elle résulte de la résolution de l’équation (3.309) où intervient l’approximation  $\bar{P}_{gg}(n) \sim \frac{c}{n-1} - b + o(n-1)$ , qui n’est valable que lorsque  $n \rightarrow 1$ . Le “col en  $n$ ”, donné par l’expression  $1 + \sqrt{c\xi/\zeta}$  doit donc être voisin de 1. On constate néanmoins que ce comportement lorsque  $\sqrt{c\xi/\zeta} \geq 1$  est, au moins qualitativement, similaire à celui de la solution exacte.

3 - Un mot encore sur la validité physique de ce résultat. On constate que, à  $\xi$  i.e.  $Q^2$  fixés, celui-ci croît avec  $\zeta$  plus rapidement que toute puissance de  $\zeta$  du fait du facteur exponentiel. Ceci contredit une conséquence de l’unitarité imposée par la mécanique quantique, qui contraint  $G_g(x, \xi)$  à ne pas croître plus vite que  $\zeta^2$ . Cette contradiction est due au fait que l’équation étudiée est obtenue dans le régime  $\alpha(\bar{Q}^2)\zeta \ll 1$ , où sont négligés des termes associés à des mécanismes physiques qui deviennent essentiels lorsque  $\zeta \rightarrow \infty$  : les partons portant une fraction  $x$  infime de l’énergie-impulsion du proton ont une densité si élevée qu’ils ne sont pas quasi libres et indépendants, mais au contraire interagissent, et ces interactions limitent leurs densités. Le phénomène non linéaire qui conduirait à cette saturation est ignoré ici.

#### 4. Application du théorème de la phase stationnaire complexe.

L'application du théorème 4.2 à la fonction  $f(t) = t + \frac{1}{t}$ , pour peu que la condition que  $\Omega$  ne contienne pas le point singulier 0, donne tout de suite la contribution du point de phase stationnaire  $t = 1$ , qui correspond à la valeur critique 2. On a ainsi démontré (3.313). Ceci permet donc de retrouver l'approximation de la fonction de répartition.

Bien sûr, cet exemple est élémentaire. D'autres exemples de théorie des champs, beaucoup plus compliqués, tiennent compte de la méthode du col pour leur résolution. Ceci est par exemple à rapprocher des résultats de G. Benarous pour l'évaluation des intégrales de Feynmann [13].