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ABSTRACT. Let F and G be generalized Hénon maps. We show that the Kato surfaces associated to the germs of F and G near infinity are biholomorphic if and only if F and G are conjugate in $\mathrm{Aut}\left(\mathbb{C}^2\right)$. This answers a question raised by Favre in the survey [5].

We also show that two generalized Hénon maps are conjugate near infinity, if and only if they are affinely conjugate.

1. INTRODUCTION

Consider a generalized Hénon map, that is $F = F_N \circ \cdots \circ F_1 \in \mathrm{Aut}\left(\mathbb{C}^2\right)$, where the F_i, $i \in \{1, \ldots, N\}$, are standard Hénon maps of the form

$$F_i \colon \mathbb{C}^2 \to \mathbb{C}^2, \qquad (z, w) \mapsto \left(P_i(z) - a_i w, z\right), \tag{1.1}$$

for P_i polynomials of degree at least 2 and $a_i \in \mathbb{C}^*$. Friedland and Milnor [17] showed that among polynomial automorphisms of $\mathbb{C}^2$, these are the only dynamically non-trivial mappings. Such automorphisms have been first studied by Fornæss and Sibony [16], Hubbard and Oberste-Vorth [18] and simultaneously by Bedford and Smillie in a long series of articles, see for example [3, 2].

The maps F and F^{-1} extend to birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with respective indeterminacy points $I^{\pm}$. The extension of F contracts the whole line at infinity onto the super-attractive point $I^- = [0 : 1 : 0]$, in homogeneous coordinates $[t : z : w]$. As for one-dimensional polynomial dynamics, it is useful to consider the dichotomy between escaping and bounded dynamics. Denote by $\Omega^+ = \{p \in \mathbb{C}^2 \mid F^n(p) \to_{n\to\infty} I^-\}$. Hubbard and Oberste-Vorth [18] gave a full analytic description of this basin of attraction, and in particular showed that F acts properly and discontinuously on Ω^+. Therefore, one can define the quotient Ω^+/F, being naturally a complex surface. Then, one can construct a canonical minimal compactification S_F of Ω^+/F.

Such surfaces obtained by compactification of the quotient by a contracting germ are known as *Kato surfaces*. They were introduced by Kato [19], later studied by Enoki [12, 13] and Nakamura [20, 21] and extensively by Dloussky with others [8, 7, 11, 9] (see the memoir [6]). The classification of Kato surfaces is deeply linked to the one of *rigid germs* by Favre [14]. Such a surface S satisfies $b_1(S) = 1$ and $b_2(S) > 0$. In particular, S is non-Kähler. In Kodaira's classification [1], they belong to the VII_0-class and are to our knowledge the only known ones of this class. Kato surfaces are characterized by the property of having a global spherical shell [6]. In the case of Hénon maps, they were first introduced by Dloussky and Oeljeklaus [10] and by Favre [14].

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In the recent survey [5], Favre asked the following question. Given a generalized Hénon map F, can one describe all the generalized Hénon maps G such that S_F is biholomorphic to S_G? It is a straightforward consequence of [14] and [18] that if two generalized Hénon maps are conjugate near infinity (that is, near I^-), then they have biholomorphic escaping set. It is also well known [6] that two conjugate germs induce the same Kato surface. Therefore, the works of Bonnot–Radu–Tanase [4] and Pal [22] can be seen as partial results towards a solution to this question. They investigated when two standard Hénon maps have biholomorphic escaping sets. However, their condition is obviously weaker than the one to be found for Kato surfaces. Indeed, on the one hand they find Hénon maps having different jacobian with biholomorphic escaping sets, and on the other hand Favre [14] showed that the germ at infinity of a Hénon map determines the jacobian of the map. Here, we prove the following.

Theorem 1.1. *Let $F = F_N \circ \cdots \circ F_1$ and G be generalized Hénon maps, where the F_i, $i \in \{1, \ldots, N\}$ are standard Hénon maps. Then, S_F and S_G are biholomorphic if and only if G is affinely conjugate to one of the $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$, for $k \in \{1, \ldots, N\}$.*

This condition is equivalent to F and G being conjugate in $\mathrm{Aut}\,(\mathbb{C}^2)$ [17].

Let us describe briefly the method of our proof. First, we show that $S_F \simeq S_G$ if and only if G is conjugate near infinity to one of the $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$. This is actually a consequence of some well known properties of Kato surfaces and a basic analysis of the Kato surface induced by a generalized Hénon map. Indeed, Dloussky [6] showed that a Kato surface S contains n compact curves, coming with n contracting germs $f_1, \ldots, f_n$, for $n = b_2(S)$. These curves are somehow cyclically ordered in the sense that one curve is obtained as the exceptional divisor of the blowing-up in one point of the previous one. The cyclic behaviour is then a consequence of taking the quotient. Dloussky [6] proved that two Kato surfaces inducing respectively germs $f_1, \ldots, f_n$, $g_1, \ldots, g_n$ are biholomorphic if and only if one of the f_i is conjugate to one of the g_j. In the case of a composition $F_N \circ \cdots \circ F_1$, where each map F_j induces separately a Kato surface, we show that each $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$ is conjugate to one of the f_j.

So, the first part of the proof reduces to showing that these are the only one which can be conjugate to a generalized Hénon map near infinity. We prove the latter by computing the configuration of self-intersection of the n curves mentioned above, in which there are exactly N curves which have particular self-intersection and correspond to the germs of the generalized Hénon maps $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$.

Next, we show that two generalized Hénon maps are conjugate near infinity if and only if they are affinely conjugate. This is the most involving part of our work. We do so by computing the normal form of the germ in the sense of Favre [14]. Such a computation was done up to degree 6 for standard Hénon maps by Bonnot–Radu–Tanase [4]. Favre gave an explicit procedure in three steps to conjugate a rigid germ to a normal form *via* an (essentially) tangent to the identity diffeomorphism. We follow this procedure and keep track of enough (exactly $2d-1$, where d is the degree of F) of the first terms of the intermediate maps. More precisely, the first step of this procedure consists in taking Böttcher coordinates, and we show that the next two steps more or less consist in cutting all the terms of degree more than $2d-1$.

Having computed this normal form, we show that this procedure is injective if the right initial choices were made, namely taking monic and centered polynomials. That

is, if two generalized Hénon maps induce the same normal form, then they are equal (not only conjugate). In some sense, we show that these $2d-1$ terms contain all the polynomials and jacobians of the standard Hénon maps in such a way that they do not "interfere". That is, the first terms are only determined by the polynomial P_N (see (1.1)) and uniquely determines it. The other polynomials only intervene after in the development. This property is also satisfied inductively and we recover the whole generalized Hénon map from its germ. Finally, we use the work of Favre [14], who gives necessary and sufficient conditions for two normal forms to be conjugate. We show in our case, that conjugate normal forms are given by affinely conjugate generalized Hénon maps.

The article is organized as follows. In Section 2, we recall some basic constructions of Kato surfaces. We show that if f and g induce Kato surfaces, then $S_{f\circ g} \simeq S_{g\circ f}$. In Section 3, we compute the self-intersection configuration of Kato surfaces induced by generalized Hénon maps. We prove our first criterion involving germs at infinity. Finally, we compute the normal form of a germ at infinity of a generalized Hénon map in Section 4 and prove Theorem 1.1.

Notations. Throughout the article, we denote by $[\![p,q]\!]$ the set of integers $p \leqslant k \leqslant q$.

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2. Kato surfaces

In this section, we recall some basic constructions of Kato surfaces. We refer the reader to Dloussky's memoir [6] or to Favre's thesis [15] for more details and proofs.

Definition 2.1. A *tower of blow-ups* is a map $\Pi = \Pi_1 \circ \cdots \circ \Pi_n$, where $\Pi_1 \colon B_1 \to (\mathbb{C}^2, 0)$ is a blow-up of the origin $O_0 = 0 \in \mathbb{C}^2$, and for each $i \in [\![2, n]\!]$, $\Pi_i \colon B_i \to B_{i-1}$ is the blow-up of some $O_{i-1} \in \Pi_{i-1}^{-1}(O_{i-2})$. We usually denote $\widehat{B} = B_n$ and $B_0 = \mathbb{C}^2$.

A contracting germ $F \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is called a *Dloussky germ* if it can be written as $F = \Pi \circ \sigma$, where $\Pi = \Pi_1 \circ \cdots \circ \Pi_n \colon \widehat{B} \to (\mathbb{C}^2, 0)$ is a tower of blow-ups and with the notations above, $\sigma \colon (\mathbb{C}^2, 0) \to \left(\widehat{B}, O_n\right)$ is a germ of biholomorphism, for some point $O_n \in \Pi_n^{-1}(O_{n-1})$.

Let $F = \Pi \circ \sigma$ be a Dloussky germ, defined on a ball $B \subset \mathbb{C}^2$ of radius $2r > 0$. Let $\varepsilon \in (0, r)$ be sufficiently small and denote by $\Sigma_\varepsilon = \{z \in \mathbb{C}^2 \mid \|z\| \in (r-\varepsilon, r+\varepsilon)\}$. The map $\sigma \circ \Pi \colon \widehat{B} \to \widehat{B}$ sends biholomorphically $\Pi^{-1}(\Sigma_\varepsilon)$ onto $\sigma(\Sigma_\varepsilon)$. Denote by B_ε the ball of radius $r+\varepsilon$ and $B_{-\varepsilon}$ the ball of radius $r-\varepsilon$. One can define the *Kato surface associated to* F to be

$$S_F = \left(\Pi^{-1}(B_\varepsilon) \setminus \sigma(B_{-\varepsilon})\right) \Big/ \sim,$$

where $\sim$ is the gluing given by $\sigma \circ \Pi \colon \Pi^{-1}(\Sigma_\varepsilon) \to \sigma(\Sigma_\varepsilon)$, that is $z \sim \sigma \circ \Pi(z)$, for $z \in \Pi^{-1}(\Sigma_\varepsilon)$ (see Figure 1). One can show that this construction only depends on the germ F and not on the other choices made [6, Proposition 3.16]. We have the following.

Proposition 2.2 (Dloussky [6, Lemma 1.10])**.** *With the notations above,* $b_1(S_F) = 1$ *and* $b_2(S_F) = n$ *is the number of blow-ups. In particular,* S_F *is non-Kähler.*

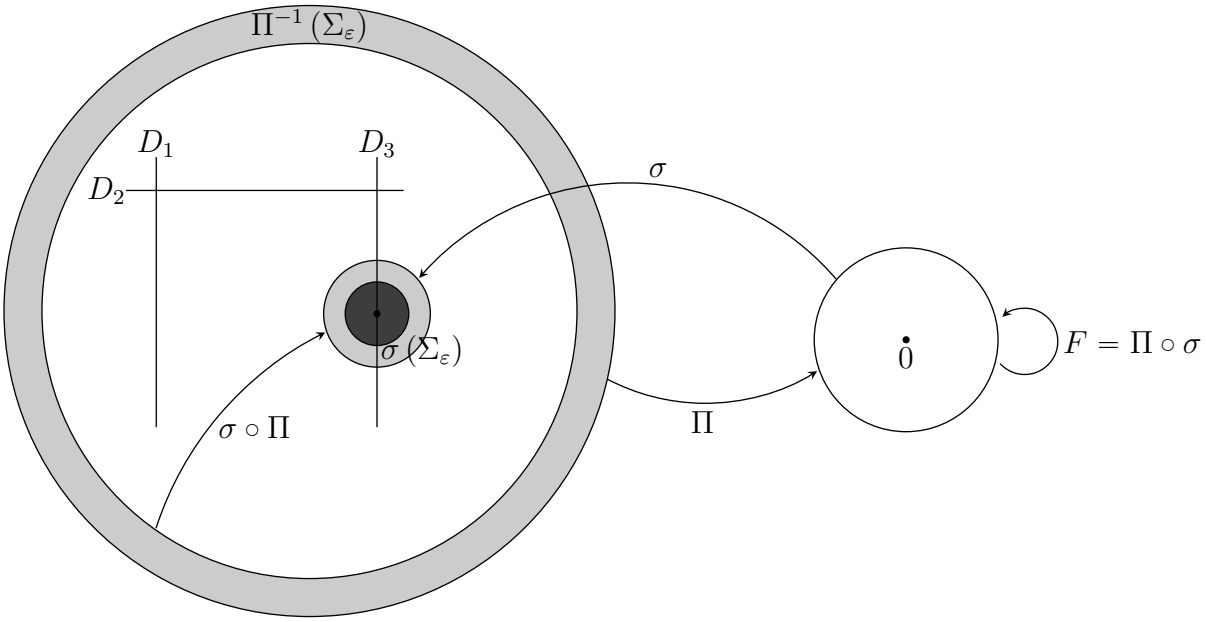


FIGURE 1. Construction of Kato surfaces

Moreover, the Kato surface S_F comes with n compact curves $D_1, \dots, D_n$, which are the respective projections of the exceptional divisor of the respective blow-ups $\Pi_1, \dots, \Pi_n$. Each of this curve comes with a contracting germ

$$F_k = \Pi_{k+1} \circ \cdots \circ \Pi_n \circ \sigma \circ \Pi_1 \circ \cdots \circ \Pi_k \colon (B_k, O_k) \to (B_k, O_k), \qquad k \in [\![1, n]\!]. \tag{2.1}$$

By definition, $F_n = \sigma \circ F \circ \sigma^{-1}$. Moreover, we have the following description of Dloussky germs inducing the same Kato surface.

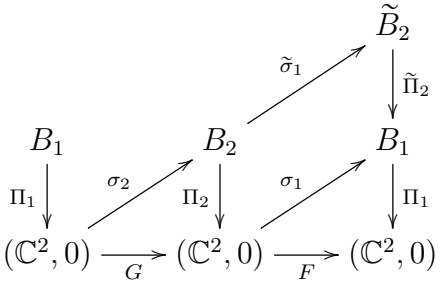
Theorem 2.3 (Dloussky [6, Subsection 3.11])**.** *Let F be as above and G be a Dloussky germ. Then, S_G is biholomorphic to S_F, if and only if G is conjugate to one of the F_k, $k \in [\![1, n]\!]$.*

In particular, we have the following. The first part of the lemma is due to Favre [15, Lemma 1.2.11], but we reprove it to introduce notations.

Proposition 2.4. *Let F, G be two Dloussky germs. Then, $G \circ F$ and $F \circ G$ are Dloussky germs and $S_{F \circ G}$ is biholomorphic to $S_{G \circ F}$.*

Proof. Let us write $F = \Pi_1 \circ \sigma_1$ and $G = \Pi_2 \circ \sigma_2$, with $\Pi_j \colon B_j \to \mathbb{C}^2$, $j \in \{1, 2\}$ towers of blow-ups. Denote by $O_1 = \sigma_1(0)$. Because blowing-up does not depend on the coordinates, we have that σ_1 lifts to a blow-up of O_1. That is, there is $\widetilde{\Pi}_2 \colon \widetilde{B}_2 \to B_1$ a tower of blow-ups of O_1 and $\widetilde{\sigma}_1 \colon B_2 \to \widetilde{B}_2$ such that $\widetilde{\Pi}_2 \circ \widetilde{\sigma}_1 = \sigma_1 \circ \Pi_2$. Let us summarize our data

in a commutative diagram.



This already shows that $F \circ G$ is a Dloussky germ, and the proof is of course similar for $G \circ F$. Moreover, by Theorem 2.3, the germ $\widetilde{\Pi}_2 \circ \widetilde{\sigma}_1 \circ \sigma_2 \circ \Pi_1$ and its conjugate induce the same Kato surface as $F \circ G$. Finally, it is easily checked that

$$\sigma_1^{-1} \circ \widetilde{\Pi}_2 \circ \widetilde{\sigma}_1 \circ \sigma_2 \circ \Pi_1 \circ \sigma_1 = G \circ F. \qquad \square$$

Let us now conclude this section with some important remarks about the n curves $D_1, \dots, D_n$ mentioned above. Consider G a Dloussky germ such that $b_2(S_G) = b_2(S_F)$. Write $G = \Pi' \circ \sigma'$, where $\Pi' = \Pi'_1 \circ \cdots \circ \Pi'_n$ is a tower of blow-ups. Denote by $E_1, \dots, E_n$ the n curves in S_G given by the exceptional divisors of $\Pi'_1, \dots, \Pi'_n$, and by G_k the germs defined similarly to (2.1).

Proposition 2.5 (Dloussky [6, Remark 3.15])**.** *Suppose there is $\varphi\colon S_F \to S_G$ a biholomorphism. Then, there is a $k \in [\![1, n]\!]$ such that $\varphi(D_j) = E_{j+k}$, F_j is conjugate to G_{j+k}, for $j \in [\![1, n-k]\!]$; $\varphi(D_j) = E_{j+k-n}$ and F_j is conjugate to G_{j+k-n}, for $j \in [\![n-k+1, n]\!]$.*

3. Self-intersection profile of Hénon maps Kato surfaces

In this section, we compute the sequence of blow-ups and the biholomorphism that make the germ of a Hénon map near I^- a Dloussky germ. The result is essentially contained in [14, Proposition 2.2] and [15, pp. 25–27] for standard Hénon maps and the first part of Proposition 2.4, due to Favre, gives it for generalized Hénon maps. It was also already observed by Dloussky and Oeljeklaus [10] for degree 2 Hénon maps. Therefore, our goal is not really this computation but the following.

Proposition 3.1. *Let $F = F_N \circ \cdots \circ F_1$ and G be two generalized Hénon maps, where $F_1, \dots, F_N$ are standard Hénon maps. Then, the Kato surfaces S_F and S_G are biholomorphic if and only if G is conjugate near infinity to one of the $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$, $k \in [\![1, N]\!]$.*

Here, we say that two generalized Hénon maps F and G are *conjugate near infinity* if their germs near the indeterminacy point I^- of F^{-1} and G^{-1} are conjugate (as germs).

Let us first consider a standard Hénon map

$$F\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad (z, w) \mapsto (P(z) - aw, z),$$

for some polynomial P of degree $d \geqslant 2$ and $a \in \mathbb{C}^*$. In affine coordinates centered at I^-, it can be written

$$F(t, w) = \left(\frac{t^d}{U(t) - awt^{d-1}}, \frac{t^{d-1}}{U(t) - awt^{d-1}} \right),$$

where $U(t) = t^d P\left(\frac{1}{t}\right)$. So, if $\Pi_1(u,v) = (uv, v)$, $\Pi_2(u,v) = (u, u^{d-1}v)$, one can write $F = \Pi_1 \circ \Pi_2 \circ \eta$, for

$$\eta(t,w) = \left(t, \frac{1}{U(t) - awt^{d-1}}\right).$$

Now, consider $\Pi_\gamma(u,v) = (u, uv+\gamma)$, for $\gamma \in \mathbb{C}$ and write $\frac{1}{U(t)} = \sum_{j=0}^{d-2} a_j t^j + t^{d-1}B(t)$, for B a holomorphic map. Note that $a_0 \neq 0$, since P is a polynomial of degree d. It is easily checked that

$$\eta = \Pi_{a_0} \circ \cdots \circ \Pi_{a_{d-2}} \circ \sigma,$$

for

$$\sigma(t,w) = \left(t, B(t) + \frac{aw}{U(t)\left(U(t) - awt^{d-1}\right)}\right). \tag{3.1}$$

It is clear that σ is a germ of biholomorphism, and since Π_1, Π_2 and the Π_{a_j} are blow-ups in standard charts, this makes $F = \Pi_1 \circ \Pi_2 \circ \Pi_3 \circ \sigma$, for $\Pi_3 = \Pi_{a_0} \circ \cdots \circ \Pi_{a_{d-2}}$, a Dloussky germ. Combined with Proposition 2.4, we obtain the following.

Lemma 3.2. *Let $F = F_N \circ \cdots \circ F_1$ be a generalized Hénon map, where the F_i are standard Hénon maps of degree d_i, $i \in [\![1, N]\!]$. Then, the germ at infinity of F is a Dloussky germ inducing a Kato surface S_F with $b_2(S_F) = \sum_{k=1}^{N}(2d_k - 1)$.*

Denote by $n = b_2(S_F)$ and by $D_1, \ldots, D_n$ the n compact curves which are the projection of the exceptional divisors in S_F. For $i \in [\![1, n]\!]$, denote by $D_i \cdot D_i$ the number of self-intersections of D_i.

Lemma 3.3. *With the notations of Lemma 3.2, the self-intersection profile of S_F, that is the n-tuple $(-D_1 \cdot D_1, \ldots, -D_n \cdot D_n)$ is*

$$(d_1, \underbrace{2, \ldots, 2}_{2d_1 - 3 \text{ times}}, 3, d_2, \underbrace{2, \ldots, 2}_{2d_2 - 3 \text{ times}}, 3, \ldots, d_N \underbrace{2, \ldots, 2}_{2d_N - 3 \text{ times}}, 3).$$

Proof. This is essentially counting and following strict transforms of exceptional divisors. Consider first a standard Hénon map and keep notations above Lemma 3.2. The map Π_2 consists of blowing up $d-1$ times a point of the strict transform of D_1. Indeed, the strict transform of D_1 is defined in coordinates by $\{v = 0\}$. Next, Π_3 blows up a point which belongs to D_d, but not to any other divisor since $a_0 \neq 0$. All the other maps of Π_3 blow up a point that is only in the last appeared exceptional divisor. Finally, $D_n = \{u = 0\}$ in the last coordinates, and by (3.1) is glued with $\{u = 0\}$ via $\sigma \circ \Pi$ in the Kato surface. Therefore, it will be blown up by Π_1, by the first blow-up of Π_2 and will be left alone after. We obtain the result for standard Hénon maps.

We let the reader check the result for generalized Hénon maps, using Proposition 2.4 and the form of the map σ in (3.1) to follow the last exceptional divisor of each standard Hénon map. □

Before concluding the section with the proof of Proposition 3.1, let us mention the following consequence of the above computation, that we will use in the next section.

Lemma 3.4. *Let $F = F_N \circ \cdots \circ F_1$ and $G = G_P \circ \cdots \circ G_1$ be generalized Hénon maps, with $F_N, \ldots, F_1, G_P, \ldots, G_1$ standard Hénon maps. Suppose that F and G are conjugate near infinity. Then, $N = P$ and $\deg F_k = \deg G_k$, for each $k \in [\![1, N]\!]$.*

Proof of Proposition 3.1. Suppose that G is conjugate to $F_k \circ \cdots \circ F_1 \circ F_N \circ \cdots \circ F_{k+1}$ near infinity. Then, Proposition 2.4 clearly implies that S_F is biholomorphic to S_G.

Conversely, suppose that S_F and S_G are biholomorphic. We apply Proposition 2.5 and use its notations. Then, we should have $D_j \cdot D_j = E_{j+k} \cdot E_{j+k}$, for $j \in [\![1, n-k]\!]$ and $D_j \cdot D_j = E_{j+k-n} \cdot E_{j+k-n}$, for $j \in [\![n-k+1, n]\!]$. Now, we use Lemma 3.3. Note that the 3 which appear in our writing (that is, not the d_i which could also be equal to 3) are the only one which are (cyclically) preceeded by a 2. Therefore, they are sent one to another by φ. Since G_n is conjugate to G, we obtain the result by applying once again Proposition 2.5. □

4. The germ at infinity of Hénon maps

In this section, we prove the following. Combined with Proposition 3.1, we obtain Theorem 1.1.

Theorem 4.1. *Two generalized Hénon maps are conjugate near infinity if and only if they are affinely conjugate.*

Our proof involves computing the normal form of a generalized Hénon map's germ near infinity in the sense of Favre [14]. We follow the three steps of his proof to obtain one particular normal form and then use his criterion to see which one are conjugate.

First, let us introduce some notations. Let $F = F_N \circ \cdots \circ F_1$ be a generalized Hénon map, where the F_i, $i \in [\![1, N]\!]$ are standard Hénon maps. That is, $F_i(z, w) = (P_i(z) - a_i w, z)$, $(z, w) \in \mathbb{C}^2$. Denote by $d_i = \deg P_i$ and $D_i = \prod_{j=1}^{i} d_j$, $i \in [\![1, N]\!]$. Note that $D_N = \deg F$. By convention, set also $D_0 = 1$. We will suppose that the polynomials P_i, $i \in [\![1, N]\!]$, are monic and centered, *i.e.* $P_i(z) = z^{d_i} + O\left(z^{d_i - 2}\right)$. In fact, replacing F by an affine conjugate, we can always do so [17]. Note $Q_i(z, w)$, $i \in [\![0, N]\!]$ the polynomials in two variables so that

$$F_i \circ \cdots \circ F_1(z, w) = (Q_i(z, w), Q_{i-1}(z, w)), \ i \in [\![1, N]\!].$$

It is easy to see that they are defined by the induction relation

$$Q_0 = z, \quad Q_1 = P_1(z) - a_1 w; \qquad Q_i = P_i \circ Q_{i-1} - a_i Q_{i-2}, \ i \in [\![2, N]\!]. \tag{4.1}$$

Denote by $V_i(t, w) = t^{D_i} Q_i\left(\frac{1}{t}, \frac{w}{t}\right)$. In affine coordinates centered at infinity, the map F is of the form

$$F(t, w) = \left(\frac{t^{D_N}}{V_N(t, w)}, \frac{t^{D_N - D_{N-1}} V_{N-1}(t, w)}{V_N(t, w)} \right).$$

To begin our way towards the normal form, we use Böttcher coordinates. Denote by $(t_n, w_n) = F^n(t, w)$ and

$$\phi(t, w) = (\varphi_w(t), w), \quad \varphi_w(t) = t \prod_{n \in \mathbb{N}} \left(V(t_n, w_n)\right)^{-1/D_N^{n+1}}.$$

Here and everywhere after, we consider the principal branch of the D_N^{n+1}-th root.

Lemma 4.2. *The map ϕ conjugates F with $F^{(1)}$ (that is $\phi \circ F \circ \phi^{-1} = F^{(1)}$) of the form*

$$F^{(1)}(x, w) = \left(x^{D_N}, \frac{A}{D_N} w x^{2D_N - 2}(1 + \eta(x, w)) + h(x) \right),$$

with $A = \prod_{i=1}^{N} a_i$, $\eta(x,w) = O(x^2, xw)$ *and*

$$h(x) = x^{D_N} \varphi_0^{-1}(x)^{-D_{N-1}} V_{N-1}\left(\varphi_0^{-1}(x), 0\right).$$

Proof. It is well known that the expression of φ_w converges (see [18] for standard Hénon maps), and $t_n = O\left(t^{D_N}\right)$, $w_n = O\left(t^{D_N - D_{N-1}}\right)$, the O being uniform in n in a neighbourhood of 0. Hence,

$$\varphi_w(t) = t\left(V_N(t,w)\right)^{-1/D_N}\left(1 + O\left(t^{D_N + D_{N-1}}\right)\right). \tag{4.2}$$

Moreover, these coordinates are designed to have $\pi_1 \circ \phi \circ F(t,w) = \varphi_w(t)^{D_N}$, π_1 being the standard projection on the first coordinate. So, the map $F^{(1)}$ is of the form

$$F^{(1)}(x,w) = \left(x^{D_N}, \lambda w x^q (1 + \eta(x,w)) + h(x)\right),$$

for some $\lambda \in \mathbb{C}^*$ and $q \in \mathbb{N}$ that we need first to identify. Note that the differentials of the F_i are given by

$$DF_i(t,w) = \begin{pmatrix} d_i t^{d_i - 1}\left(1 + O\left(t^2, tw\right)\right) & a_i t^{2d_i - 1}\left(1 + O\left(t^2, tw\right)\right) \\ \left(d_i - 1\right) t^{d_i - 2}\left(1 + O\left(t^2, tw\right)\right) & a_i t^{2d_i - 2}\left(1 + O\left(t^2, tw\right)\right) \end{pmatrix}.$$

Therefore, using (4.2) and the fact that $V_N(t,w) = 1 + O\left(t^2, tw\right)$,

$$\begin{aligned} \det\left(D\left(\phi \circ F \circ \phi^{-1}\right)(x,w)\right) &= \prod_{i=1}^{N} \det\left(DF_i\left(F_{i-1} \circ \cdots \circ F_1(x,w)\right)\right)\left(1 + O\left(x^2, xw\right)\right) \\ &= \prod_{i=1}^{N} a_i \left(x^{D_{i-1}}\right)^{3d_i - 3}\left(1 + O\left(x^2, xw\right)\right) \\ &= A x^{3D_N - 3}\left(1 + O\left(x^2, xw\right)\right). \end{aligned}$$

Now, since $\det\left(DF^{(1)}(x,w)\right) = \lambda D_N x^{q + D_N - 1}\left(1 + \eta(x,w) + w\frac{\partial \eta}{\partial w}(x,w)\right)$, we have $\lambda = \frac{A}{D_N}$, $q = 2D_N - 2$ and $\eta(x,w) = O\left(x^2, xw\right)$. Finally, $h(x) = \pi_2 \circ \phi \circ F \circ \phi^{-1}(x,0)$, where π_2 is the standard projection on the second coordinate. Thus, remarking that $\frac{\varphi_0^{-1}(x)^{D_N}}{V_N\left(\varphi_0^{-1}(x), 0\right)} = x^{D_N}$, we obtain the form wanted. □

Next, denote by

$$\varphi_{0,i}(t) = t\left(V_i(t,0)\right)^{-1/D_i} \text{ and } \psi_i = \varphi_{0,i} \circ \varphi_{0,i-1}^{-1}, \quad i \in [\![1, N]\!]. \tag{4.3}$$

By (4.2), we have $\varphi_0(t) = \varphi_{0,N}(t)\left(1 + O\left(t^{D_N + D_{N-1}}\right)\right)$ and by Lemma 4.2,

$$h(x) = x^{D_N}\left(\psi_N^{-1}(x)\right)^{-D_{N-1}}\left(1 + O\left(x^{D_N + D_{N-1}}\right)\right). \tag{4.4}$$

We continue our way towards the normal form.

Lemma 4.3. *Keep the notations of Lemma 4.2. The germ $F^{(1)}$ is conjugate to the germ*

$$F^{(2)}(x,y) = \left(x^{D_N}, \frac{A}{D_N} y x^{2D_N - 2} + \widetilde{h}(x)\right),$$

with $\widetilde{h}(x) = h(x)\left(1 + O\left(x^{D_N + D_{N-1}}\right)\right)$.

Proof. We use the work of Favre [14, pp.491–494]. He shows that we can find a map Ψ of the form $\Psi(x,w) = (x, w(1+\psi(x,w)))$ such that $\Psi \circ F^{(1)} = F^{(2)} \circ \Psi$. Moreover, he gives an explicit expression for ψ in the form of a series. Define the operator

$$T\psi(x,w) = (1+\eta(x,w))\,\psi \circ F^{(1)}(x,w) + h(x)\int_0^1 (1+\eta_1(x,wt))\,\frac{\partial \psi}{\partial w} \circ F^{(1)}(x,wt)dt,$$

for $\eta_1 = \eta + w\frac{\partial \eta}{\partial w}$. Then, $\psi = \sum_{k\in\mathbb{N}} T^k\eta$. Obviously, if $f = O\left(x^2, xw\right)$, then $Tf = O\left(x^2\right)$. Actually, the composition with $F^{(1)}$ makes degree grow far faster. Therefore, $\psi(x,w)$ is a $O\left(x^2, xw\right)$. Now, this gives

$$\widetilde{h}(x) = \pi_2 \circ F^{(2)} \circ \Psi(x,0) = \pi_2 \circ \Psi \circ F^{(1)}(x,0) = h(x)\left(1+\psi\left(x^{D_N}, h(x)\right)\right)$$
$$\widetilde{h}(x) = h(x)\left(1+O\left(x^{D_N+D_{N-1}}\right)\right).$$

□

We finish the computation of the normal form with the third step of Favre [14, pp.496–498].

Lemma 4.4. *Keep the notations of Lemma 4.3 and write $\widetilde{h}(x) = \sum_{n\in\mathbb{N}^*} b_n x^n$. Then, $F^{(2)}$ is conjugate with the normal form*

$$F^{(3)}(x,y) = \left(x^{D_N}, \frac{A}{D_N} y x^{2D_N-2} + g(x)\right),$$

where

$$g(x) = \sum_{n=1}^{2D_N-2} b_n x^n + \frac{D_N}{A} b_{2D_N-1} x^{D_N} + \widetilde{b}_{2D_N} x^{2D_N},$$

with $\widetilde{b}_{2D_N} = 0$ if $A \neq D_N$ and $\widetilde{b}_{2D_N} = b_{2D_N}$ if $A = D_N$.

Now, we will show the following.

Proposition 4.5. *The map $F \mapsto F^{(3)}$, obtained by combining Lemmas 4.2, 4.3 and 4.4, is injective.*

That is, if two generalized Hénon maps are given by monic and reduced polynomials as in the beginning of the section, and if the process of the lemmas gives the same normal form, then they are equal. Note that the statement is *not* up to conjugacy (neither for F nor for $F^{(3)}$). That's why we have to specify from where we obtain our normal form. For clarity's sake, we begin the proof by some lemmas, where we still work on one map F.

Lemma 4.6. *Recall the maps ψ_i from* (4.3) *and denote by $U_i(t) = t^{d_i} P_i\left(\frac{1}{t}\right)$. Then,*

$$\psi_i(x)^{D_i} = \frac{x^{D_i}}{U_i\left(x^{D_{i-1}}\right) - a_i x^{D_i}\left(\psi_{i-1}^{-1}(x)\right)^{-D_{i-2}}}, \quad i \in [\![2, N]\!].$$

Proof. This comes from the induction relation between the V_i. Note that

$$\psi_i(x)^{D_i} = \frac{\left(\varphi_{0,i-1}^{-1}(x)\right)^{D_i}}{V_i\left(\varphi_{0,i-1}^{-1}(x), 0\right)} = \frac{1}{Q_i\left(\frac{1}{\varphi_{0,i-1}^{-1}(x)}, 0\right)},$$

by definition of the V_i. Now, using (4.1),

$$\psi_i(x)^{D_i} = \frac{1}{P_i\left(Q_{i-1}\left(\frac{1}{\varphi_{0,i-1}^{-1}(x)}, 0\right)\right) - a_i Q_{i-2}\left(\frac{1}{\varphi_{0,i-1}^{-1}(x)}, 0\right)}.$$

Coming back from Q to V , we get

$$\psi_i(x)^{D_i} = \frac{1}{P_i\left(\left(\varphi_{0,i-1}^{-1}(x)\right)^{-D_{i-1}} V_{i-1}\left(\varphi_{0,i-1}^{-1}(x), 0\right)\right) - a_i\left(\varphi_{0,i-1}^{-1}(x)\right)^{-D_{i-2}} V_{i-2}\left(\varphi_{0,i-1}^{-1}(x), 0\right)}.$$

Now, in the denominator we recognize $\varphi_{0,i-1}^{-D_{i-1}}$ and $\varphi_{0,i-2}^{-D_{i-2}}$. Therefore,

$$\psi_i(x)^{D_i} = \frac{1}{P_i\left(\frac{1}{x^{D_{i-1}}}\right) - a_i\left(\psi_{i-1}^{-1}(x)\right)^{-D_{i-2}}} = \frac{x^{D_i}}{U_i\left(x^{D_{i-1}}\right) - a_i x^{D_i}\left(\psi_{i-1}^{-1}(x)\right)^{-D_{i-2}}}. \qquad \square$$

Lemma 4.7. *We have the following approximation of the ψ_i.*

$$\psi_i(x) = x\left(1 + \frac{a_i}{D_i} x^{D_i - D_{i-2}} + O\left(x^{2D_{i-1}}\right)\right), \quad i \in [\![2, N]\!],$$
$$\psi_1(x) = x\left(1 + O\left(x^2\right)\right).$$

In ψ_i, the term $a_i x^{D_i - D_{i-2}}$ is of course also a $O\left(x^{2D_{i-1}}\right)$ if $d_i \geqslant 3$.

Proof. Since $\psi_1(x) = x\left(V_1(x,0)\right)^{-1/d_1} = x\left(U_1(x)\right)^{-1/d_1}$, the expansion of ψ_1 is just a consequence of P_1 being monic and centered. Let i be greater or equal to 2 and suppose that the lemma is proved up to rank $i-1$. By Lemma 4.6, since ψ_i is tangent to the identity,

$$\psi_i(x) = x\left(U_i\left(x^{D_{i-1}}\right) - a_i x^{D_i}\left(\psi_{i-1}^{-1}(x)\right)^{-D_{i-2}}\right)^{-1/D_i} = x\left(1 + \frac{a_i}{D_i} x^{D_i - D_{i-2}} + O\left(x^{2D_{i-1}}\right)\right).$$

Above, we used that P_i is centered and monic, so that $U_i(x) = 1 + O\left(x^2\right)$ and

$$D_i + D_{i-1} - D_{i-2} - D_{i-3} \geqslant 2D_{i-1}.$$

We conclude by induction. $\square$

Proof of Proposition 4.5. Let F and G be two generalized Hénon maps, defined by

$$F = F_N \circ \cdots \circ F_1, \qquad G = G_N \circ \cdots \circ G_1;$$
$$F_i(z,w) = \left(P_i^{(1)}(z) - a_i^{(1)} w, z\right), \quad G_i(z,w) = \left(P_i^{(2)}(z) - a_i^{(2)} w, z\right),$$

yielding to the same normal form $(x,y) \mapsto \left(x^{D_N}, \lambda y x^{2D_N - 2} + g(x)\right)$ by Lemmas 4.2, 4.3 and 4.4. Note that Lemma 3.4 implies that indeed F and G are the composition of the same amount of standard Hénon maps, and that $\deg(F_i) = \deg(G_i)$. Denote by $\varphi_{0,i}^{(j)}$, $\psi_i^{(j)}$, for $j = 1$ (respectively $j = 2$) the maps defined in (4.3) for F (respectively for G). Denote by $h^{(1)}, \widetilde{h}^{(1)}$ (respectively $h^{(2)}, \widetilde{h}^{(2)}$) the maps obtained in Lemmas 4.2 and 4.3 for F (respectively G). If $\widetilde{h}^{(j)}(z) = \sum_{n \in \mathbb{N}} b_n^{(j)} z^n$, for $j \in \{1, 2\}$, we have by Lemma 4.4

$$\begin{aligned} g(x) &= \sum_{n=1}^{2D_N - 2} b_n^{(1)} x^n + \frac{D_N}{A} b_{2D_N - 1}^{(1)} x^{D_N} + \widetilde{b}_{2D_N}^{(1)} x^{2D_N} \\ &= \sum_{n=1}^{2D_N - 2} b_n^{(2)} x^n + \frac{D_N}{A} b_{2D_N - 1}^{(2)} x^{D_N} + \widetilde{b}_{2D_N}^{(2)} x^{2D_N}. \end{aligned}$$

Now, by (4.4) and Lemma 4.3, $\widetilde{h}^{(j)}(x) = x^{D_N}\left(\left(\psi_N^{(j)}\right)^{-1}(x)\right)^{-D_{N-1}} + O\left(x^{2D_N}\right)$. Using Lemma 4.7, we infer that $b_{D_N}^{(j)} = 0$. Therefore, if

$$g(x) = \sum_{n=1}^{2D_N-2} c_n x^n + \widetilde{c}_{2D_N} x^{2D_N},$$

$b_n^{(1)} = c_n = b_n^{(2)}$, for $n \in [\![1, 2D_N - 2]\!]$, $n \neq D_N$, and $b_{2D_N-1}^{(1)} = \lambda c_{D_N} = b_{2D_N-1}^{(2)}$. In other words, $\widetilde{h}^{(1)}(x) = \widetilde{h}^{(2)}(x) + O\left(x^{2D_N}\right)$.

Next, recall that $\widetilde{h}^{(j)}(x) = x^{D_N}\left(\left(\psi_N^{(j)}\right)^{-1}(x)\right)^{-D_{N-1}} + O\left(x^{2D_N}\right)$. It follows that we have $\psi_N^{(1)}(x) = \psi_N^{(2)}(x)\left(1 + O\left(x^{D_N+D_{N-1}}\right)\right)$. We will show that if

$$\psi_i^{(1)}(x) = \psi_i^{(2)}(x)\left(1 + O\left(x^{D_i+D_{i-1}}\right)\right), \tag{4.5}$$

for $i \geqslant 2$, then $P_i^{(1)} = P_i^{(2)}$, $a_i^{(1)} = a_i^{(2)}$ and $\psi_{i-1}^{(1)}(x) = \psi_{i-1}^{(2)}(x)\left(1 + O\left(x^{D_{i-1}+D_{i-2}}\right)\right)$. This is a consequence of Lemmas 4.6 and 4.7. Indeed, if (4.5) holds for some $i \geqslant 2$, by Lemma 4.6,

$$\frac{x^{D_i}}{\left(\psi_i^{(j)}(x)\right)^{D_i}} = U_i^{(j)}\left(x^{D_{i-1}}\right) - a_i^{(j)} x^{D_i}\left(\left(\psi_{i-1}^{(j)}\right)^{-1}(x)\right)^{-D_{i-2}}.$$

Therefore,

$$U_i^{(1)}\left(x^{D_{i-1}}\right) - a_i^{(1)} x^{D_i}\left(\left(\psi_{i-1}^{(1)}\right)^{-1}(x)\right)^{-D_{i-2}} = U_i^{(2)}\left(x^{D_{i-1}}\right) - a_i^{(2)} x^{D_i}\left(\left(\psi_{i-1}^{(2)}\right)^{-1}(x)\right)^{-D_{i-2}},$$

up to order $D_i + D_{i-1} - 1$. Since the $a_i^{(j)} x^{D_i}\left(\left(\psi_{i-1}^{(j)}\right)^{-1}(x)\right)^{-D_{i-2}}$ have no terms multiples of D_{i-1} up to D_i in their expansion by Lemma 4.7, we obtain $U_i^{(1)} = U_i^{(2)}$. Moreover, $\psi_{i-1}^{(j)}$ are tangent to the identity. Hence, $a_i^{(1)} = a_i^{(2)}$ and $\psi_{i-1}^{(1)}(x) = \psi_{i-1}^{(2)}(x)\left(1 + O\left(x^{D_{i-1}+D_{i-2}}\right)\right)$.

Using the above argument by induction, we obtain $P_i^{(1)} = P_i^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, for each $i \in [\![2, N]\!]$, and $\psi_1^{(1)}(x) = \psi_1^{(2)}(x)\left(1 + O\left(x^{d_1+1}\right)\right)$. Since $\psi_1^{(j)}(x) = x\left(U_1^{(j)}(x)\right)^{-1/d_1}$, we also get $P_1^{(1)} = P_1^{(2)}$. Finally, we have $A = \prod_{i=1}^N a_i^{(1)} = \prod_{i=1}^N a_i^{(2)}$ because $\lambda = \frac{A}{D_N}$, so that $a_1^{(1)} = a_1^{(2)}$. Hence, $F = G$ and the proposition is proved.

□

End of proof of Theorem 4.1. Again, we use the work of Favre. He shows [14, p. 482] that two normal forms like in Lemma 4.4 are conjugate if and only if $g_2(x) = bg_1(\zeta x)$, for some $b \in \mathbb{C}^*$ and $\zeta^{D_N-1} = 1$. Identifying the coefficients in $x^{D_N-D_{N-1}}$, we get $b = \zeta^{D_{N-1}-1}$ if both come from the process of the Lemmas, starting with generalized Hénon maps. It is quite easy to see that this is equivalent to $h_2(x) = \zeta^{D_{N-1}-1} h_1(\zeta x) + O\left(x^{2D_N}\right)$. By Proposition 4.5, it is enough to find one generalized Hénon map H, affinely conjugate to F, that can be written as a composition of monic and centered Hénon maps, and that yields to the normal form with g_2. Consider

$$H = \theta \circ F \circ \theta^{-1}, \quad \theta\colon (z, w) \mapsto \left(\zeta z, \zeta^{D_{N-1}} w\right).$$

That is,

$$H(z,w) = \left(\zeta Q_N\left(\zeta^{-1}z, \zeta^{-D_{N-1}}w\right), \zeta^{D_{N-1}} Q_{N-1}\left(\zeta^{-1}z, \zeta^{-D_{N-1}}w\right)\right)$$
$$H(z,w) = \left(R_N(z,w), R_{N-1}(z,w)\right).$$

Note that

$$H = \theta_N \circ F_N \circ \theta_{N-1}^{-1} \circ \theta_{N-1} \circ F_{N-1} \circ \theta_{N-2}^{-1} \circ \cdots \circ \theta_1 \circ F_1 \circ \theta_0^{-1},$$

with $\theta_i(z,w) = \left(\zeta^{D_i} z, \zeta^{D_{i-1}} w\right)$, and $D_{-1} = D_{N-1}$. In this form, we have

$$\theta_i \circ F_i \circ \theta_{i-1}^{-1}(z,w) = \left(\zeta^{D_i} P_i\left(\zeta^{-D_{i-1}} z\right) - a_i \zeta^{D_i - D_{i-2}} w, z\right),$$

which is a standard Hénon map with monic and centered polynomial. Moreover, if we denote by $W_i(t,w) = t^{D_i} R_i\left(\frac{1}{t}, \frac{w}{t}\right)$, they satisfy

$$W_N(t,0) = t^{D_N} R_N\left(\frac{1}{t}, 0\right) = t^{D_N} \zeta Q_N\left(\frac{1}{\zeta t}, 0\right) = V_N(\zeta t, 0).$$

Similarly, $W_{N-1}(t,0) = V_{N-1}(\zeta t, 0)$. If we denote by $\varphi_{0,i}^{(2)}$, $\psi_i^{(2)}$ the maps defined by (4.3) with respect to the W_i, we obtain $\varphi_{0,i}^{(2)}(t) = \zeta^{-1}\varphi_{0,i}(\zeta t)$, for $i \in \{N-1, N\}$. Therefore, $\psi_N^{(2)}(t) = \zeta^{-1}\psi_N(\zeta t)$. Using (4.4), the map $h^{(2)}$ defined by Lemma 4.2 applied to H can be written

$$h^{(2)}(x) = x^{D_N} \zeta^{D_{N-1}} \psi_N^{-1}(\zeta x)^{-D_{N-1}} + O\left(x^{2D_N}\right) = b h_1(\zeta x) + O\left(x^{2D_N}\right) = h_2(x) + O\left(x^{2D_N}\right).$$

With our first remarks, this concludes the proof. □

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